On the L^p Aleksandrov problem for negative p

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Abstract

Huang, Lutwak, Yang, and Zhang introduced the L^p integral curvature and posed the corresponding L^p Aleksandrov problem, the natural L^p extension of the classical integral curvature and Aleksandrov problem respectively. The problem asks about the existence of a convex body with prescribed L^p integral curvature measure. For the case of given even measures, the question will be solved for $p \in (-1, 0)$. Furthermore, a sufficient measure concentration condition will be provided for the case of $p \leq -1$, again provided that the given measure is even.

1 Introduction

The integral curvature measure, also known as the first curvature measure introduced by Federer [19] for sets of positive reach, was first defined by Aleksandrov [2]. The corresponding classical Aleksandrov problem asks about constructing a convex body (a compact convex set with nonempty interior) with prescribed integral curvature. This is a type of Minkowski problem, a famous and influential question in Brunn-Minkowski theory with ties to many other fields in mathematics, including functional analysis, differential geometry, and nonlinear partial differential equations. The Minkowski problem asks about the existence of a convex body with predetermined surface area measure. The problem was solved by Minkowski himself [44] for the polytope case using a variational argument. Aleksandrov [1] and Fenchel-Jessen [20] also provided complete solutions with similar variational approaches. Information on the regularity of the solution can be found in Cheng-Yau [16], Caffarelli [11], Nirenberg [46], Pogorelov [51], and Trudinger-Wang [54].

When the given measure has density $f : S^{n-1} \to (0, \infty)$, the Aleksandrov problem amounts to solving the following Monge-Ampère-type partial differential equation

$$\det\left(\nabla_{ij}^2 h + h\delta_{ij}\right) = \frac{f \cdot \left(|\nabla h|^2 + h^2\right)^{\frac{n}{2}}}{h},$$

where ∇h is the gradient of h, $\nabla_{ij}^2 h$ is the Hessian of h, and δ_{ij} is the identity matrix with respect to an orthonormal frame on S^{n-1}. The classical Aleksandrov problem was solved by Aleksandrov [2] first for polytopes and then generalized via an approximation argument. Oliker [49] gave an alternate solution to the existence question using mass transport for the polytope case and then extended it to more general shapes with the same approximation approach. Huang-Lutwak-Yang-Zhang [30] provided another solution to the existence problem for even measures with a direct variational proof. Results on the regularity of the solutions can be found in Guan-Li [24], Oliker [47, 48], and Li-Sheng-Wang [35] for more general problems.

Huang-Lutwak-Yang-Zhang [30] introduced the L^p integral curvature measure and solved the corresponding L^p Aleksandrov problem completely for $p > 0$ and obtained partial results for $p < 0$. Zhao [58] proved existence of a solution for origin-symmetric discrete measures and $p \in (-1, 0)$. The goals of this paper are to extend the two aforementioned results by: (i) completely proving existence of a solution for origin-symmetric measures and $p \in (-1, 0)$ and (ii) obtaining a sufficient measure concentration condition for existence in the case of origin-symmetric measures and $p \leq -1$. To discuss in detail the L^p Aleksandrov problem, we will first provide some background information on L^p Brunn-Minkowski theory.

We define a convex body in $\mathbb{R}^n$ to be a compact convex n-dimensional set. The set $\mathcal{K}_o^n$ is the class of convex bodies in $\mathbb{R}^n$ with the origin in their interiors. For a compact convex subset $K \subset \mathbb{R}^n$, we define its support function $h_K : \mathbb{R}^n \to \mathbb{R}$ by $h_K(y) = \max\{x \cdot y : \ x \in K\}$. The support function is homogeneous of degree 1 and convex, and it uniquely defines a convex body.

The L^p Brunn-Minkowski theory is an extension of the classical Brunn-Minkowski theory. It was first initiated by Firey, when he introduced the Minkowski-Firey L^p-combination, $K +_p t \cdot L \in \mathcal{K}_o^n$, for $K, L \in \mathcal{K}_o^n$ and small nonzero $t \in \mathbb{R}$. This L^p-sum is defined by

$$K +_p t \cdot L = \left\{ x \in \mathbb{R}^n : \ x \cdot v \leq (h_K^p(v) + t h_L^p(v))^{\frac{1}{p}} \ \ \forall v \in S^{n-1} \right\}.$$

However, the theory was brought to life and actively researched when Lutwak [39] discovered the concept of the L^p surface area measure, which subsequently led to the L^p Minkowski problem. The L^p surface area measure $S_p(K, \cdot)$ for each $K \in \mathcal{K}_o^n$ can be defined as the Borel measure on S^{n-1} that satisfies the following equation

$$\frac{d}{dt} V(K +_p t \cdot L) \bigg|_{t=0} = \frac{1}{p} \int_{S^{n-1}} h_L(u)^p \ dS_p(K, u),$$

for all $L \in \mathcal{K}_o^n$.

The corresponding L^p Minkowski problem asks: *For all $p \in \mathbb{R}$, what are the necessary and sufficient conditions on a given Borel measure μ on S^{n-1} so that there exists a $K \in \mathcal{K}_o^n$ with $\mu = S_p(K, \cdot)$?* Notice that when $p = 1$, the L^p surface area measure becomes the classical surface area measure, and hence the L^1 Minkowski problem is the same as the classical Minkowski problem. See [44, 1, 20, 16, 11, 46, 51, 54]. For the $p > 1$ case, a solution for even measures was provided by Lutwak [39], and then Chou-Wang [17] proved existence for general measures. More results on this can be found in [15, 14, 32]. Another important but largely unsolved case of the L^p Minkowski problem is the case of $p = 0$, where the $S_0(K, \cdot)$ is the cone volume measure of K with total measure equal to $V(K)$. For more information on this measure, see [5, 7, 27, 37, 38, 45, 50, 55]. The L^0 Minkowski problem is also known as the logarithmic Minkowski problem. The existence for even measures was proved in Böröczky-Lutwak-Yang-Zhang [8], and progress for more general cases can be found in [4, 53, 59, 13]. The $p = -n$ case, also largely unsolved, is called the centro-affine Minkowski problem first posed by Chou-Wang [17]. Results on this case can be found in [33, 36, 60]. The solution to the L^p Minkowski problem, together with L^p-affine isoperimetric inequalities of convex bodies [40, 26], are critical to establishing sharp affine Sobolev inequalities of functions which are stronger than the classical Euclidean Sobolev inequalities [18, 25, 41, 42, 56, 34].

Now the L^p Aleksandrov problem arose in Huang-Lutwak-Yang-Zhang [29], where the concept of dual curvature measures $\widetilde{C}_q(K,\cdot)$ and related variational formulas were discovered. The dual Minkowski problem, which analogously asks about the existence and uniqueness of a convex body with predetermined dual curvature measure, interpolates between some previously disconnected questions mentioned earlier. The $q=0$ case of the dual Minkowski problem becomes the classical Aleksandrov problem, and the $q=n$ case is the logarithmic Minkowski problem. For progress on this problem, see [3, 6, 9, 12, 22, 28, 31, 35, 43, 57, 58]. Another extension of the classical Aleksandrov problem was discovered recently by Böröczky-Lutwak-Yang-Zhang-Zhao [10], when they first posed the Gauss image problem. In the case of one of the submeasures being the spherical Lebesgue measure, the Gauss image problem becomes the classical Aleksandrov problem.

The L^p integral curvature comes from a variational formula in Huang-Lutwak-Yang-Zhang [30] for a certain entropy integral. For each $K \in \mathcal{K}_o^n$, define its entropy $\mathcal{E}$ by

$$\mathcal{E}(K) = -\int_{S^{n-1}} \log h_K(v)\ dv.$$

Then for each $p \neq 0$ and $K \in \mathcal{K}_o^n$, we define (see Huang-Lutwak-Yang-Zhang [30]) the L^p integral curvature measure, $J_p(K,\cdot)$, of K as the Borel measure on S^{n-1} that satisfies

$$\frac{d}{dt}\mathcal{E}(K \hat{+}_p t\cdot L)\bigg|_{t=0} = \frac{1}{p}\int_{S^{n-1}} \rho_L(u)^{-p}\ dJ_p(K,u)$$

for all $L \in \mathcal{K}_o^n$, where the L^p harmonic combination is defined as $K \hat{+}_p t\cdot L = (K^* +_p t\cdot L^*)^*$, and K^* is the polar of K. It turns out that the L^p integral curvature measure is related to the classical integral curvature measure in the following way

$$dJ_p(K,\cdot) = \rho_K^p\ dJ(K,\cdot).$$

Observe that when $p=0$, $J_0(K,\cdot) = J(K,\cdot)$, the classical case.

The L^p Aleksandrov problem asks about the existence of a convex body with predetermined L^p integral curvature. More specifically:

Problem. Fix a nonzero $p \in \mathbb{R}$. What are the necessary and sufficient conditions on a given Borel measure μ on S^{n-1} so that there exists a convex body $K \in \mathcal{K}_o^n$ with $\mu = J_p(K,\cdot)$?

It was shown that if μ has density f, this problem amounts to solving the Monge-Ampère-type partial differential equation

$$\det\left(\nabla_{ij}^2 h + h\delta_{ij}\right) = \frac{f\cdot\left(|\nabla h|^2 + h^2\right)^{\frac{n}{2}}}{h^{1-p}},$$

where ∇h is the gradient of h (unknown function), $\nabla_{ij}^2 h$ is the Hessian of h, and δ_{ij} is the identity matrix with respect to an orthonormal frame on S^{n-1}. Huang-Lutwak-Yang-Zhang [30] completely solved existence for when $p > 0$.

Theorem (Huang-Lutwak-Yang-Zhang 2018)**.** *Suppose $p \in (0, \infty)$ and μ is a finite Borel measure on S^{n-1}. Then there exists $K \in \mathcal{K}_o^n$ such that μ is the L^p integral curvature measure of K if and only if μ is not concentrated on any great subsphere.*

Furthermore, Huang-Lutwak-Yang-Zhang [30] solved existence under some strong conditions for the origin symmetric case and when $p < 0$. More specifically,

Theorem (Huang-Lutwak-Yang-Zhang 2018)**.** *Suppose $p \in (-\infty, 0)$ and μ is a finite, even, nonzero Borel measure on S^{n-1} that vanishes on all great subspheres of S^{n-1}. Then there exists $K \in \mathcal{K}_o^n$ such that μ is the L^p integral curvature measure of K.*

Note that this result excludes many shapes, including polytopes. Zhao [58] addressed part of this gap by proving existence for origin symmetric polytopes and $p \in (-1, 0)$.

Theorem (Zhao 2019)**.** *Suppose $p \in (-1, 0)$ and μ is a finite, even, discrete, nonzero Borel measure on S^{n-1} . Then there exists an origin symmetric polytope $K \in \mathcal{K}_o^n$ such that μ is the L^p integral curvature measure of K if and only if μ is not concentrated on any great subsphere of S^{n-1}.*

One goal of this paper is to extend the result by Zhao [58] by completely proving existence for the origin-symmetric case of the L^p Aleksandrov problem, for $p \in (-1, 0)$.

Theorem 1. *Let $-1 < p < 0$ and μ be a nonzero even finite Borel measure on S^{n-1}. Then there exists an origin symmetric convex body $K \in \mathbb{R}^n$ such that $\mu = J_p(K, \cdot)$ if and only if μ is not concentrated on any lower dimensional subspace.*

For the remaining negative index cases ($p \leq -1$), we will weaken the assumptions on the $p < 0$ existence result by Huang-Lutwak-Yang-Zhang [30] from completely no concentration to requiring some measure concentration condition. More specifically, we will show the following:

Theorem 2. *Let $p \leq -1$ and μ be a nonzero even finite Borel measure on S^{n-1}. Suppose, on all great subspheres $\xi \subset S^{n-1}$, that*

$$\frac{\mu(\xi)}{\mu(S^{n-1})} \leq C(n)^p,$$

where $C(n) = \exp\left[\frac{1}{2}\left(\psi\left(\frac{n}{2}\right) - \psi\left(\frac{1}{2}\right)\right)\right]$ is a constant depending only on n, and ψ is the digamma function. Then there exists a $K \in \mathcal{K}_e^n$ such that $\mu = J_p(K, \cdot)$.

The approach for both of these results will be variational. We will first convert the existence question into an optimization problem and then proceed to prove the existence of an optimizer.

2 Background Information

In this section, we will provide some basic background on the theory of convex bodies and explain the L^p integral curvature measure and the L^p Aleksandrov problem in more detail. For further information and reference, see Schneider [52] and Gruber [23].

2.1 General Information on Convex Bodies

We let S^{n-1} denote the unit sphere centered about the origin o in $\mathbb{R}^n$, and B^n denote the unit ball centered about o in $\mathbb{R}^n$. We will call the volume of the unit ball ω_n and the surface area of the unit ball $o_n = n\omega_n$. We define a convex body in $\mathbb{R}^n$ as a compact convex set with non-empty interior. The set $\mathcal{K}_o^n$ is the class of convex bodies in $\mathbb{R}^n$ with the origin in their interiors, and $\mathcal{K}_e^n$ is the class of origin-symmetric convex bodies in $\mathbb{R}^n$. Define $C(S^{n-1})$ to be the set of continuous functions on the unit sphere, and $C^+(S^{n-1})$ to be the set of strictly positive continuous functions on the unit sphere. Let K and L be compact convex sets, and define their Minkowski sum by

$$K + L = \{k + l : \ k \in K, \ l \in L\}$$

Suppose $K \subset \mathbb{R}^n$ is a compact, convex set. Define its support function $h_K : \mathbb{R}^n \to \mathbb{R}$ by

$$h_K(y) = \max\{x \cdot y : \ x \in K\},$$

where $x \cdot y$ is the standard inner product of x and y in $\mathbb{R}^n$. Note that the support function is convex and homogeneous of degree 1. Furthermore, the support function uniquely defines a convex body. Now suppose K contains the origin in its interior, and define the radial function of K, $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$, by

$$\rho_K(x) = \max\{\lambda : \ \lambda x \in K\}.$$

Every compact star-shaped (with respect to the origin) set is uniquely determined by its radial function. We say that a sequence of convex bodies $\{K_i\}$ converges to a compact convex set $K \in \mathbb{R}^n$ if

$$\sup\left\{|h_{K_i}(v) - h_K(v)| : \ v \in S^{n-1}\right\} \to 0, \ \text{as } i \to \infty,$$

or when $o \in \text{int } K$ (the interior of K), $\{K_i\}$ converges to K if

$$\sup\left\{|\rho_{K_i}(u) - \rho_K(u)| : \ u \in S^{n-1}\right\} \to 0, \ \text{as } i \to \infty.$$

Also note that the boundary of K, which we denote by ∂K, is related to the radial function in the following way,

$$\partial K = \left\{\rho_K(u)u : \ u \in S^{n-1}\right\}.$$

It follows from these definitions that the support function and radial function are related by

$$h_K(v) = \max_{u \in S^{n-1}} (u \cdot v)\, \rho_K(u), \ \forall v \in S^{n-1}$$

and

$$\frac{1}{\rho_K(u)} = \max_{v \in S^{n-1}} \frac{(u \cdot v)}{h_K(v)}, \ \forall u \in S^{n-1}.$$

Define the polar body $K^* \in \mathcal{K}_o^n$ of $K \in \mathcal{K}_o^n$ by

$$K^* = \{x \in \mathbb{R}^n : \ x \cdot y \leq 1, \ \forall y \in K\}.$$

Thus, from the definition of the polar body and the relationship between the support and radial functions, we can deduce the following two useful equalities

$$\rho_K(x) = \frac{1}{h_{K^*}(x)}$$

and

$$h_K(x) = \frac{1}{\rho_{K^*}(x)},$$

for all $x \in \mathbb{R}^n$. Furthermore, observe that $(K^*)^* = K$.

For every function $f \in C^+(S^{n-1})$, define the Wulff shape (also known as the Aleksandrov body) associated with f by

$$[f] = \left\{x \in \mathbb{R}^n : \ x \cdot v \leq f(v) \ \forall v \in S^{n-1}\right\} \in \mathcal{K}_o^n.$$

One can check that for each $f \in C^+(S^{n-1})$, we have that $h_{[f]} \leq f$, and $[h_K] = K$ for every $K \in \mathcal{K}_o^n$. Now let $\rho \in C^+(\Omega)$, where $\Omega \subseteq S^{n-1}$ is closed. Then $\{\rho(u)u : \ \forall u \in \Omega\}$ is a compact set. Now define $\langle \rho \rangle$ by

$$\langle \rho \rangle = \operatorname{conv} \{\rho(u)u : \ u \in \Omega\} \in \mathcal{K}_o^n.$$

So if $K \in \mathcal{K}_o^n$, we observe that $\langle \rho_K \rangle = K$.

To begin stating some necessary facts in L^p Brunn-Minkowski theory, we first start by defining the L^p Minkowski combination. For every nonzero $p \in \mathbb{R}$, $K, L \in \mathcal{K}_o^n$, and $a, b \geq 0$, define $a \cdot K +_p b \cdot L \in \mathcal{K}_o^n$ by

$$a \cdot K +_p b \cdot L = [(a \cdot h_K^p + b \cdot h_L^p)^{\frac{1}{p}}],$$

and for the case of $p = 0$, define

$$a \cdot K +_o b \cdot L = [h_K^a h_L^b].$$

Now we use the polar body to define the L^p harmonic combination of $K, L \in \mathcal{K}_o^n$. Again, fix $p \in \mathbb{R}$ and let $a, b \geq 0$. Define the L^p harmonic combination $a \cdot K \hat{+}_p b \cdot L \in \mathcal{K}_o^n$ by

$$a \cdot K \hat{+}_p b \cdot L = (a \cdot K^* +_p b \cdot L^*)^*.$$

Define the supporting hyperplane with outer unit normal $v \in S^{n-1}$ tangent to $K \in \mathcal{K}_o^n$ by

$$H_K(v) = \{x \in K : \ x \cdot v = h_K(v)\}.$$

For $\sigma \subset \partial K$, we define the spherical image $\boldsymbol{\nu}_K(\sigma) \subseteq S^{n-1}$ to be the following set

$$\boldsymbol{\nu}_K(\sigma) = \left\{v \in S^{n-1} : \ x \in H_K(v) \text{ for some } x \in \sigma\right\}.$$

And for $\eta \subset S^{n-1}$, define the reverse spherical image $\boldsymbol{x}_K(\eta) \subseteq \partial K$ to be

$$\boldsymbol{x}_K(\eta) = \{x \in \partial K : \ x \in H_K(v) \text{ for some } v \in \eta\}.$$

Call the set of all $x \in \partial K$ for which $\boldsymbol{\nu}_K(\{x\})$ consists of more than one element, $\sigma_K \subset \partial K$. Then one can show that $\mathcal{H}^{n-1}(\sigma_K) = 0$ (see Schneider [52]), where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. The spherical image map is defined as follows: $\nu_K : \partial K \backslash \sigma_K \to S^{n-1}$ with

$$\nu_K(x) = \boldsymbol{\nu}_K(\{x\}) \ \forall x \in \partial K \backslash \sigma_K.$$

Note that this function is continuous (see Schneider [52]). We similarly define $\eta_K \subset S^{n-1}$ to be the set of all $v \in S^{n-1}$ such that $\boldsymbol{x}_K(\{x\})$ contains more than one element. We also have that η_K is $\mathcal{H}^{n-1}$ measure 0 (see Schneider [52]). Define the reverse spherical image map $x_K : S^{n-1} \backslash \eta_K \to \partial K$ by

$$x_K(v) = \boldsymbol{x}_K(\{v\}) \ \forall v \in S^{n-1} \backslash \eta_K.$$

This function is also continuous (see Schneider [52]). The radial map for $K \in \mathcal{K}_o^n$, $r_K : S^{n-1} \to \partial K$ is defined by

$$r_K(u) = \rho_K(u)u \ \forall u \in S^{n-1}.$$

For $\omega \subset S^{n-1}$, the radial Gauss image of ω is a subset of S^{n-1} defined by

$$\boldsymbol{\alpha}_K(\omega) = \boldsymbol{\nu}_K(r_K(\omega)).$$

Rewriting this for $u \in S^{n-1}$,

$$\boldsymbol{\alpha}_K(\{u\}) = \left\{v \in S^{n-1} : \ r_K(u) \in H(K, v)\right\}.$$

Observe that $\boldsymbol{\alpha}_K(\cdot)$ is positive homogeneous of degree 0. Now define the reverse radial Gauss image of $\eta \subset S^{n-1}$ to be

$$\boldsymbol{\alpha}^*_K(\eta) = r_K^{-1}(\boldsymbol{x}_K(\eta)),$$

which for $u \in S^{n-1}$ can can be rewritten as

$$\boldsymbol{\alpha}^*_K(\eta) = \left\{u \in S^{n-1} : \ v \cdot (\rho_K(u)u) = h_K(v), \text{ for some } v \in \eta\right\}.$$

2.2 L^p Integral Curvature

We define a geometric measure on a convex body first introduced by Aleksandrov [2]. The integral curvature of $K \in \mathcal{K}_o^n$ is a Borel measure on S^{n-1} defined as

$$J(K, \omega) = \mathcal{H}^{n-1}(\boldsymbol{\alpha}_K(\omega)),$$

for every Borel $\omega \subset S^{n-1}$. Observe that the total integral curvature of any $K \in \mathcal{K}_o^n$ is the surface area of S^{n-1}, i.e. $J(K, S^{n-1}) = o_n$, and it is positively homogeneous of degree 0. The integral curvature is also known as the first curvature measure, where the latter family of measures was first introduced by Federer [19] for sets of positive reach. The classical Aleksandrov problem asks: *What are the necessary and sufficient conditions on a Borel measure μ on S^{n-1} so that $\mu = J(K,\cdot)$ for some $K \in \mathcal{K}_o^n$?*

Aleksandrov [2] solved the problem completely, and his theorem with the necessary and sufficient conditions is stated below.

Theorem (Aleksandrov 1942)**.** *Suppose μ is a finite Borel measure on S^{n-1}. Then $\mu = J(K,\cdot)$ for some $K \in \mathcal{K}_o^n$ if and only if $|\mu| = o_n$ and*

$$\mu(\omega) < \mathcal{H}^{n-1}(S^{n-1}\backslash\omega^*) \tag{2.1}$$

for each convex $\omega \subset S^{n-1}$ and where $\omega^* = \{v \in S^{n-1} : \ v \cdot u \le 0 \ \forall u \in \omega\}$.

Inequality (2.1) is also known as the Aleksandrov condition. Aleksandrov also proved that the solution, if it exists, is unique up to scaling. The integral curvature measure for $K \in \mathcal{K}_o^n$ can also be defined as the unique Borel measure on S^{n-1} that satisfies the following variational formula:

$$\frac{d}{dt}\mathcal{E}(K \hat{+}_0 t\cdot L)\bigg|_{t=0} = -\int_{S^{n-1}} \log(\rho_L(u))\ dJ(K,u),$$

for each $L \in \mathcal{K}_o^n$. Subsequently, Huang-Lutwak-Yang-Zhang [30] defined the L^p integral curvature by the following variational formula. For each $K \in \mathcal{K}_o^n$ and $p \neq 0$, the L^p integral curvature of K is the unique Borel measure on S^{n-1} that satisfies

$$\frac{d}{dt}\mathcal{E}(K \hat{+}_p t\cdot L)\bigg|_{t=0} = \frac{1}{p}\int_{S^{n-1}} \rho_L(u)^{-p}\ dJ_p(K,u),$$

for each $L \in \mathcal{K}_o^n$. It turns out the L^p integral curvature measure is related to the classical integral curvature measure in the following way

$$dJ_p(K,\cdot) = \rho_K^p\ dJ(K,\cdot).$$

Huang-Lutwak-Yang-Zhang [30] first posed the L^p Aleksandrov problem:

Problem. Fix $p \in \mathbb{R}$. What are the necessary and sufficient conditions on a given Borel measure μ on S^{n-1} so that there exists a convex body $K \in \mathcal{K}_o^n$ with $\mu = J_p(K,\cdot)$?

The case of $p = 0$ is the classical Aleksandrov problem. The $p > 0$ case for existence was completely proved in Huang-Lutwak-Yang-Zhang [30]. Additionally, Huang-Lutwak-Yang-Zhang [30] solved existence under the strong condition of no measure concentration on any great subspheres, for the origin symmetric case of $p < 0$. Zhao [58] extended the result for $p \in (-1,0)$ to existence for origin symmetric polytopes. We aim to extend the result by Zhao [58] for $p \in (-1,0)$ by proving existence for any even measure not completely concentrated on any great subsphere. We will also weaken the assumptions of the $p < 0$ existence result by Huang-Lutwak-Yang-Zhang [30] to a nonzero measure concentration condition requirement for origin-symmetric measures.

3 Optimization Problems

The purpose of this paper is to prove existence for the L^p Aleksandrov problem for $p < 0$ cases and even Borel measures μ on S^{n-1}. To solve the $-1 < p < 0$ case for even measures, we consider the following optimization problem. For any nonzero, finite Borel measure μ on S^{n-1} and $p \neq 0$, define

$$\widetilde{\Phi}_p(Q) = \exp\left(\frac{1}{o_n}\mathcal{E}(Q)\right) \cdot \left(\int_{S^{n-1}} \rho_Q^{-p}(u)\, d\mu(u)\right)^{-\frac{1}{p}}.$$

We consider the maximization problem $\sup\left\{\widetilde{\Phi}_p(Q) :\ Q \in \mathcal{K}_e^n\right\}$.

We first present a variational formula of the entropy integral. This result was proved in Corollary 4.6 of [29].

Lemma 1 (Huang-Lutwak-Yang-Zhang 2016)**.** *Let $\Omega \subset S^{n-1}$ be a closed set that is not contained in any closed hemisphere, $g : \Omega \to \mathbb{R}$ be continuous, $K \in \mathcal{K}_o^n$, and $\delta > 0$ be sufficiently small. Define $\rho_t(v) = \rho_K(v) \cdot e^{tg(v)}$ for each $t \in (-\delta, \delta)$. Then*

$$\frac{d}{dt}\left(\mathcal{E}(\langle \rho_t \rangle)\right)\Big|_{t=0} = -\int_{S^{n-1}} g(u)\, dJ(K, u). \tag{3.1}$$

We now apply this result to prove that the solution to the maximization problem is also a solution to the L^p Aleksandrov problem.

Lemma 2. *Suppose $p \neq 0$ and μ is an even Borel measure on S^{n-1}. If $K \in \mathcal{K}_e^n$ satisfies*

$$o_n = \int_{S^{n-1}} \rho_K^{-p}(u)\, d\mu(u) \tag{3.2}$$

and $\widetilde{\Phi}_p(K) = \sup\left\{\widetilde{\Phi}_p(Q) :\ Q \in \mathcal{K}_e^n\right\}$, then $\mu = J_p(K, \cdot)$.

Proof. Define the functional $\widetilde{\Psi}_p :\ C^+(S^{n-1}) \to \mathbb{R}$ by

$$\widetilde{\Psi}_p(f) = \exp\left(\frac{1}{o_n}\mathcal{E}(\langle f \rangle)\right) \cdot \left(\int_{S^{n-1}} f^{-p}(u)\, d\mu(u)\right)^{-\frac{1}{p}}.$$

Then $\widetilde{\Psi}_p(f)$ is homogeneous of degree 0.

Notice that $\rho_{\langle f \rangle} \geq f$, and so

$$\left(\int_{S^{n-1}} \rho_{\langle f \rangle}^{-p}(v)\, d\mu(v)\right)^{-\frac{1}{p}} \geq \left(\int_{S^{n-1}} f^{-p}(u)\, d\mu(u)\right)^{-\frac{1}{p}}.$$

This combined with the fact that $\langle \rho_{\langle f \rangle} \rangle = \langle f \rangle$ implies $\widetilde{\Psi}_p(f) \leq \widetilde{\Psi}_p(\rho_{\langle f \rangle})$. Thus, in searching for the function that maximizes $\widetilde{\Psi}_p$, we can restrict to considering just radial functions of sets in $\mathcal{K}_e^n$. Thus,

$$\sup\left\{\widetilde{\Psi}_p(f) :\ f \in\ C^+(S^{n-1})\right\} = \sup\left\{\widetilde{\Phi}_p(Q) :\ Q \in \mathcal{K}_e^n\right\}.$$

For $g \in C^+(S^{n-1})$ and $t \in (-\delta, \delta)$ where $\delta > 0$ is sufficiently small, define again

$$\rho_t(v) = \rho_K(v) \cdot e^{tg(v)}.$$

Thus, by (3.1), the definition of $\widetilde{\Psi}_p$, and (3.4), we have that

$$\begin{aligned}
0 &= \frac{d}{dt}\left(\widetilde{\Psi}_p(\rho_t)\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(\exp\left(\frac{1}{o_n}\mathcal{E}(\langle\rho_t\rangle)\right)\cdot\left(\int_{S^{n-1}}\rho_t^{-p}(u)\,d\mu(u)\right)^{-\frac{1}{p}}\right)\Bigg|_{t=0} \\
&= -\frac{1}{o_n}\left(\int_{S^{n-1}} g(u)\,dJ(K,u)\right)\cdot\exp\left(\frac{1}{o_n}\mathcal{E}(\langle\rho_K\rangle)\right)\cdot\left(\int_{S^{n-1}}\rho_K^{-p}(u)\,d\mu(u)\right)^{-\frac{1}{p}} \\
&+ \left(\int_{S^{n-1}}\rho_K^{-p}(u)\cdot g(u)\,d\mu\,(u)\right)\cdot\exp\left(\frac{1}{o_n}\mathcal{E}(\langle\rho_K\rangle)\right)\cdot\left(\int_{S^{n-1}}\rho_K^{-p}(u)\,d\mu(u)\right)^{-\frac{1}{p}-1}.
\end{aligned}$$

Then by (3.2), $\rho_K^{-p}\,d\mu = dJ(K,\cdot)$. Therefore, $\mu = \rho_K^p\;J(K,\cdot) = J_p(K,\cdot)$. □

For the $p < -1$ case, we consider the following optimization problem. For any nonzero, finite Borel measure μ on S^{n-1} and $p \neq 0$, define

$$\Phi_p(Q) = -\frac{1}{p}\log\left(\int_{S^{n-1}}\rho_Q^{-p}(v)\,d\mu(v)\right) + \frac{1}{o_n}\mathcal{E}(Q),$$

where o_n is the surface area of S^{n-1}. We prove that the optimizer of this functional is also a solution to the L^p Aleksandrov problem.

Lemma 3. *Let $p, q \neq 0$ and μ an even Borel measure on S^{n-1}. If $K \in \mathcal{K}_e^n$ satisfies*

$$o_n = \int_{S^{n-1}}\rho_K^{-p}(u)\,d\mu(u) \tag{3.3}$$

and the maximization problem $\Phi_p(K) = \sup\{\Phi_p(Q):\ Q \in \mathcal{K}_e^n\}$, then $\mu = J_p(K,\cdot)$.

Proof. Define the functional $\Psi_p:\ C^+(S^{n-1}) \to \mathbb{R}$ by

$$\Psi_p(f) = -\frac{1}{p}\log\left(\int_{S^{n-1}} f^{-p}(u)\,d\mu(u)\right) + \frac{1}{o_n}\mathcal{E}(\langle f\rangle).$$

Then $\Psi_p(f)$ is homogeneous of degree 0.

Notice that $\rho_{\langle f\rangle} \geq f$, and so

$$-\frac{1}{p}\log\left(\int_{S^{n-1}}\rho_{\langle f\rangle}^{-p}(v)\,d\mu(v)\right) \geq -\frac{1}{p}\log\left(\int_{S^{n-1}} f^{-p}(u)\,d\mu(u)\right).$$

This combined with the fact that $\langle\rho_{\langle f\rangle}\rangle = \langle f\rangle$ implies $\Psi_p(f) \leq \Psi_p(\rho_{\langle f\rangle})$. Thus, in searching for the function that maximizes Ψ_p, we can restrict to considering just radial functions of sets in $\mathcal{K}_e^n$. Thus,

$$\sup\left\{\Psi_p(f): \ f\in \ C^+(S^{n-1})\right\}=\sup\left\{\Phi_p(Q): \ Q\in\mathcal{K}_e^n\right\}.$$

For $g\in C^+(S^{n-1})$ and $t\in(-\delta,\delta)$ where $\delta>0$ is sufficiently small, define

$$\rho_t(v)=\rho_K(v)e^{tg(v)} \tag{3.4}$$

Thus, by (3.1), the definition of Ψ_p, and (3.4),

$$\begin{aligned}
0 &= \frac{d}{dt}\left(\Psi_p(\rho_t)\right)\Big|_{t=0}\\
&= \frac{d}{dt}\left(\frac{1}{o_n}\mathcal{E}(\langle\rho_t\rangle)-\frac{1}{p}\log\left(\int_{S^{n-1}}\rho_t^{-p}(u)\,d\mu(u)\right)\right)\Big|_{t=0}\\
&= -\frac{1}{o_n}\int_{S^{n-1}} g(u)\,dJ(K,u)\\
&+\left(\int_{S^{n-1}}\rho_K^{-p}(u)\cdot g(u)\,d\mu(u)\right)\cdot\left(\int_{S^{n-1}}\rho_K^{-p}(u)\,d\mu(u)\right)^{-1}.
\end{aligned}$$

Then by (3.3), $\rho_K^{-p}\,d\mu=dJ(K,\cdot)$. Therefore, $\mu=\rho_K^{p}\ dJ(K,\cdot)=dJ_p(K,\cdot)$. □

4 Existence of Solutions

We will prove the existence of a solution to the optimization problem in Lemma 2 for $p\in(-1,0)$, under the assumption of origin symmetry. First, we begin with showing the continuity of the optimization function $\widetilde{\Phi}_p$.

Lemma 4. *Suppose μ is an even Borel measure on S^{n-1} that is not concentrated on any great subspheres. Let $Q_l\in\mathcal{K}_e^n$ be a sequence of origin-symmetric convex bodies such that Q_l converges to Q^0, an origin-symmetric compact convex set, in the Hausdorff metric. Then, for $p<0$, after possibly taking a subsequence $\{Q_{l_k}\}$,*

$$\lim_{k\to\infty}\widetilde{\Phi}_p(Q_{l_k})=\widetilde{\Phi}_p(Q^0).$$

Proof. Since $\widetilde{\Phi}_p$ is homogeneous of degree 0, we rescale so that $\max_{u\in S^{n-1}}\rho_{Q_l}(u)=1$. For each $l\in\mathbb{N}$, define $u_l\in S^{n-1}$ to be the unit vector that satisfies $\rho_{Q_l}(u_l)=1$. Since Q_l is origin-symmetric, observe that

$$|u_l\cdot v|\le h_{Q_l}(v)\le 1,$$

for each $v\in S^{n-1}$. Then

$$\left|\log\left(h_{Q_l}(v)\right)\right|\le-\log|u_l\cdot v|. \tag{4.1}$$

We apply the generalized dominated convergence theorem to prove the convergence of $\mathcal{E}(Q_l)$ to $\mathcal{E}(Q^0)$. The generalized dominated convergence theorem is stated as follows (see

Folland [21]): *Let* $\{f_l\}_{l=1}^{\infty}$ *be a sequence of measurable functions and* $\{g_l\}_{l=1}^{\infty}$ *be a sequence of nonnegative measurable functions. Suppose that* $|f_l| \leq |g_l|$ *for all* l, $\{f_l\}_{l=1}^{\infty}$ *converges pointwise almost everywhere to* f *and* $\{g_l\}_{l=1}^{\infty}$ *converges pointwise almost everywhere to* g, *and* $\lim_{l\to\infty}\int_{S^{n-1}} g_l \to \int_{S^{n-1}} g < \infty$. *Then* $\lim_{l\to\infty}\int_{S^{n-1}} f_l = \int_{S^{n-1}} f$.

From Inequality (4.1), we let $g_l(v) = -\log|u_l \cdot v|$ and $f_l(v) = \log(h_{Q_l}(v))$ for all $v \in S^{n-1}$. Recall the assumption that $Q_l \to Q^0$ in the Hausdorff metric, and so $h_{Q_l} \to h_{Q^0}$ pointwise. Since $h_{Q^0} > 0$ almost everywhere, we have that $\log h_{Q_l} \to \log h_{Q^0}$ almost everywhere. The assumption that $Q_l \to Q^0$ in the Hausdorff metric and Bolzano-Weierstrass implies that there exists a subsequence such that $\rho_{Q_{l_k}}(u_{l_k})u_{l_k} \to \rho_{Q^0}(u_0)u_0$, which means that $u_{l_k} \to u_0$, and so $|u_{l_k} \cdot v| \to |u_0 \cdot v|$ for all $v \in S^{n-1}$. Since $|u_0 \cdot v| > 0$ almost everywhere, $\log|u_{l_k} \cdot v| \to \log|u_0 \cdot v|$ almost everywhere. Since S^{n-1} is rotationally symmetric, we have that $\lim_{k\to\infty}\int_{S^{n-1}} \log|u_{l_k} \cdot v| \ dv = \int_{S^{n-1}} \log|u_0 \cdot v| \ dv < \infty$. Hence, by the generalized dominated convergence theorem,

$$\lim_{k\to\infty}\int_{S^{n-1}} \log\left(h_{Q_{l_k}}(u)\right)\, du = \int_{S^{n-1}} \log\left(h_{Q^0}(u)\right)\, du. \tag{4.2}$$

To prove convergence of the radial function terms, notice that since $Q_l \to Q^0$ in the Hausdorff metric, $\rho_{Q_l} \to \rho_{Q^0}$ pointwise. Since $p < 0$, this implies that $\rho_{Q_l}^{-p} \to \rho_{Q^0}^{-p}$. Recall also the assumption that $\rho_{Q_l} \leq 1$. Applying again the fact that $p < 0$, we have $\rho_{Q_l}^{-p} \leq 1$. Hence, by the dominated convergence theorem,

$$\lim_{l\to\infty}\int_{S^{n-1}} \rho_{Q_l}^{-p}(u)\, d\mu(u) = \int_{S^{n-1}} \rho_{Q^0}^{-p}(u)\, d\mu(u). \tag{4.3}$$

Equations 4.2 and 4.3 with the definition of $\widetilde{\Phi}_p$ imply that

$$\lim_{k\to\infty}\widetilde{\Phi}_p(Q_{l_k}) = \widetilde{\Phi}_p(Q^0).$$

□

Lemma 5. *Let* L *be a* k*-dimensional origin-symmetric convex body that spans a subspace called* ξ, *and consider a corresponding cylindrical thickening for* $t > 0$, $L^t = L + tB^{n-k}$ *where* B^{n-k} *is the unit ball in the complementary subspace of* ξ. *Then there exists an* $r > 0$ *such that*

$$\frac{\exp\left(\frac{1}{o_n}\mathcal{E}\left(\left(rB^k\right)^t\right)\right)}{\exp\left(\frac{1}{o_n}\mathcal{E}\left(rB^k\right)\right)} \leq \frac{\exp\left(\frac{1}{o_n}\mathcal{E}\left(L^t\right)\right)}{\exp\left(\frac{1}{o_n}\mathcal{E}\left(L\right)\right)}. \tag{4.4}$$

Proof. Observe that Inequality (4.4) is equivalent to

$$\left(\frac{\exp\left(\frac{1}{o_n}\mathcal{E}\left(L\right)\right)}{\exp\left(\frac{1}{o_n}\mathcal{E}\left(rB^k\right)\right)}\right) \leq \left(\frac{\exp\left(\frac{1}{o_n}\mathcal{E}\left(L^t\right)\right)}{\exp\left(\frac{1}{o_n}\mathcal{E}\left(\left(rB^k\right)^t\right)\right)}\right).$$

Or after rewriting the right side of the above inequality,

$$\exp\left(\frac{-1}{o_n}\left[\int_{S^{n-1}}\log\left(\left(h_L+th_{B^{n-k}}\right)(u)\right)\,du-\int_{S^{n-1}}\log\left(\left(h_{rB^k}+th_{B^{n-k}}\right)(u)\right)\right]\,du\right). \quad (4.5)$$

We now compare the derivatives of the two integrals

$$\frac{d}{dt}\int_{S^{n-1}}\log\left(\left(h_L+th_{B^{n-k}}\right)(u)\right)\,du=\int_{S^{n-1}}\left(h_L+th_{B^{n-k}}\right)^{-1}(u)\cdot h_{B^{n-k}}(u)\,du \quad (4.6)$$

and

$$\frac{d}{dt}\int_{S^{n-1}}\log\left(\left(h_{rB^k}+th_{B^{n-k}}\right)(u)\right)\,du=\int_{S^{n-1}}\left(h_{rB^k}+th_{B^{n-k}}\right)^{-1}(u)\cdot h_{B^{n-k}}(u)\,du. \quad (4.7)$$

If $rB^k\subset L$, we notice that from (4.6) and (4.7), the exponent of (4.5) increases at every point $t>0$. Therefore, we have the desired inequality. □

Let L be a k-dimensional origin-symmetric convex body that spans a k-dimensional subspace called ξ. Henceforth, we denote the ratio of the support function terms of $\widetilde{\Phi}_p(L^t)$ and $\widetilde{\Phi}_p(L)$ by Δ_1,

$$\begin{aligned}\Delta_1(L,t)&:=\frac{\exp\left(\frac{1}{o_n}\mathcal{E}(L^t)\right)}{\exp\left(\frac{1}{o_n}\mathcal{E}(L)\right)}\\&=\exp\left(\frac{-1}{o_n}\int_{S^{n-1}}\left(\log\left(h_L+th_{B^{n-k}}\right)-\log\left(h_L(u)\right)\right)\,du\right)\end{aligned}$$

and the ratio of the radial function terms of $\widetilde{\Phi}_p(L^t)$ and $\widetilde{\Phi}_p(L)$ by Δ_2. For L and μ such that $\int_{S^{n-1}}\rho_L^{-p}(u)\,d\mu(u)\neq 0$, define

$$\begin{aligned}\Delta_2(L,t)&:=\left(\frac{\int_{S^{n-1}}\rho_{L^t}^{-p}(u)\,d\mu(u)}{\int_{S^{n-1}}\rho_L^{-p}(u)\,d\mu(u)}\right)^{-\frac{1}{p}}\\&=\left(\frac{\int_{S^{n-1}}\rho_{L+tB^{n-k}}^{-p}(u)\,d\mu(u)}{\int_{S^{n-1}}\rho_L^{-p}(u)\,d\mu(u)}\right)^{-\frac{1}{p}}.\end{aligned}$$

We will now obtain an estimate for the differential of $\Delta_1(L,t)$ in the limit as $t\to 0^+$.

Lemma 6. *Let $-1<p<0$, μ be a nonzero even finite Borel measure on S^{n-1} that is not completely concentrated on any lower dimensional subspace, and $L\in\mathcal{K}_e^n$ span a $k<n$ dimensional subspace called ξ, and $t>0$. Then there exists a function $\tilde{\Delta}_1(L,t)$ such that $\Delta_1(L,t)\geq\tilde{\Delta}_1(L,t)$ with*

$$\lim_{t\to 0^+}\tilde{\Delta}_1(L,t)=1 \quad (4.8)$$

and

$$\lim_{t\to 0^+}\frac{d}{dt}\tilde{\Delta}_1(L,t)\gtrsim\log(t). \quad (4.9)$$

Proof. By Lemma 5, there exists an $R > 0$ such that $\Delta_1(L,t) \geq \Delta_1(RB^k, t)$. We will denote $\tilde{\Delta}_1(L,t) \coloneqq \Delta_1(RB^k, t)$. Then we have

$$\begin{aligned}
\tilde{\Delta}_1(L,t) &= \exp\left(-\frac{1}{o_n}\left[\int_0^{\frac{\pi}{2}} \left(\cos^{k-1}\phi\right)\left(\sin^{n-k-1}\phi\right)\log\left(R\cos\phi + t\sin\phi\right)\, d\phi \right.\right.\\
&\qquad \left.\left. - \int_0^{\frac{\pi}{2}} \left(\cos^{k-1}\phi\right)\left(\sin^{n-k-1}\phi\right)\log\left(R\cos\phi\right)\, d\phi\right]\right)\\
&= \exp\left(-\frac{1}{o_n}\int_0^{\frac{\pi}{2}} \left(\cos^{k-1}\phi\right)\left(\sin^{n-k-1}\phi\right)\log\left(\frac{(R^2+t^2)^{\frac{1}{2}}}{R}\cdot\frac{\cos(\phi-\Delta)}{\cos\phi}\right)\, d\phi\right),
\end{aligned}$$

where we use the identity $(R^2+t^2)^{\frac{1}{2}}\cdot\cos(\phi-\Delta) = R\cdot\cos(\phi) + t\cdot\sin(\phi)$, with $\Delta = \arctan\left(\frac{t}{R}\right)$ and o_n is the surface area of S^{n-1}. So,

$$\begin{aligned}
\lim_{t\to 0^+}\tilde{\Delta}_1(L,t) = \lim_{t\to 0^+}\exp&\left(-\frac{1}{o_n}\log\left(\frac{(R^2+t^2)^{\frac{1}{2}}}{R}\right)\int_0^{\frac{\pi}{2}}\left(\cos^{k-1}\phi\right)\left(\sin^{n-k-1}\phi\right)\, d\phi\right.\\
&\left. + \frac{1}{o_n}\int_0^{\frac{\pi}{2}}\left(\cos^{k-1}\phi\right)\left(\sin^{n-k-1}\phi\right)\log\left(\frac{\cos\phi}{\cos(\phi-\Delta)}\right)\, d\phi\right).
\end{aligned} \tag{4.10}$$

We focus on the limit of the second integral. Notice that we can bound the absolute value of the integrand

$$\begin{aligned}
\left|\left(\cos^{k-1}\phi\right)\left(\sin^{n-k-1}\phi\right)\log\left(\frac{\cos\phi}{\cos(\phi-\Delta)}\right)\right| &\leq \left|\log\left(\frac{\cos\phi}{\cos(\phi-\Delta)}\right)\right|\\
&= \left|\log\left(\cos\phi\right) - \log\left(\cos(\phi-\Delta)\right)\right|\\
&\leq \left|\log\left(\cos\phi\right)\right| + \left|\log\left(\cos(\phi-\Delta)\right)\right|\\
&= -\log\left(\cos\phi\right) - \log\left(\cos(\phi-\Delta)\right).
\end{aligned} \tag{4.11}$$

For sufficiently small $t > 0$, we observe that $-\log\left(\cos(\phi-\Delta)\right) \leq -\log\left(\cos\phi\right) + 1$ for each $\phi \in \left[0, \frac{\pi}{2}\right]$. Hence, the left side of (4.11) is bounded by an integrable function in the interval $\left[0, \frac{\pi}{2}\right]$. So by the dominated convergence theorem, we can switch the order of the limit and the integral in (4.10). We then have (4.8)

$$\begin{aligned}
\lim_{t\to 0^+}\tilde{\Delta}_1(L,t) &= \exp(0)\\
&= 1.
\end{aligned}$$

Now we take the derivative

$$
\begin{aligned}
\frac{d}{dt}\tilde{\Delta}_1(L,t) = \exp\Bigg(&-\frac{1}{o_n}\log\left(\frac{(R^2+t^2)^{\frac{1}{2}}}{R}\right)\int_0^{\frac{\pi}{2}}\left(\cos^{k-1}\phi\right)\left(\sin^{n-k-1}\phi\right)\,d\phi \\
&-\frac{1}{o_n}\int_0^{\frac{\pi}{2}}\left(\cos^{k-1}\phi\right)\left(\sin^{n-k-1}\phi\right)\log\left(\frac{\cos(\phi-\Delta)}{\cos\phi}\right)\,d\phi\Bigg) \\
&\cdot\Bigg\{\frac{-1}{o_n}\cdot\left(\frac{R^2}{R^2+t^2}\right)^{\frac{1}{2}}\cdot\frac{1}{2}\cdot\frac{(R^2+t^2)^{-\frac{1}{2}}}{R}\cdot 2t\cdot\int_0^{\frac{\pi}{2}}\left(\cos^{k-1}\phi\right)\left(\sin^{n-k-1}\phi\right)\,d\phi \\
&-\frac{1}{o_n}\int_0^{\frac{\pi}{2}}\left(\cos^{k-1}\phi\right)\left(\sin^{n-k-1}\phi\right)\left(\frac{\cos\phi}{\cos(\phi-\Delta)}\right)\left(\frac{\sin(\phi-\Delta)}{\cos\phi}\right)\Delta'\,d\phi\Bigg\}.
\end{aligned}
\tag{4.12}
$$

Performing an integration by parts on the last integral of (4.12),

$$
\begin{aligned}
&-\frac{1}{o_n}\int_0^{\frac{\pi}{2}}\left(\cos^{k-1}\phi\right)\left(\sin^{n-k-1}\phi\right)\left(\frac{\sin(\phi-\Delta)}{\cos(\phi-\Delta)}\right)\Delta'\,d\phi \\
&=\frac{1}{o_n}\left(\cos^{k-1}\phi\right)\left(\sin^{n-k-1}\phi\right)\Delta'\cdot\log\left(\cos(\phi-\Delta)\right)\Big|_0^{\pi/2} \\
&-\frac{1}{o_n}\int_0^{\frac{\pi}{2}}(k-1)\left(\cos^{k-2}\phi\right)\left(\sin^{n-k}\phi\right)\Delta'\,\log\left(\cos(\phi-\Delta)\right)d\phi \\
&+\frac{1}{o_n}\int_0^{\frac{\pi}{2}}(n-k-1)\left(\cos^{k}\phi\right)\left(\sin^{n-k-2}\phi\right)\Delta'\,\log\left(\cos(\phi-\Delta)\right)d\phi.
\end{aligned}
$$

Hence, in the limit $\lim_{t\to 0^+}$, we have (4.9)

$$
\begin{aligned}
\lim_{t\to 0^+}\frac{d}{dt}\tilde{\Delta}_1(L,t) &= \lim_{t\to 0^+}\frac{1}{o_n}\left(\cos^{k-1}\frac{\pi}{2}\right)\left(\sin^{n-k-1}\frac{\pi}{2}\right)\Delta'\log\left(\cos\left(\frac{\pi}{2}-\Delta\right)\right)+C \\
&\gtrsim \lim_{t\to 0^+}\log t,
\end{aligned}
$$

where C is a finite negative constant. The inequality above is from the fact that the right side of the equality is minimized when $k=1$. □

We will now obtain an estimate for the differential of $\Delta_2(L,t)$ in the limit as $t\to 0^+$.

Lemma 7. *Let $-1<p<0$, μ be a nonzero even finite Borel measure on S^{n-1} that is not completely concentrated on any lower dimensional subspace, and $L\in\mathcal{K}_e^n$ span a $k<n$ dimensional subspace called ξ, and $t>0$. Suppose also that $\int_{S^{n-1}}\rho_L^{-p}(u)\,d\mu(u)\neq 0$. Then there exists a function $\tilde{\Delta}_2(L,t)$ such that $\Delta_2(L,t)\geq\tilde{\Delta}_2(L,t)$ with*

$$
\lim_{t\to 0^+}\tilde{\Delta}_2(L,t)=1 \tag{4.13}
$$

and

$$
\lim_{t\to 0^+}\frac{d}{dt}\tilde{\Delta}_2(L,t)\sim t^{-p-1}. \tag{4.14}
$$

Proof. Recall the definition of $\Delta_2(L,t)$,

$$\Delta_2(L,t) = \left(\frac{\int_{S^{n-1}} \rho_{L+tB^{n-k}}^{-p}(u)\, d\mu(u)}{\int_{S^{n-1}} \rho_L^{-p}(u)\, d\mu(u)} \right)^{-\frac{1}{p}},$$

with $\int_{S^{n-1}} \rho_L^{-p}(u)\, d\mu(u) \neq 0$. Recall also that ξ is the k-dimensional subspace such that $L \subset \xi$. Then for sufficiently small $t > 0$,

$$\begin{aligned}
\Delta_2(L,t) &\geq \left(\frac{\int_\xi \rho_L^{-p}(u)\, d\mu(u) + \int_{S^{n-1}\backslash\xi} t^{-p}\, d\mu(u)}{\int_{S^{n-1}} \rho_L^{-p}(u)\, d\mu(u)} \right)^{-\frac{1}{p}} \\
&= \left(\frac{\int_\xi \rho_L^{-p}(u)\, d\mu(u) + t^{-p} \cdot \mu(S^{n-1}\backslash\xi)}{\int_\xi \rho_L^{-p}(u)\, d\mu(u)} \right)^{-\frac{1}{p}} \\
&= \left(1 + \frac{t^{-p} \cdot \mu(S^{n-1}\backslash\xi)}{\int_\xi \rho_L^{-p}(u)\, d\mu(u)} \right)^{-\frac{1}{p}} \\
&=: \tilde{\Delta}_2(L,t).
\end{aligned}$$

So we have $\tilde{\Delta}_2(L,t) \to 1$ as $t \to 0^+$. And differentiating $\tilde{\Delta}_2(L,t)$, we have that

$$\frac{d}{dt}\tilde{\Delta}_2(L,t) = \left(1 + \frac{t^{-p} \cdot \mu(S^{n-1}\backslash\xi)}{\int_\xi \rho_L^{-p}(u)\, d\mu(u)} \right)^{-\frac{1}{p}-1} \cdot t^{-p-1} \cdot \left(\frac{\mu(S^{n-1}\backslash\xi)}{\int_\xi \rho_L^{-p}(u)\, d\mu(u)} \right),$$

which means that μ not concentrated on ξ implies that $\frac{d}{dt}\tilde{\Delta}_2(L,t) \to \infty$ as $t \to 0^+$, at rate $\sim t^{-p-1}$. □

We combine Lemmas 6 and 7 to obtain the existence of a solution to the optimization problem $\sup\left\{ \widetilde{\Phi}_p(Q) :\ Q \in \mathcal{K}_e^n \right\}$.

Lemma 8. *Let $-1 < p < 0$ and μ be a nonzero even finite Borel measure on S^{n-1} that is not completely concentrated on any lower dimensional subspace. Then there exists $K \in \mathcal{K}_e^n$ such that $\widetilde{\Phi}_p(K) = \sup\left\{ \widetilde{\Phi}_p(Q) :\ Q \in \mathcal{K}_e^n \right\}$.*

Proof. Suppose $\{Q_l\}$ is a maximizing sequence, i.e.

$$\lim_{l\to\infty} \widetilde{\Phi}_p(Q_l) = \sup\left\{ \widetilde{\Phi}_p(Q) :\ Q \in \mathcal{K}_e^n \right\}.$$

Since $\widetilde{\Phi}_p$ is homogeneous of degree 0, we can rescale every term of the sequence so that $\max_{u\in S^{n-1}} \rho_{Q_l}(u) = 1$. Then by the Blaschke selection theorem, after taking a subsequence, we can assume there exists an origin-symmetric compact convex set Q^0 such that $Q_l \to Q^0$ in the Hausdorff metric. By Lemma 4, we have that after taking a subsequence,

$$\lim_{l\to\infty} \widetilde{\Phi}_p(Q_l) = \widetilde{\Phi}_p(Q^0).$$

Now to show nondegeneracy of the limit, i.e. that $o \in \text{int } Q^0$, we proceed by contradiction. Suppose there exists a k-dimensional subspace ξ such that $Q^0 \subseteq \xi$ and $\text{span}(Q^0) = \xi$ with $k < n$. Taking advantage again of the fact that $\widetilde{\Phi}_p$ is homogeneous of degree 0, we can rescale Q^0 so that $\min_{u \in \xi \cap S^{n-1}} \rho_{Q^0}(u) = 1$.

First consider the case of $\int_{S^{n-1}} \rho_{Q^0}^{-p}(u)\, d\mu(u) = 0$. Observe that this implies

$$\begin{aligned}\widetilde{\Phi}_p(Q^0) &= \exp\left(\frac{1}{o_n}\mathcal{E}(Q^0)\right) \cdot \left(\int_{S^{n-1}} \rho_{Q^0}^{-p}(u)\, d\mu(u)\right)^{\frac{-1}{p}} \\ &= 0 \\ &< \widetilde{\Phi}_p(B^n).\end{aligned}$$

This a contradiction to the assumption that Q_l is a maximizing sequence.

From here on, we investigate the case of $\int_{S^{n-1}} \rho_{Q^0}^{-p}(u)\, d\mu(u) \neq 0$. Now consider the same cylindrical thickening of Q^0 as in Lemma 5, given by $K^t = Q^0 + tB^{n-k}$, where B^{n-k} is the $(n-k)$-dimensional unit ball in the complementary subspace of ξ, and $t > 0$. We will reach a contradiction by showing that $\widetilde{\Phi}_p(K^t) > \widetilde{\Phi}_p(Q^0)$ for sufficiently small $t > 0$. By Lemmas 6 and 7, we observe that there exist functions $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ such that

$$\frac{\widetilde{\Phi}_p(K^t)}{\widetilde{\Phi}_p(Q^0)} = \Delta_1(Q^0, t) \cdot \Delta_2(Q^0, t) \geq \tilde{\Delta}_1(Q^0, t) \cdot \tilde{\Delta}_2(Q^0, t) \to 1$$

as $t \to 0^+$. Furthermore,

$$\frac{d}{dt}\left(\tilde{\Delta}_1(t) \cdot \tilde{\Delta}_2(t)\right) = \frac{d}{dt}\tilde{\Delta}_1(t) \cdot \tilde{\Delta}_2(t) + \frac{d}{dt}\tilde{\Delta}_2(t) \cdot \tilde{\Delta}_1(t). \tag{4.15}$$

Recall that $\lim_{t \to 0^+} \frac{d}{dt}\tilde{\Delta}_1(t) \gtrsim \log(t)$, $\frac{d}{dt}\tilde{\Delta}_2(t) \sim t^{-p-1}$, $\tilde{\Delta}_1(t) \to 1$, and $\tilde{\Delta}_2(t) \to 1$ as $t \to 0^+$, which implies that (4.15) will be positive for small t if $-1 < p < 0$. Therefore, there exists a small $t_0 > 0$ such that

$$\frac{\widetilde{\Phi}_p(K^t)}{\widetilde{\Phi}_p(Q^0)} > 1, \tag{4.16}$$

which is a contradiction to the assumption that Q_l is a maximizing sequence. □

We now prove the existence of a solution to the optimization problem in Lemma 3 for $p \leq -1$, under the assumptions of a measure concentration condition and origin symmetry. Namely, we will provide a sufficient condition on how the given measure μ can be distributed along S^{n-1} to guarantee a solution to the aforementioned optimization problem. This is an expansion to the $p < 0$ existence result by Huang-Lutwak-Yang-Zhang [30], since, for example, some discrete measures can now be included.

Lemma 9. *Let $p \leq -1$ and μ be a nonzero even finite Borel measure on S^{n-1}. Suppose, on all great subspheres $\xi \subset S^{n-1}$, that*

$$\frac{\mu(\xi)}{\mu(S^{n-1})} \leq C(n)^p,$$

where $C(n) = \exp\left(\frac{1}{2}\left(\psi\left(\frac{n}{2}\right) - \psi\left(\frac{1}{2}\right)\right)\right)$ *is a constant depending only on* n*, and* ψ *is the digamma function. Then there exists a* $K \in \mathcal{K}_e^n$ *such that* $\Phi_p(K) = \sup\{\Phi_p(Q) : \ Q \in \mathcal{K}_e^n\}$.

Proof. Suppose $\{Q_l\}$ is a maximizing sequence for Φ_p, i.e.

$$\lim_{l\to\infty} \Phi_p(Q_l) = \sup\{\Phi_p(Q) : \ Q \in \mathcal{K}_e^n\}.$$

Since Φ_p is homogeneous of degree 0, we can rescale every term of the sequence so that

$$\left(\int_{S^{n-1}} \log\left(h_{Q_l}(u)\right)\, du\right) = 0.$$

We now prove that Q_l is uniformly bounded. Define $v_l \in S^{n-1}$ by $\rho_{Q_l}(v_l) := \max_{u\in S^{n-1}} \rho_{Q_l}(u)$. Then

$$\begin{aligned}
0 &= -\int_{S^{n-1}} \log\left(h_{Q_l}(u)\right)\, du \\
&\geq \int_{S^{n-1}} \log\left(\rho_{Q_l}(v_l)\,|v_l\cdot u|\right)\, du \\
&= \int_{S^{n-1}} \log\left(\rho_{Q_l}(v_l)\right)\, du + \int_{S^{n-1}} \log\left(|v_l\cdot u|\right)\, du \\
&= \log\left(\rho_{Q_l}(v_l)\right)\cdot o_n + \int_{S^{n-1}} \log\left(|v_l\cdot u|\right)\, du.
\end{aligned}$$

Thus,

$$\log\left(\rho_{Q_l}(v_l)\right) \leq -\int_{S^{n-1}} \log\left(|v_l\cdot u|\right)\, du, \tag{4.17}$$

and so since the right side of Inequality (4.17) is finite, we conclude that $\rho_{Q_l}(v_l)$ uniformly is bounded. Thus, $\forall l$, $Q_l \subset MB^n$, for some $M > 0$.

By Blaschke's selection theorem, Q_l converges to some origin-symmetric compact convex set $Q_0 \subset \mathbb{R}^n$. Now we show nondegeneracy of the limit, i.e. that Q_0 contains the origin in its interior. Proceed by contradiction, and assume $\exists u_0 \in S^{n-1}$ such that $h_{Q_0}(\pm u_0) = 0$. Now $\forall \delta > 0$, define $\omega_\delta(u_0) := \{v \in S^{n-1} : \ |v\cdot u_0| > \delta\}$, and note that $\rho_{Q_l} \to 0$ uniformly on ω_δ.

Now

$$\begin{aligned}
\Phi_p(Q_l) &= -\frac{1}{p}\log\left(\int_{S^{n-1}} \rho_Q^{-p}(v)\, d\mu(v)\right) \\
&= -\frac{1}{p}\log\left(\int_{\omega_\delta} \rho_Q^{-p}(v)\, d\mu(v) + \int_{S^{n-1}\setminus\omega_\delta} \rho_Q^{-p}(v)\, d\mu(v)\right) \\
&\leq -\frac{1}{p}\log\left(\left(\sup_{\omega_\delta} \rho_{Q_l}^{-p}(v)\right)\mu(\omega_\delta) + M^{-p}\mu(S^{n-1}\setminus\omega_\delta)\right) \\
&= -\frac{1}{p}\log\left(\left(\sup_{\omega_\delta} \rho_{Q_l}^{-p}(v) - M^{-p}\right)\mu(\omega_\delta) + M^{-p}\mu(S^{n-1})\right).
\end{aligned}$$

Taking the limit $l\to\infty$, we have

$$\lim_{l\to\infty} \Phi_p(Q_l) \le -\frac{1}{p}\log\left(M^{-p}\mu(S^{n-1}) - M^{-p}\mu(\omega_\delta)\right)$$
$$= -\frac{1}{p}\log\left(M^{-p}\mu(S^{n-1}) - M^{-p}\mu(S^{n-1})\cdot\frac{\mu(\omega_\delta)}{\mu(S^{n-1})}\right).$$

Take the limit $\delta \to 0^+$ to obtain

$$\lim_{l\to\infty} \Phi_p(Q_l) \le -\frac{1}{p}\log\left(M^{-p}\mu(S^{n-1}) - M^{-p}\mu(S^{n-1})\cdot\frac{\mu(S^{n-1}\backslash\xi)}{\mu(S^{n-1})}\right)$$
$$= -\frac{1}{p}\log\left(M^{-p}\mu(S^{n-1})\cdot\frac{\mu(\xi)}{\mu(S^{n-1})}\right),$$

where ξ is a great subsphere of S^{n-1}. Suppose $C(n) = \exp\left(\frac{-1}{o_n}\int_{S^{n-1}}\log|v_0\cdot u|\ du\right)$, then

$$\lim_{l\to\infty} \Phi_p(Q_l) \le -\frac{1}{p}\log\left(M^{-p}\cdot\mu(S^{n-1})\cdot\exp\left(\frac{-p}{o_n}\int_{S^{n-1}}\log|v_0\cdot u|\ du\right)\right). \tag{4.18}$$

Recall that we rescaled Q_l so that $\left(\int_{S^{n-1}}\log(h_{Q_l}(u))\,du\right) = 0$. Then

$$0 \ge \int_{S^{n-1}}\log(M\cdot|v_0\cdot u|)\,du.$$

Thus,

$$0 \ge o_n\cdot\log(M) + \int_{S^{n-1}}\log|v_0\cdot u|\ du.$$

And so

$$M \le \exp\left(\frac{-1}{o_n}\int_{S^{n-1}}\log|v_0\cdot u|\ du\right).$$

Applying this to Inequality (4.18), we have

$$\lim_{l\to\infty} \Phi_p(Q_l) \le -\frac{1}{p}\log(\mu(S^{n-1}))$$
$$= \Phi_p(B^n).$$

This contradicts the assumption that $\{Q_l\}$ is a maximizing sequence.

We will now calculate $C(n)$. First notice that

$$C\,(n) = \exp\left(\frac{-1}{o_n}\int_{S^{n-1}}\log|v_0\cdot u|\ du\right)$$
$$= \exp\left(-\frac{2\cdot o_{n-1}}{o_n}\int_0^{\frac{\pi}{2}}\left(\sin^{n-2}\phi\right)\log\left(\cos\phi\right)\ d\phi\right)$$
$$= \lim_{q\to 0}\exp\left[\frac{1}{q}\log\left(\frac{2\cdot o_{n-1}}{o_n}\int_0^{\frac{\pi}{2}}\left(\sin^{n-2}\phi\right)\left(\cos^{-q}\phi\right)\ d\phi\right)\right],$$

where o_n is the surface area of S^{n-1}. Focusing on the exponent, we apply a change of variables to obtain

$$\begin{aligned}
&\lim_{q\to 0}\frac{1}{q}\log\left(\frac{2\cdot o_{n-1}}{o_n}\int_0^{\frac{\pi}{2}}\left(\sin^{n-2}\phi\right)\left(\cos^{-q}\phi\right)\,d\phi\right)\\
&=\lim_{q\to 0}\frac{1}{q}\log\left(\frac{o_{n-1}}{o_n}\int_0^{1}(1-t)^{\frac{n-3}{2}}\,t^{\frac{-q-1}{2}}\,dt\right)\\
&=\lim_{q\to 0}\frac{1}{q}\log\left(\frac{o_{n-1}}{o_n}B\left(\frac{1-q}{2},\frac{n-1}{2}\right)\right)\\
&=\lim_{q\to 0}\frac{1}{q}\log\left(\frac{o_{n-1}}{o_n}\cdot\frac{\Gamma\left(\frac{1-q}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-q}{2}\right)}\right)\\
&=\lim_{q\to 0}\frac{1}{q}\log\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\cdot\frac{\Gamma\left(\frac{1-q}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-q}{2}\right)}\right).
\end{aligned}$$

Applying L'Hopital's rule, we have

$$\begin{aligned}
&\lim_{q\to 0}\frac{1}{q}\log\left(\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\cdot\frac{\Gamma\left(\frac{1-q}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-q}{2}\right)}\right)\\
&=\frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\cdot\lim_{q\to 0}\frac{\frac{-1}{2}\Gamma\left(\frac{n-q}{2}\right)\Gamma'\left(\frac{1-q}{2}\right)\Gamma\left(\frac{n-1}{2}\right)+\frac{1}{2}\Gamma\left(\frac{1-q}{2}\right)\Gamma\left(\frac{n-1}{2}\right)\Gamma'\left(\frac{n-q}{2}\right)}{\left[\Gamma\left(\frac{n-q}{2}\right)\right]^2}\\
&=\frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma'\left(\frac{n}{2}\right)-\Gamma\left(\frac{n}{2}\right)\Gamma'\left(\frac{1}{2}\right)}{\left[\Gamma\left(\frac{n}{2}\right)\right]^2}\\
&=\frac{1}{2}\left(\psi\left(\frac{n}{2}\right)-\psi\left(\frac{1}{2}\right)\right),
\end{aligned}$$

where ψ is the digamma function. □

We will now make some remarks on $C(n)$. For even n, $\psi\left(\frac{n}{2}\right)=\sum_{i=1}^{\frac{n}{2}-1}\frac{1}{i}-\gamma$, and for odd n, $\psi\left(\frac{n}{2}\right)=-\gamma-2\ln 2+\sum_{i=1}^{\frac{n-1}{2}}\frac{2}{2i-1}$ (where γ is the Euler-Mascheroni constant). So,

$$\exp\left[\frac{1}{2}\left(\psi\left(\frac{n}{2}\right)-\psi\left(\frac{1}{2}\right)\right)\right]=\begin{cases}\exp\left(\frac{1}{2}\left(2\ln 2+\sum_{i=1}^{\frac{n}{2}-1}\frac{1}{i}\right)\right) & ,\ k \text{ even}\\ \exp\left(\sum_{i=1}^{\frac{n-1}{2}}\frac{1}{2i-1}\right) & ,\ k \text{ odd}\end{cases}.$$

Notice that for all $n\in\mathbb{N}$, $0<C(n)^p\leq 1$. For large n, $C(n)^p=\exp\left[\frac{p}{2}\left(\psi\left(\frac{n}{2}\right)-\psi\left(\frac{1}{2}\right)\right)\right]=O(n^{\frac{p}{2}})$. Since $p<0$, we observe that the measure concentration bound approaches 0 as $n\to\infty$.

From Lemmas 2 and 8, we have the following.

Theorem 3. *Let $-1<p<0$ and μ be a nonzero even finite Borel measure on S^{n-1}. Then there exists an origin symmetric convex body $K\in\mathbb{R}^n$ such that $\mu=J_p(K,\cdot)$ if and only if μ is not completely concentrated on any lower dimensional subspace.*

Proof. Necessity is obvious, and sufficiency follows from combining Lemmas 2 and 8. □

Lemmas 3 and 9 imply the following.

Theorem 4. *Let $p \leq -1$ and μ be a nonzero even finite Borel measure on S^{n-1}. Suppose, on all great subspheres $\xi \subset S^{n-1}$, that*

$$\frac{\mu(\xi)}{\mu(S^{n-1})} \leq C(n)^p,$$

where $C(n) = \exp\left[\frac{1}{2}\left(\psi\left(\frac{n}{2}\right) - \psi\left(\frac{1}{2}\right)\right)\right]$ is a constant depending only on n, and ψ is the digamma function. Then there exists a $K \in \mathcal{K}_e^n$ such that $\mu = J_p(K, \cdot)$.

Proof. Combine Lemmas 3 and 9. □

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