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The pentagram map, Poncelet polygons, and commuting difference operators

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ABSTRACT

The pentagram map takes a planar polygon P to a polygon P' whose vertices are the intersection points of consecutive shortest diagonals of P . This map is known to interact nicely with Poncelet polygons, that is, polygons which are simultaneously inscribed in a conic and circumscribed about a conic. A theorem of Schwartz states that if P is a Poncelet polygon, then the image of P under the pentagram map is projectively equivalent to P . In the present paper, we show that in the convex case this property characterizes Poncelet polygons: if a convex polygon is projectively equivalent to its pentagram image, then it is Poncelet. The proof is based on the theory of commuting difference operators, as well as on properties of real elliptic curves and theta functions.

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1. Introduction

It is a classical result of Clebsch [Cle71] that every planar pentagon P is projectively equivalent to the pentagon P' formed by intersections of diagonals of P . More precisely, if we label the vertices of P and P' in such way that the k th vertex of P' is opposite to the k th vertex of P (see Figure 1), then there is a projective transformation that takes P to P' and respects the labelings.

Furthermore, as was proved by Schwartz [Sch07], Clebsch's theorem is true for all Poncelet polygons with an odd number of vertices. Recall that a *Poncelet polygon* is a polygon which is inscribed in a conic and circumscribed about another conic. In particular, any pentagon is Poncelet, whereas for n -gons with $n \geq 6$ being Poncelet is a non-trivial restriction. Poncelet polygons owe their name to Poncelet and his famous 'porism' which states that if there exists an n -gon inscribed in a conic C_1 and circumscribed about a conic C_2 , then any point of C_1 is a vertex of such an n -gon (see Figure 2).

Schwartz's generalization of Clebsch's theorem is as follows. Let P be an n -gon with odd n , and let P' be the polygon whose vertices are the intersections of consecutive shortest diagonals of P , that is, diagonals connecting second-nearest vertices. Label the vertices of P' as in Clebsch's theorem: the k th vertex of P' is opposite to the k th vertex of P (see Figure 3). Assume that P is Poncelet. Then there is a projective transformation that carries P to P' and respects the labelings (a weaker result saying that if P is Poncelet, then P' is circumscribed about a conic was known to Darboux, see [DR14, Theorem 2.1]). The goal of the present paper is to show that in the convex setting the converse is also true. More precisely, we prove the converse of Schwartz's theorem for a broader class of *weakly convex* polygons. Weak convexity is a technical condition (see Definition 3.1) which, in particular, holds for truly convex polygons.

THEOREM A. *Let P be a weakly convex closed polygon with an odd number of vertices. Let also P' be the polygon whose vertices are the intersections of consecutive shortest diagonals of P , labeled as in Figure 3. Assume that there is a projective transformation that carries P to P' and respects the labelings. Then P (and, hence, P') is a Poncelet polygon.*

Combining Theorem A with Schwartz's theorem, we obtain the following characterization of weakly convex Poncelet polygons.

COROLLARY 1.1. *Let P be a weakly convex closed polygon with an odd number of vertices. Let also P' be the polygon whose vertices are the intersections of consecutive shortest diagonals of P , labeled as in Figure 3. Then P is Poncelet if and only if it is projectively equivalent to P' .*

The map taking the polygon P in Figure 3 to P' is known as the *pentagram map*. It was defined by Schwartz in 1992 [Sch92] but became especially popular in the last decade thanks to the discovery that it is a discrete integrable system [OST10, OST13, Sol13], and also because of its connections with cluster algebras [Gli11, GSTV16, GP16, KV15, FM16]. As the pentagram map commutes with projective transformations, it is usually considered as a dynamical system on the space of polygons modulo projective equivalence. Our result can thus be viewed as a description

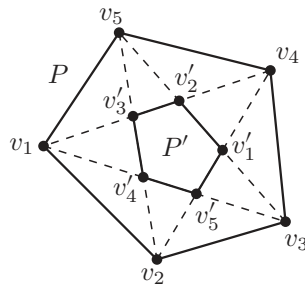


FIGURE 1. Pentagons $P = v_1v_2v_3v_4v_5$ and $P' = v'_1v'_2v'_3v'_4v'_5$ are projectively equivalent.

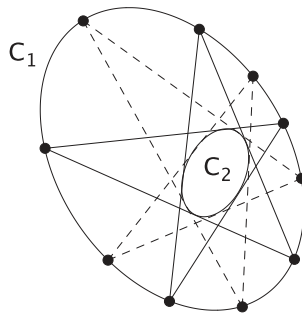


FIGURE 2. Every point of C_1 is a vertex of a pentagon inscribed in C_1 and circumscribed about C_2 .

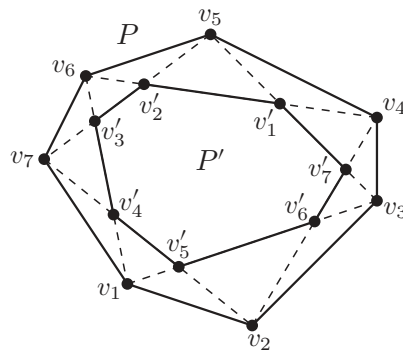


FIGURE 3. A convex polygon P is Poncelet if and only if it is projectively equivalent to P' .

of fixed points of the pentagram map, which has been an open question since Schwartz's first paper [Sch92].

Remark 1.2. Note that the pentagram map can be considered either on labeled polygons (i.e. polygons with labeled vertices) or on unlabeled polygons. Theorem A describes fixed points of the pentagram map on the space of projective equivalence classes of *labeled* polygons, where the labeling rule is depicted in Figure 3. Although this is not the only possible labeling, it is the only one for which the pentagram map commutes with the action of the dihedral group and, hence, the most symmetric of all labelings. A more common, non-symmetric labeling is given by the rule $v'_k := (v_{k-1}, v_{k+1}) \cap (v_k, v_{k+2})$. One can easily see that the only fixed points of the pentagram map with this non-symmetric labeling are regular polygons (again, assuming that the number of vertices is odd). The problem of describing the fixed points of the pentagram map

for an *arbitrary* labeling can also be approached using the techniques of the present paper, but due to the break of symmetry, one should not expect an answer as nice as for the symmetric labeling.

Theorem A also has an interpretation in terms of billiards. Indeed, if the conic C_1 circumscribed about a Poncelet polygon P is confocal to the inscribed conic C_2 (which can always be arranged by applying a suitable projective transformation), then P can be viewed as a closed trajectory of a billiard ball in the domain bounded by C_1 . Conversely, any closed billiard trajectory in a conic is a Poncelet polygon. Thus, Corollary 1.1 establishes a correspondence between fixed points of the pentagram map and periodic billiard trajectories in conics. Also note that, as shown in [LT07], the fact that a closed billiard trajectory in a conic is projectively equivalent to its pentagram image is essentially a corollary of integrability of the corresponding billiard system. At the same time, we show that if P is projectively equivalent to its pentagram image then the vertices of P are contained in a conic. Thus, one may hope to combine our results with the approach of [LT07] to show that for *any* integrable billiard the impact points of a periodic trajectory are contained in a conic and, hence, shed some light on the Birkhoff conjecture which says that the only integrable billiards are those in conics.

It is also an interesting question whether this correspondence between periodic billiard trajectories in conics and fixed points of the pentagram map extends to higher dimensions. There exists numerous generalizations of the pentagram map to higher-dimensional spaces [GSTV16, KS13, KS16, FB19] and one may wonder if their fixed points are related to periodic trajectories of billiards in multidimensional quadrics.

Remark 1.3. We do not know whether Theorem A is true with no convexity-type assumptions, but it is certainly not true over complex numbers, as demonstrated by the following example. Let $\lambda := \exp(2\pi i/7)$, where $i = \sqrt{-1}$, and let P be a heptagon in $\mathbb{C}^2$ with vertices $v_k := (\lambda^{2k}, \lambda^{3k})$. Then a direct computation shows that there exists a projective (in fact, even affine) transformation ϕ taking P to P' (see also Remark 4.3 for a conceptual proof). Moreover, for any vertex v of P and v' of P' , the map ϕ can be chosen to take v to v' . This means that the projective equivalence class of P is fixed by the pentagram map, regardless of the labeling convention used to define the map. However, P is not Poncelet. Moreover, it is not even inscribed in a conic. Indeed, the vertices of P lie on a semi-cubical parabola $y^2 = x^3$, which has at most six intersection points with any conic. Thus, there exists no conic which contains all seven vertices of P .

Theorem A being not true over $\mathbb{C}$ is one of the reasons one should not expect it to have any kind of ‘elementary’ proof, as such a proof would be valid over any field. Another reason is that the theorem is not true for non-closed polygons (see Remark 1.4). Again, if there was a local (i.e. involving only a few adjacent vertices) geometric construction producing inscribed and circumscribed conics for P based on the projective equivalence between P and P' , such a construction would work whether P is closed or not.

We now outline the scheme of the proof of Theorem A. As a first step, we prove the theorem under an additional assumption that the polygon P is *self-dual*. Recall that the dual of a polygon is the polygon in the dual projective plane whose vertices are the sides of the initial one. We label the vertices of the dual polygon as shown in Figure 4. A polygon is self-dual if it is projectively equivalent to its dual.

Under this additional assumption of self-duality, Theorem A is true in a more general setting of *twisted polygons*, that is, polygons that are closed only up to a projective transformation. More precisely, a twisted n -gon is a sequence $v_k \in \mathbb{P}^2$ such that $v_{k+n} = \psi(v_k)$ for a certain projective transformation ψ , called the *monodromy*.

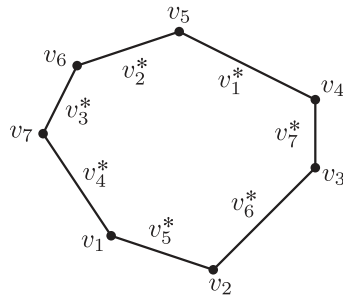


FIGURE 4. The k th vertex of the dual polygon is opposite to the k th vertex of the initial one.

THEOREM B. *Let P be a weakly convex twisted n -gon with odd n , and let P' be as in Theorem A. Assume that P is self-dual and projectively equivalent to P' . Then P is a Poncelet polygon.*

Remark 1.4. Theorem B is not true without the self-duality assumption (in other words, Theorem A is not true for non-closed polygons). As an example, consider a polygon P in $\mathbb{R}^2$ whose vertices are given by $v_k := (4^k, 8^k)$. This is a twisted n -gon for any n , weakly convex and projectively equivalent to its pentagram image P' . However, P is not Poncelet (cf. Remark 1.3).

The proof of Theorem B is based on the theory of commuting difference operators, elliptic curves, and theta functions. Given the result of the theorem, the appearance of elliptic curves is not surprising, as their connection to Poncelet polygons is well-known. However, we are not given *a priori* that P is Poncelet, so there should be some other source where the elliptic curve is coming from. In our approach, that source is the theory of commuting difference operators. Namely, we show that a twisted polygon P is projectively equivalent to its pentagram image P' if and only if certain associated difference operators commute (see § 4). The general theory then states that the joint spectrum of those operators is a Riemann surface Γ , called the *spectral curve* (see, e.g., [KN03]). Using further that the operators in question are of special form and, in particular, dual to each other (which is a reflection of self-duality of P), we show that the genus of Γ is at most one, that is, Γ is rational or elliptic (see § 5.1). This is one of the few places in the proof where we use weak convexity in an essential way. Without that assumption, one only seems to be able to conclude that the genus is at most two. In particular, it seems possible to construct non-weakly-convex counterexamples to Theorems A and B using genus-2 theta functions.

The next step is to show that our upper bound on the genus of Γ implies that P is Poncelet (as *a priori* there is no connection between the elliptic curve Γ and that which is classically associated with a Poncelet polygon). More precisely, we show that elliptic spectral curves correspond to generic Poncelet polygons, whereas rational curves correspond to their degenerations (such as the regular polygon). To that end, we express coordinates of vertices of P in terms of certain meromorphic functions on the spectral curve Γ . One ends up with elementary functions or theta functions, depending on whether the curve Γ is rational or elliptic. In both cases, using relations between those functions (e.g. Riemann's relation in the elliptic case), one shows that P is a Poncelet polygon, so Theorem B holds (see § 6.1 for the rational case and § 6.2 for the elliptic case). Furthermore, in the elliptic case the spectral curve Γ turns out to be isogenous to the elliptic curve attached to P due to its Poncelet property.

After that, we proceed to prove Theorem A. To that end, we first show that the self-duality assumption of Theorem B is not too restrictive. Namely, given a polygon that is projectively equivalent to its pentagram image, it can be what is called *rescaled* so that it becomes self-dual. This rescaling (closely related to the notion of *spectral parameter*) is a one-parametric group

of transformations of the moduli space of twisted polygons which preserves weak convexity but not closedness. Thus, starting with a closed polygon as in Theorem A, we rescale it to a non-necessarily closed, but self-dual, weakly convex polygon, which is the setting of Theorem B. In this way, we conclude that a weakly convex closed polygon projectively equivalent to its pentagram image is Poncelet up to rescaling. The last step is to show that this rescaling must actually be trivial. To that end, we show that no non-trivial rescaling of a weakly convex Poncelet polygon is closed. In the rational case, this is proved by an elementary argument (see § 7.2), whereas the elliptic case requires careful analysis of the real part of the spectral curve (see § 7.3). In the latter case, the proof once again essentially relies on weak convexity.

In addition to proofs of Theorems A and B, the paper contains an appendix (Appendix A) where we establish an auxiliary result on correspondence between dual difference operators and dual polygons. Although that result seems to be well-known, we could not find a proof in the literature that does not rely on a computation. Thus, we provide a proof here.

We tried to make the exposition self-contained. In particular, we do not assume that the reader is familiar with the general theory of integrable systems or commuting difference operators. Only basic knowledge of Riemann surfaces is assumed.

2. Background results: polygons, difference operators, and corner invariants

This section is an overview of mostly well-known results on the relation between difference operators and polygons. Namely, we give an introduction to difference operators in § 2.1, after which we connect them to polygons in § 2.2. Note that although the description of the moduli space of polygons in terms of difference operators is well-known, our point of view is slightly different from the standard one. In particular, we identify the space of polygons with a certain *quotient* of the space of third-order difference operators, as opposed to the standard approach in which one identifies polygons with a certain *subspace* of that space. In that respect, our approach is close to that of [CO19]. In addition, we provide, in § 2.3, another description of the polygon space, in terms of so-called corner invariants. Note that although corner invariants *per se* are not used heavily in the paper, they are needed to define weakly convex polygons and rescaling. Rescaling is also defined in § 2.3, whereas weakly convex polygons are discussed in § 3.

2.1 A primer on difference operators

This section is a brief introduction to the elementary theory of difference operators. Our terminology mainly follows that of [VM79]. Let $\mathbb{R}^\infty$ be the vector space of bi-infinite sequences of real numbers. For $\xi \in \mathbb{R}^\infty$ and any $k \in \mathbb{Z}$, let $\xi_k \in \mathbb{R}$ be the k th entry of the sequence ξ , so that $\xi = (\xi_k)_{k \in \mathbb{Z}}$. Let also $m_- \leq m_+$ be integers. A linear operator $\mathcal{D}: \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ is called a *difference operator supported in* $[m_-, m_+] := \{m_-, \dots, m_+\}$ if it can be written as

$$(\mathcal{D}\xi)_k = \sum_{j=m_-}^{m_+} a_k^j \xi_{k+j}, \quad (1)$$

where $a_k^j \in \mathbb{R}$ for every $k \in \mathbb{Z}$ and every $j \in [m_-, m_+]$. In matrix terms, this can be rewritten as

$$\mathcal{D}\xi = \begin{pmatrix} \ddots & & & & & \\ & a_{k-1}^{m_-} & \dots & a_{k-1}^{m_+} & & \\ & & a_k^{m_-} & \dots & a_k^{m_+} & \\ & & & a_{k+1}^{m_-} & \dots & a_{k+1}^{m_+} \\ & & & & \ddots & \\ & & & & & \ddots \end{pmatrix} \xi, \quad (2)$$

so difference operators can be equivalently described as those whose matrices are finite band (i.e. have only finitely many non-zero diagonals). Furthermore, denoting, for every j , the sequence of a_k^j by a^j , formula (1) can be rewritten as

$$\mathcal{D} = \sum_{j=m_-}^{m_+} a^j T^j, \quad (3)$$

where $T: \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ is the left shift operator $(T\xi)_k = \xi_{k+1}$, and each $a^j \in \mathbb{R}^\infty$ acts on $\mathbb{R}^\infty$ by term-wise multiplication.

The *order* of difference operator (1) is the number $\text{ord } \mathcal{D} := m_+ - m_-$. Difference operator (1) is called *properly bounded* if $a_k^{m_-} \neq 0$ and $a_k^{m_+} \neq 0$ for every $k \in \mathbb{Z}$. Clearly, for a properly bounded difference operator $\mathcal{D}$ one has $\dim \text{Ker } \mathcal{D} = \text{ord } \mathcal{D}$. Sequences $\xi^1, \dots, \xi^d \in \text{Ker } \mathcal{D}$, where $d := \text{ord } \mathcal{D}$, form a basis in $\text{Ker } \mathcal{D}$ if and only if the associated *difference Wronskian*

$$W_k := \begin{vmatrix} \xi_k^1 & \dots & \xi_k^d \\ \vdots & \ddots & \vdots \\ \xi_{k+d-1}^1 & \dots & \xi_{k+d-1}^d \end{vmatrix},$$

(where $|M|$ denotes the determinant of the matrix M) is non-vanishing for some $k \in \mathbb{Z}$. This is equivalent to non-vanishing of W_k for any k due to the relation

$$W_{k+1} = (-1)^d \frac{a_{k-m_-}^{m_-}}{a_{k-m_-}^{m_+}} W_k. \quad (4)$$

Along with $\mathbb{R}^\infty$, difference operators naturally act on the space $(\mathbb{R}^d)^\infty$ of bi-infinite sequences of vectors in $\mathbb{R}^d$. The case $d = \text{ord } \mathcal{D}$ is of particular interest. Let $V \in (\mathbb{R}^d)^\infty$, where $d = \text{ord } \mathcal{D}$, be a solution of the difference equation $\mathcal{D}V = 0$. Define scalar sequences $\xi^1, \dots, \xi^d \in \mathbb{R}^\infty$ by setting ξ_k^j to be equal to the j th coordinate of V_k . We say that V is a *fundamental solution* if the sequences $\xi^1, \dots, \xi^d \in \mathbb{R}^\infty$ form a basis in $\text{Ker } \mathcal{D}$. As follows from the Wronskian criterion, a solution of $\mathcal{D}V = 0$ is fundamental if and only if the vectors $V_k, \dots, V_{k+d-1}$ are linearly independent for some (equivalently, for all) $k \in \mathbb{Z}$.

A difference operator $\mathcal{D}$ is *n-periodic* if its coefficients a_k^j are n -periodic in the index k . This is equivalent to saying that $\mathcal{D}$ commutes with the n th power of the shift operator: $\mathcal{D}T^n = T^n \mathcal{D}$, so the kernel of an n -periodic operator $\mathcal{D}$ is invariant under the action of T^n . The finite-dimensional operator $T^n|_{\text{Ker } \mathcal{D}}$ is called the *monodromy* of $\mathcal{D}$. Note that the eigenvectors of the monodromy are exactly scalar quasi-periodic solutions of the equation $\mathcal{D}\xi = 0$, that is, solutions which belong to the space

$$Q^n(z) := \{\xi \in \mathbb{R}^\infty \mid \xi_{k+n} = z\xi_k\} \quad (5)$$

for some $z \in \mathbb{R}^*$.

The monodromy can also be understood in terms of fundamental solutions. Namely, note that any two fundamental solutions $V, V' \in (\mathbb{R}^d)^\infty$ of $\mathcal{D}$ are related by $V' = AV$, where $A \in \text{GL}_d(\mathbb{R})$ acts on $(\mathbb{R}^d)^\infty$ by term-wise multiplication. Furthermore, if V is a fundamental solution of an n -periodic operator $\mathcal{D}$, then so is $T^n V$, which means that $T^n V = AV$ for some $A \in \text{GL}_d(\mathbb{R})$. In other words, we have $V_{k+n} = AV_k$ for every $k \in \mathbb{Z}$, which means that the fundamental solution of a periodic difference operator is always quasi-periodic. Furthermore, the matrix A can be easily seen to be the transpose of the monodromy matrix M of $\mathcal{D}$, written in the basis of $\text{Ker } \mathcal{D}$ associated with the fundamental solution V . In particular, this implies that the Wronskian of an n -periodic operator $\mathcal{D}$ satisfies $W_{k+n} = (\det M)W_k$. Combined with (4), the latter formula gives the following expression for the determinant of the monodromy, which is used several times

throughout the paper:

$$\det M = (-1)^{nd} \prod_{k=1}^n \frac{a_k^{m_-}}{a_k^{m_+}}. \quad (6)$$

The *dual* of the operator (3) is defined by

$$\mathcal{D}^* := \sum_{j=l}^m T^{-j} a^j = \sum_{j=l}^m \tilde{a}^j T^{-j},$$

where $\tilde{a}_k^j = a_{k-j}^j$. In other words, $\mathcal{D}^*$ is the formal adjoint of $\mathcal{D}$ with respect to the l^2 inner product on $\mathbb{R}^\infty$, that is, $\langle \xi, \mathcal{D}\eta \rangle = \langle \mathcal{D}^*\xi, \eta \rangle$ whenever at least one of these inner products is well-defined. In the periodic case, the duality between $\mathcal{D}$ and $\mathcal{D}^*$ can also be understood as follows. If $\mathcal{D}$ is an n -periodic operator, then $\mathcal{D}^*$ is n -periodic as well. Furthermore, the formula

$$\langle \xi, \eta \rangle := \sum_{k=1}^n \xi_k \eta_k \quad (7)$$

defines an inner product on the space $Q^n(1)$ of n -periodic sequences, and the restrictions of $\mathcal{D}$ and $\mathcal{D}^*$ to $Q^n(1)$ are dual to each other with respect to this inner product. More generally, for every $z \in \mathbb{R}^*$, the restriction of $\mathcal{D}$ to $Q^n(z)$ is dual to the restriction of $\mathcal{D}^*$ to $Q^n(z^{-1})$ with respect to the pairing between $Q^n(z)$ and $Q^n(z^{-1})$ given by the same formula (7). As a corollary, we have

$$\dim \operatorname{Ker} \mathcal{D}^*|_{Q^n(z^{-1})} = \dim \operatorname{Ker} \mathcal{D}|_{Q^n(z)}.$$

In particular, a non-zero number $z \in \mathbb{R}^*$ is an eigenvalue of the monodromy of $\mathcal{D}$ if and only if z^{-1} is an eigenvalue of the monodromy of $\mathcal{D}^*$.

2.2 Difference operators and polygons

In this section we describe the space of projective equivalence classes of planar polygons as a certain quotient of third-order difference operators.

DEFINITION 2.1. A *polygon* in $\mathbb{RP}^2$ is a bi-infinite sequence of points $v_k \in \mathbb{RP}^2$ satisfying the following *3-in-a-row* condition: for every $k \in \mathbb{Z}$ the points v_{k-1}, v_k, v_{k+1} are in general position.

Polygons modulo projective transformations can be encoded by means of properly bounded third-order difference operators, that is, operators of the form

$$\mathcal{D} = aT^j + bT^{j+1} + cT^{j+2} + dT^{j+3}, \quad (8)$$

where $a, b, c, d \in \mathbb{R}^\infty$ are such that $a_k \neq 0, d_k \neq 0$ for any $k \in \mathbb{Z}$.

PROPOSITION 2.2 (Cf. [OST10, Proposition 4.1]). *For any $j \in \mathbb{Z}$, there is a one-to-one correspondence between projective equivalence classes of planar polygons and properly bounded difference operators $\mathcal{D}$ supported in $[j, j+3]$, considered up to the action $\mathcal{D} \mapsto \lambda \circ \mathcal{D} \circ \mu^{-1}$, where $\lambda, \mu \in (\mathbb{R}^*)^\infty$ are sequences of non-zero real numbers, acting on $\mathbb{R}^\infty$ by term-wise multiplication.*

Proof. Given a properly bounded difference operator $\mathcal{D}$ supported in $[j, j+3]$, consider its fundamental solution V , which is a sequence of non-zero vectors in $\mathbb{R}^3$ (see § 2.1). Each term V_k of that sequence determines a point $v_k \in \mathbb{RP}^2$ with homogeneous coordinates given by V_k . Furthermore, because V is a fundamental solution, the vectors V_k, V_{k+1}, V_{k+2} are linearly independent, and, thus, the sequence $v_k \in \mathbb{RP}^2$ satisfies the 3-in-a-row condition. Note also that because the fundamental solution V is unique up to a linear transformation $V \mapsto AV$, it follows that the

polygon $\{v_k\}$ is well-defined up to projective equivalence. Thus, with each properly bounded difference operator $\mathcal{D}$ supported in $[j, j+3]$ one can associate a polygon $\{v_k\}$, defined up to a projective transformation. Conversely, given a polygon $\{v_k\}$, one can revert this construction to obtain a properly bounded difference operator $\mathcal{D}$ supported in $[j, j+3]$. To that end, one first lifts every point $v_k \in \mathbb{P}^2$ to a vector $V_k \in \mathbb{R}^3$, and then finds an operator $\mathcal{D}$ whose fundamental solution is given by V . Since the lifts V_k of the points v_k are unique up to a transformation $V_k \mapsto \mu_k V_k$, whereas the choice of an operator $\mathcal{D}$ with a given fundamental solution V is unique up to $\mathcal{D} \mapsto \lambda \circ \mathcal{D}$, where $\lambda \in (\mathbb{R}^*)^\infty$, it follows that the operator $\mathcal{D}$ corresponding to a given polygon is defined up to the action $\mathcal{D} \mapsto \lambda \circ \mathcal{D} \circ \mu^{-1}$, as desired. $\square$

In what follows, we are interested in closed and, more generally, twisted polygons. A closed n -gon is a polygon satisfying $v_{k+n} = v_k$ for every $k \in \mathbb{Z}$. For such a polygon, the corresponding difference operator $\mathcal{D}$ can be chosen to be n -periodic. The converse is, however, not true: polygons corresponding to periodic operators are, in general, not closed but twisted. Indeed, if $\mathcal{D}$ is a periodic operator, then its fundamental solution $V \in (\mathbb{R}^3)^\infty$ is, in general, not periodic, but satisfies $V_{k+n} = AV_k$, where $A \in \mathrm{GL}_3(\mathbb{R})$ is the transposed monodromy of $\mathcal{D}$. Therefore, the corresponding polygon satisfies $v_{k+n} = \psi(v_k)$, where $\psi \in \mathrm{PGL}_3(\mathbb{R})$ is the projective transformation determined by the linear operator A .

DEFINITION 2.3. A *twisted n -gon* in $\mathbb{RP}^2$ is a polygon $\{v_k\}$ which satisfies $v_{k+n} = \psi(v_k)$ for some projective transformation $\psi \in \mathrm{PGL}_3$, called the *monodromy of the polygon*.

The above construction (see the proof of Proposition 2.2) allows one to identify the space of projective equivalence classes of twisted n -gons with an appropriate quotient of the space of n -periodic properly bounded difference operators supported in $[j, j+3]$. Under this identification, closed polygons correspond to those operators whose monodromy is a scalar multiple of the identity (furthermore, one can arrange that the monodromy of an operator corresponding to a closed polygon is exactly the identity).

Remark 2.4. One can adapt the proof of Proposition 2.2 to show that for every $j \in \mathbb{Z}$ there is a one-to-one correspondence between projective equivalence classes of twisted planar n -gons and properly bounded n -periodic difference operators $\mathcal{D}$ supported in $[j, j+3]$, considered up to the action $\mathcal{D} \mapsto \lambda \circ \mathcal{D} \circ \mu^{-1}$, where $\lambda, \mu \in (\mathbb{R}^*)^\infty$ are sequences of non-zero real numbers which are n -quasi-periodic and have the same monodromy. In other words, one has $\lambda, \mu \in (\mathbb{R}^*)^\infty \cap Q^n(z)$ for some $z \in \mathbb{R}^*$ where the space $Q^n(z)$ is defined by (5). We do not use the result in this form in the paper, so we omit the proof.

We also mention that when n is not divisible by three, the above-described action of quasi-periodic sequences on difference operators admits a section given by n -periodic operators of the form $T^j + bT^{j+1} + aT^{j+2} - T^{j+3}$. This is essentially the content of [OST10, Proposition 4.1]. As a result, the entries of periodic sequences a, b constitute a global coordinate system on the space of projective equivalence classes of twisted planar n -gons. Such a coordinate system is no longer available when n is divisible by three, however our description of the twisted n -gons space as a quotient remains valid.

Dual difference operators correspond to projectively dual polygons. Recall that the dual of a polygon is the polygon in the dual projective plane whose vertices are the sides of the initial one. Note that although there is, in general, no canonical way to label the vertices of the dual polygon, polygons with odd number of vertices admit one particular labeling which is more symmetric than the others. This labeling is depicted in Figure 4. For closed polygons, such labeling makes

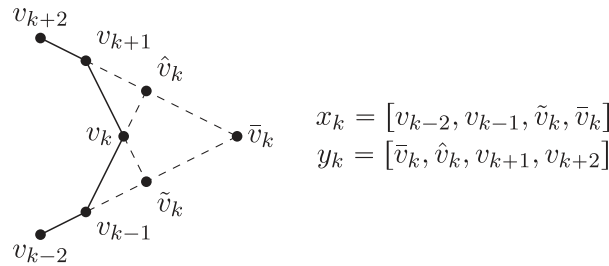


FIGURE 5. The definition of corner invariants.

projective duality an involution. For twisted polygons, it is only an involution up to the action of the monodromy, but still an actual involution on projective equivalence classes.

DEFINITION 2.5. Let P be a closed or twisted n -gon with odd n . Then the k th vertex of its *dual polygon* P^* is the side of P which joins the vertices with indices $k + (n - 1)/2$, $k + (n + 1)/2$. A polygon P is called *self-dual* if it is projectively equivalent to its dual polygon P^* .

Remark 2.6. Closed self-dual polygons are studied in [FT09]. In that paper, polygons which are self-dual in the sense of Definition 2.5 are called n -self-dual (where n is the number of vertices).

With our definition of duality, we have the following result.

PROPOSITION 2.7. Let n be odd. Consider a n -periodic properly bounded difference operator supported in $[(n - 3)/2, (n + 3)/2]$ and its dual operator $\mathcal{D}^*$. Then the polygons corresponding to $\mathcal{D}$ and $\mathcal{D}^*$ are dual to each other in the sense of Definition 2.5.

Proof. This follows from more general Proposition A.1 in Appendix A. $\square$

2.3 Corner invariants and rescaling

Another description of the space of polygons modulo projective transformations is by means of so-called *corner invariants* [Sch08, OST10]. To every vertex v_k of a polygon, one associates two cross-ratios x_k, y_k , as shown in Figure 5. The definition of the cross-ratio that we use is

$$[t_1, t_2, t_3, t_4] := \frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_3)(t_2 - t_4)}.$$

Remark 2.8. Note that the definition of x_k, y_k requires somewhat more than the 3-in-a-row condition. However, we do not need to care about this, because in this paper we only deal with weakly convex polygons (see Definition 3.1 below) for which the numbers x_k, y_k are well-defined by definition.

Clearly, the sequences x_k, y_k only depend on the projective equivalence class of the polygon. Furthermore, in the twisted case these sequences are n -periodic, and $\{x_1, y_1, \dots, x_n, y_n\}$ is a coordinate chart on an open dense subset of twisted n -gons modulo projective transformations. Therefore, because the pentagram map preserves the space of twisted polygons and commutes with projective transformations, it can be written in terms of the (x, y) coordinates. The following formulas were obtained in [OST10].

PROPOSITION 2.9. One has

$$x'_k = x_k \frac{1 - x_{k-1}y_{k-1}}{1 - x_{k+1}y_{k+1}}, \quad y'_k = y_{k+1} \frac{1 - x_{k+2}y_{k+2}}{1 - x_k y_k},$$

where x_k, y_k are the corner invariants of the polygon P , and x'_k, y'_k are corner invariants of its pentagram image P' .

Here we label P' as in [OST10]. The labeling used in Figure 3 leads to the same formulas with a certain shift of indices. Although we never use these explicit formulas, we use the following corollary: the pentagram map, with any labeling of vertices, commutes with the 1-parametric group of transformations R_s given by

$$R_s: x_k \mapsto sx_k, \quad y_k \mapsto s^{-1}y_k. \quad (9)$$

These transformations are known as *rescaling*. They were introduced in [OST10] to prove that the pentagram map is a completely integrable system.

We now discuss the relation between two representations of the space of polygons: in terms of difference operators, and in terms of corner invariants.

PROPOSITION 2.10. *Assume we are given a polygon P defined by a difference operator (8) supported in $[j, j+3]$. Then the corner invariants of P are given by*

$$x_{k+j+2} = \frac{c_k a_{k+1}}{b_k b_{k+1}}, \quad y_{k+j+2} = \frac{d_k b_{k+1}}{c_k c_{k+1}}. \quad (10)$$

Proof. The proof is a computation following the lines of the proof of Lemma 4.5 in [OST10]. $\square$

Formulas (10) allow one to describe rescaling operation (9) in terms of difference operators as follows.

COROLLARY 2.11. *In terms of difference operators, rescaling (9) can be defined as*

$$aT^j + bT^{j+1} + cT^{j+2} + dT^{j+3} \mapsto aT^j + bT^{j+1} + s(cT^{j+2} + dT^{j+3}).$$

Proof. Formulas (10) show that multiplying c and d coefficients by s is equivalent to multiplying x variables by s and y variables by s^{-1} , which is exactly rescaling (9). $\square$

Remark 2.12. Note that because there are many operators corresponding to a given polygon, there are also many different ways to define the rescaling on operators. For example, the following formula defines the same operation on polygons as the formula provided above:

$$aT^j + bT^{j+1} + cT^{j+2} + dT^{j+3} \mapsto aT^j + s^{-1/3}bT^{j+1} + s^{1/3}cT^{j+2} + dT^{j+3}.$$

3. Weakly convex polygons

In this section we define weakly convex polygons and describe their properties needed to prove Theorems B and A.

DEFINITION 3.1. A polygon is *weakly convex* if its corner invariants are well-defined and satisfy

$$x_k > 0, \quad y_k > 0, \quad x_k y_k < 1.$$

PROPOSITION 3.2. *Convex polygons are weakly convex.*

Proof. For convex polygons, the collinear points $v_{k-2}, v_{k-1}, \tilde{v}_k, \bar{v}_k$ in Figure 5 are distinct and their cyclic order is exactly as shown. Thus, x_k is well-defined and $0 < x_k < 1$. Likewise, we have $y_k \in (0, 1)$. The result follows. $\square$

Remark 3.3. More generally, all corner invariants of a polygon satisfy $x_k, y_k \in (0, 1)$ if and only if any five consecutive vertices of that polygon form a convex pentagon (where by a convex pentagon in $\mathbb{RP}^2$ we mean a pentagon which is convex in a suitable affine chart). Thus, all

polygons satisfying this ‘5-in-a-row’ condition are weakly convex. The geometric meaning of the general weak convexity condition is not that clear. However, it turns out to be really convenient for the purposes of the present paper.

The following is an exhaustive list of properties of weakly convex polygons needed for our purposes.

PROPOSITION 3.4. *Assume that P is a closed or twisted weakly convex n -gon, where $n \geq 3$ is odd.*

1. *The corresponding third-order n -periodic difference operator (8) can be chosen in such a way that for all $k \in \mathbb{Z}$ we have*

$$a_k, c_k > 0, \quad b_k, d_k < 0. \quad (11)$$

2. *For any difference operator (8) corresponding to P and satisfying (11), consider the operators $\mathcal{D}_l := aT^j + bT^{j+1}$, $\mathcal{D}_r := cT^{j+2} + dT^{j+3}$. Then the monodromies z_l, z_r of these operators (which are real numbers because the operators are of first order) satisfy $0 < z_l < z_r$. In particular, we have $\text{Ker } \mathcal{D}_l \cap \text{Ker } \mathcal{D}_r = 0$.*
3. *Any polygon obtained from P by means of rescaling (9) with $s > 0$ is weakly convex.*
4. *Any polygon obtained from P by means of rescaling (9) with $s < 0$ has monodromy with distinct eigenvalues.*

Proof. To prove the first statement, consider the corner invariants $x_k, y_k > 0$ of P , and let

$$\mathcal{D} := T^j - T^{j+1} + x_{k+j+2}T^{j+2} - x_{k+j+2}y_{k+j+2}x_{k+j+3}T^{j+3}.$$

Then, by Proposition 2.10, the polygon associated with $\mathcal{D}$ is P . Furthermore, the signs of coefficients of $\mathcal{D}$ satisfy (11), as needed.

To prove the second statement, consider an arbitrary operator (8) representing P and satisfying (11), along with the associated operators $\mathcal{D}_r, \mathcal{D}_l$. Then, by formula (6), the monodromies of those operators are given by

$$z_l = - \prod_{k=1}^n \frac{a_k}{b_k}, \quad z_r = - \prod_{k=1}^n \frac{c_k}{d_k},$$

so $z_l, z_r > 0$ due to (11) and n being odd. Further, using formulas (10), we obtain

$$\frac{z_l}{z_r} = \prod_{k=1}^n \frac{a_k d_k}{b_k c_k} = \prod_{k=1}^n x_k y_k,$$

where x_k, y_k are the corner invariants of P . However, because P is weakly convex, we have $x_k y_k < 1$ and, thus, $z_l < z_r$. This, in turn, implies $\text{Ker } \mathcal{D}_l \cap \text{Ker } \mathcal{D}_r = 0$, because non-zero elements of the kernel of $\mathcal{D}_l$ are sequences with monodromy z_l , whereas non-zero elements of the kernel of $\mathcal{D}_r$ are sequences with monodromy $z_r \neq z_l$. Thus, the second statement is proved.

The third statement is obvious from the definitions of weak convexity and rescaling, so we proceed to the fourth statement. Let P_{sd} be a polygon obtained from P by means of rescaling with $s < 0$. Then the corner invariants $\hat{x}_k, \hat{y}_k$ of P_{sd} satisfy $\hat{x}_k, \hat{y}_k < 0$, $\hat{x}_k \hat{y}_k < 1$. To show that the monodromy of such a polygon has distinct eigenvalues, we use a result from [Izo16, Appendix A] which states that the monodromy of a twisted n -gon with corner invariants $\hat{x}_k, \hat{y}_k$ is conjugate

to the product $L_1 \cdots L_n$, where

$$L_k := \begin{pmatrix} 1 & 0 & 1 \\ 1 - \hat{x}_k \hat{y}_k & 0 & 1 \\ 0 & -\hat{y}_k & 0 \end{pmatrix}.$$

Note that because $\hat{y}_k < 0$ and $\hat{x}_k \hat{y}_k < 1$, the matrices L_k are non-negative. Furthermore, the product of at least three such matrices is positive, so the matrix $M := L_1 \cdots L_n$ is positive. Therefore, by the Perron–Frobenius theorem, M has a real positive eigenvalue z_1 such that its any other eigenvalue z satisfies $|z| < z_1$. Furthermore, because $\hat{x}_k < 0$ and n is odd, we have $\det M = \hat{x}_1 \cdots \hat{x}_n (\hat{y}_1 \cdots \hat{y}_n)^2 < 0$, so the product of two other eigenvalues z_2, z_3 of M is negative, which means that they are real and distinct. The result follows. $\square$

4. Polygons fixed by the pentagram map and commuting difference operators

In this section we show that a closed or twisted polygon P projectively equivalent to its pentagram image P' gives rise to commuting difference operators. This is the first step in the proof of both Theorems A and B. In addition, in the self-dual case (i.e. in the setting of Theorem B) we show that those commuting operators are negative duals of each other.

Let $P = \{v_k\}$ be a closed or twisted n -gon with odd n , and let $P' = \{v'_k\}$ be the image of P under the pentagram map, labeled as in Figure 3. Then, as explained in § 2.2, one can encode P' by means of a difference operator $\mathcal{D}$ of the form

$$\mathcal{D} = aT^{(n-3)/2} + bT^{(n-1)/2} + cT^{(n+1)/2} + dT^{(n+3)/2}. \quad (12)$$

The coefficients of this operator are related to the polygon P' by means of the equation

$$a_k V'_{i+(n-3)/2} + b_k V'_{k+(n-1)/2} + c_k V'_{k+(n+1)/2} + d_k V'_{k+(n+3)/2} = 0,$$

where V'_k are lifts of the vertices v'_k of P' .

PROPOSITION 4.1. *The vector*

$$V_k := a_k V'_{k+(n-3)/2} + b_k V'_{k+(n-1)/2} = -c_k V'_{k+(n+1)/2} - d_k V'_{k+(n+3)/2}$$

is the lift of the vertex v_k of P .

Proof. Indeed, we have

$$V_k \in \text{span}(V'_{k+(n-3)/2}, V'_{k+(n-1)/2}) \cap \text{span}(V'_{k+(n+1)/2}, V'_{k+(n+3)/2}),$$

which means that the projection of V_k to $\mathbb{P}^2$ is the intersection point of the lines $(v'_{k+(n-3)/2}, v'_{k+(n-1)/2})$ and $(v'_{k+(n+1)/2}, v'_{k+(n+3)/2})$. By the definition of the pentagram map with our labeling convention, this is exactly the vertex v_k of P , as desired. $\square$

Now, as in Proposition 3.4, consider the operators

$$\mathcal{D}_l := aT^{(n-3)/2} + bT^{(n-1)/2}, \quad \mathcal{D}_r := \mathcal{D} - \mathcal{D}_l = cT^{(n+1)/2} + dT^{(n+3)/2}. \quad (13)$$

By Proposition 4.1, these operators take the lifts V'_k of the vertices of P' to the lifts $\pm V_k$ of the vertices of P .

PROPOSITION 4.2. *Assume that a closed or twisted n -gon (where $n \geq 5$ is arbitrary) P is projectively equivalent to its pentagram image P' .*

1. One can choose the n -periodic operator $\mathcal{D}$ of the form (12) associated with P in such a way that the corresponding operators $\mathcal{D}_l, \mathcal{D}_r$ given by (13) commute:

$$\mathcal{D}_l \mathcal{D}_r = \mathcal{D}_r \mathcal{D}_l. \quad (14)$$

2. Furthermore, if P is weakly convex and n is odd, then $\mathcal{D}$ can be chosen to satisfy the alternating signs condition (11).
3. If, on top of that, P is self-dual, then $\mathcal{D}_l, \mathcal{D}_r$ may be chosen to be negative duals of each other (up to multiplication by T^{-n}):

$$\mathcal{D}_l^* = -T^{-n} \mathcal{D}_r.$$

Equivalently, the operator $\mathcal{D} = \mathcal{D}_l + \mathcal{D}_r$ can be chosen to be anti-self-dual (again, up to multiplication by T^{-n}):

$$\mathcal{D}^* = -T^{-n} \mathcal{D}.$$

Proof of Proposition 4.2. We begin with the first statement. Take an arbitrary n -periodic operator $\tilde{\mathcal{D}}$ of the form (12) representing the polygon P . Then, because P' is projectively equivalent to P , there is a fundamental solution V' of $\tilde{\mathcal{D}}$ such that the projection of V'_k to $\mathbb{P}^2$ is the k th vertex of P' . Consider the projective transformation taking P' to P . Any lift $A \in \mathrm{GL}_3(\mathbb{R})$ of this projective transformation will then take the sequence V' to a sequence of lifts of vertices of P . On the other hand, by Proposition 4.1, lifts of vertices of P are given by the sequence $\tilde{\mathcal{D}}_l V'$. Thus, there is an n -periodic sequence μ of non-zero real numbers such that $AV' = \mu \tilde{\mathcal{D}}_l V'$, where A acts on sequences of vectors by term-wise multiplication. Let $\mathcal{D} := \mu \tilde{\mathcal{D}}$. Then the operator $\mathcal{D}$ still satisfies $\mathcal{D}V' = 0$ and, hence, represents the same polygons P and P' . Furthermore, the corresponding operator $\mathcal{D}_l$ satisfies $AV' = \mathcal{D}_l V'$. Applying the operator $\mathcal{D}$ to both sides, we obtain $\mathcal{D}\mathcal{D}_l V' = 0$. Also taking into account that $\mathcal{D}V' = 0$, this can be rewritten as

$$(\mathcal{D}\mathcal{D}_l - \mathcal{D}_l\mathcal{D})V' = 0. \quad (15)$$

At the same time, we have

$$\mathcal{D}\mathcal{D}_l - \mathcal{D}_l\mathcal{D} = (\mathcal{D}_l + \mathcal{D}_r)\mathcal{D}_l - \mathcal{D}_l(\mathcal{D}_l + \mathcal{D}_r) = \mathcal{D}_r\mathcal{D}_l - \mathcal{D}_l\mathcal{D}_r = [\mathcal{D}_r, \mathcal{D}_l],$$

so (15) gives

$$[\mathcal{D}_r, \mathcal{D}_l]V' = 0. \quad (16)$$

Now it remains to note that the commutator $[\mathcal{D}_r, \mathcal{D}_l]$ is of the form $\alpha T^{n-1} + \beta T^n + \gamma T^{n+1}$, so (16) is equivalent to $\alpha_k V'_{k+n-1} + \beta_k V'_{k+n} + \gamma_k V'_{k+n+1} = 0$, which, in view of the 3-in-a-row condition for P' (which holds because it holds for P), gives $\alpha_k = \beta_k = \gamma_k = 0$ and, thus, $[\mathcal{D}_r, \mathcal{D}_l] = 0$, as desired.

To prove the second statement, one repeats the same argument, with the only modification that the initial operator $\tilde{\mathcal{D}}$ should be chosen to satisfy (11), which can be done by the first statement of Proposition 3.4. Then the coefficients a, b, c, d of the operator $\mathcal{D} = \mu \tilde{\mathcal{D}}$ satisfy $\mathrm{sgn}(a_k) = -\mathrm{sgn}(b_k) = \mathrm{sgn}(c_k) = -\mathrm{sgn}(d_k)$. Furthermore, we claim that a_k are all of the same sign. Indeed, using explicit formulas (13) for $\mathcal{D}_l$ and $\mathcal{D}_r$ and equating the coefficients of T^{n-1} in (14), we obtain

$$a_k c_{k+(n-3)/2} = c_k a_{k+(n+1)/2}. \quad (17)$$

Furthermore, we have $\mathrm{sgn} c_j = \mathrm{sgn} a_j$ for any j , so taking the signs of both sides of (17) we obtain

$$\mathrm{sgn} a_{k+(n+1)/2} = \mathrm{sgn} c_{k+(n-3)/2} = \mathrm{sgn} a_{k+(n-3)/2},$$

which means that the sequence $\operatorname{sgn} a_k$ is 2-periodic. However, because the period n of the sequence a_k is an odd number, it follows that $\operatorname{sgn} a_k = \operatorname{const}$. Now, multiplying $\mathcal{D}$ by -1 is necessary, we can arrange that $a_k > 0$ for all k , so that $\mathcal{D}$ satisfies (11), as needed.

To prove the third statement, we consider the operator $\mathcal{D}$ constructed above and show that if P is self-dual, then $\mathcal{D}$ can be replaced with another operator, which has all the properties of $\mathcal{D}$ and is, in addition, anti-self-dual. To that end, observe that if P is self-dual, then the operators $\mathcal{D}$ and $\mathcal{D}^*$ represent the same polygon, so

$$\mathcal{D}^* = \alpha T^{-n} \mathcal{D} \beta^{-1} \quad (18)$$

for certain sequences α, β of non-zero real numbers. Taking the duals, we obtain

$$\mathcal{D} = \beta^{-1} T^n \mathcal{D}^* \alpha = \alpha \beta^{-1} \mathcal{D} \alpha \beta^{-1},$$

which implies $\beta = \pm \alpha$. Further, because $\mathcal{D}$ satisfies (11), the corresponding coefficients of $\mathcal{D}^*$ and $T^{-n} \mathcal{D}$ are of opposite sign, so (18) implies $\operatorname{sgn}(\alpha_k) = -\operatorname{sgn}(\beta_k) = \operatorname{const}$, and we must have $\beta = -\alpha$. Therefore,

$$\mathcal{D}^* = -\alpha T^{-n} \mathcal{D} \alpha^{-1},$$

where $\operatorname{sgn}(\alpha_k) = \operatorname{const}$ and without loss of generality we can assume $\alpha_k > 0$. Furthermore, because both operators $\mathcal{D}^*$, $T^{-n} \mathcal{D}$ are n -periodic, the sequence α is quasi-periodic, that is, $\alpha_{k+n} = z \alpha_k$ for some $z \in \mathbb{R}^*$ (actually, $z \in \mathbb{R}_+$). Now, let $\gamma_k := \sqrt{\alpha_k}$. Then the sequence γ is also quasi-periodic, so the operator $\mathcal{D}' := \gamma \mathcal{D} \gamma^{-1}$ is n -periodic. Moreover, it has all the properties of $\mathcal{D}$ and is anti-self-dual. Thus, the proposition is proved. $\square$

Remark 4.3. It is easy to see from the proof that the converse of the first statement is also true: if $\mathcal{D}_r$ and $\mathcal{D}_l$ commute, then P is projectively equivalent to P' . For instance, consider the polygon P from Remark 1.3. The vertices of that polygon can be lifted to vectors $V_k := (\lambda^{2k}, \lambda^{3k}, 1)$, where $\lambda := \exp(2\pi i/7)$. The sequence V_k is annihilated by a difference operator with constant coefficients, namely by $\mathcal{D} := aT^{(n-3)/2} + bT^{(n-1)/2} + cT^{(n+1)/2} + dT^{(n+3)/2}$, where $a, b, c, d \in \mathbb{C}$ are such that the roots of the corresponding characteristic equation $a + bx + cx^2 + dx^3 = 0$ are λ^2 , λ^3 , and 1. Therefore, the associated operators $\mathcal{D}_r$ and $\mathcal{D}_l$ also have constant coefficients and hence commute. Thus, the polygon P is indeed projectively equivalent to its pentagram image P' .

5. The spectral curve

The results of this section are central to the proof of Theorem B (and are also used to derive Theorem A from Theorem B). Namely, in § 5.1 we consider the joint spectrum of the commuting difference operators $\mathcal{D}_l$, $\mathcal{D}_r$ constructed above (see § 4), the so-called *spectral curve*, and show that the genus g of that curve is at most one. We note that this estimate on the genus is not predicted by the general theory of commuting difference operators. It seems that the best bound one can get from the general theory is $g \leq 2$. Proving the $g \leq 1$ estimate requires somewhat more careful analysis of the field of meromorphic functions. Also note that even the $g \leq 1$ result is still insufficient to prove Theorem B. Another important ingredient of the proof is the so-called *eigenvector function*, which encodes the joint eigenvectors of the commuting operators $\mathcal{D}_l$, $\mathcal{D}_r$. We study that function in § 5.2 and, in particular, prove that it has very few poles.

5.1 The spectral curve and a bound on its genus

In this section, we construct the spectral curve associated with commuting difference operators $\mathcal{D}_l$, $\mathcal{D}_r$ given by Proposition 7.1 and discuss its properties, in particular prove that its genus is at most one.

Remark 5.1. Note that instead of defining the spectral curve using commuting difference operators, we could have done this using the Lax representation, as in [Sol13]. However, at the end of the day these two definitions turn out to be equivalent to each other (see Remark 5.14). Furthermore, even if we defined the spectral curve using the Lax representation, we would still need commuting difference operators to establish the properties of the curve that we need. Thus, all in all, these two approaches are equivalent, and our choice is just a matter of convenience.

Remark 5.2. A different notion of a spectral curve corresponding to a difference operator (and, hence, a polygon) is defined in [Kri15]. In our terminology, it is the spectral curve corresponding to commuting difference operators $\hat{\mathcal{D}}$, T^n where $\hat{\mathcal{D}}$ is a difference operator supported in $[0, 3]$ associated with a given polygon. As the operator $\hat{\mathcal{D}}$ does not, generally speaking, commute with $\mathcal{D}_l$ and $\mathcal{D}_r$, the corresponding spectral curve seems to have no relation to ours.

Assume that P is a weakly convex twisted n -gon, self-dual, and projectively equivalent to its pentagram image. Then, by Proposition 4.2, we have an n -periodic operator $\mathcal{D}_l = aT^{(n-3)/2} + bT^{(n-1)/2}$ which commutes with its dual. For notational convenience, we define

$$\mathcal{D}_+ := \mathcal{D}_l, \quad \mathcal{D}_- := \mathcal{D}_l^* = -T^{-n}\mathcal{D}_r.$$

Periodicity of these operators means that they also commute with T^n . Therefore, we have a whole algebra $\mathcal{A}$ of commuting operators, generated by $\mathcal{D}_+$, $\mathcal{D}_-$, and T^n (to preserve the left–right symmetry, it is natural to include T^{-n} in $\mathcal{A}$ too, so that $\mathcal{A} = \mathbb{C}[\mathcal{D}_+, \mathcal{D}_-, T^{\pm n}]$). To such an algebra $\mathcal{A}$, one can always associate an algebraic curve. This curve may be constructed using any two generic elements of $\mathcal{A}$. As such elements, we pick the operators T^n and the product $\mathcal{D}_+\mathcal{D}_- = \mathcal{D}_-\mathcal{D}_+$. This choice is motivated by a particularly simple form of the operator $\mathcal{D}_+\mathcal{D}_-$. Namely, that operator is self-dual and supported in $[-1, 1]$:

$$\mathcal{D}_+\mathcal{D}_- = T^{-1}\alpha + \beta + \alpha T, \quad (19)$$

where α, β are n -periodic sequences, and $\alpha_k \neq 0$ for any k .

DEFINITION 5.3. The *affine spectral curve* Γ_a is the joint spectrum of T^n and $\mathcal{D}_+\mathcal{D}_-$:

$$\Gamma_a := \{(z, w) \in \mathbb{C}^* \times \mathbb{C} \mid \exists \xi \in \mathbb{R}^\infty : \xi \neq 0, T^n \xi = z\xi, \mathcal{D}_+\mathcal{D}_- \xi = w\xi\}.$$

In other words, a point $(z, w) \in \mathbb{C}^* \times \mathbb{C}$ is in Γ_a if and only if w is an eigenvalue of the restriction of $\mathcal{D}_+\mathcal{D}_-$ to the space $Q^n(z)$ defined by (5).

To obtain an explicit equation of the affine spectral curve Γ_a , take a basis $e^1, \dots, e^n$ in $Q^n(z)$ determined by the condition $e_k^j = \delta_k^j$ for $k = 1, \dots, n$. In this basis, the matrix of the operator $\mathcal{D}_+\mathcal{D}_-$ is almost tridiagonal, with two additional elements in the upper-right and bottom-left corners:

$$\begin{pmatrix} \beta_1 & \alpha_1 & & & \alpha_n z^{-1} \\ \alpha_1 & \beta_2 & \alpha_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_{n-2} & \beta_{n-1} & \alpha_{n-1} \\ \alpha_n z & & & \alpha_{n-1} & \beta_n \end{pmatrix}. \quad (20)$$

The affine spectral curve Γ_a is the zero locus of the characteristic polynomial of (20), which, up to the factor $\alpha_1 \dots \alpha_n$, reads

$$p(z, w) = z^{-1} + q(w) + z \quad (21)$$

for a certain polynomial $q(w)$ of degree n . In particular, the spectral curve is algebraic, as predicted by the general theory of commuting difference operators.

PROPOSITION 5.4. *The affine spectral curve Γ_a is irreducible.*

Proof. This curve is the zero locus of the polynomial (21), which is irreducible whenever $q(w)$ is non-constant. $\square$

We now define the *spectral curve* Γ as the Riemann surface corresponding to the affine curve Γ_a . In other words, Γ is the unique Riemann surface biholomorphic to Γ_a away from a finite number of points. The existence of such a Riemann surface is guaranteed by Riemann's theorem. It can be obtained from Γ_a by means of normalization (which we actually explicitly construct in Remark 5.23), followed by compactification. As Γ_a is irreducible (Proposition 5.4), it follows that Γ is connected. Furthermore, the Riemann surface Γ comes equipped with:

- two meromorphic functions z and w , obtained from coordinate functions on Γ_a , and satisfying the equation $p(z, w) = 0$, with p given by (21);
- a holomorphic involution $\sigma: \Gamma \rightarrow \Gamma$, coming from the involution $(z, w) \mapsto (z^{-1}, w)$ on Γ_a , and satisfying $\sigma^*w = w$ and $\sigma^*z = z^{-1}$.

PROPOSITION 5.5. *The degrees of the functions w and z on Γ are equal to 2 and n , respectively.*

Proof. As the polynomial $q(w)$ in (21) has degree n , the equation $p(z, w) = 0$ has n solutions in terms of w for generic z . Thus, the degree of z on Γ is n . Likewise, the number of solutions of $p(z, w) = 0$ in terms of z is 2 for generic w , so the degree of w is 2. $\square$

As w is a function of degree 2, and σ is a non-trivial involution preserving w , it follows that σ interchanges the two points in any level set of w . In particular, the fixed points set of σ coincides with the set of branch points of w , that is, points where $dw = 0$. At the end of this section we show that the number of such branch points is at most four, which implies, by the Riemann–Hurwitz formula, that the genus of Γ is at most one. However, first we need to discuss in detail the analytic properties of the functions z and w , as well as of some other functions on Γ which we introduce in the following.

PROPOSITION 5.6. *The Riemann surface Γ is obtained from the normalization of Γ_a by adding two points $Z_{\pm}$, interchanged by the involution σ . The point Z_+ is a zero of order n for the function z , whereas Z_- is its pole of order n . Both points are simple poles of the function w .*

Proof. Let $\Gamma_n \subset \Gamma$ be the normalization of Γ_a . This set can be described as the preimage of Γ_a under the map $(z, w): \Gamma \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Also note that the image of Γ under the latter map is precisely the closure of Γ_a in $\mathbb{P}^1 \times \mathbb{P}^1$, which consists of Γ_a and the points $(0, \infty)$ and (∞, ∞) . Thus, the image of $\Gamma \setminus \Gamma_n$ under the map (z, w) is two points $(0, \infty)$ and (∞, ∞) . This means, first, that any point in $\Gamma \setminus \Gamma_n$ is a pole of w , and second, that there are at least two such points. However, because w has degree 2 (Proposition 5.5), it follows that $\Gamma \setminus \Gamma_n$ consists of exactly two points, and that these points are simple poles of w . Furthermore, at one of these points, which we denote by Z_+ , we have $z = 0$, whereas at the other, which we call Z_- , we have $z = \infty$. Finally, note that because all points of Γ except $Z_{\pm}$ belong to Γ_n , it follows that z does not have zeros

TABLE 1. The orders of the functions $z, w, s, \mu_{\pm}$ at the points $Z_{\pm}, S_{\pm} \in \Gamma$. The order of these functions at any other point of Γ is zero.

Function	Degree	Order at Z_+	Order at Z_-	Order at S_+	Order at S_-
z	n	n	$-n$	0	0
w	2	-1	-1	1	1
s	3	-2	2	1	-1
μ_+	$(n-1)/2$	$(n-3)/2$	$-(n-1)/2$	1	0
μ_-	$(n-1)/2$	$-(n-1)/2$	$(n-3)/2$	0	1

or poles except for $Z_{\pm}$. Thus, Z_+ is a zero of z of order n , whereas Z_- is a pole of order n , as desired. $\square$

Denote also by $S_{\pm}$ the two zeros of the function w on Γ . *A priori*, these two points may coincide, but later we show that they are distinct (see the proof of Proposition 5.18). Table 1 summarizes information about the orders of the functions z and w at the points $Z_{\pm}, S_{\pm}$ (recall that the *order* of a meromorphic function f at a point X is equal to m if f has a zero of order m at X , $-m$ if f has a pole of order m at X , and zero otherwise). Also note that the order of z and w at any other point of Γ is equal to zero. The table also contains information about functions s and $\mu_{\pm}$, which we introduce in the following.

PROPOSITION 5.7. *The pair $\mathcal{D}_+\mathcal{D}_-, T^n$ of commuting difference operators is of rank one, which means that the generic common eigenspace of these operators is one-dimensional.*

Proof. As follows from the explicit form (21) for the characteristic polynomial of the matrix (20), for generic z that matrix has distinct eigenvalues and, hence, one-dimensional eigenspaces. $\square$

For a generic point $(z, w) \in \Gamma_a$, let $\xi \in \mathbb{R}^{\infty}$ be the corresponding common eigenvector of $\mathcal{D}_+\mathcal{D}_-$ and T^n , normalized by the condition $\xi_0 = 1$.

PROPOSITION 5.8. *The components ξ_k of ξ extend to meromorphic functions on Γ . The corresponding vector-function ξ on Γ satisfies the equations*

$$T^n \xi = z\xi, \quad \mathcal{D}_+\mathcal{D}_-\xi = w\xi. \quad (22)$$

Proof. For $(z, w) \in \Gamma_a$, let $\eta = (\eta_1, \dots, \eta_n)$ be the first row of the comatrix of $L - w\text{Id}$, where L is given by (20). Extend η to a bi-infinite sequence by the rule $\eta_{k+n} = z\eta_k$. Then η is a common eigenvector of T^n and $\mathcal{D}_+\mathcal{D}_-$: $T^n\eta = z\eta$, $\mathcal{D}_+\mathcal{D}_-\eta = w\eta$. Furthermore, the components of η are, by construction, rational functions of z and w . Thus, to obtain the desired function ξ , it remains to normalize η : $\xi_k = \eta_k/\eta_0$. Note that $\eta_0 = z^{-1}\eta_n$ does not vanish identically on Γ_a , because η_n is a polynomial in z, w which is linear in z and, hence, cannot be divisible by the defining polynomial of Γ_a . Thus, ξ is a well-defined rational vector-function of z, w and, hence, a meromorphic function on Γ . $\square$

We call the vector-function ξ the *eigenvector function*. Its analytic properties are studied in detail in §5.2.

Remark 5.9. Note that at every point $X \in \Gamma \setminus \{Z_{\pm}\}$, the vector-function ξ is meromorphic in the following strong sense: there exists a local holomorphic function f such that $f\xi$ is holomorphic at X . Moreover, the function f can be chosen in such a way that $(f\xi)(X)$ does not vanish. Therefore, the value of the function ξ at any point $X \in \Gamma \setminus \{Z_{\pm}\}$ determines a direction in the infinite-dimensional projective space $\mathbb{P}^{\infty}$, regardless of whether the components of ξ are finite or infinite

(note also that this direction does not change if we replace our particular normalization $\xi_0 = 1$ by any other normalization). This is not true, however, at the points $Z_{\pm}$. At those points, the components ξ_k of ξ are still meromorphic, but the order of the pole of ξ_k is an unbounded function of k (see Proposition 5.21), so there exists no f such that $f\xi$ is holomorphic. In particular, the value of ξ at $Z_{\pm}$ does not determine any direction.

We now show that every operator $\mathcal{L}$ from the commutative algebra $\mathcal{A} = \langle \mathcal{D}_+, \mathcal{D}_-, T^{\pm n} \rangle$ gives rise to a meromorphic function $f_{\mathcal{L}}$ on Γ , which is holomorphic everywhere except possibly the points $Z_{\pm}$ and satisfies $\mathcal{L}\xi = f_{\mathcal{L}}\xi$. In particular, the assignment $\mathcal{L} \mapsto f_{\mathcal{L}}$ is a homomorphism from $\mathcal{A}$ to the algebra of meromorphic functions on Γ which are holomorphic in $\Gamma \setminus \{Z_{\pm}\}$. We already have $f_{T^{\pm n}} = z^{\pm 1}$ and $f_{\mathcal{D}_+ \mathcal{D}_-} = w$, so it remains to construct the functions $f_{\mathcal{D}_+}$ and $f_{\mathcal{D}_-}$ (of course, one of them determines the other, because their product must be equal to w). We denote these functions by μ_+ and μ_- .

PROPOSITION 5.10. *There exist meromorphic functions μ_+, μ_- on Γ which are holomorphic in $\Gamma \setminus \{Z_{\pm}\}$ and satisfy $\mathcal{D}_{\pm}\xi = \mu_{\pm}\xi$. Furthermore, we have $\mu_+\mu_- = w$.*

Proof. By Proposition 5.7, a generic common eigenspace of $\mathcal{D}_+ \mathcal{D}_-$ and T^n is one-dimensional, and is therefore generated by the vector ξ , evaluated at the corresponding point of the Riemann surface Γ . For this reason, because the operator $\mathcal{D}_+$ commutes with $\mathcal{D}_+ \mathcal{D}_-$ and T^n , at generic points of Γ we must have

$$\mathcal{D}_{\pm}\xi = \mu_{\pm}\xi \quad (23)$$

for certain numbers $\mu_{\pm} \in \mathbb{C}$ depending on the point of Γ . Furthermore, because the left-hand side of (23) is a meromorphic vector-function on Γ , and so is ξ , it follows that the functions $\mu_{\pm}$ also extend to meromorphic functions on the whole of Γ . Moreover, given a point $X \in \Gamma \setminus \{Z_{\pm}\}$, renormalizing ξ if necessary, we can assume that $\xi(X)$ is finite and non-zero (see Remark 5.9). However, then (23) implies that the functions $\mu_{\pm}$ are holomorphic at X . Finally, the equation $\mu_+\mu_- = w$ follows directly from (23) and the second of (22). $\square$

PROPOSITION 5.11. *We have $\sigma^*\mu_+ = \mu_-$.*

Remark 5.12. The existence of the involution σ on Γ is due to the invariance of the algebra $\mathcal{A} = \langle \mathcal{D}_+, \mathcal{D}_-, T^{\pm n} \rangle$ under operator duality: $\mathcal{A} = \mathcal{A}^* := \{\mathcal{L}^* \mid \mathcal{L} \in \mathcal{A}\}$. Thus, because $\mathcal{D}_+^* = \mathcal{D}_-$, it is only natural that $\sigma^*\mu_+ = \mu_-$.

Proof of Proposition 5.11. It suffices to show that $\mu_+(X_+) = \mu_-(X_-)$, where $X_{\pm}$ is a generic pair of points interchanged by σ . As the points $X_{\pm}$ are generic, one can assume that the vectors $\xi(X_{\pm})$ are finite. Under this assumption, we have $\mathcal{D}_{\pm}\xi(X_{\pm}) = \mu_{\pm}(X_{\pm})\xi(X_{\pm})$. Furthermore, we have $\xi(X_{\pm}) \in Q^n(z_{\pm}^{\pm 1})$, where $z_+ := z(X_+) = z(X_-)^{-1}$. Thus, using the pairing (7) between $Q^n(z_+)$ and $Q^n(z_+^{-1})$, we obtain

$$\mu_+(X_+) \langle \xi(X_+), \xi(X_-) \rangle = \langle \mathcal{D}_+\xi(X_+), \xi(X_-) \rangle = \langle \xi(X_+), \mathcal{D}_-\xi(X_-) \rangle = \mu_-(X_-) \langle \xi(X_+), \xi(X_-) \rangle.$$

Thus, to complete the proof, it suffices to show that $\langle \xi(X_+), \xi(X_-) \rangle \neq 0$. To that end, observe that $\xi(X_+)$ belongs to the kernel of the operator $(\mathcal{D}_+ \mathcal{D}_- - w_0)|_{Q^n(z_+)}$, where $w_0 := w(X_{\pm})$, and, in the generic case, spans that kernel. Thus, the orthogonal complement to $\xi(X_+)$ with respect to the pairing (7) is the image of the dual operator

$$((\mathcal{D}_+ \mathcal{D}_- - w_0)|_{Q^n(z_+)})^* = (\mathcal{D}_+ \mathcal{D}_- - w_0)|_{Q^n(z_+^{-1})}.$$

However, for generic z_+ the operator $\mathcal{D}_+\mathcal{D}_-$ has simple spectrum on $Q^n(z_+^{-1})$ and is, therefore, diagonalizable, which in particular implies

$$\operatorname{Im}(\mathcal{D}_+\mathcal{D}_- - w_0)|_{Q^n(z_+^{-1})} \cap \operatorname{Ker}(\mathcal{D}_+\mathcal{D}_- - w_0)|_{Q^n(z_+^{-1})} = 0.$$

Therefore, we have $\langle \xi(X_+), \xi(X_-) \rangle \neq 0$, as desired. $\square$

Now, define a meromorphic function s on Γ by the formula

$$s := \frac{\mu_+}{z\mu_-}. \quad (24)$$

This function does not correspond to any difference operator $\mathcal{L} \in \mathcal{A} = \mathbb{C}[\mathcal{D}_+, \mathcal{D}_-, T^{\pm n}]$ but can be thought of as corresponding to a pseudo-difference operator $T^{-n}\mathcal{D}_+\mathcal{D}_-^{-1}$. Accordingly, the function s satisfies the equation

$$(\mathcal{D}_+ - sT^n\mathcal{D}_-)\xi = 0. \quad (25)$$

Recall that the operator on the left-hand side encodes the family of polygons obtained from P by means of rescaling (9).

PROPOSITION 5.13. *The function s has degree 3. There exist three distinct points on Γ at which $s = -1$. The function z takes three distinct values at those points.*

Proof. We first show that $\deg s \leq 3$. Let $X_1, \dots, X_m \in \Gamma$ belong to the level set $s = s_0$. Then, for generic $s_0 \in \mathbb{C}$, these points correspond to distinct points on the affine spectral curve Γ_a . This, in particular, means that the vectors $\xi(X_1), \dots, \xi(X_m) \in \operatorname{Ker}(\mathcal{D}_+ - s_0T^n\mathcal{D}_-)$ are linearly independent (as joint eigenvectors of T^n and $\mathcal{D}_+\mathcal{D}_-$ corresponding to distinct eigenvalues). However,

$$\dim \operatorname{Ker}(\mathcal{D}_+ - s_0T^n\mathcal{D}_-) = \operatorname{ord}(\mathcal{D}_+ - s_0T^n\mathcal{D}_-) = 3,$$

so $m \leq 3$, and the degree of s is at most 3, as desired.

We now show that there exist three distinct points on Γ at which $s = -1$, which, in turn, implies that the degree of s is exactly 3. As the polygon P , given by the operator $\mathcal{D}_+ - T^n\mathcal{D}_-$, is weakly convex, it follows from the fourth statement of Proposition 3.4 that the monodromy of $\mathcal{D}_+ + T^n\mathcal{D}_-$ has simple spectrum. This means that there exist three distinct numbers z_1, z_2, z_3 such that the operator $\mathcal{D}_+ + T^n\mathcal{D}_-$ has non-trivial (and, hence, one-dimensional) kernel on $Q^n(z_k)$. Let ξ^k be the generator of that kernel. Then, because the operator $\mathcal{D}_+\mathcal{D}_-$ commutes with $\mathcal{D}_+ + T^n\mathcal{D}_-$ and T^n , it follows that ξ^k is also an eigenvector of $\mathcal{D}_+\mathcal{D}_-$, corresponding to some eigenvalue w_k . Then the three points (z_k, w_k) belong to the affine spectral curve Γ_a and, thus, give rise to at least three points $X_1, X_2, X_3 \in \Gamma \setminus \{Z_{\pm}\}$ with $z(X_k) = z_k$, $w(X_k) = w_k$. We now claim that $s(X_1) = s(X_2) = s(X_3) = -1$. Indeed, the vector ξ_k spans the (z_k, w_k) joint eigenspace of T^n and $\mathcal{D}_+\mathcal{D}_-$. Therefore, at each of the points X_k , we have $\xi(X_k) = c_k\xi^k$, where $c_k \in \mathbb{C}$ (here we assume here that the vectors $\xi(X_k)$ are finite, which can be always arranged by multiplying ξ by an appropriate meromorphic function, see Remark 5.9). Thus, by construction of the vectors ξ^k , we have

$$(\mathcal{D}_+ + T^n\mathcal{D}_-)\xi(X_k) = c_k(\mathcal{D}_+ + T^n\mathcal{D}_-)\xi^k = 0.$$

On the other hand,

$$(\mathcal{D}_+ + T^n\mathcal{D}_-)\xi(X_k) = (\mu_+ + z\mu_-)|_{X_k}\xi(X_k),$$

so

$$(\mu_+ + z\mu_-)|_{X_k} = 0. \quad (26)$$

Note also that X_k cannot be a common zero of μ_+ and μ_- , because that would imply $\xi(X_k) \in \text{Ker } \mathcal{D}_+ \cap \text{Ker } \mathcal{D}_-$, which is not possible by the second statement of Proposition 3.4 (the latter applies to $\mathcal{D}_\pm$ because $\mathcal{D}_+ = \mathcal{D}_l$, $\mathcal{D}_- = -T^{-n}\mathcal{D}_r$). Furthermore, $\mu_\pm$ cannot have a pole at X_k by Proposition 5.10. However, then (26) implies $s(X_k) = -1$, as desired. $\square$

Remark 5.14. It follows from Proposition 5.13 that the function s has the following meaning. Fix some generic $s_0 \in \mathbb{C}$. Then there are three points X_1, X_2, X_3 in Γ with $s = s_0$. Furthermore, the vectors $\xi(X_k) \in Q^n(z(X_k))$ belong to $\text{Ker } (\mathcal{D}_+ - s_0 T^n \mathcal{D}_-)$. Thus, $z(X_1), z(X_2), z(X_3)$ is the spectrum of the monodromy of $\mathcal{D}_+ - s_0 T^n \mathcal{D}_-$. In other words, if we consider a meromorphic mapping $\Gamma \rightarrow \mathbb{C}^2$ given by the functions (z, s) , then its image belongs to the algebraic curve

$$\Gamma'_a := \{(z, s) \in \mathbb{C}^* \times \mathbb{C} \mid z \text{ is an eigenvalue of the monodromy of } \mathcal{D}_+ - s T^n \mathcal{D}_-\}.$$

Using also that $\deg z = n$ and $\deg s = 3$, it is easy to show that the mapping $\Gamma \rightarrow \Gamma'_a$ is generically biholomorphic. Thus, Γ'_a is just another affine model of the spectral curve Γ . This model can be thought of as the joint spectrum of the operators T^n and $T^{-n}\mathcal{D}_+\mathcal{D}_-^{-1}$ (the latter is well-defined on a generic eigenspace of T^n). Furthermore, because the operator $\mathcal{D}_+ - s T^n \mathcal{D}_-$ corresponds to the polygon $R_s(P)$, where R_s is the rescaling action (9), it follows that Γ'_a can be regarded as the graph of the spectrum for the monodromy of $R_s(P)$. As explained in [Izo16], this definition of the spectral curve coincides with the one used in [Sol13] to prove algebraic integrability of the pentagram map. Thus, as a Riemann surface, our spectral curve is isomorphic to that of [Sol13].

We are now in a position to prove the main result of this section.

PROPOSITION 5.15. *The genus g of Γ satisfies $g \leq 1$.*

Proof. The function w on Γ is a 2-fold ramified covering of the Riemann sphere whose branch points coincide with fixed points of the involution σ . To estimate the number of such fixed points, note that from Proposition 5.11 and formula (24) it follows that $\sigma^*s = s^{-1}$. Thus, at each fixed point of σ we must have $s = \pm 1$. Furthermore, by Proposition 5.13, the function z takes three distinct values at points of Γ where $s = -1$, and because the set $s = -1$ is invariant under the involution σ which takes z to z^{-1} , it follows that those values must be of the form $\pm 1, z_0, z_0^{-1}$, where $z_0 \neq \pm 1$. Thus, σ must have exactly one fixed point at the level set $s = -1$. In addition, it may have up to three fixed points at the level set $s = 1$, which is up to four fixed points in total. Now, the desired inequality for the genus follows from the Riemann–Hurwitz formula. $\square$

Remark 5.16. In fact, because the values of z at points where $s = -1$ are the eigenvalues of the monodromy of $\mathcal{D}_+ + T^n \mathcal{D}_-$, it follows from formula (6) that they are of the form $-1, z_0, z_0^{-1}$. Another way to see this is to notice that by Proposition 5.11 at fixed points of σ we must have $\mu_+ = \mu_-$ and thus $z = s^{-1}$ (here we use that the functions $\mu_\pm$ do not have common zeros and also do not have poles in $\Gamma \setminus \{Z_\pm\}$). This also implies that if Γ has genus 1, then $z = 1$ at points where $s = 1$. In other words, all eigenvalues of the monodromy of the polygon P are equal to one. Later on, we show that this monodromy is, in fact, the identity. In other words, if the spectral curve is elliptic, then in the setting of Theorem B the polygon P must be closed (see Remark 6.9).

Remark 5.17. Note that without weak convexity (used to prove Proposition 5.13) we would not be able to say that there is just one fixed point of the involution σ at the level $s = -1$. In that case, nothing seems to prevent σ from having six fixed points, which means that Γ may be a genus-2 curve. Thus, it should be possible to construct a counterexample to Theorems A and B in the non-weakly-convex case using genus-2 curves and their associated genus-2 theta functions.

Another way to obtain the estimate $g \leq 2$ is to use the existence on Γ of meromorphic functions of degree 2 and 3 (namely, w and s). However, this does not guarantee the $g \leq 1$ estimate obtained above.

We finish this section with two additional results on the spectral curve which are useful later on.

PROPOSITION 5.18. *The degrees of the functions $\mu_{\pm}, s$ and their orders at points $Z_{\pm}, S_{\pm}$ are as listed in Table 1.*

Proof. As for any $\lambda \in \mathbb{C}^*$ the degree of operators $\mathcal{D}_{\pm} - \lambda$ is $(n-1)/2$, an argument analogous to that we used to show that $\deg s \leq 3$ (see the proof of Proposition 5.13) gives $\deg \mu_{\pm} \leq (n-1)/2$. Further, let $d_{\pm} := \text{ord}_{Z_{\pm}} \mu_{\pm}$. Then, because $\text{ord}_{Z_{+}} w = -1$ (see Table 1), the equation $\mu_{+}\mu_{-} = w$ implies

$$d_{+} + d_{-} = -1. \quad (27)$$

Furthermore, using that $\text{ord}_{Z_{+}} z = n$ and (24), we obtain

$$d_{+} - d_{-} = n + \text{ord}_{Z_{+}} s, \quad (28)$$

so

$$d_{-} = -\frac{1}{2}(n+1) - \frac{1}{2}\text{ord}_{Z_{+}} s, \quad (29)$$

and because $\deg \mu_{-} \leq (n-1)/2$, we must have $d_{-} \geq -(n-1)/2$, which implies $\text{ord}_{Z_{+}} s \leq -2$. On the other hand, we know that $\deg s = 3$, and from (29) it follows that $\text{ord}_{Z_{+}} s$ is even. Thus, we must have $\text{ord}_{Z_{+}} s = -2$, which, along with (29) implies $\text{ord}_{Z_{+}} \mu_{-} = d_{-} = -(n-1)/2$ and thus $\deg \mu_{-} = (n-1)/2$. Similarly, adding up (27) and (28), we get $\text{ord}_{Z_{+}} \mu_{+} = d_{+} = (n-3)/2$. Analogously, replacing the point Z_{+} with Z_{-} , we find the orders of $\mu_{\pm}$ and s at Z_{-} , as well as the degree of μ_{+} (one can also use that $\sigma^* \mu_{+} = \mu_{-}$ and $\sigma(Z_{+}) = Z_{-}$).

It now remains to find the orders of the functions $\mu_{\pm}$ and s at the points $S_{\pm}$. To that end, we first show that $S_{+} \neq S_{-}$. Assume, for the sake of contradiction, that $S_{+} = S_{-} = S$. Then S is a double zero of the function w . Furthermore, we have $\mu_{+}\mu_{-} = w$, and both μ_{+} and μ_{-} are holomorphic and have at worst a simple zero at S (indeed, these functions have degree $(n-1)/2$ and zeros of order $(n-3)/2$ at the points Z_{+} and Z_{-} , respectively). Thus, both μ_{+} and μ_{-} must have a simple zero at S . However, then, from the definition (24) of the function s , it follows that it does not have a zero at S . Furthermore, s cannot have zeros at other points of $\Gamma \setminus \{Z_{\pm}\}$, because the only zero of μ_{+} in that domain is the point S , but this means that s has just two zeros counting with multiplicities, which is impossible because $\deg s = 3$. Therefore, we must have that $S_{+} \neq S_{-}$.

Now, the relation $\mu_{+}\mu_{-} = w$ implies that at both points $S_{\pm}$ one of the functions $\mu_{\pm}$ has a simple zero, whereas the second does not have a zero or a pole. Without loss of generality, assume that $\mu_{+}(S_{+}) = 0$ and, thus, $\text{ord}_{S_{+}} \mu_{+} = 1$. Then $\text{ord}_{S_{+}} \mu_{-} = 0$, and by formula (24) we obtain $\text{ord}_{S_{+}} s = 1$. Furthermore, from $\sigma^* \mu_{+} = \mu_{-}$ and $\sigma(S_{+}) = S_{-}$ it follows that $\text{ord}_{S_{-}} \mu_{+} = 0$, $\text{ord}_{S_{-}} \mu_{-} = 1$, and $\text{ord}_{S_{-}} s = -1$. Finally, note that the functions $\mu_{\pm}$ and s do not have zeros or poles other than the points $Z_{\pm}, S_{\pm}$, because for each of them the total number of zeros and poles at those points (counting with multiplicities) coincides with the degree. Thus, the proposition is proved. $\square$

PROPOSITION 5.19. *The affine spectral curve Γ_a is a nodal curve (i.e. all its singularities are double points).*

Proof. The affine spectral curve Γ_a is the zero locus of the characteristic polynomial $p(z, w) = z + z^{-1} + q(w)$ of the matrix (20). Computing the differential of that polynomial, we get that $(z_0, w_0) \in \Gamma_a$ is singular if and only if $z_0 = \pm 1$ and w_0 is a multiple root of $p(z_0, w)$. Furthermore, computing the Hessian, we get that a singular point $(z_0, w_0) \in \Gamma_a$ is a double point if and only if w_0 is a double root of $p(z_0, w)$. However, for $z_0 = \pm 1$ the matrix (20) is symmetric (equivalently, the restriction of the operator $\mathcal{D}_+\mathcal{D}_-$ to $Q^n(\pm 1)$ is self-adjoint), so the multiplicity of the root w_0 of its characteristic polynomial is equal to the dimension of the corresponding eigenspace, which is

$$\dim \operatorname{Ker} (\mathcal{D}_+\mathcal{D}_- - w_0)|_{Q^n(\pm 1)} \leq \operatorname{ord} (\mathcal{D}_+\mathcal{D}_- - w_0) = 2.$$

Thus, indeed all singular points of Γ_a are double points. $\square$

Remark 5.20. It is easy to see that the genus of the normalization Γ of Γ_a is equal to $n - d - 1$ where d is the number of double points of Γ_a . Furthermore, as can be seen from the proof of Proposition 5.19, double points of Γ_a correspond to double roots of the polynomials $q(w) \pm 2$. The polynomial q is of degree n , so each of those polynomials may have at most $(n - 1)/2$ double roots. Therefore, Γ is rational when each of the polynomials $q(w) \pm 2$ has precisely $(n - 1)/2$ double roots (and, in addition, one simple root). Likewise, Γ is elliptic when one of the polynomial $q(w) \pm 2$ has $(n - 1)/2$ double roots, whereas the second has $(n - 3)/2$ double roots. Using Remark 5.16 one can show that it is the polynomial $q(w) + 2$ that has $(n - 3)/2$ double roots.

5.2 The eigenvector function

In this section, we study in detail the analytic properties of the meromorphic vector-function ξ constructed in Proposition 5.8. This will allow us to obtain analytic formulas for coordinates of vertices of the polygon P (see § 6.1 for the rational case and § 6.2 for the elliptic case). We keep all the notation of § 5.1.

PROPOSITION 5.21. *We have $\operatorname{ord}_{Z_{\pm}} \xi_k = \pm k$.*

Proof. Let $d_k := \operatorname{ord}_{Z_+} \xi_k - k$. We need to show that $d_k = 0$ for every $k \in \mathbb{Z}$. Note that $d_0 = 0$ because $\xi_0 = 1$. Thus, it suffices to show that d_k is a constant sequence. Also note that because $\xi_{k+n} = z\xi_k$ and z has a zero of order n at Z_+ (see Table 1), the sequence d_k is n -periodic. Thus, if it is not constant, then there must exist $k \in \mathbb{Z}$ such that $d_{k-1} > d_k \leq d_{k+1}$. However, because ξ is the eigenvector of the operator (19) with eigenvalue w , we have

$$\alpha_{k-1}\xi_{k-1} + \alpha_k\xi_{k+1} = (w - \beta_k)\xi_k. \quad (30)$$

As α is a non-vanishing sequence, the order of the left-hand side at Z_+ can be bounded as

$$\begin{aligned} \operatorname{ord}_{Z_+} (\alpha_{k-1}\xi_{k-1} + \alpha_k\xi_{k+1}) &\geq \min(\operatorname{ord}_{Z_+} \xi_{k-1}, \operatorname{ord}_{Z_+} \xi_{k+1}) \\ &= \min(d_{k-1} + k - 1, d_{k+1} + k + 1) \geq d_k + k. \end{aligned}$$

On the other hand, because $\operatorname{ord}_{Z_+} w = -1$, the order of the right-hand side of (30) is $d_k + k - 1$. Thus, d_k must be a constant sequence, as desired. $\square$

We now proceed to describe the behavior of ξ away from the points $Z_{\pm}$. We begin with the following preliminary lemma.

LEMMA 5.22. *Assume that $X_{\pm} \in \Gamma \setminus \{Z_{\pm}\}$ are distinct points such that $w(X_+) = w(X_-)$ (equivalently, $\sigma(X_+) = X_-$). Then the directions (i.e. points in $\mathbb{P}^{\infty}$, see Remark 5.9) determined by the values of ξ at $X_{\pm}$ are distinct from each other.*

Proof. Without loss of generality, assume that the vectors $\xi(X_{\pm})$ are finite and non-zero (if not, we multiply ξ by an appropriate meromorphic function, see Remark 5.9). One then needs to show that these vectors are linearly independent. To that end, recall that $T^n \xi(X_{\pm}) = z(X_{\pm}) \xi(X_{\pm})$. Thus, if $z(X_+) \neq z(X_-)$, then the vectors $\xi(X_{\pm})$ are independent as eigenvectors of T^n corresponding to distinct eigenvalues. Therefore, it suffices to consider the case $z(X_+) = z(X_-)$. In that case, we have

$$z(X_+) = z(X_-) = z(\sigma(X_-))^{-1} = z(X_+)^{-1},$$

so $z(X_{\pm}) = \pm 1$. Suppose for the sake of contradiction that the corresponding vectors $\xi(X_{\pm})$ are linearly dependent. Then, without loss of generality, we can assume that $\xi(X_+) = \xi(X_-)$ (this can be always arranged by multiplying ξ by an appropriate meromorphic function). We use the notation $\xi_0 := \xi(X_{\pm})$, $z_0 := z(X_{\pm}) = \pm 1$, $w_0 := w(X_{\pm})$. Note that because $w(X_+) = w(X_-)$, the differential of w does not vanish at $X_{\pm}$, so w can be taken as a local parameter near those points. Then, differentiating the relation $(T^n - z)\xi = 0$ with respect to w at $X_{\pm}$, we obtain

$$(T^n - z_0)\xi'(X_+) = z'(X_+)\xi_0, \quad (T^n - z_0)\xi'(X_-) = z'(X_-)\xi_0.$$

Taking a linear combination of these equations, we obtain $(T^n - z_0)\hat{\xi} = 0$, where $\hat{\xi} := z'(X_+)\xi'(X_-) - z'(X_-)\xi'(X_+)$. In other words, we have $\hat{\xi} \in Q^n(z_0)$. Similarly, using the equation $(\mathcal{D}_+ \mathcal{D}_- - w)\xi = 0$, we obtain

$$(\mathcal{D}_+ \mathcal{D}_- - w_0)\hat{\xi} = \lambda \xi_0, \tag{31}$$

where $\lambda := z'(X_+) - z'(X_-)$. Note also that $\lambda \neq 0$. Indeed, $\lambda = 0$ would mean that the two branches of the curve Γ_a given by the functions $z(w)$ near $X_{\pm}$ are tangent to each other. This is, however, not possible, because Γ_a is a nodal curve (Proposition 5.19). Thus, because $\lambda \neq 0$ and $\hat{\xi} \in Q^n(z_0)$, it follows from (31) that the operator $(\mathcal{D}_+ \mathcal{D}_-)|_{Q^n(z_0)}$ has a non-trivial Jordan block. This is, however, not possible, because $z_0 = \pm 1$, and thus $\mathcal{D}_+ \mathcal{D}_-$ is self-adjoint on $Q^n(z_0)$. Thus, it must be that the vectors $\xi(X_{\pm})$ are linearly independent, as desired. $\square$

Remark 5.23. In the elliptic case (i.e. when the genus of Γ is 1), one can also prove Lemma 5.22 as follows. The vectors $\xi(X_{\pm})$ are common eigenvectors of the operators $\mathcal{D}_+$, $\mathcal{D}_-$, T^n , with the corresponding eigenvalues given by the values of the function μ_+ , μ_- , z at the points $X_{\pm}$. Thus, it follows that $\xi(X_{\pm})$ are independent as long as at least one of the functions μ_+ , μ_- , z separates X_+ from X_- . Assume that this is not the case, which means that $\mu_{\pm}(X_+) = \mu_{\pm}(X_-)$ and $z(X_+) = z(X_-)$. Then we have

$$\mu_-(X_+) = \mu_+(\sigma(X_+)) = \mu_+(X_-) = \mu_+(X_+).$$

Along with $z(X_{\pm}) = \pm 1$, this gives $s(X_{\pm}) = \pm 1$. However, in the elliptic case four of the six points where $s = \pm 1$ are fixed by σ (which would force $X_+ = X_-$), whereas the remaining two points are separated by the function z (by Proposition 5.15). Thus, the functions $\mu_{\pm}$, z indeed separate any pair of points on Γ , which proves Lemma 5.22. As a byproduct, we also obtain the following result: the functions $\mu_{\pm}$, z define an embedding $\Gamma \setminus \{Z_{\pm}\} \hookrightarrow \mathbb{C}^3$. In other words, if we view $\mu_{\pm}$ and z as rational functions on Γ_a , then these functions provide a resolution of singularities.

Lemma 5.22 also admits an infinitesimal version, corresponding to the case when $X_+ = X_- = X$ is a branch point of w (equivalently, a fixed point of σ). In this case, the role of $\xi(X_{\pm})$ is played by the vectors $\xi(X)$, $\xi'(X)$, where the derivative is taken with respect to a local parameter near X . Note that upon renormalization of ξ , its derivative changes as $\xi' \mapsto f\xi' + f'\xi$, so the direction

of ξ' is well-defined modulo the direction of ξ . In particular, linear independence of ξ and ξ' is well-defined.

LEMMA 5.24. Assume that $X \in \Gamma$ is a branch point of w (equivalently, a fixed point of σ). Then the directions determined by the values of ξ and ξ' at X are distinct from each other.

Proof. Renormalizing ξ if necessary, we can assume that its value at X is finite and non-zero. Then, differentiating the equation $(T^n - z)\xi = 0$ with respect to a local parameter near X , we obtain

$$(T^n - z(X))\xi'(X) = z'(X)\xi(X). \quad (32)$$

Also note that because Γ_a is a nodal curve, it follows that the mapping $(z, w): \Gamma \setminus \{Z_\pm\} \rightarrow \mathbb{C}^2$ is an immersion, so at a branch point of w we must have $z' \neq 0$. However, then (32) implies that the vectors $\xi(X)$ and $\xi'(X)$ are linearly independent, as desired. $\square$

Remark 5.25. Differentiating $(\mathcal{D}_+\mathcal{D}_- - w)\xi = 0$ at X and using that $w'(X) = 0$, we obtain $(\mathcal{D}_+\mathcal{D}_- - w(X))\xi'(X) = 0$. Thus, Lemma 5.24 means that $\xi(X)$ and $\xi'(X)$ form a basis of solutions for the equation $(\mathcal{D}_+\mathcal{D}_- - w(X))\xi = 0$.

PROPOSITION 5.26. The function ξ_1 has g poles in $\Gamma \setminus \{Z_\pm\}$, where $g \in \{0, 1\}$ is the genus of Γ .

Proof. Let $u \in \bar{\mathbb{C}}$, and let $X_\pm$ be the two preimages of u under the function $w: \Gamma \rightarrow \bar{\mathbb{C}}$. Recall that the trace of a meromorphic function f on Γ under w is a meromorphic function on $\bar{\mathbb{C}}$ is defined by $(\text{tr}_w f)(u) := f(X_+) + f(X_-)$. Let

$$\zeta(u) := \begin{vmatrix} \xi_0(X_+) & \xi_1(X_+) \\ \xi_0(X_-) & \xi_1(X_-) \end{vmatrix}^2.$$

Note that $\xi_0 \equiv 1$ by the definition of the eigenvector function ξ , so $\zeta(u) = (\xi_1(X_+) - \xi_1(X_-))^2$. This means that $\zeta := 2\text{tr}_w(\xi_1^2) - (\text{tr}_w \xi_1)^2$, so in particular ζ is meromorphic (i.e. rational). To understand the behavior of that function, fix a point $u_0 \in \bar{\mathbb{C}}$. Let $\Sigma := w(\{X \in \Gamma \mid dw(X) = 0\}) \subset \bar{\mathbb{C}}$ be the set of critical values of w (this set contains two or four points depending on the genus of Γ). Then the following cases are possible.

Case 1: $u_0 \notin \Sigma$ is finite, and ξ_1 is finite at both preimages $X_\pm$ of u_0 under w . In this case, $\zeta(u_0)$ is the squared Wronskian of the solutions $\xi(X_\pm)$ of equation $\mathcal{D}_+\mathcal{D}_-\eta = u_0\eta$. By Lemma 5.22, these solutions are independent, so $\zeta(u_0)$ is finite and non-zero.

Case 2: $u_0 \notin \Sigma$ is finite, ξ_1 has a pole of order d at one of the preimages $X_\pm$ of u_0 (say, X_+), and is finite at the other preimage. In this case, the function $(u - u_0)^{2d}\zeta(u)$ is finite at u_0 and is equal to the squared Wronskian of linearly independent solutions $((w - u_0)^d\xi)(X_+)$, $\xi(X_-)$ of $\mathcal{D}_+\mathcal{D}_-\eta = u_0\eta$. Thus, ζ has a pole of order $2d$ at u_0 .

Case 3: $u_0 \notin \Sigma$ is finite, and ξ_1 has poles at both preimages $X_\pm$ of u_0 . This is not possible, since after renormalizing ξ we would obtain

$$\begin{pmatrix} \xi_0(X_+) & \xi_1(X_+) \\ \xi_0(X_-) & \xi_1(X_-) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

which would mean that the Wronskian of $\xi(X_\pm)$ vanishes.

Case 4: $u_0 = \infty$ (in which case we also have $u \notin \Sigma$). In this case $X_\pm = Z_\pm$, so ζ has a pole of order 2 at u_0 by Proposition 5.21.

All in all, the function ζ does not vanish in $\bar{\mathbb{C}} \setminus \Sigma$, whereas the number of its poles in that domain is twice the number of poles of ξ_1 in $\{X \in \Gamma \mid dw(X) \neq 0\}$ (counting with multiplicities).

Now, consider $u_0 \in \Sigma$, and let $X \in \Gamma$ be the unique point such that $w(X) = u_0$. Then there exists a parameter t near X such that the function w can be locally written as $t \mapsto u_0 + t^2$. So $\zeta(u)$ near u_0 can be written as

$$\zeta(u) = \left| \begin{array}{cc} \xi_0(t) & \xi_1(t) \\ \xi_0(-t) & \xi_1(-t) \end{array} \right|^2,$$

where $t = \sqrt{u - u_0}$. Then at $t = 0$ we have

$$\zeta(u) \sim t^2 \left| \begin{array}{cc} \xi'_0(0) & \xi'_1(0) \\ \xi_0(0) & \xi_1(0) \end{array} \right|^2, \quad (33)$$

up to a constant factor and higher-order terms. Thus, when $u_0 \in \Sigma$, we have the following two cases.

Case 5: $u_0 \in \Sigma$, and ξ_1 is finite at the preimage X of u_0 . In this case, in view of Remark 5.25, the determinant in (33) is the Wronskian of two independent solutions of $\mathcal{D}_+\mathcal{D}_-\eta = u_0\eta$, so $\zeta(u) \sim t^2 = u - u_0$ and, thus, has a simple zero at u_0 .

Case 6: $u_0 \in \Sigma$, and ξ_1 has a pole of order d at the preimage X of u_0 . In this case, renormalizing ξ as in Case 2, we obtain that ζ has a pole of order $2d - 1$ at u_0 .

In the latter case, one can regard a pole of order $2d - 1$ as a pole of order $2d$ that collided with a simple zero. With this understanding, the number of zeros of ζ is equal to the number of branch points of w , whereas the number of poles of ζ is twice the number of poles of ξ_1 (with some zeros and poles possibly cancelling each other out). In addition, because the number of zeros of ζ is equal to the number of its poles, it follows that the number of poles of ξ_1 is half the number of branch points of w , which is $2g + 2$. Furthermore, because Z_+ is not a pole of ξ_1 , whereas Z_- is its pole of order 1 (see Proposition 5.21), it follows that the number of poles of ξ_1 in $\Gamma \setminus \{Z_\pm\}$ is exactly g , as desired. $\square$

COROLLARY 5.27. *In the rational case, all functions ξ_k are holomorphic in $\Gamma \setminus \{Z_\pm\}$, whereas in the elliptic case all of them have at worst a simple pole at one and the same point X_p , and no other poles.*

Proof. First note that $\xi_0 \equiv 1$ by construction. Furthermore, in the rational case ξ_1 is holomorphic in $\Gamma \setminus \{Z_\pm\}$ by Proposition 5.26. Thus, both ξ_0 and ξ_1 are holomorphic in that domain. At the same time, by (30) we have

$$\xi_{k+1} = \frac{1}{\alpha_k}((w - \beta_k)\xi_k - \alpha_{k-1}\xi_{k-1}),$$

and because w is holomorphic in $\Gamma \setminus \{Z_\pm\}$ (Table 1), it follows by induction that so are all ξ_k with $k \geq 0$. Analogously, using (30) to express ξ_{k-1} , we get that ξ_k with $k < 0$ are holomorphic in $\Gamma \setminus \{Z_\pm\}$ too. This proves the corollary in the rational case.

In the elliptic case, the argument is similar, but now Proposition 5.26 implies that ξ_1 has a single pole in $\Gamma \setminus \{Z_\pm\}$. Denoting that pole by X_p , we obtain by induction that all ξ_k are holomorphic in $\Gamma \setminus \{Z_\pm, X_p\}$, as desired. $\square$

6. Proof of Theorem B: a self-dual polygon fixed by the pentagram map is Poncelet

In this section, we prove Theorem B: any weakly convex self-dual twisted odd-gon P projectively equivalent to its pentagram image P' is Poncelet. To that end, we use the results of § 5 to obtain

explicit formulas for coordinates of vertices of P (see §6.1 for the case $g = 0$ and §6.2 for the case $g = 1$) and, hence, show that P is a Poncelet polygon.

6.1 The rational case: degenerate Poncelet polygons

In this section, we prove Theorem B in the case when the genus of Γ is zero, that is, when Γ is a rational curve. In that case, we show that P is a degenerate Poncelet polygon in the sense that the corresponding inscribed and circumscribed conics are not in general position. We keep the notation of the previous two sections.

PROPOSITION 6.1. *The set $s^{-1}(1) := \{X \in \Gamma \mid s(X) = 1\}$ consists of either one or three points.*

Proof. This set is invariant under the involution σ and contains exactly one fixed point of that involution (see the proof of Proposition 5.15). Thus, it must contain odd number of points, and because $\deg s = 3$, it follows that $|s^{-1}(1)| = 1$ or $|s^{-1}(1)| = 3$. $\square$

We consider the cases $|s^{-1}(1)| = 1$ and $|s^{-1}(1)| = 3$ separately. First, assume that $|s^{-1}(1)| = 3$. Denote points in $s^{-1}(1)$ by A, B, C , where A and B are switched by σ , whereas C is fixed by σ .

PROPOSITION 6.2. *The vectors $\xi(A), \xi(B), \xi(C)$ form a basis of $\text{Ker}(\mathcal{D}_+ - T^n \mathcal{D}_-)$.*

Remark 6.3. Note that the vectors $\xi(A), \xi(B), \xi(C)$ are finite because, by Corollary 5.27, the vector-function ξ is holomorphic in $\Gamma \setminus \{Z_{\pm}\}$, and $A, B, C \neq Z_{\pm}$ because $s(Z_+) = \infty$ and $s(Z_-) = 0$ (Table 1).

Proof of Proposition 6.2. We have $\xi(A), \xi(B), \xi(C) \in \text{Ker}(\mathcal{D}_+ - T^n \mathcal{D}_-)$ by (25), so it suffices to show that these vectors are linearly independent. To that end, recall that they are eigenvectors of the operator $\mathcal{D}_+ \mathcal{D}_-$. Furthermore, the eigenvalue $w(C)$ corresponding to $\xi(C)$ is distinct from the eigenvalue $w(A) = w(B)$ corresponding to the other two vectors. Thus, it suffices to prove the independence of $\xi(A)$ and $\xi(B)$. However, that follows from Lemma 5.22. $\square$

Now recall that the polygon corresponding to the operator $\mathcal{D}_+ - T^n \mathcal{D}_-$ is P . Thus, by Proposition 6.2, the vertices of P (defined up to a projective transformation) are given by $(\xi_k(A) : \xi_k(B) : \xi_k(C)) \in \mathbb{P}^2$. To explicitly compute the coordinates of vertices, we identify Γ with $\bar{\mathbb{C}}$. Note that because automorphisms of $\bar{\mathbb{C}}$ act transitively on triples of points, the map $u : \Gamma \rightarrow \bar{\mathbb{C}}$ may be chosen in such a way that $u(Z_+) = 0$, $u(Z_-) = \infty$, and $u(C) = 1$. Then the involution σ , written in terms of u , is $u \mapsto u^{-1}$, whereas the points A, B are identified with r and r^{-1} , where $r \in \mathbb{C}^* \setminus \{\pm 1\}$. Furthermore, from Proposition 5.21 and Corollary 5.27 we obtain $\xi_k(u) = c_k u^k$, where $c_k \neq 0$ is a constant. Therefore, the vertices of P are given by

$$v_k = (r^k : r^{-k} : 1). \quad (34)$$

Thus, the polygon P is inscribed in a conic with homogeneous equation

$$x_1 x_2 = x_3^2. \quad (35)$$

Furthermore, because P is self-dual, it is also circumscribed and, hence, Poncelet. Thus, the proof of Theorem B in the case when the spectral curve is rational and $|s^{-1}(1)| = 3$ is complete.

Remark 6.4. A direct calculation shows that the conic inscribed in the polygon (34) is

$$x_1 x_2 = \left(\frac{1}{2} + \frac{1}{4}(r + r^{-1})\right) x_3^2. \quad (36)$$

The conics (35) and (36) are tangent to each other at two points $(1 : 0 : 0)$ and $(0 : 1 : 0)$. In particular, they are not in general position (instead of four intersections we have two intersections of multiplicity 2).

We now consider the case $|s^{-1}(1)| = 1$. To begin with, notice that this case can be thought of as a limit of the case $|s^{-1}(1)| = 3$, with the points A, B, C colliding and forming a single point $D \in s^{-1}(1)$. This observation leads to the following version of Proposition 6.2.

PROPOSITION 6.5. *The vectors $\xi(D), \xi'(D), \xi''(D)$ form a basis of $\text{Ker}(\mathcal{D}_+ - T^n \mathcal{D}_-)$, where the derivatives are taken with respect to any local parameter near D .*

Proof. First, note that because the set $s^{-1}(1)$ consists of a single point D , the latter must be an order 2 branch point of the function s . In other words, we have $s'(D) = s''(D) = 0$. Thus, differentiating the equation $(\mathcal{D}_+ - sT^n \mathcal{D}_-)\xi = 0$ at the point D twice, we obtain

$$(\mathcal{D}_+ - T^n \mathcal{D}_-)\xi'(D) = (\mathcal{D}_+ - T^n \mathcal{D}_-)\xi''(D) = 0.$$

Thus, we have $\xi(D), \xi'(D), \xi''(D) \in \text{Ker}(\mathcal{D}_+ - T^n \mathcal{D}_-)$, and it suffices to show that these vectors are linearly independent. To that end, we differentiate the equation $(\mathcal{D}_+ \mathcal{D}_- - w)\xi = 0$ twice at D . Using that D is a branch point of w and, thus, $w'(D) = 0$, we obtain

$$(\mathcal{D}_+ \mathcal{D}_- - w(D))\xi'(D) = 0, \quad (\mathcal{D}_+ \mathcal{D}_- - w(D))\xi''(D) = w''(D)\xi(D).$$

Furthermore, because the degree of the function w is 2, D is an order 1 branch point for w , so $w''(D) \neq 0$, which means that $\xi(D), \xi'(D)$ are eigenvectors of $\mathcal{D}_+ \mathcal{D}_-$, whereas $\xi''(D)$ is not. Furthermore, the vectors $\xi(D)$ and $\xi'(D)$ are linearly independent by Lemma 5.24. Thus, $\xi(D), \xi'(D), \xi''(D)$ are indeed independent, as desired. $\square$

We now find the vertices of the polygon P in the same fashion as in the case $|s^{-1}(1)| = 3$. Namely, choose an identification $u: \Gamma \rightarrow \mathbb{P}^1$ in such a way that $u(Z_+) = 0$, $u(Z_-) = \infty$, and $u(D) = 1$. Then, as in the case $|s^{-1}(1)| = 3$, we obtain $\xi_k(u) = c_k u^k$, where $c_k \neq 0$ is a constant. In particular, at the point D we obtain $\xi_k = c_k$, $\xi'_k = k c_k$, $\xi''_k = k(k-1)c_k$, so up to a projective transformation the vertices of P are given by

$$v_k = (k : k^2 : 1). \quad (37)$$

These points belong to a conic

$$x_2 x_3 = x_1^2, \quad (38)$$

so P is inscribed and, hence, Poncelet. Thus, the proof of Theorem B in the rational case is complete.

Remark 6.6. A direct calculation shows that the conic inscribed in the polygon (37) is

$$x_2 x_3 = x_1^2 + \frac{1}{4} x_3^2. \quad (39)$$

This is an even more degenerate case: the conics (38) and (39) intersect each other at one single point $(0 : 1 : 0)$, of multiplicity 4.

Remark 6.7. Degenerate Poncelet polygons (34) and (37) correspond to degenerations of an elliptic curve to Abelian groups $\mathbb{C}^*$ and $\mathbb{C}$, respectively. Indeed, removing the tangency points with the inscribed conic (36) from the circumscribed conic (35) we obtain an affine curve (a hyperbola) which is naturally isomorphic to $\mathbb{C}^*$. The vertices of the Poncelet polygon (34) are uniformly spaced on that hyperbola with respect to the $\mathbb{C}^*$ group structure. This should be compared with the case of genuine Poncelet polygons which are images of uniformly spaced points on an elliptic curve under a double covering map from the elliptic curve to the circumscribed conic. Thus, degenerate Poncelet polygons (34) correspond to degenerations of an elliptic curve to $\mathbb{C}^*$. Likewise, removing from (38) the tangency point with (39) we obtain a parabola which identifies

with $\mathbb{C}$, and points (37) are again uniformly spaced. Thus, degenerate Poncelet polygons (37) correspond to degenerations of an elliptic curve to $\mathbb{C}$.

6.2 The elliptic case: genuine Poncelet polygons

In this section, we complete the proof of Theorem B in the case when the spectral curve Γ has genus one, that is, is elliptic. The argument is similar to the rational case, but instead of elementary expressions (34) and (37), we obtain formulas for vertices of P in terms of theta functions.

Recall that in the elliptic case the involution σ on Γ has four fixed points, at three of which we have $s = 1$, whereas at the fourth we have $s = -1$ (see the proof of Proposition 5.15). Denote those points by A, B, C, D , where $s(D) = -1$, and $s(A) = s(B) = s(C) = 1$.

PROPOSITION 6.8. *The directions in $\mathbb{P}^\infty$ determined by the values of the vector-function ξ at the points A, B, C (see Remark 5.9) are linearly independent.*

Proof. They are eigendirections of the operator $\mathcal{D}_+\mathcal{D}_-$ corresponding to distinct eigenvalues $w(A), w(B), w(C)$. $\square$

As in the rational case, it follows that the vectors $\xi(A), \xi(B), \xi(C)$ form a basis of $\text{Ker}(\mathcal{D}_+ - T^n\mathcal{D}_-)$ (as usual, one may need to renormalize these vectors to ensure that they are finite, see Remark 5.9). Therefore, the vertices of the corresponding polygon P (defined up to projective transformation) are given by $(\xi_k(A) : \xi_k(B) : \xi_k(C)) \in \mathbb{P}^2$.

Remark 6.9. As $z(A) = z(B) = z(C) = 1$ (see Remark 5.16), it follows that the infinite vectors $\xi(A), \xi(B), \xi(C)$ are n -periodic, so the polygon P is closed.

To explicitly compute the coordinates of vertices of P , we identify Γ with $\mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice. Without loss of generality, assume that Λ is spanned by 1 and τ , where τ is in the upper half-plane. Furthermore, one can choose an identification between Γ and $\mathbb{C}/\Lambda$ in such a way that the point $D \in \Gamma$ is identified with $d := (1 + \tau)/2$. Then σ , understood as an involution in $\mathbb{C}/\Lambda$, is simply $u \mapsto -u$. Thus, A, B, C must coincide with the remaining order 2 points in $\mathbb{C}/\Lambda$, namely 0, $1/2$, $\tau/2$. Without loss of generality, assume that $A = 0$, $B = 1/2$, $C = \tau/2$.

We express the vertices of P using theta functions. Recall (see, e.g., [Mum83]) that the *theta function* corresponding to the lattice $\Lambda = \langle 1, \tau \rangle$ is defined by

$$\theta(u) := \sum_{k \in \mathbb{Z}} \exp(\pi i(2ku + k^2\tau))$$

where $i = \sqrt{-1}$, and the dependence of θ on τ is suppressed for notational convenience. We also need *theta functions with (half-integer) characteristics*, defined by

$$\begin{aligned} \theta_{00}(u) &:= \theta(u), & \theta_{01}(u) &:= \theta(u + 1/2), & \theta_{10}(u) &:= \exp(\pi i(u + \tau/4))\theta(u + \tau/2), \\ \theta_{11}(u) &:= \exp(\pi i(u + \tau/4 + 1/2))\theta(u + (1 + \tau)/2). \end{aligned}$$

Choose complex numbers $x_p, z_\pm \in \mathbb{C}$ whose images in $\mathbb{C}/\Lambda$ are the points $X_p, Z_\pm \in \Gamma$ (see Corollary 5.27 for the definition of X_p). Note that the points $Z_\pm$ are interchanged by the involution σ , so $z_+ + z_- = 0$ modulo Λ . Therefore, without loss of generality one can assume that $z_+ + z_- = 1 + \tau$. Fix $z_\pm$ satisfying the latter condition, and let $\delta := z_+ - z_-$.

PROPOSITION 6.10. *Up to a projective transformation, the vertices of the polygon P are given by*

$$v_k = (\theta_{00}(k\delta + d - x_p) : \theta_{01}(k\delta + d - x_p) : \theta_{10}(k\delta + d - x_p)). \quad (40)$$

Before proving this proposition, we recall standard properties of the theta function $\theta(u)$. It is easily seen from its definition that the theta function is holomorphic in $\mathbb{C}$, even, periodic with period one, and quasi-periodic with period τ :

$$\theta(-u) = \theta(u), \quad \theta(u+1) = \theta(u), \quad \theta(u+\tau) = \exp(-\pi i(2u+\tau))\theta(u).$$

In addition, one can show using the argument principle that the theta function has a unique simple zero at the point $d = (1+\tau)/2$, and no other zeros in the fundamental parallelogram spanned by 1 and τ . These properties allow one to express any meromorphic function on $\mathbb{C}/\Lambda$ in terms of the theta function. The construction is based on the following well-known result: there exists a meromorphic function with zeros at $p_1, \dots, p_m \in \mathbb{C}/\Lambda$ and poles at $q_1, \dots, q_m \in \mathbb{C}/\Lambda$ if and only if $\sum p_k = \sum q_k$. Thus, assume that we are given a collection of points with this property. Then the expression

$$f(u) := \frac{\prod_{k=1}^m \theta(u - p_k + d)}{\prod_{k=1}^m \theta(u - q_k + d)} \quad (41)$$

defines a meromorphic function on $\mathbb{C}$ which can be easily seen to be periodic with respect to both 1 and τ (here we regard p_k and q_k as points in $\mathbb{C}$ and assume that they are chosen in such a way that $\sum p_k = \sum q_k$ exactly, and not just modulo Λ). Therefore, this function can be viewed as a meromorphic function on $\mathbb{C}/\Lambda$. Furthermore, the only zeros of $f(u)$ in $\mathbb{C}/\Lambda$ are p_k , whereas its only poles are q_k . As zeros and poles determine a meromorphic function up to a constant factor, it follows that any meromorphic function on $\mathbb{C}/\Lambda$ with zeros at $p_1, \dots, p_m$ and poles $q_1, \dots, q_m$ can be written as (41) times a constant.

Proof of Proposition 6.10. Using Proposition 5.21 and Corollary 5.27, we obtain

$$\xi_k(u) = c_k \frac{(\theta(u - z_+ + d))^k \theta(u - x_p + k\delta + d)}{(\theta(u - z_- + d))^k \theta(u - x_p + d)}, \quad (42)$$

where c_k is a non-zero constant, and the term containing δ is found by equating the sum of zeros with the sum of poles. Note that because we are only interested in the direction of the vector ξ , we may multiply all ξ_k by $\theta(u - x_p + d)$, which results in

$$\tilde{\xi}_k(u) = c_k \frac{(\theta(u - z_+ + d))^k}{(\theta(u - z_- + d))^k} \theta(u - x_p + k\delta + d).$$

These are no longer meromorphic functions on $\mathbb{C}/\Lambda$, but still meromorphic functions on $\mathbb{C}$. Furthermore, in contrast to ξ_k , the functions $\tilde{\xi}_k$ are always finite at the points $0, 1/2, \tau/2 \in \mathbb{C}$ corresponding to $A, B, C \in \Gamma$, so the vertices of P are given by $(\tilde{\xi}_k(0) : \tilde{\xi}_k(1/2) : \tilde{\xi}_k(\tau/2))$. Also note that the values of the constants c_k do not affect the latter expression, so one can assume that $c_k = 1$. Under this assumption, we obtain

$$\tilde{\xi}_k(0) = \frac{(\theta(d - z_+))^k}{(\theta(d - z_-))^k} \theta(k\delta + d - x_p) = \theta(k\delta + d - x_p),$$

where the last equality follows from $d - z_- = -(d - z_+)$ and $\theta(-u) = \theta(u)$. Similarly, we have

$$\begin{aligned} \tilde{\xi}_k(1/2) &= \frac{(\theta(1/2 + d - z_+))^k}{(\theta(1/2 + d - z_-))^k} \theta(1/2 + k\delta + d - x_p) \\ &= \frac{(\theta(-1/2 + d - z_+))^k}{(\theta(1/2 + d - z_-))^k} \theta(1/2 + k\delta + d - x_p) = \theta(1/2 + k\delta + d - x_p), \end{aligned}$$

where the second last equality follows from 1-periodicity of θ , and the last equality follows from $d - z_- = -(d - z_+)$ and $\theta(-u) = \theta(u)$. Finally,

$$\begin{aligned}\tilde{\xi}_k(\tau/2) &= \frac{(\theta(\tau/2 + d - z_+))^k}{(\theta(\tau/2 + d - z_-))^k} \theta(\tau/2 + k\delta + d - x_p) = \frac{(\theta(1/2 + \tau - z_+))^k}{(\theta(1/2 + \tau - z_-))^k} \theta(\tau/2 + k\delta + d - x_p) \\ &= \exp(\pi ki(2z_+ - 1 - \tau)) \frac{(\theta(1/2 - z_+))^k}{(\theta(1/2 + \tau - z_-))^k} \theta(\tau/2 + k\delta + d - x_p) \\ &= \exp(\pi ki\delta) \theta(\tau/2 + k\delta + d - x_p),\end{aligned}$$

where the second equality uses the definition $d = (1 + \tau)/2$, the third uses the formula for $\theta(u + \tau)$, whereas the last uses that θ is even along with the relation $\delta = 2z_+ - 1 - \tau$.

Now, to complete the proof it remains to rewrite the obtained formulas using theta functions with characteristics. We have $\tilde{\xi}_k(0) = \theta_{00}(k\delta + d - x_p)$, $\tilde{\xi}_k(1/2) = \theta_{01}(k\delta + d - x_p)$, whereas $\tilde{\xi}_k(\tau/2) = \theta_{10}(k\delta + d - x_p)$ up to a factor not depending on k . As the latter factor does not affect the projective equivalence class of P , one obtains the desired formulas for vertices. $\square$

Remark 6.11. Note that the functions ξ_k may, but not necessarily do, have poles at X_p (see Corollary 5.27). However, formula (42) is valid anyway. Indeed, if ξ_k does not have a pole at X_p , then its only pole is the point Z_- (which is of order k), whereas its only zero is the point Z_+ (which is also of order k). Thus, we must have $kz_+ = kz_-$ modulo Λ , that is, $k\delta \in \Lambda$. However, then the factor $\theta(u - x_p + k\delta + d)/\theta(u - x_p + d)$ in (42) is a non-vanishing holomorphic function, so the analytic properties (i.e. zeros and poles) of the right-hand side of (42) are the same as for the left-hand side, which means that these functions coincide for a suitable value of c_k .

Now, to prove that P is Poncelet it suffices to establish the following.

PROPOSITION 6.12. *The image of the map $\Phi: \mathbb{C} \rightarrow \mathbb{CP}^2$ given by*

$$\Phi(u) := (\theta_{00}(u) : \theta_{01}(u) : \theta_{10}(u)) \quad (43)$$

is a conic.

Proof. First, note that the functions θ_{00} , θ_{01} , θ_{10} have no common zeros, so the mapping Φ is well-defined. Further, following [Mum83], define the following operators $\mathcal{S}, \mathcal{T}$ on holomorphic functions on $\mathbb{C}$:

$$(\mathcal{S}f)(u) := f(u + 1), \quad (\mathcal{T}f)(u) = \exp(\pi i(2u + \tau))f(u + \tau).$$

Then

$$\mathcal{S}\theta_{jk} = (-1)^j \theta_{jk}, \quad \mathcal{T}\theta_{jk} = (-1)^k \theta_{jk}. \quad (44)$$

In particular, we have $\mathcal{S}^2\theta_{jk} = \theta_{jk}$, $\mathcal{T}^2\theta_{jk} = \theta_{jk}$, which means that

$$\theta_{jk}(u + 2) = \theta_{jk}(u), \quad \theta_{jk}(u + 2\tau) = \exp(-4\pi i(u + \tau))\theta_{jk}(u). \quad (45)$$

From the latter it follows that Φ descends to a holomorphic mapping $\mathbb{C}/2\Lambda \rightarrow \mathbb{CP}^2$, so the image of Φ is an algebraic curve. To find the degree of that curve, one needs to find the number of its intersections with a generic line. Clearly, that number can be found as $m/\deg \Phi$, where m is the number of zeros of a generic linear combination of θ_{00} , θ_{01} , θ_{10} in the fundamental parallelogram of the lattice $2L$, whereas $\deg \Phi$ is the degree of Φ , when the latter is regarded as a mapping $\mathbb{C}/2\Lambda \rightarrow \mathbb{CP}^2$. The number m can be easily computed using quasi-periodicity relations (45) and the argument principle. That number is equal to four. Further, note that the functions θ_{00} , θ_{01} , θ_{10} are even, so $\Phi(-x) = \Phi(x)$, which means that $\deg \Phi \geq 2$. Therefore, the degree of the image

of Φ is either two or one, that is, the image of Φ is a conic or a straight line. However, it cannot be a straight line, because the functions $\theta_{k,j}$ are linearly independent by (44). This, the image of Φ is a conic. $\square$

Thus, we conclude that the vertices (40) of the polygon P lie on a conic. As P is self-dual, it is also circumscribed about a conic and, hence, Poncelet. Thus, Theorem B is proved.

Remark 6.13. One can also explicitly describe the image of the mapping (43) and, hence, the conic circumscribed about P using Riemann's relation

$$\sum_{j,k \in \{0,1\}} \theta_{jk}(\alpha_1) \theta_{jk}(\alpha_2) \theta_{jk}(\alpha_3) \theta_{jk}(\alpha_4) = 2 \theta_{00}(\beta_1) \theta_{00}(\beta_2) \theta_{00}(\beta_3) \theta_{00}(\beta_4),$$

where $\beta_1 := (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/2$, $\beta_2 := (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)/2$, $\beta_3 := (\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)/2$, $\beta_4 := (\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4)/2$. Taking $\alpha_1 = 0$, $\alpha_2 = u$, $\alpha_3 = v$, $\alpha_4 = u + v$, we obtain the identity

$$\begin{aligned} & -\theta_{00}(0) \theta_{00}(u) \theta_{00}(v) \theta_{00}(u+v) + \theta_{01}(0) \theta_{01}(u) \theta_{01}(v) \theta_{01}(u+v) \\ & + \theta_{10}(0) \theta_{10}(u) \theta_{10}(v) \theta_{10}(u+v) = 0, \end{aligned} \quad (46)$$

which, after a further substitution $v = 0$, becomes

$$-\theta_{00}^2(0) \theta_{00}^2(u) + \theta_{01}^2(0) \theta_{01}^2(u) + \theta_{10}^2(0) \theta_{10}^2(u) = 0. \quad (47)$$

Thus, the conic circumscribed about P is given by

$$-\theta_{00}^2(0) x_1^2 + \theta_{01}^2(0) x_2^2 + \theta_{10}^2(0) x_3^2 = 0. \quad (48)$$

Similarly, the conic inscribed in P is

$$-\theta_{00}^2(\delta/2) x_1^2 + \theta_{01}^2(\delta/2) x_2^2 + \theta_{10}^2(\delta/2) x_3^2 = 0. \quad (49)$$

Indeed, let $t_k := k\delta + d - x_p$, $m := k + 1/2$, and $t'_m := (t_k + t_{k+1})/2$. Then, as follows from (47), the point

$$v'_m := \left(\frac{\theta_{00}(0) \theta_{00}(t'_m)}{\theta_{00}(\delta/2)} : \frac{\theta_{01}(0) \theta_{01}(t'_m)}{\theta_{01}(\delta/2)} : \frac{\theta_{10}(0) \theta_{10}(t'_m)}{\theta_{10}(\delta/2)} \right)$$

belongs to the conic (49). Furthermore, the tangent line to (49) at v'_m passes through the vertices v_k and v_{k+1} of P . Indeed, that is equivalent to the relation

$$\begin{aligned} & -\theta_{00}(0) \theta_{00}(\delta/2) \theta_{00}(t'_m) \theta_{00}(t_{m \pm 1/2}) + \theta_{01}(0) \theta_{01}(\delta/2) \theta_{01}(t'_m) \theta_{01}(t_{m \pm 1/2}) \\ & + \theta_{10}(0) \theta_{10}(\delta/2) \theta_{10}(t'_m) \theta_{10}(t_{m \pm 1/2}) = 0, \end{aligned}$$

which is a particular case of (46) corresponding to $u = t_{m \pm 1/2}$, $v = \mp \delta/2$. Thus, indeed, the polygon P is circumscribed about the conic (49).

Remark 6.14. Note that formula (40) describes a *family* of polygons, parametrized by x_p . Our argument shows that all these polygons are inscribed in one and the same conic (48) and circumscribed about one and the same conic (49). Thus, polygons (40) form what is called a *Poncelet family*, that is, a family of polygons inscribed in the same conic and circumscribed about the same conic (recall that every Poncelet polygon is a member of such a family by Poncelet's porism). Also note that the expression (40) is periodic in x_p with the periods given by the lattice 2Λ . Thus, the Poncelet family containing our polygon P is parametrized by the elliptic curve $\mathbb{C}/2\Lambda$, which is a four-to-one covering of the spectral curve $\Gamma = \mathbb{C}/\Lambda$. As a corollary, the Poncelet family containing P contains four polygons projectively equivalent to P : one of those polygons is P , whereas the other three can be obtained from P by replacing x_p in formula (40) with

$x_p + 1$, $x_p + \tau$, and $x_p + 1 + \tau$. This quadruple of polygons admits a geometric description when the circumscribed conic C_1 and inscribed one C_2 are confocal. In this case, these polygons can be obtained from P by means of reflection with respect to the common symmetry axes of C_1, C_2 .

This argument also shows that the spectral curve is the same for all polygons in a Poncelet family. Using a different approach, this was earlier proved in [Sch15]. Formulas for Poncelet families similar to (40) are given in [Ves88].

Remark 6.15. Note that because the polygon P is closed (Remark 6.9), the expression (40) must be n -periodic in k . Therefore, we must have $n\delta \in 2\Lambda$. Another way to see this is to consider the function $(s-1)\mu_-$ on Γ . Using Table 1 and the fact that $s(A) = s(B) = s(C) = 1$, we conclude that this function has simple zeros at A, B, C , a zero of order $(n-3)/2$ at Z_- , and a pole of order $(n+3)/2$ at Z_+ . Thus, we have $0 + 1/2 + \tau/2 + (n-3)/2 \cdot z_- = (n+3)/2 \cdot z_+ \pmod{\Lambda}$, which implies $n/2 \cdot \delta = n/2 \cdot (z_+ - z_-) = 1/2 + \tau/2 - 3/2 \cdot (z_+ + z_-) = -2d = 0 \pmod{\Lambda}$, and thus $n\delta \in 2\Lambda$, as desired.

Also note that formula (40) still defines a Poncelet polygon if $n\delta \in \Lambda \setminus 2\Lambda$. It is then a *twisted* n -gon, which can also be viewed as a closed $2n$ -gon. Such twisted Poncelet polygons do not arise in our setting, because they are not fixed points of the pentagram map.

7. Proof of Theorem A: a closed polygon fixed by the pentagram map is Poncelet

In this section, we derive Theorem A from Theorem B. To that end, we first show, in § 7.1, that the self-duality assumption of Theorem B is not very restrictive. Namely, any polygon satisfying all the assumptions of the theorem except for possibly self-duality, can be transformed, by means of rescaling (9) with $s > 0$, into a self-dual polygon. From that we conclude that a polygon as in Theorem A (i.e. weakly convex, closed, and projectively equivalent to its pentagram image) must be Poncelet up to rescaling (9) with $s > 0$. Thus, to show that the polygon is actually Poncelet, we need to prove that the rescaling is trivial, that is, corresponds to $s = 1$. To that end, we show that if a weakly convex Poncelet polygon is rescaled in a non-trivial way, then the resulting polygon cannot be closed. This is done separately in the rational (see § 7.2) and elliptic (see § 7.3) cases. In the rational case we have a very simple explicit description of the corresponding degenerate Poncelet polygons (see § 6.1), so in that case the proof is completely elementary. As for the elliptic situation, in that case the proof relies on the study of the real part of the corresponding elliptic curve and location of various special points within that real part.

7.1 Self-duality up to rescaling

PROPOSITION 7.1. *Assume that a closed or twisted weakly convex polygon P is projectively equivalent to its pentagram image P' . Then one can choose the n -periodic operator $\mathcal{D}$ of the form (12) associated with P in such a way that the corresponding commuting operators $\mathcal{D}_l, \mathcal{D}_r$ given by (13) satisfy*

$$\mathcal{D}_r = -s_0 T^n \mathcal{D}_l^* \quad (50)$$

for certain $s_0 \in \mathbb{R}_+$.

Proof. Let $\mathcal{D}$ be an n -periodic operator corresponding to P such that the corresponding operators $\mathcal{D}_l, \mathcal{D}_r$ commute, and, moreover, the coefficients of $\mathcal{D}$ satisfy the alternating signs condition (11) (such $\mathcal{D}$ exists by Proposition 4.2). Then the operator $T^{-n} \mathcal{D}_l \mathcal{D}_r$ has the form

$$T^{-n} \mathcal{D}_l \mathcal{D}_r = \alpha T^{-1} + \beta + \gamma T. \quad (51)$$

Moreover, from the alternating signs condition we have $\alpha_k, \gamma_k > 0$ for all $k \in \mathbb{Z}$. Therefore, the operator (51) can be symmetrized. Namely, there exists a positive quasi-periodic sequence λ such that the operator $\lambda T^{-n} \mathcal{D}_l \mathcal{D}_r \lambda^{-1}$ is self-dual. That sequence can be found from the equation $\lambda_{k+1}/\lambda_k = \sqrt{\gamma_k/\alpha_{k+1}}$. Thus, conjugating $\mathcal{D}$ by λ if needed, we may assume that the operator (51) is self-dual, meaning that

$$T^{-n} \mathcal{D}_l \mathcal{D}_r = T^n \mathcal{D}_r^* \mathcal{D}_l^*. \quad (52)$$

We now show that under that assumption we must have (50). Let z_l, z_r be the monodromies of $\mathcal{D}_l, \mathcal{D}_r$ respectively. Then, by the second statement of Proposition 3.4, we have $0 < z_l < z_r$. Furthermore, because $\mathcal{D}_l$ and $\mathcal{D}_r$ commute, it follows that the kernels of both of them are contained in $\text{Ker } \mathcal{D}_l \mathcal{D}_r$. Thus, the spectrum of the monodromy of $\mathcal{D}_l \mathcal{D}_r$ is $\{z_l, z_r\}$. Moreover, we have

$$\text{Ker } (\mathcal{D}_l \mathcal{D}_r)|_{Q^n(z_l)} = \text{Ker } \mathcal{D}_l, \quad \text{Ker } (\mathcal{D}_l \mathcal{D}_r)|_{Q^n(z_r)} = \text{Ker } \mathcal{D}_r.$$

Similarly, using that the monodromy of $\mathcal{D}_l^*$ and $\mathcal{D}_r^*$ is given by z_l^{-1} and z_r^{-1} , respectively, we conclude that the monodromy of $\mathcal{D}_r^* \mathcal{D}_l^*$ is $\{z_l^{-1}, z_r^{-1}\}$, which, in view of (52) and the inequality $0 < z_l < z_r$ implies $z_l = z_r^{-1}$. Furthermore, we have

$$\text{Ker } \mathcal{D}_l^* = \text{Ker } (\mathcal{D}_r^* \mathcal{D}_l^*)|_{Q^n(z_l^{-1})} = \text{Ker } (\mathcal{D}_l \mathcal{D}_r)|_{Q^n(z_l^{-1})} = \text{Ker } (\mathcal{D}_l \mathcal{D}_r)|_{Q^n(z_r)} = \text{Ker } \mathcal{D}_r,$$

thus,

$$\mathcal{D}_l^* = T^{-n} \mu \mathcal{D}_r \quad (53)$$

for a certain n -periodic sequence μ of non-zero real numbers. Taking the dual of both sides, we also obtain $\mathcal{D}_r^* = T^{-n} \mathcal{D}_l \mu^{-1}$, so

$$\mathcal{D}_l^* \mathcal{D}_r^* = T^{-2n} \mu \mathcal{D}_r \mathcal{D}_l \mu^{-1} = T^{-2n} \mu \mathcal{D}_l \mathcal{D}_r \mu^{-1}.$$

At the same time, we have

$$\mathcal{D}_l^* \mathcal{D}_r^* = \mathcal{D}_r^* \mathcal{D}_l^* = T^{-2n} \mathcal{D}_l \mathcal{D}_r,$$

so μ commutes with $\mathcal{D}_l \mathcal{D}_r$. However, that is only possible if μ is a constant sequence $\mu_k = c$. Thus, (53) implies (50), with $s_0 = -c^{-1}$. Furthermore, because the coefficient of the highest degree term in $\mathcal{D}_r$ is a sequence of negative numbers, whereas the coefficient of the coefficient of the highest degree term in $\mathcal{D}_l^*$ is a sequence of positive numbers, (50) can only be satisfied for $s_0 > 0$, as desired. $\square$

COROLLARY 7.2. *Assume that a closed or twisted weakly convex polygon P is projectively equivalent to its pentagram image P' . Then there exists a polygon P_{sd} with the same properties which is, in addition, self-dual (and, hence, Poncelet by Theorem B), such that $P = R_{s_0}(P_{\text{sd}})$ where R_{s_0} is the rescaling (9) with $s = s_0 > 0$.*

Proof. Take the operator $\mathcal{D}$ provided by Proposition 7.1. It has the form $\mathcal{D} = \mathcal{D}_l - s_0 T^n \mathcal{D}_l^*$, where $s_0 \in \mathbb{R}_+$. Consider also the operator $\mathcal{D}_{\text{sd}} = \mathcal{D}_l - T^n \mathcal{D}_l^*$, and the associated polygon P_{sd} . Then, by Corollary 2.11, we have $P = R_{s_0}(P_{\text{sd}})$. In particular, P_{sd} is projectively equivalent to its pentagram image (because the pentagram map commutes with rescaling) and weakly convex (by the third statement of Proposition 3.4). Furthermore, we have $\mathcal{D}_{\text{sd}}^* = -T^{-n} \mathcal{D}_{\text{sd}}$, so P_{sd} is self-dual, as desired. $\square$

7.2 End of proof in the rational case

Let P be a weakly convex closed polygon projectively equivalent to its pentagram image P' , as in Theorem A. Then, by Corollary 7.2, there exists a generally speaking twisted polygon P_{sd} such

that $P = R_{s_0}(P_{\text{sd}})$ for some $s_0 > 0$, and P_{sd} is self-dual. Consider the spectral curve associated with P_{sd} , constructed in the proof of Theorem B. In this section, we prove Theorem A in the case when the genus of Γ is zero, that is, when Γ is rational. To that end, we show that $s_0 = 1$, so $P = P_{\text{sd}}$ and, hence, Poncelet.

As we know from § 6.1, in the rational case the vertices of P_{sd} are given by (34) or (37). In the case (34), the associated difference operator reads

$$\mathcal{D}_{\text{sd}} = T^{(n-3)/2} - aT^{(n-1)/2} + aT^{(n+1)/2} - T^{(n+3)/2}, \quad (54)$$

where a is such that the roots of the corresponding characteristic polynomial $1 - ax + ax^2 - x^3$ are r, r^{-1} , and 1 (note that since the polygon P_{sd} is real, a must be real too, so we must have $|r| = 1$). Indeed, the kernel of such an operator is spanned by the sequences r^k , r^{-k} , and a constant sequence, so the associated polygon is precisely (34). Likewise, in the case (37), the associated difference operator is also of the form (54), with $a = 3$. Thus, because the polygon P_{sd} is defined by the operator (54), the polygon $P = R_{s_0}(P_{\text{sd}})$ is defined by

$$\mathcal{D} = T^{(n-3)/2} - aT^{(n-1)/2} + s_0(aT^{(n+1)/2} - T^{(n+3)/2}).$$

The kernel of this operator is spanned by the sequences x_1^k, x_2^k, x_3^k , where x_1, x_2, x_3 are the roots of the characteristic polynomial $h(x) := 1 - ax + s_0(ax^2 - x^3)$ (note that we do not need to consider the case of multiple roots, because in that case the monodromy of $\mathcal{D}$ is not diagonalizable, and the polygon P cannot be closed). Moreover, because P is closed, we must have $x_1^n = x_2^n = x_3^n$, so $|x_1| = |x_2| = |x_3| = \lambda$, where $\lambda > 0$ is a real number. Thus, the roots of the polynomial $h(\lambda x) = 1 - a\lambda x + s_0(a\lambda^2 x^2 - \lambda^3 x^3)$ must all have absolute value 1. Also taking into account that this polynomial is real, and that $s_0\lambda^3 > 0$, we conclude that the roots of $h(\lambda x)$ are of the form $1, \alpha, \bar{\alpha}$, where $|\alpha| = 1$. However, this yields $s_0\lambda^3 = 1$ and $s_0\lambda^2 = \lambda$, so $s_0 = 1$. Therefore, the polygon P coincides with P_{sd} and, hence, Poncelet. Thus, the proof of Theorem A in the rational case is complete.

Remark 7.3. One can also give a more concrete description of P , as follows. As the vertices of P are given by (34) (with (37) being impossible due to closedness of P), and P is a closed n -gon, it follows that $r^n = 1$. Thus, applying a linear transformation to (34), we obtain a polygon whose vertices have affine coordinates $\cos(2\pi mk/n)$, $\sin(2\pi mk/n)$, where $2\pi m/n = \arg r$. In particular, if $m = 1$, then P is a regular n -gon.

7.3 End of proof in the elliptic case

In this section, we prove Theorem A in the case when the genus of Γ is one, that is, when Γ is elliptic. As in the rational case, we show that $s_0 = 1$, so $P = P_{\text{sd}}$ and, hence, Poncelet. We keep the notation of §§ 5.1 and 6.2.

Recall that a *real structure* on a Riemann surface Γ is an anti-holomorphic involution $\rho: \Gamma \rightarrow \Gamma$. The *real part* $\Gamma_{\mathbb{R}}$ of Γ (with respect to the real structure ρ) is then defined as the set of fixed points of ρ : $\Gamma_{\mathbb{R}} := \{X \in \Gamma \mid \rho(X) = X\}$. A meromorphic function f on Γ is called a *real function* if $\rho^* f = \bar{f}$. Real functions take real values at real points (i.e. points in $\Gamma_{\mathbb{R}}$).

In our case, the spectral curve Γ is endowed with a real structure $\rho: \Gamma \rightarrow \Gamma$ induced by the involution $(z, w) \mapsto (\bar{z}, \bar{w})$ on the affine spectral curve Γ_a .

PROPOSITION 7.4. *The functions $z, w, \mu_{\pm}, s$, and ξ on Γ are real (see § 5.1 for the definition of those functions).*

Proof. The functions z, w are real by construction of the real structure ρ . To prove that the vector-function ξ is real, note that it is defined by (22) up to a scalar factor. Taking the complex

conjugate of those equations and then applying ρ^* , we get that $\rho^*\bar{\xi} = f\xi$ for a certain meromorphic function f . However, then the normalization condition $\xi_1 = 1$ implies $f = 1$, as desired. Now, the reality of the functions $\mu_{\pm}$ follows from (23), whereas the reality of s follows from its definition (24). $\square$

COROLLARY 7.5. *The points $Z_{\pm}, S_{\pm}, A, B, C, D \in \Gamma$ are real (see §5.1 for the definition of $Z_{\pm}, S_{\pm}$ and §6.2 for the definition of A, B, C, D).*

Proof. As z is a real function (Proposition 7.4), the involution ρ takes zeros of z to zeros of z . However, the only zero of z is Z_+ (Table 1), so the latter must be real. Analogously, Z_- is real as the only pole of z , S_+ is real as the only simple zero of s , S_- is real as the only simple pole of s , whereas D is real as the only point where both s and z are equal to -1 (see Remark 5.16). To show that A, B, C are real, observe that they constitute the set of points where $s = 1$, so ρ takes the set $\{A, B, C\}$ to itself. Further, note that the values of the function w at A, B, C are eigenvalues of a self-adjoint operator $(\mathcal{D}_+\mathcal{D}_-)|_{Q^n(1)}$ and, hence, real. Furthermore, those values are distinct, because A, B, C are branch points of w , whereas $\deg w = 2$. However, if, say, $\rho(A) = B$, then we must have $w(B) = \bar{w}(A)$, which is not possible because $w(A), w(B)$ are real and distinct. Thus, ρ cannot permute $\{A, B, C\}$ and, thus, preserves each of them. $\square$

COROLLARY 7.6. *The real part $\Gamma_{\mathbb{R}}$ of Γ consists of two disjoint circles.*

Proof. The real part of any Riemann surface consists a finite number of disjoint circles (ovals). Furthermore, because the genus of Γ is one, the number of connected components of $\Gamma_{\mathbb{R}}$ is at most two by Harnack's theorem. At the same time, the number of connected components is non-zero because the real part $\Gamma_{\mathbb{R}}$ of Γ is not empty (by Corollary 7.5). Thus, it remains to determine whether the number of connected components is one or two. These cases can be distinguished by counting the number of real points of order 2 on Γ . Namely, if Γ is identified with $\mathbb{C}/\Lambda$ in such a way that 0 is a real point, then $\Gamma_{\mathbb{R}}$ is a subgroup of Γ isomorphic to S^1 if $\Gamma_{\mathbb{R}}$ is connected, and $S^1 \times \mathbb{Z}_2$ if $\Gamma_{\mathbb{R}}$ has two components. Thus, the number of real order 2 points in $\Gamma_{\mathbb{R}}$ is 2^m , where m is the number of components of Γ . Identifying Γ with $\mathbb{C}/L$ as in §6.2, we see that the order 2 points are A, B, C, D , which are all real. Thus, $m = 2$. $\square$

This argument also shows that one of the components of $\Gamma_{\mathbb{R}}$ contains the point D and one of the points $\{A, B, C\}$, whereas the second component of $\Gamma_{\mathbb{R}}$ contains the remaining two points. Without loss of generality, assume that C and D are located in the same component. Denote that component by $\Gamma_{\mathbb{R}}^0$.

PROPOSITION 7.7. *We have $Z_{\pm}, S_{\pm} \in \Gamma_{\mathbb{R}}^0$.*

Proof. The function z is real-valued on $\Gamma_{\mathbb{R}}^0$ and satisfies $z(C) = 1$, $z(D) = -1$. Thus, there must be at least two points on $\Gamma_{\mathbb{R}}^0$ where z changes sign. But the only points which have this property are $Z_{\pm}$ (Table 1). Similarly, $s(C) = 1$, $s(D) = -1$, so the function s should also change sign at two points. Moreover, these cannot be the points $Z_{\pm}$, because at those points s has a zero and a pole of order 2. Thus, we must have $S_{\pm} \in \Gamma_{\mathbb{R}}^0$, as desired. $\square$

PROPOSITION 7.8. *The cyclic order of the points $C, D, Z_{\pm}, S_{\pm}$ on $\Gamma_{\mathbb{R}}^0$ is as shown in Figure 6.*

The proof is based on the following two lemmas.

LEMMA 7.9. *We have $z(S_+) \in (0, 1)$.*

Proof. Without loss of generality, assume that the vector $\xi(S_+)$ is finite and non-zero (see Remark 5.9). Then, using the definition of the function μ_+ and the fact that $\mu_+(S_+) = 0$

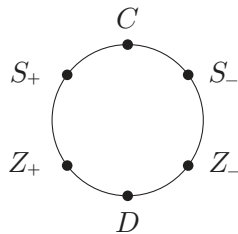


FIGURE 6. Location of the points $C, D, Z_{\pm}, S_{\pm}$ in the component $\Gamma_{\mathbb{R}}^0$ of the real part of the spectral curve.

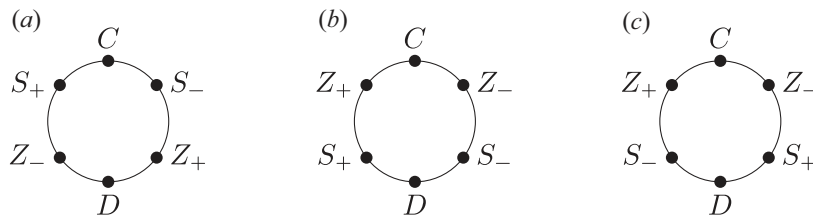


FIGURE 7. Impossible locations of the points $C, D, Z_{\pm}, S_{\pm}$ on $\Gamma_{\mathbb{R}}^0$.

(Table 1), we obtain $\mathcal{D}_+\xi(S_+) = \mu_+(S_+)\xi(S_+) = 0$. Therefore, $\xi(S_+)$ spans the kernel of the operator $\mathcal{D}_+$, whereas $z(S_+)$ is the monodromy of that operator. Thus, by the second statement of Proposition 3.4, the number $z(S_+)$ is positive and is less than the monodromy of $\mathcal{D}_r = -T^n\mathcal{D}_+^*$, but the monodromy of the latter operator is the same as the monodromy of $\mathcal{D}_+^*$, which is $z(S_+)^{-1}$. Thus, we obtain $0 < z(S_+) < z(S_+)^{-1}$, and the result follows. $\square$

LEMMA 7.10. *The only point in $\Gamma_{\mathbb{R}}^0$ where $z = 1$ is the point C .*

Proof. Assume that $X \in \Gamma_{\mathbb{R}}^0$ and $z(X) = 1$. Then the latter condition, in particular, implies $X \neq Z_{\pm}$. Therefore, without loss of generality, we may assume that the vector $\xi(X)$ is finite and non-zero (if not, we renormalize ξ , see Remark 5.9). Under this assumption, using the inner product (7) on $Q^n(\pm 1)$, we obtain

$$\mu_+(X) \langle \xi(X), \xi(X) \rangle = \langle \mathcal{D}_+\xi(X), \xi(X) \rangle = \langle \xi(X), \mathcal{D}_-\xi(X) \rangle = \mu_-(X) \langle \xi(X), \xi(X) \rangle.$$

Furthermore, because the vector $\xi(X)$ is real, it follows that $\langle \xi(X), \xi(X) \rangle > 0$ and, thus, $\mu_+(X) = \mu_-(X)$. Thus, using formula (24) for the function s , we obtain $s(X) = z(X)^{-1} = 1$ (here we use that the value $\mu_+(X) = \mu_-(X)$ is finite and non-zero, which is true because the functions $\mu_{\pm}$ do not have common zeros or poles; Table 1). Furthermore, recall that the set of points where $s = 1$ consists of the point C , plus points A and B which do not belong to $\Gamma_{\mathbb{R}}^0$. The result follows. $\square$

Proof of Proposition 7.8. The restriction of the involution σ to $\Gamma_{\mathbb{R}}^0$ preserves the points C, D , interchanges Z_+ with Z_- , and interchanges S_+ with S_- . For this reason, the only possible locations of those points on $\Gamma_{\mathbb{R}}^0$ are those depicted in Figure 6, as well as those depicted in Figure 7. Assume that $C, D, Z_{\pm}, S_{\pm}$ are located as in Figure 7(a). Then, because $z(S_+) \in (0, 1)$ by Lemma 7.9, whereas Z_- is a pole of z , there must be a point X in the open arc (S_+, Z_-) such that $z(X) = 1$ or $z(X) = 0$ (here and in the following (X, Y) denotes an open arc going from X to Y in counter-clockwise direction). However, the former is impossible by Lemma 7.10, whereas the latter is impossible because the only zero of z is the point Z_+ . Thus, the points cannot be located as in Figure 7(a). Further, because $z(D) = -1$, whereas the only points where

z changes sign are $Z_{\pm}$, in Figures 7(b) and 7(c) we must have $z(S_+) < 0$, which is impossible by Lemma 7.9. Thus, the points $C, D, Z_{\pm}, S_{\pm}$ are located as in Figure 6. $\square$

Now recall that the elliptic curve Γ is associated with a Poncelet n -gon P_{sd} and we have a closed n -gon $P = R_{s_0}(P_{\text{sd}})$, where $s_0 > 0$. Our aim is to show that $s_0 = 1$.

PROPOSITION 7.11. *There is a point $X_0 \in \Gamma_{\mathbb{R}}^0$ such that $s(X_0) = s_0$ and $z(X_0) = s_0^{-n/3}$.*

Proof. The function s has one simple pole and one double pole in $\Gamma_{\mathbb{R}}^0$ (Table 1). Therefore, the degree of the mapping $s: \Gamma_{\mathbb{R}}^0 \rightarrow \mathbb{RP}^1$ is equal to ± 1 (depending on the orientations). In particular, this mapping is surjective, so there exists $X_0 \in \Gamma_{\mathbb{R}}^0$ such that $s(X_0) = s_0$. To show that $z(X_0) = s_0^{-n/3}$, recall that the polygon P associated with the operator $\mathcal{D}_+ - s_0 T^n \mathcal{D}_-$ is closed. Therefore, the monodromy of that operator has the form λId . At the same time, because $\mathcal{D}_- = \mathcal{D}_+^*$, the explicit form of that operator is

$$\mathcal{D}_+ - s_0 T^n \mathcal{D}_- = aT^{(n-3)/2} + bT^{(n-1)/2} - s_0 \tilde{a}T^{(n+1)/2} - s_0 \tilde{a}T^{(n+3)/2},$$

where the sequences $\tilde{a}, \tilde{b}$ coincide with a, b up to a shift of indices. Thus, by formula (6), the determinant of the monodromy of this operator is s_0^{-n} . Thus, we have $\lambda = s_0^{-n/3}$, and the result follows. $\square$

We now show that $X_0 = C$, which implies $s_0 = 1$ and, thus, proves Theorem A. To that end, note that because $s(X_0) = s_0$ is finite and positive, X_0 must be located in the open arc (S_-, S_+) (see Figure 6). At the same time, because the function s is equal to one at C , has a pole at S_- , and does not take values $0, 1, \infty$ in (S_-, C) , it follows that $s > 1$ in (S_-, C) . Furthermore, the same argument applied to the function z shows that $z > 1$ in (Z_-, C) and, in particular, in (S_-, C) . However, then X_0 cannot belong to (S_-, C) , because it is not possible that both $s(X_0) = s_0$ and $z(X_0) = s_0^{-n/3}$ are greater than 1. Analogously, s and z are both less than 1 in (C, S_+) , so X_0 cannot belong there either. Therefore, we must have $X_0 = C$, which implies $s_0 = 1$, but this means that the polygon P is the same as the polygon P_{sd} and, hence, Poncelet. Thus, Theorem A is proved.

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Appendix A. Duality of difference operators and polygons

The goal of this appendix is to prove that polygons corresponding to dual difference operators are dual to each other. This seems to be a well-known result, and it explicitly appears as Proposition 4.4.3 in [MOST14]. Here we give a different proof, based on interpretation of difference operators as infinite matrices.

PROPOSITION A.1. *Let $\mathcal{D}$ be a properly bounded difference operator supported in $[m_-, m_+]$, and let $P = \{v_k\}$ be the corresponding polygon in $\mathbb{P}^{d-1}$, where $d = m_+ - m_-$ is the order of $\mathcal{D}$. Then the dual operator $\mathcal{D}^*$ corresponds to a polygon $P^* = \{v_k^*\}$ in the dual space $(\mathbb{P}^{d-1})^*$ whose k th vertex v_k^* is the hyperplane in $\mathbb{P}^{d-1}$ spanned by the vertices $v_{k+m_-+1}, \dots, v_{k+m_+-1}$ of P .*

Proof. Let $\mathcal{D} = \sum_{j=m_-}^{m_+} a^j T^j$. Then one can interpret $\mathcal{D}$ as a finite-band matrix (2) whose non-zero diagonals have labels $m_-, \dots, m_+$. (Here and in what follows, the k th diagonal of an infinite

matrix is the collection of its entries a_{ij} such that $j - i = k$. In other words, the diagonals are labeled from southwest to northeast, with the main diagonal labeled by 0.) Note that even though infinite matrices do not form an algebra, any infinite matrix can be multiplied by a finite band matrix.

LEMMA A.2. *There exists an infinite matrix $\mathcal{L}$ such that:*

1. $\mathcal{D}\mathcal{L} = \mathcal{L}\mathcal{D} = 0$;
2. the diagonals of $\mathcal{L}$ with labels $-m_+ + 1, \dots, -m_- - 1$ vanish;
3. none of the entries of $\mathcal{L}$ on the diagonals with labels $-m_+$ and $-m_-$ vanish.

Remark A.3. One can think of infinite matrices as formal Laurent series in terms of the shift operator T , with coefficients given by sequences. In this language, Lemma A.2 states the existence of $\mathcal{L}$ of the form $\sum_{j=-\infty}^{-m_+} b^j T^j + \sum_{j=-m_-}^{+\infty} b^j T^j$, where $b_k^{-m_+} \neq 0$, $b_k^{-m_-} \neq 0$ for any $k \in \mathbb{Z}$.

Proof of Lemma A.2. The infinite matrix $\mathcal{D}$ can be regarded as an element of two groups: the group GL_{∞}^+ of invertible infinite matrices with finitely many non-zero diagonals below the main diagonal, and the group GL_{∞}^- of invertible infinite matrices with finitely many non-zero diagonals above the main diagonal. Denote by $\hat{\mathcal{D}}^{-1}, \check{\mathcal{D}}^{-1}$ the inverses of $\mathcal{D}$ in these two groups, and set $\mathcal{L} := \hat{\mathcal{D}}^{-1} - \check{\mathcal{D}}^{-1}$. Then we clearly have $\mathcal{D}\mathcal{L} = \mathcal{L}\mathcal{D} = 0$. To see that $\mathcal{L}$ is of desired form, write $\mathcal{D}$ as $a^{m_-} T^{m_-} (1 + \dots)$, where the dots denote terms of higher order in T . Then the inverse of $(1 + \dots)$ in GL_{∞}^+ can be computed using the Taylor series $(1 + x)^{-1} = 1 - x + \dots$. Thus, the inverse of $\mathcal{D}$ in GL_{∞}^+ reads $\hat{\mathcal{D}}^{-1} = (1 + \dots)^{-1} T^{-m_-} (a^{m_-})^{-1}$ and, hence, is of the form $\sum_{j=-m_-}^{+\infty} b^j T^j$ with $b_k^{-m_-} \neq 0$. Likewise, $\check{\mathcal{D}}^{-1}$ is of the form $\sum_{j=-\infty}^{-m_+} b^j T^j$ with $b_k^{-m_+} \neq 0$. The result follows. $\square$

We now finish the proof of Proposition A.1. Let $V = \{V_k \in \mathbb{R}^d\}$ be a sequence of lifts of vertices v_k of P such that $\mathcal{D}V = 0$. Then any scalar sequence $\xi \in \mathrm{Ker} \mathcal{D}$ can be obtained from V by means of term-wise application of a linear functional. In particular, because $\mathcal{D}\mathcal{L} = 0$, this applies to columns of the matrix $\mathcal{L}$. Thus, the j th column of $\mathcal{L}$ is of the form $W_j(V_k)$ for a certain linear functional $W_j \in (\mathbb{R}^d)^*$. Furthermore, because the diagonals of $\mathcal{L}$ with labels $-m_+ + 1, \dots, -m_- - 1$ vanish, it follows that W_j annihilates $V_{j+m_-+1}, \dots, V_{j+m_+-1}$. Moreover, because $\mathcal{L}$ has a non-vanishing diagonal, we have $W_j \neq 0$. Therefore, the projection of W_j to $(\mathbb{P}^{d-1})^* = \mathbb{P}(\mathbb{R}^d)^*$ is exactly the hyperplane spanned by the vertices $v_{j+m_-+1}, \dots, v_{j+m_+-1}$ of P . Thus, to complete the proof, it suffices to show that the sequence of W_j is annihilated by $\mathcal{D}^*$. To that end, note that because $\mathcal{L}\mathcal{D} = 0$, the rows of $\mathcal{L}$ are annihilated by $\mathcal{D}^*$. However, those rows are of the form $W_j(V_k)$, and because V_k span $\mathbb{R}^d$, it follows that the sequence W_j is annihilated by $\mathcal{D}^*$, as desired. $\square$

Remark A.4. It is also easy to see that the matrix $\mathcal{L}$ provided by Lemma A.2 is unique up to a constant factor. It takes a particularly simple form when the polygon P is closed. To show that, assume for simplicity that $m_- = 0$, so that the operator $\mathcal{D}$ is supported in $[0, d]$. Furthermore, assume that $\mathcal{D}$ is n -periodic and has trivial monodromy (in particular, the polygon P corresponding to $\mathcal{D}$ is closed). Then, as shown in [Kri15], there exists an n -periodic operator $\mathcal{R}$ supported in $[0, n - d]$ such that $\mathcal{R}\mathcal{D} = \mathcal{D}\mathcal{R} = 1 - T^n$ (the operator $\mathcal{R}$ is closely related to the so-called *Gale dual* of $\mathcal{D}$). Using that, one can find the inverses of $\mathcal{D}$ in $\mathrm{GL}_{\infty}^{\pm}$ as

$$\begin{aligned} \hat{\mathcal{D}}^{-1} &= \mathcal{R}(\widehat{1 - T^n})^{-1} = \mathcal{R}(1 + T^n + T^{2n} + \dots), \\ \check{\mathcal{D}}^{-1} &= \mathcal{R}(\widetilde{1 - T^n})^{-1} = -\mathcal{R}T^{-n}(\widetilde{1 - T^{-n}})^{-1} = -\mathcal{R}T^{-n}(1 + T^{-n} + T^{-2n} + \dots) \\ &= -\mathcal{R}(T^{-n} + T^{-2n} + \dots). \end{aligned}$$

As a result, one obtains

$$\mathcal{L} = \hat{\mathcal{D}}^{-1} - \check{\mathcal{D}}^{-1} = \mathcal{R} \sum_{j=-\infty}^{+\infty} T^{jn}.$$

REFERENCES

- Cle71 A. Clebsch, *Ueber das ebene fünfeck*, Math. Ann. **4** (1871), 476–489.
- CO19 C. H. Conley and V. Ovsienko, *Lagrangian configurations and symplectic cross-ratios*, Math. Ann. **375** (2019), 1105–1145.
- DR14 V. Dragović and M. Radnović, *Bicentennial of the great Poncelet theorem (1813–2013): current advances*, Bull. Aust. Math. Soc. **51** (2014), 373–445.
- FB19 R. Felipe and G. M. Beffa, *The pentagram map on Grassmannians*, Ann. Inst. Fourier **69** (2019), 421–456.
- FM16 V. V. Fock and A. Marshakov, *Loop groups, clusters, dimers and integrable systems*, in *Geometry and quantization of moduli spaces* (Springer, 2016), 1–65.
- FT09 D. Fuchs and S. Tabachnikov, *Self-dual polygons and self-dual curves*, Funct. Anal. Math. **2** (2009), 203–220.
- GSTV16 M. Gekhtman, M. Shapiro, S. Tabachnikov and A. Vainshtein, *Integrable cluster dynamics of directed networks and pentagram maps*, Adv. Math. **300** (2016), 390–450.
- Gli11 M. Glick, *The pentagram map and Y-patterns*, Adv. Math. **227** (2011), 1019–1045.
- GP16 M. Glick and P. Pylyavskyy, *Y-meshes and generalized pentagram maps*, Proc. Lond. Math. Soc. **112** (2016), 753–797.
- Izo16 A. Izosimov, *Pentagrams, inscribed polygons, and Prym varieties*, Electron. Res. Announc. Math. Sci. **23** (2016), 25–40.
- KV15 R. Kedem and P. Vichitkunakorn, *T-systems and the pentagram map*, J. Geom. Phys. **87** (2015), 233–247.
- KS13 B. Khesin and F. Soloviev, *Integrability of higher pentagram maps*, Math. Ann. **357** (2013), 1005–1047.
- KS16 B. Khesin and F. Soloviev, *The geometry of dented pentagram maps*, J. Eur. Math. Soc. **18** (2016), 147–179.
- Kri15 I. M. Krichever, *Commuting difference operators and the combinatorial Gale transform*, Funct. Anal. Appl. **49** (2015), 175–188.
- KN03 I. M. Krichever and S. P. Novikov, *Two-dimensionalized Toda lattice, commuting difference operators, and holomorphic bundles*, Russian Math. Surveys **58** (2003), 473–510.
- LT07 M. Levi and S. Tabachnikov, *The Poncelet grid and billiards in ellipses*, Amer. Math. Monthly **114** (2007), 895–908.
- MOST14 S. Morier-Genoud, V. Ovsienko, R. E. Schwartz and S. Tabachnikov, *Linear difference equations, frieze patterns, and the combinatorial Gale transform*, Forum Math. Sigma **2** (2014), E22.
- Mum83 D. Mumford, *Tata Lectures on Theta I*, Progress in Mathematics, vol. 28 (Birkhäuser, 1983).
- OST10 V. Ovsienko, R. Schwartz and S. Tabachnikov, *The pentagram map: a discrete integrable system*, Comm. Math. Phys. **299** (2010), 409–446.
- OST13 V. Ovsienko, R. Schwartz and S. Tabachnikov, *Liouville–Arnold integrability of the pentagram map on closed polygons*, Duke Math. J. **162** (2013), 2149–2196.
- Sch92 R. Schwartz, *The pentagram map*, Exp. Math. **1** (1992), 71–81.
- Sch07 R. Schwartz, *The Poncelet grid*, Adv. Geom. **7** (2007), 157–175.

- Sch08 R. Schwartz, *Discrete monodromy, pentagrams, and the method of condensation*, J. Fixed Point Theory Appl. **3** (2008), 379–409.
- Sch15 R. Schwartz, *The pentagram integrals for Poncelet families*, J. Geom. Phys. **87** (2015), 432–449.
- Sol13 F. Soloviev, *Integrability of the pentagram map*, Duke Math. J. **162** (2013), 2815–2853.
- VM79 P. Van Moerbeke and D. Mumford, *The spectrum of difference operators and algebraic curves*, Acta Math. **143** (1979), 93–154.
- Ves88 A. P. Veselov, *Integrable discrete-time systems and difference operators*, Funct. Anal. Appl. **22** (1988), 83–93.

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