



An overview of the ANCF approach, justifications for its use, implementation issues, and future research directions

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Abstract

This paper presents an overview of the finite-element (FE) *absolute nodal coordinate formulation* (ANCF), provides justifications for its use, and discusses issues relevant to its proper computer implementation and interpretation of its numerical results. The paper discusses future research directions for using ANCF finite elements in new areas such as soft tissues and materials relevant to broader areas of computational engineering and science. Selection of coordinates, definitions of forces and moments, geometric interpretation of the position gradients, and noncommutativity of finite rotations are among the topics discussed. To address concerns associated with finite-rotation noncommutativity and definition of moments in flexible-body dynamics, the paper demonstrates that the interpolation order is not preserved when the finite-rotation sequence is changed. Position gradients, on the other hand, are unique and preserve the highest interpolation order. It is shown that, while the *spin tensor* used to define the ANCF generalized forces due to moment application is associated with a rigid frame defined by the *polar decomposition theorem*, explicit polar decomposition of the matrix of position-gradient vectors is not required. ANCF elements have features that distinguish them from conventional finite elements and make them suited for large-displacement analysis of multibody systems (MBS). Their displacement fields, which allow increasing interpolation order without increasing number of nodes or using noncommutative finite rotations, are the basis for developing lower-dimension *consistent rotation-based formulations* (CRBF) without lowering the interpolation order. Nonetheless, the continuum-kinematic description of fully parameterized ANCF elements cannot be ignored when interpreting the ANCF numerical results. This issue is particularly important when comparing ANCF results with solutions obtained using semi-continuum conventional beam and plate models and simplified analytical approaches.

Keywords Absolute nodal coordinate formulation (ANCF) · Computer implementation · Noncommutativity of finite rotations · Consistent rotation-based formulations (CRBF) · Position-gradient vectors

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1 Introduction

Multibody system (MBS) research is focused on developing general formulations and computer algorithms for the nonlinear dynamic analysis of systems that consist of interconnected rigid and deformable bodies. MBS algorithms and general-purpose computer software are used in a wide range of research and industry applications, some of which are not known to the formulation or software developers. For this reason, simplifications, linearization, and numerical adjustments to solve fundamental formulation problems are avoided when developing MBS algorithms to achieve high degree of generality and robustness required for using such algorithms in new application areas unknown to the software developers. Furthermore, MBS equations of motion do not assume the simple structure based only on mass and stiffness matrices, as it is the case with conventional commercial FE software. The geometric nonlinearities in MBS applications are not limited to nonlinearities that arise only from using nonlinear strain–displacement relationships and/or material nonlinearities. In MBS applications, in addition to such elastic nonlinearities, geometric nonlinearities are also attributed to high-speed spinning motion and highly nonlinear kinematic constraint equations that describe mechanical joints and specified motion trajectories. The nonlinear kinematic constraint equations lead to geometric nonlinearities that pose serious challenges when solving the system equations of motion since the penalty approach is not recommended for modeling mechanical joints. Instead, MBS algorithms aim at solving system of *differential/algebraic equations* (DAEs). Furthermore, the *incremental-rotation solution procedures* widely used in commercial FE software for solving nonlinear large-displacement problems are avoided in developing general-purpose MBS algorithms. Avoiding simplifications, linearization, and numerical adjustments to solve fundamental formulation problems has been the motivation for using particular *finite element* (FE) formulations for the deformation analysis in MBS applications. For the small-deformation problems, the FE *floating frame of reference* (FFR) formulation has been widely used. For large-deformation problems, the *absolute nodal co-ordinate formulation* (ANCF), focus of this overview article, has emerged as a popular approach for implementation in MBS algorithms.

ANCF finite elements were introduced to alleviate known limitations of conventional finite elements and address problems associated with using noncommutative finite rotations as nodal coordinates. Recent research activities demonstrate growing interest in adopting ANCF finite elements to solve challenging problems in wide range of engineering and science applications. Because of the difficulty of having a comprehensive list of all ANCF investigations, examples that demonstrate the scope of ANCF applications are cited in this paper and briefly discussed in Sect. 7 [1–211]. Nonetheless, verification and validation of ANCF results require proper implementation of its finite elements and solution procedures.

ANCF was introduced in 1996 in a technical report intended to clarify differences between the FE co-rotational approach and approaches used in MBS algorithms [212]. Richard Schwertassek of the German Aerospace Agency (DLR) reviewed the technical report, recognized the ANCF potential, and used it as the basis of a diploma thesis written by a student from University of Stuttgart, Stefan von Dombrowski, whose research was supervised by Schwertassek and the author of this article. The diploma thesis by von Dombrowski, which documented the first set of ANCF results [213], showed superior ANCF performance in solving static problems, in comparison with commercial FE software. Without von Dombrowski's results, it was likely that ANCF research could have been significantly delayed due to lack of recognition of the potential of this approach.

This paper presents an overview of the ANCF approach and provides justifications for using its finite elements which have specific features that distinguish them from conven-

tional finite elements and make them suited for MBS large-displacement analysis. The paper provides background materials in Sect. 2 and reviews the ANCF displacement field and discusses some of its important features in Sect. 3. The position-gradients and their relationships to rotations and moments are discussed in Sect. 4 to provide basic materials required for the ANCF definitions of forces and moments.

Section 5 provides justifications for using the ANCF approach that include avoiding non-commutative finite rotations which are not directly associated with moments or deformation modes, and their independent interpolation is not consistent with the continuum-mechanics description. Section 5 also explains the need for using ANCF position gradients which preserve the interpolation order; ensure continuity of rotation, strain, and stress fields; allow using non-incremental solution procedures; lead to constant mass matrix and zero centrifugal and Coriolis forces; capture coupled deformation modes and geometric stiffening effects that cannot be captured by existing FE approaches; serve as the basis for developing lower-order consistent rotation-based formulations (CRBF); allow using general continuum-mechanics approach for structural elements; capture accurately the reference-configuration geometry; lead to the same displacement fields for straight and curved geometries; achieve higher degree of element-interface conformity; allow for defining more general boundary conditions and constraint equations; and can be used to simultaneously control the motion and shape.

In Sect. 6, implementation issues critical for sound interpretation and verification of the ANCF numerical results are discussed. The continuum kinematic description of fully parameterized ANCF elements cannot be ignored when interpreting the ANCF numerical results and comparing these results with results of conventional semi-continuum beam and plate models. Among the implementation issues discussed in Sect. 6 are the need for both FFR and ANCF approaches; ANCF Cholesky coordinates and sparse matrix implementation; preprocessor linear constraints; numerical integration of the stress forces; fully parameterized and gradient-deficient elements; definition of forces and moments; gradient transformation versus conventional vector transformation; coordinate reduction; use of explicit and implicit numerical integration methods; and choice of the reference solutions.

In Sect. 7, and before identifying future research directions in Sect. 8 to address challenges in different areas of computational science and engineering, past ANCF research activities are briefly reviewed. The research directions discussed in Sect. 8 include element technology; description of shell geometry; modeling composite structures; motion, shape, and stress control; geometric stiffening and mode coupling; locking alleviation; evaluation of the assumptions of the classical approaches; developing new implicit numerical integration methods; implementation of thermo-elasticity, viscoelasticity, and plasticity models in MBS algorithms; developing contact-mechanics algorithms; design optimization and sensitivity analysis; and application areas and result verifications. Concluding remarks are provided in Sect. 9. This paper is intended as an overview paper to provide assessment of and justification for using ANCF finite elements; the paper is not intended as a comprehensive review paper of ANCF contributions.

2 Background

This section provides background materials to explain the need for the departure from the conventional FE approach when considering MBS applications. The section discusses issues related to using higher-order elements, noncommutativity of the finite rotations, definition of moments, and infinitesimal-rotation finite elements.

2.1 Higher-Order Elements

Higher-order elements do not always imply FE meshes with higher dimensions or larger number of coordinates. Convergence to correct and smoother solutions may require using large number of low-order elements as compared to higher-order elements [214, 215]. For example, large number of linear or bilinear elements is required to describe bending deformations and such large number of elements does not achieve the desired rotation and stress continuity at the nodes. On the other hand, higher-order elements based on cubic interpolations can describe bending deformations with much smaller number of elements. Using higher-order elements is common in the FE literature as evident by using the 20-node solid element in commercial FE software as an alternative to the 4-node solid element which exhibits poor performance in some applications. The ANCF approach allows increasing interpolation order without increasing number of nodes or using noncommutative finite rotations. Such an approach also allows developing *consistent rotation-based formulations* (CRBF) which are the basis for lower-dimension ANCF/CRBF finite elements.

2.2 Noncommutativity of Finite Rotations

Finite rotations are not commutative; are not associated directly with deformation modes; cannot be treated as vectors, and therefore, cannot be interpolated; and should be treated as generalized coordinates since they are not directly associated with Cartesian moments. To address these issues and to highlight the concerns regarding finite-rotation noncommutativity and definition of moments in flexible-body dynamics, the paper demonstrates that the order of interpolation is not preserved when the sequence of rotations changes. In fact, linear interpolation of one set of Euler angles associated with a rotation sequence leads to infinite-order interpolation of another set of Euler angles associated with another rotation sequence. The fact that finite rotations should be interpreted as generalized coordinates and are not directly associated with particular deformation modes is clear from the $X_3 - X_1 - X_3$ rotation sequence used by Euler himself to study the gyroscopic motion. Excluding the X_2 axis in the rotation sequence does not imply that there are no moments associated with this axis. Moments are directly associated with the angular acceleration vector and not with the second time derivatives of finite rotations. On the other hand, the order of interpolation of the unique ANCF position gradients is preserved if the coordinate lines are changed. The paper also explains other concerns regarding use of finite rotations to describe deformation modes in computational mechanics to provide justification for using position-gradient vectors.

2.3 Forces and Moments

It is demonstrated in this paper that the definition of the *spin tensor* used to define ANCF generalized forces due to Cartesian-moment applications is associated with a rigid frame defined by the *polar decomposition theorem*. The definition of this rigid frame is particularly important in developing more general and nonlinear torsion formulations. It is shown that the definition of the generalized ANCF forces in terms of the applied Cartesian moments does not require explicit definition of the symmetric stretch tensor $\mathbf{U}$ that appears in the polar decomposition of the matrix of position-gradient vectors $\mathbf{J} = \mathbf{R}\mathbf{U}$, where $\mathbf{R}$ is the orthogonal matrix that defines the orientation of the rigid frame whose spin tensor is used to define the ANCF generalized forces.

2.4 Structural Finite Elements

Despite significant research efforts devoted to overcoming limitations of conventional approaches, with more emphasis on accurate geometric description, some of the conventional finite elements have been recently abandoned in commercial FE software because of their obvious geometric shortcomings. For example, conventional structural finite elements such as beams, plates, and shells employ infinitesimal or finite rotations as nodal coordinates [214–218]. Such a kinematic description, however, has known limitations and lacks a linear mapping to computational geometry methods [219–222]. Using infinitesimal-rotation nodal coordinates, in the case of large-displacement analysis, requires using incremental-rotation solution procedures or the FFR approach [214, 223, 224]. Use of independent interpolation of finite rotations, on the other hand, raises concerns regarding consistency with the continuum-mechanics description in which the rotations are determined from the position field [225–228]. While the isogeometric analysis (IGA) has been proposed to address known geometry shortcomings of conventional FE approaches, there are concerns regarding using the IGA approach in MBS applications, as has been discussed in the literature [214, 229].

3 ANCF continuum description

The ANCF continuum position field employs nodal coordinates that include position-gradient vectors. This position field allows describing arbitrary rigid-body displacements, captures systematically cross-section deformations, eliminates the need for using infinitesimal or finite rotations as nodal coordinates, and is related by a linear map to computational-geometry methods.

3.1 Kinematic Description

The global position vector of an arbitrary point on an ANCF finite element can be written as $\mathbf{r}(\mathbf{x}, t) = \mathbf{S}(\mathbf{x}) \mathbf{e}(t)$, where $\mathbf{S}(\mathbf{x})$ is the FE shape-function matrix that depends on the FE spatial coordinates $\mathbf{x} = [x_1 \ x_2 \ x_3]^T$, and $\mathbf{e}(t)$ is the vector of element nodal coordinates that depend on time t in case of dynamics. At a given node k , absolute position and position-vector gradients define the vector of nodal coordinates as $\mathbf{e}^k = [\mathbf{r}^{kT} \ \mathbf{r}_{x_1}^{kT} \ \mathbf{r}_{x_2}^{kT} \ \mathbf{r}_{x_3}^{kT}]^T$, where $\mathbf{r}_{x_l} = \partial \mathbf{r} / \partial x_l$, $l = 1, 2, 3$, are position-gradient vectors. For an arbitrary point, the position vector $\mathbf{r}(\mathbf{x}, t) = \mathbf{S}(\mathbf{x}) \mathbf{e}(t)$, the velocity vector $\dot{\mathbf{r}}(\mathbf{x}, t) = \mathbf{S}(\mathbf{x}) \dot{\mathbf{e}}(t)$, and the acceleration vector $\ddot{\mathbf{r}}(\mathbf{x}, t) = \mathbf{S}(\mathbf{x}) \ddot{\mathbf{e}}(t)$ can be defined and used to formulate the equations of motion of the ANCF element. Using 12 coordinates per node in the case of three-dimensional fully parameterized ANCF elements offers clear geometric and computational advantages that allow for conveniently describing reference-configuration geometry, capturing deformation modes that cannot be captured by lower-order elements, and obtaining lower-dimension FE meshes in many applications as has been demonstrated in the literature. Therefore, using higher-order elements, as previously mentioned, does not always imply an FE mesh with larger number of degrees of freedom.

3.2 Position and Displacement Gradients

Position-gradient vectors are different from displacement-gradient vectors. Position-gradient vectors are tangent to the coordinate lines x_l , $l = 1, 2, 3$, and therefore, they have different

geometric interpretation. Using position-gradient vectors allows imposing boundary conditions that cannot be conveniently imposed when using FE formulations that employ rotations as nodal coordinates. In addition to the lack of a linear map to computational geometry methods, rotation coordinates do not directly define stretches at the nodes. This is particularly important in the definition of joints and forces since zero displacement at a point does not always imply zero strains at this point. Furthermore, constraints on the position gradients automatically impose constraints on the stresses; this is an important issue in the motion and shape control of soft robots and materials.

3.3 Fully Parameterized and Gradient-Deficient Elements

There are two types of ANCF finite elements; *fully parameterized* and *gradient-deficient*. Fully parameterized elements are elements that employ complete set of element spatial coordinates allowing for developing, at the integration points, a square position-gradient matrix whose columns are all position-gradient vectors obtained by differentiation of the position vector $\mathbf{r}$ with respect to the element spatial coordinates. Gradient-deficient elements, on the other hand, do not employ complete set of spatial coordinates, and consequently, the position-gradient matrix is not square. An example of a fully parameterized element is the three-dimensional ANCF beam element for which the position vector can be written as $\mathbf{r} = \mathbf{r}(x_1, x_2, x_3)$. In this case, one can define, at integration points regardless of the nodal coordinates used, a complete and square position-gradient matrix $\mathbf{J} = [\mathbf{r}_{x_1} \quad \mathbf{r}_{x_2} \quad \mathbf{r}_{x_3}]$, allowing the use of general continuum mechanics approaches for the formulation of the stress forces. In the case of the gradient deficient ANCF cable element, only one spatial coordinate x_1 is used, and the position vector is written as $\mathbf{r} = \mathbf{r}(x_1)$. Fully parameterized elements have several advantages that include: (1) providing more flexibility and versatility in the definition of the reference-configuration geometry; (2) allowing use of general continuum-mechanics approaches for formulating the stress forces; and (3) capturing coupled deformation modes and geometric stiffening effects that cannot be captured by gradient-deficient elements. Gradient deficient elements have the advantage of being more efficient because they do not capture some deformation modes characterized by high frequencies.

3.4 Gradient Transformation

In case of discontinuities, gradient transformation is used instead of the conventional vector transformation to preserve the geometric meaning of position-gradient vectors as tangent to coordinate lines. Proper application of the gradient transformation ensures the correct application of the connectivity conditions when two elements with different orientations are connected at a node. If the spatial coordinates of one element are $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^T$ and the spatial coordinates of the second element are $\mathbf{y} = [y_1 \quad y_2 \quad y_3]^T$, then

$$\begin{aligned} \partial \mathbf{r} / \partial \mathbf{x} &= [\mathbf{r}_{x_1} \quad \mathbf{r}_{x_2} \quad \mathbf{r}_{x_3}] = (\partial \mathbf{r} / \partial \mathbf{y}) (\partial \mathbf{y} / \partial \mathbf{x}) \\ &= [\mathbf{r}_{y_1} \quad \mathbf{r}_{y_2} \quad \mathbf{r}_{y_3}] (\partial \mathbf{y} / \partial \mathbf{x}) \end{aligned} \quad (1)$$

That is, unlike the conventional finite elements, the gradient transformation is used instead of the conventional vector transformation to preserve the geometric meaning of the position gradients. In the preceding equation, the matrix of gradient transformation $\partial \mathbf{y} / \partial \mathbf{x}$ is a constant matrix defined in the initial configuration.

3.5 ANCF Element Example

An example of an element that will be used in this paper to provide justifications for using the ANCF approach and discuss implementation issues is the three-dimensional fully parameterized beam element. The element has two nodes, and each node has 12 coordinates. The vector of nodal coordinates of the three-dimensional ANCF beam element is $\mathbf{e} = [\mathbf{e}^{1T} \quad \mathbf{e}^{2T}]^T$, where $\mathbf{e}^k = [\mathbf{r}^{kT} \quad \mathbf{r}_{x_1}^{kT} \quad \mathbf{r}_{x_2}^{kT} \quad \mathbf{r}_{x_3}^{kT}]^T$, $k = 1, 2$ is the node number, $\mathbf{r}^k$ is the global position vector of node k , $\mathbf{r}_{x_l}^k = (\partial \mathbf{r} / \partial x_l)^k$, $l = 1, 2, 3$, is the position-gradient vector evaluated at node k , and $\mathbf{x} = [x_1 \quad x_2 \quad x_3]^T$ is the vector of element spatial coordinates. The element displacement field can be written as $\mathbf{r}(\mathbf{x}, t) = \mathbf{S}(\mathbf{x}) \mathbf{e}(t)$, where t is time and $\mathbf{S}$ is the element shape-function matrix defined as [214, 230]

$$\mathbf{S}^j = [s_1 \mathbf{I} \quad s_2 \mathbf{I} \quad s_3 \mathbf{I} \quad s_4 \mathbf{I} \quad s_5 \mathbf{I} \quad s_6 \mathbf{I} \quad s_7 \mathbf{I} \quad s_8 \mathbf{I}], \quad (2)$$

where the shape functions s_i , $i = 1, 2, \dots, 8$ are defined as

$$\left. \begin{aligned} s_1 &= 1 - 3(\xi)^2 + 2(\xi)^3, \quad s_2 = l(\xi - 2(\xi)^2 + (\xi)^3), \\ s_3 &= l\eta(1 - \xi), \quad s_4 = l\zeta(1 - \xi), \quad s_5 = 3(\xi)^2 - 2(\xi)^3, \\ s_6 &= l(-(\xi)^2 + (\xi)^3), \quad s_7 = l\xi\eta, \quad s_8 = l\xi\zeta \end{aligned} \right\} \quad (3)$$

and $\xi = x_1/l$, $\eta = x_2/l$, $\zeta = x_3/l$ and l is the length of the element. While the three-dimensional ANCF beam element allows stretch of the cross-section, the cross-section remains planar. The element can also be used for accurate description of arbitrary cross-section geometry including tapered beams. This element allows formulating the elastic forces using general continuum-mechanics approach since a complete set of gradient vectors can be defined at the integration points.

4 Position gradients and moment definition

The role of the position gradients in the definition of forces, moments, and rotations is important for correct ANCF implementation. Rigid-body concepts, such as equipollent systems of forces, are not directly applicable to flexible body dynamics. Furthermore, proper definitions of the forces and moment and proper interpretation of the rotations allow performing credible comparative studies required for result verifications and evaluation of the FE performance and convergence. Analytical solutions and conventional FE approaches employ simplifying assumptions that are not applicable when more general formulations are used to solve large-displacement problems.

4.1 Generalized Forces

A force vector acting at a point on a finite element leads to generalized forces associated with the ANCF position and gradient coordinates. For example, if a force vector $\mathbf{F}$ acts at a point P defined by the spatial coordinates $\mathbf{x}_P$, the virtual work of this force vector is defined as $\delta W = \mathbf{F}^T \delta \mathbf{r} = \mathbf{F}^T \mathbf{S}(\mathbf{x}_P) \delta \mathbf{e}$. This equation can be used to show that the generalized force vector $\mathbf{S}^T(\mathbf{x}_P) \mathbf{F}$ has components associated with the nodal position-gradients of the element. This fact can be demonstrated using the displacement field of the three-dimensional ANCF beam element presented in the preceding section.

4.2 Moment Definition

The definitions of moments and implementation of torsional springs, dampers, and motors require defining the relationship between the spin tensor and time derivatives of the ANCF nodal coordinates. This relationship will be developed in a later section of this paper. In rigid-body dynamics, the moments are directly associated with the angular acceleration vector and not with the second time derivative of finite rotations. Therefore, when a Cartesian moment vector is applied on an ANCF element, the concept of the angular velocity of a rigid frame of reference needs to be used. Using the polar decomposition theorem [214, 225–228], the matrix of position-gradient vectors can be written as

$$\mathbf{J} = \mathbf{J}(x_1, x_2, x_3) = [\mathbf{r}_{x_1} \quad \mathbf{r}_{x_2} \quad \mathbf{r}_{x_3}] = \mathbf{R}\mathbf{U}, \quad (4)$$

where $\mathbf{R}$ is an orthogonal matrix, and $\mathbf{U}$ is a symmetric positive-definite stretch matrix. It follows that the velocity-gradient tensor $\mathbf{L}$ can be written as [214, 225–228]

$$\mathbf{L} = \dot{\mathbf{J}}\mathbf{J}^{-1} = (\dot{\mathbf{R}}\mathbf{U} + \mathbf{R}\dot{\mathbf{U}})\mathbf{U}^{-1}\mathbf{R}^T = (\dot{\mathbf{R}}\mathbf{R}^T) + (\mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{R}^T). \quad (5)$$

Because of the orthogonality of $\mathbf{R}$, $\mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T$ is a skew-symmetric matrix and $\dot{\mathbf{R}}\mathbf{R}^T + (\dot{\mathbf{R}}\mathbf{R}^T)^T = \mathbf{0}$. It follows that $\mathbf{L} + \mathbf{L}^T = \dot{\mathbf{J}}\mathbf{J}^{-1} + (\dot{\mathbf{J}}\mathbf{J}^{-1})^T = 2\mathbf{D}$, where $\mathbf{D} = \mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{R}^T$ is the symmetric rate of deformation tensor. This analysis shows that the spin tensor $\mathbf{W}$ is associated with a rigid frame whose orientation is defined at a point by the orthogonal matrix $\mathbf{R}$ which appears in the polar decomposition theorem $\mathbf{J} = \mathbf{R}\mathbf{U}$. Implementation of moment vectors and torsional spring–damper–motor elements in MBS algorithms requires developing the relationship between the spin tensor and time derivatives of the vector of nodal coordinates $\mathbf{e}$. The analysis presented in a later section demonstrates that this relationship can be established without explicitly defining the polar decomposition or the stretch matrix $\mathbf{U}$.

5 Justifications for using ANCF elements

ANCF elements were introduced to alleviate conventional-element shortcomings rooted in the geometric description. Some important features which provide justifications and explanation for the need to further develop ANCF finite elements are discussed in this section. The justifications for using the ANCF approach include avoiding noncommutative finite rotations which are not directly associated with moments or deformation modes and their independent interpolation is not consistent with mechanics principles and continuum-mechanics description. This section also explains the need for using the position gradients which preserve the interpolation order; ensure continuity of rotation, strain, and stress fields; allow using nonincremental solution procedures, lead to constant mass matrix and zero centrifugal and Coriolis forces; capture coupled deformation modes and geometric stiffening effects that cannot be captured using existing FE approaches; serve as the basis for developing lower-order consistent rotation-based formulations (CRBF); allow using general continuum-mechanics approach for structural elements; capture accurately the reference-configuration geometry; lead to the same displacement fields for straight and curved geometries; achieve higher degree of element-interface conformity; allow defining more general boundary conditions and constraint equations; and can be used to simultaneously control the motion and shape.

5.1 Interpolation of Finite Rotations

In general, interpolation of noncommutative finite rotations violates mechanics principles, while interpolation of position-gradient vectors is allowed. This issue can be further clarified if one selects the $X_3 - X_1 - X_3$ rotation sequence used by Euler in his study of the gyroscopic motion. While Euler did not consider a rotation about the local X_2 axis, the set $X_3 - X_1 - X_3$ of Euler angles completely defines the configuration of a coordinate system in space [231]. The set $X_3 - X_1 - X_3$ of Euler angles can be written in terms of another set of Euler angles that employs a different rotation sequence by equating the elements of the transformation matrices obtained using the two sets of angles; an example of this other sequence is the sequence $X_1 - X_2 - X_3$ more widely used in computational mechanics and vehicle dynamics. Linear interpolation of the angles of the first sequence does not imply linear interpolation of the angles of the second sequence if one set of angles is written in terms of the other set using the transformation matrices expressed in terms of the two sets of angles that describe the same physical configuration [231]. In fact, the order of interpolation of the second set can be infinite, demonstrating that linear interpolation of the angles based on one sequence is not applicable to another set of angles if the sequence of rotation is slightly changed. To clarify this point, let ϕ , θ , and ψ be the angles associated with the sequence $X_3 - X_1 - X_3$ used by Euler, and θ_1 , θ_2 , and θ_3 the angles associated with the sequence $X_1 - X_2 - X_3$. By developing Euler transformation matrices using these two sequences, one can show that $\sin \theta_2 = \sin \theta \sin \phi$. That is, if θ and ϕ are linearly interpolated, interpolation order of θ_2 is infinite; demonstrating dependence of the interpolation order on the selected sequence.

Nonetheless, linear interpolation of the unique position gradients defined by differentiation with respect to one set of coordinate lines $\mathbf{x}$ implies linear interpolation of position gradients defined by differentiation with respect to another set of coordinate lines $\mathbf{y}$ since the relationship between the two sets of coordinate lines in the reference configuration is constant. In general, when using the position-gradient vectors, the highest order of interpolation is preserved since $\partial \mathbf{r} / \partial \mathbf{x} = (\partial \mathbf{r} / \partial \mathbf{y}) (\partial \mathbf{y} / \partial \mathbf{x})$ and $\partial \mathbf{y} / \partial \mathbf{x}$ is a constant transformation, as discussed previously. In the case of rigid frame of reference, the columns of the transformation matrix that defines the orientation of this reference frame are position-gradient vectors that define unit vectors along the axes of the reference frame. Interpolation of the position-gradient vectors preserves the definition of gradient vectors as tangent to coordinate lines. Interpolation of finite rotations does not preserve this gradient definition; that is, the columns of the transformation matrix obtained by interpolation of the noncommutative finite rotations are not necessarily tangent to the coordinate lines defined by the finite elements spatial coordinates.

5.2 Finite Rotations and Deformation Modes

In general, finite rotations are not directly associated with specific deformation modes. For example, simple torsion due to shear is not associated with a single Green–Lagrange shear strain. The case of simple torsion discussed in the literature assumes a single rotation of a cross-section, leading to two nonzero Green–Lagrange shear strains ε_{12} and ε_{13} attributed to the change of the orientation of the beam longitudinal fibers due to torsion [232–235]. In this simplified torsion analysis, the cross-section is assumed planar, but not rigid since strains are not zero everywhere on the cross-section. In more general three-dimensional deformation scenarios, the finite rotations are not commutative and cannot be directly related to deformation modes. Even in the case of Euler–Bernoulli beam, the curvature and twist of the beam centerline are not, in general, exact differentials and the curvature or twist cannot be integrated and written in terms of a single angle in the case of general displacement [231].

5.3 Consistency with Continuum-Mechanics Description

ANCF coordinates are consistent with the kinematic description used in the general continuum-mechanics theory. ANCF elements employ as nodal coordinates position gradients, which are different from displacement gradients and have clear geometric meaning as tangents to coordinate lines [236]. When using a material-point approach in which the position field is defined by the vector $\mathbf{r} = \mathbf{r}(x_1, x_2, x_3, t)$, the vector $\mathbf{r}$ completely defines the continuum configuration and allows for defining a complete matrix of position-gradient vectors at an arbitrary point. Using the special description $\mathbf{r} = \mathbf{r}(x_1, t)$ to define a beam centerline and introducing a set of Euler angles $\boldsymbol{\theta} = \boldsymbol{\theta}(x_1, t)$ to define the orientation of the beam cross-section not only lead to a semi-continuum model in which a complete set of position-gradient vectors cannot be defined, but also leads to previously mentioned problems related to the interpolation of the finite rotations, their noncommutativity, and lack of interpretation as deformation modes in the large-displacement analysis. In continuum mechanics, the position field $\mathbf{r} = \mathbf{r}(x_1, x_2, x_3, t)$ defines the rotation field and leads to the general continuum-mechanics partial-differential equations of equilibrium regardless of the type of coordinates used when approximation methods are introduced.

5.4 Interpolation Order and Degrees of Continuity and Conformity

ANCF displacement fields allow increasing the order of interpolation without increasing the number of nodes or using noncommutative finite rotations. Using higher-order elements, which can be developed by keeping the same number of nodes, does not always imply a mesh with larger number of degrees of freedom since coarser meshes can be developed when higher-order elements are used and such meshes can lead to higher degree of smoothness and can have better convergence characteristics as compared to meshes based on lower-order elements [215], particularly in case of bending problems in which continuity of the rotation field allow obtaining desired smooth solutions that ensure accurate curvature representation. The partial-differential equation of bending vibration is of fourth order in the spatial beam coordinate, and therefore, smoothness of higher derivatives in such bending problems ensures consistency with the equations used in elementary strength of material and vibration theories [253].

Use of position gradients as nodal coordinates ensures the continuity of the rotation, strain, and stress fields at the nodal points of the ANCF elements. Such degree of continuity cannot be achieved using conventional elements since continuity of the rotations does not imply strain and stress continuity at the nodal points. Higher degree of continuity is important in many problems including the above-mentioned bending problems. While cubic polynomials are required to achieve consistency with the partial differential equation of bending vibration and consistency with the fact that the curvature vector is defined by the second-order derivative, full conformity of cubic surfaces of solid elements requires using forty eight nodal coordinates of the four nodes of an ANCF surface. This important property is automatically achieved by the ANCF solid elements and is not achieved by conventional brick elements including the 20-node brick element.

5.5 Nonincremental Solution Procedure

The co-rotational procedure was introduced to circumvent the problems arising from using infinitesimal-rotation structural finite elements such as beams, plates, and shells. Such elements, due to using infinitesimal rotations as nodal coordinates, lead to kinematic linearization, cannot describe exact rigid-body motion, and do not lead to zero strains under arbitrary

rigid-body displacements. Such an incremental-rotation solution procedure is not suited for MBS applications characterized by highly nonlinear geometric nonlinearities due to high-speed spinning motion and highly nonlinear kinematic constraint equations, as previously mentioned. This was the main motivation for introducing the small-deformation FE/FFR formulations in the early 1980s. In the FE/FFR formulation, the large body displacements are not described using the element nodal coordinates to avoid linearization of the large reference rotation. Unlike the co-rotational approach which assigns a co-rotational frame for each element, in the FFR formulation, one floating frame of reference is introduced for the entire mesh and this frame of reference is uniquely defined using the FFR *reference conditions* which do not have equivalent in the co-rotational approach. The FE/FFR formulation is implemented in most widely used commercial MBS software which adopt algorithms based on nonincremental solution procedures.

Use of the incremental co-rotational approach can also be avoided when solving large-deformation MBS problems. Because ANCF elements do not employ infinitesimal or finite rotations as nodal coordinates, such elements do not require using incremental-rotation procedures and do not require introducing a local mesh frame as in the FE/FFR formulation. The resulting ANCF equations can be solved nonincrementally, and therefore, linearization of the kinematic equations is avoided.

5.6 Constant Inertia and Optimum Sparse Matrix Structure

Three-dimensional ANCF structural elements, such as beams and plates, lead to constant mass matrix. Such a constant mass matrix cannot be obtained when using the co-rotational approach with conventional beam, plate, and shell elements that employ infinitesimal or finite rotations as nodal coordinates. Consequently, the equations of motion assume a simpler form since the vectors of Coriolis and centrifugal forces in the case of ANCF elements are zero. In case of large deformation problems, regardless of the formulation used, the strain–displacement relationship is nonlinear and the stress forces are highly nonlinear in the coordinates. However, the mass matrix for both fully parameterized and gradient-deficient elements always assume the constant and symmetric form $\mathbf{M} = \int_V \rho \mathbf{S}^T \mathbf{S} dV$, where ρ and V are, respectively, mass density and volume of the element. Because the mass matrix is constant, a Cholesky decomposition of this matrix can be defined as $\mathbf{M} = \mathbf{L}\mathbf{L}^T$, where $\mathbf{L}$ is a lower-triangular matrix. This Cholesky decomposition can be used to write the mesh nodal coordinate vector in terms of ANCF Cholesky coordinates. The mass matrix associated with the Cholesky coordinates is an identity matrix leading to an optimum sparse matrix structure for the MBS equations of motion.

5.7 General Continuum Mechanics and Coupled Modes

ANCF structural finite elements allow using both general continuum-mechanics approach and classical beam and plate theories. Such elements can also be used to define more general and accurate shear-deformable elements based on general definitions of shear strains used in continuum-mechanics. Therefore, they are more general than elements based on simplified beam and plate theories such as Timoshenko beam and Mindlin plate theories [233]. Furthermore, planar and spatial ANCF structural elements, such as beams and plates, capture deformation modes that are not captured by conventional beam and plate elements. For example, ANCF structural elements capture the deformation of the beam cross-section and do not require using ad hoc approaches to describe cross-section deformations, as it is the case with the conventional elements [237].

5.8 Consistent Rotation-Based Formulations (CRBF)

Geometrically accurate lower-dimension infinitesimal-rotation finite elements can be developed using the ANCF displacement fields leading to ANCF/CRBF finite elements that preserve reference-configuration geometry and order of the interpolation polynomials [238–240]. Such ANCF/CRBF elements, which can be used with the *floating frame of reference* (FFR) formulation for efficient small-deformation analyses, cannot be developed using the displacement fields of conventional finite elements which replace the polynomial coefficients from the outset by infinitesimal-rotation nodal coordinates. To preserve the reference-configuration geometry when using infinitesimal rotations as nodal coordinates, a two-step procedure is used. In the first step, an ANCF element is defined using position and position-gradient coordinates. In the second step, the displacement gradients are expressed in terms of finite or infinitesimal rotations. For example, the position-gradient vectors at a node k can be written as [238–240]

$$\begin{bmatrix} \mathbf{r}_{x_1}^k & \mathbf{r}_{x_2}^k & \mathbf{r}_{x_3}^k \end{bmatrix} = \mathbf{A}^k(\boldsymbol{\theta}^k) \mathbf{J}_o^k, \quad k = 1, 2. \quad (6)$$

In this equation, $\mathbf{A}^k(\boldsymbol{\theta}^k)$ is an orthogonal transformation matrix written in terms of three orientation parameter $\boldsymbol{\theta}^k$, and $\mathbf{J}_o^k$ is the matrix of position-gradient vectors in the reference-configuration geometry. Applying the preceding equation to the ANCF displacement field reduces the number of coordinates at a node to six, three positions and three rotations; and defines a finite-rotation ANCF/CRBF element that does not impose restrictions on the rotation within the element, preserves the reference-configuration geometry, maintains order of interpolation of the original ANCF element, and eliminates stretch degrees of freedom at the nodes. An infinitesimal-rotation ANCF/FFR element can be systematically obtained from the ANCF/CRBF element by writing $\mathbf{A}^k = \mathbf{I} + \tilde{\boldsymbol{\theta}}^k$, where in this case, $\tilde{\boldsymbol{\theta}}^k$ is a skew-symmetric matrix associated with the infinitesimal rotations $\boldsymbol{\theta}^k$ [241]. Therefore, in case of ANCF/FFR elements, the following condition is applied to the nodal coordinates of the original ANCF element:

$$\begin{bmatrix} \mathbf{r}_{x_1}^k & \mathbf{r}_{x_2}^k & \mathbf{r}_{x_3}^k \end{bmatrix} = (\mathbf{I} + \tilde{\boldsymbol{\theta}}^k) \mathbf{J}_o^k, \quad k = 1, 2. \quad (7)$$

ANCF/FFR elements can be used with the FFR formulation to develop geometrically accurate elements that preserve the reference-configuration geometry, which cannot be accurately preserved using existing rotation-based FE formulations. In the ANCF/FFR procedure, the product $\tilde{\boldsymbol{\theta}}^k \mathbf{J}_o^k$ in the preceding equation leads to *constant geometric coefficients* defined by the elements of the matrix $\mathbf{J}_o^k$. These constant geometric coefficients are embedded in the shape-function matrix of the element and enter into the definition of the inertia and elastic forces.

5.9 Reference-Configuration Geometry

The ANCF position field can be written as $\mathbf{r}(\mathbf{x}, t) = \mathbf{S}(\mathbf{x}) \mathbf{e}(t) = \mathbf{S}(\mathbf{x}) (\mathbf{e}_o + \mathbf{e}_d(t))$, where $\mathbf{e}_o$ and $\mathbf{e}_d$ are, respectively, the nodal coordinates in the reference configuration and vector of nodal displacements. This position field allows describing complex curved geometry by proper choice of $\mathbf{e}_o$, which defines position of material points in the reference configuration. Having the position gradients as nodal coordinates allows for this geometry description, which cannot be conveniently described using conventional rotation-based elements

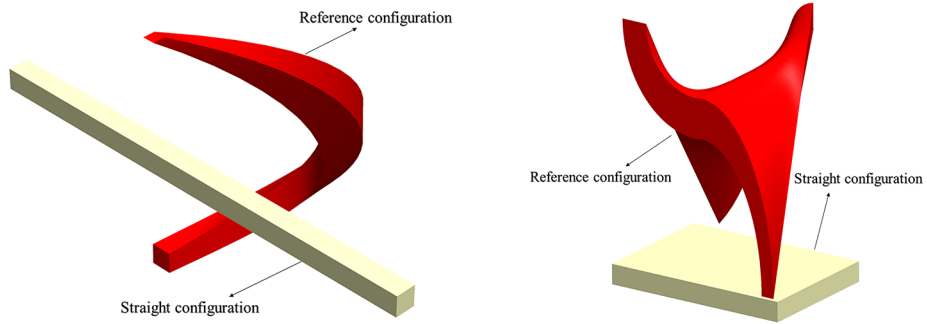


Fig. 1 ANCF beam and plate element geometries

which often assume rigid cross-sections and cannot accurately describe complex and smooth reference-configuration geometry without using large number of elements.

Furthermore, the same position field can be used for straight and curved beams and for plates and shells. That is, no distinction is made between shell and plate or curved- and straight-beam displacement fields. This is again due to the fact that the vector of the ANCF element nodal coordinates can be written as $\mathbf{e}(t) = \mathbf{e}_o + \mathbf{e}_d(t)$, which leads to $\mathbf{r}(\mathbf{x}, t) = \mathbf{S}(\mathbf{x}) \mathbf{e}(t) = \mathbf{S}(\mathbf{x}) (\mathbf{e}_o + \mathbf{e}_d(t))$, as previously discussed. The reference-configuration geometry of curved beams and shells can be conveniently described by the vector $\mathbf{e}_o$ and the same shape-function matrix $\mathbf{S}$ for straight element. In case of curved geometry, only the vector $\mathbf{e}_o$ is changed to define the initial geometry. For example, at a node k , one can write

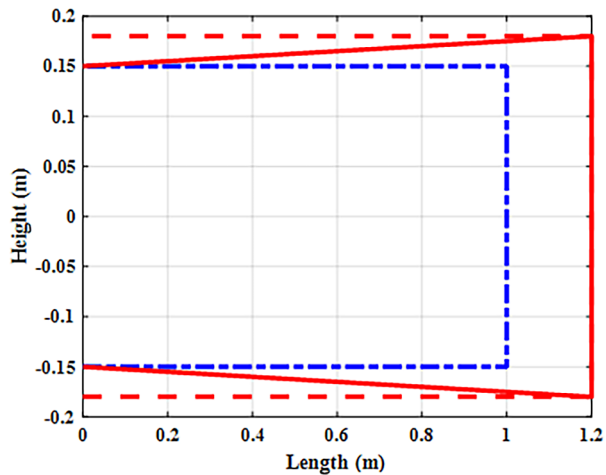
$$\begin{aligned} \mathbf{J}_o^k &= [(\mathbf{r}_{x_1}^k)_o \quad (\mathbf{r}_{x_2}^k)_o \quad (\mathbf{r}_{x_3}^k)_o] \\ &= [(\alpha_1 \hat{\mathbf{r}}_{x_1}^k)_o \quad (\alpha_2 \hat{\mathbf{r}}_{x_2}^k)_o \quad (\alpha_3 \hat{\mathbf{r}}_{x_3}^k)_o], \quad k = 1, 2 \end{aligned} \quad (8)$$

where $\alpha_l, l = 1, 2, 3$, are stretch constants and $\hat{\mathbf{r}}_{x_l}^k, l = 1, 2, 3$, are unit vectors that define the orientation of the gradient vectors at the node. By changing the length and orientation of the position-gradient vectors at the nodes, complex geometric shapes can be obtained. Figure 1 shows examples of reference-configuration geometries that can be obtained using two ANCF spatial fully parameterized beam elements and two ANCF fully parameterized plate elements. The geometries shown in this figure are obtained by changing the orientation and/or length of the position-gradient vectors at the nodal points, that is, the shape manipulation to obtain the geometries presented in this figure is accomplished using the vector of nodal coordinates $\mathbf{e}_o$ only.

5.10 General Boundary Conditions, MBS Constraints, and Model Assembly

ANCF position and gradient coordinates are independent nodal coordinates, and therefore, more general boundary conditions and MBS joint constraint formulations can be defined. For example, in conventional FE brick elements, only position constraints can be imposed on the nodal coordinates, and such constraints do not imply zero strains at the nodal points. In the case of ANCF elements different boundary conditions can be conveniently applied including fixing the position and achieving zero strains by using the nodal coordinates directly. For example, fully and partially clamped joints can be defined. The fully-clamped joint imposes constraints on the position and the position gradients and ensures zero strains at the nodal points. Figure 2 shows the effect of the boundary conditions on the thermal

Fig. 2 Effect of boundary conditions on thermal expansion [242] (--- Initial configuration, — Fully clamped under thermal load, --- Partially clamped thermal load)



expansion of a rectangular structure modeled using planar ANCF beam elements [242]. The figure shows in case of fully constrained boundary conditions at the first end of the beam, the beam thermally expands to a tapered geometry, while in the case of imposing constraints on the node position, the beam maintains its rectangular shape.

ANCF elements also allow formulating MBS joint constraints, considered highly non-linear, as linear constraint equations that can be formulated at a preprocessing stage. This allows eliminating dependent variables before the start of the dynamic simulation. For example, a planar revolute joint or spatial spherical joint between two elements i and j at two nodes k and l , respectively, can be written as linear system of equations in the ANCF coordinates as $\mathbf{r}^{ik} = \mathbf{r}^{jl}$. This simple and linear joint formulation can be achieved because ANCF elements can describe arbitrarily large displacements. The linear equations $\mathbf{r}^{ik} = \mathbf{r}^{jl}$ can be used to eliminate a number of dependent variables equal to the number of constraint equations at preprocessing stage; leading in some applications to significant reduction in the problem dimension and to a numerically stable solution since nonlinear algebraic constraint equations, which cannot be eliminated at a preprocessing stage, must be satisfied at the position, velocity, and acceleration levels. Furthermore, when joint coordinates are not used, the constraint equations at the position level are satisfied using an iterative Newton–Raphson algorithm, and therefore, such nonlinear constraint equations, not only increase the problem dimensions and require introducing Lagrange multipliers, but also require a more elaborate solution procedure to ensure numerical convergence and stability.

Another example of an MBS joint which is formulated in the literature using highly nonlinear constraint equations and can be formulated as linear ANCF constraint equations is the spatial revolute (pin) joint. In rigid-body dynamics, this spatial joint is formulated using five highly nonlinear constraint equations that cannot be eliminated at a preprocessing stage. When using ANCF elements, on the other hand, this spatial joint can be formulated using six linear constraint equations. In the case of a spatial revolute joint between two elements i and j at the two nodes k and l , respectively, the six ANCF linear equations of this joint can be written as

$$\mathbf{r}^{ik} = \mathbf{r}^{jl}, \quad (\partial \mathbf{r}^{ik} / \partial x_\alpha) = (\partial \mathbf{r}^{jl} / \partial x_\alpha), \quad (9)$$

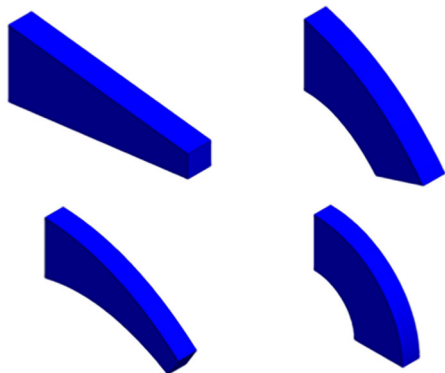
where x_α , $\alpha = 1, 2, 3$, denotes the direction of the joint axis. Because the transformation between coordinate lines is constant, as previously discussed, the choice of the joint axis is

Fig. 3 Tracked vehicle [249]

not limited to only axes along element coordinate lines. Using ANCF linear joint constraint equations allows developing efficient computer models of many MBS applications, example of which are chain drives commonly used in tracked vehicles as the one shown in Fig. 3. For such systems, ANCF elements can be used to obtain efficient solutions for the stresses of the chain links. Furthermore, large scale models with different joint types can be assembled at a preprocessing stage using the concept of the *ANCF reference node*. For example, vehicle chassis and all tires can be represented at a preprocessing stage as one model with constant mass matrix despite the fact that the tires can have arbitrarily large rotation with respect to the chassis and axles [243].

5.11 Motion, Shape, and Stress Control

In the control of soft robots, using position gradients allows developing a more general inverse dynamics problem using MBS algorithms to simultaneously control the motion and shape. Simultaneous control of motion and shape can be difficult to achieve using conventional elements. Specified ANCF gradient trajectories allows developing an inverse dynamics problem to determine the actuation forces. Figure 4 shows an example of simultaneous control of the motion and shape by determining the control forces from the inverse dynamics problem and apply these control force to obtain the desired motion and geometry [244]. As discussed in [244], one can specify all or a subset of the following rheonomic constraints at

Fig. 4 Motion and shape control [244]

a node k of the ANCF mesh:

$$\mathbf{r}^k(t) = \mathbf{f}_p(t), \quad \mathbf{r}_{xl}^k(t) = \mathbf{f}_g(t), \quad k = 1, 2, \dots, n_n, l = 1, 2, 3, \quad (10)$$

where $\mathbf{f}_p$ and $\mathbf{f}_g$ are specified functions that define the desired motion and geometry. The vector of constraint equations that describe desired motion and shape can be written in a vector form as $\mathbf{C}_s(\mathbf{q}, t) = [C_{s1} \ C_{s2} \ \dots \ C_{sn_{cs}}]^T = \mathbf{0}$, where n_{cs} is the number of equations, t is time, and $\mathbf{q}$ is the vector of all system coordinates. If the system is subjected to other joint constraints defined by the vector $\mathbf{C}_m$, the system equations of motion can be written as $\mathbf{M}\ddot{\mathbf{q}} = \mathbf{Q}_e + \mathbf{Q}_c$, where $\mathbf{M}$ is the system mass matrix, $\mathbf{Q}_e$ is the vector of applied forces, and $\mathbf{Q}_c$ is the vector of constraint forces. The vector of constraint forces can be written as [244]

$$\mathbf{Q}_c = -\mathbf{C}_q^T \boldsymbol{\lambda} = -\mathbf{C}_{sq}^T \boldsymbol{\lambda}_s - \mathbf{C}_{mq}^T \boldsymbol{\lambda}_m = -[\mathbf{C}_{sq}^T \ \mathbf{C}_{mq}^T] \begin{bmatrix} \boldsymbol{\lambda}_s \\ \boldsymbol{\lambda}_m \end{bmatrix}. \quad (11)$$

In this equation, $\mathbf{C}_q = \partial \mathbf{C} / \partial \mathbf{q} = [\mathbf{C}_{sq}^T \ \mathbf{C}_{mq}^T]^T$ is the constraint Jacobian matrix, $\boldsymbol{\lambda} = [\boldsymbol{\lambda}_s^T \ \boldsymbol{\lambda}_m^T]^T$ is the vector of Lagrange multipliers, $\mathbf{C}_{sq} = \partial \mathbf{C}_s / \partial \mathbf{q}$ and $\mathbf{C}_{mq} = \partial \mathbf{C}_m / \partial \mathbf{q}$ are, respectively, the Jacobian matrices of the constraint equations $\mathbf{C}_s$ and $\mathbf{C}_m$; while $\boldsymbol{\lambda}_s$ and $\boldsymbol{\lambda}_m$ are Lagrange multipliers associated, respectively, with the constraint equations $\mathbf{C}_s$ and $\mathbf{C}_m$. The resulting system of differential and algebraic equations (DAEs) can be solved to determine the system coordinates, velocities, accelerations, and Lagrange multipliers. The actuation forces used to control the motion and geometry are defined as $-\mathbf{C}_{sq}^T \boldsymbol{\lambda}_s$. Developing the motion and geometry control procedure described in this section using elements that employ noncommutative finite rotations as nodal coordinates is not simple since local shape manipulations and stretches cannot be efficiently achieved using generalized coordinates that may lack clear geometric interpretation.

6 Implementation issues

While the FE/FFR formulation was introduced in the early 1980s, the FE/FFR formulation was not implemented correctly in leading commercial MBS software, resulting in developing inaccurate research and industry computer models over a period that spans more than quarter a century. This wide-spread FFR implementation problem negatively impacted the credibility of the industry durability investigations and virtual prototyping [245]. Given the wide use of the FE/FFR formulation by a large number of industry sectors; loss of resources, efforts, and time as well as the negative impact on the industry computer-aided engineering (CAE) process and accident investigations have been significant.

Implementation of both FE/FFR and ANCF approaches can enhance the software capabilities and generality. To avoid the implementation problem encountered with the FE/FFR formulation, ANCF implementation issues are discussed in this section. Among the implementation issues discussed are need for both FFR and ANCF approaches; use of ANCF Cholesky coordinates and sparse matrix implementation; preprocessor linear constraints; numerical integration of the stress forces; fully parameterized and gradient-deficient elements; definition of the forces and moments; gradient transformation versus conventional vector transformation; coordinate reduction; use of explicit and implicit numerical integration methods; and choice of the reference solutions. Understanding implementation issues discussed in this section will contribute to developing more general and robust large-displacement algorithms required for the computer simulations of large class of systems which contain components made of softer materials. Such systems include automotive, aerospace, robotic, machine, biological, bio-mechanics, etc.

6.1 FFR and ANCF Approaches

The FFR and ANCF approaches are fundamentally different in concept and implementation. Nonetheless, implementation of both formulations can enhance the software capabilities and generality for solving wide range of engineering and science applications that include both stiff and soft materials. In the FFR approach, a local reference frame is introduced and uniquely defined by the *FFR reference conditions*. While the FFR formulation is suited for small-deformation analysis, it can be used in some large-deformation applications if the deformation shape is simple by using nonlinear strain–displacement relationships when formulating the stress forces. In this case, the concept of linear modes can still be applied and the deformation coordinates can be expressed in terms of the modal coordinates. Nonetheless, the application of the FFR formulation to large-deformation problems has limited scope and cannot be used in general-purpose MBS software as a general large-deformation approach for systems with components experiencing significant geometry changes.

The implementation of the FE/FFR formulation in general-purpose MBS algorithm is necessary since stiff components can fail due to small deformations that lead to high stresses. The FE/FFR formulation allows creating a local linear problems and using linear strain–displacement relationship to efficiently solve small-deformation problems by eliminating insignificant high-frequency modes. In addition to the FE/FFR formulation, implementing the ANCF approach in new generation of MBS algorithms is recommended to be able to handle a wide range of applications in which small-deformation assumptions are no longer applicable or in applications, such as chain drives, in which the ANCF constraint formulations can offer a computational advantage even in the case of small-deformation problems. For ANCF meshes, the local frame is not required; using such a local mesh frame makes the ANCF approach conceptually equivalent to the FE/FFR approach. It is also important to distinguish between the FE/FFR formulation and the co-rotational formulation widely used in the commercial FE software. The fundamental differences between the two formulations are discussed in the literature [246].

6.2 Cholesky Coordinates and Sparse Matrix Implementation

Sparse matrix techniques are used in general-purpose MBS algorithms to achieve efficient computer simulations of large scale and complex systems that consist of many components and joints. While ANCF meshes can have large number of nodal coordinates, the mesh mass matrix is constant regardless of the magnitude of displacements. Using this feature, significant improvement in the computational efficiency can be achieved by using the ANCF Cholesky coordinates [214]. Using the ANCF Cholesky transformation leads to an identity inertia matrix and to an optimum sparse matrix structure of the acceleration equations. The Cholesky transformation can be determined in a preprocessor computer program before the start of the dynamic simulation. The constant and symmetric ANCF mass matrix $\mathbf{M}$ can be written in terms of its Cholesky factors as $\mathbf{M} = \mathbf{L}\mathbf{L}^T$ [247, 248], where $\mathbf{L}$ is a lower triangular matrix. The equation of unconstrained motion of the ANCF mesh can be written as $\mathbf{M}\ddot{\mathbf{e}} = \mathbf{Q}$, where in this equation $\mathbf{e}$ is the element of the mesh nodal coordinates and $\mathbf{Q}$ is the vector of all forces including the stress forces. Introducing the Cholesky coordinates $\mathbf{e}_C$, using the coordinate transformation $\mathbf{e} = \mathbf{L}^{-1^T} \mathbf{e}_C$ in the equation of motion $\mathbf{M}\ddot{\mathbf{e}} = \mathbf{Q}$, and premultiplying by $\mathbf{L}^{-1}$, one obtains

$$\left[\mathbf{L}^{-1} (\mathbf{L}\mathbf{L}^T) \mathbf{L}^{-1^T} \right] \ddot{\mathbf{e}}_C = \mathbf{L}^{-1} \mathbf{Q}. \quad (12)$$

The matrix $\left[\mathbf{L}^{-1} (\mathbf{L}\mathbf{L}^T) \mathbf{L}^{-1T} \right]$ is an identity matrix, and therefore, the Cholesky accelerations can be determined from the equation $\ddot{\mathbf{e}}_C = \mathbf{L}^{-1}\mathbf{Q}$. Using Cholesky coordinates and sparse matrix techniques is recommended to improve computational efficiency, particularly in case of large ANCF meshes. It is important, however, to have a consistent implementation of the Cholesky transformation, particularly in case of constrained systems in which the joint constraint equations must also be written in terms of the Cholesky coordinates. The ANCF nodal coordinates can always be recovered using the transformation $\mathbf{e} = \mathbf{L}^{-1T} \mathbf{e}_C$.

6.3 Linear Constraint Formulations

Nonlinear constraint equations that describe mechanical joints can significantly increase the dimensions and complexity of the solution procedure of the MBS problem. In the MBS augmented formulation, the technique of Lagrange multipliers is used and the joint constraint equations must be satisfied at the position, velocity, and acceleration solution steps. In the recursive approach, nonlinear constraint equations, which are eliminated at every solution step, increase the complexity of formulating the velocity-transformation matrix used to write the system variables in terms of the independent variables. In some applications, such as flexible chains, the ANCF approach allows formulating linear constraint equations of mechanical joints, normally formulated as nonlinear constraints even in the case of rigid bodies. The ANCF linear constraint equations can be defined and eliminated at preprocessing stage, leading to significant reduction in the model dimension as demonstrated in the literature using tracked vehicles [249]. The two chains of the tracked vehicle can have hundreds of pin joints that can be eliminated at preprocessing, making the numerical solution of the problem in the main solver more efficient and stable because iterative Newton–Raphson methods and numerical tolerances are not required for linear constraints. Examples of joints that are modeled by highly nonlinear constraint equations in rigid-body dynamics, but can be modeled as ANCF linear equations are the spherical and revolute joints, as previously discussed. The MBS software can be designed to allow eliminating these constraints before the start of the dynamic simulations, defining meshes with lower dimensions.

6.4 Numerical Integration of the Stress Forces

Nonlinear strain–displacement relationships, which define the Green–Lagrange strain tensor $\boldsymbol{\varepsilon} = (\mathbf{J}^T \mathbf{J} - \mathbf{I}) / 2$ where $\mathbf{J} = \begin{bmatrix} \mathbf{r}_{x_1} & \mathbf{r}_{x_2} & \mathbf{r}_{x_3} \end{bmatrix}$ is the matrix of position-gradient vectors, must be used with ANCF finite elements to ensure zero strains in the case of arbitrarily large rigid-body displacements. Consequently, the stress forces are highly nonlinear in the ANCF coordinates, and closed-form expressions cannot, in general, be obtained for such stress forces. Furthermore, ANCF displacement fields allow using arbitrary geometric shapes for the cross-sections that can be described numerically using spline functions. In this case, both the mass matrix and the elastic forces must be evaluated using numerical integration. The constant mass matrix can be evaluated once in advance of the dynamic simulations and can be used to determine the Cholesky transformation that accounts for arbitrary cross-section geometry. The elastic forces, on the other hand, must be evaluated at every time step by quadrature numerical integration. Use of parallel computer architecture can significantly improve the computational efficiency, particularly in case of meshes with large number of finite elements. Therefore, code parallelization is an important consideration when developing ANCF algorithms.

The numerical evaluation of integrals of functions is covered in detail in textbooks on numerical methods [247, 248]. For example, in one-dimensional problems such as in the

case of Euler–Bernoulli beam theory, if the function $f(x)$ in the integral $I = \int_a^b f(x)dx$ over the interval $[a, b]$ is not simple, one must resort to numerical methods to evaluate the integral. In the Gaussian quadrature formulas, the integral is evaluated by approximating the function $f(x)$ by a polynomial $P_n(x)$ defined at unequally spaced base points. The integral is written in terms of the function evaluated at these base points multiplied by *weight factors* or *weight coefficients* [246, 247]. By changing the integration domain to $\xi \in [-1, 1]$ using relationship $dx = Jd\xi$ and $f(x)J = g(\xi)$, the integral I can be written in terms of the function at the base points and weight coefficients as $I = \sum_{i=1}^m w_i g(\xi_i)$, where w_i , $i = 1, 2, \dots, m$, are the weight factors. In case of three-dimensional integrals used for spatial fully parameterized elements, one has

$$\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 g(\xi, \eta, \zeta) d\xi d\eta d\zeta = \sum_i \sum_j \sum_k w_i w_j w_k g(\xi_i, \eta_j, \zeta_k). \quad (13)$$

In the case of cross-sections defined numerically by spline functions, the algorithm must be designed to allow increasing number of integration points in the transverse direction even if linear interpolation is used for the transverse coordinates to capture more accurately details of cross-section geometry. For example, if the transverse coordinates are x_2 and x_3 , a cross-section with arbitrary shape can be defined using the function $x_3 = f(x_2)$, which can be defined numerically using cubic spline function. If the shape of the cross-section is not simple, the number of integration points along x_2 coordinate must be increased to capture complexity of the cross-section geometry. The determinant J must be evaluated at the integration points using the spline data.

6.5 Fully Parameterized and Gradient-Deficient Elements

An element is considered fully parameterized if its displacement field is written in terms of a complete set of spatial coordinates regardless of the nodal coordinates used. That is, for spatial ANCF elements, the displacement field is developed in terms of the three coordinates x_1, x_2 , and x_3 ; and for planar fully parameterized elements, the displacement field is developed in terms of the two coordinates x_1 and x_2 , regardless of whether all the gradient vectors are used as nodal coordinates. Using full set of coordinates allows defining tangent vectors in all directions and a complete set of gradients at the integration points. That is, the matrix of position-gradient vectors $\mathbf{J}$ is a square matrix allowing the use of general continuum mechanics approaches. Classical beam and plate theories can also be used with fully parameterized finite elements. Because a square nonsingular coordinate-line transformation $\partial \mathbf{y} / \partial \mathbf{x}$ can be developed for fully parameterized elements, discontinuities at the interface of the elements due differences in their orientation can be systematically described. Fully parameterized elements also offer more flexibility and versatility for describing reference-configuration geometry since having all position-gradient vectors allows for conveniently defining stress-free stretches, shearing, and bending. Therefore, implementation of fully parameterized elements is recommended despite some numerical issues due to coupling between deformation modes. When implementing fully parameterized elements, it is important to distinguish between discontinuities and curvatures when defining nodal position-gradient vectors. A curved beam, for example, has a continuous coordinate line (no sharp change in directions and no intersections). In this case, no gradient transformation is required for the element assembly when applying connectivity conditions. In case of discontinuities, however, the coordinate lines are not continuous (intersections or sharp edges) and

using gradient transformation is necessarily for proper application of the element connectivity conditions. The software must be designed to distinguish between these two different cases and treat them separately for efficient implementation to avoid unnecessary large number of arithmetic operations.

Gradient-deficient elements, on the other hand, do not have complete set of spatial coordinates. An example of these elements is the three-dimensional ANCF thin plate element which has a displacement field expressed in terms of the two coordinates x_1 and x_2 ; that is, $\mathbf{r} = [r_1 \ r_2 \ r_3]^T = \mathbf{r}(x_1, x_2)$, where $\mathbf{r}$ is a three-dimensional vector. In this case, in which only two gradient vectors $\mathbf{r}_{x_1}$ and $\mathbf{r}_{x_2}$ can be determined, general continuum-mechanics approaches that require square and nonsingular matrix of position-gradient vectors cannot be used, and classical beams and plate theories are the only options available for these elements. It is important to note that the vector $\mathbf{r}_{x_1} \times \mathbf{r}_{x_2} / |\mathbf{r}_{x_1} \times \mathbf{r}_{x_2}|$ is not a position-gradient vector since this vector is not obtained by differentiation with respect to a coordinate line. That is, in case of thin plates, the matrix

$$\mathbf{J}_{def} = [\mathbf{r}_{x_1} \quad \mathbf{r}_{x_2} \quad (\mathbf{r}_{x_1} \times \mathbf{r}_{x_2} / |\mathbf{r}_{x_1} \times \mathbf{r}_{x_2}|)] \quad (14)$$

is not equivalent to the matrix $\mathbf{J} = [\mathbf{r}_{x_1} \quad \mathbf{r}_{x_2} \quad \mathbf{r}_{x_3}]$ of the fully parameterized elements. The matrix $\mathbf{J}_{def}$ captures only the membrane strains since $\varepsilon_{33} = \varepsilon_{13} = \varepsilon_{23} = 0$, demonstrating the need for classical plate theories that employ the curvature for defining the strain energy. Bending deformations are due to the moment of the membrane stresses and using $\mathbf{J}_{def}$ in the absence of the x_3 coordinate in the displacement field does not capture the bending moments. The concept is similar to Euler–Bernoulli beam theory in which only the x_1 coordinate is used to define the displacement field, while the bending moment is introduced by integrating $x_2 \sigma_{11} dA$ over the area A of the beam cross-section, where σ_{11} is the axial stress.

6.6 Force and Moment Definitions

The definition of the generalized forces due to applications of force and moment vectors influences the accuracy of the results used in comparative numerical studies aimed at evaluation of element performance and/or merit of using an approach. In some problems, simplified analytical solutions based on linearization assumptions are not ideal as basis of comparison or verification of the results. This is particularly true in case of large-deformation problems because finite rotations are not directly associated with moment vectors [231]. Unlike rigid-body dynamics, in flexible-body dynamics, the force is not a sliding vector and the moment is not a free vector. Some of these concepts can be demonstrated using the three-dimensional ANCF beam element previously introduced. If a force couple, defined by two equal force vectors $\mathbf{F}_1 = \mathbf{F}$ and $\mathbf{F}_2 = -\mathbf{F}$ acting at x_3 and $-x_3$, is considered at $\xi = 1$, one has $s_1 = s_2 = s_3 = s_4 = s_6 = 0$, $s_5 = 1$, $s_7 = l\eta = x_2$, and $s_8 = l\zeta = x_3$. In this case, $\delta \mathbf{r}(\xi = 1) = \delta \mathbf{r}^2 + x_2 \delta \mathbf{r}_{x_2}^2 + x_3 \delta \mathbf{r}_{x_3}^2$. If the couple is defined at $\eta = 0$, the virtual work of the couple can be written as

$$\begin{aligned} \delta W &= \mathbf{F}_1^T \delta \mathbf{r}(\xi = 1, \eta = 0, \zeta) + \mathbf{F}_2^T \delta \mathbf{r}(\xi = 1, \eta = 0, -\zeta) \\ &= (2x_3 \mathbf{F}^T) \delta \mathbf{r}_{x_3}^2 \end{aligned} \quad (15)$$

This equation defines generalized forces associated with the transverse coordinate $\mathbf{r}_{x_3}^2$ and not with finite rotation coordinates.

The definition of the generalized forces associated with the position-gradient coordinates due to a torque application can be defined using the *velocity gradient tensor* $\mathbf{L}$, which can

be written as $\mathbf{L} = (\partial \dot{\mathbf{r}} / \partial \mathbf{r}) = (\partial \dot{\mathbf{r}} / \partial \mathbf{x}) (\partial \mathbf{x} / \partial \mathbf{r}) = \dot{\mathbf{J}} \mathbf{J}^{-1}$. As previously discussed, the tensor $\mathbf{L}$ can be written as the sum of symmetric and skew-symmetric tensors as $\mathbf{L} = \mathbf{D} + \mathbf{W} = (1/2) (\mathbf{L} + \mathbf{L}^T) + (1/2) (\mathbf{L} - \mathbf{L}^T)$, where $\mathbf{D} = (1/2) (\mathbf{L} + \mathbf{L}^T)$ is the rate of deformation tensor and $\mathbf{W} = (1/2) (\mathbf{L} - \mathbf{L}^T)$ is the spin tensor. Using ANCF finite elements, the velocity gradient tensor can be written in terms of the gradient vectors at an arbitrary point using outer products as [250]

$$\mathbf{L} = \dot{\mathbf{J}} \mathbf{J}^{-1} = [\dot{\mathbf{r}}_1 \quad \dot{\mathbf{r}}_2 \quad \dot{\mathbf{r}}_3] \begin{bmatrix} \mathbf{J}_{R1}^{-1} \\ \mathbf{J}_{R2}^{-1} \\ \mathbf{J}_{R3}^{-1} \end{bmatrix} = \sum_{l=1}^3 (\dot{\mathbf{r}}_{x_l} \otimes \mathbf{J}_{Rl}^{-1}), \quad (16)$$

where $\mathbf{J}_{Rl}^{-1}$ denotes the l th row of the inverse of the matrix of position-gradient vectors $\mathbf{J}$, and $\mathbf{r}_{x_l} = \partial \mathbf{r} / \partial x_l$, $l = 1, 2, 3$. The spin tensor can be obtained from the velocity gradient tensor as $\mathbf{W} = \tilde{\boldsymbol{\omega}} = (1/2) \sum_{l=1}^3 (\dot{\mathbf{r}}_{x_l} \otimes \mathbf{J}_{Rl}^{-1} - \mathbf{J}_{Rl}^{-1} \otimes \dot{\mathbf{r}}_{x_l})$. Using this equation, one can show that the angular velocity vector can be written in terms of the ANCF gradients as $\boldsymbol{\omega} = (\sum_{l=1}^3 \mathbf{J}_{Rl}^{-1} \times \dot{\mathbf{r}}_{x_l}) / 2$, which can be written as

$$\boldsymbol{\omega} = (1/2) \left(\sum_{l=1}^3 \mathbf{J}_{Rl}^{-1} \times (\partial \mathbf{r}_{x_l} / \partial \mathbf{e}) \right) \dot{\mathbf{e}} = \mathbf{G}(\mathbf{e}) \dot{\mathbf{e}}, \quad (17)$$

where the matrix $\mathbf{G}$ is defined as

$$\mathbf{G} = \mathbf{G}(\mathbf{e}) = (1/2) \left(\sum_{l=1}^3 \mathbf{J}_{Rl}^{-1} \times (\partial \mathbf{r}_{x_l} / \partial \mathbf{e}) \right). \quad (18)$$

The angular velocity vector can be defined using infinitesimal Cartesian rotation vector $\delta \boldsymbol{\pi}$ as $\boldsymbol{\omega} = \delta \boldsymbol{\pi} / \delta t$, and the ANCF generalized moment $\mathbf{Q}_M$ associated with the ANCF coordinates can be defined as the result of an external Cartesian moment $\mathbf{M}_M$ using the relationship $\delta \boldsymbol{\pi} = \mathbf{G} \delta \mathbf{e}$ as $\mathbf{M}_M^T \delta \boldsymbol{\pi} = \mathbf{M}_M^T \mathbf{G} \delta \mathbf{e}$, which shows that

$$\mathbf{Q}_M = \mathbf{G}^T \mathbf{M}_M = (1/2) \left(\sum_{l=1}^3 \mathbf{J}_{Rl}^{-1} \times (\partial \mathbf{r}_{x_l} / \partial \mathbf{e}) \right)^T \mathbf{M}_M. \quad (19)$$

In the ANCF implementation, the effect of a moment vector is considered using the above equation. In the case of rigid-body motion, one can show that the analysis presented in this section leads to the definition of the angular velocity used in rigid-body dynamics.

6.7 Gradient Transformation

In the computer implementation, distinction is made between curved and discontinuous structures, as previously discussed. In case of curved structures, coordinate lines x_1 , x_2 , and x_3 are continuous everywhere. In this case, two elements connected at a node have the same position-gradient vectors at this node, and there is no need for applying coordinate transformation when the elements are assembled. In the case of discontinuities, elements can have different orientations in the reference configuration because coordinate lines x_1 , x_2 , and/or x_3 are not continuous. In this case, gradient transformation must be applied at the discontinuity nodes when the elements are assembled. The gradient transformation is not the same

as the conventional vector transformation used in the conventional FE formulation. Such a difference must be observed to preserve the geometric meaning of position-gradient vectors as tangent to coordinate lines and to ensure proper application of the connectivity conditions when two elements with different orientations are connected at a node. For example, if the spatial coordinates of one element is $\mathbf{x} = [x_1 \ x_2 \ x_3]^T$ and the position-gradient vectors of this element at the node are defined as $\partial \mathbf{r} / \partial \mathbf{x} = [\mathbf{r}_{x_1} \ \mathbf{r}_{x_2} \ \mathbf{r}_{x_3}]$; and the spatial coordinates of the second element are $\mathbf{y} = [y_1 \ y_2 \ y_3]^T$ and its position gradients at the same node are $\partial \mathbf{r} / \partial \mathbf{y} = [\mathbf{r}_{y_1} \ \mathbf{r}_{y_2} \ \mathbf{r}_{y_3}]$, then $\partial \mathbf{r} / \partial \mathbf{x} = (\partial \mathbf{r} / \partial \mathbf{y}) (\partial \mathbf{y} / \partial \mathbf{x})$, where the constant matrix $\partial \mathbf{y} / \partial \mathbf{x}$ defines the relationship between the coordinate lines $\mathbf{x}$ and $\mathbf{y}$. The generality and efficiency of the ANCF implementation depends on the proper geometric interpretation of the position-gradient vectors.

6.8 Coordinate Reduction

ANCF finite elements have constant mass matrix, but highly nonlinear elastic forces since nonlinear strain–displacement relationships are used to ensure accurate description of the rigid-body displacements. While large-deformation elements can also be used in the analysis of small deformation, the computational efficiency becomes an issue because of the high-frequency modes. Furthermore, there is no need for using ANCF finite elements to solve small-deformation problems which can be efficiently solved using FFR formulation implemented in most commercial MBS software. In the large-displacement small-deformation FFR analysis, the number of coordinates can be reduced by defining a local linear problem that allows using component mode synthesis methods. The FFR modal approach can also be used in case of large deformation if the deformation shape remains simple and nonlinear strain–displacement relationship is used. Nonetheless, the FFR formulation has been mainly used to solve small-deformation problems.

ANCF finite elements can still be used with the FFR formulation by introducing a mesh or body coordinate system. In this case, the mass matrix becomes nonlinear function of the coordinates. ANCF/FFR finite elements that consistently employ rotations as nodal coordinates are recommended for such ANCF/FFR implementation [241]. These elements allow for consistently accounting for the reference configuration geometry.

6.9 Explicit and Implicit Numerical Integration

Both explicit and implicit numerical integration methods are recommended for implementation in order to solve problems with different degrees of stiffness and also to verify the accuracy of the solutions. Explicit methods such as the explicit predictor–corrector multi-step Adams–Bashforth method described in [247] do not include numerical damping, and therefore, high-frequency modes are not filtered out. For this reason, the solutions obtained using accurate explicit methods which have well-designed error check and order-selection criteria and the magnitude of the error is controlled by varying both the step size and order of the method are often used as the reference numerical solutions to verify numerical results obtained using other integration methods. Nonetheless, accurate explicit methods are not suited for solving stiff differential equations characterized by widely separated eigenvalues. Consequently, implementation of implicit integration methods is recommended. However, widely-adopted implicit integration methods should be used with care since such methods include numerical damping which is not physical damping and can lead to filtering out significant modes [251]. Therefore, solutions obtained using implicit methods need to be carefully checked if such methods include numerical damping. For example, displacement

and/or stress oscillations due to initial conditions should not die out in the absence of physical damping. It can be shown that this is not the case when using some commercial FE software that employ implicit integration methods. Furthermore, distinguishing the effect of physical damping can be difficult in the presence of artificial numerical damping.

6.10 Reference Solutions

Higher-order FE formulations are more general than simplified analytical formulations or formulations based on lower-order of interpolations. Higher-order FE formulations, however, can be systematically simplified to obtain the analytical solutions if the simplifying assumptions and force and moments are properly defined. However, using simplified analytical solutions or solutions obtained using less general FE formulations, which do not capture deformation modes captured by more general formulations, may be misleading when used to examine the performance and convergence of the finite elements. Simplified Euler–Bernoulli or Timoshenko beam formulations, for example, do not capture the stretch of the beam cross-section, and such a stretch can absorb energy during the deformation as the result of the coupling between different displacement modes when more general FE formulations are used. In some commercial FE software, for example, ad hoc approaches, used to account for the beam cross-section deformation, suffer from known limitations, and therefore, such ad hoc approaches are not suited to verify the results of or evaluate the more general FE formulations [237].

7 ANCF research activities

During the past two decades, there has been growing interest in adopting ANCF finite elements to solve new problems and/or obtain more accurate solutions of problems which have been previously solved using less general FE approaches. Furthermore, because ANCF solution procedures do not require using incremental-rotation procedures, ANCF finite elements have been implemented in computational MBS algorithms designed to solve problems with high geometric nonlinearities due to the spinning motion and highly nonlinear constraint functions. ANCF investigations, which have been focused on different research topics, contributed to developing the ANCF approach and to demonstrating its use in different engineering and science applications. In this section, and before discussing future research directions, some past ANCF research activities are briefly discussed. The literature has also articles of review nature that can be consulted for more detailed review of ANCF research [44, 49, 63, 112, 136]

7.1 Element Technology

Large number of ANCF investigations has been focused on *element technology* [22], an area critical to the development of any new FE approach. The research on developing new ANCF elements and enhancing their performance to be able to solve wide range of applications covered different element types. A large number of investigations have been focused on developing beam and cable elements and enhancing their performance [20, 25, 26, 28, 36, 48, 68, 72, 77, 84, 85, 94, 96, 102, 107, 110, 111, 113–116, 125, 130, 131, 134, 135, 137, 138, 147, 148, 151, 154, 162, 163, 166, 171, 177, 179, 181, 183, 199, 202, 203]. ANCF plate and shell elements, including thin and triangular elements, have also been the subject of a large number of investigations [1, 16, 21, 23, 27, 29, 69, 70, 92, 99, 104, 105, 108, 118, 119, 127,

144–146, 150, 174, 182, 191]. ANCF solid elements, which can be used to develop detailed models and ensure continuity of the position, gradient, rotation, and stress fields have been the subject of several investigations, examples of which are [129, 205]. The research in the element technology area has been focused on introducing new element displacement fields based on new interpolation functions or using different sets of nodal coordinates. Some of the new elements were developed to solve particular applications. The research in the element technology area covered both fully parameterized and gradient-deficient elements. Furthermore, some research investigations focused on developing rational ANCF (RANCF) elements that describe circular and conic geometry accurately [100, 101].

7.2 Modeling Issues

Past ANCF research has been concerned with fundamental modeling issues that require adjustment or enhancement of the element performance. The research in this area covered large number of topics that include modeling discontinuities [3]; nanowires [35]; modeling soft tissues and bio-mechanics systems [82, 122–124]; textile materials [95, 176]; electromechanical systems [121]; tire modeling [143]; geometric representation [54]; thin and flexible structures [8, 9, 17]; and fluid and sloshing problems [100, 139, 159].

7.3 Control, Robotics, Mechanisms, and Actuation

Control and flexible and soft robots have been active areas of ANCF research. The research in this area includes modeling robots and mechanisms with flexible links [52, 56, 65, 90, 173, 209]; and developing control systems [45, 65, 93]. Some investigations in this area, as well as in the area of space structure, have been focused on developing actuation forces required to obtain the desired motion and control the robot or system dynamics and stability.

7.4 Formulation and Implementation Issues

The formulations and model implementations have been the focus of several ANCF studies. The goal of the research in this area is to develop new and efficient procedures and framework for implementing the ANCF approach and alleviating numerical locking. Examples of research topics in this area are new formulations, solution procedures, and model implementations [35, 40, 64, 78, 83, 97, 155–157, 167, 193, 204, 208]; locking alleviation [37, 126, 132]; composite-structure formulation [51, 61, 98, 120, 152]; parallel computations and software developments [79, 169]; joint formulations [59, 89, 165, 168]; and formulation validation, experimentation, and comparative studies [13, 19, 87, 142, 188, 189]. Formulations of the contact forces using ANCF finite elements have been also presented in the literature [5–7, 57]. Improving the efficiency and introducing new numerical procedures to solve problems in which ANCF finite elements are used have been the focus of several investigations [58, 80, 81, 106, 140].

7.5 Constitutive Models and Elastic-Force Formulations

The choice of the constitutive models and formulation of the elastic forces play an important role in determining the efficiency and accuracy of flexible-body algorithms. Therefore, this area of research will remain active as new models are being developed and tested. For example, ANCF fully parameterized structural elements such as beams and plates allow using general continuum mechanics approach or classical beam and plate theories for formulating

the elastic forces. The choice of the elastic force formulation can have a significant effect on the mode coupling and how the results are interpreted. For this reason, ANCF formulation of the elastic forces and their efficient implementation have been the subject of several investigations [41–43, 53, 73, 108, 134, 152, 160, 165, 198]. Damping models that account for energy dissipation when using ANCF elements were proposed [12, 39, 86, 164, 207]. Plasticity and thermo-elasticity models have also been used with ANCF finite elements [14, 18, 92, 153] as well as composite-structure models [51, 61, 98, 120, 152, 195, 206, 210].

7.6 Applications

In some of the applications considered in the ANCF research area, components made of very soft materials are considered. Some other applications include components spinning or rotating at very high speeds. Examples of application problems that have been solved using ANCF elements are belt drives [12, 13, 76]; rotating structures, shafts, beams, strings, and plates [4, 15, 24, 31, 38, 172, 175, 211]; vehicle components [74]; wind-turbine blades [2, 3], nuclear-reactor applications [8]; catenary and power-transmission lines [30, 46, 47, 50, 88, 141, 149, 171]; cables, tethers, ropes, and cable/pulley systems [10, 32, 33, 56, 61, 62, 71, 75, 200]; and space applications and deployable structures [52, 66, 91, 135, 180, 196, 197, 210].

8 Future research directions and challenges

In this section, future research directions and challenges that need to be addressed for further developing and enhancing the performance of ANCF elements are discussed. The list of research directions and challenges is not inclusive and only reflects the interest of the author in future ANCF developments. Nonetheless, given the wide range of applications and new problems in which ANCF elements are used, new research directions and challenges can emerge in response to the need for developing virtual prototyping approaches and solution procedures to solve the new problems that have not been previously or properly solved. Among the research directions discussed in this section are element technology; description of shell geometry; modeling composite structures; motion, shape, and stress control; geometric stiffening and mode coupling; locking alleviation; evaluation of the assumptions of the classical approaches; developing new implicit numerical integration methods; implementation of thermo-elasticity, viscoelasticity, and plasticity models in MBS algorithms; developing contact-mechanics algorithms; design optimization and sensitivity analysis; and application areas and result verifications

8.1 Element Technology

Despite significant research efforts devoted to developing new ANCF elements, the development of more elements is expected to continue as more problems emerge. *Element technology* is one of the most important areas for further developing any FE approach because of the need for large element library to allow for accurate modeling of new and challenging problems. For this reason, the area of element technology has been the focus of large number of investigations since the inception of the FE method. Without a diverse base of finite elements that allow for solving different problems, the scope of the applicability of the method remains limited. For example, capturing warping of beam and plate cross-sections requires using higher-order interpolation in the transverse directions, and therefore, new elements are

needed to properly capture cross-section warping and torsion. Development, testing, and enhancing performance, including new *locking alleviation techniques*, of such new elements, which may include additional nodal points, nodal coordinates, and/or derivatives, represent an open research area that is worthy of pursuing to have varieties of elements that are suited for different problems.

Another open element-technology research area is the development of *rational ANCF elements* that capture more accurately circular and conic sections. The new rational elements can include beam, plate, and solid elements; and can serve as the foundation for developing lower-order rational ANCF/CRBF and ANCF/FFR elements without lowering the order of interpolation [238–240]. The ANCF/CRBF and ANCF/FFR elements can be used to obtain efficient solutions for curved beams, plates, and shell problems as will be further discussed in this section.

8.2 Shell and Curved Geometries

As discussed in the preceding section, the ANCF analysis of plates and shells is an active area of research motivated by the fact that conventional FE approaches suffer from known limitations that include reliance on rotations for describing plate and shell geometries, use of different displacement fields for plates and shells, difficulties of accurate representation of reference-configuration geometries, and misinterpretation of the forces and moments due to using rotation-based displacement fields. These problems become more serious in case of large deformation and rotation of plates and shells. ANCF finite elements provide an alternative and systematic approach for modeling plates and shells with complex geometries. By changing length and orientation of the ANCF gradient vectors, arbitrary reference-configuration geometries can be systematically obtained without the need for using rotations. More accurate interpretations of the forces and moments and their dependence on the plate and shell geometry can be made.

While ANCF elements allow for defining more general models and boundary conditions for plate and shell problems, more efficient nonlinear models can still be developed for complex reference-configuration geometries using ANCF/CRBF elements which employ finite rotations as nodal coordinates. The ANCF/CRBF elements employ six coordinates per node, can be used in large-deformation and rotation problems, allow for developing nonlinear models, and preserve the reference-configuration geometries [238–240]. Such elements, however, lead to less coupled-deformation modes and do not allow for the cross-section deformations. ANCF/FFR elements employ infinitesimal rotations based on a consistent formulation that preserves the reference-configuration geometry [241]. In case of small-deformation of plate and shell structures, ANCF/FFR elements lead to constant stiffness matrix and can be efficient in solving small-deformation problems for curved beams and shell problems while accurately describing reference-configuration geometries using constant *geometric coefficients* embedded in the element displacement field. Using ANCF/CRBF and ANCF/FFR elements to solve curved beam and shell problems, comparing with existing formulations, and providing new interpretation and definitions of forces and moments are open research areas that will require large number of investigations.

8.3 Composite Structures

Composite-material formulations can be conveniently developed for different types of ANCF elements such as beams, plates, and solids, particularly when using fully parameterized elements. One important feature of ANCF elements that will contribute to making

research progress in the area of composite structure is the higher degree of conformity provided by these elements. Using position and position-gradients as nodal coordinates allows bonding composite layers in a manner that cannot be achieved by using existing FE formulations. This feature is particularly important in case of curved beams and shells since accurate representation of the reference-configuration geometry has significant effect on the solution accuracy. Composite-structure models can be developed using ANCF elements using layers with different material properties and dimensions and modeled using different constitutive models. Research in this area covers a wide range of technologically important applications that include robots, tires, space structures, etc. Fully parameterized ANCF elements which use complete set of spatial coordinates, including coordinate along the element thickness, allow developing composite models by changing the material properties along the coordinates that enter into the integration of the elastic forces. Because complete set of position gradients can be determined at the integration points, developing such composite-material models is straightforward.

8.4 Motion, Shape, and Stress Control

Motion and shape control of flexible and soft materials and robots is an emerging research and development area that spans a wide range of applications. Developing effective actuation strategies for the control of the motion and oscillations requires proper definitions of the forces and moments in flexible-body dynamics. In the case of soft materials, the deformation can be large and using conventional incremental-rotation procedures which amount to linearization leads to control laws based on linearized dynamic models. ANCF position-gradient vectors allow to simultaneously control the motion and shape [244]. An inverse-dynamics problem can be formulated to define the actuation forces required to obtain the desired motion and shape. Control of both motion and shape is an emerging area with new challenges. Constraints on the position-gradients in the inverse-dynamics problem impose constraints on the motion, shape, and stresses simultaneously. Using MBS algorithms, the constraint forces that produce the desired motion can be determined using an inverse problem. In the case of the augmented formulation, for example, the constraint forces are determined using Lagrange multipliers and can be used to determine the actuation forces. Nonetheless, proper interpretation of the actuation forces is necessary in order to develop effective control strategy, as previously discussed. The motion, shape, and stress control of soft robots and materials using MBS algorithms is a research area that still not well-developed and open for investigations focused on the control force definitions; type, number, and arrangements of soft-material actuators; spillover problems and number of degrees of freedom used in both inverse- and forward-dynamics problems; and effect of choice of the constitutive model used in the virtual prototyping on actuation-force definition.

8.5 Geometric Stiffening and Mode Coupling

The geometric stiffening resulting from the coupling between bending and axial deformations has been the subject of several investigations. It has been shown that using linearized equations of motion of rotating blades leads to unstable solution if the angular velocity of the rotating blade becomes equal to the frequency of the first bending mode of the blade. In general, as discussed in the literature, linearization of the highly nonlinear MBS equations can lead to wrong dynamics and stability results, and therefore, the linearization approaches implemented in commercial MBS software to study system stability need to be reevaluated in view of the well-known rotating blade example, extensively studied in the literature.

While the coupling between axial and bending deformations of rotating blades has been the focus of large number of investigations, other mode couplings that include cross-section stretch, shear, and warping have not been the focus of many studies, and therefore, the effect of such mode couplings on the system dynamics and stability is less understood. Accurate representation of such mode couplings particularly in the large-displacement analysis requires developing nonlinear mathematical models using more general FE formulations. Such new models and formulations will eliminate the need for using ad hoc approaches to overcome the limitations of the displacement fields of elements that employ infinitesimal or finite rotation [237]. Because ANCF finite elements capture deformation modes that cannot be captured using conventional elements, such elements can be used to evaluate mode-coupling effects that have not been thoroughly investigated.

8.6 Locking Alleviation

For approximately five decades, extensive research has focused on solving the locking problems associated with conventional finite elements. This research was motivated by the fact that locking leads to deterioration of the element performance and to poor convergence. Depending on the loading, problem, and element used; many conventional elements suffer from locking including isoparametric elements. Some of these elements may perform well in some applications, while they exhibit serious locking in other applications. Reduced and selective integration methods have been used to alleviate locking of conventional finite elements. However, use of some of these locking alleviation techniques is equivalent to reduction of the order of interpolation due to decrease of the number of integration points [215]. When lower-order elements, such as linear and bilinear elements, are used with such locking alleviation method, there can be problems due to further reducing number of integration points in the process of computing the stress forces. Such numerical problems can have a negative impact on the quality of the solution obtained.

It is also important to distinguish between locking and higher stiffness that results from the coupling between deformation modes when using ANCF finite elements. There are open research problems in this area which is relevant to the element technology area previously discussed in this section. The effect of the form of the constitutive model on the frequency content in the solution needs to be further analyzed [237]. Some constitutive models or coupling between deformation modes can lead to higher stiffness that can be misinterpreted as locking. For example, Poisson effect introduces coupling between longitudinal and transverse stretches and such a coupling can lead to high stiffness which can be misinterpreted and cannot be captured using reference solutions based on simplified analytical models. Conventional FE formulations, based on ad hoc approaches to capture some coupled deformation modes, can produce inaccurate results [237]. Therefore, research in this area using ANCF elements allows investigating the effect of coupled deformation modes without the need for adopting such ad hoc approaches. While several investigations have been devoted to study ANCF locking as discussed in the preceding section, more studies will be needed as new elements are introduced.

8.7 Evaluation of the Classical Approaches

Beam and plate elastic-force formulations are based on theories such as Euler–Bernoulli, Timoshenko, Kirchhoff, and Mindlin theories [232, 233]. These theories employ simplifying assumptions and are not based on general continuum-mechanics formulation of the elastic forces [214, 225–228]. One of the reasons for adopting these simplified theories

is due to the limitations of the geometric description used in conventional finite elements which do not allow for defining a complete set of position-gradient vectors, required for using more general continuum-mechanics approaches for describing accurately reference-configuration geometry and for defining the Green–Lagrange strain tensor based on nonlinear strain-displacement relationship. Future research investigations can be focused on providing assessment of these theories and their constitutive relationships in view of the more general formulations. For example, in Timoshenko beam theory, the beam cross-section is assumed rigid and the shear is defined by a rotation of the cross-section with respect to the plane normal to the beam centerline. In the case of large displacements, the shear cannot be directly defined in terms of an angle. Furthermore, in the three-dimensional large-displacement analysis, finite rotations are not commutative, and angles cannot be directly associated with shear deformations. ANCF finite elements can be used in future investigations to assess the assumptions used in the classical approaches including plate bending which is often formulated in terms of curvature vectors instead of the general continuum-mechanics approach. Evaluation of the assumptions of the classical approaches will contribute to developing accurate reference solutions that can be used to verify the numerical results and to better understanding the source of higher stiffness and effect of the coupling between different modes of deformation that cannot be captured using the simplified classical approaches.

8.8 Implicit Numerical Integration

Temporal numerical integration plays a critical role in the accuracy, efficiency, and robustness of MBS solution procedures. Explicit numerical integration methods, such as the predictor–corrector multistep Adams–Bashforth methods have been used in MBS software because of their accuracy and robustness in solving nonstiff differential equations [247]. By varying the step size and order and using well-designed error check criterion, the solutions obtained using the explicit Adams–Bashforth methods are accurate and have been used as the reference solutions to check the results of the MBS simulations. Nonetheless, explicit methods often fail in the case of stiff differential equations, which have widely separated eigenvalues. The predictor–corrector Adams–Bashforth method is not an implicit method since iterations are not performed at a fixed time to achieve convergence. In the predictor–corrector Adams–Bashforth method, if the error criterion is not satisfied, the time step is shortened and the function is reevaluated. That is, iterations are not performed at a fixed time point.

Because accurate explicit methods that employ sophisticated error-check criteria to change the order and step size often fail in the case of stiff differential equations and because MBS equations can be stiff in some applications, it is recommended to implement implicit integration methods, as previously discussed. However, because most widely used implicit integration methods, such as generalized alpha time-integration method, lead to high-frequency dissipation, their use in MBS algorithms can lead to filtering out modes that can have significant effect on the solution accuracy. Furthermore, because these implicit methods have artificial numerical damping, it becomes difficult to distinguish the effect of the physical damping, as previously discussed [251]. An open and important research area is the development of accurate and efficient implicit time-integration methods that do not include numerical damping and allow for solving stiff MBS equations. This open research area can be challenging, but pursuing such research can significantly enhance the robustness and accuracy of the new generation of MBS algorithms that employ ANCF finite elements as the basis for the large-deformation analysis.

8.9 Thermo-Elasticity, Viscoelasticity, and Plasticity

The scope of the MBS research has been limited in many technological areas due to limitations of the conventional finite elements. The deformation analysis has been, for the most part, focused on small-deformation problems using the FFR formulation. The incremental-rotation procedures commonly used in conventional FE approaches are not normally adopted in MBS algorithms due to the geometric nonlinearities that result from spinning motion and highly nonlinear algebraic constraint equations. Incremental-rotation procedures cannot accurately solve general high-speed spinning motion problems over a long simulation period, and consequently, non-incremental rotation procedures are adopted in developing MBS algorithms. Because accurate description of the large reference motion of flexible bodies using non-incremental-rotation procedures is necessary, the ANCF approach has been used to solve MBS large-deformation problems. Nonetheless, application of this approach to important mechanics areas such as thermo-elasticity, viscoelasticity, and plasticity has been limited [252]. When using ANCF finite elements in these research areas, the displacement formulations of some problems may require using only deformation modes. For example, in the case of viscoelasticity, one must ensure that the rigid body modes are not damped out by using proper constitutive models. Because only linear damping models have been developed for ANCF finite elements, application of more general viscoelasticity constitutive models that better reflect the mechanism of energy dissipation will contribute to the evaluation of the assumptions of the linear damping models. Given the significant effect of the damping in many engineering and science applications, such investigations can define limitations of linear damping models and their effect on the control actuation forces of soft robots.

ANCF displacement formulation of the thermo-elasticity problems is another open research area, particularly important in emerging scientific fields such as soft robots. Thermal expansion does not contribute to rigid-body displacement and in case of constrained motion is not stress-free process due to the restrictions imposed on the boundary of the deformable bodies. Therefore, proper integration of *Lagrange–D'Alembert principle* and thermo-elasticity approaches is a new research area which has a wide scope of applications and will lead to new results that will shed light on the effect of temperature on articulated mechanical systems (AMS).

Similarly, simple plasticity formulations have been used for MBS applications, this is despite extensive research in this area by the computational mechanics community [252]. Plasticity is crucial in identifying component failure, and therefore, accurate plasticity analysis is critical for credible durability investigations performed by the industry. Integration of metal and soil plasticity formulations with MBS algorithms has been limited despite the wide range of applications that include vehicles negotiating uneven soft terrains and robot systems subjected to high forces or working in high temperature environment that can lead to plastic deformations. Space applications is another area in which the terrain and environmental conditions can lead to plastic deformation whose effect cannot be ignored.

8.10 Contact-Mechanics Algorithms

Contact and impact are an important problems in many MBS applications that include tires, robots, belt and chain drives, leaf springs, etc. Some components subjected to contact and impact forces can be very flexible and can have complex reference-configuration geometry. In biomechanics and biological applications, modeling soft-tissue contact is an important issue for developing accurate virtual biological models. Design of actuation forces of flexible

and soft robots must consider the contact of the robot with the environment and the impact forces on the robot end effector. This is particularly important in space applications which consist of long and very flexible arms. Tire/soil interaction of Mars and space rovers is another example in which the contact forces cannot be ignored. Therefore, developing efficient and accurate contact algorithms is necessary for building virtual models that contribute to better understanding of the dynamics and stability of a wide class of technological systems. In solving contact problems, theories of curves and surfaces is fundamental since contact frames and contact surfaces must be determined even if the assumptions of nonconformal contact are used. Because ANCF finite elements ensure higher degree of continuity and can describe accurately the geometry, such elements can be used to develop new MBS contact algorithms capable of modeling very flexible bodies. While ANCF finite elements have been used to model contact problems, there are still research problems that need to be addressed in case of MBS contact involving very flexible bodies subjected to motion constraints.

8.11 Design Optimization and Sensitivity Analysis

In the design optimization and design sensitivity analysis, it is desirable to use an analysis method that defines accurate change in the geometry when changing the design parameters. The fact that the shape of a component modeled using ANCF elements can be changed in the reference-configuration by adjusting the vector of nodal coordinates $\mathbf{e}_o$ that defines the reference-configuration geometry makes the design process and shape optimization much simpler since in many applications developing a new FE mesh can be avoided. This simple and straightforward approach for changing the component shape cannot be achieved using conventional FE formulations that employ rotations as nodal coordinates. Stretch and change of orientation of the ANCF position-gradient vectors allow for efficient local shape manipulation that cannot be performed using conventional FE approaches. To further elaborate on this important ANCF feature, recall that the position vector $\mathbf{r}$ of the material points of an element can be written as $\mathbf{r} = \mathbf{S}(\mathbf{e}_o + \mathbf{e}_d)$. The transverse gradient vectors of a beam or plate element in the reference configuration can always be written as $(\alpha_l \hat{\mathbf{r}}_{x_l})_o$, $l = 2, 3$, where $(\alpha_l)_o$ and $(\hat{\mathbf{r}}_{x_l})_o$ are, respectively, the magnitude of and unit vector along the position-gradient vector $\mathbf{r}_{x_l}$ in the reference configuration. If the magnitudes $(\alpha_l)_o$, $l = 2, 3$, which define the dimensions of the cross-section, are used as design parameters, the derivative of the position vector with respect to these design parameters, $\partial \mathbf{r} / \partial \alpha_l = \mathbf{S}(\partial \mathbf{e}_o / \partial \alpha_l)$, $l = 2, 3$, takes a simple form since $\mathbf{r}$ is linear in α_l . The shape optimization due to the change of the orientation of the position-gradient vectors at the nodes becomes also simple because $(\hat{\mathbf{r}}_{x_l})_o$, $l = 2, 3$, can be written in terms of constant orientation parameters, and the design sensitivity analysis derivatives can be obtained in a straightforward manner without the need for remeshing. Developing new ANCF-based design optimization and design sensitivity analysis algorithms is a promising and new research area that can have impact on the efficiency of the CAE process.

8.12 Applications and Result Verifications

MBS virtual prototyping has been focused, for the most part, on rigid-body or small-deformation analyses. The small-deformation flexible-body analysis is often performed using the FE/FFR formulation which allows for obtaining efficient solutions for MBS applications with stiff components. Existing general-purpose MBS computer programs do not have well-developed capabilities based on continuum-based approaches for modeling large-deformation problems and very flexible bodies. This shortcoming limits using these

computer programs in many important technological applications. Examples of these large-deformation and very-flexible body problems are liquid sloshing, tires, leaf springs, cables, belt drives, very flexible chain drives, soft robots, biological tissues, etc. Furthermore, the literature lacks large-deformation benchmark examples that are based on more general formulations that relax the assumptions of the analytical solutions and less general conventional FE formulations. Use of such simplified analytical solutions and less general FE solutions as reference solutions to verify the results of more general formulations can lead to wrong interpretations. The solutions obtained using the more general solutions are better candidates as reference solutions. Therefore, an important research area is developing new benchmark examples using more general FE approaches that include more deformation modes that capture more realistically the behavior of the physical systems and allow for more objective comparison with experimental results.

9 Summary

While large-deformation formulations can be used to solve small-deformation problems, the large- and small-deformation problems are treated in general MBS algorithms using two fundamentally different approaches to achieve software robustness and computational efficiency. The ANCF and FFR approaches are used in MBS algorithms because these two approaches do not suffer from fundamental problems associated with other FE formulations and they allow for using nonincremental solution procedures. Nonetheless, proper implementation of the ANCF approach is necessary to avoid problems similar to those encountered with the FFR implementation which resulted in developing inaccurate models and waste of significant resources, time, and efforts for approximately three decades [245].

This paper presents an overview of the ANCF approach, provides justifications for its use, and discusses implementation issues critical for interpretation and verification of the numerical results. The paper also discusses future research directions and addresses modeling issues related to selection of coordinates, definitions of forces and moments, geometric interpretation of the position gradients, and noncommutativity of finite rotations. It is demonstrated that the interpolation order is not preserved when finite-rotation sequence is changed, while position gradients are unique and preserve the highest interpolation order. The spin tensor, associated with a rigid frame defined by the polar decomposition theorem, is used to define ANCF generalized forces due to moment application. Nonetheless, defining the spin tensor does not require explicit polar decomposition of the matrix of position-gradient vectors. As discussed in the paper, ANCF elements have features that distinguish them from conventional elements and make them suited for large-displacement MBS analysis. Their displacement fields allow increasing interpolation order without increasing number of nodes or using noncommutative finite rotations; and allow developing lower-dimension CRBF elements without lowering the interpolation order. Nonetheless, the continuum description of fully parameterized ANCF elements need to be considered when interpreting the numerical results, particularly when comparing ANCF results with solutions of conventional beam and plate models and simplified analytical approaches.

As discussed in the paper, use of higher-order elements is common in the FE literature to alleviate shortcoming of lower-order elements. For example, because the 4-node solid element is not adequate for all applications, the 20-node solid element is implemented in commercial FE software. Nonetheless, some of the higher-order elements implemented in commercial FE software do not offer the degree of continuity and conformity at the interface surfaces offered by ANCF elements. Because of the growing interest in developing new

ANCF finite elements and using these elements to solve a wide range of technological applications and science problems, developing new computer algorithms, and implementing more efficient procedures for the computations of the elastic forces [253–295]; observing the implementation issues discussed in this paper can contribute to enhancing the efficiency of the algorithms and to avoiding the problems encountered with the FFR implementation.

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Data Availability All the data used in this investigation are presented in the paper.

Declarations

Competing interests The authors declare no competing interests.

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