EXPANDING THE REALS BY CONTINUOUS FUNCTIONS ADDS NO COMPUTATIONAL POWER

URI ANDREWS, JULIA F. KNIGHT, RUTGER KUYPER,
JOSEPH S. MILLER, AND MARIYA I. SOSKOVA

Abstract. We study the relative computational power of structures related to the ordered field of reals, specifically using the notion of generic Muchnik reducibility. We show that any expansion of the reals by a continuous function has no more computing power than the reals, answering a question of Igusa, Knight, and Schweber [7]. On the other hand, we show that there is a certain Borel expansion of the reals that is strictly more powerful than the reals and such that any Borel quotient of the reals reduces to it.

1. Introduction

We would like to compare the computational power of algebraic structures. For countable structures, Muchnik reducibility provides a useful way to do this: if $\mathcal{A}$ and $\mathcal{B}$ are countable structures (in computable languages), then $\mathcal{A}$ is *Muchnik reducible* to $\mathcal{B}$ ($\mathcal{A} \leqslant_w \mathcal{B}$) provided that every copy of $\mathcal{B}$ computes a copy of $\mathcal{A}$. Schweber, in [9], introduced a generalization of this reducibility that allows us to compare the computational power of structures of arbitrary cardinality.

Definition 1.1 (Generic Muchnik reducibility)**.** For a pair of structures $\mathcal{A}$ and $\mathcal{B}$ (not necessarily countable) in V, we say that $\mathcal{A}$ is *generically Muchnik reducible* to $\mathcal{B}$, and we write $\mathcal{A} \leqslant_w^* \mathcal{B}$, if for any generic extension $V[G]$ of the set-theoretic universe V in which both structures are countable, we have $V[G] \models \mathcal{A} \leqslant_w \mathcal{B}$.

In other words, we collapse cardinals so that the structures $\mathcal{A}$ and $\mathcal{B}$ become countable, and then we apply the standard tools of computability theory to study them. It follows from Shoenfield's absoluteness theorem [18] that generic Muchnik reducibility is set-theoretically robust:

Lemma 1.2 (Schweber [9])**.** *If $\mathcal{A} \leqslant_w^* \mathcal{B}$, then $\mathcal{A} \leqslant_w \mathcal{B}$ in every forcing extension that makes $\mathcal{A}$ and $\mathcal{B}$ countable.*

After the initial paper [9], there have been several further papers in which generic Muchnik reducibility is used to compare the computational power of structures related to the real numbers. Here are some of these structures.

Date: September 8, 2022.

2010 *Mathematics Subject Classification.* Primary 03D30; Secondary 03D45, 03E40.

Andrews was partially supported by grant DMS #1600228 from the National Science Foundation. Knight was partially supported by grant DMS #1800692 from the National Science Foundation. Miller was partially supported by grant #358043 from the Simons Foundation. Soskova was partially supported by grant DMS #1762648 from the National Science Foundation. Soskova and Miller were partially supported by grant DMS #2053848 from the National Science Foundation .

- *Cantor space*: This is represented in [9] by $\mathcal{W} = (\mathcal{P}(\omega) \cup \omega, \mathcal{P}(\omega), \omega, S, \in)$, where S is the successor function on ω. Another representation, possibly more natural, is $\mathcal{C} = (2^\omega, (R_n)_{n\in\omega})$, where for $\alpha \in 2^\omega$, $\alpha \in R_n$ if and only if $\alpha(n) = 1$. Clearly, $\mathcal{W} \equiv^*_w \mathcal{C}$.
- *The ordered field of reals*: This is $\mathcal{R} = (\mathbb{R}, +, \times, <)$.
- *The non-standard reals*: $\mathcal{R}^*$ is an ω-saturated extension of $\mathcal{R}$. Note that this structure is not unique, although it becomes unique after collapse of cardinals.
- *Baire space*: This is the structure $\mathcal{B} = (\omega^\omega, (R_{n,m})_{n,m\in\omega})$, where $f \in R_{n,m}$ if and only if $f(n) = m$.

The structures listed above all fall into one of two generic Muchnik degrees. In [9], it is shown that $\mathcal{W} \leqslant^*_w \mathcal{R}$. Igusa and Knight [6] showed, using a result of Macintyre and Marker [11], that $\mathcal{R}^* \equiv^*_w \mathcal{W}$. Downey, Greenberg, and Miller [3] showed that $\mathcal{R} \equiv^*_w \mathcal{B}$. Thus, we have that

$$\mathcal{C} \equiv^*_w \mathcal{W} \equiv^*_w \mathcal{R}^* \leqslant^*_w \mathcal{R} \equiv^*_w \mathcal{B}.$$

Finally, we know from [3] and [6] that the above inequality $\leqslant^*_w$ is strict, so

$$\mathcal{C} \equiv^*_w \mathcal{W} \equiv^*_w \mathcal{R}^* <^*_w \mathcal{R} \equiv^*_w \mathcal{B}.$$

In the present paper, we investigate structures of the form $\mathcal{R}_f = (\mathcal{R}, f)$, the ordered field of reals expanded by a function f. Igusa, Knight, and Schweber [7] showed that if f is analytic, then $\mathcal{R}_f \equiv^*_w \mathcal{R}$. They asked whether this remains true for arbitrary continuous functions f. They believed that the answer should be negative, witnessed possibly by something like Brownian motion, with complicated level sets as studied in Allen, Bienvenu, and Slaman [1]. Here, we show that the answer to the question is actually positive.

Theorem 1.3. *If f is a continuous function (of any arity) on $\mathcal{R}$, then $\mathcal{R}_f \leqslant^*_w \mathcal{R}$.*

Below, we give a brief outline of the proof of Theorem 1.3. Recall that for a countable family of sets $S \subseteq \mathcal{P}(\omega)$, an *enumeration* is a relation $E \subseteq \omega^2$ such that the sets $E_n = \{x : (n, x) \in E\}$ are exactly those in S. For a countable family of functions $F \subseteq \omega^\omega$, an *enumeration* is a function $P : \omega^2 \to \omega$ such that the functions $P_n(x) = P(n, x)$ are exactly those in F.

Definition 1.4 (Turing ideal, jump ideal)**.**

(1) A *Turing ideal* is a family of sets $S \subseteq \mathcal{P}(\omega)$ that is closed under disjoint union and Turing reducibility.
(2) A *jump ideal* is a Turing ideal that is closed under Turing jump.

After collapse, the old $\mathcal{P}(\omega)$ becomes a countable jump ideal S, and the family F of functions $f \in \omega^\omega$ that are present in the old ω^ω is the family of functions computable from elements of S. Downey, Greenberg, and Miller [3] give an important characterization of the two generic Muchnik degrees discussed above, that of $\mathcal{C}$ and that of $\mathcal{R}$. After collapse, computing a copy of $\mathcal{C}$ is equivalent to computing an enumeration of S, while computing a copy of $\mathcal{R}$ is equivalent to computing an enumeration of F. The first step in the proof of our main theorem is to explore what computational power is given by an enumeration of F, but not given by an enumeration of S.

Let S be a countable jump ideal, and let E be an enumeration of S. We may think of E as a function taking each index n to the set E_n. We consider the

companion function J taking n to the set $(E_0 \oplus \ldots \oplus E_n)'$. More formally, J will be an enumeration of a subfamily of S, with specified indices.

Definition 1.5 (Running jump)**.** Let E be an enumeration of a jump ideal S. The *running jump for* E is the relation $J \subseteq \omega^2$ such that $(n, x) \in J$ if and only if $x \in (E_0 \oplus \ldots \oplus E_n)'$.

In Section 2, we prove the following result, which we believe is of independent interest.

Theorem 1.6. *Let $S \subseteq \mathcal{P}(\omega)$ be a countable jump ideal, and let $F \subseteq \omega^\omega$ be the family of all functions computable from sets in S. From an arbitrary enumeration P of F, we can compute an enumeration E of S together with the running jump for E.*

After collapse, let S be the family of sets in the old $\mathcal{P}(\omega)$, and let $F \subseteq \omega^\omega$ be the family of functions computable from sets in S. From a copy of $\mathcal{R}$ (now a countable structure), we can compute an enumeration P of F (this is immediate from the fact that $\mathcal{R} \equiv^*_w \mathcal{B}$). Applying Theorem 1.6, we get an enumeration E of S together with the running jump J for E. In Section 3, we show that the combination of E and the running jump J provides the information needed to build a copy of $\mathcal{R}_f$.

Theorem 1.7. *Suppose $f : \mathcal{R}^k \to \mathcal{R}$ is continuous. After collapse, let E be an enumeration of the sets in the old $\mathcal{P}(\omega)$, let J be the running jump for E, and let $\mathcal{R}_f = (\mathcal{R}, f)$. Then there is a copy of $\mathcal{R}_f$ computable from $E \oplus J$.*

We further show that continuous expansions of Cantor space also have generic Muchnik degree bounded by the degree of $\mathcal{R}$. Denote by $\mathcal{C}_f$ the expansion of $\mathcal{C}$ by a function f on Cantor space. The following is a direct consequence of the more generally phrased Theorem 3.2.

Theorem 1.8. *If f is a continuous function (of any arity) on $\mathcal{C}$, then $\mathcal{C}_f \leqslant^*_w \mathcal{R}$.*

In Section 3, we also give examples of continuous functions f such that $\mathcal{C}_f \equiv^*_w \mathcal{R}$. An interesting question arises: are there generic Muchnik degrees strictly between the degree of $\mathcal{C}$ and the degree of $\mathcal{R}$ and if so, can they be obtained as continuous expansions of $\mathcal{C}$? This problem is treated in detail in the upcoming paper by Andrews, Miller, Schweber, and M. Soskova [2]. They show that there is a degree strictly between $\mathcal{C}$ and $\mathcal{R}$. Later Gura[1] exhibited a whole hierarchy of such degrees. On the other hand, Andrews, Miller, Schweber, and M. Soskova [2] show a dichotomy result for expansions of $\mathcal{C}$ by closed predicates: they lie either in the degree of $\mathcal{C}$ or in the degree of $\mathcal{R}$. Recall that continuous functions have closed graphs, hence this answers our original questions.

In Section 4, we investigate continuous expansions of Baire space $\mathcal{B}$. We show that, unlike with $\mathcal{R}$, there is a way to continuously expand $\mathcal{B}$ to get a structure of strictly higher complexity. Let $(\mathcal{B}, \oplus, ')$ be the structure of Baire space with adjoined join and jump functions (appropriately defined for members of ω^ω). We show that this structure has a very powerful generic Muchnik degree.

Theorem 1.9. *There is a continuous expansion of Baire space in the generic Muchnik degree of $(\mathcal{B}, \oplus, ')$.*

(1) $(\mathcal{B}, \oplus, ') >^*_w \mathcal{B}$.

[1]Unpublished.

(2) *Every Borel expansion (even Borel quotient) of $\mathcal{B}$ is generic Muchnik reducible to $(\mathcal{B}, \oplus, ')$.*

We refer to the generic Muchnik degree of the structure $(\mathcal{B}, \oplus, ')$ as the *Borel complete degree*. In Section 4, we show that as a consequence of this, Theorem 1.3 cannot be strengthened to Borel expansions of $\mathcal{R}$—there are such expansions of $\mathcal{R}$ in the Borel complete degree. As with $\mathcal{C}$, it is natural to ask if continuous expansions of $\mathcal{B}$ can lie strictly between $\mathcal{B}$ and the Borel complete degree. Assuming $\mathbf{\Delta^1_2}$-Wadge determinacy, Andrews, Miller, Schweber, and M. Soskova [2] prove a dichotomy result for closed (equivalently, continuous) expansions of $\mathcal{B}$: they have the same generic Muchnik degree either as $\mathcal{B}$ or as the Borel complete degree. They also show that there are generic Muchnik degrees strictly between these two.

2. Results on enumerations

We turn to the proof of Theorem 1.6. Recall the statement.

Theorem 1.6. *Let $S \subseteq \mathcal{P}(\omega)$ be a countable jump ideal, and let $F \subseteq \omega^\omega$ be the family of functions computable from sets in S. From an arbitrary enumeration P of F, we can compute an enumeration E of S together with the running jump for E.*

Proof. We will compute an enumeration E in stages, so that at any stage we have determined finitely many bits of E. At stage s, we will have instructions for each column E_n with $n < s$. These instructions either will be to copy some element of P (or rather a set associated to that element in a fixed way that will be made precise) or will be an index for a computation that describes how we should complete the column. In the former case, we will say that n is a *copy column* and in the latter case, we will say that n is a *fix column*.

The difficulty in the construction, of course, comes from computing the relation J that gives us the running jump for E. The core idea is that the jump ideal contains sets computing the settling time functions for the running jumps. Recall that A' has a standard representation as an A-c.e. set W^A. A *settling time function* for A' is a function $s\colon \omega \to \omega$ such that $n \in A'$ if and only if $n \in W^{A\upharpoonright s(n)}_{s(n)}$, i.e. n is the stage $s(n)$ approximation to the set W^A and references the oracle A only on numbers less than n. If a jump ideal S contains A, then the least (with respect to majorizing) settling time function $s_{A'}$ is computable from A', hence it is in F. To define J_i, we make an initial guess towards a P-index for a settling time function of $(E_0 \oplus \cdots \oplus E_i)'$: we guess that P_0 is such a function. We then try to compute $(E_0 \oplus \cdots \oplus E_i)'$ using our current guess of that function and the columns $E_0, \ldots, E_i$. If the guess is incorrect, we will notice this after a finite amount of time. We will see that the settling time function predicted that some natural number $x \notin (E_0 \oplus \cdots \oplus E_i)'$, but now, after computing more steps than our guess assured us would be enough, we see that $x \in (E_0 \oplus \cdots \oplus E_i)'$. At that point, we would like to revise our guess at the settling time function by moving to the next possible function in our enumeration P of F, until we eventually hit the right one.

We need a way to deal with the injury that happens when we discover that our guess is incorrect. We will have already specified finitely many bits of E, and of J. We have already announced that $x \notin (E_0 \oplus \cdots \oplus E_i)'$, misled by the wrong guess. This means that our assignment of copy columns and fix columns is no longer consistent with J being the running jump of E. To remedy this problem, we ensure during the construction that there is always a way to extend E_i that is compatible

with J_i and that we can compute such an extension. To achieve this, we use the Low Basis Theorem by Jockusch and Soare [8]. We state it here in a relativized, uniform manner.

Theorem 2.1 (The Low Basis Theorem)**.** *Let $T \subseteq 2^{<\omega}$ be an infinite X-computable tree. There is an infinite path Y through T such that $(X \oplus Y)' \leqslant_T X'$. Moreover, an index for this reduction can be obtained uniformly in an index from the reduction witnessing that $T \leqslant_T X$.*

As a result, J will remain correct. The trade-off is that this action interferes with the requirement that E is an enumeration of S. In particular, we will have to turn more columns into fix columns (whose role is simply to be filled in and keep J consistent). Indeed, as a copy column, E_n was trying to enumerate some element X of S, but after injury, we abandon this goal and instead let E_n fill in its column in a way that preserves J. Our only option is to enumerate X at position E_j for some fresh j.

The approach described above ensures that we compute J correctly, and that if every requirement copying some element X of S is injured only finitely often, then E is an enumeration of S. Intuitively, the injury should only happen finitely often because we eventually find the right settling time function. However, there is a complication. Let us fix a set X and assume that all higher priority requirements are no longer injured after stage s. At stage s, we find out that our guess towards the settling time function for $(E_0 \oplus \cdots \oplus E_i)'$ is incompatible with our assignment of E_i as a copy column and transition to the escape strategy explained above. We let E_i, $E_{i+1}, \ldots, E_{j-1}$ be fix columns and we begin filling them in with sets that are low with respect to previous columns, i.e., E_{i+k} is low over $E_0 \oplus \cdots \oplus E_{i+k-1}$. Note that this will make it easy to fill in the columns $J_i, \ldots, J_{j-1}$ as $(E_0 \oplus \cdots \oplus E_{i+k})'$ will be computable from $(E_0 \oplus \cdots \oplus E_{i-1})'$. We start copying X at position $j > i$, where j is a column for which we have not yet defined any values. We now need to guess at the settling time function for $(E_0 \oplus \cdots \oplus E_{j-1} \oplus X)'$. This settling time function could be vastly different from the one for $(E_0 \oplus \cdots \oplus E_{i-1} \oplus X)'$. Indeed, even though we were able to select the fix columns so that we can control the corresponding running jump columns, we cannot even guarantee that $E_0 \oplus \ldots E_{i-1} \oplus X \equiv_T E_0 \oplus \cdots \oplus E_{j-1} \oplus X$. An easy counterexample can be derived from the existence of two low sets whose join is $\varnothing'$: it could be that both X and $E_i \oplus \cdots \oplus E_{j-1}$ are low over $E_0 \oplus \cdots \oplus E_{i-1}$ but $(E_0 \oplus \cdots \oplus E_{j-1} \oplus X)' \equiv_T (E_0 \oplus \cdots \oplus E_{i-1} \oplus X)''$. Thus, there is no guarantee that we ever guess the settling time function correctly.

Luckily, there is an easy fix for this problem. Instead of guessing at the settling time function for $(E_0 \oplus \cdots \oplus E_{i-1} \oplus X)'$ and then for $(E_0 \oplus \cdots \oplus E_{j-1} \oplus X)'$ after the first injury, as we were previously doing, we will guess at a function that encodes the settling time function for all possible situations that we might end up in. More precisely, we guess at a function g with the property that if $\Phi_e(((E_0 \oplus \cdots \oplus E_{i-1})' \oplus X)')\downarrow = Z'$ for some set Z, then the e-th column $g^{[e]}$ of g is the settling time function for Z'. Since the Low Basis Theorem is uniform, we will know an index e such that $\Phi_e(((E_0 \oplus \cdots \oplus E_{i-1})' \oplus X)') = (E_0 \oplus \cdots \oplus E_{j-1} \oplus X)'$. If our current guess towards g is $\tilde{g}$, we modify the construction to use $\tilde{g}^{[e]}$ as the current guess at the settling time function, and if it turns out that $\tilde{g}^{[e]}$ is incorrect, then we give up on $\tilde{g}$ completely and move on to the next guess for g.

Construction. Our requirements are as follows. We have an ambient requirement:

- Q: for all $x, i \in \omega$, $x \in J_i$ if and only if $x \in (E_0 \oplus \cdots \oplus E_i)'$,

and we have countably many copy requirements:

- R_i: for every $i \in \omega$ there exists a $j \in \omega$ such that $E_j = \hat{P}_i = \{n : P_i(n) = 1\}$.

The ambient requirement Q will not be subject to injury, and the requirements R_i are given a priority order of order type ω.

The construction will proceed in stages. At stage s, we will construct $E[s]$ and $J[s]$ such that, ultimately, $E = \bigcup_{s<\omega} E[s]$ and $J = \bigcup_{s<\omega} J[s]$. We will omit $[s]$ in the construction; unless explicitly mentioned, all objects being constructed are evaluated at the current stage.

For every $i \in \omega$, we have a module M_i that is building J_i and a module N_i that is building E_i. They share two parameters: $q_i \in \omega \cup \{\text{fix}\}$ and $c_i \in \omega$. If $q_i \in \omega$, then we are in the case that i is currently a copy column; our current guess for the function g described above is P_{q_i}, and c_i indicates that M_i is currently copying $\hat{P}_{c_i}$. If $q_i = \text{fix}$, then we are in the case that E_i is a fix column and c_i is an index for a computation; we will endeavor to make $(E_0 \oplus \cdots \oplus E_i)' = \Phi_{c_i}((E_0 \oplus \cdots \oplus E_{i-1})')$. At any stage, let $m_i < i$ be greatest such that $q_{m_i} \neq \text{fix}$. If there is no such natural number m_i, then we let $m_i = -1$, and we let $J_{-1} = \varnothing'$. We have access to $\varnothing'$, having fixed an index for it in the enumeration P. Then by composing the computations $\Phi_{c_{m_i+1}}$ through Φ_{c_i}, we see how we intend to compute J_i from J_{m_i}, baring further injury.

At stage s, we attempt to determine the value of $J(m)$ for every $m = \langle i, j \rangle$ with $i \leqslant s$ and $j \leqslant s$. We first call M_0 on input x for every $x \leqslant s$, then M_1 on input x for every $x \leqslant s$ and so forth.

The M-module. We describe the module M_i on input x, that is, we describe how we determine the value of $J_i(x)$. Whenever we call the module M_i on input x, we first recursively call M_{i-1} on every input $\leqslant x$. (Note that both N and M modules may call an M module on numbers larger than s, so this instruction is not redundant. It ensures that if there is a mistake in our guess at an earlier column, then that mistake gets discovered. This will ultimately allow us to prove that every stage of the construction terminates in finite time.) If $J_i(x)$ is already defined, we do nothing further. Otherwise, we have two cases to consider:

Case 1. Suppose that $q_i \in \omega$. That is, the i-th column is currently a copy column. Then using the indices c_k for $k \in (m_i, i)$, we can find a computation which (assuming no injury) will describe the columns $E_{m_i+1}, \ldots, E_{i-1}$ from $(E_0 \oplus \cdots \oplus E_{m_i})'$, which (again assuming no injury) equals J_{m_i}. Thus we can find an index e so that (again assuming no injury) we will have $(E_0 \oplus \cdots \oplus E_i)' = \Phi_e((J_{m_i} \oplus E_i)')$. We then let $t = P_{q_i}(\langle e, x \rangle)$. If the requirement's guess is correct about q_i, then this t bounds the settling time function for $(E_0 \oplus \cdots \oplus E_i)'$ at x. We thus call the modules N_k on input x' for each $k \leqslant i$ and $x' \leqslant t$. Once these all return, we will know the values of $E_k(x')$ for each $k \leqslant i$ and $x' \leqslant t$. We then run the enumeration of $(E_0 \oplus \cdots \oplus E_i)'$ (as a set c.e. in $E_0 \oplus \cdots \oplus E_i$) for t stages to see if it enumerates x in this time. This determines whether we want to set $J_i(x)$ to be 0 or 1. We are almost ready to declare that $J_i(x)$ is this value, but before we can do this, we have to ask permission of every previous column; we will describe the process of asking for permission below. If they all give permission, then we declare $J_i(x)$ to be the value determined above.

Case 2. Suppose that $q_i = \text{fix}$. Then the i-th column is a fix column, and c_i gives a parameter for a computation of J_i from J_{i-1}. Thus, we simply call M_{i-1} for every x for which J_{i-1} is undefined until we see a computation giving $\Phi_{c_i}^{J_{i-1}}(x) \downarrow$.

Define a computable increasing function α so that for all sets A, B, we have $z \in B$ if and only if $\alpha(z) \in (A \oplus B)'$. We say that x is a coding number if and only if $x = \alpha(z)$ for some z. Note that if $x = \alpha(z)$ is a coding number for z then whether x enters $(A \oplus B)'$ when this set is enumerated computably in $A \oplus B$ will depend solely on whether $z \in B$. If $\Phi_{c_i}^{J_{i-1}}(x) = 0$, we ask for permission from every previous column to make $J_i(x) = 0$. If $\Phi_{c_i}^{J_{i-1}}(x) = 1$ and $x = \alpha(z)$ is a coding number, then we ask for permission to set $E_i(z) = 1$ and $J_i(x) = 1$. If $\Phi_{c_i}^{J_{i-1}}(x) = 1$ and x is not a coding number, then we search for a confirmation set. This means that we call the modules $N_0, \ldots, N_i$ on increasing inputs until we have assigned enough of $E_0 \oplus \cdots \oplus E_i$ so that the previous columns give permission to define $J_i(x)$ to be 1.

The N-module. We now describe the modules N_i on input z. Our goal is to determine whether $E_i(z)$ is 0 or 1.

Case 1. Suppose that $q_i \in \omega$. Then the goal of E_i is to copy $\hat{P}_{c_i}$. We ask permission to set $E_i(z) = \hat{P}_i(z)$ and do so if every previous column gives permission.

Case 2. E_i is a fix column. Call $M_i(\alpha(z))$. If this assigns $J_i(\alpha(z)) = 0$, then assign $E_i(z) = 0$. (The fact that previous columns gave permission to assign $J_i(\alpha(z)) = 0$ will imply that they also give permission to make $E_i(z) = 0$.) Note that if $M_i(\alpha(z))$ assigns $J_i(\alpha(z)) = 1$, then it also assigns $E_i(z) = 1$.

Permissions. We now describe the process of asking for permission from previous columns. We either want to assign $J_i(x)$ for some i and x, or $E_i(z)$ for some i and z (or both) and we need permission from each previous column. Let m be the largest so that q_m is defined. We describe how the k-th column determines whether to give permission. Let $\sigma_{k+1}, \ldots, \sigma_m$ be the fragments of $E_{k+1}, \ldots, E_m$ determined so far including the requested assignment, and let $\tau_{k+1}, \ldots, \tau_m$ be the fragments of $J_{k+1}, \ldots, J_m$ determined so far including the requested assignment.

Let T be the tree of possible ways to complete $\sigma_{k+1}, \ldots, \sigma_m$ so that for every j, x with $j \in (k, m]$ and $\tau_j(x) = 0$, we do not put x into the running jump. The k-th column wants to give permission if and only if:

- For every pair j, x with $j \in (k, m]$ and $\tau_j(x) = 1$, $x \in (E_0 \oplus \cdots \oplus E_k \oplus \sigma_{k+1} \oplus \cdots \oplus \sigma_j)'$. Here $E_0, \ldots, E_k$ are not just the partial fragments determined so far, rather they are the sets as they would be determined assuming all our guesses are correct. In other words, over the first k columns, the σ's have enough information already encoded to put x into the correct running jump.
- T has an infinite path.

Note that these two conditions are true if and only if two bits in the set $(E_0 \oplus \cdots \oplus E_k)'$ have fixed specific values—the first condition is equivalent to a fixed bit being 1 and the second to a fixed bit being 0. Thus, in order to determine whether or not to give permission, we call the module M_k on these bits. If these modules both return the correct value confirming the condition, then we give permission. Otherwise, if the first condition is incorrect (the module returns 0 to show that the first condition fails), then we simply deny permission but do not declare injury.

If the module returns a value signifying that T does not have a path, then we declare *injury* which means that for all $j \in [i, m]$, we set q_j to be fix. Let S be the tree of possible ways to complete the columns $E_i, \ldots, E_m$ so that for every j, x with $j \in (k, m]$ and $J_j(x) = 0$, we do not put x into the running jump. Since the $(i-1)$st column gave permission to the current configuration, there is some y such

that $J_{i-1}(y) = 0$ and this confirms that S has a path. Thus, by the effectiveness of the Low Basis Theorem, we can set $c_i, \dots, c_m$ to be so that there is some path through S so which would make $\Phi_{c_j}^{J_{j-1}} = J_j$ for each $j \in [i, m]$. We then complete the stage (though we may have not succeeded in the goal of assigning any value of J).

Lastly, whether we declared injury or not during the stage, before finishing the stage, we take the least m so that q_m is undefined and the least i so that currently no column has $c_j = i$ and define c_m to be i. If this is the l-th time that we define some c_j to be i since we last assigned some $c_{j'}$ to be $i - 1$, then we assign q_j to be l. This completes the construction.

Verification. We say that the i-th column is correct at stage s if for all $j \leqslant i$, the parameters q_j and c_j will never be changed at a stage $t > s$.

Lemma 2.2. *Suppose that the i-th column is correct at stage s. Define the sets A_j, B_j for $j \leqslant i$ inductively as follows: $A_{-1} = \varnothing$ and $B_{-1} = \varnothing'$. If the j-th column has parameters q_j, c_j with $q_j = \text{fix}$, then define B_j as $\Phi_{c_j}(B_{j-1})$. Define $z \in A_j$ if and only if $\alpha(z) \in B_j$. If $q_j \in \omega$, then define $A_j = \hat{P}_{c_j}$ and $B_j = (A_0 \oplus \cdots \oplus A_j)'$. Then for each $j \leqslant i$, $E_j = A_j$ and $J_j = B_j$. In particular, $J_j = (E_0 \oplus \cdots \oplus E_j)'$.*

Proof. We prove the result by induction on j. It is clearly true for $j = -1$. If $q_j = \text{fix}$, then since $J_{j-1} = B_{j-1} = (A_0 \oplus \cdots \oplus A_{j-1})'$, it follows that when we determined the final value of the parameter c_j, we were looking at a tree S that was accurately determined by previous columns (which do not get injured again or else c_j would be modified) to be infinite. Thus, by the Low Basis Theorem, S has an infinite low path and the sequence of Turing functionals we determined, one of which is Φ_{c_j}, is correct. Note that when we ask for permission to define $J_j(x)$ or $E_j(z)$, it cannot end in injury by our assumption that the j-th column is already correct. When we run the N_j module on z, if we want to make $E_j(z) = 1$, then that permission will be given because changing a bit in E cannot result in a failure of the permission for the first reason (i.e., it can only cause injury). If we want to make $E_j(z) = 0$, then we do not even ask for permission. When we run the M_j module on x, Φ_{c_j} will eventually converge on x. If M_j wants to define $J_j(x) = 0$, it will be given permission (or else j is not correct). If M_j wants to make $J_j(x) = 1$ and $x = \alpha(z)$ then we will get permission to enumerate z in E_j and that is all we need as a confirmation set, hence M_j will be granted permission to define $J_j(x)$. If x is not a coding number then we are assured that eventually we will find a confirmation set by the correctness of Φ_{c_j} and all previous columns and the fact that N_j always gets permission to define E_j, as we just argued. Thus, we will build $E_j = A_j$ and $J_j = B_j$ for some sets A_j and B_j so that $(A_0 \oplus \cdots \oplus A_j)' = B_j$.

If $q_j \in \omega$, then since the column is correct, it follows that $E_j = \hat{P}_{c_j} = A_j$. Note that since, in Case 1 of the M_j module, we only ever request to place $J_j(x) = 1$ if we already see enough of $E_0 \oplus \cdots \oplus E_j$ defined, it follows that when we ask for permission to assign $J_j(x)$, we either get permission or some column declares injury. Since the columns are all correct by assumption, the latter case is impossible. Thus J_j defines a sequence in 2^ω. Since q_j is correct, it is never injured, so $(A_0 \oplus \cdots \oplus A_{j-1} \oplus A_j)'$ cannot contain an x so that $J_j(x) = 0$. Thus $J_j \supseteq (A_0 \oplus \cdots \oplus A_j)'$. Similarly, since we only ever put $J_j(x) = 1$ after we have already seen a fragment $\sigma \preceq E_j$ such that $x \in (A_0 \oplus \cdots \oplus A_{j-1} \oplus \sigma)'$, it follows that $J_j = (A_0 \oplus \cdots \oplus A_j)' = B_j$. $\square$

Lemma 2.3. *If $i-1$ is correct at stage s and $q_i = \mathrm{fix}$, then i is also correct at stage s.*

Proof. Let t be the stage when c_i was last redefined. It follows that the first i columns were correct at the stage t. Hence, the tree S that we considered at that stage was determined correctly and the Low Basis Theorem allowed us to compute the functional Φ_{c_i} such that $\Phi_{c_i}((E_0 \oplus \cdots \oplus E_{i-1})') = B = (E_0 \oplus \cdots \oplus E_{i-1} \oplus A)'$ for some set A, and the fragment of E_i that has already been determined agrees with A. By Lemma 2.2, if i is correct at stage t then eventually $E_i = A$ (and $J_i = B$). Using the same proof as above, we can argue that unless i is injured, whenever it defines a value for $E_i(z)$ from now on, it agrees with A. The only reason for i not being correct at stage s is that a request for permission ends in injury. This request can only come from M_i and only for a bit $J_i(x)$ to be given value 0, given the assumption that s is correct for $i-1$. (As we argued in the previous proof, requests for $J_i(x)$ to be given value 1 do not end in a failure of the second kind and so do not cause injury.) On the other hand, M_i will only ask for such a permission if it has already seen that $\Phi_{c_i}^{J_{i-1}}(x) = 0$. The tree T of possible extensions of $\sigma \preceq E_i$ that make $J_i(x) = 0$ contains an infinite path, namely A. So injury will not occur. □

Lemma 2.4. *Every stage terminates.*

Proof. Note that when we call M_j on input x, we may have to call N_k for $k \leqslant j$, and when we call N_j on input x, we may call M_k for $k \leqslant j$. One fear is that we may call M_j from N_j and N_j from M_j. The only reason that we call M_j from N_j is if j is a fix column. In this case, N_j calls an instance M_j on input x where x is a coding number, so this instance of M_j does not call N_j. Thus there are no loops in the module calls.

The other fear is in M_j calls for fix columns j. This makes a sequence of $M_{(j-1)}$-calls and waits to see a computation converge. Notice that if the sequence of $M_{(j-1)}$-calls were infinite, then we would determine the full $j-1$ first columns during this stage. Thus, the stage ends if $j-1$ is not correct at stage s. If $j-1$ is correct at stage s, then Lemma 2.3 shows that j is correct at stage s, and Lemma 2.2 shows that we eventually assign the appropriate values of J_j, and thus the stage ends. □

Lemma 2.5. *For every i, there is a stage s so that the i-th column is correct at stage s.*

Proof. Let $s_0 > i$ be a stage such that $i-1$ is correct. Thus, at stage $t > s_0$, each of the columns $j < i$ successfully determines $J_j(x)$ for every $x \leqslant t$. By the definition of correctness, if the i-th column is not correct at stage s_0, then there is a $t > s_0$ at which q_i becomes redefined to fix. But then by Lemma 2.3, the i-th column is correct at stage t. □

Lemma 2.6. *For every n there is a stage s and a column i so that the i-th column is correct at stage s, $q_i \in \omega$, and $c_i = n$ at stage s.*

Proof. Let s_0 be the first stage at which the lemma is true for every $m < n$. Let $X = \hat{P}_n$. Let i be the largest such that $q_i \in \omega$ is defined at stage s_0. By the final action for every stage, all $t > s_0$ end with some column $k(t)$ having $q_k \in \omega$ and $c_k = n$. If this $k(t)$ is constant from some stage onward, then we have that this column must be correct and we are done. Thus, we need to see why k cannot change

infinitely often. Suppose this were the case. Then let t be a stage where $q_{k(t)}$ is the P-index of a function g such that for every e, if $\Phi_e(((E_0 \oplus \cdots \oplus E_i)' \oplus X)')\downarrow = Z'$ for some set Z, then the e-th column $g^{[e]}$ of g bounds the settling time function for Z'.

Since every column between i and $k(t)$ is a fix column, we know that the $(k(t)-1)$st column is correct at stage t. Also, for every stage greater than t where we call $M_{k(t)}$ on any input, we will get the correct response because g bounds the appropriate settling time function. Hence the $k(t)$-th column is not injured, so it is also correct at stage t. □

Lemma 2.7. *The set E gives an enumeration of S, and J is the running jump of E.*

Proof. By Lemma 2.6, every set in S is enumerated in E. By Lemma 2.5, every column is eventually correct. By Lemma 2.2, this means that the columns up to the i-th column each either copy elements of S or are low over elements of S, thus are in S. Similarly, Lemma 2.2 shows that the i-th column of J is the running jump of the first i columns of E. □

This completes the proof of Theorem 1.6. □

Note that we did not build an injective enumeration in the previous proposition, but any enumeration E for which we can compute the running jump can easily be turned into an enumeration E^* without repetitions, using the running jump. Furthermore, we do not lose the running jump, because we also have an index telling us how to compute J_i^* from J_{i*}, where i^* is the least j such that $E_0, \ldots, E_j$ contains i different elements.

Finally, let us remark that it is essential that we started with an enumeration of the *functions*, but ended up with an enumeration of the *sets*. We cannot strengthen Theorem 1.6 by assuming that we are only given an enumeration of the *sets* in S; this follows from Lemma 3.3 and the fact mentioned above that $\mathcal{C} <_w^* \mathcal{B}$. On the other hand, we cannot strengthen Theorem 1.6 to give us an enumeration of the *functions* computable from elements of S, along with the running jump of this enumeration; this follows by Proposition 4.6 and Theorem 4.15.

3. Continuous expansions of the reals

We will now show that continuous expansions of $\mathcal{R}$ are generic Muchnik reducible to Baire space.

Theorem 1.7. *Suppose $f : \mathcal{R}^k \to \mathcal{R}$ is continuous. After collapse, let E be an enumeration of the sets in the old $\mathcal{P}(\omega)$, let J be the running jump for E, and let $\mathcal{R}_f = (\mathcal{R}, f)$. Then there is a copy of $\mathcal{R}_f$ computable from E and J.*

Proof. We need a good way of representing the elements in our copy of $\mathcal{R}_f$. We may think of each element as the sum of an integer z and a remainder e in the interval $[0, 1]$, where e has a binary expansion corresponding to some E_i. (Here, we identify E_i with χ_{E_i}.) Recall that a dyadic rational[2] is one of form $\frac{z}{2^k}$, where $z \in \mathbb{Z}$ and $k \in \omega$. Let D be the set of dyadic rationals. If e is a dyadic rational in the interval $[0, 1]$, then e has two binary expansions. Even fixing the binary expansion, our enumeration may have more than one index i for the same set E_i. To resolve

[2]We do not restrict to the interval $[0, 1]$.

this issue we note the following obvious claim and use the properties of the running jump.

Claim 1: Using oracle E_i, and fixing z, we can apply a uniformly effective procedure to enumerate the true statements $q < z + E_i$, $z + E_i < q$, where q is a dyadic rational.

Using E and J, we can effectively say whether $z + E_i = q$, where q is a dyadic rational. We can also effectively say whether $z + E_i = z' + E_j$. Thus, we can effectively build a structure $(R, <, \{q\}_{q \in D})$ with the ordering, plus constants q for the dyadic rationals, and with the elements we want. The constants are dense in the ordering, and we have a many-one function F, computable from E and J, taking each pair $\langle z, i \rangle$ to the appropriate $r \in R$ such that the statements from Claim 1 are true.

Claim 2: Using E and J, we can effectively expand the structure $(R, <, \{q\}_{q \in D})$, adding $+$, $\times$, and f.

We prove Claim 2 as follows. Recall that the set $T = \mathrm{Th}(\mathcal{R}_f)$ is one of the columns in E, so we can fix it as a parameter. Using T, we can enumerate the sentences of T saying that f maps I into I', where I is a k-fold product of closed intervals with dyadic rational endpoints and I' is a closed interval with dyadic rational endpoints. Using a tuple $\bar{r} = \langle z_j + E_{i_j} : i < k \rangle$, we can enumerate the set $\mathcal{S}$ of k-fold products of closed intervals with dyadic rational endpoints which contain $\bar{r}$. We want $f(\bar{r}) = F(z' + E_m)$, where $z' + E_m$ lies in all of the intervals I' with dyadic rational endpoints that, according to the theory T, contain $f(I)$ for some $I \in \mathcal{S}$. For the given $\bar{r}$, the set of I' is c.e. in T and the E_{i_j}. Given E_m, and the jump of $T \oplus \bigoplus_{j<n} E_{i_j} \oplus E_m$, we can determine whether a given $z' + E_m$ is an appropriate F-pre-image of $f(\bar{r})$. Thus, using E and J, we can effectively compute $f(\bar{r})$ from $\bar{r}$.

Similarly, since the operations $+$ and $\times$ are also continuous, we can effectively compute $r + r'$ and $r \times r'$ from r and r'. □

In the same way, we can prove that $(\mathcal{R}, \{f_n\}_{n \in \omega}) \leqslant^*_w \mathcal{R}$, where $\{f_n\}_{n \in \omega}$ is a countable family of continuous functions on $\mathcal{R}$.

Theorem 3.1. *Let $f_1, f_2, \ldots$ be continuous functions (of any arities) on $\mathcal{R}$. Then $(\mathcal{R}, \{f_n\}_{n \in \omega}) \equiv^*_w \mathcal{R}$.*

In Downey, Greenberg, and Miller [3] and Igusa, Knight, and Schweber [7], it is shown that $(\mathbb{R}, +, <)$ and $(\mathbb{R}, \{q\}_{q \in \mathbb{Q}}, <)$ are both generic Muchnik equivalent to $\mathcal{R}$. Using the techniques given above, and the fact that both of these structures are generic Muchnik above $\mathcal{B}$, it is now straightforward to show that $(\mathbb{R}, +, <)$ and $(\mathbb{R}, \{q\}_{q \in \mathbb{Q}}, <)$ expanded with a countable number of continuous functions are equivalent to the original structures. In particular, $(\mathbb{R}, +, <) \equiv^*_w (\mathbb{R}, \{q\}_{q \in \mathbb{Q}}, <) \equiv^*_w \mathcal{R}$ now follows directly, because $\times$ and $+$ are continuous.

Recall that two other structures that have been studied are Cantor space $\mathcal{C}$ and Baire $\mathcal{B}$ space and for them we know that $\mathcal{C} \equiv^*_w \mathcal{R}^*$, and Baire space $\mathcal{B} \equiv^*_w \mathcal{R}$. We may consider expanding either structure by continuous functions. Let us start with $\mathcal{C}$.

Theorem 3.2. *Let $f_1, f_2, \ldots$ be continuous functions (of any arities) on 2^ω. Then $(\mathcal{C}, \{f_i\}_{i \in \omega}) \leqslant^*_w \mathcal{B}$.*

Proof. After collapse, given a copy of $\mathcal{B}$, we can compute an enumeration E of the family of sets in the old $\mathcal{P}(\omega)$, together with the associated running jump enumeration J by Theorem 1.6. Given E, we immediately compute a copy $\mathcal{A}$ of $\mathcal{C}$. Computable in E and J, we have a many-one function I taking all indices i for sets E_i to elements $a \in \mathcal{C}$ such that if $E_i = E_j$, then $I(i) = I(j)$. We let $R_n(a)$ hold if and only if $n \in E_i$ for any and all i such that $I(i) = a$. For simplicity, consider a single continuous function f on $\mathcal{C}$, of a single variable x. The fact that f is continuous means that $\mathrm{Th}(\mathcal{C}, f)$ includes sentences $\varphi_{\sigma,\tau}$, saying, for certain $\sigma, \tau \in 2^{<\omega}$, that f maps the sequences extending σ to sequences extending τ. For $a \in \mathcal{A}$, we say that a *extends* σ if, for all $k \in dom(\sigma)$, $\sigma(k) = 1$ implies $R_k(a)$ and $\sigma(k) = 0$ implies $\neg R_k(a)$.

We let $f(a) = b$ where for all sentences $\varphi_{\sigma,\tau}$ in $\mathrm{Th}(\mathcal{C}, f)$, if a extends σ, then b extends τ. For each $a \in \mathcal{A}$, let T_a be the set of τ such that for some sentence $\varphi_{\sigma,\tau} \in \mathrm{Th}(\mathcal{C}, f)$, a extends σ. Assuming that E_j is the theory and $I(i) = a$, we can effectively enumerate T_a using E_i and E_j. For any $s > i, j$, we can use J_s to see if there is $k \leqslant s$ with $I(k) = b$ such that b extends τ for all $\tau \in T_a$. For sufficiently large s, we will find an appropriate k and b. We let $f(a) = b$. □

Recall that $\mathcal{C}$ itself is not equivalent to $\mathcal{B}$, so not all continuous expansions of $\mathcal{C}$ are equivalent to $\mathcal{B}$. However, some simple-looking expansions of $\mathcal{C}$ turn out to be equivalent to $\mathcal{B}$. In particular, let $\oplus\colon 2^\omega \times 2^\omega \to 2^\omega$ denote the join operator and $\sigma\colon 2^\omega \to 2^\omega$ denote the *shift*, i.e., $\sigma(b_0 b_1 b_2 b_3 \cdots) = b_1 b_2 b_3 \cdots$. Both are natural continuous functions on Cantor space. We show that Cantor space with join, and Cantor space with shift, are both equivalent to Baire space.

Lemma 3.3. *If X can compute an enumeration E of the sets in a jump ideal I, and the set $Fin_E = \{n : E_n \text{ is finite}\}$ is c.e. in X, then X computes an enumeration P of the functions in I.*

Proof. First, we show that X computes an enumeration F of the infinite sets in I. The construction of F is straightforward: F_n initially copies E_n, and if at any stage s we see that E_n is finite, because n enters the c.e. in X set Fin_E, we make F_n cofinite by adding all elements $\geqslant s$.

If A is an infinite set, then let f_A be the function such that $f_A(n)$ is one less than the distance between the $(n+1)$-st and $(n+2)$-nd element of A; in other words, $f_A(n) = p_A(n+1) - p_A(n) - 1$, where p_A is the principal function of A. It is straightforward to check that the functions computable from sets in I are exactly the functions of the form f_A where A is an infinite set in I. We can compute an enumeration P of these functions from the enumeration F. □

Proposition 3.4. $(\mathcal{C}, \oplus) \equiv_w^* (\mathcal{C}, \sigma) \equiv_w^* \mathcal{B}$.

Proof. That $(\mathcal{C}, \oplus) \leqslant_w^* \mathcal{B}$ and $(\mathcal{C}, \sigma) \leqslant_w^* \mathcal{B}$ follows directly from Theorem 3.2. For the other direction, we claim that, after collapse, every copy of $(\mathcal{C}, \oplus)$ or $(\mathcal{C}, \sigma)$ computes an enumeration E of the sets in the old $\mathcal{P}(\omega)$ and enumerates the set Fin_E of the indices of finite sets in E. By Lemma 3.3 and the characterization given in Downey, Greenberg, and Miller [3] this shows that $(\mathcal{C}, \oplus) \geqslant_w^* \mathcal{B}$ and $(\mathcal{C}, \sigma) \geqslant_w^* \mathcal{B}$.

For the structure $(\mathcal{C}, \sigma)$, the claim follows from the fact that a set A is finite if and only if $\sigma^n(A) = 0^\omega$ for some $n \in \omega$. For $(\mathcal{C}, \oplus)$, the claim follows from the fact that every finite set can be generated by taking joins of $0^\omega = (0000\cdots)$ and $10^\omega = (1000\cdots)$. We show by induction on n that for any σ of length 2^n, the

sequence $\sigma 0^\omega$ can be obtained from applying the operation join to 0^ω and 10^ω. For σ of length 1 the statement is obviously true. Suppose the statement is true for all τ with $|\tau| = 2^n$. Let σ be a string of length 2^{n+1}. Let τ_0 be the string with length 2^n, such that $\tau_0(k) = \sigma(2k)$ and τ_1 be the string with length 2^n, such that $\tau_1(k) = \sigma(2k+1)$. Note that $\tau_0 0^\omega \oplus \tau_1 0^\omega = \sigma 0^\omega$, so if E_1 and E_2 are the expressions for $\tau_0 0^\omega$ and $\tau_1 0^\omega$, then $(E_1) \oplus (E_2)$ is the expression for $\sigma 0^\omega$. □

Another interesting relationship between Cantor space and Baire space is given by the notion of the *jump of a structure*. There have been several approaches to this notion: A. Soskova and Soskov [19, 20] use Moschovakis extensions [15] and a coding of the forcing relation for Π_1 formulas, Puzarenko [16] and Stukachev [23] proposed a definition that works well with Σ-reducibility, and Montalbán investigated a definition that expands the original structure by a complete set of relations defined by computable infinitary Π_1 formulas [12]. The relationships among these three approaches are discussed by Montalbán in [13]. In [13], Montalbán uses relations defined by computable infinitary Σ_1 formulas. This is the approach that we use, lifting it to uncountable structures.

Definition 3.5 (Jump of a structure)**.** Let $\mathcal{A}$ be a structure in a computable language. The *jump* of $\mathcal{A}$ is the structure $\mathcal{A}' = (\mathcal{A}, P_0, P_1, \dots)$, where $\{P_n\}_{n<\omega}$ is a listing of the relations on $\mathcal{A}$ defined by computable infinitary Σ_1 formulas (without parameters). (Note that we have a computable list of the formulas that define these relations.)

Proposition 3.6. *$\mathcal{C}' \equiv^*_w \mathcal{B}$.*

Proof. We know that after collapse, every copy of $\mathcal{B}$ computes an enumeration E of the family of sets that is the old powerset of ω, together with the running jump. We claim that this computes a copy $\mathcal{A}'$ of $\mathcal{C}'$. We assign constants a effectively to indices for the distinct sets E_i. Note that equality can be determined using the running jump (although we could even have assumed that our original enumeration were injective). Also using the running jump, we can determine the truth of a given computable Σ_1 formula $\varphi(\bar{a}) = \bigvee_j (\exists \bar{u}_j)\ \alpha_j(\bar{a}, \bar{u}_j)$ (where α_j is finitary quantifier-free). To see this, we note that the elementary first-order theory of $\mathcal{C}$ is computably axiomatizable, with effective elimination of quantifiers. So we have an effective procedure, using the sets $\bar{a}$, for deciding whether a given disjunct $(\exists \bar{u}_j)\ \alpha_j(\bar{a}, \bar{u}_j)$ is true. With the running jump, we can say whether there is some disjunct that is true.

After collapse, a copy $\mathcal{A}$ of $\mathcal{C}'$ with universe ω computes an enumeration E of the sets in the old $\mathcal{P}(\omega)$. In particular, for the n-th element x_n of $\mathcal{A}$, we enumerate the set E_n so that $x_n = \chi(E_n)$. Further, the jump has predicates P_k which define the computable Σ_1 formula which says that $x(m) = 1$ for some $m > k$. We can thus enumerate from $\mathcal{A}$ the indices n so that for some k, $\mathcal{A} \models \neg P_k(x_n)$. Thus, computable in $\mathcal{A}$, we have an enumeration of all the sets in the old $\mathcal{P}(\omega)$ and Fin_E is c.e. in $\mathcal{A}$. Thus by Lemma 3.3 and the characterization given in Downey, Greenberg, Miller [3], this shows that $\mathcal{C}' \geqslant^*_w \mathcal{B}$. □

4. The Borel complete degree

We have seen that $\mathcal{B} \equiv^*_w \mathcal{R}$ is generic Muchnik above every continuous expansion of $\mathcal{R}$ or $\mathcal{C}$. In this section we prove that continuous expansions of Baire space can be strictly more complicated. This can be seen as a consequence of the fact that

the projections of closed subsets of Baire space can be quite complicated—indeed, they are exactly the $\mathbf{\Sigma^1_1}$ classes. The upper bound of all continuous expansions of $\mathcal{B}$ is an interesting generic Muchnik degree in its own right. We call it the *Borel complete* degree because, as we will see, it bounds all Borel structures (in fact, all Borel quotients).

We consider structures with universe equal to Baire space, although we could equally well consider structures with universe equal to Cantor space or $\mathbb{R}$. For Baire space, we have the topology generated by the basic open neighborhoods $N_\sigma = \{f \in \omega^\omega : f \supseteq \sigma\}$, where $\sigma \in \omega^{<\omega}$. The *Borel* subsets of Baire space are the members of the σ-algebra generated by these basic open neighborhoods.

Definition 4.1 (Borel structure, congruence relation, Borel quotient)**.**

(1) For a Polish space X, a *Borel structure (on X)* has the form $\mathcal{A} = (D, (R_n)_{n\in\omega})$, where $D \subseteq X$ is a Borel set, and for all n, R_n is a Borel relation on X.
(2) For a structure $\mathcal{A} = (D, (R_n)_{n\in\omega})$, a binary relation E on D is a *congruence relation* if it is an equivalence relation and for all n, if R_n is k-ary and $\bar{a}, \bar{b}$ are k-tuples such that $a_i E b_i$ for $i < k$, then $R_n(\bar{a})$ iff $R_n(\bar{b})$. From a congruence relation E on $\mathcal{A}$, we get a well-defined *quotient structure* $\mathcal{A}/_E$ with universe equal to the set of equivalence classes $D/_E$ and relations $R_n/_E$ that hold of a k-tuple $a_0/_E, \dots, a_{k-1}/_E$ iff R_n holds of $a_0, \dots, a_{k-1}$.
(3) A *Borel quotient* is the quotient of a Borel structure $\mathcal{A}$ by a Borel congruence relation.

Borel structures were first studied by H. Friedman in unpublished notes, see Steinhorn [22]. More work on Borel structures can be found in Steinhorn [21] and Louveau [10]. Some recent work can be found in Hjorth and Nies [5] and Montalbán and Nies [14], although they use the name Borel *structure* for what we call a Borel *quotient*.

Examples of Borel structures include $(\mathcal{B}, \oplus, ')$, $(\mathcal{C}, \oplus, ')$, and $\mathcal{R}$. An example of a Borel quotient is the Turing degrees with $\oplus$ and $'$. The Büchi automatic structures are also Borel quotients [4]. Another class of examples consists of the automorphism groups of countable structures that are algebraic in the sense that the language consists just of operation symbols.

Definition 4.2. A structure $\mathcal{A}$ is *Borel complete* if it is a Borel structure that is generic Muchnik above every other Borel structure. The generic Muchnik degree of such a structre is also called *Borel complete*.

Below, we give an example of a *continuous* expansion of $\mathcal{B}$ that turns out to have Borel complete degree. First, we know what the join and jump are, for sets; we define these operations on functions as follows.

Definition 4.3 (Join and jump for functions)**.**

(1) For functions $f, g \in \omega^\omega$, the *join* is the function h such that $h(2n) = f(n)$ and $h(2n+1) = g(n)$. We write $f \oplus g$ for the join.
(2) For $f \in \omega^\omega$, we have a jump $\{e : \varphi_e^f(e)\downarrow\}$. We define the jump of the function f to be the characteristic function of this set.

Example 4.4. Consider the following subclass of ω^ω:

$$P = \{(f \oplus g) \oplus h \colon h \text{ is the settling time function for } f' \text{ and } g = f'\}.$$

Note that it is Π^0_1 to verify that h is the settling time function for f', and given h, it is Π^0_1 to verify that $g = f'$. Therefore, P is a Π^0_1 class in ω^ω, hence closed. Let $F\colon \mathcal{B} \to \mathcal{B}$ be a continuous function such that $P = F^{-1}(0^\omega)$. Because P is a Π^0_1 class, we can even take F to be computable. The structure $(\mathcal{B}, \oplus, F)$ is a continuous expansion of $\mathcal{B}$ with strictly higher generic Muchnik degree. Towards proving this, we need to understand what kind of computational power is present in copies of $(\mathcal{B}, \oplus, F)$ after we collapse the continuum.

Proposition 4.5. *Let I be a countable jump ideal. Let $(\mathcal{B}_I, \oplus, F)$ be the restriction of $(\mathcal{B}, \oplus, F)$ to the funtions in I. Any copy of $(\mathcal{B}_I, \oplus, F)$ computes an enumeration of the functions in I along with join and jump as functions on indices.*

Proof. A copy $\mathcal{A}$ of $(\mathcal{B}_I, \oplus, F)$ gives us a natural enumeration $\{f_n\}_{n\in\omega}$ of the functions in I such that $\oplus^{\mathcal{A}}$ is exactly a function that takes two indices to the index of the join. To find the jump of f_n, search for $m, j \in \omega$ such that $F^{\mathcal{A}}((n \oplus^{\mathcal{A}} m) \oplus^{\mathcal{A}} j)$ is the index of 0^ω. Then $f_m = f'_n$. □

In Theorem 4.15, we will prove that for a sufficiently rich ideal I, it is strictly easier to compute an enumeration of the functions in I than to compute an enumeration of the functions in I along with join and jump as functions on indices. First, we make some easy observations about the difficulty of the latter task, before defining how rich we want our ideals to be.

Proposition 4.6. *For a countable ideal I and a set X, the following are equivalent:*

1. *X computes an enumeration of the sets, or the functions, in I, with join and jump as functions on indices,*
2. *X computes an enumeration of the sets, or the functions, in I, with running jump as a function on indices,*
3. *X computes an enumeration of the sets, or the functions, in I with the corresponding running double jump function, not on indices,*
4. *X computes an enumeration of the functions in I, with the running jump function, not on indices.*

Proof. We first show that (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1) for functions. Suppose X computes an enumeration E of the functions in I, with join and jump as functions on indices. We can find an index for $(E_0 \oplus \ldots \oplus E_n)'$, so we have the running jump as a function on indices. Having an index j for $(E_0 \oplus \ldots \oplus E_n)'$, we can find an index k for $(E_0 \oplus \ldots \oplus E_j)'$, so we can compute E''_n. In particular, we can also compute the running jump function. Finally, having the running jump function, and two indices i and j, we can search for an index k such that $E_i \oplus E_j = E_k$, which we computably do using the running jump function; and given an index i, we can search for indices j and k such that E_k is the jump of E_i with settling time function E_j, which we can also do computably using the running jump function. Using the same argument, we can show that (4) ⇒ (1) ⇒ (2) ⇒ (3) holds with (1), (2) and (3) for sets.

Finally, we show that (3) for sets is equivalent to (3) for functions. Clearly, the version for functions implies the version for sets. Suppose that X computes an enumeration E of the sets in I along with the running double jump. From an infinite set A, we obtain a natural function $f_A(n) = p_A(n+1) - p_A(n) - 1$, where p_A is the principal function of A as we did in the proof of Lemma 3.3. Once again we note that every function $f\colon \omega \to \omega$ is of the form f_A for some infinite

set A. Using the running double jump we can effectively determine whether E_i is infinite. If E_i is infinite, then we let $F_i = f_{E_i}$, and if E_i is finite, we let F_i be the identity function. Now, note that $(E_0 \oplus \cdots \oplus E_i)''$ can compute an index e such that $\Phi_e(E_0 \oplus \cdots \oplus E_i) = (F_0 \oplus \cdots \oplus F_i)$, so $(E_0 \oplus \cdots \oplus E_i)''$ can also compute $(F_0 \oplus \cdots \oplus F_i)''$. Thus, we can compute the running double jump on the enumeration F of the functions. □

Corollary 4.7. $(\mathcal{B}, \oplus, ') \equiv^*_w (\mathcal{C}, \oplus, ')$.

Now it is not hard to show, using the standard encoding of Cantor space in the real numbers, that there is also a Borel expansion of $\mathcal{R}$ in the same generic Muchnik degree as $(\mathcal{B}, \oplus, ')$.

Proposition 4.6 also allows us to easily adapt the proof of Proposition 3.6 to give an additional example of a structure in the Borel complete degree.

Corollary 4.8. $(\mathcal{B}, \oplus, ') \equiv^*_w \mathcal{B}'$.

We turn toward the second main result of the paper, which says that adding further Borel relations to $(\mathcal{B}, \oplus, ')$ does not increase the generic Muchnik degree.

Theorem 4.9. *If $\mathcal{B}^*$ is a Borel expansion of $\mathcal{B}$, then $\mathcal{B}^* \leqslant^*_w (\mathcal{B}, \oplus, ')$.*

To extend this result from Borel expansions of $\mathcal{B}$ to arbitrary Borel structures, we use the following general observation about quotients.

Observation 4.10. *For any structure $\mathcal{A}$ and any congruence relation E on $\mathcal{A}$, $\mathcal{A}/_E \leqslant^*_w (\mathcal{A}, E)$. That is, from a copy of $\mathcal{A}$, with the congruence relation added, we can compute a copy of the quotient structure.*

Hence, we get the following.

Corollary 4.11. *If $\mathcal{B}^*$ is a Borel expansion of $\mathcal{B}$ and E is a Borel congruence relation on $\mathcal{B}^*$, then $\mathcal{B}^*/_E \leqslant^*_w (\mathcal{B}, \oplus, ')$.*

In other words, $(\mathcal{B}, \oplus, ')$ is Borel complete. Recall that we started this section promising that the *continuous* expansion $(\mathcal{B}, \oplus, F)$ of $\mathcal{B}$ has Borel complete degree.

Corollary 4.12. $(\mathcal{B}, \oplus, ') \equiv^*_w (\mathcal{B}, \oplus, F)$.

Proof. Since $(\mathcal{B}, \oplus, F)$ is a Borel expansion of $\mathcal{B}$, we have $(\mathcal{B}, \oplus, F) \leqslant^*_w (\mathcal{B}, \oplus, ')$. On the other hand, $(\mathcal{B}, \oplus, ') \leqslant^*_w (\mathcal{B}, \oplus, F)$ by Proposition 4.5. □

Toward the proof of Theorem 4.9, we note that that the ideal of sets from the ground model satisfies a very strong closure property, much stronger than we have needed so far.

Definition 4.13 (Hyper-Scott ideal)**.** We say that I is a *hyper-Scott* ideal if it is a Turing ideal, and whenever $T \subseteq \omega^{<\omega}$ is a tree in I, if T has an infinite path, then it has an infinite path computable from a set in I.

Proposition 4.14. *If I is the ideal sets from the ground model, then it is a hyper-Scott ideal.*

Proof. This is a very simple application of Shoenfield absoluteness [18]. Arguing directly, if $T \subseteq \omega^{<\omega}$ is a tree in the ground model with no path, then in the ground model there is a rank function $\rho \colon T \to \omega_1$ witnessing that T is well-founded. But ρ also witnesses that T is well-founded in the extension. □

Proof of Theorem 4.9. Let $\mathcal{B}^* = (\mathcal{B}, (R_n)_{n\in\omega})$, where the R_n's are Borel relations. These relations may have different arity. However, we can code a finite sequence $(f_0, \dots, f_r)$ of functions by a single function f, where $f_i(k) = m$ iff $f(r \cdot k + i) = m$. With this in mind, we may suppose that the relations R_n are all unary. Furthermore, we can combine the relations R_n into one relation R such that $R(nf)$ holds if and only if $R_n(f)$ holds, where $nf(k) = \langle n, f(k)\rangle$. Thus, we may suppose that we have just one unary Borel relation $R \subseteq \omega^\omega$.

Let I be the ideal of sets from the ground model. After collapse, any copy of $(\mathcal{B}, \oplus, ')$ can compute an enumeration of the functions in I along with join and jump as functions on indices of the enumeration. Since R is Borel in the ground model, it is $\mathbf{\Delta^1_1}$. Hence there are trees $T, S \subseteq \omega^{<\omega}$, both in I, such that

$$R(f) \iff (\exists h)\ f \oplus h \in [T] \iff (\forall h)\ f \oplus h \notin [S].$$

Note that $f \oplus h \in [T]$ and $f \oplus h \in [S]$ can be checked using $(f \oplus h \oplus T)'$ and $(f \oplus h \oplus S)'$, which in turn can be uniformly computed from the indices of f, h, and the functions computing T and S. Therefore, by searching for a function h in I that witnesses either $R(f)$ or its negation, we can effectively determine if $R(f)$ holds for any function f from our enumeration of the functions in I. Now it is straightforward to compute a copy of $\mathcal{B}^* = (\mathcal{B}, R)$. □

Finally, we show that $(\mathcal{B}, \oplus, ')$ is strictly stronger than $\mathcal{B}$. Note that this does not follow immediatly from the fact that $\mathcal{B}' \equiv^*_w (\mathcal{B}, \oplus, ')$ because the jump of a structure is not neccessarily strictly above the structure itself; both Montalbán [12] and Puzarenko [17] give fixed points for the jump operator on structures, the former under an additional set theoretic assumption.

Theorem 4.15. *Assume that I is a hyper-Scott ideal. Then there is an enumeration of the functions in I that does not compute an enumeration of the functions in I along with functions $f\colon \omega^2 \to \omega$ and $g\colon \omega \to \omega$ on indices of the enumeration that interpret join and jump, respectively.*

Proof. We produce an enumeration $\mathcal{F}$ of the functions in I by forcing. Our forcing partial order consists of conditions $q \in (\omega^\omega)^{<\omega}$ that are sequences of functions from I. We say that a condition p extends q (written as $p \preceq q$) if p extends q as a sequence.

Consider Turing functionals $\mathcal{E}$, f, and g. We want to diagonalize against $\mathcal{E}^{\mathcal{F}}$ being an enumeration of the functions in I such that $f^{\mathcal{F}}\colon \omega^2 \to \omega$ takes any two indices of functions in $\mathcal{E}^{\mathcal{F}}$ to an index of their join, and $g^{\mathcal{F}}\colon \omega \to \omega$ takes every index of a function in $\mathcal{E}^{\mathcal{F}}$ to an index of its jump. Assume, for a contradiction, that there is a condition q that forces this outcome. Fix an index e such that $e \in (T \oplus h)'$ if and only if h is *not* a path through T, where T is treated as a tree under an effective bijection between trees in $\omega^{<\omega}$ and elements of ω^ω. We will use the symbol $\not\perp$ to denote *compatability*. In particular, for $\sigma \in (\omega^{<\omega})^{<\omega}$ and a condition p, $\sigma \not\perp p$ if each component σ_n is an initial segment of the corresponding component p_n.

Claim: For any tree T in I, the following $\Pi^1_1[q \oplus T]$ statement is equivalent to "T has an infinite path":

$$(*) \qquad \begin{aligned}&(\forall p \sqsupseteq q)(\forall n)[\text{ if } (\forall \sigma \not\perp p)[\ \mathcal{E}^\sigma_n \not\perp T\ ],\\ &\qquad\qquad \text{then } (\exists m, k)(\exists \tau \not\perp p)[\ g^\tau(f^\tau(n, m)) = k \text{ and } \mathcal{E}^\tau_k(e) = 0\ ]\ ].\end{aligned}$$

(Here p is understood to range over extensions of q in $(\omega^\omega)^{<\omega}$, though not necessarily forcing conditions—the new functions that are listed need not be in I. Also, σ and τ are intended to be finite, hence they are essentially number quantifiers.)

Note that if we can prove this claim, then we have the necessary contradiction, because it is $\Sigma^1_1[q]$-complete to determine whether a q-computable tree $T \subseteq \omega^{<\omega}$ has an infinite path.

To prove the claim, let T be a tree in I. First, let us assume that $(*)$ holds. Let $p \preceq q$ force the specific location of T in the list $\mathcal{E}^{\mathcal{F}}$, and let this location be n. Therefore, it is true that $(\forall \sigma \npreceq p)[\, \mathcal{E}^\sigma_n \npreceq T \,]$. Otherwise, we could refute our choice of p. Since $(*)$ holds, we have

$$(\exists m, k)(\exists \tau \npreceq p)[\, g^\tau(f^\tau(n,m)) = k \text{ and } \mathcal{E}^\tau_k(e) = 0 \,].$$

Let $r \preceq p$ extend τ. Then r forces that $e \notin (T \oplus \mathcal{E}^{\mathcal{F}}_m)'$, so $\mathcal{E}^{\mathcal{F}}_m$ is a path on T. Therefore, T has an infinite path.

Now assume that T has an infinite path. Because I is a hyper-Scott ideal, T has a path h in I. Also, because I is a hyper-Scott ideal, if $(*)$ fails, it fails for a $p \in I$. So it is enough to consider $p \preceq q$ and $n \in \omega$ such that $(\forall \sigma \npreceq p)[\, \mathcal{E}^\sigma_n \npreceq T \,]$. Note that p forces $\mathcal{E}^{\mathcal{F}}_n$ to be T, because q forces it to be total and it cannot be incompatible with T. Take $r \preceq p$ forcing the specific location of h in $\mathcal{E}^{\mathcal{F}}$, and let this location be m. Now take $\tau \npreceq r$ and k such that $g^\tau(f^\tau(n,m)) \downarrow = k$ and $\mathcal{E}^\tau_k(e) \downarrow$. Such a τ exists because q forces $g^{\mathcal{F}}$, $f^{\mathcal{F}}$, and $\mathcal{E}^{\mathcal{F}}$ to be total. Our choice of r ensures that $\mathcal{E}^\tau_k(e) = (T \oplus h)'(e) = 0$, because h is a path on T. Therefore, $(*)$ holds.

We have proved that any tree T in I has an infinite path if and only if $(*)$ holds. This is a contradiction, hence any condition can be extended to a condition that diagonalizes against $\mathcal{E}$, f, and g. □

Theorem 4.16. $(\mathcal{B}, \oplus, ') >^*_w \mathcal{B}$.

Note that, as a function on either $\mathcal{B}$ or $\mathcal{C}$, the jump $'$ is of Baire class 1, i.e., a pointwise limit of continuous functions. Indeed it is a pointwise limit of a computable sequence of computable functions. In terms of the effective Borel hierarchy, the graph of $'$ is Π^0_2, in other words effective G_δ. So as mentioned above, there is an expansion of the real numbers by continuous functions and a unary function of Baire class 1 (or a Π^0_2 binary relation) which is in the Borel complete degree. This puts a severe limitation on any potential strengthening of Theorem 1.7.

References

1. Kelty Allen, Laurent Bienvenu, and Theodore A. Slaman, *On zeros of Martin-Löf random Brownian motion*, J. Log. Anal. **6** (2014), Paper 9, 34. MR 3311552
2. Uri Andrews, Joseph S. Miller, Noah D. Schweber, and Mariya I. Soskova, *Complexity profiles and generic Muchnik reducibility*, Submitted. (https://people.math.wisc.edu/ jmiller/Papers/GenericMuchnik3.pdf).
3. Rod Downey, Noam Greenberg, and Joseph S. Miller, *Generic Muchnik reducibility and presentations of fields*, Israel J. Math. **216** (2016), no. 1, 371–387. MR 3556972
4. Greg Hjorth, Bakhadyr Khoussainov, Antonio Montalbán, and André Nies, *From automatic structures to Borel structures*, Proceedings of the Twenty-Third Annual IEEE Symposium on Logic in Computer Science (LICS 2008), 2008, pp. 431–441.
5. Greg Hjorth and André Nies, *Borel structures and Borel theories*, J. Symbolic Logic **76** (2011), no. 2, 461–476. MR 2830412
6. Gregory Igusa and Julia F. Knight, *Comparing two versions of the reals*, J. Symb. Log. **81** (2016), no. 3, 1115–1123. MR 3569122

7. Gregory Igusa, Julia F. Knight, and Noah D. Schweber, *Computing strength of structures related to the field of real numbers*, J. Symb. Log. **82** (2017), no. 1, 137–150. MR 3631279
8. Carl G. Jockusch, Jr. and Robert I. Soare, Π^0_1 *classes and degrees of theories*, Trans. Amer. Math. Soc. **173** (1972), 33–56. MR 316227
9. Julia F. Knight, Antonio Montalbán, and Noah D. Schweber, *Computable structures in generic extensions*, J. Symb. Log. **81** (2016), no. 3, 814–832. MR 3569106
10. Alain Louveau, *Classifying Borel structures*, Set Theory and the Continuum (meeting at Berkeley, 1989), MSRI Publication, 1989, pp. 103–112. MR 1107320
11. Angus Macintyre and David Marker, *Degrees of recursively saturated models*, Trans. Amer. Math. Soc. **282** (1984), no. 2, 539–554. MR 732105
12. Antonio Montalbán, *Notes on the jump of a structure*, Mathematical theory and computational practice, Lecture Notes in Comput. Sci., vol. 5635, Springer, Berlin, 2009, pp. 372–378. MR 2545911
13. ———, *A fixed point for the jump operator on structures*, J. Symbolic Logic **78** (2013), no. 2, 425–438. MR 3145189
14. Antonio Montalbán and André Nies, *Borel structures: a brief survey*, Effective mathematics of the uncountable, Lect. Notes Log., vol. 41, Assoc. Symbol. Logic, La Jolla, CA, 2013, pp. 124–134. MR 3205056
15. Yiannis N. Moschovakis, *Abstract first order computability. I, II*, Trans. Amer. Math. Soc. **138** (1969), 427–464. MR 244045
16. Vadim G. Puzarenko, *On a certain reducibility on admissible sets*, Sibirsk. Mat. Zh. **50** (2009), no. 2, 415–429. MR 2531766
17. ———, *Fixed points for the jump operator*, Algebra Logika **50** (2011), no. 5, 615–646, 693, 695. MR 2907410
18. Joseph R. Shoenfield, *The problem of predicativity*, Essays on the foundations of mathematics, Magnes Press, Hebrew Univ., Jerusalem, 1961, pp. 132–139. MR 0164886
19. Alexandra A. Soskova, *A jump inversion theorem for the degree spectra*, Computation and logic in the real world, Lecture Notes in Comput. Sci., vol. 4497, Springer, Berlin, 2007, pp. 716–726. MR 2646285
20. Alexandra A. Soskova and Ivan N. Soskov, *A jump inversion theorem for the degree spectra*, J. Logic Comput. **19** (2009), no. 1, 199–215. MR 2475650
21. C. I. Steinhorn, *Borel structures and measure and category logics*, Model-theoretic logics, Perspect. Math. Logic, Springer, New York, 1985, pp. 579–596. MR 819547
22. Charles Steinhorn, *Borel structures for first-order and extended logics*, Harvey Friedman's research on the foundations of mathematics, Stud. Logic Found. Math., vol. 117, North-Holland, Amsterdam, 1985, pp. 161–178. MR 835258
23. Alexey I. Stukachev, *A jump inversion theorem for the semilattices of Sigma-degrees [translation of mr2586684]*, Siberian Adv. Math. **20** (2010), no. 1, 68–74. MR 2655896

(Andrews, Miller, Soskova) Department of Mathematics, University of Wisconsin–Madison, Madison, USA
Email address: andrews@math.wisc.edu
Email address: jmiller@math.wisc.edu
Email address: msoskova@math.wisc.edu

(Knight) Department of Mathematics, University of Notre Dame, South Bend, USA
Email address: Julia.F.Knight.1@nd.edu

(Kuyper) Metaswitch, Wellington, New Zealand
Email address: mail@rutgerkuyper.com