Commun. Math. Phys. 384, 699–732 (2021)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-021-04084-0

Communications in **Mathematical Physics**

The Speed of a Random Front for Stochastic Reaction–Diffusion Equations with Strong Noise

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Received: 13 May 2019 / Accepted: 25 March 2021

Abstract: We study the asymptotic speed of a random front for solutions $u_t(x)$ to stochastic reaction–diffusion equations of the form

$$\partial_t u = \frac{1}{2}\partial_x^2 u + f(u) + \sigma\sqrt{u(1-u)}\dot{W}(t,x),\ t \geq 0,\ x \in \mathbb{R},$$

arising in population genetics. Here, f is a continuous function with $f(0) = f(1) = 0$, and such that $|f(u)| \leq K|u(1-u)|^\gamma$ with $\gamma \geq 1/2$, and $\dot{W}(t,x)$ is a space-time Gaussian white noise. We assume that the initial condition $u_0(x)$ satisfies $0 \leq u_0(x) \leq 1$ for all $x \in \mathbb{R}$, $u_0(x) = 1$ for $x < L_0$ and $u_0(x) = 0$ for $x > R_0$. We show that when $\sigma > 0$, for each $t > 0$ there exist $R(u_t) < +\infty$ and $L(u_t) < -\infty$ such that $u_t(x) = 0$ for $x > R(u_t)$ and $u_t(x) = 1$ for $x < L(u_t)$ even if f is not Lipschitz. We also show that for all $\sigma > 0$ there exists a finite deterministic speed $V(\sigma) \in \mathbb{R}$ so that $R(u_t)/t \to V(\sigma)$ as $t \to +\infty$, almost surely. This is in dramatic contrast with the deterministic case $\sigma = 0$ for nonlinearities of the type $f(u) = u^m(1-u)$ with $0 < m < 1$ when solutions converge to 1 uniformly on $\mathbb{R}$ as $t \to +\infty$. Finally, we prove that when $\gamma > 1/2$ there exists $c_f \in \mathbb{R}$, so that $\sigma^2 V(\sigma) \to c_f$ as $\sigma \to +\infty$ and give a characterization of c_f. The last result complements a lower bound obtained by Conlon and Doering (J Stat Phys 120(3–4):421–477, 2005) for the special case of $f(u) = u(1-u)$ where a duality argument is available.

1. Introduction

Reaction–diffusion equations of the form

$$\partial_t u = \frac{1}{2}\partial_x^2 u + f(u), \tag{1.1}$$

with $f(0) = f(1) = 0$, are often used to model biological invasions and other spreading phenomena, with one steady state, say, $u \equiv 1$ invading another, $u \equiv 0$, or vice versa.

Under very mild assumptions on $f(u)$, such as, for instance, that $f(u)$ is Lipschitz on $[0, 1]$ and either $f(u) > 0$ for $u \in (0, 1)$, or there exists $\theta \in (0, 1)$ so that $f(u) \le 0$ for $u \in (0, \theta)$ and $f(u) > 0$ for $u \in (\theta, 1)$, such equations admit traveling wave solutions of the form $u_t(x) = U(x - ct)$ such that

$$-cU' = \frac{1}{2}U'' + f(U),\ U(-\infty) = 1,\ U(+\infty) = 0. \tag{1.2}$$

Note that, in the probabilistic spirit of the present paper, the subscript t denotes the time dependence of the function $u_t(x)$ rather than a time derivative, common to the PDE literature. It is easy to see that

$$c\int_{\mathbb{R}} |U'(x)|^2 dx = \int_0^1 f(z)dz, \tag{1.3}$$

thus c has the same sign as

$$I[f] := \int_0^1 f(u)du, \tag{1.4}$$

so that if $I[f] > 0$ then the steady state $u \equiv 1$ is more stable, and invades the "less stable" steady state $u \equiv 0$, and if $I[f] < 0$ then the opposite happens, while if $I[f] = 0$ then (1.1) has a time-independent solution. It is also well-known that traveling wave solutions to (1.1) determine the spreading speed for the solutions of the Cauchy problem. More precisely, let $u_t(x)$ be the solution to (1.1) with an initial condition $u_0(x)$ such that $0 \le u_0(x) \le 1$ for all $x \in \mathbb{R}$, and there exist $L_0 \le R_0$ so that $u_0(x) = 1$ for $x < L_0$ and $u_0(x) = 0$ for $x > R_0$. There exists a function $m(t)$ such that

$$|m(t) - c_* t| = o(t) \text{ as } t \to +\infty, \tag{1.5}$$

so that

$$|u_t(x + m(t)) - U_{c_*}(x)| = o(1) \text{ as } t \to +\infty. \tag{1.6}$$

Here, depending on the nature of the nonlinearity $f(u)$, the spreading speed c_* may be either the speed of the unique traveling wave, or the minimal speed of a traveling wave if traveling waves are not unique. The latter happens for the class of the Fisher-KPP nonlinearities, such that f is Lipschitz, $f(0) = f(1) = 0$, $f(u) > 0$ for all $u \in (0, 1)$, and $f(u) \le f'(0)u$ for all $u \in [0, 1]$. In that case, we have

$$c_* = \sqrt{2f'(0)}. \tag{1.7}$$

Much more precise results than (1.5)–(1.6) on the convergence of the solutions to the Cauchy problem to traveling waves are available, and we refer to the classical papers [AW78,Bra78,Bra83] for the basic results, and to [NRR18,Rob13] and references therein for more recent developments. We also point out the relation

$$c_* = \lim_{t\to+\infty} \int_{\mathbb{R}} f(u_t(x))dx = \int_{\mathbb{R}} f(U_{c_*}(x))dx, \tag{1.8}$$

that can be obtained simply by integrating (1.1) and (1.2) in space.

Note that if $f'(0)$ blows up, then the speed of propagation may also tend to infinity, as can be seen from (1.7). For Hölder nonlinearities such that $f(u) \sim u^p$ with $p \in (0, 1)$, it was shown in [AE86] that solutions become instantaneously strictly positive everywhere: $u(t, x) \ge ct^{1/(1-p)}$ for $t \ll 1$. In particular, if we approximate such nonlinearity by a sequence of Lipschitz nonlinearities f_n, then the corresponding spreading speeds $c_*^{(n)}$ blow up as $n \to +\infty$.

1.1. Reaction–diffusion equations with noise. The physical and biological systems modeled by reaction–diffusion equations are often subject to noise. In this paper, we study solutions $u_t(x)$, to the stochastic reaction–diffusion equations of the form

$$\partial_t u = \frac{1}{2}\partial_x^2 u + f(u) + \sigma\sqrt{u(1-u)}\dot{W}(t,x) \tag{1.9}$$

where $\dot{W}(t,x)$ is a space-time Gaussian white noise, and $\sigma > 0$ measures its strength. Our interest is in the effect of the noise term on the spreading speed. Since traveling waves will no longer maintain a fixed shape due to the noise, we will refer instead to the speed of the random front, which is defined below.

Let us give an motivation for the noise term in (1.9) similar to that given by Fisher in his pioneering work [Fis37]. See also [Shi88]. Imagine that two populations, type A and type B, move in a Brownian way along $\mathbb{R}$, and let $u_t(x)$ is the proportion of the population of type A at time t at position x. When an individual of type A meets an individual of type B, it can be converted into type B, and vice versa, and the outcome is partially random. The function $f(u)$ in (1.9) describes the deterministic evolution of the population of type A, due to these interactions, and it is natural to assume that $f(0) = f(1) = 0$ since there are no interactions when one type is absent. The random term in (1.9) accounts for the stochastic aspect of the interactions. We assume that for each such meeting we have a mean-zero random variable affecting the outcome, and these random variables are i.i.d. By the central limit theorem, the sum of such variables would be approximately Gaussian. The independence of the variables means that the random input should be independent for different values of t and x, giving rise to the space-time noise $\dot{W}(t,x)$. The rate of such meetings at a given site x and time t would be proportional to $u_t(x)(1-u_t(x))$, which is the variance of the noise at (t,x). Thus we should multiply the white noise $\dot{W}(t,x)$ by the standard deviation $\sqrt{u_t(1-u_t)}$. This leads to the noise term in (1.9).

As we have mentioned, we are interested in the long time speed of a random front for the solutions to (1.9). To this end, we define the left and the right edge of the solution as follows. Given a function $h(x)$ such that $0 \le h(x) \le 1$ for all $x \in \mathbb{R}$, with $h(x) \to 1$ as $x \to -\infty$ and $h(x) \to 0$ as $x \to +\infty$, we set

$$\begin{aligned} L(h) &= \inf\{x \in \mathbb{R} : h(x) < 1\} \\ R(h) &= \sup\{x \in \mathbb{R} : h(x) > 0\}. \end{aligned} \tag{1.10}$$

In the absence of the noise, when $\sigma = 0$, and for Lipschitz nonlinearities $f(u)$, we have $L(u_t) = -\infty$ and $R(u_t) = +\infty$ for all $t > 0$. This, however, is not necessarily the case in the presence of the noise. In order to make this claim precise, we assume that

$$\begin{aligned} &f \text{ is continuous on } [0,1] \text{ and there exists} \\ &K_f > 0 \text{ such that } |f(u)| \le K_f\sqrt{|u(1-u)|}. \end{aligned} \tag{1.11}$$

As for the initial condition $u_0(x)$, we will assume that

$$\begin{aligned} &0 \le u_0(x) \le 1 \text{ for all } x \in \mathbb{R}, \\ &\text{and both } L(u_0) \text{ and } R(u_0) \text{ are finite.} \end{aligned} \tag{1.12}$$

We will denote by $\mathcal{C}_I$ the set of continuous functions satisfying (1.12). In addition $\widehat{\mathcal{B}}_I$ will denote the space of functions on $\mathbb{R}$ taking values in $[0,1]$ and $\widehat{\mathcal{C}}_I$ will denote the space of continuous functions on $\mathbb{R}$ taking values in $[0,1]$.

We say that u_t has a speed $V(\sigma)$ if the following limit exists:

$$V(\sigma) = \lim_{t\to\infty} \frac{R(u_t)}{t}.$$

We prove the following theorem in Sect. 2.

Theorem 1.1. *Let $f(u)$ satisfy (1.11) and $u_0(x)$ be as in (1.12), then (1.9) with an initial condition $u_0(x)$ has a solution $u_t(x)$ taking values in $\widehat{\mathcal{C}}_I$ for $t > 0$. The solution is unique in law. Moreover, $L(u_t)$ and $R(u_t)$ are almost surely finite for all $t \geq 0$ and the solution has a speed $V(\sigma) \in \mathbb{R}$.*

We see that the noise has a very strong slowdown effect: $V(\sigma)$ is finite for all $\sigma > 0$ even if $f(u)$ is Hölder with an exponent $m \geq 1/2$, and not Lipschitz, such as, for instance $f(u) = u^m(1-u)$, for which, as we have mentioned, the speed of the front is infinite when $\sigma = 0$.

Most of the papers dealing with (1.9), such as Mueller and Sowers [MS95] have treated the Fisher-KPP nonlinearity $f(u) = u(1-u)$, and small noise, where σ is close to 0. Mueller, Mytnik, and Quastel [MMQ11] studied the behavior of $V(\sigma)$ as $\sigma \downarrow 0$ and verified some conjectures of Brunet and Derrida [BD97] and [BD00]. Less attention has been devoted to $V(\sigma)$ for large or intermediate values of σ, but Conlon and Doering [CD05] proved that for $f(u) = u(1-u)$ there exists an asymptotic velocity $V(\sigma) > 0$ for solutions u to (1.9) for all $\sigma > 0$, and that

$$\liminf_{\sigma\to\infty} \sigma^2 V(\sigma) \geq 1. \tag{1.13}$$

Note that (1.13) differs from (1.7) in [CD05] because the diffusivity in that paper is taken to be 1 rather than 1/2 as chosen here. To formulate our main result, we note that a rescaling of (1.9), discussed in detail in Sect. 4.1 below allows us to move the noise coefficient into the nonlinearity, and obtain the rescaled equation

$$\partial_t v = \frac{1}{2}\partial_x^2 v + \sigma^{-4} f(v) + \sqrt{v(1-v)}\,\dot{W}(t,x). \tag{1.14}$$

Here v is a rescaling of u which we specify later. Later we will use the results of Tribe [Tri95], and Mueller and Tribe [MT97] for (1.14) with $f = 0$, a version of a continuous voter model, or a stepping stone model in population genetics:

$$\partial_t w = \frac{1}{2}\partial_x^2 w + \sqrt{w(1-w)}\,\dot{W}(t,x). \tag{1.15}$$

By Theorem 1 of [MT97], we know that the law of $w_t(L(w_t) + x)$ converges weakly to a stationary distribution as $t \to \infty$. We denote the expectation with respect to the stationary distribution of w by $\mathbb{E}_{w,st}$, where "st" is an abbreviation for "stationary". For the next theorem we need an assumption on f which is slightly stronger than (1.11): we assume that

$$\begin{aligned} &f \text{ is continuous on } [0,1] \text{ and there exists } \widetilde{K}_f > 0 \\ &\quad \text{s.t. } |f(u)| \leq \widetilde{K}_f |u(1-u)|^{\gamma} \text{ for some } \gamma \in (1/2, 1]. \end{aligned} \tag{1.16}$$

Theorem 1.2. *Suppose that* u_0 *satisfies* (1.12) *and* f *satisfies* (1.16). *Then we have*

$$\lim_{\sigma\to\infty} \sigma^2 V(\sigma) = c_f, \tag{1.17}$$

where

$$c_f \equiv \mathbb{E}_{w,st}\left[\int_{\mathbb{R}} f(w(x))\,dx\right] \tag{1.18}$$

and

$$|c_f| < \infty. \tag{1.19}$$

Note that Lemma 2.1 of [Tri95] shows that

$$\lim_{t\to\infty} \mathbb{E}_w\left[\int_{\mathbb{R}} w_t(x)(1-w_t(x))\,dx\right] = 1. \tag{1.20}$$

Let us explain why c_f is finite, at least for Lipschitz f that satisfy (1.16) with $\gamma = 1$. Using Theorem 1 of [MT97] and the Skorokhod representation theorem to switch to the probability space where w_t converges to w almost surely, we get

$$|c_f| = \left|\mathbb{E}_{w,st}\Big[\int f(w(x))dx\Big]\right| = \left|\mathbb{E}_w\Big[\int \lim_{t\to\infty} f(w_t(x))dx\Big]\right|.$$

Then by (1.16) and the Fatou lemma we have

$$\begin{aligned}|c_f| &\le \widetilde{K}_f\mathbb{E}_w\Big[\int \lim_{t\to\infty} w_t(x)(1-w_t(x))dx\Big]\\ &\le \widetilde{K}_f \lim_{t\to\infty} \mathbb{E}_w\Big[\int w_t(x)(1-w_t(x))dx\Big] < +\infty.\end{aligned}$$

For general f satisfying (1.16), we show that (1.19) holds in Lemma 3.4.

In particular, as a consequence of Theorem 1.2 and (1.20), we get that for the Fisher-KPP nonlinearity $f(u) = u(1-u)$, we have

$$\lim_{\sigma\to\infty} \sigma^2 V(\sigma) = 1,$$

giving a matching upper bound to the lower bound (1.13) of Conlon and Doering in [CD05], after adjusting for the different diffusivities adopted in the present paper and in [CD05].

We also see the slowdown due to strong noise in Theorem 1.2 even for Lipschitz nonlinearities. As discussed in Sect. 4.1 below, the large noise asymptotics in (1.17) corresponds to the speed of the front for solutions of (1.14) that is $V^{(v)}(\sigma) \sim c_f/\sigma^4$. However, solutions of the corresponding equation without the noise

$$\partial_t v = \frac{1}{2}\partial_x^2 v + \sigma^{-4} f(v) \tag{1.21}$$

spread with the speed $\bar{V}(\sigma) = c_*/\sigma^2$, where c_* is the speed of the traveling wave for (1.21) with $\sigma = 1$, so that $V^{(v)}(\sigma) \ll \bar{V}(\sigma)$ for $\sigma \gg 1$, and the noise slows down the propagation.

Let us also point out that expression (1.17)–(1.18) for the front speed $V(\sigma)$ is a direct analog of (1.8) except now the role of the traveling wave is played by the invariant measure of $w_t(x)$. One may conjecture that instead of the convergence to a traveling wave in shape, as in (1.6) that happens in the deterministic case, here, in the limit $\sigma \to +\infty$, the law of $u_t(x)$ after rescaling converges, as $t \to +\infty$, in the frame moving with the speed $V(\sigma)$, to the invariant distribution of $w_t(x)$.

Another interesting observation is that the noise, despite its symmetry with respect to $u = 0$ and $u = 1$ can change the direction of the invasion. One may construct a nonlinearity f such that $I(f)$ given by (1.4) has a different sign than c_f, meaning that that the speed of propagation for $\sigma = 0$, in the absence of the noise, may have a different sign than $V(\sigma)$ for large $\sigma \gg 1$, changing the direction of the invasion, because of the noise.

The paper is organized as follows. The proof of Theorem 1.1 is in Sect. 2. Section 3 contains some auxiliary results on solutions to (1.15). They are used later in the proof of Theorem 1.2, presented in Sect. 4 for the upper bound, and in Sect. 5 for the matching lower bound on the speed $V(\sigma)$ for $\sigma \gg 1$.

2. The Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Existence of a solution to (1.9) follows by a rather standard argument. To prove the uniqueness, we use Girsanov's theorem. In order to be able to apply this theorem, we need to have an a priori bound showing that for any solutions to (1.9) taking values in $\widehat{\mathcal{B}}_I$ for all $t \geq 0$ with $R(u_0) < +\infty$, $L(u_0) > -\infty$, we have $-\infty < L(u_t) < R(u_t) < +\infty$ for all $t \geq 0$, almost surely.

2.1. Existence of a solution. We first show that (1.9) has a mild solution. The notion of a mild solution to (1.9) follows the standard definition, see Walsh [Wal86]. We interpret (1.9) as a shorthand for the mild form,

$$u_t(x) = \int_{\mathbb{R}} G_t(x-y)u_0(y)dy + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)f(u_s(y))dyds \\ + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-y)\sqrt{u_s(y)(1-u_s(y))}W(dyds), \tag{2.1}$$

where $u_0(x)$ is the given initial condition. Here,

$$G_t(x) = (2\pi t)^{-1/2} \exp\left(-x^2/(2t)\right),$$

is the fundamental solution of the heat equation

$$\partial_t G = \frac{1}{2}\partial_x^2 G.$$

In what follows, with some abuse of notation $\{G_t\,, t \geq 0\}$ will also denote the corresponding semigroup, that is,

$$G_t\phi(x) = \int_{\mathbb{R}} G_t(x-y)\phi(y)\,,dy,\ \ t > 0, \tag{2.2}$$

for any function ϕ for which the above integral is well-defined.

Almost sure existence and uniqueness of mild solutions to SPDEs of the form

$$\partial_t u = \frac{1}{2}\partial_x^2 u + f(u) + a(u)\dot{W}(t,x) \tag{2.3}$$

is standard [Wal86] when the coefficients are Lipschitz continuous functions of u. Because in our case $f(u)$ may be not Lipschitz, and $a(u) = \sqrt{u(1-u)}$ is not Lipschitz, one needs to be slightly more careful. Solutions to (1.9) are constructed as follows. Let the initial condition u_0 satisfy (1.12). We approximate $f(u)$ and $a(u)$ by Lipschitz functions $f_n(u)$ and $a_n(u)$ such that

$$f_n(0) = f_n(1) = a_n(0) = a_n(1) = 0,$$

and construct the corresponding solutions $u_t^n(x)$ using the standard theory. The comparison principle implies that $u_t^n(x)$ take values in [0, 1], see [Shi94] and [Mue91]. The proof of Theorem 2.6 of [Shi94], on pp. 436–437 of that paper, shows that the sequence $u_t^n(x)$ is tight. Note that in [Shi94] the tightness (and therefore existence of a solution) is proved for the processes taking values in a space of unbounded functions. We can follow the proof of Theorem 2.6 of [Shi94] line by line to show that in our situation the tightness holds for the sequence of function-valued processes with functions taking values in [0, 1]. Passing to the limit $n \to +\infty$ we obtain a mild solution $u_t(x)$ to (1.9) taking values in [0, 1]. This proves existence of a solution.

Remark 2.1. (Comparison principle) Using the above existence proof it is easy to show existence of two solutions $u_t^{(1)}(x), u_t^{(2)}(x)$ to (1.9) with initial values $g_1 \leq g_2$, respectively, defined on the same probability space, and such that that with probability 1, we have $u_t^{(1)}(x) \leq u_t^{(2)}(x)$ for all t, x. To show this, we approximate $f(u)$ and $a(u) = \sqrt{u(1-u)}$ by Lipschitz functions $f_n(u)$ and $a_n(u)$ such that

$$f_n(0) = f_n(1) = a_n(0) = a_n(1) = 0,$$

and construct the corresponding solutions $u_t^{1,n}(x), u^{2,n}$ with initial values $g_1 \leq g_2$, respectively. The comparison principle implies that $u_t^{1,n}(x) \leq u_t^{2,n}(x)$ for all t, x (see [Shi94]) and all n. Using again the proof of Theorem 2.6 of [Shi94], we see that the sequence of pairs of processes $(u^{(1,n)}, u^{(2,n)})$ is tight. Any limit point will preserve the inequality $u^{(1)} \leq u^{(2)}$. So we can always construct a pair of solutions $u^{(1)}, u^{(2)}$ to (1.9) on the same probability space with initial data $g_1 \leq g_2$, respectively, coupled in a way so that $u^{(1)} \leq u^{(2)}$. After the proof of the weak uniqueness for (1.9) in Theorem 1.1 is finished, we get that both $u^{(1)}$ and $u^{(2)}$ are unique in law (but not necessarily as a pair!).

2.2. Uniqueness via the Girsanov theorem. In order to prove uniqueness in law of the solution to (1.9), we will use a version of the Girsanov theorem that will allow us to compare the laws of the solution $u_t(x)$ to (1.9) and $w_t(x)$, the solution to (1.15), which corresponds to $f = 0$ in (1.9), with the same initial condition $w_0(x) = u_0(x)$. Recall that we have set $\sigma = 1$, including in (1.15). Let $\mathbb{P}_{t,u}$ be the measure induced on the canonical path space up to time t by u, and $\mathbb{P}_{t,w}$ be the measure induced by w, also up to time t. We also define the corresponding expectations $\mathbb{E}_{t,u}$ and $\mathbb{E}_{t,w}$, and write $\mathbb{P}_u$ for $\mathbb{P}_{\infty,u}$, and likewise $\mathbb{P}_w$ for $\mathbb{P}_{\infty,w}$. We will not use the subscripts in the situations when it is clear which probability measure is used.

In [Daw78], Dawson gives a version of Girsanov's theorem which applies to $\mathbb{P}_{t,u}$ and $\mathbb{P}_{t,w}$. We will use its variant, Theorem IV.1.6 in [Per02]. In such theorems, the change of measure always involves an exponential term which must be a martingale. In our situation, let

$$Z_t = \int_0^t \int_{\mathbb{R}} \frac{f(w_s(x))}{\sqrt{w_s(x)(1-w_s(x))}} W(dx, ds) - \frac{1}{2} \int_0^t \int_{\mathbb{R}} \frac{f(w_s(x))^2}{w_s((x)1-w_s(x))} dxds. \tag{2.4}$$

Here, and elsewhere we adopt the convention in the integrands that

$$\frac{f(u)}{\sqrt{u(1-u)}} = 0 \text{ if } u = 0 \text{ or } u = 1.$$

Then Girsanov's theorem for stochastic PDE [Daw78,Per02] says that

$$\frac{d\mathbb{P}_{t,u}}{d\mathbb{P}_{t,w}} = e^{Z_t}, \tag{2.5}$$

as long as

$$\int_0^t \int_{\mathbb{R}} \frac{f(u_s(x))^2}{u_s(x)(1-u_s(x))} dxds < +\infty, \quad \mathbb{P}_u\text{-almost surely.} \tag{2.6}$$

In particular, if (2.6) holds then (2.5) and uniqueness in law for solutions of (1.15) (see [Shi88]) imply immediately that the solution to (1.9) is unique in law. For the moment, as we do not have any information on the support of $f(u_s(x))$, we can not conclude that (2.6) holds. The bulk of the rest of this section is to show that (2.6) holds for any solution to (1.9) taking values in $\widehat{\mathcal{B}}_I$ for all $t \geq 0$ and such that $R(u_0) < +\infty$ and $L(u_0) > -\infty$.

First, we make a much simpler observation that allows us to use Girsanov's theorem to eliminate the drift on a finite interval. Fix and arbitrary $b > 0$ and let v^b denote a solution to a modified version of (1.9), with the nonlinearity set to zero on the interval $[-10b, 10b]$:

$$\begin{aligned} \partial_t v_t^b(x) = \frac{1}{2}\partial_x^2 v_t^b(x) + f(v_t^b(x))\mathbf{1}_{\{x\in(-\infty,-10b)\cup(10b,\infty)\}} \\ +\sqrt{v_t^b(x)(1-v_t^b(x))}\dot{W}(t,x). \end{aligned} \tag{2.7}$$

We again write this equation in the mild form:

$$\begin{aligned} v_t^b(x) &= G_t v_0^b(x) \\ &\quad + \int_0^t \int_{\mathbb{R}} G_{t-s}(x-z) f(v_s^b(z))\mathbf{1}_{\{z\in(-\infty,-10b)\cup(10b,\infty)\}}\, dz + N_t^b(x), \end{aligned} \tag{2.8}$$

where

$$N_t^b(x) = \int_0^t \int_{\mathbb{R}} G_{t-s}(x-z)\sqrt{v_s^b(z)(1-v_s^b(z))}\,W(ds,dz). \tag{2.9}$$

Let $\mathbb{P}_{t,v^b}$ be the measure induced on the canonical path space up to time t by v^b, with the corresponding expectation $\mathbb{E}_{t,v^b}$, and $\mathbb{P}_{\infty,v^b}$ be $\mathbb{P}_{v^b}$. Note that by (1.11) we have

$$\int_0^t \int_{\mathbb{R}} \frac{f(u_s(x))^2\mathbf{1}_{\{x\in(-10b,10b)\}}}{u_s(x)(1-u_s(x))} dxds \leq 20bK_f^2 t < +\infty, \quad \mathbb{P}_u\text{-almost surely.} \tag{2.10}$$

Thus we can use Girsanov's theorem for stochastic PDE [Daw78,Per02] to get

$$\frac{d\mathbb{P}_{t,u}}{d\mathbb{P}_{t,v^b}} = e^{Z_t^b}, \tag{2.11}$$

where

$$\begin{aligned} Z_t^b &= \int_0^t \int_{\mathbb{R}} \frac{f(v_s^b(x))\mathbf{1}_{\{x\in(-10b,10b)\}}}{\sqrt{v_s^b(x)(1-v_s^b(x))}} W(dx,ds) \\ &\quad - \frac{1}{2}\int_0^t \int_{\mathbb{R}} \frac{f(v_s^b(x))^2\mathbf{1}_{\{x\in(-10b,10b)\}}}{v_s^b((x)1-v_s^b(x))} dxds. \end{aligned} \tag{2.12}$$

2.3. A bound on the front speed. The next step is to get the following bound on the speed of the front of u.

Lemma 2.2. *Let $u_t(x)$ be a solution to (1.9) taking values in $\widehat{\mathcal{B}}_I$ for all $t \geq 0$ such that the initial condition $u_0(x)$ satisfies (1.12). Then, for all $T > 0$, both $\sup_{t\leq T} R(u_t)$ and $\inf_{t\leq T} L(u_t)$ are almost surely finite. Moreover, for all $T \geq 0$ there exists $C_T > 0$ so that for all $b \geq 4\sqrt{T}(T\|f\|_\infty \vee 1)$ we have, with $R_0 := R(u_0)$, $L_0 = L(u_0)$:*

$$\begin{aligned} &\mathbb{P}\big(\sup_{0\leq t\leq T}(R(u_t)-R_0) > b\big) \\ &\quad +\mathbb{P}\big(\sup_{0\leq t\leq T}(L_0-L(u_t)) > b\big) \leq C_T \exp\Big(-\frac{b^2}{100T}\Big). \end{aligned} \tag{2.13}$$

Since

$$|R(u(t)) - L(u(t)| \leq (R(u_t)-R_0)_+ + (L_0 - L(u_t))_+ + |R_0 - L_0|,$$

an immediate consequence is

Corollary 2.3. *Under conditions of Lemma 2.2, we have, for each $T \geq 0$:*

$$\begin{aligned} &\mathbb{E}\big[\sup_{0\leq t\leq T}(R(u_t)-R_0)_+\big] < +\infty, \\ &\mathbb{E}\big[\sup_{0\leq t\leq T}(L_0-L(u_t))_+\big] < +\infty, \\ &\mathbb{E}\big[\sup_{0\leq t\leq T}|R(u_t)-L(u_t)|\big] < +\infty. \end{aligned} \tag{2.14}$$

In other words, the length of the interface of any solution to (1.9) has a finite expectation.

2.3.1. Bounds on the martingale with the cut-off The proof of Lemma 2.2 relies on a priori bounds on the propagation of v^b, solution to (2.7). First, we need to control the modulus of continuity of the martingale $N_t^b(\cdot)$ defined in (2.9).

Lemma 2.4. *Let $v_t^b(x)$ be a solution to (2.7) taking values in $\widehat{\mathcal{B}}_I$ for all $t \geq 0$, such that the initial condition $v_0^b(x)$ satisfies (1.12). Then, for all $p \geq 1$, there exists $C(p) > 0$ so that for all $t \geq s \geq 0$, and $x, y \in [b/2, 9b]$ we have*

$$\begin{aligned}
&\mathbb{E}\big[|N_t^b(x) - N_t^b(y)|^{2p}\big] \\
&\quad \leq C(p)(|x-y| \wedge t^{1/2})^{p-1} t^{1/2} \int_{\mathbb{R}} (G_t(x-z) + G_t(y-z)) \\
&\qquad \times (v_0^b(z) + t\|f\|_\infty \mathbf{1}_{\{z\in(-\infty,-10b)\cup(10b,\infty)\}})\, dz, \qquad (2.15)\\
&\mathbb{E}\big[|N_t^b(x) - N_s^b(x)|^{2p}\big] \\
&\quad \leq C(p)|t-s|^{(p-1)/2} t^{1/2} \int_{\mathbb{R}} (G_t(x-z) + G_s(x-z)) \qquad (2.16)\\
&\qquad \times (v_0^b(z) + t\|f\|_\infty \mathbf{1}_{\{z\in(-\infty,-10b)\cup(10b,\infty)\}})\, dz.
\end{aligned}$$

Proof. The proof follows the lines of the proof of Lemma 3.1 in [Tri95]. We only verify (2.15). Note that

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} (G_{t-s}(x-z) - G_{t-s}(y-z))^2\, dz\, ds \\
&\quad \leq C(|x-y| \wedge t^{1/2}) \ \ \forall t > 0, x, y \in \mathbb{R}. \qquad (2.17)
\end{aligned}$$

Burkholder's and Hölder's inequalities give

$$\begin{aligned}
&\mathbb{E}\big[|N_t^b(x) - N_t^b(y)|^{2p}\big] \\
&\leq C(p)\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}} (G_{t-s}(x-z) - G_{t-s}(y-z))^2 v_s^b(z)(1-v_s^b(z))\, dz\, ds\right)^p\right] \\
&\leq C(p)(|x-y| \wedge t^{1/2})^{p-1} \\
&\quad \times \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} (G_{t-s}(x-z) - G_{t-s}(y-z))^2 \big(v_s^b(z)(1-v_s^b(z))\big)^p\, dz\, ds\right] \\
&\leq C(p)(|x-y| \wedge t^{1/2})^{p-1} \\
&\quad \times \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} (G_{t-s}(x-z) - G_{t-s}(y-z))^2 v_s^b(z)\, dz\, ds\right] \\
&\leq C(p)(|x-y| \wedge t^{1/2})^{p-1} \qquad (2.18)\\
&\quad \times \mathbb{E}\left[\int_0^t (t-s)^{-1/2} \int_{\mathbb{R}} (G_{t-s}(x-z) + G_{t-s}(y-z)) v_s^b(z)\, dz\, ds\right].
\end{aligned}$$

We used the fact that $0 \leq v^b \leq 1$ in the third inequality above. Note that

$$\begin{aligned}
\mathbb{E}[v_s^b(x)] &= G_s v_0^b(x) + \mathbb{E}\Big[\textstyle\int_0^s \int_{\mathbb{R}} G_{s-r}(x-z) f(v_r^b(z)) \mathbf{1}_{\{z\in(-\infty,-10b)\cup(10b,\infty)\}}\, dz\, dr\Big] \quad (2.19)\\
&\leq G_s v_0^b(x) + \|f\|_\infty \textstyle\int_0^s \int_{\mathbb{R}} G_{s-r}(x-z) \mathbf{1}_{\{z\in(-\infty,-10b)\cup(10b,\infty)\}}\, dz\, dr.
\end{aligned}$$

We substitute this bound into the right side of (2.18) and use the semi-group property of G_t to get

$$
\begin{aligned}
\mathbb{E}\big[|N_t^b(x) - N_t^b(y)|^{2p}\big] &\le C(p)(|x-y| \wedge t^{1/2})^{p-1} \\
&\quad \times \Big\{ \int_0^t (t-s)^{-1/2} \Big(\int_{\mathbb{R}} (G_t(x-z) + G_t(y-z)) v_0^b(z)\, dz \\
&\quad + \int_0^s \|f\|_\infty \int_{\mathbb{R}} (G_{t-r}(x-z) + G_{t-r}(y-z)) \\
&\quad \times \mathbf{1}_{\{z\in(-\infty,-10b)\cup(10b,\infty)\}})\, dz\, dr \Big)\, ds \Big\} \\
&\le C(p)(|x-y| \wedge t^{1/2})^{p-1} t^{1/2} \Big(\int_{\mathbb{R}} (G_t(x-z) + G_t(y-z)) v_0^b(z)\, dz \\
&\quad + \|f\|_\infty \int_0^t \int_{\mathbb{R}} (G_{t-r}(x-z) + G_{t-r}(y-z)) \mathbf{1}_{\{z\in(-\infty,-10b)\cup(10b,\infty)\}}\, dz\, dr \Big).
\end{aligned}
$$

Since $x, y \in (b/2, 9b)$ and $z \ge 10b$ we have

$$
\int_{z\ge 10b} G_r(x-z)\, dz \le \int_{z\ge 10b} G_t(x-z)\, dz, \quad \forall x \in (b/2, 9b), \quad 0 \le r \le t, \tag{2.20}
$$

and thus we get

$$
\begin{aligned}
\mathbb{E}\big[|N_t^b(x) - N_t^b(y)|^{2p}\big] \le C(p)(|x-y| \wedge |t-s|^{1/2})^{p-1} \\
\times t^{1/2} \int_{\mathbb{R}} (G_t(x-z) + G_t(y-z)) \\
\times \Big(v_0^b(z) + t\|f\|_\infty \mathbf{1}_{\{z\in(-\infty,-10b)\cup(10b,\infty)\}} \Big)\, dz,
\end{aligned} \tag{2.21}
$$

which is (2.15). The proof of (2.16) goes along similar lines. □

A corollary of Lemma 2.4 is a bound on the size of $N_s^b(x)$.

Lemma 2.5. *Let $v_t^b(x)$ be a solution to (2.7), taking values in $\widehat{\mathcal{B}}_I$ for all $t \ge 0$, and the initial condition $v_0^b(x)$ satisfies (1.12). Then, for all $t > 0$, there exists C such that*

$$
\begin{aligned}
&\mathbb{P}\big(|N_s^b(x)| \ge \varepsilon \text{ for some } x \in (b/2, 9b), s \in [0,t]\big) \\
&\quad \le C\varepsilon^{-20}(t \vee t^{22}) \int_{\mathbb{R}} \int_{\mathbb{R}} G_t(x-z) \\
&\qquad \times \Big(v_0^b(z) + t\|f\|_\infty \mathbf{1}_{\{z\in(-\infty,-10b)\cup(10b,\infty)\}} \Big)\, dz \mathbf{1}_{\{x\in(b/2,9b)\}}\, dx.
\end{aligned} \tag{2.22}
$$

Proof. The proof goes exactly as the second part of the proof of Lemma 3.1 in [Tri95] (on p. 295) while taking $v_0^b(z) + t\|f\|_\infty \mathbf{1}_{\{z\in(-\infty,-10b)\cup(10b,\infty)\}}$ instead of f and $(b/2, 9b)$ instead of (A, ∞) there. □

2.3.2. The support of the solution with a cut-off Now, we prove the following lemma.

Lemma 2.6. *Let $v_t^b(x)$ be a solution to (2.7) taking values in $\widehat{\mathcal{B}}_I$ for all $t \geq 0$ such that the initial condition $v_0^b(x)$ satisfies (1.12) with $R(v_0^b) \leq 0$. Then, for all $t > 0$ there exists $C(t, \|f\|_\infty) > 0$ so that for all $b \geq 4\sqrt{t}(t\|f\|_\infty \vee 1)$ we have*

$$\mathbb{P}\big(\sup_{0\leq s\leq t}\sup_{x\in[b,2b]} v_s^b(x) > 0\big) \leq C(t, \|f\|_\infty)\exp\Big(-\frac{b^2}{50t}\Big). \tag{2.23}$$

Proof. We will follow the proof of Proposition 3.2 in [Tri95]. Let us take a function

$$\psi \in L^1(\mathbb{R}) \cap C(\mathbb{R})$$

such that $0 \leq \psi(x) \leq 1$ for all $x \in \mathbb{R}$ and $\{x : \psi(x) > 0\} = (0, b)$, and set $\psi_b(x) = \psi(x - b)$. For simplicity of notation, we define

$$\langle h, g\rangle = \int_{\mathbb{R}} h(x)g(x)\,dx$$

for any functions h, g such that the integral above exists.

Fix $t > 0$ and let $\phi_s^\lambda(x)$, $0 \leq s \leq t$, $x \in \mathbb{R}$ be the unique non-negative bounded solution to the backward in time problem

$$-\partial_s\phi_s^\lambda = \frac{1}{2}\Delta\phi_s^\lambda - \frac{1}{4}(\phi_s^\lambda)^2 + \lambda\psi_b, \tag{2.24}$$

with the terminal condition $\phi_t^\lambda(x) \equiv 0$. A similar equation to (2.24) but with different function ψ_b in the right side appears in the proof of Proposition 3.2 in [Tri95]. As $\psi_b(x) \geq 0$ for all $x \in \mathbb{R}$, the maximum principle implies existence of the solution to (2.24) and that $\phi_s^\lambda(x) \geq 0$ for all $0 \leq s \leq t$ and $x \in \mathbb{R}$. The maximum principle also implies that

$$\phi_s^\lambda(x) \leq \lambda\int_0^{t-s}\int G_r(x-y)\psi_b(y)dy\,dr, \quad s \leq t.$$

Since $\int G_r(x-y)dx = 1$ and ψ_b is integrable by assumptions on ψ, we get that $\phi_s^\lambda(x)$ is integrable for all $0 \leq s \leq t$. Next, note that the function

$$\zeta_t(x) = \begin{cases} \frac{\alpha}{(x-b)^2}, & x < b, \\ \frac{\alpha}{(x-2b)^2}, & x > 2b, \end{cases}$$

satisfies, in the region $x < b$, where $\psi_b(x) \equiv 0$:

$$\partial_t\zeta - \frac{1}{2}\Delta\zeta + \frac{1}{4}\zeta^2 - \lambda\psi_b = -\frac{1}{2}\frac{2\cdot 3\alpha}{(x-b)^4} + \frac{\alpha^2}{4(x-b)^4} = \frac{\alpha(\alpha-12)}{(x-b)^4} \geq 0,$$

provided that we take $\alpha \geq 12$. As $\zeta_t(x) = +\infty$ at $x = b$, the maximum principle implies that, for α sufficiently large, we have

$$\phi_s^\lambda(x) \leq \frac{\alpha}{(b-x)^2}, \quad \text{for all } x < b,\, s \leq t, \text{ and } \lambda > 0. \tag{2.25}$$

Similarly, again for α large enough, we get

$$\phi_s^\lambda(x) \le \frac{\alpha}{(2b-x)^2}, \quad \text{for all } x > 2b,\ s \le t,\ \text{and } \lambda > 0. \tag{2.26}$$

Now, given any $b \ge 4t^{1/2}$, we may use the fundamental solution for the heat equation on the half-lines $x < b - t^{1/2}$, $x > 2b + t^{1/2}$ together with the upper bound in (2.25) on $\phi_s^\lambda(x)$ at $x = b - t^{1/2}$, and $x = 2b + t^{1/2}$ to conclude that there exists $\alpha_1 > 0$ such that

$$\phi_s^\lambda(x) \le \frac{\alpha_1}{t} \exp\Big(- \frac{(b-x)^2}{20t} \Big), \\ \text{for all } b \ge 4t^{1/2},\ x < b - 2t^{1/2},\ s \le t,\ \text{and } \lambda > 0, \tag{2.27}$$

and

$$\phi_s^\lambda(x) \le \frac{\alpha_1}{t} \exp\Big(- \frac{(2b-x)^2}{20t} \Big), \\ \text{for all } b \ge 4t^{1/2},\ x > 2b + 2t^{1/2},\ s \le t,\ \text{and } \lambda > 0. \tag{2.28}$$

Next, by Itô's formula, we get, for any $0 \le s \le t$:

$$\begin{aligned} &\exp\Big(- \langle v_s^b, \phi_s^\lambda \rangle - \lambda \int_0^s \langle v_{s'}^b, \psi_b \rangle\, ds' \Big) = \exp\big(- \langle v_0^b, \phi_0^\lambda \rangle \big) \\ &+ \int_0^s \exp\Big(- \langle v_{s'}^b, \phi_{s'}^\lambda \rangle - \lambda \int_0^{s'} \langle v_r^b, \psi_b \rangle\, dr \Big) \\ &\times \Big(\langle v_{s'}^b, -\partial_s \phi_{s'}^\lambda - \frac{1}{2}\Delta \phi_{s'}^\lambda - \lambda \psi_b \rangle - \langle f(v_{s'}^b) \mathbf{1}_{(-\infty,-10b)\cup(10b,\infty)}, \phi_{s'}^\lambda \rangle \\ &+ \frac{1}{2} \langle v_{s'}^b (1 - v_{s'}^b), (\phi_{s'}^\lambda)^2 \rangle \Big)\, ds' \\ &+ M_s^{\phi^\lambda, \psi_b}, \end{aligned}$$

where $s \mapsto M_s^{\phi^\lambda, \psi_b}$, $s \le t$, is a local martingale. In fact, M^{ϕ^λ, ψ_b} is a square integrable martingale: this follows easily from integrability of $(\phi^\lambda)^2$. Then we get

$$\begin{aligned} &\exp\Big(- \langle v_s^b, \phi_s^\lambda \rangle - \lambda \int_0^s \langle v_{s'}^b, \psi_b \rangle\, ds' \Big) \\ &= \exp(-\langle v_0^b, \phi_0^\lambda \rangle) + \int_0^s \exp\Big(- \langle v_{s'}^b, \phi_{s'}^\lambda \rangle - \lambda \int_0^{s'} \langle v_r^b, \psi_b \rangle\, dr \Big) \\ &\times \big(\langle -f(v_{s'}^b) \mathbf{1}_{(-\infty,-10b)\cup(10b,\infty)}, \phi_{s'}^\lambda \rangle + \langle -\frac{1}{4} v_{s'}^b \\ &+ \frac{1}{2} v_{s'}^b (1 - v_{s'}^b), (\phi_{s'}^\lambda)^2 \rangle \big)\, ds' + M_s^{\phi^\lambda, \psi_b}. \end{aligned} \tag{2.29}$$

Note that (2.25) implies that for $b > R_0$ we have a uniform bound

$$|\langle v_0^b, \phi_0^\lambda \rangle| \le c_0, \tag{2.30}$$

with a constant c_0 that does not depend on λ. Now we define the stopping times

$$\tau_b = \inf\left\{t \geq 0 : \exists x \in [b/2, 3b] \text{ s.t. } v_t^b(x) \geq \frac{1}{2}\right\}, \quad \rho_b = \inf\{t \geq 0 : \langle v_t^b, \psi_b\rangle > 0\}.$$

Note that we have

$$\langle v^b_{t\wedge\tau_b}, \phi^\lambda_{t\wedge\tau_b}\rangle + \lambda \int_0^{t\wedge\tau_b} \langle v_s^b, \psi_b\rangle\, ds \to +\infty \text{ as } \lambda \to +\infty, \tag{2.31}$$

almost surely on the event $\{\rho_b < t \wedge \tau_b)$, thus

$$\mathbb{P}(\rho_b < t\wedge\tau_b) \leq \lim_{\lambda\to+\infty} \mathbb{E}\Big[1 - \exp\Big(-\langle v_{t\wedge\tau_b}, \phi^\lambda_{t\wedge\tau_b}\rangle - \lambda\int_0^{t\wedge\tau_b}\langle v_s, \psi_b\rangle\, ds\Big)\Big] \tag{2.32}$$

On the other hand, taking the expectation in (2.29) with $s = t \wedge \tau_b$, we get

$$\begin{aligned}&\mathbb{E}\Big[1 - \exp\Big(-\langle v^b_{t\wedge\tau_b}, \phi^\lambda_{t\wedge\tau_b}\rangle - \lambda\int_0^{t\wedge\tau_b}\langle u_s, \psi_b\rangle\, ds\Big)\Big] \leq \mathbb{E}\big[1 - \exp\big(-\langle v_0^b, \phi_0^\lambda\rangle\big)\big]\\ &+\mathbb{E}\Big[\int_0^t \Big(\|f\|_\infty \langle \mathbf{1}_{(-\infty,-10b)\cup(10b,\infty)}, \phi_s^\lambda\rangle + \langle \frac{1}{4} v_s^b \mathbf{1}_{(-\infty,b/2)\cup(3b,\infty)}, (\phi_s^\lambda)^2\rangle\Big)\, ds\Big].\end{aligned} \tag{2.33}$$

Note that for each $0 \leq s \leq t$ and $x \in \mathbb{R}$ the family $\phi_s^\lambda(x)$ is increasing in λ. Moreover, for $s < t$ and $x > b$ we have $\phi_s^\lambda(x) \to +\infty$ as $\lambda \to +\infty$, while for $x < b$, the limit $\phi_s^\infty(x)$ is finite because of (2.25). Passing to the limit $\lambda \to +\infty$ in (2.33), using the bound in (2.32) and since $v_s^b(x) \leq 1$ for all $s \geq 0$, $x \in \mathbb{R}$, we get

$$\begin{aligned}\mathbb{P}(\rho_b < t\wedge\tau_b) &\leq \mathbb{E}\big[1 - \exp\big(-\langle v_0^b, \phi_0^\infty\rangle\big)\big]\\ &+\int_0^t \Big(\|f\|_\infty \langle \mathbf{1}_{(-\infty,-10b)\cup(10b,\infty)}, \phi_s^\infty\rangle + \frac{1}{4}\langle \mathbf{1}_{(-\infty,b/2)\cup(3b,\infty)}, (\phi_s^\infty)^2\rangle\Big)\, ds.\end{aligned} \tag{2.34}$$

Recalling (2.27)–(2.28), we have

$$\begin{aligned}\mathbb{P}(\rho_b < t\wedge\tau_b) &\leq \frac{C}{t}\int_{-\infty}^0 e^{-(b-x)^2/(20t)}dx\\ &+\frac{\|f\|_\infty}{t}\int_0^t\Big(\int_{10b}^\infty e^{-(2b-x)^2/(20t)}\,dx + \int_{-\infty}^{-10b} e^{-(b-x)^2/(20t)}\,dx\Big)\,ds\\ &+\frac{C}{t^2}\int_0^t\Big(\int_{-\infty}^{b/2} e^{-(b-x)^2/(10t)}dx + \int_{3b}^\infty e^{-(2b-x)^2/(10t)}\,dx\Big)\,ds\\ &\leq \frac{C}{b}\exp\Big(-\frac{b^2}{40t}\Big) + \frac{C\|f\|_\infty}{b}\exp\Big(-\frac{b^2}{t}\Big) \leq \frac{C(\|f\|_\infty + 1)}{t^{1/2}}\exp\Big(-\frac{b^2}{40t}\Big).\end{aligned} \tag{2.35}$$

We used the assumption that $R_0 = 0$ in the first term in the right side above. To estimate the integrals in (2.35), we used the standard Gaussian estimate

$$\int_y^\infty \exp(-x^2/2)dx \leq y^{-1}\exp(-y^2/2)$$

along with a few changes of variables.

Now we need to estimate

$$\mathbb{P}(\tau_b \le t) = \mathbb{P}\Big(\exists x \in [b/2, 3b], s \le t : G_s v_0^b(x) + \int_0^s \left(\int_{10b}^{\infty} + \int_{-\infty}^{-10b}\right) G_{s-r}(x-z) f(v_r^b(z))\, dz\, dr + N_s^b(x) \ge 1/2\Big). \tag{2.36}$$

It is easy to check that since $b \ge 4\sqrt{t}(t\|f\|_\infty \vee 1)$

$$\begin{aligned} G_s v_0^b(x) &\le \int_{-\infty}^0 G_s(x-z)\, dz \le \int_{-\infty}^0 G_t(x-z)\, dz \le \int_{-\infty}^0 G_t(b/2-z)\, dz \\ &\le \int_{-\infty}^0 G_1(2-z)\, dz \le 1/10, \ \forall s \le t, x \in [b/2, 3b]. \end{aligned} \tag{2.37}$$

Similarly, we have

$$\begin{aligned} t\int_{10b}^{\infty} G_{s-r}(z-x) f(v_r^b(z)\, dz &\le t\|f\|_\infty \int_{10b}^{\infty} G_t(z-3b)\, dz \\ &\le t\|f\|_\infty \int_{\frac{7b}{\sqrt{t}}}^{\infty} G_1(z)\, dz, \\ &\le t\|f\|_\infty \int_{28(t\|f\|_\infty \vee 1)}^{\infty} G_1(z)\, dz \le 0.05, \ \forall r \le s \le t, x \in [b/2, 3b]. \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} &t\int_{-\infty}^{-10b} G_{s-r}(z-x) f(v_r^b(z)\, dz \\ &\le t\|f\|_\infty \int_{-\infty}^{-10b} G_t(z-b/2)\, dz \le t\|f\|_\infty \int_{\frac{10b}{\sqrt{t}}}^{\infty} G_1(z)\, dz, \\ &\le t\|f\|_\infty \int_{40(t|f\|_\infty \vee 1)}^{\infty} G_1(z)\, dz \le 0.05, \ \forall r \le s \le t, x \in [b/2, 3b]. \end{aligned} \tag{2.39}$$

Altogether substituting the last inequalities into (2.36) we get

$$\begin{aligned} \mathbb{P}(\tau_b \le t) &\le \mathbb{P}\Big(\exists x \in [b/2, 3b], s \le t : N_s^b(x) \ge 0.3\Big) \\ &\le C \cdot (t \vee t^{22}) \int_{\mathbb{R}}\int_{\mathbb{R}} G_t(x-z)(v_0^b(z) \\ &\quad + t\|f\|_\infty \mathbf{1}_{\{z\in(-\infty,-10b)\cup(10b,\infty)\}})\, dz \mathbf{1}_{\{x\in(b/2,9b)\}})\, dx \\ &\le C(t, \|f\|_\infty) \exp\Big(-\frac{b^2}{50t}\Big), \ \forall t > 0, x \in [b/2, 3b], \end{aligned} \tag{2.40}$$

where the second inequality follows by Lemma 2.5 and in the last one we used simple Gaussian bounds. By combining (2.40) with (2.35) we are done. □

2.3.3. The proof of Lemma 2.2 Now we are ready to prove Lemma 2.2. Note that Lemma 2.6 implies a similar result for $u_t(x)$.

Lemma 2.7. *Let $u_t(x)$ be a solution to (1.9) taking values in $\widehat{\mathcal{B}}_I$ for all $t \geq 0$ such that the initial condition $u_0(x)$ satisfies (1.12) with $R(u_0) \leq 0$. Then, for all $T > 0$ there exists $C_T > 0$ so that for all $b \geq 4\sqrt{T}(T\|f\|_\infty \vee 1)$ we have*

$$\mathbb{P}\Big(\sup_{0\leq t\leq T}\sup_{x\in[b,2b]} u_t(x) > 0\Big) \leq C(T, \|f\|_\infty)\exp\Big(-\frac{b^2}{100T}\Big). \tag{2.41}$$

Proof. By Girsanov's theorem we have

$$\begin{aligned}\mathbb{P}_u\Big(\sup_{0\leq t\leq T}\sup_{x\in[b,2b]} u_t(x) > 0\Big) &\leq \mathbb{E}_{v^b}\big[e^{Z_T^b}\mathbf{1}_{\{\sup_{0\leq t\leq T}\sup_{x\in[b,2b]} v_t^b(x)>0\}}\big]\\ &\leq \Big(\mathbb{E}_{v^b}\big[e^{2Z_T^b}\big]\Big)^{1/2}\Big(\mathbb{P}_{v^b}\Big(\sup_{0\leq t\leq T}\sup_{x\in[b,2b]} v_t^b(x) > 0\Big)\Big)^{1/2},\end{aligned} \tag{2.42}$$

where Z^b was defined in (2.12). By (2.10) the quadratic variation of

$$t \mapsto \int_0^t\int_{\mathbb{R}} \frac{f(v_s^b(x))\mathbf{1}_{\{x\in(-10b,10b)\}}}{\sqrt{v_s^b(x)(1-v_s^b(x))}}W(dx,ds)$$

is

$$\int_0^t\int_{\mathbb{R}} \frac{f(v_s^b(x))^2\mathbf{1}_{\{x\in(-10b,10b)\}}}{v_s^b((x)1-v_s^b(x))}dxds \leq 20bK_f^2 t,\ \forall t\geq 0,\ \ \mathbb{P}_{v^b}\text{-a.s.}$$

Thus from (2.12) we can easily get

$$\mathbb{E}_{v^b}\big[e^{2Z_T^b}\big] \leq e^{40bK_f^2 T}, \tag{2.43}$$

and combining this with (2.42) and Lemma 2.6 we obtain (2.41). □

Now, the conclusions of Lemma 2.2 follow essentially immediately. The bound (2.13) on

$$\mathbb{P}\big(\sup_{0\leq t\leq T}(R(u_t) - R_0) > b\big)$$

in Lemma 2.2 is a simple consequence of Lemma 2.7, by adding up the Gaussian estimate (2.41) over the intervals $[b, 2b]$, $2b, 4b]$, etc. The finiteness of $\sup_{t\leq T} R(u_t)$ follows from (2.13). The corresponding bounds on $L(u_t)$ follow by repeating the arguments used in the proof of Lemmas 2.4–2.7 for $1-u(-x)$ instead of $u(x)$.

2.3.4. Uniqueness of the solution So far, we have shown that both $R(u_t)$ and $L(u_t)$ are $\mathbb{P}_u$-a.s. finite for any solution to (1.9) taking values in $\widehat{\mathcal{B}}_I$ for all $t \geq 0$ such that the initial condition $u_0(x)$ satisfies (1.12). As a consequence, (2.6) holds for any such solution to (1.9). As we have discussed in Sect. 2.2, it follows that we may apply Girsanov's theorem to immediately deduce uniqueness in law of the solution to (1.9) that satisfies the above conditions.

2.4. Existence of the speed. The last ingredient in the proof of Theorem 1.1 is the existence of the speed.

Lemma 2.8. *There exists a deterministic constant* $V(\sigma) \in (-\infty, +\infty)$ *so that the limit*

$$V(\sigma) = \lim_{t\to+\infty} \frac{R(u_t)}{t} \tag{2.44}$$

exists almost surely.

Proof. The proof goes along the lines of the proof of the corresponding result in [CD05]. First, we show that the limit $V(\sigma)$ in (2.44) exists and $V(\sigma) < \infty$. Let us set $b(m) = R(u_m)$, for $m = 0, 1, 2, \ldots$, and note that by Corollary 2.3 we have

$$\mathbb{E}\left[(b(1) - b(0))_+\right] < \infty. \tag{2.45}$$

Then, as in the proof of Lemma 5.1 in [CD05] we can use the subadditive ergodic theorem to deduce that there exists a constant $c(\sigma) \in [-\infty, \infty)$, such that

$$\lim_{m\to+\infty} \frac{b(m)}{m} = c(\sigma). \tag{2.46}$$

Using Lemma 2.2, we get (see Lemma 5.3 in [CD05] for the same argument) that for all $m = 1, 2, \ldots$

$$\begin{aligned}&\mathbb{P}\Big(\sup_{0\le s\le 1} \{b(s+m) - b(m), b(m+1) - b(s+m)\} > \sqrt{m}\Big)\\ &\quad\le C(\sigma)\exp(-m/50).\end{aligned} \tag{2.47}$$

Then by the Borel-Cantelli lemma we get that in fact,

$$\lim_{t\to+\infty} \frac{b(t)}{t} = c(\sigma). \tag{2.48}$$

and thus $V(\sigma) = c(\sigma) < \infty$.

To show that $V(\sigma) > -\infty$, one needs to consider equation for $1 - u_t(-x)$ and repeat the above argument. □

3. The Interface in the Voter Model

Girsanov's theorem connecting solutions to the rescaled equation (1.14) and to the voter model (1.15) not only allows us to deduce uniqueness in the law for the solutions to the former problem but also obtain the asymptotics on their front speed in Theorem 1.2. As a preliminary step, in this section, we make some observations about the latter. To begin, we rephrase Lemma 4.2(a) of [Tri95], putting it into a form more directly useful for our purposes. Let $w_t(x)$ be the solution to (1.15) with an initial condition $w_0(x)$ satisfying (1.12). Recall that we denote by $\mathbb{P}_w$ the measure induced on the canonical path space $C([0, +\infty); C(\mathbb{R}))$ by w, and by $\mathbb{E}_w$ we denote the corresponding expectation. Recall that two random processes X_t and Y_t are said to be coupled if they can be defined on the same probability space. We assume throughout the rest of the paper that f satisfies assumption (1.16).

Lemma 3.1. *Given $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that for all $T \geq T_\varepsilon$ there is a coupling of processes $(w_t, B_t : t \geq 0)$ where B a standard Brownian motion started at 0, such that*

$$\mathbb{P}_w\Big(\sup_{0\leq t\leq T} \big|R(w_t) - B_t\big| \vee \big|L(w_t) - B_t\big| \geq T^{1/2}\varepsilon\Big) \leq \varepsilon.$$

The following lemma shows that another good measure of the location of the interface is

$$M_t := \int_0^t \int_{\mathbb{R}} \sqrt{w_s(x)(1 - w_s(x))}\,W(dx, ds). \tag{3.1}$$

Lemma 3.2. *Let B be the Brownian motion from Lemma 3.1. Given $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that for all $T \geq T_\varepsilon$ we have*

$$\mathbb{P}_w\Big(\sup_{0\leq t\leq T} \big|M_t - B_t\big| \geq 4T^{1/2}\varepsilon\Big) \leq \varepsilon.$$

Proof. Let us define

$$\Xi(w_t) := \int_{-\infty}^0 \big[w_t(x) - 1\big]dx + \int_0^\infty w_t(x)dx. \tag{3.2}$$

Clearly, we have

$$|\Xi(w_t)| \leq R(w_t)_+ + L(w_t)_-,$$

and thus $\Xi(w_t)$ is almost surely a finite functional of w_t by Theorem 1.1. As $w_t(x) = 1$ for $x < L(w_t)$ and $w_t(x) = 0$ for $x > R(w_t)$, we have

$$\Xi(w_t) = \int_{L(w_t)\wedge 0}^0 [w_t(x) - 1]dx + \int_0^{R(w_t)\vee 0} w_t(x)dx,$$

thus

$$\begin{aligned} L(w_t) &= \int_{L(w_t)\wedge 0}^0 [-1]dx + \int_0^{L(w_t)\vee 0} dx \\ &\leq \int_{L(w_t)\wedge 0}^0 [w_t(x) - 1]dx + \int_0^{R(w_t)\vee 0} w_t(x)dx = \Xi(w_t), \end{aligned}$$

and likewise

$$\begin{aligned} R(w_t) &= \int_{R(w_t)\wedge 0}^0 [-1]dx + \int_0^{R(w_t)\vee 0} dx \\ &\geq \int_{L(w_t)\wedge 0}^0 [w_t(x) - 1]dx + \int_0^{R(w_t)\vee 0} w_t(x)dx = \Xi(w_t). \end{aligned}$$

We conclude that

$$L(w_t)) \leq \Xi(w_t) \leq R(w_t). \tag{3.3}$$

Next, let $\theta(x)$ be a smooth monotonically decreasing function such that $\theta(x) = 1$ for $x < -2$ and $\theta(x) = 0$ for $x > -1$, and set $\theta_n(x) = \theta(nx)$. Then for

$$\zeta_n(x) := w_t(x) - \theta_n(x)$$

we have

$$\Xi(w_t) = \lim_{n\to\infty} \Xi_n(t), \quad \Xi_n(t) = \int_{-\infty}^{\infty} \zeta_n(x)dx.$$

The function $\zeta_n(t, x)$ satisfies

$$\partial_t \zeta_n = \frac{1}{2}\partial_x^2 \zeta_n + \frac{1}{2}\partial_x^2 \theta_n + \sqrt{w(1-w)}\dot{W}(t, x). \tag{3.4}$$

Integrating in t and x gives

$$\Xi_n(w_t) = \Xi_n(w_0) + \int_0^t \int_{\mathbb{R}} \sqrt{w_s(y)(1 - w_s(y))}W(dyds). \tag{3.5}$$

Passing to the limit $n \to +\infty$, we arrive at

$$\Xi(w_t) = \Xi(w_0) + \int_0^t \int_{\mathbb{R}} \sqrt{w_s(y)(1 - w_s(y))}W(dyds) = \Xi(w_0) + M_t. \tag{3.6}$$

As $\Xi(w_0) < +\infty$ and is not random, the conclusion of the present lemma follows from (3.3) and Lemma 3.1 by taking T_ε sufficiently large. □

For any metric space $\mathbf{E}$, we denote by $\mathbb{D}_{\mathbf{E}}$ the space of càdlàg functions $[0, \infty) \to \mathbf{E}$ equipped with the Skorohod topology. Define the rescaled functionals

$$L_t^a = \frac{1}{a}L(w_{a^2t}), \; R_t^a = \frac{1}{a}R(w_{a^2t}), \;\; M_t^a = \frac{1}{a}M_{a^2t}\,.$$

As a consequence of Lemmas 3.1 and 3.2, we conclude that

$$(L^a, R^a, M^a) \Rightarrow (B, B, B) \text{ in } \mathbb{D}_{\mathbb{R}^3}, \quad \text{as } a \to \infty,$$

where B is a standard Brownian motion starting at 0 and $\Rightarrow$ denotes convergence in law.

As in the application of the Girsanov theorem in the proof of Theorem 1.1, we will make use of the functionals

$$A_t^f := \int_0^t \int_{\mathbb{R}} \frac{f(w_s(x))^2}{w_s(x)(1 - w_s(x))}\, dx\, ds, \tag{3.7}$$

$$M_t^f := \int_0^t \int_{\mathbb{R}} \frac{f(w_s(x))}{\sqrt{w_s(x)(1 - w_s(x))}} W(dx, ds), \tag{3.8}$$

and their rescaled versions

$$M_t^{f,a} = \frac{1}{a}M_{a^2t}, \;\; A_t^{f,a} = \frac{1}{a^2}A_{a^2t}^f, \;\; a > 0.$$

The difference in the scaling of these two functionals comes from the fact that M_t is, roughly, a Brownian motion on large time scales, and A_t is deterministic to the leading order on large time scales. Note that both A_t and M_t are almost surely finite if f satisfies assumption (1.11), since the interface of w_t has a finite length almost surely. However, we will need the stronger assumption (1.16) in Lemma 3.5 below.

Let us now recall Theorem 1 of [MT97].

Theorem 3.3. ([MT97]) *There exists a unique stationary measure* μ *on* $\mathcal{C}_I$ *for* (1.15). *Furthermore, for each* $u_0 \in \mathcal{C}_I$, *the law of* $w_t(x + L_t)$ *converges in total variation to* μ *as* $t \to \infty$. *In addition, the moment of the width of the interface* $\mathbb{E}_{w,st}[(R(w) - L(w))^p]$ *is finite if* $0 \le p < 1$, *and infinite for* $p \ge 1$.

The following estimate is a consequence of the second part of Theorem 3.3.

Lemma 3.4. *For any* $\eta \in (0, 1]$, *we have*

$$\mathbb{E}_{w,st}\left[\int_{\mathbb{R}} (w(x)(1 - w(x)))^\eta \, dx\right] < \infty. \tag{3.9}$$

Note that this result fails at $\eta = 0$: according to Theorem 3.3, the length of the interface has an infinite expectation under the stationary distribution of w.

Proof. For $\eta = 1$ the result is known (see Lemma 2.1(a) in [Tri95]), so we assume that $\eta \in (0, 1)$. Let ℓ be the length of the interface of w under the stationary distribution. By applying Hölder's and Young's inequalities we get

$$\begin{aligned}\mathbb{E}_{w,st}\Big[\int_{\mathbb{R}} (w(x)(1 - w(x)))^\eta \, dx\Big] &\le \mathbb{E}_{w,st}\Big[\Big(\int_{\mathbb{R}} (w(x)(1 - w(x))) \, dx\Big)^\eta \ell^{1-\eta}\Big] \\ &\le C_\alpha \mathbb{E}_{w,st}\Big[\Big(\int_{\mathbb{R}} (w(x)(1 - w(x))) \, dx\Big)^{\alpha\eta}\Big] \\ &\quad + C_\alpha \mathbb{E}_{w,st}\Big[\ell^{\frac{\alpha(1-\eta)}{\alpha-1}}\Big],\end{aligned}$$

for any $\alpha > 1$. We take $\alpha = 2/\eta$ and get

$$\begin{aligned}&\mathbb{E}_{w,st}\Big[\int_{\mathbb{R}} (w(x)(1 - w(x)))^\eta \, dx\Big] \\ &\le C_\alpha \mathbb{E}_{w,st}\Big[\Big(\int_{\mathbb{R}} (w(x)(1 - w(x))) \, dx\Big)^2\Big] + C_\alpha \mathbb{E}_{w,st}\big[\ell^\gamma\big],\end{aligned}$$

with $\gamma = (1 - \eta)/(1 - \eta/2)$. Since $\gamma < 1$, by Theorem 3.3 we get $\mathbb{E}_{w,st}[\ell^\gamma] < \infty$. In addition, Lemma 2.1(d) in [Tri95] implies that

$$\mathbb{E}_{w,st}\Big[\Big(\int_{\mathbb{R}} (w(x)(1 - w(x))) \, dx\Big)^2\Big] < \infty,$$

and we are done. □

Lemma 3.5. *Let* f *satisfy assumption* (1.16), *then we have convergence in law*

$$(M^{f,a}, A^{f,a}) \Rightarrow \{B_t^f, Dt), \ t \ge 0\}, \tag{3.10}$$

in $\mathbb{D}_{\mathbb{R}^2}$, *as* $a \to \infty$. *Here* $\{B_t^f, t \ge 0\}$ *is a Brownian motion with variance* D

$$D = \mathbb{E}_{w,st}\left[\int_{\mathbb{R}} \frac{f(w(x))^2}{w(x)(1 - w(x))} \, dx\right] < \infty. \tag{3.11}$$

Note that $D < +\infty$ because of Lemma 3.4 and assumption (1.16) on f.

Proof. Since w has a unique stationary distribution on the space $\mathcal{C}_I$ of continuous functions h such that $-\infty < L(h) < R(h) < +\infty$, by the ergodic theorem we have

$$\lim_{a\to\infty} a^{-2} A^f_{a^2 t} = t\mathbb{E}_{w,st}\left[\int_{\mathbb{R}} \frac{f(w(x))^2}{w(1-w(x))}\,dx\right] = Dt, \tag{3.12}$$

uniformly on compact sets in t. Recall that $\mathbb{E}_{w,st}$ denotes the expectation with respect to the stationary measure of w on $\mathcal{C}_I$. Since

$$M^{f,a}_t = \tilde{B}_{A^{f,a}_t},\ \ t \ge 0, \tag{3.13}$$

for some standard Brownian motion $\tilde{B}$, it follows from (3.12) that

$$M^{f,a}_{\cdot} \Rightarrow \{B^f_t, t \ge 0\} := \{\tilde{B}_{Dt}, t \ge 0\}, \tag{3.14}$$

where $\tilde{B}_{Dt}$ is a Brownian motion with variance D. □

Define

$$A_t := \int_0^t \int_{\mathbb{R}} w_s(x)(1 - w_s(x))\,dx\,ds, \tag{3.15}$$

and its rescaled version

$$A^a_t = \frac{1}{a^2} A_{a^2 t},\ \ a > 0.$$

Corollary 3.6. *We have convergence in law*

$$(L^a, R^a, M^a, A^a, M^{f,a}, A^{f,a}) \Rightarrow \{(B_t, B_t, B_t, t, B^f_t, Dt),\ t \ge 0\}, \tag{3.16}$$

in $\mathbb{D}_{\mathbb{R}^5}$, *as* $a \to \infty$. *Here,* B_t *is a standard Brownian motion,* B^f_t *is a Brownian motion with variance* D *and their correlation is given by*

$$\langle B_{\cdot}, B^f_{\cdot}\rangle_t = c_f t,\ \ t \ge 0, \tag{3.17}$$

with c_f *as in* (1.18).

Proof. It only remains to check the correlation:

$$\langle M^{f,a}_{\cdot}, M^a_{\cdot}\rangle_t = a^{-2}\int_0^{a^2 t}\int_{\mathbb{R}} f(w_s(x))\,dx\,ds \Rightarrow t\mathbb{E}_{w,st}\Big[\int_{\mathbb{R}} f(w(x))\,dx\Big] = c_f t,\ \ t \ge 0,$$

as $a \to \infty$, exactly as in (3.12). □

4. The Proof of Theorem 1.2: The Upper Bound on the Speed for $c_f \ge 0$

In this section, we prove the upper bound on the front speed in Theorem 1.2 for non-negative c_f.

Proposition 4.1. *Suppose that* u_0 *satisfies* (1.12) *and* f *satisfies* (1.16), *and that* $c_f \ge 0$, *then*

$$\limsup_{\sigma\to\infty} \sigma^2 V(\sigma) \le c_f\,.$$

4.1. Rescaling. First, we show via a rescaling how to pass from (1.9) to (1.14). Consider the rescaled function

$$v_t(x) = u_{\sigma^{-4}t}(\sigma^{-2}x).$$

To get an equation for $v_t(x)$, we use the mild form (2.1) and the relations

$$\begin{aligned} G_{a^2t}(bx) &= b^{-1}G_{(a^2t/b^2)}(x) \\ W^{a,b}(dyds) &:= a^{-1}b^{-1/2}W(bdy, a^2ds) \stackrel{\mathcal{D}}{=} W(dyds) \\ ab^{1/2}W^{a,b}(dyds) &= W(bdy, a^2ds), \end{aligned} \tag{4.1}$$

that hold for any $a, b > 0$. Here, $\stackrel{\mathcal{D}}{=}$ means equality in distribution. From (2.1), for any $a, b > 0$, we get

$$\begin{aligned} u_{(a^2t)}(bx) &= \int_{\mathbb{R}} G_{a^2t}(bx-y)u_0(y)dy + \int_0^{a^2t}\int_{\mathbb{R}} G_{a^2t-s}(bx-y)f(u_s(y))dyds \\ &\quad + \sigma\int_0^{a^2t}\int_{\mathbb{R}} G_{a^2t-s}(bx-y)\sqrt{u_s(y)(1-u_s(y))}W(dyds) =: I + II + III. \end{aligned}$$

We make the change of variables $s = a^2s'$, $y = by'$ and use (4.1). For the term I we have

$$I = b\int_{\mathbb{R}} G_{a^2t}(bx-by')u_0(by')dy' = \int_{\mathbb{R}} G_{a^2t/b^2}(x-y')u_0(by')dy'. \tag{4.2}$$

The second term can be rewritten as

$$\begin{aligned} II &= \int_0^{a^2t}\int_{\mathbb{R}} G_{a^2t-s}(bx-y)f(u_s(y))dyds \\ &= \int_0^t\int_{\mathbb{R}} G_{a^2t-a^2s'}(bx-by')f(u_{a^2s'}(by'))ba^2dy'ds' \\ &= a^2\int_0^t\int_{\mathbb{R}} G_{a^2(t-s')/b^2}(x-y')f(u_{a^2s'}(by'))dy'ds', \end{aligned} \tag{4.3}$$

and changing variables, the last term is

$$\begin{aligned} III &= \sigma\int_0^{a^2t}\int_{\mathbb{R}} G_{a^2t-s}(bx-y)\sqrt{u_s(y)(1-u_s(y))}W(dyds) \\ &= \sigma\int_0^t\int_{\mathbb{R}} G_{a^2t-a^2s'}(bx-by')\sqrt{u_{a^2s'}(by')(1-u_{a^2s'}(by'))}W(bdy', a^2ds') \\ &= \sigma\int_0^t\int_{\mathbb{R}} b^{-1}G_{a^2(t-s')/b^2}(x-y')\sqrt{u_{a^2s'}(by')(1-u_{a^2s'}(by'))}ab^{1/2}W^{a,b}(dy'ds'). \\ &= ab^{-1/2}\sigma\int_0^t\int_{\mathbb{R}} G_{a^2(t-s')/b^2}(x-y')\sqrt{u_{a^2s'}(by')(1-u_{a^2s'}(by'))}W^{a,b}(dy'ds'). \end{aligned} \tag{4.4}$$

We take

$$a = \sigma^{-2}, \quad b = \sigma^{-2},$$

so that $ab^{-1/2}\sigma = 1$ and $a^2/b^2 = 1$. Defining $v_t(x) := u_{(a^2t)}(bx)$ and putting together the above terms, we see that $v_t(x)$ satisfies

$$v_t(x) = \int_{\mathbb{R}} G_t(x-y)u_0(y)dy + \sigma^{-4}\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)f(v_s(y))dyds \\ + \int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)\sqrt{v_s(y)(1-v_s(y))}W^{a,b}(dyds). \tag{4.5}$$

Since the solution v to (4.5) is unique in law, and since W and $W^{a,b}$ are equal in law, we see that v is the unique weak solution to (1.14) with the initial condition $v_0(x) = u_0(\sigma^{-2}x)$. Thus in general our scaling changes the initial data. However, if $u_0(x) = \mathbf{1}(x \le 0)$, then clearly $v_0(x) = u_0(x)$.

Now it is clear that the conclusion of Proposition 4.1 would follow if we show that

$$\limsup_{\sigma\to\infty} \sigma^4 V^{(v)}(\sigma) \le c_f, \tag{4.6}$$

where

$$V^{(v)}(\sigma) = \lim_{t\to+\infty} \frac{R(v_t)}{t}.$$

Let us also note that the rescaled Girsanov functional (2.4) takes the form

$$Z_t = \sigma^{-4}\int_0^t\int_{\mathbb{R}} \frac{f(w_s(x))}{\sqrt{w_s(x)(1-w_s(x))}}W(dx,ds) - \frac{1}{2}\sigma^{-8}\int_0^t\int_{\mathbb{R}} \frac{f(w_s(x))^2}{w_s((x)1-w_s(x))}dxds \\ = \sigma^{-4}M_t^f - \frac{1}{2}\sigma^{-8}A_t^f. \tag{4.7}$$

4.2. Time steps for the upper bound. Note that for the upper bound on $V(\sigma)$ and $V^{(v)}(\sigma)$, we may assume without loss of generality that the initial condition $u_0(x) = v_0(x) = \mathbf{1}(x \le 0)$, by the comparison principle (see Remark 2.1) and translation invariance in law. We will define a sequence of stopping times $0 = \tau_0 \le \tau_1 \le \cdots$, and a sequence $v_t^{(m)}(x)$ of solutions to (1.14) for $t \ge \tau_m$, with the initial conditions $v_{\tau_m}^{(m)}(x) \ge v_{\tau_m}(x)$ at $t = \tau_m$. The comparison principle (see again Remark 2.1) will imply that $v_t^{(m)}(x) \ge v_t(x)$ for $t \ge \tau_m$. Moreover, we will choose $v_t^{(m)}$ so that for each $m = 0, 1, 2, \ldots$ the following conditions hold almost surely:

$$v_t(x) \le v_t^{(m)}(x), \quad \text{for } t \ge \tau_m,\ x \in \mathbb{R} \tag{4.8}$$

$$R\big(v_{\tau_m}^{(m)}\big) = m\lambda_1\sigma^4, \tag{4.9}$$

$$v_{\tau_m}^{(m-1)}(x) \le v_{\tau_m}^{(m)}(x), \quad \text{for } x \in \mathbb{R}, \tag{4.10}$$

with the constant λ_1 to be specified later. It follows from (4.8), that for all $m = 0, 1, 2, \ldots$ and for all $t \geq \tau_m$, we have

$$R(v_t) \leq R\big(v_t^{(m)}\big), \tag{4.11}$$

almost surely. Thus, to bound $R(v_t)$ from above, it suffices to bound $R(v_t^{(m)})$.

Let us inductively construct τ_k and $v_t^{(k)}(x)$ for $k = 0, 1, \ldots,$ with the convention $v_t^{(-1)}(x) = 0$. Fix $T_0 > 0$ and $N \in \mathbb{N}$, to be specified later, and start with $\tau_0 = 0$ and $v_t^{(0)}(x) = v_t(x)$, so that (4.8), (4.9), and (4.10) hold for $m = 0$ automatically. Suppose that we have defined τ_m and $v_t^{(m)}$ for $t \geq \tau_m$ and $0 \leq m \leq k$, and assume that (4.8), (4.9) and (4.10) hold for $0 \leq m \leq k$. Given $v_t^{(k)}(x)$, defined for $t \geq \tau_k$ and $x \in \mathbb{R}$, we set

$$M_t^{f,v,k} := \int_{\tau_k}^{t} \int_{\mathbb{R}} \frac{f\big(v_s^{(k)}(x)\big)}{\sqrt{v_s^{(k)}(x)(1 - v_s^{(k)}(x))}} W(dx, ds), \tag{4.12}$$

and

$$\tau_{k+1} = \inf\left\{t \in [\tau_k, \tau_k + T_0\sigma^8] : R\big(v_t^{(k)}\big) = (k+1)\lambda_1\sigma^4 \ \text{ or } \ \frac{1}{\sigma^4} M_t^{f,v,k} \geq N\right\}, \tag{4.13}$$

with $\tau_{k+1} = \tau_k + T_0\sigma^8$ if the above set is empty. Then, we define $v_t^{(k+1)}(x)$ for $t \geq \tau_{k+1}$, and $x \in \mathbb{R}$ as the solution to (1.14) with the initial condition

$$v_{\tau_{k+1}}^{(k+1)}(x) = \mathbf{1}(x \leq (k+1)\lambda_1\sigma^4).$$

Note that for $m = k + 1$, (4.8) and (4.10) hold by the comparison principle (see Remark 2.1). (4.9) holds by construction.

For convenience, we write

$$\Delta\tau_m = \tau_{m+1} - \tau_m$$

and note that $\{\Delta\tau_m\}$ are i.i.d. random variables for $m \geq 0$.

4.3. A good event and its consequences (for the upper bound). To get an upper bound on $V_f^{(v)}(\sigma)$, it suffices to get an appropriate lower bound on τ_m as $m \to \infty$. Let us define the event

$$G_m = \{\Delta\tau_m = T_0\sigma^8\}, \quad m \geq 0. \tag{4.14}$$

Proposition 4.1 is a consequence of the following lemma.

Lemma 4.2. *Let $\varepsilon \in (0, \min(10^{-1}, c_f^{-2}))$ be arbitrary and set $\delta_\varepsilon = \varepsilon/10$. There exist $\bar{T}_\varepsilon$, N_ε, and σ_ε such that for $T_0 = \bar{T}_\varepsilon$, $N = N_\varepsilon$ and any $\sigma \geq \sigma_\varepsilon$, $m \geq 0$, and*

$$\lambda_1 = (c_f + \delta_\varepsilon)\bar{T}_\varepsilon, \tag{4.15}$$

we have

$$\lambda_2 := \mathbb{P}_v\big(G_m\big) \geq 1 - \delta_\varepsilon. \tag{4.16}$$

Note that λ_2 does not depend on m since $\Delta\tau_m$ are i.i.d for $m \geq 1$. We will prove Lemma 4.2 in the next section. Now we are ready to give

Proof of Proposition 4.1. Given $\varepsilon \in (0, 1/10)$, let $\bar{T}_\varepsilon$, N_ε and σ_ε be as in Lemma 4.2, and take an arbitrary $\sigma \geq \sigma_\varepsilon$. Then by Lemma 4.2, we have

$$\lambda_2 \geq 1 - \delta_\varepsilon, \tag{4.17}$$

and by (4.16) and the definition of $G^{(m)}$ with $T_0 = \bar{T}_\varepsilon$, we get

$$\mathbb{E}_v[\Delta\tau_m] \geq \bar{T}_\varepsilon \lambda_2 \sigma^8.$$

The strong law of large numbers implies that we have, $\mathbb{P}_v$ almost surely,

$$\lim_{m\to\infty} \frac{\tau_m}{m} \geq \bar{T}_\varepsilon \lambda_2 \sigma^8. \tag{4.18}$$

Since $R(v_{\tau_m}^{(m)}) = m\lambda_1\sigma^4$, we have that, also $\mathbb{P}_v$ almost surely,

$$\limsup_{m\to\infty} \frac{R(v_{\tau_m}^{(m)})}{\tau_m} = \limsup_{m\to\infty} \frac{m\lambda_1\sigma^4}{\tau_m} \leq \frac{\lambda_1\sigma^4}{\bar{T}_\varepsilon\lambda_2\sigma^8} = \frac{\lambda_1}{\bar{T}_\varepsilon\lambda_2}\sigma^{-4}. \tag{4.19}$$

Furthermore, since by definition, for $\tau_m \leq t \leq \tau_{m+1}$ we have

$$R\left(v_t^{(m)}\right) \leq (m+1)\lambda_1\sigma^4.$$

Hence, we get that, $\mathbb{P}_v$ almost surely, we have, using (4.17) and (4.18),

$$\begin{aligned} V^{(v)}(\sigma) &\leq \limsup_{m\to\infty} \sup_{\tau_m \leq t \leq \tau_{m+1}} \frac{R\left(v_t^{(m)}\right)}{t} \leq \limsup_{m\to\infty} \frac{\lambda_1(m+1)\sigma^4}{\tau_m} \\ &\leq \frac{\lambda_1\sigma^4}{\lambda_2\bar{T}_\varepsilon\sigma^8} \leq \frac{(c_f + \delta_\varepsilon)}{(1-\delta_\varepsilon)}\sigma^{-4} \leq (c_f + \sqrt{\varepsilon})\sigma^{-4}. \end{aligned} \tag{4.20}$$

Note that (4.20) holds for any $\sigma \geq \sigma_\varepsilon$, and, since ε is arbitrary small, we are done. This finishes the proof of Proposition 4.1. □

4.4. Proof of Lemma 4.2. As G_m are i.i.d., it suffices to set $m = 0$. We fix $\varepsilon \in (0, 1/10)$, let $\delta_\varepsilon = \varepsilon/10$, take $\bar{T}_\varepsilon$ sufficiently large, so that

$$2\exp\left(-\frac{\delta_\varepsilon^2\bar{T}_\varepsilon}{2}\right) \leq \frac{\varepsilon}{100}, \tag{4.21}$$

set $\lambda_1 = (c_f + \delta_\varepsilon)\bar{T}_\varepsilon$, and let $N_\varepsilon > (2+\delta_\varepsilon)\bar{T}_\varepsilon D$ be sufficiently large (its value will be determined later in the proof). We define the stopping time

$$\xi^\varepsilon = \inf\{t \geq 0 : M_t^{f,\sigma^4} \geq N_\varepsilon\}. \tag{4.22}$$

Then by Girsanov's theorem, we have, with Z_t as in (4.7):

$$\mathbb{P}_v(G_0^c) = \mathbb{E}_w\left[\exp\left(Z_{\sigma^8(\bar{T}_\varepsilon \wedge \xi^\varepsilon)}\right)\mathbf{1}_{G_0^c}\right]$$

$$
\begin{aligned}
&= \mathbb{E}_w\Big[\exp\Big(\sigma^{-4}\Big(M^f_{\sigma^8(\bar{T}_\varepsilon\wedge\xi^\varepsilon)} - \frac{1}{2}\sigma^{-4}A^f_{\sigma^8(\bar{T}_\varepsilon\wedge\xi^\varepsilon)}\Big)\Big) \\
&\quad \times \mathbf{1}\big(R(w_t) \geq \lambda_1\sigma^4 \text{ for some } t \leq \sigma^8\bar{T}_\varepsilon \text{ or } \sigma^{-4}M^f_t \geq N_\varepsilon \text{ for some } t \leq \sigma^8\bar{T}_\varepsilon\big)\Big] \\
&= \mathbb{E}_w\left[\exp\left(M^{f,\sigma^4}_{\bar{T}_\varepsilon\wedge\xi^\varepsilon} - \frac{1}{2}A^{f,\sigma^4}_{\bar{T}_\varepsilon\wedge\xi^\varepsilon}\right) \times \mathbf{1}\left(R^{\sigma^4}_t \geq (c_f+\delta_\varepsilon)\bar{T}_\varepsilon \text{ for some } t \leq \bar{T}_\varepsilon \text{ or } \xi^\varepsilon \leq \bar{T}_\varepsilon\right)\right] \\
&\leq \mathbb{E}_w\Big[\exp\left(M^{f,\sigma^4}_{\bar{T}_\varepsilon\wedge\xi^\varepsilon} - \frac{1}{2}A^{f,\sigma^4}_{\bar{T}_\varepsilon\wedge\xi^\varepsilon}\right)\mathbf{1}\left(\xi^\varepsilon \leq \bar{T}_\varepsilon\right)\Big] \\
&\quad + \mathbb{E}_w\Big[\exp\left(M^{\sigma^4}_{\bar{T}_\varepsilon\wedge\xi^{N,\sigma}} - \frac{1}{2}A^{f,\sigma^4}_{\bar{T}_\varepsilon\wedge\xi^{N,\sigma}}\right) \\
&\quad \times \mathbf{1}\left(R^{\sigma^4}_t \geq (c_f+\delta_\varepsilon)\bar{T}_\varepsilon \text{ for some } t \leq \bar{T}_\varepsilon\right)\mathbf{1}\left(\xi^\varepsilon > T_\varepsilon\right)\Big] \\
&=: I^\varepsilon_1 + I^\varepsilon_2\,.
\end{aligned}
$$

We first bound I^ε_1:

$$
I^\varepsilon_1 = \mathbb{E}_w\left[\exp\left(M^{f,\sigma^4}_{\xi^\varepsilon\wedge\bar{T}_\varepsilon} - \frac{1}{2}A^{f,\sigma^4}_{\xi^\varepsilon\wedge\bar{T}_\varepsilon}\right)\mathbf{1}\left(\xi^\varepsilon \leq \bar{T}_\varepsilon\right)\right] \leq e^{N_\varepsilon}\mathbb{P}_w\Big(\sup_{0\leq t\leq\bar{T}_\varepsilon} M^{f,\sigma^4}_t \geq N_\varepsilon\Big). \tag{4.23}
$$

Let $\mathbb{P}^B$ and $\mathbb{P}^{B^f}$ be the measures induced on the canonical path space by the standard Brownian motion B and by the Brownian motion with variance D, respectively, and $\mathbb{E}^B$ and $\mathbb{E}^{B^f}$ be the corresponding expectations. Then by Lemma 3.5 we have

$$
\limsup_{\sigma\to\infty} I^\varepsilon_1 \leq e^{N_\varepsilon}\mathbb{P}^{B^f}\Big(\sup_{0\leq t\leq\bar{T}_\varepsilon} B^f_t \geq N_\varepsilon\Big) \leq e^{N_\varepsilon}2\mathbb{P}^B\left(\sqrt{D}B_{\bar{T}_\varepsilon} \geq N_\varepsilon\right) \leq 2e^{N_\varepsilon}e^{-N^2_\varepsilon/(2\bar{T}_\varepsilon D)}, \tag{4.24}
$$

where the second inequality follows by the reflection principle and the last inequality follows by a simple bound on Gaussian tail probabilities. By choosing N_ε sufficiently large, we get

$$
\limsup_{\sigma\to\infty} I^\varepsilon_1 \leq \varepsilon/100.
$$

Thus, there exists σ_ε, such that for all $\sigma \geq \sigma_\varepsilon$ we have

$$
I^\varepsilon_1 \leq \varepsilon/50. \tag{4.25}
$$

Next, we bound I^ε_2. Let $\mathbb{P}^{B^f,B}$ be the measure induced on the canonical path space by the zero-mean Brownian motions B^f, B, such that B^f has variance D, B has variance 1, and the covariance of B^f and B is c_f, and let $\mathbb{E}^{B^f,B}$ be the corresponding expectation. We use again Corollary 3.6, properties of weak convergence, the dominated convergence theorem (we can switch to the Skorohod space if needed) to get

$$
\begin{aligned}
&\limsup_{\sigma\to\infty} I^\varepsilon_2 \qquad (4.26)\\
&\quad = \limsup_{\sigma\to\infty}\mathbb{E}_w\Big[\exp\big(M^{f,\sigma^4}_{\bar{T}_\varepsilon} - \frac{1}{2}A^{f,\sigma^4}_{\bar{T}_\varepsilon}\big)
\end{aligned}
$$

$$\times \mathbf{1}(\sup_{0\le t\le \bar T_\varepsilon} R_t^{\sigma^4} \ge (c_f+\delta_\varepsilon)\bar T_\varepsilon)\mathbf{1}(\sup_{0\le t\le \bar T_\varepsilon} M_t^{f,\sigma^4} < N_\varepsilon)\Big]$$
$$\le \mathbb{E}^{B^f,B}\Big[e^{B^f_{\bar T_\varepsilon}-\frac12 D\bar T_\varepsilon)}\mathbf{1}\Big(\sup_{0\le t\le \bar T_\varepsilon} B_t \ge (c_f+\delta_\varepsilon)\bar T_\varepsilon\Big)\mathbf{1}\Big(\sup_{0\le t\le \bar T_\varepsilon} B_t^f \le N_\varepsilon\Big)\Big]$$
$$\le \mathbb{E}^{B^f,B}\Big[e^{B^f_{\bar T_\varepsilon}-\frac12 D\bar T_\varepsilon}\mathbf{1}(\sup_{0\le t\le \bar T_\varepsilon} B_t \ge (c_f+\delta_\varepsilon)\bar T_\varepsilon)\Big]$$
$$= \mathbb{P}^B\Big(\sup_{0\le t\le \bar T_\varepsilon}(B_t+c_f t) \ge (c_f+\delta_\varepsilon)\bar T_\varepsilon\Big). \tag{4.27}$$

In the last equality we used the Girsanov theorem, since under the $\exp(B^f_{\bar T_\varepsilon} - \frac12 D\bar T_\varepsilon)$ change of measure, B is a Brownian motion with the drift $2c_f$ (recall that the covariance of B^f and B is c_f). Now it is easy to get

$$\mathbb{P}^B\Big(\sup_{0\le t\le \bar T_\varepsilon}(B_t+c_f t) \ge (c_f+\delta_\varepsilon)\bar T_\varepsilon\Big) \le \mathbb{P}^B\Big(\sup_{0\le t\le \bar T_\varepsilon} B_t \ge \delta_\varepsilon\bar T_\varepsilon\Big) \le 2e^{-\delta_\varepsilon^2\bar T_\varepsilon/2} \le \varepsilon/100, \tag{4.28}$$

where in the second inequality we again used reflection principle and a bound on Gaussian tail, and the last inequality follows from (4.21). Hence, there is σ_ε such that for all $\sigma \ge \sigma_\varepsilon$, we have

$$I_2^\varepsilon \le \varepsilon/50.$$

Combining the above estimates, we get that for $\sigma \ge \sigma_\varepsilon$ we have

$$\mathbb{P}_v(G_0^c) \le 2\varepsilon/50 \le \varepsilon/10, \tag{4.29}$$

so that

$$\mathbb{P}_v(G_0) \ge 1-\varepsilon/10. \tag{4.30}$$

This finishes the proof of Lemma 4.2. □

5. Proof of Theorem 1.2: The Lower Bound on the Speed for $c_f > 0$

We now prove the lower bound on $V(\sigma)$ for positive c_f.

Proposition 5.1. *Suppose that u_0 satisfies (1.12), f satisfies (1.16), and $c_f > 0$, then*

$$\liminf_{\sigma\to\infty} \sigma^2 V(\sigma) \ge c_f\,.$$

The proof of Proposition 5.1 follows a similar strategy to that of Proposition 4.1. As in the proof of the upper bound, using the comparison principle (see Remark 2.1) and shift invariance in law, we may assume without loss of generality that $u_0(x) = v_0(x) = \mathbf{1}(x \le 0)$.

5.1. Time steps for the lower bound. We start with the definition of the time steps. The main difference with the proof of the upper bound is that we will sometimes update "backwards", and that the "good events" will be when the stopping time happens before a fixed time length rather than when the stopping times happen at a deterministic time steps, as in (4.14) in the proof of the upper bound. We will define stopping times $0 = \tau_0 \leq \tau_1 \leq \cdots$, and a sequence $v_t^{(m)}(x)$ of random processes, which will be solutions to (1.14), for $t \geq \tau_m$, such that for each $m = 0, 1, 2, \ldots$ the following conditions will hold almost surely:

$$v_t(x) \geq v_t^{(m)}(x), \qquad \text{for } t \geq \tau_m,\ x \in \mathbb{R} \tag{5.1}$$

$$v_{\tau_m}^{(m-1)}(x) \geq v_{\tau_m}^{(m)}(x), \qquad \text{for } x \in \mathbb{R}. \tag{5.2}$$

Given (5.1) and (5.2), it would follow almost surely for all $m = 0, 1, 2, \ldots$ and for all $t \geq \tau_m$, that

$$L(v_t) \geq L(v_t^{(m)}). \tag{5.3}$$

Thus, to bound $L(v_t)$ from below, it would suffice to bound $L(v_t^{(m)})$.

We now describe the induction, starting with $\tau_0 = 0$, and $v_t^{(0)}(x) = v_t(x)$, so that (5.1) holds for $m = 0$. Also define $v_t^{(-1)}(x) = 1$, so that (5.2) holds. Let us fix some constants $\tilde{\lambda}_1, \widetilde{T}_0, N > 0$, to be specified later. Suppose that we have defined τ_m for $0 \leq m \leq k$ and $v_t^{(m)}$ for $t \geq \tau_m$ and $0 \leq m \leq k$, and that (5.1) and (5.2) hold for $0 \leq m \leq k$. To define τ_{k+1}, we consider, as in (4.12),

$$M_t^{f,v,k} := \int_{\tau_k}^t \int_{\mathbb{R}} \frac{f(v_s^{(k)}(x))}{\sqrt{v_s^{(k)}(x)(1 - v_s^{(k)}(x))}} W(dx, ds), \tag{5.4}$$

and set

$$\tau_{k+1} = \inf\left\{t \in [\tau_k, \tau_k + \widetilde{T}_0\sigma^8] : |L\big(v_t^{(k)}\big) - L\big(v_{\tau_k}^{(k)}\big)| \geq \tilde{\lambda}_1\sigma^4 \ \text{ or } \ \frac{1}{\sigma^4} M_t^{f,v,k} \geq N\right\} \tag{5.5}$$

with the convention $\tau_{k+1} = \tau_k + \widetilde{T}_0\sigma^8$ if the above set is empty.

We then let $v_t^{(k+1)}(x)$ for $t \geq \tau_{k+1}$, $x \in \mathbb{R}$ be the solution to (1.14) with the initial condition

$$v_{\tau_{k+1}}^{(k+1)}(x) = \begin{cases} \mathbf{1}(x \leq L(v_{\tau_{k+1}}^{(k)})), & \text{if } \tau_{k+1} < \tau_k + \widetilde{T}_0\sigma^8, \\ \mathbf{1}(x \leq L(v_{\tau_k}^{(k)}) - \tilde{\lambda}_1\sigma^4), & \text{if } \tau_{k+1} = \tau_k + \widetilde{T}_0\sigma^8. \end{cases}$$

Then for $m = k + 1$, the comparison principle (see Remark 2.1) gives (5.1), and (5.2) is true by definition.

As before, we write

$$\Delta\tau_k = \tau_{k+1} - \tau_k$$

and

$$\Delta L_k = L\big(v_{\tau_{k+1}}^{(k)}\big) - L\big(v_{\tau_k}^{(k)}\big) \tag{5.6}$$

Note that

$$\{(\Delta\tau_k, \Delta L_k)\}$$

are i.i.d. random variables.

5.2. A good event and its consequences for lower bound. We define the “good” events

$$\widetilde{G}_0^{(m)} = \{\Delta\tau_m < \widetilde{T}_0\sigma^8\}.$$

and

$$\widetilde{G}_0^{(1,m)} = \Big\{\Delta\tau_m < \widetilde{T}_0\sigma^8,\ \Delta L_m = \tilde{\lambda}_1\sigma^4,\ \sup_{\tau_m \le t \le \tau_{m+1}} \frac{1}{\sigma^4} M_t^{f,v,m} < N\Big\},$$
$$\widetilde{G}_0^{(2,m)} = \Big\{\Delta\tau_m < \widetilde{T}_0\sigma^8,\ \Delta L_m = -\tilde{\lambda}_1\sigma^4,\ \sup_{\tau_m \le t \le \tau_{m+1}} \frac{1}{\sigma^4} M_t^{f,v,m} < N\Big\},$$
$$\widetilde{G}_0^{(3,m)} = \Big\{\Delta\tau_m < \widetilde{T}_0\sigma^8,\ \sup_{\tau_m \le t \le \tau_{m+1}} \frac{1}{\sigma^4} M_t^{f,v,m} = N\Big\}.$$

To get a lower bound on $V^{(v)}(\sigma)$ we need a lower bound on ΔL_m as $m \to \infty$. To this end the following lemma will be helpful.

Lemma 5.2. *Let* $\varepsilon \in (0, \min(1/10, c_f^{-2}, c_f/2))$ *be arbitrary and* $\delta_\varepsilon = \varepsilon/10$. *There exist* T_ε^*, N_ε *and* σ_ε *so that for all* $\sigma \ge \sigma_\varepsilon$, $m \ge 0$, $\widetilde{T}_0 = T_\varepsilon^*$, $N = N_\varepsilon$ *and*

$$\tilde{\lambda}_1 = (c_f - \delta_\varepsilon)T_\varepsilon^*, \tag{5.7}$$

we have

$$\mathbb{P}_v\big(\widetilde{G}_0^{(1,m)}\big) \ge 1 - \varepsilon/50, \tag{5.8}$$
$$\mathbb{P}_v\big(\widetilde{G}_0^{(2,m)}\big) \le \varepsilon/20, \tag{5.9}$$
$$\mathbb{P}_v\big(\widetilde{G}_0^{(3,m)}\big) \le \varepsilon/50. \tag{5.10}$$

We postpone the proof of this lemma and first give

Proof of Proposition 5.1. Let us take $\varepsilon \in (0, \min(10^{-1}, c_f^{-2}, c_f/2))$, and choose T_ε^*, N_ε and σ_ε as in Lemma 5.2, and consider an arbitrary $\sigma \ge \sigma_\varepsilon$. Lemma 5.2 implies that

$$\mathbb{P}_v(\widetilde{G}_0^{(1,m)}) - \mathbb{P}_v(\widetilde{G}_0^{(2,m)}) - \mathbb{P}_v(\widetilde{G}_0^{(3,m)}) - \big(1 - \mathbb{P}_v(\widetilde{G}_0^{(m)})\big) \ge 1 - \varepsilon/5, \tag{5.11}$$

for all $\sigma \ge \sigma_\varepsilon$, so that for all $m \ge 0$ we have

$$\begin{aligned}\mathbb{E}_v[\Delta L_m] &\ge \tilde{\lambda}_1\sigma^4\Big(\mathbb{P}_v(\widetilde{G}_0^{(1,m)}) - \mathbb{P}_v(\widetilde{G}_0^{(2,m)}) - P_v(\widetilde{G}_0^{(3,m)}) - \big(1 - P_v(\widetilde{G}_0^{(m)})\big)\Big)\\ &\ge \tilde{\lambda}_1\sigma^4(1 - \varepsilon/5).\end{aligned} \tag{5.12}$$

Then using the strong law of large numbers, we have that $\mathbb{P}_v$ almost surely,

$$\lim_{m\to\infty} \frac{L(v_{\tau_m}^{(m)})}{m} \ge \tilde{\lambda}_1(1 - \varepsilon/5)\sigma^4,$$

for all $\sigma \ge \sigma_\varepsilon$. Since $\tau_m \le mT_\varepsilon^*\sigma^8$, we have that $\mathbb{P}_v$ almost surely,

$$\liminf_{m\to\infty} \frac{L(v_{\tau_m}^{(m)})}{\tau_m} = \liminf_{m\to\infty} \frac{L(v_{\tau_m}^{(m)})}{m}\frac{m}{\tau_m} \ge \frac{\tilde{\lambda}_1(1 - \varepsilon/5)\sigma^4}{T_\varepsilon^*\sigma^8} = \frac{\tilde{\lambda}_1(1 - \varepsilon/5)}{T_\varepsilon^*}\sigma^{-4}. \tag{5.13}$$

Furthermore, since for $\tau_m \le t \le \tau_{m+1}$ we have

$$L\big(v_t^{(m)}\big) \ge L\big(v_{\tau_m}^{(m)}\big) - \tilde{\lambda}_1\sigma^4,$$

and $\Delta\tau_m \le T_\varepsilon^*\sigma^8$, it follows that, P_v almost surely and since $\varepsilon < 10^{-1}$,

$$\begin{aligned} V^{(v)}(\sigma) &\ge \liminf_{m\to\infty} \inf_{\tau_m\le t\le\tau_{m+1}} \frac{L\big(v_t^{(m)}\big)}{t} \ge \liminf_{m\to\infty} \frac{L\big(v_{\tau_m}^{(m)}\big) - \tilde{\lambda}_1\sigma^4}{\tau_m + T_\varepsilon^*\sigma^8} \ge \frac{\tilde{\lambda}_1(1-\varepsilon/5)}{T_\varepsilon^*}\sigma^{-4} \\ &= \frac{(c_f - \varepsilon/10)T_\varepsilon^*(1-\varepsilon/5)}{T_\varepsilon^*}\sigma^{-4} \ge \Big(c_f - \frac{\varepsilon}{5}(c_f+1)\Big)\sigma^{-4} \\ &\ge (c_f - \sqrt{\varepsilon})\sigma^{-4}. \end{aligned}$$

In the second inequality above we used (5.13) and the fact that $\tau_m \to \infty$, $\mathbb{P}_v$-a.s. since $\Delta\tau_m \ge 0$, not identically zero and i.i.d. Since ε was chosen to be arbitrary small we are done. □

5.3. Proof of Lemma 5.2. As $(\Delta_{\tau_m}, \Delta L_m)$ are i.i.d., the events $G_0^{(m)}$ are also i.i.d., hence we only need to prove (5.8)–(5.10) for $m = 0$ and write

$$\widetilde{G}_0 = \widetilde{G}_0^{(0)},\ \widetilde{G}_0^{(i)} = \widetilde{G}_0^{(i,0)},\ i = 1, 2, 3.$$

Fix $\varepsilon \in (0, 10^{-1})$, let $\delta_\varepsilon = \varepsilon/10$, and let T_ε^* be sufficiently large so that

$$\mathbb{P}^B\Big(B_1 \ge -\delta_\varepsilon\sqrt{T},\ \inf_{0\le t\le 1} B_t > -(c_f - \delta_\varepsilon)\sqrt{T}\Big) \ge 1 - \varepsilon/100, \quad \forall T \ge T_\varepsilon^*. \quad (5.14)$$

We consider $N_\varepsilon > (2+\delta_\varepsilon)T_\varepsilon^* D$ sufficiently large, with a precise value to be specified later, and define the stopping time

$$\xi^\varepsilon = \inf\{t \ge 0 : M_t^{f,\sigma^4} \ge N_\varepsilon\}.$$

Then by Girsanov's theorem, and since

$$\Big\{\sup_{0\le t\le\Delta\tau_0} \frac{1}{\sigma^4}M_t^f < N_\varepsilon\Big\} \supset \Big\{\sup_{0\le t\le T_\varepsilon^*\sigma^8} \frac{1}{\sigma^4}M_t^f < N_\varepsilon\Big\} = \{\xi^\varepsilon > T_\varepsilon^*\},$$

we have

$$\begin{aligned} \mathbb{P}_v\big(\widetilde{G}_0^{(1)}\big) &= \mathbb{E}_w\Big[\exp\Big(Z_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)}\Big)\mathbf{1}_{\widetilde{G}_0^{(1)}}\Big] \\ &\ge E_w\bigg[\exp\bigg(\sigma^{-4}\big(M^f_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)} - \frac{1}{2}\sigma^{-4}A^f_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)}\big)\bigg) \\ &\quad \times \mathbf{1}\Big(L(w(t)) \ge \tilde{\lambda}_1\sigma^4 \text{ for some } 0 \le t \le \sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)\Big) \\ &\quad \times \mathbf{1}\Big(L(w(t)) > -\tilde{\lambda}_1\sigma^4 \text{ for all } 0 \le t \le \sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)\Big) \\ &\quad \times \mathbf{1}\bigg(\frac{1}{\sigma^4}M_t^f < N_\varepsilon \text{ for all } 0 \le t \le \sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)\bigg)\bigg] \end{aligned}$$

$$\geq \mathbb{E}_w \left[\exp\left(M^{f,\sigma^4}_{T^*_\varepsilon} - \frac{1}{2} A^{f,\sigma^4}_{T^*_\varepsilon} \right) \mathbf{1}\left(L^{\sigma^4}_t > \tilde{\lambda}_1 \text{ for some } 0 \leq t \leq T^*_\varepsilon \right) \right.$$
$$\left. \times \mathbf{1}\left(L^{\sigma^4}_t > -\tilde{\lambda}_1 \text{ for all } 0 \leq t \leq T^*_\varepsilon \right) \times \mathbf{1}\Big(\sup_{0\leq t\leq T^*_\varepsilon} M^{f,\sigma^4} < N_\varepsilon \Big) \right] =: J^\varepsilon_1 . \tag{5.15}$$

Next, passing to the limit $\sigma \to +\infty$, we obtain, using the weak convergence in Corollary 3.6:

$$\liminf_{\sigma\to\infty} J^\varepsilon_1 = \liminf_{\sigma\to\infty} \mathbb{E}_w \left[\exp\left(M^{f,\sigma^4}_{T^*_\varepsilon} - \frac{1}{2} A^{f,\sigma^4}_{T^*_\varepsilon} \right) \right.$$
$$\times \mathbf{1}\left(L^{\sigma^4}_t > (c_f - \delta_\varepsilon) T^*_\varepsilon \text{ for some } 0 \leq t \leq T^*_\varepsilon \right)$$
$$\left. \times \mathbf{1}\left(L^{\sigma^4}_t > -(c_f - \delta_\varepsilon) T^*_\varepsilon \text{ for all } 0 \leq t \leq T^*_\varepsilon \right) \times \mathbf{1}\Big(\sup_{0\leq t\leq T^*_\varepsilon} M^{f,\sigma^4} < N_\varepsilon \Big) \right]$$
$$\geq \mathbb{E}^{B^f,B}\Big[e^{B^f_{T^*_\varepsilon} - \frac{1}{2} D T^*_\varepsilon} \mathbf{1}\Big(\sup_{0\leq t\leq T^*_\varepsilon} B_t > (c_f - \delta_\varepsilon) T^*_\varepsilon \Big) \mathbf{1}\Big(\inf_{0\leq t\leq T^*_\varepsilon} B_t > -(c_f - \delta_\varepsilon) T^*_\varepsilon \Big)$$
$$\times \mathbf{1}\Big(\sup_{0\leq t\leq T^*_\varepsilon} B^f_t < N_\varepsilon \Big) \Big]. \tag{5.16}$$

We rewrite this, using Girsanov's theorem for correlated Brownian motions with a drift, as

$$\liminf_{\sigma\to\infty} J^\varepsilon_1 \geq \mathbb{E}^{B^f,B}\Big[e^{B^f_{T^*_\varepsilon} - \frac{1}{2} D T^*_\varepsilon} \mathbf{1}\Big(\sup_{0\leq t\leq T^*_\varepsilon} B_t > (c_f - \delta_\varepsilon) T^*_\varepsilon \Big)$$
$$\times \mathbf{1}\Big(\inf_{0\leq t\leq T^*_\varepsilon} B_t > -(c_f - \delta_\varepsilon) T^*_\varepsilon \Big) \Big]$$
$$- \mathbb{E}^{B^f,B}\Big[e^{B^f_{T^*_\varepsilon} - \frac{1}{2} D T^*_\varepsilon} \mathbf{1}\Big(\sup_{0\leq t\leq T^*_\varepsilon} B^f_t \geq N_\varepsilon \Big) \Big]$$
$$= \mathbb{P}^B\Big(\sup_{0\leq t\leq T^*_\varepsilon} (B_t + c_f t) \geq (c_f - \delta_\varepsilon) T^*_\varepsilon, \inf_{0\leq t\leq T^*_\varepsilon} (B_t + c_f t) > -(c_f - \delta_\varepsilon) T^*_\varepsilon \Big)$$
$$- \mathbb{P}^{B^f}\Big(\sup_{0\leq t\leq T^*_\varepsilon} B^f_t + Dt \geq N_\varepsilon \Big). \tag{5.17}$$

The first term in the right side can be bounded as

$$\mathbb{P}^B\Big(\sup_{0\leq t\leq T^*_\varepsilon} (B_t + c_f t) \geq (c_f - \delta_\varepsilon) T^*_\varepsilon, \inf_{0\leq t\leq T^*_\varepsilon} (B_t + c_f t) > -(c_f - \delta_\varepsilon) T^*_\varepsilon \Big)$$
$$\geq \mathbb{P}^B\Big(B_{T^*_\varepsilon} + c_f T^*_\varepsilon \geq (c_f - \delta_\varepsilon) T^*_\varepsilon, \inf_{0\leq t\leq T^*_\varepsilon} B_t > -(c_f - \delta_\varepsilon) T^*_\varepsilon \Big)$$
$$= \mathbb{P}^B\Big(B_{T^*_\varepsilon} \geq -\delta_\varepsilon T^*_\varepsilon, \inf_{0\leq t\leq T^*_\varepsilon} B_t > -(c_f - \delta_\varepsilon) T^*_\varepsilon \Big)$$
$$= \mathbb{P}^B\Big(B_1 \geq -\delta_\varepsilon \sqrt{T^*_\varepsilon}, \inf_{0\leq t\leq 1} B_t > -(c_f - \delta_\varepsilon) \sqrt{T^*_\varepsilon} \Big) \geq 1 - \varepsilon/100, \tag{5.18}$$

where the last inequality follows by (5.14). The second term in the right side of (5.17) can be bounded using the reflection principle for Brownian motion, bounds on tails of Gaussian probabilities and by choosing $N_\varepsilon \geq (2+\delta_\varepsilon)T_\varepsilon^* D$ sufficiently large, so that

$$\begin{aligned}\mathbb{P}^{B^f}\Big(\sup_{0\le t\le T_\varepsilon^*} B_t^f + Dt \geq N_\varepsilon\Big) &\leq \mathbb{P}^{B^f}\Big(\sup_{0\le t\le T_\varepsilon^*} B_t^f \geq N_\varepsilon - DT_\varepsilon^*\Big)\\ \leq 2\mathbb{P}^B\Big(\sqrt{D}B_{T_\varepsilon^*} \geq N_\varepsilon - DT_\varepsilon^*\Big) &\leq 2\exp\Big(-\frac{(N_\varepsilon - DT_\varepsilon^*)^2}{2T_\varepsilon^* D}\Big) \leq \varepsilon/100. \qquad (5.19)\end{aligned}$$

Combining (5.15)–(5.19) we get that for N_ε sufficiently large we have

$$\mathbb{P}_v\big(\widetilde{G}_0^{(1)}\big) \geq 1 - \varepsilon/50, \qquad (5.20)$$

which is (5.8).

Next, we bound $\mathbb{P}_v(\widetilde{G}_0^{(2)})$. Again, using Girsanov's theorem we write

$$\begin{aligned}\mathbb{P}_v(\widetilde{G}_0^{(2)}) &= \mathbb{E}_w\Big[\exp\Big(Z_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)}\Big)\mathbf{1}_{\widetilde{G}_0^{(2)}}\Big]\\ &\leq \mathbb{E}_w\Big[\exp\Big(\sigma^{-4}(M^f_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)} - \frac{1}{2}\sigma^{-4}A^f_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)}\Big)\\ &\quad\times \mathbf{1}\Big(L(w_t) \leq -\tilde{\lambda}_1\sigma^4 \text{ for some } 0\leq t\leq \sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)\Big)\\ &\quad\times \mathbf{1}\Big(\frac{1}{\sigma^4}M_t^f < N_\varepsilon \text{ for all } 0\leq t\leq \sigma^8 T_\varepsilon^*\Big)\Big]\\ &\quad+ \mathbb{E}_w\Big[\exp\Big(\sigma^{-4}(M^f_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)} - \frac{1}{2}\sigma^{-4}A^f_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)})\Big)\times\mathbf{1}\Big(\sup_{0\le t\le\sigma^8 T_\varepsilon^*}\frac{1}{\sigma^4}M_t^f \geq N_\varepsilon\Big)\Big]\\ &\leq \mathbb{E}_w\Big[\exp\Big(M^{f,\sigma^4}_{T_\varepsilon^*} - \frac{1}{2}A^{f,\sigma^4}_{T_\varepsilon^*}\Big)\times\mathbf{1}\Big(\inf_{0\le t\le T_\varepsilon^*} L_t^{\sigma^4} \leq -\tilde{\lambda}_1\Big)\times\mathbf{1}\Big(\sup_{0\le t\le T_\varepsilon^*} M^{f,\sigma^4} < N_\varepsilon\Big)\Big]\\ &\quad+ \mathbb{E}_w\Big[\exp\Big(\sigma^{-4}(M^f_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)} - \frac{1}{2}\sigma^{-4}A^f_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)}\Big)\\ &\quad\times\mathbf{1}\Big(\sup_{0\le t\le\sigma^8 T_\varepsilon^*}\frac{1}{\sigma^4}M_t^f \geq N_\varepsilon\Big)\Big] =: J^\varepsilon_{2,1} + J^\varepsilon_{2,2}.\end{aligned}$$

The term $J^\varepsilon_{2,2}$ is exactly as I_1^ε in (4.23), thus, as in (4.25) we have, by choosing N_ε sufficiently large:

$$J^\varepsilon_{2,2} \leq \varepsilon/50, \qquad (5.21)$$

for all σ sufficiently large. As for $J^\varepsilon_{2,1}$, proceeding similarly to (5.16), we obtain

$$\begin{aligned}\limsup_{\sigma\to\infty} J^\varepsilon_{2,1} &\leq \mathbb{E}^{B^f,B}\Big[e^{B^f_{T_\varepsilon^*} - \frac{1}{2}DT_\varepsilon^*}\mathbf{1}\Big(\inf_{0\le t\le T_\varepsilon^*} B_t \leq -(c_f - \delta_\varepsilon)T_\varepsilon^*\Big)\\ &\quad\times\mathbf{1}\Big(\sup_{0\le t\le T_\varepsilon^*} B_t^f \leq N_\varepsilon\Big)\Big]\\ &\leq \mathbb{P}^B\Big(\inf_{0\le t\le T_\varepsilon^*}(B_t + c_f t) \leq -(c_f - \delta_\varepsilon)T_\varepsilon^*\Big)\\ &\leq \mathbb{P}^B\Big(\inf_{0\le t\le 1} B_t \leq -(c_f - \delta_\varepsilon)\sqrt{T_\varepsilon^*}\Big)\end{aligned}$$

$$\leq \varepsilon/100. \tag{5.22}$$

Here, the last inequality follows from (5.14). Combining (5.21) and (5.22) we see that for N_ε sufficiently large we have

$$\liminf_{\sigma\to\infty} J_2^\varepsilon \leq 3\varepsilon/100, \tag{5.23}$$

and (5.9) follows.

To bound $\widetilde{G}_0^{(3)}$, once again by Girsanov's theorem and recalling the definition of $J_{2,2}^\varepsilon$, we obtain

$$\begin{aligned}\mathbb{P}_v(\widetilde{G}_0^{(3)}) &= \mathbb{E}_w\Big[\exp\Big(Z_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)}\Big)\mathbf{1}_{\widetilde{G}_0^{(3)}}\Big]\\ &\leq \mathbb{E}_w\left[\exp\left(\sigma^{-4}(M^f_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)}-\frac{1}{2}\sigma^{-4}A^f_{\sigma^8(T_\varepsilon^*\wedge\xi^\varepsilon)})\right)\right.\\ &\quad\left.\times\mathbf{1}\Big(\sup_{0\leq t\leq T_\varepsilon^*}M^{f,\sigma^4}\geq N_\varepsilon\Big)\right]=J_{2,2}^\varepsilon\leq\varepsilon/50,\end{aligned}$$

where the last inequality follows from (5.21) for N_ε sufficiently large and all σ sufficiently large. Thus (5.10) follows, and the proof of Lemma 5.2 is complete. □

Proof of Theorem 1.2. Now, we can complete the proof of Theorem 1.2. From Propositions 4.1 and 5.1 we obtain the conclusion of Theorem 1.2 for $c_f > 0$. If $c_f < 0$, consider $\tilde{u}(x) = 1 - u(-x)$ instead of $u(x)$. Then c_f turns into $-c_f > 0$ for $\tilde{u}$. Thus by Propositions 4.1, 5.1 we get the upper and lower bounds on the speed of $\tilde{u}$ which gives us the lower and upper bounds respectively on the speed of u with $c_f < 0$. This proves Theorem 1.2 for $c_f < 0$.

If $c_f = 0$, the upper bound on the speed of u follows from Propositions 4.1. Switching again to $\tilde{u}(x) = 1-u(-x)$, we get the upper bound on the speed of $\tilde{u}$ by Propositions 4.1. This, in turn, gives the lower bound on the speed of u. The proof is complete. □

Acknowledgements The work of LM and LR was supported by a US-Israel BSF grant. LR was supported by NSF Grant DMS-1613603 and ONR Grant N00014-17-1-2145, and CM was supported by a Simons Grant.

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Communicated by M. Hairer