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SHORT COMMUNICATION

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Nominal geometry and force measures for solids and fluids

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Abstract

Understanding solid- and fluid-inertia forces and their coupling with the gravity potential in complex motion scenarios is necessary for evaluating system stability and identifying root causes of system failure and accidents. Because solids and fluids have an infinite number of degrees of freedom and distributed inertia and elasticity, having meaningful qualitative and quantitative nominal measures of the kinematics and forces will contribute to a better understanding of the system dynamics. This paper proposes developing new continuum-based *nominal measures* for the characterization of the oscillations and forces. By using a material-point approach, these new nominal measures, which have their roots in the continuum-mechanics partial-differential equations of equilibrium and *Frenet geometry*, are independent of the formulation or generalized coordinates used to develop the dynamic equations of motion. The paper proposes a *data-driven-science* approach to define a nominal continuum space-curve geometry with nominal curvature and torsion; a nominal *instantaneous motion plane* (IMP), which contains the resultant of all forces including the inertia forces; and a nominal *instantaneous zero-force axis* (IZFA) along which the resultant of all forces vanishes. While using the material-point approach eliminates the need for introducing moment equations associated with orientation coordinates, the IMP and IZFA concepts can be used to define the instantaneous axis of significant moment components, which can lead to accidents such as in the case of vehicle rollovers.

KEYWORDS

nominal geometry and force measures, instantaneous motion plane, instantaneous zero-force axis, Frenet geometry, nominal continuum mechanics measures

1 | INTRODUCTION

Solid- and fluid-inertia forces and their coupling with the gravity potential in complex motion scenarios are not well understood due to the lack of accurate nominal measures of the continuum kinematics and forces. Developing such nominal measures will contribute to a better understanding of the system dynamics and stability and to identifying the root causes of system failure and accidents. These nominal measures are particularly important in developing operation and safety guidelines for the transportation of hazardous materials (HAZMAT), where the effect of liquid-sloshing oscillations on vehicle dynamics and stability has not been thoroughly investigated and is not well understood. Solids and fluids have an infinite number of degrees of freedom and have distributed inertia and elasticity, and consequently, their dynamics is governed by high-dimensional nonlinear models. Having meaningful qualitative and quantitative low-order nominal measures of the kinematics and forces, which have dimensions independent of the model dimension, will contribute to a better understanding of the system dynamics.

Three-dimensional *recorded motion trajectories* (RMT) of continuum points can be used to define *inertia forces* whose magnitudes and directions can be more conveniently analyzed quantitatively or qualitatively by using the new nominal geometry and force measures. Such nominal measures, which can be designed to capture the effects of the actual forces and reference-configuration geometry, can also contribute to developing credible virtual prototyping approaches for accident reconstructions. The new nominal measures for the characterization of continuum oscillations and forces have their roots in the *continuum-mechanics partial-differential equations of equilibrium* and *Frenet geometry* of differential calculus.[1–5] The use of a *material-point approach* allows developing general procedures independent of the formulation or the type of generalized coordinates used to develop the dynamic equations. This allows defining a general *inverse problem* to account for the actual forces by defining new *force integrals* and *nominal motion trajectories*. Using the material-point approach and equating the continuum inertia force to the sum of the applied, stress, and constraint forces lead to identifying an *instantaneous zero-force axis* (IZFA) along which no continuum forces are applied and an *instantaneous motion plane* (IMP) that includes the resultant of all forces. In the case of vehicle dynamics, an IMP with normal along the yaw axis leads to predominantly yaw moment, an IMP with normal along the pitch axis leads to predominantly pitch moment, and an IMP with normal along the roll axis leads to predominantly roll moment; such a roll moment can cause rollover.

The new nominal measures, which are given clear physical interpretation using new concepts such as *Frenet angles* for describing curve geometry, can contribute to a better understanding of the effect of the coupling between gravity potential, inertia forces, and reference-configuration geometry on the continuum oscillations. *Frenet bank angle*, in particular, defines IMP *superelevation*. In the case of general motion scenarios, the centrifugal forces do not lie, in general, in a plane parallel to the horizontal plane, as it is normally assumed when developing operation and safety guidelines such as *balance speeds*. Therefore, the analysis of RMT using new *data-driven-science* (DDS) *multibody system* (MBS) algorithms can shed light on new precise force balance and balance–speed definitions. This short paper explains the procedure for developing the nominal continuum-based geometry and force measures using a material-point approach independent of the type of formulation or generalized coordinates used to develop the continuum dynamic equations of motion. The new nominal measures can be considered as a generalization of force concepts used in particle and rigid-body mechanics.[6–9]

2 | BACKGROUND

The material-point approach used in this investigation is based on the partial differential equations of equilibrium of a continuum, which has an infinite number of degrees of freedom, $\rho\ddot{\mathbf{r}} = \mathbf{f}_b + \mathbf{f}_s + \mathbf{f}_e$,[10,11] and Frenet geometry that defines the Frenet frame transformation $\mathbf{A}_f = [\mathbf{t}\ \ \mathbf{n}\ \ \mathbf{b}]$,[1–5] where ρ is the mass density; $\mathbf{f}_b$, $\mathbf{f}_s$, and $\mathbf{f}_e$ are, respectively, vectors of body, stress, and other applied forces that may include contact and constraint forces; and $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ are, respectively, the tangent, normal, and binormal vectors of a space curve. To define a general *inverse problem* and develop the continuum nominal force measures, a material-point approach is adopted regardless of the generalized coordinates used to formulate the dynamic equations. This approach is to be distinguished from writing $\mathbf{r} = \mathbf{r}(\mathbf{q})$, where $\mathbf{q}$ is the vector of generalized coordinates, which can have orientation parameters giving rise to the definition of moments.[6–8,12] For example, in the case of the *absolute nodal coordinate formulation* (ANCF), finite rotations are not used as nodal coordinates.[11] For ANCF finite elements, the global position of an arbitrary point on an element of a body i can be written as $\mathbf{r}^i(\mathbf{x}, t) = \mathbf{S}^i(\mathbf{x})\mathbf{e}^i(t)$, where $\mathbf{S}^i(\mathbf{x})$ is the FE shape-function matrix that depends on the FE spatial coordinates $\mathbf{x} = [x_1\ \ x_2\ \ x_3]^{\mathrm{T}}$ and $\mathbf{e}^i(t)$ is the vector of element nodal coordinates that depends on time t. At a given node k, absolute position and position-vector gradients define the vector of nodal coordinates as $\mathbf{e}^{ik} = [\mathbf{r}^{ik\mathrm{T}}\ \mathbf{r}_{x_1}^{ik\mathrm{T}}\ \mathbf{r}_{x_2}^{ik\mathrm{T}}\ \mathbf{r}_{x_3}^{ik\mathrm{T}}]^{\mathrm{T}}$. For an arbitrary point, the position vector $\mathbf{r}^i(\mathbf{x}, t) = \mathbf{S}^i(\mathbf{x})\mathbf{e}^i(t)$, the velocity vector $\dot{\mathbf{r}}^i(\mathbf{x}, t) = \mathbf{S}^i(\mathbf{x})\dot{\mathbf{e}}^i(t)$, and the acceleration vector $\ddot{\mathbf{r}}^i(\mathbf{x}, t) = \mathbf{S}^i(\mathbf{x})\ddot{\mathbf{e}}^i(t)$ that account for all the forces acting on the system can be recorded using MBS simulations.

Therefore, regardless of the formulation and generalized coordinates used, MBS simulations of low- and high-fidelity continuum models allow recording absolute position, velocity, and acceleration of the FE integration points to define the continuum *inverse problem* developed in this paper. A three-dimensional space curve can be written in a parametric form as $\mathbf{r} = \mathbf{r}(t)$, where t is the curve parameter, which can be considered time.[1–5] In general, the motion–trajectory curve $\mathbf{r} = \mathbf{r}(t)$ is a general three-dimensional curve with non-zero curvature and torsion. The acceleration vector of a point can be written as $\ddot{\mathbf{r}} = \ddot{s}\mathbf{t} + (\dot{s}^2/R)\mathbf{n}$, where $\mathbf{t} = \mathbf{r}_s = \partial\mathbf{r}/\partial s$ is the unit tangent to the curve, $\mathbf{n}$ is the curve normal, R is the radius of curvature, and s is the curve arc length. As discussed in Refs. 13, 14, the normal vector, which defines centrifugal-force direction, can be written in terms of Frenet angles as

$$\mathbf{n} = [-\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi \quad \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi \quad -\cos\theta\sin\phi]^{\mathrm{T}}, \tag{1}$$

where ψ, θ, and ϕ are, respectively, *Frenet curvature*, *vertical-development*, and *bank angles*. The gravity component along the normal vector is $m_p g\cos\theta\sin\phi$, where $m_p = \rho dV$ is an infinitesimal mass and g is the gravity constant. Absolute position, velocity, and accelerations of FE integration points lie in the $\mathbf{t} - \mathbf{n}$ IMP, in which the resultant of all other forces acting on the continuum point lies. The *Frenet binormal vector* $\mathbf{b}$ defines the IZFA at this point.[15] The IMP and IZFA definitions demonstrate that, when RMT are used, Frenet geometry has a clear physical interpretation associated with the actual forces applied to the system.

3 | NOMINAL GEOMETRY AND FORCE MEASURES

Point trajectories can be used to define nominal trajectories and forces that have a clear physical interpretation. Using motion trajectories recorded at FE integration points, continuum inertia forces can be computed as

$$\mathbf{F}_i = \int_V \rho \ddot{\mathbf{r}} dV = \int_V \rho (\ddot{s} \mathbf{r}_s + (\dot{s}^2/R)\mathbf{n}) dV \tag{2}$$

which can be written as $\mathbf{F}_i = \mathbf{F}_{it} + \mathbf{F}_{ic} = F_{it}\mathbf{t}_c + F_{ic}\mathbf{n}_c$, where

$$\left.\begin{aligned} \mathbf{F}_{it} &= F_{it}\mathbf{t}_c = \int_V \rho(\ddot{s}\,\mathbf{r}_s) dV \\ \mathbf{F}_{ic} &= F_{ic}\mathbf{n}_c = \int_V \rho((\dot{s}^2/R)\mathbf{n}) dV \end{aligned}\right\} \tag{3}$$

$\mathbf{t}_c$ is a nominal continuum tangent vector and $\mathbf{n}_c$ is a nominal continuum normal vector. The $\mathbf{t}_c - \mathbf{n}_c$ plane defines the continuum nominal IMP and the normal to this plane, $\mathbf{b}_c = \mathbf{t}_c \times \mathbf{n}_c$, defines the nominal IZFA. The scalars F_{it} and F_{ic} can be used with proper integration methods to define continuum nominal curve geometric properties such as radius of curvature R_c, curvature κ_c, and arc length s_c following a procedure similar to the one described in the literature.[13,14] Furthermore, Frenet angles can be used to define coordinates of the continuum nominal space curve $\mathbf{r}_c = \mathbf{r}_c(s_c)$.[9] These new formulations, and numerical procedures will define new continuum *nominal force and geometry measures*. The centrifugal inertia force can be used to determine the *nongyroscopic inertia moment* associated with wheel/road or wheel/rail contact points,[9] thereby introducing a new continuum-based force balance to quantify the effect of the continuum oscillations on the system's nonlinear dynamics and stability. In the case of vehicle dynamics, an IZFA parallel to the roll axis produces a predominantly roll moment that can cause rollover, an IZFA parallel to the pitch axis produces a predominantly rocking moment, and an IZFA parallel to the yaw axis produces a predominantly sway moment. Furthermore, the continuum-nominal tangent vector $\mathbf{t}_c$ can be written in terms of the continuum-nominal Frenet angles as[9,13,14]

$$\mathbf{t}_c = d\mathbf{r}_c/ds_c = [\cos\psi_c \cos\theta_c \quad \sin\psi_c \cos\theta_c \quad \sin\theta_c]^{\mathrm{T}}, \tag{4}$$

where subscript c refers to the nominal-continuum variables. Knowing nominal tangent vector $\mathbf{t}_c$, numerical integration can be used to determine the continuum-nominal space curve. The continuum-nominal bank angle ϕ_c defines the continuum IMP superelevation.[9,13,14]

The force integrals obtained using the material-point approach used in this paper should be distinguished from the generalized forces obtained when the system generalized coordinates are used. For example, the integration of both sides of the equation $\rho\ddot{\mathbf{r}} = \mathbf{f}_b + \mathbf{f}_s + \mathbf{f}_e$ over the volume leads to three-dimensional vectors regardless of the number of generalized coordinates used. Internal forces, which are equal in magnitude and of opposite directions, such as elastic forces, will cancel when performing the integration. The effect of such internal forces, however, on bodies that are in contact with the solids and fluids is taken into consideration since the RMT account for the effect of all applied forces including the contact forces.

4 | SUMMARY

Geometry plays a fundamental role in developing credible prototyping algorithms and in the definition of continuum oscillations and forces. Continuum inertia forces are equal to the sum of the applied, stress, and constraint forces. The IMP includes the resultant of all forces, and the IMP normal vector defines the moment axis of the resultant force. An IMP normal vector parallel to the roll axis leads to predominantly roll moment, an IMP normal vector parallel to the pitch axis leads to predominantly pitch moment, and an IMP normal vector parallel to the yaw axis leads to predominantly yaw moment. Furthermore, the IMP orientation defines the effect of *gravity potential* on the force balance. Using the DDS material-point approach, the continuum-mechanics partial-differential equations of equilibrium, and RMT at FE integration points, an *inverse problem* can be solved and used to define nominal inertia force integrals and identify nominal continuum tangential and centrifugal inertia forces. A nominal IMP and IZFA can be defined for the continuum, and nominal IMP superelevation that enters into the definition of the force balance and its dependence on the gravity force can be determined. The material-point approach proposed for the analysis of the motion trajectories is independent of the formulation and generalized coordinates used to develop the dynamic equations of motion.

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CONFLICT OF INTEREST

The author declares no conflict of interest.

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REFERENCES

1. Farin G. *Curves and Surfaces for CAGD, A Practical Guide*. 5th ed. Morgan Kaufmann, Publishers; 1999.
2. Gallier J. *Geometric Methods and Applications: For Computer Science and Engineering*. Springer; 2011.
3. Goetz A. *Introduction to Differential Geometry*. Addison Wesley; 1970.
4. Kreyszig E. *Differential Geometry*. Dover Publications; 1991.
5. Rogers DF. *An Introduction to NURBS with Historical Perspective*. Academic Press; 2001.
6. Greenwood DT. *Principles of Dynamics*. 2nd ed. Prentice Hall; 1988.
7. Roberson RE, Schwertassek R. *Dynamics of Multibody Systems*. Springer-Verlag; 1988.
8. Shabana AA. *Computational Dynamics*. 3rd ed. Wiley & Sons; 2010.
9. Shabana AA. *Mathematical Foundation of Railroad Vehicle Systems: Geometry and Mechanics*. Wiley & Sons; 2021.

10. Bonet J, Wood RD. *Nonlinear Continuum Mechanics for Finite Element Analysis*. Cambridge University Press; 1997.
11. Shabana AA. *Computational Continuum Mechanics*. 2nd ed. Cambridge University Press; 2012.
12. Goldstein H. *Classical Mechanics*. Addison-Wesley; 1950.
13. Shabana AA, Ling H. Characterization and quantification of railroad spiral-joint discontinuities. *Mech Based Des Struct Mach*. 2022; 50(2022):1-25.
14. Shabana AA. Frenet oscillations and Frenet-Euler angles: curvature singularity and motion-trajectory analysis. *Nonlinear Dyn*. 2021;106: 1-19.
15. Shabana A. Frenet force analysis of motion trajectories and instantaneous motion plane. Report. University of Illinois at Chicago; 2022.

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