EXTENSIONS OF TWO CONSTRUCTIONS OF AHMAD

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Abstract. In her 1990 thesis, Ahmad showed that there is a so-called "Ahmad pair", i.e., there are incomparable Σ^0_2-enumeration degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ such that every enumeration degree $\mathbf{x} < \mathbf{a}_0$ is $\leq \mathbf{a}_1$. At the same time, she also showed that there is no "symmetric Ahmad pair", i.e., there are no incomparable Σ^0_2-enumeration degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ such that every enumeration degree $\mathbf{x}_0 < \mathbf{a}_0$ is $\leq \mathbf{a}_1$ and such that every enumeration degree $\mathbf{x}_1 < \mathbf{a}_1$ is $\leq \mathbf{a}_0$.

In this paper, we first present a direct proof of Ahmad's second result. We then show that her first result cannot be extended to an "Ahmad triple", i.e., there are no Σ^0_2-enumeration degrees $\mathbf{a}_0$, $\mathbf{a}_1$ and $\mathbf{a}_2$ such that both $(\mathbf{a}_0, \mathbf{a}_1)$ and $(\mathbf{a}_1, \mathbf{a}_2)$ are an Ahmad pair. On the other hand, there is a "weak Ahmad triple", i.e., there are pairwise incomparable Σ^0_2-enumeration degrees $\mathbf{a}_0$, $\mathbf{a}_1$ and $\mathbf{a}_2$ such that every enumeration degree $\mathbf{x} < \mathbf{a}_0$ is also $\leq \mathbf{a}_1$ or $\leq \mathbf{a}_2$; however neither $(\mathbf{a}_0, \mathbf{a}_1)$ nor $(\mathbf{a}_0, \mathbf{a}_2)$ is an Ahmad pair.

1. Introduction

Enumeration reducibility is a positive reducibility between sets of natural numbers. It arises naturally as a notion of relative computability for partial functions and has applications in effective mathematics, especially in computable topology, in computable model theory and in group theory.

We associate an algebraic presentation of this reducibility as a degree structure. The structure of the enumeration degrees is a partial order with least upper bound and a jump operator (just like its more famous cousin, the structure of the Turing degrees). In this article we focus on structural properties of its local substructure—the degree structure of the enumeration degrees of the Σ^0_2-sets, which can be defined also as those enumeration degrees below the degree $\mathbf{0}'_e$. Here, $\mathbf{0}'_e$ is the enumeration degree of the complement $\overline{K}$ of the halting problem $K = \{e \mid \varphi_e(e)\downarrow\}$. The Σ^0_2-enumeration degrees can be viewed as the counterpart in enumeration reducibility of either the c.e. Turing degrees or the Turing degrees $\leq \mathbf{0}'$, i.e., the Δ^0_2-Turing degrees. Both analogies are imperfect, but reasonable in certain respects. We refer the reader to [14] for more information on current trends in research on the enumeration degrees.

One of the common questions about a degree structure viewed as a partial order is that of the complexity of its first-order theory. For most degree structures commonly being considered, the theory turns out to be as complicated as possible: global structures like the Turing degrees or the enumeration degrees have theories that are computably isomorphic to the theory of second-order arithmetic, while local structures usually have theories that are equivalent to the theory of first-order arithmetic. We then wonder about the fragments of the first-order theory, identified by restricting sentences to a certain quantifier complexity. We find that decidability

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breaks down at level 3, i.e., the $\exists\forall\exists$-fragment is not decidable. On the other hand the $\exists$- and often even the $\forall\exists$-fragment is decidable.

For the Σ^0_2-enumeration degrees, the first of these questions has been completely settled: The full first-order theory was shown to be undecidable by Slaman and Woodin [13], and equivalent to full first-order arithmetic by Ganchev and Soskova [4].

As for the second question, the $\exists$-fragment is easily seen to be decidable, whereas Kent [5] showed the $\exists\forall\exists$-fragment to be undecidable. The decidability of the $\forall\exists$-fragment remains open.

The decidability of the $\forall\exists$-fragment can be rephrased algebraically as (uniformly effectively) deciding the following

Question 1.1. For any given finite partial orders $\mathcal{P}$ and $\mathcal{Q}_i \supseteq \mathcal{P}$ (for $i \leq n$), can any embedding of $\mathcal{P}$ into the Σ^0_2-enumeration degrees be extended to an embedding of $\mathcal{Q}_i$ for some $i \leq n$ (where i may depend on the particular embedding of $\mathcal{P}$)? (Without loss of generality, we will from now on assume that any finite partial order is *bounded*, i.e., has a least element 0 and a greatest element 1.)

Two major subproblems of Question 1.1 have been shown to be decidable:

- Lempp, Slaman and Sorbi [8] showed that the above question is decidable for $n = 0$, i.e., given any finite partial orders $\mathcal{P} \subseteq \mathcal{Q}$, it is decidable whether any embedding of $\mathcal{P}$ into the Σ^0_2-enumeration degrees can be extended to an embedding of $\mathcal{Q}$.
- Lempp and Sorbi [10] showed that all finite lattices can be embedded, even preserving 0 and 1. (The lattice embeddings question can be seen as a disjunction of extending embeddings to certain one-point extensions $\mathcal{Q}_i$ of a finite lattice $\mathcal{P}$ viewed as a partial order.)

As noted earlier, the Σ^0_2-enumeration degrees are often compared to the c.e. Turing degrees. Both are dense structures with full first-order theories as complicated as the theory of first-order arithmetic. For the c.e. Turing degrees this was proved by Slaman and Woodin (unpublished, see Nies, Shore and Slaman [11]); for the c.e. Turing degrees we have that in addition the $\exists$-fragment is decidable, whereas Lempp, Nies and Slaman [7] showed the $\exists\forall\exists$-fragment to be undecidable. However, the lattice embeddings problem for the c.e. Turing degrees remains one of the main open problems dating back to the 1960's (see Lempp, Lerman and Solomon [6] for the most recent update), and thus the decidability of the $\forall\exists$-theory of the c.e. Turing degrees remains wide open as well.

An important algebraic difference between the c.e. Turing degrees and the Σ^0_2-enumeration degrees was discovered by Ahmad in her Ph.D. thesis [1] (see Ahmad and Lachlan [2, Corollary 3.2]): There are incomparable Σ^0_2-enumeration degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ (called an "Ahmad pair") such that any degree $\mathbf{x} < \mathbf{a}_0$ is also $< \mathbf{a}_1$. (This makes $\mathbf{a}_0$ "non-splitting", i.e., join-irreducible, and thus cannot happen in the c.e. Turing degrees by the Sacks Splitting Theorem [12].) More interestingly even, Ahmad also showed (see Ahmad and Lachlan [2, Theorem 3.3]) that this phenomenon is *not* symmetric: For any two incomparable Σ^0_2-enumeration degrees $\mathbf{a}_0$ and $\mathbf{a}_1$, there is either a degree $\mathbf{x}_0 < \mathbf{a}_0$ which is $\not\leq \mathbf{a}_1$, or there is a degree $\mathbf{x}_1 < \mathbf{a}_1$ which is $\not\leq \mathbf{a}_0$.

In the language of Question 1.1, Ahmad's results can be rephrased as stating that not every embedding of $\mathcal{P} = \{0, a_0, a_1, 1\}$ with incomparable a_0 and a_1 can be extended to an embedding of $\mathcal{Q}_0 = \{0, x_0, a_0, a_1, 1\}$ where $0 < x_0 < a_0$ and $x_0 \not\leq a_1$, but that every embedding of $\mathcal{P}$ can be extended to an embedding of either $\mathcal{Q}_0$ or of $\mathcal{Q}_1 = \{0, a_0, x_1, a_1, 1\}$ where $0 < x_1 < a_1$ and $x_1 \not\leq a_0$.

In this paper, we prove two extensions of Ahmad's results in different directions, thus adding to our toolbox toward our ultimate goal, deciding the $\forall\exists$-theory of the Σ^0_2-enumeration degrees. Again in the language of Question 1.1, our first result can be rephrased as stating that every embedding of $\mathcal{P} = \{0, a_0, a_1, a_2, 1\}$ with incomparable a_0, a_1 and a_2 can be extended to an embedding of $\mathcal{Q}_0 = \{0, x_0, a_0, a_1, a_2, 1\}$ where $0 < x_0 < a_0$ and $x_0 \not\leq a_1$ or an embedding of $\mathcal{Q}_1 = \{0, x_1, a_0, a_1, a_2, 1\}$ where $0 < x_1 < a_1$ and $x_1 \not\leq a_2$ (leaving the relationship between x_0 and a_2, and between x_1 and a_0, unspecified so as to not have too many cases); a similar formulation can be found for our second result.

We first present, in Section 2, a direct proof of Ahmad's result that there is no symmetric Ahmad pair. (Currently, the only published proof in the literature is indirect and hard to modify.) In Section 3, we show that there is no Ahmad triple, i.e., there are no Σ^0_2-degrees $\mathbf{a}_0$, $\mathbf{a}_1$ and $\mathbf{a}_2$ such that both $(\mathbf{a}_0, \mathbf{a}_1)$ and $(\mathbf{a}_1, \mathbf{a}_2)$ form an Ahmad pair. On the other hand, in Section 4, we also show that there is a weak Ahmad triple, i.e., there are pairwise incomparable Δ^0_2-enumeration degrees $\mathbf{a}_0$, $\mathbf{a}_1$ and $\mathbf{a}_2$ such that every enumeration degree $\mathbf{x} < \mathbf{a}_0$ is also $\leq \mathbf{a}_1$ or $\leq \mathbf{a}_2$; however, neither $(\mathbf{a}_0, \mathbf{a}_1)$ nor $(\mathbf{a}_0, \mathbf{a}_2)$ forms an Ahmad pair. We should add here that Kent (personal communication around 2006) identified the existence of an Ahmad triple and of a "cupping Ahmad pair" (i.e., an Ahmad pair whose join is $\mathbf{0}'_e$) as the two main initial obstacles toward a decision procedure for the $\forall\exists$-theory of the Σ^0_2-enumeration degrees.

It is worth pointing out that the first two results are specific to the Σ^0_2-enumeration degrees. Lempp, Slaman, and Soskova [9] have shown that every finite distributive lattice L can be embedded as an interval of Π^0_2-enumeration degrees $[\mathbf{a}, \mathbf{b}]$ so that for every enumeration degree $\mathbf{x} < \mathbf{b}$ we have that $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ or $\mathbf{x} < \mathbf{a}$. Embedding the diamond in such a way shows that symmetric Ahmad pairs are possible in general.

2. A Direct Proof that there is no Symmetric Ahmad Pair

In this section, we will present a direct proof of the following

Theorem 2.1 (Ahmad [1] (see Ahmad and Lachlan [2, Corollary 3.2]))**.** *There is no* symmetric Ahmad pair *in the Σ^0_2-enumeration degrees, i.e., there are no incomparable Σ^0_2-degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ such that every enumeration degree $\mathbf{x}_0 < \mathbf{a}_0$ is $\leq \mathbf{a}_1$, and every enumeration degree $\mathbf{x}_1 < \mathbf{a}_1$ is $\leq \mathbf{a}_0$.*

To show that the degrees of a pair of sets A_0 and A_1 is not an Ahmad pair, we need to build a set $X_0 <_e A_0$ such that $X_0 \not\leq_e A_1$. Cooper's density proof [3] builds precisely such a set X_0 assuming that in addition $A_1 <_e A_0$. Under this additional assumption, we can build $X_0 = \Phi_0(A_0)$ as follows: We satisfy two types of requirements. The first type ensures that for every e, we have $X_0 \neq \Gamma_e(A_1)$ by threatening to code A_0 into the e-th column of X_0. The second type of requirement ensures that for every i, we have $A_0 \neq \Gamma_i(X_0)$ by threatening to make $\Gamma_i(X_0) = \Gamma_i(X_0^{[\leq i]} \cup \mathbb{N}^{[>i]})$, which (assuming $X_0^{[\leq i]}$ is computable) is a c.e. set. Here $X^{[\leq n]} = \{\langle m, x\rangle \mid m \leq n\}$ and $X^{[>n]} = \{\langle m, x\rangle \mid m > n\}$. To make this idea work, the construction uses a *good approximation* to the set A_0, i.e., a uniformly computable sequence of finite sets $\{A_{0,s}\}_{s<\omega}$ such that

(1) for every n, there is a stage s such that $A_0 \upharpoonright n \subseteq A_{0,s} \subseteq A_0$; and
(2) for every n, there is a stage s such that for every $t \geq s$, if $A_{0,t} \subseteq A_0$ then $A_0 \upharpoonright n \subseteq A_{0,t}$.

Stages at which $A_{0,s} \subseteq A_0$ are called *good*. The sets reducible to A_0, namely, X_0, A_1, $\Gamma_e(A_1)$, and $\Gamma_i(X_0)$, are approximated with *correct* approximations—good approximations whose good stages include all good stages for the approximation of A_0. If we restrict our attention to good stages for A_0, then two sets with correct approximations are the same if and only the length of agreement between them measured at such stages is unbounded. So, if we enumerate elements into $X_{0,s}$ only by enumerating axioms of the form $\langle x, A_{0,s}\rangle$ into Φ_0, then we ensure that X_0 gains new elements only at good stages for A_0, and this allows us to limit the activity of each requirement to a finite set.

If A_1 is not bounded by A_0, then this construction fails: It can be that there are infinitely many good stages for A_0 that are bad for A_1, causing us to falsely assume that a requirement of the first type requires attention again and again and forcing it to contribute an infinite set to X_0. This, in turn, is in critical conflict with the second type of requirements, which depend on the assumption that each column in X_0 is finite. The problem we see should not surprise us, because Ahmad pairs do exist. Ahmad's original proof of Theorem 2.1 uses the Gutteridge operator to show that if A_0 and A_1 form an Ahmad pair, then A_0 and all sets bounded by A_0 have eventually correct approximations with respect to the approximation to A_1, and so we can build $X_1 <_e A_1$ with $X_1 \not\leq_e A_0$ using essentially the same construction as the one described above. The proof is ingenious, though difficult to modify. We give a direct construction, using the priority method and a tree of strategies. The main idea is to build the sets X_0 and X_1 in a more entangled way so that our failure to prove that $X_1 <_e A_0$ allows us to switch off unwanted axioms enumerated into X_0 and avoid the problem described above.

Assume that A_0 and A_1 are incomparable Σ^0_2-sets. (If A_0 and A_1 are comparable, then their enumeration degrees do not form an Ahmad pair by definition.)

We fix approximations for A_0 and A_1 so that $\{A_{0,s} \oplus A_{1,s}\}_{s<\omega}$ is a good approximation to $A_0 \oplus A_1$. So even though we cannot ensure that good stages for A_0 are good for A_1 or vice versa, we may at least ensure that there are infinitely many common good stages.

2.1. **Requirements.** The construction builds an enumeration operator Φ_1, attempting to satisfy the following requirements for each enumeration operator Γ_1 and each enumeration operator Δ_1:

$$\mathcal{R}_{\Gamma_1} : \Phi_1(A_1) \neq \Gamma_1(A_0)$$
$$\mathcal{S}_{\Delta_1} : A_1 \neq \Delta_1(\Phi_1(A_1)).$$

If some $\mathcal{R}_{\Gamma_1}$ requirement fails then we will construct an enumeration operator Φ_0 satisfying the following subrequirements for each of the enumeration operators Γ_0 and Δ_0:

$$\mathcal{R}_{\Gamma_1,\Gamma_0} : \Phi_0(A_0) = \Gamma_0(A_1) \implies A_0 = \Psi(A_1) \text{ (for a } \Psi \text{ built by us)}$$
$$\mathcal{S}_{\Gamma_1,\Delta_0} : A_0 \neq \Delta_0(\Phi_0(A_0)).$$

Clearly, satisfying either group of requirements, namely, all $\mathcal{R}_{\Gamma_1}$- and all $\mathcal{S}_{\Delta_1}$-requirements, or, for some fixed Γ_1, all $\mathcal{R}_{\Gamma_1,\Gamma_0}$- and all $\mathcal{S}_{\Gamma_1,\Delta_0}$-requirements, will suffice since $A_0 \not\leq_e A_1$.

We will denote $\Phi_i(A_i)$ by X_i (for $i < 2$) whenever the operator Φ_i is clear from the context.

2.2. **Tree of strategies.** Order each of the types of requirements and subrequirements in a priority of order type ω. We have four types of strategies: an $\mathcal{R}_{\Gamma_1}$-strategy α, an $\mathcal{R}_{\Gamma_1,\Gamma_0}$-strategy β, an $\mathcal{S}_{\Delta_1}$-strategy γ, and an $\mathcal{S}_{\Gamma_1,\Delta_0}$-strategy δ.

The root of the tree of strategies T is $\emptyset$, an $\mathcal{R}_{\Gamma_1}$-strategy working on the highest-priority $\mathcal{R}_{\Gamma_1}$-requirement. An $\mathcal{R}_{\Gamma_1}$-strategy has only one outcome $\langle 0\rangle$ and is immediately followed by an $\mathcal{R}_{\Gamma_1,\Gamma_0}$-strategy, working on the highest-priority $\mathcal{R}_{\Gamma_1,\Gamma_0}$-requirement.

An $\mathcal{R}_{\Gamma_1,\Gamma_0}$-strategy β has outcomes $\langle 2k\rangle$, $\langle 2k+1, \text{old}\rangle$, $\langle 2k+1, \text{off}\rangle$, and $\langle 2k+1, \text{new}\rangle$ for all $k \in \omega$, ordered as follows:

$$\langle 0\rangle <_L \langle 1, \text{old}\rangle <_L \langle 1, \text{off}\rangle <_L \langle 1, \text{new}\rangle <_L \langle 2\rangle <_L \cdots$$

For every $k < \omega$, the nodes $\beta\hat{\ }\langle 2k\rangle$ and $\beta\hat{\ }\langle 2k+1, \text{off}\rangle$ are $\mathcal{S}_{\Gamma_1,\Delta_0}$-strategies working on the highest-priority $\mathcal{S}_{\Gamma_1,\Delta_0}$-requirement that is not assigned to any of β's predecessors. The nodes $\beta\hat{\ }\langle 2k+1, \text{old}\rangle$ and $\beta\hat{\ }\langle 2k+1, \text{new}\rangle$ are $\mathcal{S}_{\Delta_1}$-strategies working on the highest-priority $\mathcal{S}_{\Delta_1}$-requirement that is not assigned to any of β's predecessors.

An S_{Δ_1}-strategy γ has outcomes $\langle k\rangle$, where $k \in \omega$ is ordered by the standard ordering on ω. Each such immediate successor of this strategy is a main $\mathcal{R}_{\Gamma_1}$-strategy, working on the highest-priority $\mathcal{R}_{\Gamma_1}$-requirement that is not assigned to any of its predecessors.

Similarly, an S_{Γ_1,Δ_0}-strategy δ has outcomes $\langle k\rangle$, where $k \in \omega$ is ordered by the standard ordering on ω. Each such immediate successor of this strategy is an $\mathcal{R}_{\Gamma_1,\Gamma_0}$-strategy, working on the highest-priority $\mathcal{R}_{\Gamma_1,\Gamma_0}$-requirement (for the same Γ_1 as δ) that is not assigned to any of its predecessors.

2.3. **Construction.** At stage 0, all strategies are in initial state: All operators associated with these strategies are empty, all parameters are undefined. At stage $s > 0$, we build a path f_s of length $\leq s$. The intention is that there will be a *true path* defined by

$$f(n) = \liminf_{f_s \succeq f\restriction n} f_s(n)$$

that correctly describes the outcomes of each strategy. The construction consists of substages t for $t < s$, where we act for some strategy $f_s \restriction t$ of length t depending on the current outcome of the strategy which acted at the previous substage starting at the root. When a strategy is activated at stage s, it first ensures that it is not missing any good stages by adjusting the approximations to A_0 and A_1: If s^- is the previous stage at which this strategy was active, then it replaces $A_{i,s}$ by $\bigcap_{u\in[s^-,s]} A_{i,u}$ for $i < 2$. We describe further actions of each strategy depending on its type below.

At the end of stage s, we initialize all strategies of *lower priority* than f_s, i.e., strategies extending or to the right of the strategies which acted at stage s.

Each $\mathcal{R}_{\Gamma_1,\Gamma_0}$-strategy β and each $\mathcal{S}_{\Gamma_1,\Delta_0}$-strategy δ works with the version of Φ_0 and X_0 determined by the longest $\mathcal{R}_{\Gamma_1}$-strategy $\alpha \prec \beta, \delta$ (we say that β and δ *work for* α); this version of Φ_0 is the set of Φ_0-axioms enumerated by all the $\mathcal{R}_{\Gamma_1,\Gamma_0}$- and $\mathcal{S}_{\Gamma_1,\Delta_0}$-strategies working for the same $\mathcal{R}_{\Gamma_1}$-strategy α.

2.3.1. $\mathcal{S}_{\Delta_1}$. We begin with the $\mathcal{S}$-strategies, as they are directly lifted from the density construction. So, let γ be an $\mathcal{S}_{\Delta_1}$-strategy. The first time a strategy is visited after initialization, the strategy receives a unique number s_γ, the stage of first visit after initialization. To keep this assignment of stages injective, we interrupt the stage s construction if s is the first stage when γ is visited: We set $f_s = \gamma$, $s_\gamma = s$, and move on to stage $s+1$. If $s_\gamma < s$ is already defined, then we consider the length $l_{\gamma,s} < s$ of the common initial segment of $A_{1,s}$ and $\Delta_{1,s}(X_{1,s})$ up to s. For every number $n \leq l_{\gamma,s}$, if $n \in \Delta_{1,s}(X_{1,s}^{[<s_\gamma]} \cup \mathbb{N}^{[\geq s_\gamma]})$, we search for the axiom $\langle n, F\rangle \in \Delta_1$ that has been valid the longest and enumerate each element of the form $\langle r, x\rangle \in F$, where $r \geq s_\gamma$ into the set $X_{1,s}$ via the axiom $\langle\langle r, x\rangle, A_{1,s}\rangle$. The outcome of the strategy is $\langle k\rangle$, where k is the standard code of the finite set D_k of all numbers for which γ has enumerated an axiom that looks valid at stage s. The only thing we assume about the coding of finite sets, in addition to its effective properties, is that $D_{k_1} \subseteq D_{k_2}$ implies $k_1 \leq k_2$.

We will be able to argue that if γ is on the true path, then γ enumerates only a finite set D into X_1, as the sequence $\{l_{\gamma,s}\}_{s \text{ is good}}$ must be bounded. At sufficiently large stages in the approximation to A_1, the outcome we select will always correspond to a superset of D. At stages that are also good (i.e., stages s such that $A_{1,s} \subseteq A_1$), we will be able to correctly identify the code of D as the correct outcome. In other words, the code of the set D will be γ's true outcome.

2.3.2. $\mathcal{S}_{\Gamma_1,\Delta_0}$. An $\mathcal{S}_{\Gamma_1,\Delta_0}$-strategy δ works similarly to the $\mathcal{S}_{\Delta_1}$-strategy. It also receives a unique number s_δ, the stage of first visit after initialization, and interrupts the stage s construction if s is the first stage when δ is visited. Otherwise, we consider the length $l_{\delta,s} < s$ of the common initial segment of $A_{0,s}$ and $\Delta_{0,s}(X_{0,s})$ up to s. For every number $n \leq l_{\delta,s}$, if $n \in \Delta_{0,s}(X_{0,s}^{[<s_\delta]} \cup \mathbb{N}^{[\geq s_\delta]})$, we search for the axiom $\langle n, F\rangle \in \Delta_0$ that has been valid the longest and enumerate each element of the form $\langle r, x\rangle \in F$, where $r \geq s_\delta$ is in the set $X_{0,s}$ via the axiom $\langle\langle r, x\rangle, A_{0,s}\rangle$. The outcome of the strategy is $\langle k\rangle$, where k is the standard code of the finite set D_k of all numbers for which δ has enumerated an axiom that looks valid at stage s.

2.3.3. $\mathcal{R}_{\Gamma_1}$. The $\mathcal{R}_{\Gamma_1}$-strategy does nothing, has only one outcome $\langle 0\rangle$, and determines the version of Φ_0 and X_0 that all the $\mathcal{R}_{\Gamma_1,\Gamma_0}$- and $\mathcal{S}_{\Gamma_1,\Delta_0}$-strategies working for the $\mathcal{R}_{\Gamma_1}$-strategy use.

2.3.4. $\mathcal{R}_{\Gamma_1,\Gamma_0}$. The $\mathcal{R}_{\Gamma_1,\Gamma_0}$-strategy β attempts to construct an enumeration operator Ψ such that $A_0 = \Psi(A_1)$ by enumerating axioms into Φ_1 and its version of Φ_0.

At the first stage after initialization, the $\mathcal{R}_{\Gamma_1,\Gamma_0}$-strategy β is assigned the parameter s_β. Note that we can assume that s_β is larger than $\max(D_k)$ for any k such that a higher-priority $\mathcal{S}$-strategy λ (which can be either an $\mathcal{S}_{\Delta_1}$- or an $\mathcal{S}_{\Gamma_1,\Delta_0}$-strategy) has $\lambda\hat{\ }k \preceq \beta$. Until its next initialization, β will only contribute numbers to the s_β-th columns of X_0 and X_1. To every element a, we assign the coding location $x_a = \langle s_\beta, a\rangle$ targeted for X_0. The coding locations m_a that we associate with a given number a but are targeted for X_1 will change more dynamically during the construction. Initially, we assign $m_a = \langle s_\beta, a\rangle$ as well.

At a stage $s > s_\beta$, β does the following. It orders the elements of $A_{0,s} \cup \Psi_s(A_{1,s})$ by age: For each $a \in A_{0,s} \cup \Psi_s(A_{1,s})$, we define its *age (at stage s)* as follows.

$$\text{age}_0^s(a) = 2k+1 \text{ for } k = \min(s+1, \mu t \forall u \in [t,s](a \in A_{0,u}))$$
$$\text{age}_1^s(a) = 2k \text{ for } k = \min(s+1, \mu t \forall u \in [t,s](a \in \Psi(A_{1,u})[u] \text{ via the same axiom}))$$
$$\text{age}^s(a) = \min\{\text{age}_0^s(a), \text{age}_1^s(a)\}.$$

Without loss of generality, we will assume that at most one element enters the approximation to A_0 or $\Psi(A_1)$ at a fixed stage s. (We can ensure this by artificially delaying the approximations if necessary.) And so, for every stage $t \le s$, there may be at most one element with $\text{age}^s(a) = t$. Furthermore, if $a \in A_{0,s} \setminus \Psi_s(A_{1,s})$, then $\text{age}^s(a)$ is odd, and if $a \in \Psi_s(A_{1,s}) \setminus A_{0,s}$, then $\text{age}^s(a)$ is even. At stage s, we will say that b is *older* than a if $\text{age}^s(b) < \text{age}^s(a)$.

If $A_{0,s} = \Psi_s(A_{1,s})$, then we exit this strategy with outcome $\langle 2(s+1)\rangle$ (this is an outcome that has not been visited so far). Since A_0 is infinite, this will only be a temporary situation. Otherwise, we pick the oldest number a such that $A_{0,s}(a) \neq \Psi_s(A_{1,s})(a)$. Let $k = \text{age}^s(a)$. We must ensure that β's effect on X_1 is computable, and so the strategy will *dump* into X_1 all elements of the form $\langle s_\beta, n\rangle \in (m_a, s]$ and assign new markers $m_{a'} = \langle s_\beta, n\rangle > s$ to all elements a' with $\text{age}^s(a') > k$. (Here, to dump an element m into X_1 means to enumerate the axiom $\langle m, \emptyset\rangle$ into Φ_1.) We have two cases depending on the parity of k.

Case 1: If k is even, i.e., if $a \in \Psi_s(A_{1,s}) \setminus A_{0,s}$, then we will be able to argue that $x_a = \langle s_\beta, a\rangle \in \Gamma_0(A_1) \setminus X_0$. The strategy selects outcome $\langle k\rangle$. While a maintains its age, we will design axioms for younger elements enumerated into X_0 by β so that their use includes a. Thus, if this is β's true outcome, they will be invalid and hence β contributes finitely much to X_0.

Case 2: If k is odd, i.e., if $a \in A_{0,s} \setminus \Psi_s(A_{1,s})$, then we would like to add an axiom for a into Ψ, but to do this we need some preparation. We will identify an axiom $\langle x_a, F_a\rangle$ in $\Gamma_0(A_1)$ and use it. Let s_k be the previous stage when β considered k:

(1) If some b with $\text{age}^s(b) < k$ has $m_b \notin \Gamma_1(A_{0,t})$ at some stage $t \in [s_k, s]$, then, since b is older than a, we may assume that we have identified F_b for b and that $F_b \subseteq A_{1,s}$. (Otherwise, b would be our choice for the oldest disagreement.) We can therefore enumerate the axiom $\langle m_b, F_b\rangle \in \Phi_1$ so that $m_b \in X_{1,s} \setminus \Gamma_1(A_{0,s})$. The outcome is $\langle k, \text{old}\rangle$. If this is the true outcome, then we do not care what happens to X_0 as strategies below this outcome will be working with new versions of this set.

(2) Otherwise, for every b with $\text{age}^s(b) < k$, we can associate a set G_b, the use of the oldest valid axiom for m_b in the set Γ_1. We enumerate into $\Phi_{0,s}$ the axiom
$$\langle x_a, \bigcup_{\text{age}^s(b)<k} G_b \cup \{b \mid \text{age}^s(b) \le k\}\rangle.$$
Next, we check whether $x_a \in \Gamma_0(A_1)$. If $x_a \notin \Gamma_0(A_{1,s})$, then we have evidence that this requirement may be satisfied by $x_a \in X_0 \setminus \Gamma_0(A_1)$. Unfortunately, we have no evidence that the effect of β on X_0 is finite, so we use the marker m_a. We will always only enumerate axioms of the form $\langle m_a, F_a\rangle$ into Φ_1, where F_a is the use of a Γ_0-axiom for x_a. The case we are in suggests that $m_a \notin X_1$.

(a) If $m_a \notin \Gamma_1(A_{0,t})$ at some stage $t \in [s_k, s]$, then we can guarantee that under this outcome, β's effect on X_0 is finite. This is because we include the use of a Γ_1-axiom for m_a in the use of every axiom we enumerate into Φ_0 for numbers a' with age larger than k. (This is true as long as a maintains its age.) We set the outcome to be $\langle k, \text{off}\rangle$.

(b) Otherwise, $m_a \in \Gamma_1(A_{0,t})$ at all stages $t \in [s_k, s]$, so we have evidence that $m_a \in \Gamma_1(A_0) \setminus X_1$. We end with outcome $\langle k, \text{new}\rangle$ and let strategies below forget about this version of X_0.

(3) Finally, if $x_a \in \Gamma_0(A_{1,t})$ at all stages $t \in [s_k, s]$ (by the same axiom at all stages since the last visit), then let $\langle x_a, F_a\rangle \in \Gamma_0$ be the axiom that has been valid the longest. Enumerate $\langle a, F_a\rangle$ into Ψ_s. We have eliminated a as a difference, and so we may proceed to pick the oldest difference once again.

2.4. **Verification.** We define the true path f in the tree of strategies as the leftmost path of strategies visited infinitely often. If $\lambda\hat{\ }o \prec f$, then we will say that λ has *true outcome o*. If s is a stage at which λ is visited, then we say that s is *λ-true*. We need to prove that f is well defined and strategies along it satisfy their requirements. We do so by showing the following properties of the construction by simultaneous induction.

Lemma 2.2. *The true path f is infinite, furthermore:*

A. If β is an $\mathcal{R}_{\Gamma_1,\Gamma_0}$-strategy and $\beta \preceq f$, then:

(1) There is a leftmost outcome o that β visits at infinitely many stages.

(2) There are finitely many values of the parameter s_β, and for each such value, $X_1^{[s_\beta]}$ is a computable set.

(3) If $o \in \{\langle 2k\rangle, \langle 2k+1, \text{off}\rangle \mid k \in \omega\}$, then $\mathcal{R}_{\Gamma_1,\Gamma_0}$ is satisfied, and for every value of s_β, the set $X_0^{[s_\beta]}$ is finite.

(4) If $o \in \{\langle 2k+1, \text{old}\rangle, \langle 2k+1, \text{new}\rangle \mid k \in \omega\}$, then $\mathcal{R}_{\Gamma_1}$ is satisfied.

B. If γ is an $\mathcal{S}_{\Delta_1}$-strategy and $\gamma \preceq f$, then:

(1) There is a leftmost outcome o that γ visits at infinitely many stages.

(2) The set D_o consists of all numbers that γ contributes to X_1.

(3) The requirement $\mathcal{S}_{\Delta_1}$ is satisfied.

C. If δ is an $\mathcal{S}_{\Gamma_1,\Delta_0}$-strategy and $\delta \preceq f$, then:

(1) There is a leftmost outcome o that δ visits at infinitely many stages.

(2) The set D_o consists of all numbers that γ contributes to X_0.

(3) The requirement $\mathcal{S}_{\Gamma_1,\Delta_0}$ is satisfied.

Proof. We will prove the statements above in turn, assuming that all statements are true for higher-priority strategies along the true path. We first note that $\mathcal{R}_{\Gamma_1}$-strategies along the true path have only one possible outcome, visited at every true stage, hence cannot cause f to be finite.

A. Let $\beta \preceq f$ be an $\mathcal{R}_{\Gamma_1,\Gamma_0}$-strategy. It follows from the definition of the true path that β is visited at infinitely many stages and initialized finitely often. There is a stage at which β is first visited after its last initialization. At this stage, s_β receives its final value, and by construction, we interrupt this stage so that no other strategy has the same parameter at any point during the construction. By construction, no strategy has so far enumerated any element into the s_β-th column of X_0 or X_1: Lower-priority strategies σ are initialized at stage s_β, so their parameter s_σ (if defined) will have higher value than s_β. Higher-priority strategies λ will not add elements to the s_β-th column of X_0 or X_1, either. To see this, note that they are either not visited at further stages, hence do not act any longer; they are $\mathcal{R}_{\Gamma_1,\Gamma_0}$-strategies with $s_\lambda < s_\beta$ and hence enumerate elements into smaller columns of X_0 or X_1; or they are $\mathcal{S}$-strategies whose true outcome is extended by β and hence by B(2) and C(2), they will not enumerate any more valid axioms into either operator Φ_0 or Φ_1. Thus β is the unique strategy that adds elements into the s_β-th column of X_0 or X_1. If $t < s_\beta$ is a previous value of the parameter s_β, then our analysis shows that no strategy can add valid axioms for numbers in the t-th columns of X_0 and X_1 after stage s_β.

Let $\Psi = \bigcup \Psi_s$ be the enumeration operator constructed by β. By assumption, $A_0 \not\leq_e A_1$, hence $\Psi(A_1) \neq A_0$. Let a be the oldest disagreement, where the age of the disagreement is defined as in the construction. This means that there is some stage s_a such that at all stages $t > s_a$, we have that if $\mathrm{age}^t(b) < \mathrm{age}^t(a)$, then $b \in A_{0,t} \cap \Psi(A_{1,t})$ with a fixed marker m_b. The age c of a remains constant, and at stages $t > s_a$, the strategy β will not visit any outcome left of the c-outcomes (which depend on the parity of c), so the marker m_a will remain fixed. Furthermore, the way β adjusts the approximation to A_0 and A_1 when visited ensures that there are infinitely many stages $t > s_a$ at which we visit β, and a is the oldest disagreement at stage t. At such stages, β will visit a c-outcome, and since there are finitely many c-outcomes (only one c-outcome $\langle c\rangle$ if c is even, and three c-outcomes $\langle c, \mathrm{old}\rangle$, $\langle c, \mathrm{off}\rangle$, and $\langle c, \mathrm{new}\rangle$ if c is odd), there is a leftmost outcome visited at infinitely many stages, proving (1). Note that if β reaches Case 2.3 infinitely often, then β also ends in Case 2.2 infinitely often because if t is such that $a \notin \Psi(A_{1,t})$, but $a \in \Psi(A_{1,t^-})$, where t^- is the previous β-true stage, then our convention ensures that $x_a \notin \Gamma_0(A_{1,t})$. All numbers greater than m_a in the s_β-th column of X_1 will be dumped into X_1, hence the s_β-th column of X_1 is cofinite, proving (2).

If $a \notin A_0$, then the age of a after stage s_a is $c = 2k$, where k is the stage such that at all $t \geq k$, we have that $a \in \Psi(A_{1,t})$ via the same axiom $\langle a, F_a\rangle$, say. As we argued above, $\langle 2k\rangle$ is β's true outcome. We prove that $x_a \in \Gamma_0(A_1) \setminus X_0$: That $x_a \notin X_0$ is clear, as by construction, any axiom that β enumerates into Φ_0 for x_a contains a in its use, and as we already argued, no other strategy enumerates valid axioms for $x_a = \langle s_\beta, a\rangle$. On the other hand, β enumerated the valid axiom $\langle a, F_a\rangle$ into Ψ because it saw that $\langle x_a, F_a\rangle \in \Gamma_0$, and since $F_a \subseteq A_1$, it follows that $x_a \in \Gamma_0(A_1)$. Up until stage s_a, there are only finitely many axioms enumerated into Φ_0 by β. After stage s_a, any axiom enumerated by β into Φ_0 will include a in its use because the age of a remains constant. It follows that all such axioms are invalid, and so β contributes a finite set to $X_0^{[s_\beta]}$.

Suppose $a \in A_0$. Then $c = 2k + 1$, and we have several cases, depending on the leftmost outcome visited infinitely often. If this is $\langle 2k + 1, \mathrm{old}\rangle$, then infinitely often after stage s_a, we visit β, and it stops at Case 2.1 of the construction, because some b with $\mathrm{age}^s(b) < k$ has $m_b \notin \Gamma_1(A_{0,t})$ at some stage t since we last considered a. There are finitely many such b, and hence the described scenario happens infinitely often with some fixed such b. As pointed out in the construction, since b is older than a, we know that $b \in \Psi(A_{1,t})$ at such stages via the same axiom $\langle b, F_b\rangle$, and so the construction ensures that $m_b \in X_1$ using this axiom. It follows that $m_b \in X_1 \setminus \Gamma_1(A_0)$.

Otherwise, there is a stage $s_b > s_a$ such that at all β-true stages $t > s_b$ at which a is the oldest disagreement, Case 2.1 does not apply. This means that for every older b, there is a fixed valid axiom $\langle m_b, G_b\rangle \in \Gamma_1$. This means that the axiom that we enumerate into Φ_0 for x_a is valid, and so $x_a \in X_0$. On the other hand, since infinitely often we are in Case 2.2, $x_a \notin \Gamma_0(A_1)$, and so $\mathcal{R}_{\Gamma_0,\Gamma_1}$ is satisfied.

If β's true outcome is $\langle 2k+1, \text{off}\rangle$, then $m_a \notin \Gamma_1(A_0)$. In this case, we already know that $\mathcal{R}_{\Gamma_0,\Gamma_1}$ is satisfied. After stage s_b, any axiom enumerated by β into Φ_0 will include some G_a in its use, where $\langle m_a, G_a\rangle \in \Gamma_1$. It follows that all such axioms are invalid, and so β contributes a finite set to $X_0^{[s_\beta]}$. This ensures that (3) is true.

Otherwise, there is a stage $s_c > s_b$ after which Case 2.2.a does not apply. The true outcome is $\langle 2k+1, \text{new}\rangle$, and $m_a \in \Gamma_1(A_{0,t})$ at all stages $t > s_c$ by the same axiom. It follows that $m_a \in \Gamma_1(A_0)$. To complete the proof, we will show that $m_a \notin X_1$. Any Φ_1-axiom enumerated for m_a has the form $\langle m_a, F_a\rangle$. Such an axiom can only be enumerated after the axiom $\langle a, F_a\rangle$ is enumerated into Ψ. Since $a \notin \Psi(A_1)$, it follows that $m_a \notin X_1$. We conclude that $\mathcal{R}_{\Gamma_1}$ is satisfied, proving (4).

B. Let $\gamma \preceq f$ be an $\mathcal{S}_{\Delta_1}$-strategy. It follows from the definition of the true path that γ is visited at infinitely many stages and initialized finitely often. There is a first stage at which γ is visited after its last initialization. This is the stage at which s_γ receives its final value, and by construction, we interrupt this stage so that no other strategy has the same parameter at any point during the construction. Lower-priority strategies σ have $s_\sigma > s_\gamma$. Higher-priority strategies are the only ones that can enumerate elements into $X_1^{[<s_\gamma]}$, so by induction, $X_1^{[<s_\gamma]}$ is a computable set. If s is a good stage in the approximation to A_1, then $A_{1,s} \subseteq A_1$ and $\Delta_1(X_{1,s}) \subseteq \Delta_1(X_1)$. Suppose that $\{l_{\gamma,s}\}_{A_{1,s}\subseteq A_1}$ is unbounded. Then we can argue that $A_1 = \Delta_1(X_1)$: If $a \in A_1$, then pick a good stage s at which $a \in A_{1,s}$ and $l_{\gamma,s} > a$. It follows that $a \in \Delta_1(X_{1,s}) \subseteq \Delta_1(X_1)$. Similarly, if $a \in \Delta_1(X_1)$, then we can pick a good stage s at which $a \in \Delta_1(X_{1,s})$ and $l_{\gamma,s} > a$. It follows that $a \in A_{1,s} \subseteq A_1$.

Furthermore, if $\{l_{\gamma,s}\}_{A_{1,s}\subseteq A_1}$ is unbounded then we can also argue that $\Delta_1(X_1) = \Delta_1(X_1^{[<s_\gamma]} \cup \mathbb{N}^{[\geq s_\gamma]})$. One inclusion follows from the fact that $X_1 \subseteq X_1^{[<s_\gamma]} \cup \mathbb{N}^{[\geq s_\gamma]}$. For the reverse inclusion, fix $n \in \Delta_1(X_1^{[<s_\gamma]} \cup \mathbb{N}^{[\geq s_\gamma]})$. Let $\langle n, F\rangle$ be the oldest valid axiom. (Note that the age of this axiom depends only on $X_1^{[<s_\gamma]}$.) Pick a good stage $s > s_\gamma$ that is greater than the age of this axiom and at which $l_{\gamma,s} > n$. At this stage, we enumerate all $\langle r, x\rangle \in F$, where $r \geq s_\gamma$, into the set $X_{1,s}$ via the axiom $\langle\langle r, x\rangle, A_{1,s}\rangle$. Since s is good, these are valid axioms, and hence $n \in \Delta_1(X_1)$. It follows that $A_1 = \Delta_1(X_1^{[<s_\gamma]} \cup \mathbb{N}^{[\geq s_\gamma]})$, contradicting the fact that A_1 is not c.e. (otherwise it would be comparable with A_0).

Thus $l_{\gamma,s}$ is bounded by some number l_γ, say, at all good stages in the construction. At good stages, the strategy γ enumerates axioms only in response to finitely many n. For each such n, we know by the fact that we are looking at a good stage that $n \in \Delta_1(A_1)$. Eventually, the oldest valid axiom will emerge, and so γ will keep selecting the same axiom $\langle n, F\rangle$ for this element, and thus ultimately γ will enumerate only finitely many elements into X_1, and all these elements will be enumerated at good stages. Let $D = D_k$ be the set of these elements. Let s_k be a stage such that at all $t \geq s_k$, we have that $D \subseteq X_1$. At all γ-true stages $t \geq s_k$, the strategy γ will have outcome $\langle m\rangle$, where $D_k \subseteq D_m$. By our choice of coding, we have that $\langle k\rangle \le \langle m\rangle$. By the adjustment that γ makes to the approximation of A_1, we know that γ is visited at infinitely many good stages for the approximation to A_1. At such stages, γ will have outcome $\langle k\rangle$. This proves (1) and (2).

To see that the requirement is satisfied, we prove that there is some element $n \le l_\gamma$ such that $A_1(n) \neq \Delta_1(X_1)(n)$. Assume that this is not the case. Pick a stage s such that all elements $n \le l_\gamma$ that are in A_1 are in $A_{1,s}$ and s is good. There are infinitely many such stages, and we visit γ at such stages. At such a γ-true stage t, we have that $l_{\gamma,t} > l_\gamma$, contradicting our choice of l_γ.

C. The case where $\delta \preceq f$ is an $\mathcal{S}_{\Gamma_1,\Delta_0}$-strategy is proved similarly to Case B. □

Lemma 2.3. *Either all requirements $\mathcal{R}_{\Gamma_1}$ and $\mathcal{S}_{\Delta_1}$ are satisfied, or there is some requirement $\mathcal{R}_{\Gamma_1}$ such that all requirements $\mathcal{R}_{\Gamma_1,\Gamma_0}$ and $\mathcal{S}_{\Gamma_1,\Delta_0}$ are satisfied.*

Proof. If there are infinitely many $\mathcal{R}_{\Gamma_1}$-strategies along the true path f, then by the construction of the tree, it follows that there are infinitely many $\mathcal{S}_{\Delta_1}$-requirements assigned to nodes on the true path, as only such strategies have immediate successors that are $\mathcal{R}_{\Gamma_1}$-strategies. Thus all $\mathcal{S}_{\Delta_1}$-requirements are assigned to nodes on the true path and hence by Lemma 2.2 are satisfied. Consider any $\mathcal{R}_{\Gamma_1}$-strategy $\alpha \prec f$. Let $\gamma \prec f$ be the next $\mathcal{S}_{\Delta_1}$-strategy along the true path. By the construction of the tree, γ's immediate predecessor is an $\mathcal{R}_{\Gamma_1,\Gamma_0}$-strategy β with true outcome $\langle 2k+1, \text{old}\rangle$ or $\langle 2k+1, \text{new}\rangle$ for some k. It follows from Lemma 2.2 that $\mathcal{R}_{\Gamma_1}$ is satisfied, thus all requirements $\mathcal{R}_{\Gamma_1}$ are satisfied.

If there are finitely many $\mathcal{R}_{\Gamma_1}$-strategies along f, then fix the longest such α. Every immediate successor of α along the true path is either an $\mathcal{R}_{\Gamma_1,\Gamma_0}$-strategy with true outcome $\langle 2k\rangle$ or $\langle 2k+1, \text{off}\rangle$ or an $\mathcal{S}_{\Gamma_1,\Delta_0}$-strategy. Hence there are infinitely many of each, and by Lemma 2.2, they are all successful. By the design of the tree, it follows that all requirements $\mathcal{R}_{\Gamma_1,\Gamma_0}$ and $\mathcal{S}_{\Gamma_1,\Delta_0}$ are satisfied. □

3. No Ahmad triple

In this section, we extend the ideas introduced in the previous section to prove our main result:

Theorem 3.1. *There is no* Ahmad triple *in the Σ^0_2-enumeration degrees, i.e., there are no Σ^0_2-degrees $\mathbf{a}_0$, $\mathbf{a}_1$, and $\mathbf{a}_2$ such that $\mathbf{a}_0 \not\le \mathbf{a}_1$ but every enumeration degree $\mathbf{x}_0 < \mathbf{a}_0$ is $\le \mathbf{a}_1$, and such that $\mathbf{a}_1 \not\le \mathbf{a}_2$ but every enumeration degree $\mathbf{x}_1 < \mathbf{a}_1$ is $\le \mathbf{a}_2$.*

3.1. **Requirements.** Suppose A_0, A_1 and A_2 are Σ^0_2-sets. The construction builds an enumeration operator Φ_0, attempting to satisfy the following requirements for each of the enumeration operators Γ_0 and Δ_0:

$$\mathcal{R}_{\Gamma_0} : \Phi_0(A_0) \neq \Gamma_0(A_1),$$
$$\mathcal{S}_{\Delta_0} : A_0 \neq \Delta_0(\Phi_0(A_0)).$$

If some $\mathcal{R}_{\Gamma_0}$-requirement fails, then we will construct an enumeration operator Φ_1 satisfying the following subrequirements for each of the enumeration operators Γ_1 and Δ_1:

$$\mathcal{R}_{\Gamma_0,\Gamma_1} : \Phi_1(A_1) \neq \Gamma_1(A_2)$$
$$\mathcal{S}_{\Gamma_0,\Delta_1} : A_1 \neq \Delta_1(\Phi_1(A_1)).$$

If some $\mathcal{R}_{\Gamma_0,\Gamma_1}$-requirement fails, then we will construct an enumeration operator Φ_2 satisfying the following subsubrequirements for each of the enumeration operators Γ_2 and Δ_2:

$$\begin{aligned}\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2} &: \Phi_2(A_0) = \Gamma_2(A_1) \implies A_0 = \Psi_0(A_1) \text{ or } A_1 = \Psi_1(A_2)\\ &\qquad\qquad (\text{for } \Psi_0 \text{ and } \Psi_1 \text{ built by us})\\ \mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2} &: A_0 \neq \Delta_2(\Phi_2(A_0)).\end{aligned}$$

We will denote $\Phi_0(A_0)$ by X_0, $\Phi_1(A_1)$ by X_1, and $\Phi_2(A_0)$ by X_2 whenever the operator Φ_i is clear from the context.

3.2. **Overview.** We first give a high-level overview of how the overall construction works, without going into the specifics of the priority tree layout and the arrangement of different outcomes. At each node α of the priority tree, there will be an active version of X_1 and X_2 (where X_0 is, of course, maintained globally). Each version of X_1 and X_2 is built in some *cone*; X_1 is built in a cone with an $\mathcal{R}_{\Gamma_0}$-requirement at the top of the cone, while X_2 is built in a cone with an $\mathcal{R}_{\Gamma_0,\Gamma_1}$-requirement at the top. These cones are nested in the sense that each node where a particular set X_2 is active is also a node where a set X_1 is active; but a cone for X_1 can contain many different X_2-cones. The setup here is typical of a non-uniform argument; the situation in our construction is perhaps slightly more complicated than a typical non-uniform argument due to having to keep track of three levels of non-uniformity. However, the overall spirit is the same: The set X_0 is maintained globally, and there will only be one version of it, i.e., every node in the tree is in the one X_0-cone. Inside each X_2-cone, we will have the active sets X_0 and X_1. Inside this X_2-cone, we actively try and satisfy the $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$- and the $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$-strategies, while leveraging on the assumption that the $\mathcal{R}_{\Gamma_0,\Gamma_1}$- and $\mathcal{R}_{\Gamma_0}$-strategies at the top of the X_2- and X_1-cones are unsuccessful. While this assumption is not violated, we stay in the X_2-cone and only consider the $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$- and the $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$-strategies.

If we ever detect that the $\mathcal{R}_{\Gamma_0,\Gamma_1}$-strategy is successful, we will exit the X_2-cone and immediately place the next $\mathcal{S}_{\Gamma_0,\Delta_1}$-strategy before starting a new X_2-cone below it. Similarly, if we ever detect that the $\mathcal{R}_{\Gamma_0}$-strategy is successful, we will end the current X_1- and X_2-cones and immediately place the next $\mathcal{S}_{\Delta_0}$-strategy before starting a new X_1- and a new X_2-cone below. In this way, depending on how many different X_1- and X_2-cones the true path of the construction crosses, we will be able to argue that along the true path, either all $\mathcal{R}_{\Gamma_0}$- and $\mathcal{S}_{\Delta_0}$-requirements are satisfied, or all $\mathcal{R}_{\Gamma_0,\Gamma_1}$- and $\mathcal{S}_{\Gamma_0,\Gamma_1}$-requirements are satisfied in some final X_1-cone, or all $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$- and $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$-requirements are satisfied in some final X_2-cone.

It remains to describe how a node α assigned to a requirement $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$ is able to either detect the success of its parent $\mathcal{R}_{\Gamma_0,\Gamma_1}$-strategy, or the success of its grandparent $\mathcal{R}_{\Gamma_0}$-strategy, or be able to leverage on the failure of both to ensure the success of its own requirement. By our experience with the $\mathcal{S}$-strategies thus far, it is important to note that α must not be too liberal with enumerating true axioms into Φ_0, Φ_1 and Φ_2; if we do not exit the current X_2-cone, we must make sure that α's contribution to X_2 is finite or at least computable. If we do not exit the current X_1-cone, then α's contribution to X_1 must be finite, while α's contribution to X_0 must be finite regardless of the true outcome. (If we do not ensure this, then along the true path, when we have the next $\mathcal{S}_{\Delta_0}, \mathcal{S}_{\Gamma_0,\Delta_1}$- or $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$-node, we will not be able to run the respective basic $\mathcal{S}$-strategy.)

With the foregoing comment in mind, consider a node α assigned to a requirement $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$. The obvious strategy is to associate each number a (targeted for A_1) with a number x_a (targeted for $X_1 = \Phi_1(A_1)$) and try to maintain that $a \in A_1$ iff $x_a \in X_1$. We build a reduction Ψ_1 which will emulate Γ_1. Since $A_1 = \Psi_1(A_2)$ cannot possibly hold, we must be able to find some x_a where $X_1(x_a) \neq \Gamma_1(A_2)(x_a)$, and hence $\mathcal{R}_{\Gamma_0,\Gamma_1}$ will be satisfied. This naive strategy will work to satisfy $\mathcal{R}_{\Gamma_0,\Gamma_1}$ in isolation; unfortunately, we may not be able to guarantee that the effect on X_1 is finite; as discussed above, if we satisfy $\mathcal{R}_{\Gamma_0,\Gamma_1}$, we stay in the X_1-cone, and we will need the strategy to enumerate only finitely many true axioms for $\Phi_1(A_1)$. Notice that if there is some $x_a \in \Gamma_1(A_2)\backslash X_1$, then this condition can be ensured, since the strategy for α will only need to enumerate further axioms putting some $x_{a'}$ into $\Phi_1(A_2)$ if the length of agreement goes up; hence all newer axioms in Φ_1 will include the number a. However, if the disagreement is witnessed by some $x_a \in X_1 \setminus \Gamma_1(A_2)$, then there is no way to prevent infinitely many elements $x_{a'}$ from being put into X_1 by the strategy. Note that the same problem applies even if we try and diagonalize X_0 and $\Gamma_0(A_1)$, or X_2 and $\Gamma_2(A_1)$. The solution to this problem is to ensure that under the problematic outcome where $x_a \in X_1 \setminus \Gamma_1(A_2)$, all future axioms enumerated by α putting some $x_{a'}$ into $X_1 = \Phi_1(A_1)$ must also include the use of certain elements in $\Gamma_0(A_1)$ and $\Gamma_2(A_1)$. If we entangle the axioms for newer $x_{a'}$ in this way correctly, then we will be able to argue that in the end, either we will be able to diagonalize X_0 and $\Gamma_0(A_1)$, or we can diagonalize X_2 and $\Gamma_2(A_1)$, or else we can force all newer $x_{a'}$-axioms enumerated by α to become invalid.

To arrange for this entangling to work properly, we will need another setup. Under the assumption that $a \in A_1 \setminus \Psi_1(A_2)$ and $x_a \in X_1 \setminus \Gamma_1(A_2)$ hold, we will need to start a *backup* $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$*-strategy*, which we will name β. Each time β sees further proof that $a \in A_1 \setminus \Psi_1(A_2)$ and $x_a \in X_1 \setminus \Gamma_1(A_2)$ hold, it will extend the reduction $A_0 = \Psi_0(A_1)$ that it builds. The basic working of β is that it associates each number b targeted for A_0 with a number y_b targeted for $X_2 = \Phi_2(A_0)$ and a number m_b targeted for $X_0 = \Phi_0(A_0)$. (For technical reasons, in the construction, the association $b \mapsto y_b$ will be fixed while the association $b \mapsto m_b$ will be dynamic, but we do not encumber ourselves with these details at this time.) The plan will be to let Ψ_0 emulate Γ_2, so that the necessary disagreement between A_0 and $\Psi_0(A_1)$ must produce a corresponding disagreement between X_2 and $\Gamma_2(A_1)$. Fix the element b so that we have either

(i) $b \in A_0 \setminus \Psi_0(A_1)$ and $y_b \in X_2 \setminus \Gamma_2(A_1)$ and $m_b \in X_0$, or
(ii) $b \in \Psi_0(A_1) \setminus A_0$ and $y_b \in \Gamma_2(A_1) \setminus X_2$ and $m_b \notin X_0$.

Since b will eventually be in one of the Σ^0_2-sets involved above, almost every axiom enumerated by the main strategy α putting some $x_{a'}$ into $X_1 = \Phi_1(A_1)$ will be able to observe and use the information provided by this number b.

Consider a future stage s when the strategy α is deciding whether or not to enumerate an axiom putting some $x_{a'}$ into $X_1 = \Phi_1(A_1)$. By our assumption on b, we must see either $y_b \in X_2[s]$ or $y_b \in \Gamma_2(A_1)[s]$. If the strategy for α sees $y_b \in X_2[s]$ but $y_b \notin \Gamma_2(A_1)[s]$, then it does not need to proceed with its strategy and does not enumerate any axiom for $x_{a'}$ into X_1 since it currently looks like $X_2 \neq \Gamma_2(A_1)$. If, however, α sees $y_b \in \Gamma_2(A_1)[s]$ but $m_b \notin \Gamma_0(A_1)$, then it also does not need to enumerate an axiom for $x_{a'}$ into X_1. This is because either $b \in A_0$, in which case both $y_b \in X_2$ and $m_b \in X_0$, or they are all three out of the respective sets, and so we must currently see $X_2 \neq \Gamma_2(A_1)$ or $X_0 \neq \Gamma_0(A_1)$. Therefore, the only time when α enumerates $x_{a'}$ into X_1 is when it sees both $y_b \in \Gamma_2(A_1)[s]$ and $m_b \in \Gamma_0(A_1)[s]$, in which case it will include the use of the latter two in the axiom for $x_{a'}$ in X_1.

Now finally assume that β is along the true path of the construction. If the first case (i) above applies to b, then almost every axiom $x_{a'}$ enumerated by the strategy for α will be invalid, since they will include the use of $y_b \in \Gamma_2(A_1)$, which was exactly what we wanted to achieve. If the second case (ii) applies and m_b is not eventually in $\Gamma_0(A_1)$, then again almost every axiom $x_{a'}$ enumerated by the strategy for α will be invalid, since they will include the use of $m_b \in \Gamma_0(A_1)$. Finally, assume that the second case (ii) applies and $m_b \in \Gamma_0(A_1)$. Then in this case, it may be possible that the strategy for α enumerates infinitely many elements into X_1, but then we would have $m_b \in \Gamma_0(A_1) \setminus X_0$, and we will exit the current X_1-cone. In this case, the current set X_1 will be irrelevant anyway.

3.3. **Tree of strategies.** Order each of the types of requirements, subrequirements and subsubrequirements in a priority of order type ω such that each $\mathcal{R}_{\Gamma_0}$-requirement precedes all the $\mathcal{R}_{\Gamma_0,\Gamma_1}$-subrequirements, and each $\mathcal{R}_{\Gamma_0,\Gamma_1}$-subrequirement precedes all the $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-subsubrequirements.

The root of the tree of strategies T is $\emptyset$, an $\mathcal{R}_{\Gamma_0}$-strategy working on the highest-priority $\mathcal{R}_{\Gamma_0}$-requirement. An $\mathcal{R}_{\Gamma_0}$-strategy has only one outcome $\langle 0\rangle$ and is immediately followed by an $\mathcal{R}_{\Gamma_0,\Gamma_1}$-strategy, working on the highest-priority $\mathcal{R}_{\Gamma_0,\Gamma_1}$-requirement. An $\mathcal{R}_{\Gamma_0,\Gamma_1}$-strategy has only one outcome $\langle 0\rangle$ and is immediately followed by an $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy, working on the highest priority $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-requirement.

An $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy α has outcomes $\langle 2k\rangle$, $\langle 2k+1,\text{old}\rangle$, $\langle 2k+1,\text{off}\rangle$, $\langle 2k+1,\text{new}\rangle$, and $\langle 2k+1,\text{backup}\rangle$ for all $k \in \omega$, ordered as follows:

$$\langle 0\rangle <_L \langle 1,\text{old}\rangle <_L \langle 1,\text{off}\rangle <_L \langle 1,\text{new}\rangle <_L \langle 1,\text{backup}\rangle <_L \langle 2\rangle <_L \cdots$$

For every $k < \omega$, the nodes $\alpha\hat{\ }\langle 2k\rangle$ are $\mathcal{S}_{\Gamma_0,\Delta_1}$-strategies working on the highest-priority $\mathcal{S}_{\Gamma_0,\Delta_1}$-requirement that is not assigned to any of α's predecessors. The nodes $\alpha\hat{\ }\langle 2k+1,\text{old}\rangle$ and $\alpha\hat{\ }\langle 2k+1,\text{off}\rangle$ are $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$-strategies working on the highest priority $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$-requirement that is not assigned to any of α's predecessors. The nodes $\alpha\hat{\ }\langle 2k+1,\text{new}\rangle$ are $\mathcal{S}_{\Delta_0}$-strategies working on the highest-priority $\mathcal{S}_{\Delta_0}$-requirement that is not assigned to any of α's predecessors. Finally, the nodes $\alpha\hat{\ }\langle 2k+1,\text{backup}\rangle$ are backup $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategies with their own outcomes (defined below).

An S_{Δ_0}-strategy γ has outcomes $\langle k\rangle$, where $k \in \omega$ ordered by the standard ordering on ω. Each such immediate successor of this strategy is a main $\mathcal{R}_{\Gamma_0}$-strategy, working on the highest-priority $\mathcal{R}_{\Gamma_0}$-requirement that is not assigned to any of its predecessors. The outcomes and immediate successors of the $\mathcal{S}_{\Gamma_0,\Delta_1}$- and $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$-strategies are defined analogously ($\mathcal{R}_{\Gamma_0,\Gamma_1}$ and $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$, respectively).

Finally, a backup $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy β has outcomes $\langle 2l,\text{off}\rangle$, $\langle 2l,\text{new}\rangle$ and $\langle 2l+1\rangle$ for all $l \in \omega$, ordered as follows:

$$\langle 0,\text{off}\rangle <_L \langle 0,\text{new}\rangle <_L \langle 1\rangle <_L \langle 2,\text{off}\rangle <_L \cdots$$

For every $l < \omega$, the nodes $\beta\hat{\ }\langle 2l,\text{off}\rangle$ and $\beta\hat{\ }\langle 2l+1\rangle$ are $\mathcal{S}_{\Gamma_0,\Delta_1}$-strategies working on the highest-priority $\mathcal{S}_{\Gamma_0,\Delta_1}$-requirement that is not assigned to any of β's predecessors. The nodes $\beta\hat{\ }\langle 2l,\text{new}\rangle$ are $\mathcal{S}_{\Delta_0}$-strategies working on the highest-priority $\mathcal{S}_{\Delta_0}$-requirement that is not assigned to any of β's predecessors.

3.4. **Construction.** This construction has many properties that are similar to the one in Section 2. At stage 0, all strategies are in initial state: All operators associated with these strategies are empty, and all parameters are undefined. At stage $s > 0$, we build a path f_s of length $\leq s$ with the intention of building a *true path* defined by

$$f(n) = \liminf_{f_s \succeq f \restriction n} f_s(n).$$

When a strategy is activated at stage s, it first adjusts the approximations to A_0, A_1, and A_2: If s^- is the previous stage at which this strategy was active, then it replaces $A_{i,s}$ by $\bigcap_{u \in [s^-,s]} A_{i,u}$ for $i \leq 2$. At the end of stage s, we initialize all strategies of *lower priority* than f_s, i.e., strategies extending or to the right of the strategies which acted at stage s.

Each strategy β works with the version of Φ_1 and X_1 determined by the longest $\mathcal{R}_{\Gamma_0}$-strategy $\alpha \prec \beta$ (we say that β *works for* α); this version of Φ_1 is the set of Φ_1-axioms enumerated by all the $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategies and $\mathcal{S}_{\Gamma_0,\Delta_1}$-strategies working for the same $\mathcal{R}_{\Gamma_0}$-strategy. Similarly, each strategy β works with the version of Φ_2 and X_2 determined by the longest $\mathcal{R}_{\Gamma_0,\Gamma_1}$-strategy $\alpha \prec \beta$; this version of Φ_2 is the set of Φ_2-axioms enumerated by all the backup $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategies and $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$-strategies working for the same $\mathcal{R}_{\Gamma_0,\Gamma_1}$-strategy.

3.4.1. $\mathcal{S}_{\Delta_0}$, $\mathcal{S}_{\Gamma_0,\Delta_1}$, $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$. The $\mathcal{S}$-strategies work precisely as in the previous construction. Let γ be such a strategy. The first time γ is visited after initialization, it receives a unique number s_γ, the stage of first visit after initialization, and stops the construction of $f_s = \gamma$ for this stage. If $s_\gamma < s$ is already defined, then we consider the length $l_{\gamma,s} < s$ of the common initial segment up to s of the sets $A_{j,s}$ and $\Delta_{i,s}(X_i)$ that are named in the corresponding requirement: For $\mathcal{S}_{\Delta_0}$, these are $A_{0,s}$ and $\Delta_{0,s}(X_{0,s})$; for $\mathcal{S}_{\Gamma_0,\Delta_1}$, these are $A_{1,s}$ and $\Delta_{1,s}(X_{1,s})$; and for $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$, these are $A_{0,s}$ and $\Delta_{2,s}(X_{2,s})$. For every number $n \leq l_{\gamma,s}$, if $n \in \Delta_{i,s}(X_{i,s}^{[<s_\gamma]} \cup \mathbb{N}^{[\geq s_\gamma]})$, then we search for the axiom $\langle n, F\rangle \in \Delta_i$ that has been valid the longest, and we enumerate each element of the form $\langle r, x\rangle \in F$, where $r \geq s_\gamma$, into the set $X_{i,s}$ via the axiom $\langle\langle r, x\rangle, A_{j,s}\rangle$. Note that this action might enumerate some number $\langle r, x\rangle$ into $X_{i,s}$ where $\langle r, x\rangle$ is already in $X_{i,s}$ via an axiom enumerated by a different node.

The outcome of the strategy is $\langle k\rangle$, where k is the standard code of the finite set D_k of all numbers for which γ has enumerated an axiom that looks valid at stage s. As before, we assume that $D_{k_1} \subseteq D_{k_2}$ implies $k_1 \leq k_2$.

3.4.2. $\mathcal{R}_{\Gamma_0}$, $\mathcal{R}_{\Gamma_0,\Gamma_1}$. The $\mathcal{R}_{\Gamma_0}$-strategy and the $\mathcal{R}_{\Gamma_0,\Gamma_1}$-strategy do nothing, have only one outcome $\langle 0\rangle$, and determine the version of Φ_1 and X_1, or Φ_2 and X_2, respectively, that substrategies work with.

3.4.3. $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$. The $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy α attempts to construct an enumeration operator Ψ_1 such that $A_1 = \Psi_1(A_2)$ by enumerating axioms into Φ_1. (It will not enumerate axioms into either Φ_0 or Φ_2, only its backup strategies will.)

At the first stage after initialization, the $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy α is assigned the parameter s_α. We can assume that s_α is larger than $\max(D_k)$ for any k such that a higher-priority $\mathcal{S}$-strategy λ has $\lambda\hat{\ }k \preceq \alpha$. Until its next initialization, α will only contribute numbers to the s_α-th columns of X_1. To every element a, we assign the coding location $x_a = \langle s_\alpha, a\rangle$ targeted for X_1.

At stage $s > s_\alpha$, α does the following. It orders the elements of $A_{1,s} \cup \Psi_{1,s}(A_{2,s})$ by age:

$$\text{age}_1^s(a) = 2k+1 \text{ for } k = \min(s+1, \mu t \forall u \in [t,s](a \in A_{1,u}))$$
$$\text{age}_2^s(a) = 2k \text{ for } k = \min(s+1, \mu t \forall u \in [t,s](a \in \Psi_1(A_{2,u})[u] \text{ via the same axiom}))$$
$$\text{age}^s(a) = \min\{\text{age}_1^s(a), \text{age}_2^s(a)\}.$$

Once again, we assume the age is defined injectively, i.e., for every stage $t \leq s$, there may be at most one element with $\text{age}^s(a) = t$. Also, if $a \in A_{1,s} \setminus \Psi_{1,s}(A_{2,s})$, then $\text{age}^s(a)$ is odd, and if $a \in \Psi_{1,s}(A_{2,s}) \setminus A_{1,s}$, then $\text{age}^s(a)$ is even.

If $A_{1,s} = \Psi_{1,s}(A_{2,s})$, then we exit this strategy with outcome $\langle 2(s+1) \rangle$. Otherwise, we pick the oldest number a such that $A_{1,s}(a) \neq \Psi_{1,s}(A_{2,s})(a)$. Let $k = \text{age}^s(a)$. We have two cases depending on the parity of k.

Case 1: If k is even, i.e., if $a \in \Psi_{1,s}(A_{2,s}) \setminus A_{1,s}$, then we will be able to argue that $x_a = \langle s_\alpha, a \rangle \in \Gamma_1(A_2) \setminus X_1$. The strategy selects outcome $\langle k \rangle$. While a maintains its age, we will design axioms for younger elements enumerated into X_1 by α so that their use includes a. Thus if this is α's true outcome, they will be invalid, and hence α contributes finitely much to X_1. Under this outcome, we do not care about X_2.

Case 2: If k is odd, i.e., if $a \in A_{1,s} \setminus \Psi_{1,s}(A_{2,s})$, then we would like to add an axiom for a into Ψ_1. We will follow a similar scheme as in the previous construction: We will add an axiom for x_a into Φ_1, wait until x_a shows up in $\Gamma_1(A_2)$, and use the (currently) valid axiom $\langle x_a, F_a \rangle$ to define an axiom for a in Ψ_1. We entangle the axiom for x_a with axioms from both Γ_0 and Γ_2. First, we consider all $z \in X_{2,s}$ such that the *age* of z in X_2 (i.e., the least $t \leq s$ such that $z \in X_{2,u}$ for all $u \in [t,s]$) is less than k. Let s_k be the previous stage when α considered k:

(1) If some such $z \in X_{2,s}$ is not in $\Gamma_2(A_{1,t})$ via the same axiom for all stages $t \in [s_k, s]$, then we have evidence that $z \in X_2 \setminus \Gamma_2(A_1)$, and so we exit with outcome $\langle k, \text{old} \rangle$. If b is younger than a, then α will always include the use of an axiom for z being in $\Gamma_2(A_2)$ in the use of the axiom for x_b being in $\Phi_1(A_1)$, so if this is the true outcome, then $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$ is satisfied and α's effect on X_1 is finitary.

(2) Otherwise, for every $z \in X_{2,s}$ that is older than a, we can associate a set G_z, the use of the oldest valid axiom for z being in $\Gamma_2(A_1)$. Next, we consider every backup strategy β (an immediate successor of α) that is not in initial state, every number $y_b^\beta = \langle s_\beta, b \rangle \in \Gamma_{2,s}(A_{1,s})$ such that the *age* of y_b^β (i.e., the least $t \leq s$ such that at all $u \in [t,s]$, we have $y_b^\beta \in \Gamma_{2,u}(A_{1,u})$ via the same axiom) is less than k, and for each such y_b^β, the coding locations $m_b^\beta < k$ that are associated with b by β. (Note that we do not restrict β to the ones that extend an older outcome than k. There will only be finitely many such backup strategies being considered due to the restriction on the age of y_b^β.)

If for some such $y_b^\beta \in \Gamma_{2,s}(A_{1,s})$, there is an $m_b^\beta < k$ such that $m_b^\beta \notin \Gamma_0(A_{1,t})$ at some stage $t \in [s_k, s]$, then we have two cases:

(a) If $b \notin A_{0,t}$ at some stage $t \in [s_k, s]$, then we have evidence that $y_b^\beta \in \Gamma_2(A_1) \setminus X_2$. We exit with outcome $\langle k, \text{off} \rangle$ and argue that if this is the true outcome, then $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$ is satisfied and α enumerates finitely many valid axioms into Φ_1 because they all must include the use of a Γ_0-axiom for all such m_b^β.

(b) If $b \in A_{0,t}$ at all $t \in [s_k, s]$, then the outcome is $\langle k, \text{new}\rangle$. We have evidence that $m_b^\beta \in X_0$ as it was either *dumped* there (i.e., we enumerated the axiom $\langle m_b^\beta, \emptyset\rangle$ into Φ_0) or its axiom has use $\{b\}$. If this is the true outcome, then we aim to show that $\mathcal{R}_{\Gamma_0}$ is satisfied by $m_b^\beta \in X_0 \backslash \Gamma_0(A_1)$. In this case, we do not care about α's effect on X_1 even though that effect will be finitary as we argued in the previous case.

(3) Otherwise, for every older $y_b^\beta \in \Gamma_{2,s}(A_{1,s})$ and each $m_b^\beta < k$ associated with b, we can associate a set $G_{m_b^\beta}$, the use of the oldest valid axiom for m_b^β in the set Γ_0. We enumerate into $\Phi_{1,s}$ the axiom for x_a whose use consists of
- all b such that $\text{age}^s(b) \leq k$ (note that this includes a),
- all G_z for older $z \in X_{2,s}$, and
- all $G_{m_b^\beta}$ where β is a substrategy of α not in initial state (at the end of this stage) with older $y_b^\beta \in \Gamma_{2,s}(A_{1,s})$, and corresponding $m_b^\beta < k$.

Next, we check whether $x_a \in \Gamma_1(A_2)$. If $x_a \notin \Gamma_{1,t}(A_{2,t})$ at some stage $t \in [s_k, s]$, then we have evidence that $\mathcal{R}_{\Gamma_0,\Gamma_1}$ may be satisfied by $x_a \in X_1 \setminus \Gamma_1(A_2)$. Unfortunately, we have no evidence that the effect of α on X_1 is finite. So we activate the backup strategy below outcome $\langle k, \text{backup}\rangle$. The backup strategy will either turn off future axioms enumerated by α or ensure that $\mathcal{R}_{\Gamma_0}$ is satisfied.

(4) Finally, if $x_a \in \Gamma_1(A_{2,t})$ (by the same axiom at all stages since the last visit), then let $\langle x_a, F_a\rangle \in \Gamma_1$ be the axiom that has been valid longest. Enumerate $\langle a, F_a\rangle \in \Psi_{1,s}$. We have now eliminated a as a difference, and so we will pick the oldest difference once again, starting over at the current substage. (This can happen at most finitely often at any substage.)

Notice that the set X_2 is only relevant under the outcomes $\langle k, \text{old}\rangle$ and $\langle k, \text{off}\rangle$ of α (corresponding to items (1) and (2a) above, respectively). Since α itself does not add axioms to $\Phi_2(A_0)$, but rather, they are only added by the backup strategies, whenever some outcome $\langle k', \text{backup}\rangle$ of α is played, the effect on each column of X_2 will be finite if α has true outcome $\langle k, \text{old}\rangle$ or $\langle k, \text{off}\rangle$. This is also the reason we have multiple backup strategies for α.

3.4.4. *Backup* $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$. The backup $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy β works with its immediate predecessor α and attempts to construct an enumeration operator Ψ_0 such that $A_0 = \Psi_0(A_1)$ by enumerating axioms into Φ_0 and its version of Φ_2.

Just like α, the strategy β is assigned the parameter s_β at the first stage after initialization. We can assume that s_β is larger than $\max(D_k)$ for any k such that a higher priority $\mathcal{S}$-strategy λ has $\lambda\hat{\ }k \preceq \beta$. The strategy β associates to every element b the coding location $y_b = \langle s_\beta, b\rangle$ targeted for X_2. To certain elements b, it will also dynamically assign a coding location m_b targeted for X_0.

At a stage $s > s_\beta$, β orders the elements of $A_{0,s} \cup \Psi_{0,s}(A_{1,s})$ by age so that if $b \in A_{0,s} \setminus \Psi_{0,s}(A_{1,s})$, then $\text{age}^s(b)$ is odd, and if $b \in \Psi_{0,s}(A_{1,s}) \setminus A_{0,s}$, then $\text{age}^s(b)$ is even.

If $A_{0,s} = \Psi_{0,s}(A_{1,s})$, then we exit this strategy with outcome $\langle 2(s+1)+1\rangle$. Otherwise, we pick the oldest number b such that $A_{0,s}(b) \neq \Psi_s(A_{1,s})(b)$. If there is no marker m_b associated with b, then we choose m_b to be the least number of the form $\langle s_\beta, n\rangle$ which has not been chosen as a marker. We also enumerate the axiom $\langle m_b, \{b\}\rangle$ into Φ_0.

Let $l = \text{age}^s(b)$. We must ensure that β's effect on X_0 is computable and so the strategy will dump into X_0 all markers which are currently associated with any number b' with $\text{age}^s(b') > l$. Furthermore, to each such b', we associate a new marker $m_{b'}$, which is the least number of the form $\langle s_\beta, n\rangle$ which has not been chosen as a marker. We also enumerate the axiom $\langle m_{b'}, \{b'\}\rangle$ into Φ_0.

We will not ensure that β's effect on X_2 is finitary or computable because if β is on the true path, then either $\mathcal{R}_{\Gamma_0}$ or $\mathcal{R}_{\Gamma_0,\Gamma_1}$ will be satisfied and X_2 will not be relevant to strategies extending β.

We now have two cases depending on the parity of l.

Case 1: If l is even, i.e., if $b \in \Psi_{0,s}(A_{1,s}) \setminus A_{0,s}$, then unlike for α, we cannot simply take the easy win $y_b \in \Gamma_2(A_1) \setminus X_2$, because we still have not guaranteed that α's effect on X_1 is finitary. For that reason, we will instead consider the marker m_b. We will argue that $b \notin A_{0,s}$ is evidence that m_b is not an element of X_0, and so we check whether $m_b \in \Gamma_0(A_1)$. Let s_l be the previous stage when α considered l:

(1) If m_b is not in $\Gamma_0(A_{1,t})$ for some $t \in [s_l, s]$ (via the same axiom), then we will be able to argue that if this is the true outcome, then α enumerates only finitely many axioms into X_1 as all but finitely many of them will include an axiom for m_b in Γ_0. In this case, $\mathcal{R}_{\Gamma_0,\Gamma_1}$ is satisfied, and α's action on X_1 is finitary. So we take outcome $\langle l, \text{off}\rangle$.

(2) Otherwise, we have evidence that $m_b \in \Gamma_0(A_1) \setminus X_0$ and so if this is the true outcome, then $\mathcal{R}_{\Gamma_0}$ is satisfied and we take the outcome $\langle l, \text{new}\rangle$. Below this outcome, we do not care anymore what happens to X_1.

Case 2: If l is odd, i.e., if $b \in A_{0,s} \setminus \Psi_{0,s}(A_{1,s})$, then we enumerate the axiom $\langle y_b, \{b\}\rangle$ into the operator Φ_2.

(1) If $y_b \notin \Gamma_2(A_{1,s})$, then, if this is the true outcome, all but finitely many axioms that α enumerates will contain the use of an axiom for y_b being in $\Gamma_2(A_1)$ which is invalid. It follows that once again, $\mathcal{R}_{\Gamma_0,\Gamma_1}$ is satisfied and α's action on X_1 is finitary. We exit with outcome $\langle l\rangle$.

(2) Finally, if $y_b \in \Gamma_2(A_{1,s})$ (by the same axiom at all stages since the last visit), then let $\langle y_b, F_b\rangle \in \Gamma_2$ be the axiom that has been valid longest. Enumerate $\langle b, F_b\rangle$ into $\Psi_{0,s}$. We have now eliminated b as a difference, and so we will pick the oldest difference once again, starting over at the current substage. (This can happen at most finitely often at any substage.)

3.5. **Verification.** We define the true path f in the tree of strategies as the leftmost path of strategies visited infinitely often. If $\lambda\hat{\ }o \preceq f$, then we will say that λ has *true outcome* o. If s is a stage at which λ is visited, we say that s is λ-*true*. We need to prove that f is well-defined and strategies along it satisfy their requirements. We do so by showing the following properties of the construction by simultaneous induction.

Lemma 3.2. *The true path f is infinite, furthermore:*

A. If α is an $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy and $\alpha \prec f$, then:

(1) There are finitely many values of the parameter s_α.

(2) There is a leftmost outcome o that α visits at infinitely many stages.

(3) *If* $o \in \{\langle 2k\rangle \mid k \in \omega\}$, *then* $\mathcal{R}_{\Gamma_0,\Gamma_1}$ *is satisfied, and for every value of* s_α, *the set* $X_1^{[s_\alpha]}$ *is finite.*
(4) *If* $o \in \{\langle 2k+1, \mathrm{old}\rangle, \langle 2k+1, \mathrm{off}\rangle \mid k \in \omega\}$, *then* $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$ *is satisfied, and for every value of* s_α, *the set* $X_1^{[s_\alpha]}$ *is finite.*
(5) *If* $o \in \{\langle 2k+1, \mathrm{new}\rangle \mid k \in \omega\}$, *then* $\mathcal{R}_{\Gamma_0}$ *is satisfied.*
(6) *If* $o \in \{\langle 2k+1, \mathrm{backup}\rangle \mid k \in \omega\}$, *then* $\mathcal{R}_{\Gamma_0,\Gamma_1}$ *is satisfied.*

B. *If* β *is a backup strategy for an* $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$*-strategy* α *and* $\beta \prec f$, *then:*
(1) *There are finitely many values of the parameter* s_β.
(2) *There is a leftmost outcome* o *that* β *visits at infinitely many stages.*
(3) *For every value of* s_β, *the set* $X_0^{[s_\beta]}$ *is computable.*
(4) *If* $o \in \{\langle 2l, \mathrm{off}\rangle, \langle 2l+1\rangle \mid l \in \omega\}$, *then* $\mathcal{R}_{\Gamma_0,\Gamma_1}$ *is satisfied, and for every value of* s_α, *the set* $X_1^{[s_\alpha]}$ *is finite.*
(5) *If* $o \in \{\langle 2l, \mathrm{new}\rangle \mid l \in \omega\}$, *then* $\mathcal{R}_{\Gamma_0}$ *is satisfied.*

C. *If* $\gamma \prec f$ *is an* $\mathcal{S}_{\Delta_0}$-, $\mathcal{S}_{\Gamma_0,\Delta_1}$-, *or* $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$*-strategy, respectively, then:*
(1) *There is a leftmost outcome* o *that* γ *visits at infinitely many stages.*
(2) *The set* D_o *consists of all numbers that* γ *contributes to* X_0, X_1, *or* X_2, *respectively.*
(3) *The requirement* $\mathcal{S}_{\Delta_0}$, $\mathcal{S}_{\Gamma_0,\Delta_1}$, *or* $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$, *respectively, is satisfied.*

Proof. A. Since $\alpha \prec f$, there is a least stage at which α is visited after its final initialization. At this stage, s_α receives its final value, proving (1). By construction, we interrupt this stage so that no other strategy has the same parameter at any point during the construction. We claim that α is the only strategy that adds elements to $X_1^{[s_\alpha]}$. By construction, no strategy has enumerated any element into $X_1^{[s_\alpha]}$ so far. Lower-priority strategies σ are initialized at stage s_α, so any future values of s_σ will be greater than s_α. Once that occurs, σ cannot add elements to $X_1^{[s_\alpha]}$. Higher-priority strategies will not add elements to $X_1^{[s_\alpha]}$, either. To see this, note that any strategy λ of higher priority that potentially adds elements into X_1 after stage s_α is either an $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy with $s_\lambda < s_\alpha$, or an $\mathcal{S}_{\Gamma_0,\Delta_1}$-strategy which will not enumerate any valid Φ_1-axioms for numbers into $X_1^{[\geq s_\alpha]}$ by inductive hypothesis and our choice of s_α. This proves our claim. Note that if $t < s_\alpha$ is a previous value of the parameter s_α, then our analysis shows that no strategy can add elements into $X_1^{[t]}$ after stage s_α. It follows that $X_1^{[t]}$ is finite.

Let $\Psi_1 = \bigcup_{s>s_\alpha} \Psi_{1,s}$ be the enumeration operator constructed by α. By assumption, $A_1 \not\leq_e A_2$, hence $\Psi_1(A_2) \neq A_1$. Let a be the oldest disagreement between $\Psi_1(A_2)$ and A_1. This means that there is some stage s_a such that at all stages $t > s_a$, we have that $\mathrm{age}^t(a)$ stabilizes, and if $\mathrm{age}^t(b) < \mathrm{age}^t(a)$, then $b \in A_{1,t} \cap \Psi_{1,t}(A_{2,t})$. Furthermore, the way α adjusts the approximation to A_1 and A_2 when visited ensures that there are infinitely many stages $t > s_a$ at which we visit α and a is the oldest disagreement at stage t. At such stages, α will visit an $\mathrm{age}^t(a)$-outcome, and since there are finitely many $\mathrm{age}^t(a)$-outcomes, there is a leftmost outcome o visited at infinitely many stages, proving (2).

To prove (3), suppose that o is $\langle 2k\rangle$. In order to show that $\mathcal{R}_{\Gamma_0,\Gamma_1}$ is satisfied, we will show that $x_a \in \Gamma_1(A_2)\backslash X_1$. Since o is $\langle 2k\rangle$, we have $a \notin A_{1,t}$ for infinitely many stages t, so $a \notin A_1$. We argued above that only α can enumerate Φ_1-axioms for x_a. By construction, the use of any such axiom contains a. So $x_a \notin X_1$. As for $\Gamma_1(A_2)$, note that $a \notin A_1$ implies that $a \in \Psi_1(A_2)$. For any Ψ_1-axiom for a, there is a corresponding Γ_1-axiom for x_a with the same use. The latter axiom witnesses that $x_a \in \Gamma_1(A_2)$ as desired.

To show that $X_1^{[s_\alpha]}$ is finite, it suffices (by our reasoning above) to show that there is a stage after which no Φ_1-axiom enumerated by α is valid. This holds because any Φ_1-axiom enumerated by α when the current outcome is o or to the right of o must contain a in its use, yet $a \notin A_1$. This completes the proof of (3).

To prove (4), first suppose that o is $\langle 2k+1, \text{old}\rangle$. In order to show that $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$ is satisfied, we will show that $X_2 \not\subseteq \Gamma_2(A_1)$. Since o is $\langle 2k+1, \text{old}\rangle$, there are infinitely many stages s such that there is some $z \in X_{2,s}$ with age less than $2k+1$ and $z \notin \Gamma_2(A_{1,t})$, where $t \in [s_k, s]$. (Recall that s_k is the last stage before s at which α considered $2k+1$.) There are only finitely many z which ever have age less than $2k+1$, so the scenario described happens infinitely often with some fixed z. It follows that $z \in X_2 \backslash \Gamma_2(A_1)$ as desired.

To show that $X_1^{[s_\alpha]}$ is finite, note that for all but finitely many stages at which we visit α, the age of z in X_2 is smaller than the age of the oldest disagreement between A_1 and $\Psi_1(A_2)$. At such stages, if α enumerates a Φ_1-axiom, its use contains the use G_z of a Γ_2-axiom for z being in $\Gamma_2(A_1)$. Since $z \notin \Gamma_2(A_1)$, it follows that α only enumerates finitely many valid Φ_1-axioms. Therefore, $X_1^{[s_\alpha]}$ is finite (as α is the only strategy which adds elements into $X_1^{[s_\alpha]}$). This completes the analysis if o is $\langle 2k+1, \text{old}\rangle$.

Next suppose o is $\langle 2k+1, \text{off}\rangle$. In this case, we will show that $\Gamma_2(A_1) \not\subseteq X_2$. By assumption on o (and the fact that there are only finitely many backup strategies β and numbers b such that the age of y_b^β in $\Gamma_2(A_1)$ is ever less than $2k+1$), there are a backup strategy β, a number b, and a number $m < 2k+1$ such that at infinitely many stages s at which we visit α, we have that

- the age of y_b^β in $\Gamma_{2,s}(A_{1,s})$ is less than $2k+1$;
- m is one of the markers m_b^β that β associates with b; and
- $m \notin \Gamma_{0,t}(A_{1,t})$ for some $t \in [s_k, s]$.

It follows that $y_b^\beta \in \Gamma_2(A_1)$ and $m \notin \Gamma_0(A_1)$. Since o is $\langle 2k+1, \text{off}\rangle$, we have $b \notin A_0$. We will show that $y_b^\beta \notin X_2$. The only Φ_2-axiom enumerated by β for y_b^β has use $\{b\}$, so it is not valid. Furthermore, one can show that β is the only strategy that adds elements into $X_2^{[s_\beta]}$. The proof is similar to that for α and $X_1^{[s_\alpha]}$: Note that while β may not be along the true path, its immediate predecessor α is along the true path and therefore so are all of its predecessors. We have shown that $y_b^\beta \in \Gamma_2(A_1) \backslash X_2$.

To show that $X_1^{[s_\alpha]}$ is finite, note that for all but finitely many stages when we visit α, the age of y_b^β in $\Gamma_2(A_1)$ is smaller than the age of the oldest disagreement between A_1 and $\Psi_1(A_2)$. At such stages, if α enumerates a Φ_1-axiom, its use contains the use G_m of a Γ_0-axiom for m. Since $m \notin \Gamma_0(A_1)$, it follows that α only enumerates finitely many valid Φ_1-axioms. Therefore, $X_1^{[s_\alpha]}$ is finite (as α is the only strategy which adds elements into $X_1^{[s_\alpha]}$). This completes the proof of (4).

To prove (5), suppose that o is $\langle 2k+1, \text{new}\rangle$. To show that $\mathcal{R}_{\Gamma_0}$ is satisfied, we will show that $X_0 \not\subseteq \Gamma_0(A_1)$. Fix β, b and m, following the analysis in the case where o is $\langle 2k+1, \text{off}\rangle$. As before, we have $m \notin \Gamma_0(A_1)$. However, since o is $\langle 2k+1, \text{new}\rangle$, we have $b \in A_0$. Since we enumerated the Φ_0-axiom $\langle m, \{b\}\rangle$ when associating m with b, it follows that $m \in X_0$. So $m \in X_0 \backslash \Gamma_0(A_1)$.

To prove (6), suppose o is $\langle 2k+1, \text{backup}\rangle$. We have $x_a \notin \Gamma_1(A_2)$. To show that $x_a \in X_1$, consider a stage s' large enough such that for all $s > s'$,

- if $\text{age}^s(b) \leq 2k+1$, then $b \in A_1$;
- if the age of z in $X_{2,s}$ is less than $2k+1$, then $z \in X_2 \cap \Gamma_2(A_1)$, and the use G_z of the oldest valid Γ_2-axiom for z has stabilized; and
- if β is a backup strategy for α and the age of (the current value of) y_b^β in $\Gamma_{2,s}(A_{1,s})$ is less than $2k+1$, then
  - y_b^β has stabilized and lies in $\Gamma_2(A_1)$;
  - each m_b^β lies in $\Gamma_0(A_1)$; and
  - the use $G_{m_b^\beta}$ for the oldest valid Γ_0-axiom for each m_b^β has stabilized.

Such s' exists because o is $\langle 2k+1, \text{backup}\rangle$. At any stage $s > s'$ at which we visit α, we would enumerate a valid Φ_1-axiom for x_a. We conclude that $x_a \in X_1 \backslash \Gamma_1(A_2)$ as desired.

B. Since $\beta \prec f$, there is a least stage at which β is visited after its final initialization. At this stage, s_β receives its final value, proving (1). By construction, we interrupt this stage so that no other strategy has the same parameter at any point during the construction. One can show that β is the only strategy that adds elements into $X_0^{[s_\beta]}$, and if t is a previous value of s_β, then no strategy adds elements into $X_0^{[t]}$ after stage s_β. It follows that $X_0^{[t]}$ is computable for every previous value t of s_β.

Let $\Psi_0 = \bigcup_{s > s_\beta} \Psi_{0,s}$ be the enumeration operator constructed by β. By assumption, $A_0 \not\leq_e A_1$, hence $\Psi_0(A_1) \neq A_0$. Let b be the oldest disagreement between $\Psi_0(A_1)$ and A_0. Following similar reasoning as that for α, there is a leftmost outcome o of the form $\langle 2l, \text{off}\rangle$, $\langle 2l, \text{new}\rangle$ or $\langle 2l+1\rangle$ which is visited at infinitely many stages, proving (2).

To prove (3), we begin by showing that m_b^β stabilizes. Once s_β and $\text{age}^s(b)$ have stabilized, the only way that m_b^β changes is if some b' with $\text{age}^{s'}(b') < 2l$ is the oldest disagreement between $A_{0,s'}$ and $\Psi_{0,s'}(A_{1,s'})$. At such a stage s', the current outcome of β would be to the left of o. This only occurs finitely often, proving that m_b^β stabilizes. Then all numbers greater than m_b^β in the s_β-th column of $\mathbb{N}$ will eventually be dumped into X_0, implying that $X_0^{[s_\beta]}$ is cofinite. As for older values of s_β, we mentioned above that the corresponding columns of X_0 are computable.

To prove (4), first note that by A(6), $\mathcal{R}_{\Gamma_0,\Gamma_1}$ is satisfied. It remains to show that for every value of s_α, the set $X_1^{[s_\alpha]}$ is finite. As reasoned above, it suffices to show that α only enumerates valid Φ_1-axioms for finitely many elements in $X_1^{[s_\alpha]}$, where s_α has stabilized. The proof differs depending on whether o is $\langle 2l, \text{off}\rangle$ or $\langle 2l+1\rangle$.

Suppose o is $\langle 2l, \text{off}\rangle$. We have $b \in \Psi_0(A_1)\backslash A_0$ and $m_b^\beta \notin \Gamma_0(A_1)$. By β's construction of Ψ_0, we have $y_b^\beta \in \Gamma_2(A_1)$ (via an axiom with the same use as a Ψ_0-axiom for b). Let t be the age of y_b^β in $\Gamma_2(A_1)$. Consider any stage s such that s_α, s_β, the age of y_b^β in $\Gamma_2(A_1)$, and m_b^β all have stabilized. Suppose α enumerates a Φ_1-axiom for some $x_{a'}$ at stage s. Then a' is the oldest disagreement between $A_{1,s}$ and $\Psi_{1,s}(A_{2,s})$. Furthermore, if $t, m_b^\beta < \text{age}^s(a')$, then the use of the Φ_1-axiom enumerated by α contains the use of a Γ_0-axiom for m_b^β, rendering it invalid (because $m_b^\beta \notin \Gamma_0(A_1)$). But there are only finitely many a' for which there is some s such that $\text{age}^s(a') \leq \max\{t, m_b^\beta\}$, so α can only enumerate valid Φ_1-axioms for finitely many $x_{a'}$ as desired.

If o is $\langle 2l+1\rangle$, the analysis proceeds similarly but with y_b^β instead of m_b^β. In this case, we have that $b \in A_0 \backslash \Psi_0(A_1)$ and $y_b^\beta \notin \Gamma_2(A_1)$. By β's construction of X_2, we have $y_b^\beta \in X_2$ via $\langle y_b^\beta, \{b\}\rangle \in \Phi_2$. Let t be the age of y_b^β in X_2. Consider any stage s such that s_α, s_β and the age of y_b^β in X_2 all have stabilized. Suppose α enumerates a Φ_1-axiom for some $x_{a'}$ at stage s. Then a' is the oldest disagreement between $A_{1,s}$ and $\Psi_{1,s}(A_{2,s})$. Furthermore, if $t < \text{age}^s(a')$, then the use of the Φ_1-axiom enumerated by α contains the use of a Γ_2-axiom for y_b^β, rendering it invalid (because $y_b^\beta \notin \Gamma_2(A_1)$). But there are only finitely many a' for which there is some s such that $\text{age}^s(a') \leq t$, so α can only enumerate valid Φ_1-axioms for finitely many $x_{a'}$ as desired. This completes the proof of (4).

To prove (5), suppose o is $\langle 2l, \text{new}\rangle$. We will show that $m_b^\beta \in \Gamma_0(A_1)\backslash X_0$. Since o is $\langle 2l, \text{new}\rangle$, we have $m_b^\beta \in \Gamma_0(A_1)$. To show that $m_b^\beta \notin X_0$, note first that the only Φ_0-axiom enumerated by β for m_b^β has use $\{b\}$, so it is not valid. Furthermore, as mentioned above, β is the only strategy that adds elements into $X_0^{[s_\beta]}$, so $m_b^\beta \notin X_0$. This completes the proof of (5).

C. We will prove (1)–(3) in the case where γ is an $\mathcal{S}_{\Delta_0}$-strategy. Then we will sketch how to modify the proof to address the $\mathcal{S}_{\Gamma_0,\Delta_1}$- and $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$-strategies.

Since $\gamma \prec f$, there is a least stage at which γ is visited after its final initialization. At this stage, s_γ receives its final value, proving (1). By construction, we interrupt this stage so that no other strategy has the same parameter at any point during the construction.

Consider the sequence of good stages s, i.e., stages at which $A_{0,s} \subseteq A_0$. This sequence is infinite because $\{A_{0,s}\}_{s\in\omega}$ is a good approximation to A_0. We claim that the length of agreement $l_{\gamma,s}$ between $A_{0,s}$ and $\Delta_0(X_{0,s})$ is bounded on this sequence. Towards a contradiction, suppose not. We begin by showing that $\Delta_0(X_0) = A_0$: First, if $a \in A_0$, then pick a good stage s such that $a \in A_{0,s}$ and $l_{\gamma,s} > a$. Then $a \in \Delta_0(X_{0,s}) \subseteq \Delta_0(X_0)$. Conversely, if $a \in \Delta_0(X_0)$, then pick a good stage s such that $a \in \Delta_0(X_{0,s})$ and $l_{\gamma,s} > a$. Then $a \in A_{0,s} \subseteq A_0$ as desired.

Next, we shall show that $\Delta_0(X_0) = \Delta_0(X_0^{[<s_\gamma]} \cup \mathbb{N}^{[\geq s_\gamma]})$. The forward inclusion is trivial. To prove the backwards inclusion, consider $n \in \Delta_0(X_0^{[<s_\gamma]} \cup \mathbb{N}^{[\geq s_\gamma]})$. Let $\langle n, F\rangle$ be the oldest Δ_0-axiom putting n into $\Delta_0(X_0^{[<s_\gamma]} \cup \mathbb{N}^{[\geq s_\gamma]})$. Pick a good stage $s > s_\gamma$ such that $F^{[<s_\gamma]}$ is permanently in X_0 and $l_{\gamma,s} > n$. When we first visit γ at some stage $s' \geq s$, we enumerate a Φ_0-axiom $\langle\langle r, x\rangle, A_{0,s'}\rangle$ for each $\langle r, x\rangle \in F^{[\geq s_\gamma]}$. By the way that γ adjusts the approximation to A_0, we have $A_{0,s'} \subseteq A_{0,s} \subseteq A_0$. Therefore, $F \subseteq X_0$, implying that $n \in \Delta_0(X_0)$. This proves the reverse inclusion.

By inductive hypotheses B(3) and C(2), $X_0^{[<s_\gamma]}$ is computable, because the only strategies that contribute elements to $X_0^{[<s_\gamma]}$ after stage s_γ are backup strategies $\beta \prec \gamma$ or $\mathcal{S}_{\Delta_0'}$-strategies $\gamma' \prec \gamma$. The equality proved in the previous paragraph then implies that $\Delta_0(X_0)$, and hence A_0, is c.e., contradicting our assumption on A_0. Therefore $l_{\gamma,s}$ is bounded on the sequence of good stages s. Fix a bound l_γ. Using this, we will prove (1) and (2). First note that γ can only contribute numbers to X_0 at good stages, because every Φ_0-axiom enumerated by γ at stage s has use $A_{0,s}$. This means that only numbers $n \leq l_\gamma$ can cause γ to enumerate valid Φ_0-axioms.

If $n \notin \Delta_0(X_0^{[<s_\gamma]} \cup \mathbb{N}^{[\geq s_\gamma]})$, then $n \notin \Delta_{0,s}(X_{0,s}^{[<s_\gamma]} \cup \mathbb{N}^{[\geq s_\gamma]})$ at good stages, so n will not cause γ to enumerate any valid Φ_0-axioms. As for $n \in \Delta_0(X_0^{[<s_\gamma]} \cup \mathbb{N}^{[\geq s_\gamma]})$, the oldest valid Δ_0-axiom for n will appear to be the oldest valid axiom at all sufficiently large good stages, because we are working with a good approximation to A_0. Therefore, at all sufficiently large good stages, γ does not enumerate any valid Φ_0-axioms not already in Φ_0. This proves that γ enumerates only finitely many elements into X_0. Let $D = D_k$ be the set of these elements. To prove (1) and (2), it remains to show that $\langle k \rangle$ is the leftmost outcome that γ visits at infinitely many stages. Consider a stage s after which D lies permanently in X_0. At any γ-true stage $s' \geq s$, γ's current outcome $\langle k' \rangle$ satisfies $D_k \subseteq D_{k'}$. This implies that $\langle k \rangle \leq_L \langle k' \rangle$. By the adjustment that γ makes to the approximation of A_0, we know that γ is visited at infinitely many good stages. At all such stages (after stage s), γ will have outcome $\langle k \rangle$. This proves (1) and (2).

Finally, to prove (3), we show that there is $n \leq l_\gamma$ such that $A_0(n) \neq \Delta_0(X_0)(n)$. Assume that this is not the case. Fix a stage s such that for all $s' \geq s$ and each $n \leq l_\gamma$ in $A_0 \cap \Delta_0(X_0)$, we have $n \in A_{0,s'} \cap \Delta_{0,s'}(X_{0,s'})$. Consider any good stage $s' \geq s$ at which we visit γ. If $n \notin A_0 \cap \Delta_0(X_0)$, then we must have $n \notin A_{0,s'} \cup \Delta_{0,s'}(X_{0,s'})$. So $l_{\gamma,s'} > l_\gamma$, contradicting our choice of l_γ.

This proves (1)–(3) in the case where γ is an $\mathcal{S}_{\Delta_0}$-strategy. As for the $\mathcal{S}_{\Gamma_0,\Delta_1}$- and $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$-strategies, most of the above proof goes through if we simply replace A_0, Δ_0, and X_0 by the appropriate sets or operators. The only nontrivial change is in proving that $X_1^{[<s_\gamma]}$ is computable (for $\mathcal{S}_{\Gamma_0,\Delta_1}$) or $X_2^{[<s_\gamma]}$ is computable (for $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$), respectively. To prove the former, apply inductive hypotheses A(3), A(4), B(4), and C(2). Note that A(5) and B(5) are not relevant because any strategy above γ with such a true outcome works with a different version of X_1. To prove the latter, apply inductive hypothesis C(2). Any backup strategy above γ works with a different version of X_2, so we are not concerned with it. ☐

Lemma 3.3. *One of the following holds:*

(1) All requirements $\mathcal{R}_{\Gamma_0}$ and $\mathcal{S}_{\Delta_0}$ are satisfied.
(2) There is some operator Γ_0 such that all requirements $\mathcal{R}_{\Gamma_0,\Gamma_1}$ and $\mathcal{S}_{\Gamma_0,\Delta_1}$ are satisfied.
(3) There are operators Γ_0 and Γ_1 such that all requirements $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$ and $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$ are satisfied.

Proof. First, suppose there are infinitely many $\mathcal{R}_{\Gamma_0}$-strategies along the true path f. By construction of the tree of strategies, there must be infinitely many $\mathcal{S}_{\Delta_0}$-strategies along f as well. Thus all $\mathcal{S}_{\Delta_0}$-strategies are assigned to nodes on the true path and hence are satisfied. To show that $\mathcal{R}_{\Gamma_0}$ is satisfied, fix an $\mathcal{R}_{\Gamma_0}$-strategy $\alpha \prec f$. Let γ be the next $\mathcal{S}_{\Delta_0}$-strategy along f. By construction of the tree of strategies, γ's immediate predecessor is either an $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy with true outcome of the form $\langle 2k+1, \text{new} \rangle$, or a backup $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy with true outcome of the form $\langle 2l, \text{new} \rangle$. In both cases, the previous lemma shows that $\mathcal{R}_{\Gamma_0}$ is satisfied.

Second, if there are only finitely many $\mathcal{R}_{\Gamma_0}$-strategies along f, fix $\alpha \prec f$ and Γ_0 such that α is an $\mathcal{R}_{\Gamma_0}$-strategy and no immediate successor of α is an $\mathcal{R}_{\Gamma_0'}$-strategy. If there are infinitely many $\mathcal{R}_{\Gamma_0,\Gamma_1}$-strategies along f, we claim that all requirements $\mathcal{R}_{\Gamma_0,\Gamma_1}$ and $\mathcal{S}_{\Gamma_0,\Delta_1}$ are satisfied. By construction of the tree of strategies, there must be infinitely many $\mathcal{S}_{\Gamma_0,\Delta_1}$-strategies along f. Thus all $\mathcal{S}_{\Gamma_0,\Delta_1}$-strategies are assigned to nodes on the true path and hence are satisfied. To show that

$\mathcal{R}_{\Gamma_0,\Gamma_1}$ is satisfied, fix an $\mathcal{R}_{\Gamma_0,\Gamma_1}$-strategy $\beta \prec f$ extending α. Let δ be the next $\mathcal{S}_{\Gamma_0,\Delta_1}$-strategy along f. By construction of the tree of strategies, δ's immediate predecessor is either an $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy with true outcome of the form $\langle 2k\rangle$, or a backup $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy with true outcome of the form $\langle 2l, \text{off}\rangle$ or $\langle 2l+1\rangle$. In each case, the previous lemma shows that $\mathcal{R}_{\Gamma_0,\Gamma_1}$ is satisfied. This proves our claim.

Finally, suppose there are only finitely many $\mathcal{R}_{\Gamma_0,\Gamma_1}$-strategies along f. Fix $\alpha' \prec f$ extending α and Γ_1 such that α' is an $\mathcal{R}_{\Gamma_0,\Gamma_1}$-strategy and no immediate successor of α' is an $\mathcal{R}_{\Gamma_0,\Gamma_1'}$-strategy. (Such α' exists because α's only immediate successor, which must lie along f, is an $\mathcal{R}_{\Gamma_0,\Gamma_1}$-strategy.) Then no immediate successor of α along f can be a backup $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy, so every immediate successor of α' along f is either an $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$-strategy with true outcome of the form $\langle 2k+1, \text{old}\rangle$ or $\langle 2k+1, \text{off}\rangle$, or an $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$-strategy. By the previous lemma and the design of the tree, all requirements $\mathcal{R}_{\Gamma_0,\Gamma_1,\Gamma_2}$ and $\mathcal{S}_{\Gamma_0,\Gamma_1,\Delta_2}$ are satisfied. □

4. A Weak Ahmad Triple

In the previous section, we saw that an Ahmad triple is not possible in the Σ^0_2-enumeration degrees. In this section, we show a positive result, the existence of what we call a *weak Ahmad triple*.

Theorem 4.1. *There are pairwise incomparable Δ^0_2-enumeration degrees $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ such that*

(1) there are Δ^0_2-degrees $\mathbf{a}_{12} \not\leq \mathbf{a}_3$ and $\mathbf{a}_{23} \not\leq \mathbf{a}_1$ with $\mathbf{a}_{12} \vee \mathbf{a}_{23} = \mathbf{a}_2$; and
(2) for every Σ^0_2-degree $\mathbf{x} < \mathbf{a}_2$, we have that either $\mathbf{x} \leq \mathbf{a}_1$ or $\mathbf{x} \leq \mathbf{a}_3$.

We call such a triple of degrees $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ a weak Ahmad triple.

4.1. **Requirements.** We will construct Δ^0_2-sets A_1, A_3, A_{12}, and A_{23} satisfying the following list of requirements for every natural number e:

$$\mathcal{N}^{12}_e : A_{12} \neq \Theta_e(A_3 \oplus A_{23});$$
$$\mathcal{N}^{23}_e : A_{23} \neq \Theta_e(A_1 \oplus A_{12});$$
$$\mathcal{R}^1_e : X = \Phi_e(A_{12} \oplus A_{23}) \Rightarrow (\exists\Gamma)[X = \Gamma(A_1 \oplus A_{12})] \text{ or } (\exists\Delta)[A_{23} = \Delta(X)];$$
$$\mathcal{R}^3_e : Y = \Psi_e(A_{12} \oplus A_{23}) \Rightarrow (\exists\Gamma)[Y = \Gamma(A_3 \oplus A_{23})] \text{ or } (\exists\Delta)[A_{12} = \Delta(Y)].$$

Then $\mathbf{a}_1 = \deg_e(A_1 \oplus A_{12})$, $\mathbf{a}_2 = \deg_e(A_{12} \oplus A_{23})$, and $\mathbf{a}_3 = \deg_e(A_3 \oplus A_{23})$ clearly satisfy clauses (1) and (2) of the theorem: Indeed, if $Z \leq_e A_{12} \oplus A_{23}$, then Z will take the role of X for some requirement $\mathcal{R}^1_e$ and the role of Y for some requirement $\mathcal{R}^3_{e'}$. If either requirement is satisfied by the first disjunct, then we know that $Z \leq_e A_1 \oplus A_{12}$ or $Z \leq_e A_3 \oplus A_{23}$, respectively. Otherwise, we have that both $A_{12} \leq_e Z$ and $A_{23} \leq_e Z$, and so $A_{12} \oplus A_{23} \equiv_e Z$. Finally, by the definition of the degrees and by density, our requirements imply that $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are pairwise incomparable: Clearly, $\mathbf{a}_2 \not\leq \mathbf{a}_j$ for each $j \in \{1, 3\}$ by (1). Similarly, $\mathbf{a}_j \leq \mathbf{a}_{4-j}$ for some $j \in \{1, 3\}$ contradicts (1). If $\mathbf{a}_j < \mathbf{a}_2$ for some $j \in \{1, 3\}$, then fix $\mathbf{x}$ with $\mathbf{a}_j < \mathbf{x} < \mathbf{a}_2$, so $\mathbf{x} \leq \mathbf{a}_{4-j}$ by (2), and in particular $\mathbf{a}_j < \mathbf{a}_{4-j}$, contradicting the last sentence.

The reader might recognize the $\mathcal{R}$-requirements to be very similar to the requirements for making an Ahmad pair. Indeed, we can think of $\mathcal{R}^1_e$ as being the strategy making (A_{23}, A_1) an Ahmad pair "relative" to A_{12}, while $\mathcal{R}^3_e$ is the strategy making

(A_{12}, A_3) an Ahmad pair "relative" to A_{23}. In fact, as we will see, the addition of A_{12} to the oracle in $\mathcal{R}^1_e$ will not pose additional difficulties, and a similar Ahmad pair strategy can be used.

4.2. **Naive description of the strategies.** We start by briefly outlining naive strategies to satisfy each requirement and then discuss how to modify them in order to avoid conflicts between them. The construction shares many similarities with the usual construction of an Ahmad pair on a tree. We will use the usual setup of a tree of strategies ordered by priority.

An $\mathcal{N}$-requirement $\mathcal{N}^{12}$, say, is satisfied using a standard Friedberg-Muchnik strategy α: It picks a witness z_α and enumerates it into A_{12}. It waits to see if this witness will ever enter the set $\Theta_\alpha(A_3 \oplus A_{23})$. It wins if this never happens, provided that z_α remains in A_{12}. If $z_\alpha \in \Theta_\alpha(A_3 \oplus A_{23})$, we say that z_α is *realized*. In that case, α can win by extracting z_α from A_{12} and ensuring that z_α will remain in $\Theta_\alpha(A_3 \oplus A_{23})$ by imposing a finite restraint on the sets A_3 and A_{23}.

Of course, this puts $\mathcal{N}^{12}$-strategies and $\mathcal{N}^{23}$-strategies in conflict, and so already we see the need for a priority ordering between strategies. This is an easy obstacle to deal with. The complexity of our construction will only be revealed once we think about the $\mathcal{R}$-strategies as well.

Consider an $\mathcal{R}$-requirement $\mathcal{R}^3$, say, and its strategy β. Its initial goal is to build the operator Γ_β so that $\Gamma_\beta(A_3 \oplus A_{23}) = Y_\beta$, where $Y_\beta = \Psi_\beta(A_{12} \oplus A_{23})$. When activated at stage s, for every natural number $n < s$, it checks whether $n \in Y_\beta \setminus \Gamma_\beta(A_3 \oplus A_{23})$, and if so, it enumerates a new axiom $\langle n, D\rangle$ into Γ_β, where D contains a *fresh* number $a_3(n)$ that we enumerate into A_3. If, on the other hand, $n \in \Gamma_\beta(A_3 \oplus A_{23}) \setminus Y_\beta$, then the strategy invalidates all valid axioms for n by extracting from A_3 the corresponding marker $a_3(n)$. A new fresh value is then picked for $a_3(n)$ to use in the next Γ-axiom. (This allows us to keep our sets Δ^0_2.)

The $\mathcal{R}^1$- and $\mathcal{R}^3$-strategies do not interfere with each other: They modify the sets A_1 and A_3, respectively; however, $\mathcal{R}^1$-strategies do not involve the set A_3, and $\mathcal{R}^3$-strategies do not involve the set A_1. $\mathcal{R}$-strategies do not interfere with higher priority $\mathcal{N}$-strategies, as our priority tree will ensure that whenever an $\mathcal{N}$-strategy imposes a restraint, all lower-priority strategies are initialized and choose all of their parameters (specifically, numbers or witnesses they might later on like to extract from some set) as *fresh numbers*, larger than any number seen in the construction so far. An $\mathcal{N}^{12}$-strategy does not directly interfere with $\mathcal{R}^1$-strategies of higher priority: Its extraction of the witness z from A_{12} may cause some x to leave X. However, our design of the axioms that are enumerated into Γ_1 will guarantee in that case that x will also leave $\Gamma_1(A_1 \oplus A_{12})$, and hence the $\mathcal{R}^1$-strategy will not even have to act in response.

The situation is quite different, unfortunately, when one considers how the extraction of z_α from A_{12} by an $\mathcal{N}^{12}$-strategy α affects a higher-priority $\mathcal{R}^3$-strategy β: In that case as well, some y may be forced out of Y_β through this extraction, and so β will react by extracting $a_3(y)$ from A_3. This action, however, might directly interfere with the restraint that α is trying to impose on A_3 in order to keep $z_\alpha \in \Theta_\alpha(A_3 \oplus A_{23})$. In order to deal with this problem, we will need to modify our strategies.

4.3. **An $\mathcal{N}$-strategy working below a single $\mathcal{R}$-strategy.** For simplicity, we describe first the actions and outcomes of an $\mathcal{N}^{12}$-strategy α with one $\mathcal{R}^3$-strategy β

of higher priority working above it. In the formal construction below, we will deal with the more general case.

The $\mathcal{N}^{12}$-strategy α will start by defining a threshold d_α to be larger than any number mentioned so far. This threshold is meant to allow the $\mathcal{R}$-strategy β enough room to satisfy its requirement. If $\Gamma_\beta(A_3 \oplus A_{23})$ *changes its value* on a number $x \leq d_\alpha$, then α will be *restarted*. So α can assume that $\Gamma_\beta(A_3 \oplus A_{23})$ does not change below d_α and hence A_3 does not change on any $a_3(n)$ for $n \leq d_\alpha$. From this point on, α (temporarily) takes over control of the operator Γ_β: It defines a *killing point* k_α as a fresh number and enumerates it into A_3. It will require that the strategy β adds this killing point to the axiom that it enumerates into Γ_β for any $x \geq d_\alpha$. Whenever α is *restarted*, all parameters that α had at the previous stage will be canceled except for the threshold d_α and the killing point k_α. The killing point k_α is extracted from A_3 (thereby invalidating all axioms that were enumerated into Γ_β for any element $x \geq d_\alpha$). Finally, a new value for k_α will be set - a fresh number, not seen in the construction so far, and this fresh number is added to A_3. Note that (assuming $\Gamma_\beta(A_3 \oplus A_{23}) \upharpoonright d_\alpha$ changes finitely often, which is something we will prove in Lemma 4.4), this restart can happen at most finitely often.

The strategy α has three outcomes, *stop* $<_L \infty <_L$ *wait*. Once it has completed its initial setup (defining thresholds and killing points), the strategy picks a witness z_α as a fresh number and enumerates it into A_{12} as before. It waits to see if z_α becomes realized, and while waiting, the strategy has its rightmost outcome *wait*. Suppose z_α enters $\Theta_\alpha(A_3 \oplus A_{23})$ via an axiom $\langle z_\alpha, E_{z_\alpha} \oplus X\rangle$. The strategy first checks if it can extract z_α without causing Γ_β-correction that will extract from A_3 some number in E_{z_α}. At this point, the strategy is willing to sacrifice all other setups that it has made so far and enumerate into A_{12} and A_{23} as many numbers as necessary in order to guarantee this, with the exception of a certain pair of finite sets R^{12} and R^{23} consisting of witnesses selected by higher-priority strategies and kept out of their corresponding set. So if it is possible to add to A_{12} and A_{23} some finite set of numbers that make the extraction of z_α essentially harmless, then the strategy extracts z_α from A_{12} and takes outcome *stop*, where it will remain forever (unless initialized or restarted).

Suppose that this is not possible. In this case, the strategy α gives up on the witness z_α (at least for now) and decides to prove that β's requirement is satisfied by initiating the construction of Δ so that $\Delta(Y_\beta) = A_{12}$. We say that α *switches* β from Γ to Δ. The fact that an extraction of z_α from A_{12} causes the extraction of a finite set F_{z_α} from Y_β can now be turned into the first axiom in Δ. We would like to have a stronger relationship: $z_\alpha \in A_{12}$ if and only if $F_{z_\alpha} \subseteq Y_\beta$. Of course, currently there might be other numbers in A_{12} and A_{23} whose extraction may also cause F_{z_α} to leave Y_β. In order to remove their influence, we will *dump* into A_{12} and A_{23}, respectively, all numbers that were ever in A_{12} or A_{23} unless they belong to a higher-priority $\mathcal{N}$-strategy (i.e., are in R^{12} or R^{23}) or a lower-priority $\mathcal{N}$-strategy $\gamma \succeq \alpha\hat{\ }\infty$: We collect those elements in P_{12} and P_{23}. The act of dumping means that we enumerate these numbers into A_{12} or A_{23}, respectively, and never again allow them to leave these sets. (Note that any numbers controlled by strategies to the right of the outcome ∞ of α, or which will be replaced by new versions, can be dumped without harm.) We begin the construction of the operator Δ by enumerating the axiom $\langle z_\alpha, F_{z_\alpha}\rangle$, along with $\langle x, \emptyset\rangle$ for every element x that is dumped into A_{12}. At the end of this stage, we will visit the outcome ∞, but before we do so, we set things up for a new round: We extract the killing point k_α from A_3 and redefine it as a fresh number. We record the parameters $\langle z_\alpha, E_{z_\alpha}, F_{z_\alpha}\rangle$ in a list W that we keep track of and then redefine the value of z_α as a new fresh number. At the next visit, the strategy α will start a new attempt at diagonalization with this new witness, but it will keep an eye on the previous witness z and its parameters E_z and F_z. If it ever sees that by dumping into A_{12} and A_{23} elements outside of $R^{12} \cup R^{23}$, it can restore $E_z \subseteq A_3$ and extract z from A_{12} (and this extraction will not cause β to extract any element from E_z back out of A_3 to correct Γ_β), then the strategy α will do so and take outcome *stop* forever.

Below the outcome ∞ of α, we will have a duplicate strategy for every requirement of lower priority than β's requirement, including the one that α failed to satisfy. These strategies will not have to worry about the strategy β any longer as its requirement is satisfied in a different way. Specifically, an $\mathcal{N}^{12}$-strategy γ will be able to employ the original Friedberg-Muchnik strategy with a couple of modifications: The witnesses that γ can use have to be the witnesses that α formerly used for its definition of Δ; these will be collected in a *stream* S^{12} that α controls. Every time α has outcome ∞, it adds one more element to the stream S^{12}. The strategy γ will wait for the stream to contain a currently unused witness z before it can carry on. It will then proceed as usual; however, it will only trust A_3 below the current killing point of α. So z will be *realized* if $z \in \Theta_\gamma(A_3 \restriction k_\alpha \oplus A_{23})$ at the current stage. If ∞ is α's true outcome, i.e., if α visits this outcome infinitely often, then β's activity is pushed away by the extraction of the infinite unbounded sequence of killing points, thereby destroying Γ_β as discussed above, but giving γ enough room to faithfully realize its witness. Note that since $\{k_{\alpha,s}\}_{s<\omega}$ is unbounded, we will still have that if the witness is never realized then it does not belong to $\Theta_\gamma(A_3 \oplus A_{23})$. Finally, if γ succeeds in realizing a witness, then it extracts it from A_{12} and declares victory with outcome *stop*. This might have an unanticipated effect on the operator Δ that α is constructing. It is possible that an axiom for some number $z' > z$ was enumerated into Δ under the assumption that z remains in A_{12}. The extraction of z from A_{12} may cause $F_{z'}$ to not be a subset of Y_β, even though $z' \in A_{12}$. To prevent complications in the operator Δ, we will in this case dump (and thus remove from the stream) all elements in the stream S^{12} that entered the stream after z did.

A similar consideration has to be incorporated when an $\mathcal{N}^{23}$-strategy δ works below $\alpha\hat{\ }\infty$. For simplicity, we may assume that δ has no $\mathcal{R}^1$-strategy working above it. The strategy δ also operates a simple Friedberg-Muchnik strategy with the additional requirement that whenever it extracts a witness from A_{23}, it must dump into A_{12} (and thus remove from the stream) all witnesses that were put into the stream after δ defined its witness.

4.4. **Strategies, parameters and the tree.** We will describe the tree of strategies $T \subseteq (\{0, \mathit{wait}, \mathit{stop}\} \cup \{\infty_i \mid i < \omega\})^{<\omega}$ (which will be a finite-branching tree). We start with a priority ordering of all requirements of order type ω. To define the tree, we will make use of two other sets defined inductively as we move down the tree. We have a set M_σ of nodes $\prec \sigma$ that have been *killed*, and a list Q_σ of requirements that need to be assigned (or reassigned) to nodes $\succeq \sigma$. The root of the tree will be assigned the highest-priority requirement, and we set $M_\emptyset = \emptyset$ and $Q_\emptyset$ to consist of all requirements. Suppose that we have assigned a requirement to a node σ in the tree. If this strategy is an $\mathcal{R}^i_j$-strategy, say, then it has only one immediate successor $\sigma\hat{}0$. We set $Q_{\sigma\hat{}0} = Q_\sigma \setminus \{\mathcal{R}^i_j\}$ and assign to $\sigma\hat{}0$ the highest-priority requirement in the list $Q_{\sigma\hat{}0}$. We set $M_{\sigma\hat{}0} = M_\sigma$.

Suppose now that σ is assigned an $\mathcal{N}$-requirement, say, an $\mathcal{N}^{12}_e$-requirement. (The case $\mathcal{N}^{23}_e$ is similar, but now conflicting with $\mathcal{R}^1$.) Let $\delta_0 \prec \delta_1 \prec \cdots \prec \delta_n$ be all initial segments of σ to which we have assigned an $\mathcal{R}^3$-strategy and which are not in M_σ. (We call such δ_j *alive at* σ.) The strategy σ has $n+3$ immediate successors,

$$\sigma\hat{}stop <_L \sigma\hat{}\infty_0 <_L \sigma\hat{}\infty_1 <_L \cdots <_L \sigma\hat{}\infty_n <_L \sigma\hat{}wait.$$

We set $Q_{\sigma\hat{}wait} = Q_{\sigma\hat{}stop} = Q_\sigma \setminus \{\mathcal{N}^{12}_e\}$ and assign to each of the nodes $\sigma\hat{}wait$ and $\sigma\hat{}stop$ the highest-priority requirement in this list. We also set $M_{\sigma\hat{}wait} = M_{\sigma\hat{}stop} = M_\sigma$. For $i \le n$, we set $Q_{\sigma\hat{}\infty_i}$ to be Q_σ, along with the requirements associated with $\delta_{i+1}, \dots, \delta_n$, and σ. We assign to $\sigma\hat{}\infty_i$ the highest-priority requirement in $Q_{\sigma\hat{}\infty_i}$. We set $M_{\sigma\hat{}\infty_i} = M_\sigma \cup \{\delta_i, \delta_{i+1}, \dots, \delta_n\}$.

Lemma 4.2. *Let h be an infinite path in the tree of strategies T. Every requirement $\mathcal{Q}$ in our priority ordering is assigned to some node $\sigma \prec h$ such that for every δ with $\sigma \prec \delta \prec h$, $\mathcal{Q}$ is not in Q_δ.*

Proof. We prove this statement by induction on the priority ordering of all requirements. Suppose that the statement is true for all requirements of higher priority than the requirement $\mathcal{Q}$, and let $\sigma' \prec h$ be least such that no requirement of higher priority than $\mathcal{Q}$ enters Q_δ where $\sigma' \preceq \delta \prec h$, or is assigned to any such δ. It follows that $\mathcal{Q}$ is assigned to some longest $\sigma \preceq \sigma'$, and we have the following two cases:
Case 1: $\mathcal{Q} = \mathcal{R}^i_j$. Fix the $\mathcal{R}^i$-strategies $\delta_0 \prec \delta_1 \cdots \prec \delta_n \prec \sigma$ that are alive at σ. By our inductive hypothesis, if there is a least strategy $\delta \prec h$ extending σ that puts $\mathcal{Q}$ into the list Q_δ, then this strategy cannot kill δ_l for any $l \le n$ by our inductive assumptions. Thus only σ (and possibly strategies extending σ) are killed by δ, and so δ will be assigned the requirement $\mathcal{Q}$. Now, since no $\mathcal{R}^i$-requirement of higher priority than $\mathcal{Q}$ will switch from Γ to Δ along h beyond δ, the requirement $\mathcal{Q}$ cannot be added to $Q_{\delta'}$ for any δ' with $\delta \prec \delta' \prec h$; as a consequence, there is also a longest $\mathcal{Q}$-strategy along h.
Case 2: $\mathcal{Q} = \mathcal{N}^{ij}_e$, and by symmetry assume $ij = 12$. Fix as usual the $\mathcal{R}^3$-strategies $\delta_0 \prec \delta_1 \cdots \prec \delta_n \prec \sigma$ alive at σ. The strategy σ cannot put $\mathcal{Q}$ into the set $Q_{\sigma\hat{}o}$ unless σ switches the outcome of δ_n from Γ to Δ along h (since, by inductive hypothesis, no δ_l can be killed along h anymore). But then $\mathcal{Q}$ must be assigned to $\sigma\hat{}o$ and cannot be added to $Q_{\delta'}$ for any δ' with $\delta \prec \delta' \prec h$. □

An $\mathcal{R}$-strategy β has only its operator Γ_β as a parameter. Initially (and after every initialization), we set $\Gamma_\beta = \emptyset$. We will also refer to $\Phi_\beta(A_{12} \oplus A_{23})$ as X_β (in the case of an $\mathcal{R}^1$-strategy, and proceed similarly in the case of an $\mathcal{R}^3$-strategy.)

An $\mathcal{N}^{12}$-strategy α extending $\mathcal{R}^3$-strategies $\beta_0 \prec \beta_1 \prec \cdots \prec \beta_n$ still alive at α has a threshold d_α, a set of killing points $k^0_\alpha < \cdots < k^n_\alpha$, a witness z_α, a list of old witnesses W_α, each component of which contains a number z, an index $i \le n$, two finite sets E_z and F_z, and enumeration operators $\Delta_{\beta_0}, \dots, \Delta_{\beta_n}$. Initially (and after every initialization), all of these parameters are undefined or empty. An $\mathcal{N}^{23}$-strategy γ has the same list of parameters with respect to all $\mathcal{R}^1$-strategies that are still alive at γ.

In addition, every strategy has two streams S^{12} and S^{23}. These streams are determined by the predecessor of every strategy. Whenever a strategy is canceled, most of the elements in its stream (except for possibly one element) will be dumped into the corresponding set A_{12} or A_{23}.

4.5. **Construction.** In our construction, we will build a sequence $\{f_s\}_{s<\omega}$. Each f_s is a node of length s on our tree of strategies. Strategies visited at stage $s+1$ will modify the values of their parameters, as well as the approximations to the sets A_{12}, A_{23}, A_1 and A_3. Since our tree is finitely branching, there is a leftmost path of nodes visited at infinitely many stages, the *true path*. The intention is that for every requirement $\mathcal{Q}$, there is a strategy along the true path that satisfies $\mathcal{Q}$.

At stage 0, we set $A_{12} = A_{23} = A_1 = A_3 = \emptyset$, and all parameters of all strategies are in initial state (either undefined or empty). All streams are empty.

At stage $s+1$, we always start by visiting the root of the tree, namely, $f_{s+1} \upharpoonright 0 = \emptyset$. We add to the streams $S^{12}_\emptyset$ and $S^{23}_\emptyset$ of the root the element s. Suppose we have built $f_{s+1} \upharpoonright k$ along with its streams $S^{12}_{f_{s+1}\upharpoonright k}$ and $S^{23}_{f_{s+1}\upharpoonright k}$. If we have added a new number to S^{12}, then we denote it by n^{12}. If we have added a new number to S^{23}, then we denote it by n^{23}. If $k = s+1$, then we are done with the construction of f_{s+1}: We initialize all strategies $\delta > f_{s+1}$, dump their streams into A_{12} and A_{23}, respectively, empty their streams (i.e., set $S^{12}_\delta = S^{23}_\delta = \emptyset$), and move on to the next stage.

Otherwise, we have four cases depending on the requirement assigned to $f_{s+1} \upharpoonright k$:

Case 1: $f_{s+1} \upharpoonright k$ is an $\mathcal{R}^3_j$-strategy β: The strategy scans all $x \leq s$.

(a) If $x \in Y_\beta \setminus \Gamma_\beta(A_3 \oplus A_{23})$, then the strategy picks a fresh marker $a_3(x)$ and enumerates it into A_3. Then it defines K^x_β as the finite set of all β-killing points that belong to an $\mathcal{N}^{12}$-strategy $\alpha \succ \beta$ with current threshold $d_\alpha \leq x$. (Note that this is a finite set as there are currently only finitely many strategies that are not in initial state.) The strategy then enumerates into Γ_β the axiom $\langle x, (\{a_3(x)\} \cup K^x_\beta) \oplus A_{23} \upharpoonright s\rangle$.

(b) If $x \in \Gamma_\beta(A_3 \oplus A_{23}) \setminus Y_\beta$, then the strategy extracts from A_3 all markers $a_3(x)$ that are in some valid axiom for x in Γ_β.

Once the scan is over, the strategy defines the stream $S^{12}_{\beta\hat{}0}$ by adding to its previous value the number n^{12} and, similarly, the stream $S^{23}_{\beta\hat{}0}$ by adding to it the number n^{23} (if they exist). Then the strategy ends the substage with outcome 0.

Case 2: $f_{s+1} \upharpoonright k$ is an $\mathcal{R}^1_j$-strategy. This case is dealt with analogously to the previous case.

Case 3: $f_{s+1} \upharpoonright k$ is an $\mathcal{N}^{12}_e$-strategy α. Fix $\beta_0 \prec \beta_1 \prec \cdots \prec \beta_n \prec \alpha$ to be the $\mathcal{R}^3$-strategies alive at α. Let K_α be the greatest lower bound of the set of all killing points k^i_γ, where γ is an $\mathcal{N}^{12}$-strategy and $\alpha \succeq \gamma\hat{}\infty_i$. Let R^{12} and R^{23} be the sets of witnesses currently used by higher-priority $\mathcal{N}^{12}$- and $\mathcal{N}^{23}$-strategies, respectively. Similarly, the sets P^i_{12} and P^i_{23} consist of the current witnesses of $\mathcal{N}^{12}$- and $\mathcal{N}^{23}$-strategies extending outcome ∞_i.

If this is the first time that α is visited after initialization, then define the threshold d_α to be fresh and large and the killing points $k^0_\alpha < \cdots < k^n_\alpha$ as fresh numbers. Enumerate every killing point into A_3. Then go to the first case which applies:

(a) If $\Gamma_{\beta_i}(A_3 \oplus A_{23}) \upharpoonright d_\alpha$ has changed since we last visited α, then cancel z_α, W_α and Δ_{β_i} for every $i \leq n$. Extract the killing points k^j_α where $j \geq i$ from A_3. Define new values for these killing points and enumerate them into A_3. Initialize all strategies of lower priority than α. Dump the streams S^{12}_α and S^{23}_α into A_{12} and A_{23}, respectively. Set $S^{12}_{\alpha\hat{\ }o} = S^{23}_{\alpha\hat{\ }o} = \emptyset$ for every possible outcome o of α and end the substage with outcome *wait*.

(b) If the previous time when we visited α, it had outcome *stop*, and α has not been initialized since, then let the outcome again be *stop*. Define $S^{12}_{\alpha\hat{\ }stop}$ and $S^{23}_{\alpha\hat{\ }stop}$ by adding to them the number n^{12} and n^{23}, respectively (if they exist).

(c) Scan the list of old witnesses W_α. For each entry $\langle z, i_z, E_z, F_z\rangle$ such that z has not yet been dumped into A_{12}, check to see whether the number i_z can be decreased: Find the least j_z such that for every $i \geq j_z$, if we enumerate back into A_3 the set E_z, into A_{12} every number $x \neq z$ such that $x \leq s$ and $x \notin R^{12} \cup \bigcup_{j<j_z} P^j_{12}$, and into A_{23} every number $x \leq s$ that is not in the set $R^{23} \cup \bigcup_{j<j_z} P^j_{23}$, then β_i will not be forced to extract from A_3 any number in E_z during Γ_{β_i}-correction. (In this case, we say that z is Γ_{β_i}*-cleared* for $i \geq j_z$.) If there are no witnesses with $j_z \leq i_z$, then move on to step (d). Otherwise, among all witnesses with $j_z \leq i_z$, pick the one with least j_z, and among these the least z. Enumerate E_z into A_3, dump into A_{12} every number $x \neq z$ such that $x \leq s$ and $x \notin R^{12} \cup \bigcup_{j<j_z} P^j_{12}$, and into A_{23} every number $x \leq s$ such that $x \notin R^{23} \cup \bigcup_{j<j_z} P^j_{23}$. If $j_z = 0$, then set $z_\alpha = z$, extract it from A_{12} and end the substage with outcome *stop*. Set $S^{12}_{\alpha\hat{\ }o} = S^{23}_{\alpha\hat{\ }o} = \emptyset$ for every possible outcome o of α and dump the elements that were in each stream into A_{12} and A_{23}, respectively.
Otherwise, if $j_z > 0$, then set $i_z = j_z - 1$. Let $F_z \subseteq Y_{\beta_{i_z}}$ be the set such that $z \in A_{12}$ if and only if $F_z \subseteq Y_{\beta_{i_z}}$. Enumerate into $\Delta_{\beta_{i_z}}$ the axiom $\langle z, F_z\rangle$. Update the record in W_α to include $\langle z, i_z, E_z, F_z\rangle$. Extract the killing points k^j_α where $j \geq i_z$ from A_3, and end the substage with outcome ∞_{i_z}. We set the streams $S^{12}_{\alpha\hat{\ }o} = S^{23}_{\alpha\hat{\ }o} = \emptyset$ for every outcome o of α that is to the right of ∞_{i_z} and dump all elements that were in those streams except z into A_{12} and A_{23}, respectively. Dump n^{12} into A_{12} (if it exists). We leave $S^{12}_{\alpha\hat{\ }o}$ and $S^{23}_{\alpha\hat{\ }o}$ unchanged for every outcome o of α that is to the left of ∞_{i_z}. We update $S^{12}_{\alpha\hat{\ }\infty_{i_z}}$ by adding the number z to it, and $S^{23}_{\alpha\hat{\ }\infty_{i_z}}$ by adding the number n^{23} (if it exists) to it.

(d) If no current witness is selected and n^{12} exists and is larger than the current witness of every $\mathcal{N}$-strategy γ with $\gamma\hat{\ }wait \preceq \alpha$ or $\gamma\hat{\ }stop \preceq \alpha$, then define $z_\alpha = n^{12}$ and enumerate it into A_{12}. Otherwise (if n^{12} is defined but too small), dump n^{12} into A_{12}. End the substage with outcome *wait*, leaving all streams of immediate successors of α unchanged.

(e) If $z_\alpha \notin \Theta_\alpha(A_3 \oplus A_{23})$ or if $z_\alpha \in \Theta_\alpha(A_3 \oplus A_{23})$ but for every valid axiom $\langle z_\alpha, E \oplus D\rangle \in \Theta_\alpha$, we have that $\max(E) \geq K_\alpha$, then end the substage with outcome *wait*. Add n^{12} into $S^{12}_{\alpha\hat{\ }wait}$ and n^{23} into $S^{23}_{\alpha\hat{\ }wait}$

(if they exist). Leave all other streams of immediate successors of α unchanged.

(f) If $z_\alpha \in \Theta_\alpha(A_3 \oplus A_{23})$ via an axiom $\langle z_\alpha, E \oplus D\rangle$ with $\max(E) < K_\alpha$, then we add $\langle z_\alpha, n+1, E, \emptyset\rangle$ to W_α and go back to step (c). (Note that the current outcome will not be *wait*, since in step (c), we are guaranteed to find some witness $j_z \leq n+1$.) We also cancel the value of the current witness so that at the next visit, if α passes through steps (a), (b) and (c), then it will go to step (d) and select a new value of the witness.

Case 4: $f_{s+1} \upharpoonright k$ is an $\mathcal{N}^{23}_e$-strategy. This case is dealt with analogously to the previous case.

4.6. **Verification.** As anticipated, we have an infinite true path f of strategies on the tree consisting of the leftmost nodes visited at infinitely many stages. Our intention is to prove that nodes along this path satisfy their requirements. In order to prove that nodes on this path are initialized only finitely often, we must consider an $\mathcal{N}$-strategy on the true path and think about how many times it can be restarted, as that is the only reason, other than just visiting a node to the left of a strategy, that causes the initialization of strategies. We will prove, in Lemma 4.4, that for every $\mathcal{R}^3$-strategy $\beta \prec f$, the set $\Gamma_\beta(A_3 \oplus A_{23})$ is Δ^0_2, and similarly, for every $\mathcal{R}^1$-strategy $\beta' \prec f$, the set $\Gamma_{\beta'}(A_1 \oplus A_{12})$ is Δ^0_2. Throughout this proof, we will phrase various interactions between strategies for the pairs $\mathcal{R}^3$ and $\mathcal{N}^{12}$. We note that the relationship between $\mathcal{R}^1$ and $\mathcal{N}^{23}$ is symmetric.

First, we point out a technical fact about streams that will be useful in the rest of the proof.

Lemma 4.3. *If n enters a stream of a strategy δ at stage s and δ was last visited or initialized at stage s^-, then $n \geq s^-$ and n is larger than all previous elements of either stream of δ.*

Proof. The proof is an easy induction on the construction. The root is never initialized, and at stage $s+1$, $n^{12}_\emptyset = n^{23}_\emptyset = s$, which is the last time the root was visited.

Suppose the statement is true about δ. If δ adds n^{12} and n^{23} to the stream of its immediate successor, then the statement clearly follows by induction, as we cannot initialize an immediate successor of δ without either visiting or initializing δ. So suppose that δ is an $\mathcal{N}^{12}$-strategy, say, and δ adds a witness z to the stream of $\delta\hat{\ }\infty_i$, as that is the only other case. In that case, the witness z was defined after $\delta\hat{\ }\infty_i$ was last visited or initialized at stage t, as whenever we initialize or visit $\delta\hat{\ }\infty_i$, we initialize and empty the stream of all strategies $\delta\hat{\ }\infty_j$ where $j > i$. At the stage when z was defined, it was defined as n^{12}_δ, which by induction is greater than or equal to the previous stage when δ was visited, and hence greater than or equal to t, and larger than any element in the stream of $\delta\hat{\ }\infty_i$. □

Lemma 4.4. *Let $\beta \prec f$ be an $\mathcal{R}^3$-strategy. Suppose that α is an $\mathcal{N}^{12}$-strategy such that $\beta \prec \alpha \prec f$ and β is alive at α. Let d_α be a threshold of α. Then there is a stage s such that after stage s, the strategy β does not modify the set $\Gamma_\beta(A_3 \oplus A_{23})$ below the threshold d_α. Thus, in particular, if β is never killed along f then the set $\Gamma_\beta(A_3 \oplus A_{23})$ is Δ^0_2.*

Proof. We prove this theorem by induction on the priority of β and α. So towards a contradiction, suppose that the statement is false for a pair of strategies $\beta \prec \alpha \prec f$, and take the pair where α has highest priority. It follows from our choice of α that there is a least stage s_α such that after stage s_α, the strategy α (and hence β as well) is not initialized and never again changes the value of its threshold d_α. At stage s_α, the strategy α picks its killing points as fresh numbers, and hence they do not interfere with any axiom for any number $n \le s_\alpha$ in Γ_β. After stage s_α, every time $\Gamma_\beta(A_3 \oplus A_{23})$ changes on some number $n \le s_\alpha$, the strategy α is restarted. It chooses all parameters anew and initializes all lower-priority strategies. This means that if β enumerated a new axiom into Γ_β for some $n \le s_\alpha$ such that $n \in \Psi_\beta(A_{12} \oplus A_{23})$, then:

(1) This axiom cannot be invalidated by any strategy γ of equal or lower priority than α, as $\mathcal{N}^{12}$-strategies are initialized or restarted and hence pick their killing points as fresh numbers larger than $a_3(n)$ and have thresholds larger than n (hence none of their killing points will be included by β in this axiom). $\mathcal{N}^{23}$-strategies have to pick their witnesses anew, from fresh streams, as their streams are emptied at the current stage. Hence these witnesses will be larger than the current stage and will not be included in the A_{23}-portion of the axiom for n.
(2) The axiom that made n enter $\Psi_\beta(A_{12} \oplus A_{23})$ uses (by our convention) only numbers smaller than the current stage and hence it will not be invalidated by any strategy of equal or lower priority than α, as these strategies are initialized and their streams are emptied. By Lemma 4.3, their streams will contain only numbers larger than the current stage, from which they will pick their witnesses.

No strategy of higher priority than α can invalidate either of these axioms, either: A higher-priority strategy extracts from A_{12} or A_{23} only at stages at which it, for the first time after initialization, has its leftmost outcome *stop*, which by our choice of s_α must happen before stage s_α. Similarly, after stage s_α, higher-priority strategies extract killing points only associated with $\mathcal{R}$-strategies that they end up killing, and since β is alive at α, this cannot be β. Of course, all A_3-markers are different, so $\mathcal{R}^3$-strategies do not interfere with each other.

It only takes finitely many stages for any number $n \le d_\alpha$ that ever enters $\Psi_\beta(A_{12} \oplus A_{23})$ to enter $\Psi_\beta(A_{12} \oplus A_{23})$ permanently, and hence after that stage, β will not need to modify $\Gamma_\beta(A_3 \oplus A_{23})$ ever again. □

The lemma above has two easy but significant corollaries. The first corollary was already anticipated by us.

Corollary 4.5. *Every strategy along the true path is initialized at most finitely often.* □

The second corollary gives us the satisfaction of $\mathcal{R}$-requirements in one case.

Corollary 4.6. *If $\beta \prec f$ is an $\mathcal{R}^3$-strategy that is alive at every successor of β along the true path, then $\Gamma_\beta(A_3 \oplus A_{23}) = \Psi_\beta(A_{12} \oplus A_{23})$. The analogous statement for $\mathcal{R}^1$-strategies holds as well.*

Proof. Fix an $\mathcal{R}^3$-strategy $\beta \prec f$. Consider the $\mathcal{N}^{12}$-strategies $\beta \prec \alpha_0 \prec \alpha_1 \prec \dots \prec \alpha_n \prec \dots \prec f$ along the true path. The sequence $\{d_{\alpha_i}\}_{i<\omega}$ of the final values of their thresholds, attained at the first true stage after the corresponding strategy stops being initialized, is an unbounded increasing sequence. By Lemma 4.4, for every i, there is a stage s_i such that at all $t \geq s_i$, the strategy β does not modify $\Gamma_\beta(A_3 \oplus A_{23})$ on numbers less than d_{α_i}, and hence, by β's design, $\Gamma_\beta(A_3 \oplus A_{23}) \upharpoonright d_{\alpha_i} = \Psi_\beta(A_{12} \oplus A_{23}) \upharpoonright d_{\alpha_i}$ at all stages $t \geq s_i$. Furthermore, the proof of Lemma 4.4 actually gives us more: If $n \leq d_{\alpha_i}$ is in $\Psi_\beta(A_{12} \oplus A_{23})$ at any β-true stage after α_i selects its last threshold d_{α_i}, then $n \in \Gamma_\beta(A_3 \oplus A_{23}) \cap \Psi_\beta(A_{12} \oplus A_{23})$. This gives us immediately that $\Gamma_\beta(A_3 \oplus A_{23}) \upharpoonright d_{\alpha_i} = \Psi_\beta(A_{12} \oplus A_{23}) \upharpoonright d_{\alpha_i}$ and, by the unboundedness of $\{d_{\alpha_i}\}_{i<\omega}$, the fact that $\Gamma_\beta(A_3 \oplus A_{23}) = \Psi_\beta(A_{12} \oplus A_{23})$. □

We next concentrate on the $\mathcal{N}$-requirements. To prove that each is eventually satisfied, we will first show that once a number is Γ_i-cleared, it will remain Γ_i-cleared at all future stages.

Lemma 4.7. *Fix an $\mathcal{N}^{12}$-strategy $\alpha \prec f$ below $\mathcal{R}^3$-strategies $\beta_0 \prec \cdots \prec \beta_n$ alive at α. If α moves a witness z from $S^{12}_{\infty_j}$ to $S^{12}_{\infty_i}$, where $i < j$, at a stage after α's last initialization, then at any future stage, if z is extracted from A_{12} and E_z is enumerated back into A_3, the strategy β_j will not change A_3 to cause z to be extracted from $\Gamma_{\beta_j}(A_3 \oplus A_{23})$.*

Proof. Suppose that at stage s, the strategy α moves z from $S^{12}_{\infty_j}$ to $S^{12}_{\infty_i}$, where $i < j$. At this stage, it dumps into A_{12} all numbers $x \leq s$ such that $x \notin \{z\} \cup R^{12} \cup \bigcup_{i' \leq i} P^{i'}_{12}$, and into A_{23} all numbers $x \leq s$ such that $x \notin R^{23} \cup \bigcup_{i' \leq i} P^{i'}_{23}$. At stage s, we see that under these circumstances, the extraction of z from A_{12} will not cause any number y that is currently in $\Psi_{\beta_j}(A_{12} \oplus A_{23})$ and that has an A_3-marker $a_3(y) \in E_z$ to leave the set $\Psi_{\beta_j}(A_{12} \oplus A_{23})$. Assuming that $(R^{12} \cup \bigcup_{i' \leq i} P^{i'}_{12}) \cap A_{12}$ as seen at the current stage s remains a subset of A_{12}, and that $(R^{23} \cup \bigcup_{i' \leq i} P^{i'}_{23}) \cap A_{23}$ as seen at stage s remains a subset of A_{23}, this will be true at future stages as well, as every strategy to the right of $\alpha\hat{\ }\infty_i$ is initialized and will select its future witnesses from its stream that is currently empty and will by Lemma 4.3 in the future only have elements larger than any number mentioned before stage s, hence not be included in any axiom in Ψ_{β_j} valid at stage s.

The only potential problem is that some strategy γ might, at a stage t, extract from A_{12} a number y that is in the set $(R^{12} \cup \bigcup_{i' \leq i} P^{i'}_{12}) \cap A_{12}$ at stage s, or from A_{23} a number y that is in the set $(R^{23} \cup \bigcup_{i' \leq i} P^{i'}_{23}) \cap A_{23}$ at stage s.

First note that if $y \in R^{12} \cup R^{23}$, then y is the current witness of an $\mathcal{N}$-strategy $\delta \prec \alpha$. Since δ has no witness at stages when it has an infinite outcome, and its current witness is not in A_{12} if it has outcome *stop*, it follows that $\delta\hat{\ }\mathit{wait} \preceq \alpha$. This means that y remains the current witness of δ at all future stages and never enters another stream. No other strategy has access to it in order to extract it at stage t.

Suppose that $y \in P^{i'}_{12}$ for $i' \leq i$. Let $\delta \succeq \alpha\hat{\ }\infty_{i'}$ be the strategy with witness y at stage s. Let γ be the strategy that extracts y at stage t. Once again, γ cannot have higher priority than α, or else α would be initialized. It follows that $\gamma = \alpha$, or else $\gamma \succeq \alpha\hat{\ }\infty_k$, where $k \leq i'$, as all other strategies of lower priority than α are initialized at the stage when y was assigned to δ as a witness and thus have streams consisting of elements larger than y by Lemma 4.3. If $\gamma = \alpha$, then at stage t, the strategy α dumps all elements less than t that are not in $\{y\} \cup R^{12} \cup R^{23}$ into A_{12} and A_{23}, respectively, in particular z will be dumped. If $\gamma \succeq \alpha\hat{\ }\infty_k$ and $k \leq i$, then consider the stage r at which y entered the stream of $\alpha\hat{\ }\infty_k$. Since at stage s, the number y is already in the stream of $\alpha\hat{\ }\infty_{i'}$, and whenever a number switches streams, all streams associated with strategies to the right are dumped, we have that at stage r, the number z is dumped (z cannot already be in a stream to the left of or equal to $\alpha\hat{}\infty_k$ at stage r, or else y would have been dumped before stage r). □

Lemma 4.8. *Every $\mathcal{N}$-requirement is satisfied.*

Proof. By symmetry, fix an $\mathcal{N}^{12}_e$-requirement, say. By Lemma 4.2, there is a strategy α along the true path that is assigned $\mathcal{N}^{12}_e$ and such that no strategy σ extending α along the path f has $\mathcal{N}^{12}_e \in Q_\sigma$. It follows that $\alpha\hat{}\mathit{wait} \prec f$ or $\alpha\hat{}\mathit{stop} \prec f$.

Let s_α be the first stage after which α is not initialized. Recall that the number K_α is defined as the greatest lower bound of the set of all killing points k^i_γ, where γ is an $\mathcal{N}^{12}$-strategy and $\alpha \succeq \gamma\hat{}\infty_i$. Every time that α is visited, this number has a larger value than at the previous visit. Furthermore, no $\mathcal{N}$-strategy of higher priority than α modifies A_3 on numbers $x \leq K_\alpha$. This is because when such an $\mathcal{N}$-strategy γ such that $\gamma\hat{}\infty_i \preceq \alpha$ has outcome ∞_i, it extracts from A_3 all killing points k^j_γ, where $j \geq i$, and then it redefines them as fresh numbers. The strategy γ cannot extract any smaller killing point without initializing α.

So $A_3 \restriction K_\alpha$ can only be modified by an $\mathcal{R}$-strategy β above α. We note that such a strategy is necessarily alive at α. Indeed, if β is not alive above at α, then it is killed by a strategy γ such that $\gamma\hat{}\infty_i \preceq \alpha$ and β is γ's j-th $\mathcal{R}^3$-strategy, where $j \geq i$. Every time that γ has outcome ∞_i, it extracts its j-th killing point from A_3, thereby invalidating all axioms in Γ_β for numbers $x \geq d_\gamma$. After stage s_α, the strategy β does not modify $\Gamma_\beta(A_3 \oplus A_{23})$ on any number $x \leq d_\gamma$, hence if it sees a valid axiom for some x that needs to be made invalid, then $x > d_\gamma$. This axiom has marker $a_3(x) > k^j_\gamma \geq K_\alpha$.

If $\alpha\hat{}\mathit{wait} \prec f$, then let $s \geq s_\alpha$ be such that α has outcome *wait* at every stage $t \geq s$. After stage s, the strategy α will select its final witness z_α. It follows from the construction that z_α never enters $\Theta_\alpha(A_3 \oplus A_{23})$ with an axiom that does not use any numbers larger than K_α. Since the values of K_α at α-true stages form an unbounded sequence, it follows that every axiom we ever see for z_α in $\Theta_\alpha(A_3 \oplus A_{23})$ is invalid at infinitely many stages. Hence $z_\alpha \notin \Theta_\alpha(A_3 \oplus A_{23})$. As no strategy other than α can extract z_α from A_{12}, and z_α is enumerated into A_{12} at the stage when it is defined, it follows that $z_\alpha \in A_{12} \setminus \Theta_\alpha(A_3 \oplus A_{23})$.

If, on the other hand, $\alpha\hat{}\mathit{stop} \prec f$, then there is a stage $s \geq s_\alpha$ such that α has outcome *stop* for the first time at stage s. At this stage, α has found a witness z that is cleared by all higher priority $\mathcal{R}^3$-strategies that are alive at α. Note that $\max(E_z) < K_\alpha$, and by Lemma 4.7, no strategy β that is alive at α will extract a marker from A_3 that is in E_z. Every number that was in A_{23} when the axiom for z in Θ_α was found is dumped into A_{23} at stage s (only elements that are in R^{23} are preserved; however, they cannot have been in R^{23} when z was realized and not be in R^{23} later unless α is initialized). It follows that $z \in \Theta_\alpha(A_3 \oplus A_{23}) \setminus A_{12}$. □

The final lemma that we present handles the case when an $\mathcal{R}$-requirement is satisfied by its backup strategy, which completes the proof.

Lemma 4.9. *Let α be an $\mathcal{N}^{12}$-strategy below the $\mathcal{R}^3$-strategies $\beta_0 \prec \cdots \prec \beta_n \prec \alpha$ alive at α. If $\alpha\hat{}\infty_i \prec f$, then $A_{12} \leq_e Y_{\beta_i}$. (Of course, a symmetric result holds for $\mathcal{N}^{23}$-strategies below $\mathcal{R}^1$-strategies.)*

Proof. Let s_{α_i} be a stage such that $\alpha\hat{}\infty_i$ is not initialized after stage s_{α_i}. After this stage, the set R^{12}_α does not change. We prove that if $x \geq s_\alpha$ is an element that is never dumped into A_{12}, then either $x \in R^{12}_\alpha$ or $x \in A_{12}$ if and only if $x \in \Delta_{\beta_i}(Y_{\beta_i})$.

Fix x and suppose that x is never dumped into A_{12} and that $x \notin R^{12}_\alpha$. Fix the least witness z of α such that $z \geq x$. Consider the stage at which z is realized and enters one of the streams of α's immediate successors. If $z > x$, then at that stage, x would be dumped into A_{12}. So suppose that $z = x$. Now consider the next stage at which α has outcome ∞_i. At that stage, some element z' enters the stream $S_{\alpha\hat{}\infty_i}$, and by our choice of s_{α_i}, we know that at this stage, x is in some stream $S_{\alpha\hat{}\infty_j}$ where $j \geq i$. If $x \neq z'$, then x would be dumped into A_{12} at this stage, so suppose that $x = z'$ and x enters the stream $S_{\alpha\hat{}\infty_i}$ at stage s. Then at this stage, we add an axiom $\langle x, F_x\rangle$ into Δ_{β_i} for x, where F_x is such that under the current circumstances at stage s, we have that $x \in A_{12}$ if and only if $F_x \subseteq Y_{\beta_i}$. As in the proof of Lemma 4.7, if some number that is in the set $(R^{12} \cup \bigcup_{j\leq i} P^j_{12}) \cap A_{12}$ at stage s is extracted from A_{12} or a number that is in the set $(R^{23}\cup\bigcup_{j\leq i} P^j_{23})\cap A_{23}$ at stage s is extracted from A_{23}, then x is dumped into A_{12}. So suppose that neither of these ever happens. Then clearly, if $x \in A_{12}$, then $F_x \subseteq Y_{\beta_i}$, as all strategies of lower priority than $\alpha\hat{}\infty_i$ are initialized at stage s. If at any stage $t > s$, we visit α and notice that $F_x \subseteq Y_{\beta_i}$ even if $x \notin A_{12}$, then the strategy α will move x to a smaller stream and initialize $\alpha\hat{}\infty_i$, contrary to our assumptions. □

Putting Corollary 4.6 and Lemma 4.9 together, we conclude the following

Corollary 4.10. *Every $\mathcal{R}$-requirement is satisfied.*

Proof. Fix an $\mathcal{R}^3_e$-requirement, say. By Lemma 4.2, let β be the longest $\mathcal{R}^3_e$-strategy along f. If β is not switched from Γ to Δ by any strategy extending β along f, then by Corollary 4.6, $\mathcal{R}^3_e$ is satisfied. If $\beta \prec \alpha\hat{}\infty_i \prec f$ and α switches β from Γ to Δ, then β is α's i-th live $\mathcal{R}^3$-strategy. By Lemma 4.9, $A_{12} \leq_e Y_\beta$, and hence $\mathcal{R}^3_e$ is once again satisfied. □

This concludes the proof of Theorem 4.1.

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