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Polygon recutting as a cluster integrable system

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Abstract

Recutting is an operation on planar polygons defined by cutting a polygon along a diagonal to remove a triangle, and then reattaching the triangle along the same diagonal but with opposite orientation. Recuttings along different diagonals generate an action of the affine symmetric group on the space of polygons. We show that this action is given by cluster transformations and is completely integrable. The integrability proof is based on interpretation of recutting as refactorization of quaternionic polynomials.

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Contents

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1 Introduction

Recent years have seen a spark of interest in discrete integrable systems, largely due to emerging connections with cluster algebras. Many of such systems are defined by iterating a certain geometric construction, with Schwartz's *pentagram map* [14] being the best known example. In the present paper we study another dynamical system of a somewhat similar nature: Adler's *polygon recutting* [1]. Given a planar polygon, its recutting ρ_i at a vertex v_i is defined as follows. Detach the triangle formed by the vertices v_{i-1}, v_i, v_{i+1} from the rest of the polygon by cutting along the diagonal $v_{i-1}v_{i+1}$. Then attach the triangle back along the same diagonal but with opposite orientation. Put differently, recutting ρ_i at a vertex v_i is reflection of v_i in the perpendicular bisector of the diagonal $v_{i-1}v_{i+1}$, see Fig. 1.

Consider the group generated by recuttings ρ_i at arbitrary vertices. The goal of the present paper is to understand the dynamics of that group as it acts on the space of polygons. It has long been known that this dynamics possesses many features of an integrable system:

- As observed in [1] recuttings ρ_i of a closed n-gon obey the relations of the affine symmetric group $\tilde{S}_n$ (also known as the affine Weyl group $\tilde{A}_{n-1}$). Specifically, one has

  1. $\rho_i^2 = \mathrm{id}$ for any vertex v_i;
  2. $\rho_i\rho_j = \rho_j\rho_i$ for any non-consecutive vertices v_i, v_j (here indices are considered modulo n, so that the vertices v_n and v_1 are thought of as consecutive);
  3. the *braid relation* $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$ for any consecutive (modulo n) vertices v_i, v_{i+1} (see [2] and Proposition A.1 below).

  These relations in particular imply that the group generated by recuttings ρ_i has polynomial growth. According to [18], this is a necessary condition for integrability of a group action. In what follows, we refer to the action of the affine symmetric group $\tilde{S}_n$ on n-gons by recutting as the *recutting action*, and to $\tilde{S}_n$ itself as the *recutting group*.
- Another result of [1] is that recuttings constitute discrete symmetries of an integrable system known as the *dressing chain* [19]. This in particular provides a Lax (or zero curvature) representation and a number of invariants (first integrals) for recutting dynamics.
- A different Lax representation (which also applies to recutting of polygons in spaces of dimension $d > 2$) is given in [2].

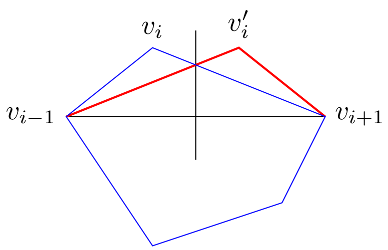

Fig. 1 Recutting ρ_i at the vertex v_i moves it to position v_i'. Other vertices remain intact

Table 1 Polygon spaces and associated structures

Space	$\mathcal{P}_n^S/S$, planar polygons closed up to similarity modulo similarities	$\mathcal{P}_n^E/E$, planar polygons closed up to isometry modulo isometries	$\mathcal{P}_n/E$, closed planar polygons modulo isometries
Recutting-invariant Poisson structure	Cluster Poisson structure	Poisson structure on quaternionic polynomials	–
Recutting invariants (first integrals)	–	Yes	Yes
Recutting is ...	... given by cluster transformations (Proposition 3.3)	... Arnold–Liouville integrable (Theorem 1.1)	... integrable in the non-Hamiltonian sense (Theorem 1.3)

- The paper [15] provides a Poisson-Lie group model for a discrete system which can be thought of as an extension of recutting.
- As shown in [16], recutting commutes with another conjecturally integrable system, the so-called *discrete bicycle transformation*. The latter has a large number of invariants which are also preserved by recutting dynamics (see Remark 4.13 below).
- The paper [8] shows that recutting has the so-called *Devron property*, a highly structured behavior of singularities common for cluster integrable systems.

The main result of the present paper is that recutting of planar polygons is indeed a completely integrable system. Moreover, it is a *cluster integrable system*, meaning that recutting at any vertex is a (Y-type) cluster transformation, and recutting invariants (first integrals) commute with respect to the log-canonical Poisson bracket associated with the corresponding quiver.

We summarize our results in Table 1. As can be seen from the table, different structures that arise in connection with recutting are defined on spaces of polygons with different periodicity conditions. The largest space that we consider is planar polygons closed up to an orientation-preserving similarity (i.e. a composition of rotations, translations, and homotheties). Such a polygon is understood as a bi-infinite sequence $v_i \in \mathbb{C}$ satisfying a quasi-periodicity condition $p_{i+n} = \psi(v_i)$ where ψ is a similarity transformation $z \mapsto az + b$, called the *monodromy* of the polygon. Let $\mathcal{P}_n^S/S$ be the space of planar n-gons closed up to similarity, modulo similarities. In Sect. 3 we interpret recutting on that space in terms of Y-type cluster mutations of a certain quiver $\mathcal{Q}_n$. Figure 2 shows the quiver $\mathcal{Q}_5$ corresponding to pentagons. The general quiver $\mathcal{Q}_n$ has a similar structure, but with $2n$ vertices. Geometrically, the variables y_i are the ratios of consecutive edges of the polygon, viewed as complex numbers, while $\bar{y}_i$ are complex conjugates of y_i. Recutting ρ_i is achieved by mutation of quiver vertices y_i, $\bar{y}_i$ followed by interchanging those vertices. This sequence of mutations is an example of a more general transformation known as a *geometric R-matrix* [11].

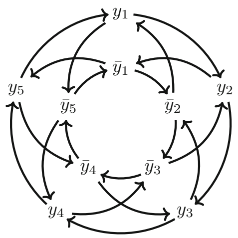

Fig. 2 The quiver $\mathcal{Q}_5$ corresponding to recutting of pentagons

Geometric R-matrices are known to satisfy braid relations, which gives yet another proof of the braid relation for recutting.

As a consequence of this cluster description, recutting on the space $\mathcal{P}_n^S/S$ of planar polygons closed up to similarity has an invariant Poisson structure, namely the standard log-canonical structure defined by the quiver $\mathcal{Q}_n$. However, we are not aware of any invariant functions (first integrals) of recutting on $\mathcal{P}_n^S/S$ besides the conjugacy class of the monodromy (whose preservation in particular means that the angle sum of the polygon is conserved) and, for even n, the sum of angles at every second vertex. To obtain additional invariants, we consider a smaller space $\mathcal{P}_n^E$ of planar polygons closed up to isometry (i.e. composition of rotations and translations). Our main result in that setting is the following:

Theorem 1.1 *The recutting action of the group $\tilde{S}_n$ on the $2n$-dimensional space $\mathcal{P}_n^E/E$ of planar n-gons closed up orientation-preserving isometry modulo said isometries is Arnold–Liouville integrable. Specifically, one has the following:*

1. *The recutting action on $\mathcal{P}_n^E/E$ has an invariant Poisson structure and $\lfloor 3n/2\rfloor + 1$ independent first integrals (invariant functions). Out of those integrals, $2\lfloor n/2\rfloor + 2$ are Casimirs, so that the number of additional integrals is $\lceil n/2\rceil - 1$, i.e. half of the dimension of symplectic leaves.*
2. *A generic joint level set of first integrals is a finite union of tori of dimension $\lceil n/2\rceil - 1$. For each such torus K, the subgroup $G_K := \{\omega \in \tilde{S}_n \mid \omega(K) \subset K\}$ of the recutting group elements preserving K has a finite index in $\tilde{S}_n$. There is a flat structure on K such that the action of G_K on K is by translations.*

We prove Theorem 1.1 in Sect. 4. Note that the second part of the theorem (quasi-periodic dynamics on tori) is a standard consequence of the first one, so most of the section is devoted to the proof of the first part (existence of invariant Poisson structure, invariants, and their independence). The idea of the proof is based on the connection between recutting and refactorization of quaternionic polynomials. Note that due to non-commutativity of the skew field $\mathbb{H}$ of quaternions, a typical polynomial over $\mathbb{H}$ can be factored into linear factors in many different ways. In particular, a generic quadratic polynomial over $\mathbb{H}$ has two different factorizations. What we show is that the map interchanging those two factorizations can be geometrically interpreted as recutting. As a result, integrability of recutting comes as a consequence of algebraic properties of quaternionic polynomials combined with some Poisson-Lie theory.

Remark 1.2 The Poisson structure on polygons closed up to isometry (coming from quaternionic polynomials) is compatible with the cluster structure on polygons closed

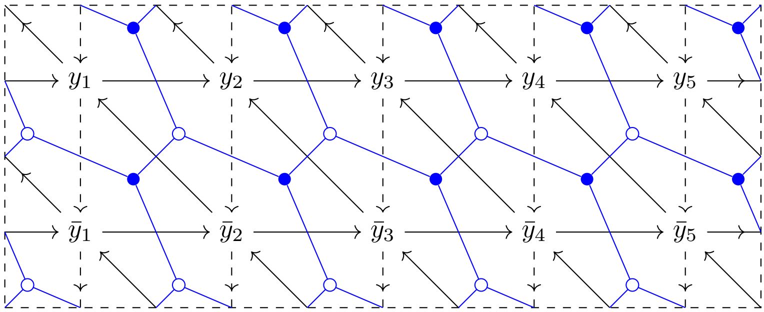

Fig. 3 Embedding of the quiver $\mathcal{Q}_5$ into a torus and its dual graph

up to similarity in the sense that the natural map $\mathcal{P}_n^E/E \to \mathcal{P}_n^S/S$ is Poisson. Furthermore, one can relate those structures as follows. As can be seen in Fig. 3, the quiver $\mathcal{Q}_n$ embeds in a torus (the quiver is represented by solid black arrows, and the opposite sides of the dashed rectangle are identified). Furthermore, by enhancing the quiver $\mathcal{Q}_n$ with obsolete arrows from each y_i to the corresponding $\bar{y}_i$ and back (dashed black arrows in the figure), one gets a quiver with bipartite dual $\mathcal{Q}_n^*$ (the blue graph in the figure). As a result, one can use the techniques of [7, 9] to build a Poisson structure on the space $\Sigma(\mathcal{Q}_n^*)$ of edge weights of $\mathcal{Q}_n^*$ modulo gauge transformations. Furthermore, there is a natural way to identify the space $\mathcal{P}_n^E/E$ with a Poisson hypersurface in $\Sigma(\mathcal{Q}_n^*)$, so that the Poisson property of the map $\mathcal{P}_n^E/E \to \mathcal{P}_n^S/S$ becomes a consequence of the Poisson property of the map from edge weights to face weights.

Interpreted in terms of the dual graph $\mathcal{Q}_n^*$, recutting becomes a certain non-local transformation of a weighted bipartite graph on a torus known as the *plabic R-matrix* [5]. Invariants of such a transformation can be constructed by using either the dimer partition function [9], or the boundary measurement matrix [7] (as shown in [12], those two approaches give the same invariants). This gives an alternative route to proving Theorem 1.1. We choose not to pursue this approach since our construction based on quaternionic polynomials seems more direct and also adapts better to the case of closed polygons that we consider next.

Our Poisson structure on polygons closed up to isometry restricts to polygons closed up to translation, so one can prove integrability of recutting in the latter setting following the lines of the proof of Theorem 1.1 (note that Theorem 1.1 does not directly imply integrability for any smaller class of polygons since it only describes the behavior of recutting on *generic* level sets of invariants). However, this approach does not work for closed polygons, since such polygons do not constitute a Poisson submanifold. One way to overcome this difficulty is by using Dirac reduction. Here we take a different approach, namely we show that although one cannot restrict the Poisson structure to closed polygons, one can still restrict the Hamiltonian vector fields generated by the invariants, which is sufficient to establish integrability in the non-Hamiltonian setting. Our main result for closed polygons is the following:

Theorem 1.3 *Assume that* $n \geq 3$. *Then the recutting action of the group* $\tilde{S}_n$ *on the* $2n-3$*-dimensional space* $\mathcal{P}_n/E$ *of closed planar* n*-gons modulo orientation-*

preserving isometries is integrable in the non-Hamiltonian sense. Specifically, one has the following:

1. *The recutting action on $\mathcal{P}_n/E$ has $\lfloor 3n/2 \rfloor - 1$ independent first integrals and a complementary number $\lceil n/2 \rceil - 2$ of independent invariant commuting vector fields tangent to level sets of first integrals.*
2. *A generic joint level set of first integrals is a finite union of tori of dimension $\lceil n/2 \rceil - 2$. For each such torus K, the subgroup $G_K := \{\omega \in \tilde{S}_n \mid \omega(K) \subset K\}$ of the recutting group elements preserving K has a finite index in $\tilde{S}_n$. There is a flat structure on K such that the action of G_K on K is given by translations.*

For example, for triangles and quadrilaterals we get $\lceil n/2 \rceil - 2 = 0$, so the orbits consist of finitely many points, cf. Remark A.3 below. For $n \geq 5$ the orbits are likely to be infinite.

Out of $\lfloor 3n/2 \rfloor - 1$ invariants, $n + 1$ have a clear geometric meaning: they are symmetric functions of the squared lengths of sides, and the area of the polygon. In addition, when n is even, one of the invariants is the sum of angles at every second vertex. The remaining invariants do not seem to have such a transparent interpretation.

We prove Theorem 1.3 in Sect. 5.

2 Polygon spaces and recutting

For the purposes of the present paper, a *polygon* is a bi-infinite sequence $(v_i \in \mathbb{C})_{i\in\mathbb{Z}}$ such that $v_i \neq v_{i+1}$ for all $i \in \mathbb{Z}$. *Recutting* ρ_j of a polygon (v_i) at a vertex v_j is defined as reflection of v_j in the perpendicular bisector of the interval (v_{j-1}, v_{j+1}). It is well-defined as long as $v_{j-1} \neq v_{j+1}$.

The space $\mathcal{P}$ of polygons carries the action of the group $S := \{\psi : \mathbb{C} \to \mathbb{C} \mid \psi(z) = \alpha z + \beta, \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}\}$ of orientation-preserving similarities, as well as of its normal subgroups $E := \{\psi : \mathbb{C} \to \mathbb{C} \mid \psi(z) = \alpha z + \beta, \alpha \in S^1, \beta \in \mathbb{C}\}$ of orientation-preserving isometries and $T := \{\psi : \mathbb{C} \to \mathbb{C} \mid \psi(z) = z + \beta, \beta \in \mathbb{C}\}$ of translations. All these actions commute with recutting.

Remark 2.1 Throughout the paper, all similarities and isometries are assumed to be orientation-preserving, so we refer to elements of S and E as simply similarities and isometries.

For a polygon $(p_i) \in \mathcal{P}$, define its *edge vectors* $z_i \in \mathbb{C}$ by $z_i := v_i - v_{i-1}$. Let also $y_i := z_{i+1}/z_i$ be the ratios of consecutive edge vectors, and let $\phi_i := \arg(y_i)$ be the angles between consecutive edge vectors (for a convex counter-clockwise oriented polygon, ϕ_i can be understood as exterior angles, so will we often refer to them in that way). Then y_i parametrize polygons modulo similarities (i.e. the action of S), edge vectors z_i parametrize polygons modulo translations (i.e. the action of T), while ϕ_i and $|z_i|$ parametrize polygons modulo isometries (i.e. the action of E). As another parametrization of polygons modulo isometries we will use the sequence of edge vectors z_i modulo simultaneous rotations.

The following results express recutting in terms of coordinates z_i and y_i.

Lemma 2.2 *Assume that a polygon (v'_i) is the image of a polygon (v_i) under recutting ρ_j at v_j. Let $z_i = v_i - v_{i-1}$ be the edge vectors of (v_i) and $z'_i := v'_i - v'_{i-1}$ be the edge vectors of (v'_i). Then*

$$\begin{aligned} z'_j + z'_{j+1} &= z_j + z_{j+1}, \\ z'_j \bar{z}'_{j+1} &= z_j \bar{z}_{j+1}, \end{aligned} \tag{1}$$

where $\bar{z}$ stands for the complex conjugate of z.

Proof The complex number $z_j + z_{j+1}$ represents the side (v_{j-1}, v_{j+1}) of the triangle (v_{j-1}, v_j, v_{j+1}). As for $z_j \bar{z}_{j+1}$, its absolute value is the product of lengths of (v_{j-1}, v_j) and (v_j, v_{j+1}), while its argument is the exterior angle of the triangle (v_{j-1}, v_j, v_{j+1}) at v_j. None of these change when the triangle is cut and then reattached with opposite orientation, hence the result. □

Corollary 2.3 *As a transformation of the space of polygons modulo translations, recutting ρ_j is given by*

$$\begin{aligned} z'_j &= \bar{z}_{j+1} \frac{z_j + z_{j+1}}{\bar{z}_j + \bar{z}_{j+1}}, \\ z'_{j+1} &= \bar{z}_j \frac{z_j + z_{j+1}}{\bar{z}_j + \bar{z}_{j+1}}, \end{aligned} \tag{2}$$

and $z'_i = z_i$ for $i \neq j, j+1$.

Proof Using Lemma 2.2 along with the relation $|z'_j| = |z_{j+1}|$, one gets

$$z_j + z_{j+1} = z'_j + z'_{j+1} = z'_j + \frac{\bar{z}_j z_{j+1}}{\bar{z}'_j} = \frac{z'_j \bar{z}'_j + \bar{z}_j z_{j+1}}{\bar{z}'_j} = \frac{z_{j+1}\bar{z}_{j+1} + \bar{z}_j z_{j+1}}{\bar{z}'_j},$$

which implies the desired formula for z'_j. The formula for z'_{j+1} now follows from any of the relations (1). Other z_i do not change under recutting ρ_j since they do not depend on the vertex v_j. □

Corollary 2.4 *As a transformation of the space of polygons modulo similarities, recutting ρ_j is given by*

$$\begin{aligned} y'_{j-1} &= \frac{y_{j-1}(1 + y_j)}{1 + \bar{y}_j^{-1}}, \\ y'_j &= \bar{y}_j^{-1}, \\ y'_{j+1} &= \frac{y_{j+1}(1 + \bar{y}_j)}{1 + y_j^{-1}}, \end{aligned} \tag{3}$$

and $y'_i = y_i$ for $i \neq j-1, j, j+1$.

Proof Straightforward calculation using (2) along with the definitions $y_i = z_{i+1}/z_i$ and $y'_i := z'_{i+1}/z'_i$ of y coordinates. □

In the rest of the paper, we only consider polygons satisfying certain periodicity-type conditions. Namely, let $\psi \in S$ be a similarity transformation. A polygon (v_i) is said to be an *n-gon closed up to ψ* (or an n-gon with *monodromy ψ*) if $p_{i+n} = \psi(v_i)$ for all $i \in \mathbb{Z}$. Recutting ρ_j of an n-gon (v_i) with monodromy ψ is a polygon with the same monodromy obtained from (v_i) by recutting at all vertices v_i with $i \equiv j \mod n$.

For a subgroup $G \subset S$, denote by $\mathcal{P}_n^G$ the space of n-gons with monodromy $\psi \in G$ (we also use the notation $\mathcal{P}_n$ for the space of closed n-gons corresponding to the trivial group G). Note that since the recutting action on $\mathcal{P}_n^G$ preserves the monodromy and commutes with similarity transformations, it descends to a self-map of the quotient $\mathcal{P}_n^G/H$ where $H \subset S$ is any subgroup normalizing G. In what follows, we will be particularly interested in the action of recutting on the spaces $\mathcal{P}_n^S/S$ and $\mathcal{P}_n^E/E$. The following results are straightforward:

Proposition 2.5 *The assignment $(v_i) \mapsto (y_i)$ taking a polygon to the ratios of its consecutive edge vectors gives a bijection*

$$\mathcal{P}_n^S\big/S \simeq \{n\text{-periodic sequences } y_i \in \mathbb{C}^*\}.$$

Written in coordinates y_i, recutting ρ_j on P_n^S/S is given by (3).

Proposition 2.6 *The assignment $(v_i) \mapsto (|z_i|, \phi_i)$ taking a polygon to its side lengths and exterior angles gives a bijection*

$$\mathcal{P}_n^E\big/E \simeq \{\text{pairs of } n\text{-periodic sequences } |z_i| \in \mathbb{R}_+,\ \phi_i \in \mathbb{R}/2\pi\mathbb{Z}\},$$

while the assignment $(v_i) \mapsto (z_i)$ taking a polygon to its edge vectors gives a bijection

$$\mathcal{P}_n^E\big/E \simeq \{\text{sequences } z_i \in \mathbb{C}^* \mid z_{i+n} = \alpha z_i \text{ for some } \alpha \in S^1\}\big/S^1,$$

where S^1 stands for the set of complex numbers of absolute value 1. *Written in terms of z_i, recutting ρ_j on $\mathcal{P}_n^E/E$ is given by* (2).

3 Recutting of polygons closed up to similarity: cluster structure

3.1 Quivers, mutations, and real structures

In this section we discuss the general notion of a Y-type (also known as $\mathcal{X}$-type) cluster mutation, with an emphasis on quivers endowed with an involution (a *real structure*). Recall that a *quiver* is a directed graph without loops or oriented cycles of length 2. For simplicity, in what follows we also prohibit multiple edges. Given a quiver $\mathcal{Q}$ with the vertex set $\{1, \dots, n\}$, denote by $Y_\mathcal{Q}$ the space of functions $\{1, \dots, n\} \to \mathbb{C}^*$. The space $Y_\mathcal{Q}$ is a complex torus of dimension n. It comes equipped with canonical coordinates $y_1, \dots, y_n$ given by evaluation of functions at vertices of $\mathcal{Q}$: for $\xi \in Y_\mathcal{Q}$,

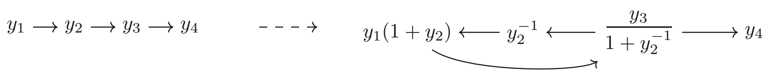

Fig. 4 A Y-mutation at y_2

one defines $y_i(\xi) := \xi(i)$. Since the variables y_i are indexed by vertices of $\mathcal{Q}$, in what follows we often identify vertices with the corresponding y variables.

The torus $Y_\mathcal{Q}$ carries a Poisson structure. In terms of y_i coordinates, it has a *log-canonical* form

$$\{y_i, y_j\} = a_{ij} y_i y_j, \tag{4}$$

where (a_{ij}) is the signed adjacency matrix of $\mathcal{Q}$, i.e.

$$a_{ij} = \begin{cases} 1, & \text{if there is an arrow (a directed edge) from vertex } i \text{ to vertex } j, \\ -1, & \text{if there is an arrow vertex } j \text{ to vertex } i, \\ 0, & \text{if the vertices } i, j \text{ are not connected by an arrow.} \end{cases}$$

The Poisson structure on $Y_\mathcal{Q}$ is natural in the following sense: any isomorphism of quivers $\mathcal{Q} \to \mathcal{Q}'$ induces a Poisson isomorphism $Y_\mathcal{Q} \to Y_{\mathcal{Q}'}$.

Given a quiver $\mathcal{Q}$, the *quiver mutation* of $\mathcal{Q}$ at its i'th vertex is the following modification of $\mathcal{Q}$:

1. For every pair of vertices j, k of $\mathcal{Q}$ such that there is an arrow from j to i and an arrow from i to k, add an arrow from j to k.
2. Reverse all arrows adjacent to the vertex i.
3. Remove all newly formed oriented cycles of length 2.

The result of a quiver mutation of $\mathcal{Q}$ is a new quiver $\mathcal{Q}'$ with the same vertex set as $\mathcal{Q}$. Note that in general a quiver mutation produces a quiver with multiple edges. This, however, does not happen for quivers relevant to the present paper.

We now define the notion of a Y-mutation. Assume that a quiver $\mathcal{Q}'$ is obtained from $\mathcal{Q}$ by means of mutation at vertex i. The corresponding *Y-mutation* is a birational map $Y_\mathcal{Q} \to Y_{\mathcal{Q}'}$ defined as follows. Let y_j be the canonical coordinates in $Y_\mathcal{Q}$, and y'_j be the canonical coordinates in $Y_{\mathcal{Q}'}$. Expressed in these coordinates, the Y-mutation $Y_\mathcal{Q} \to Y_{\mathcal{Q}'}$ at i is given by

$$y'_j = \begin{cases} y_j^{-1}, & \text{if } j = i, \\ y_j(1+y_i^{-1})^{-1}, & \text{if there is an arrow from vertex } i \text{ to vertex } j, \\ y_j(1+y_i), & \text{if there is an arrow from vertex } j \text{ to vertex } i, \\ y_j, & \text{in all other cases.} \end{cases}$$

A Y-mutation $Y_\mathcal{Q} \to Y_{\mathcal{Q}'}$ is a Poisson map.

We depict Y-mutations as shown in Fig. 4. The labels at vertices of the initial quiver $\mathcal{Q}$ are the corresponding Y-variables y_i while the labels at vertices of the mutated quiver $\mathcal{Q}'$ are pull-backs of the corresponding Y-variables y'_i by the Y-mutation map $Y_\mathcal{Q} \to Y_{\mathcal{Q}'}$.

Now assume we have sequence of quivers $\mathcal{Q} \to \cdots \to \tilde{\mathcal{Q}}$ where each quiver is obtained from the previous one by mutation. Suppose also that we have an isomorphism $\psi: \tilde{\mathcal{Q}} \to \mathcal{Q}$. Then the composition $Y_{\mathcal{Q}} \to \cdots \to Y_{\tilde{\mathcal{Q}}}$ of Y-mutations, followed by the map $Y_{\tilde{\mathcal{Q}}} \to Y_{\mathcal{Q}}$ induced by the isomorphism ψ, is a birational Poisson map of $Y_{\mathcal{Q}}$ onto itself. We call such a map a (Y-type) *cluster transformation*. Put differently, a cluster transformation is such a sequence of mutations which, after permutation of vertices, restores the initial quiver.

We now add a real structure to the picture. Let $\tau: \mathcal{Q} \to \mathcal{Q}$ be an involution (i.e. a graph automorphism such that $\tau^2 = \mathrm{id}$). Then τ defines a *real structure* (i.e. an anti-holomorphic involution) $\hat{\tau}$ on $Y_{\mathcal{Q}}$ by the rule $\hat{\tau}(\xi) := \overline{\tau^*\xi}$. In terms of coordinates y_i, the involution $\hat{\tau}$ is given by $\hat{\tau}^* y_i = \bar{y}_{\tau(i)}$. The *real part* $Y^{\mathbb{R}}_{\mathcal{Q}}$ of $Y_{\mathcal{Q}}$ is the fixed point set of the involution $\hat{\tau}$. It is a real manifold whose complexification is the complex torus $Y_{\mathcal{Q}}$ (in particular, $\dim_{\mathbb{R}} Y^{\mathbb{R}}_{\mathcal{Q}} = \dim_{\mathbb{C}} Y_{\mathcal{Q}}$ is the number of vertices of $\mathcal{Q}$). A function $\xi \in Y_{\mathcal{Q}}$ belongs to $Y^{\mathbb{R}}_{\mathcal{Q}}$ if and only if it takes real values at vertices fixed by τ and complex conjugate values at vertices switched by τ. The manifold $Y^{\mathbb{R}}_{\mathcal{Q}}$ is parametrized by y_i's subject to relations $\bar{y}_i = y_{\tau(i)}$ (in particular, y_i is real if the vertex i is fixed by τ).

Proposition 3.1 *The Poisson structure on $Y_{\mathcal{Q}}$ restricts to its real part $Y^{\mathbb{R}}_{\mathcal{Q}}$.*

The proof is based on the following general lemma, which is also used later in the paper.

Lemma 3.2 *Let V be a real vector space endowed with a polynomial Poisson structure, and let $\sigma: V \to V$ be a linear Poisson involution. Define an anti-linear involution $\bar{\sigma}$ on $V_{\mathbb{C}} := V \otimes \mathbb{C}$ by $\bar{\sigma}(x) := \sigma(\bar{x})$, where σ is extended from V to $V_{\mathbb{C}}$ by $\mathbb{C}$-linearity. Let $V_\sigma := \mathrm{Fix}(\bar{\sigma})$ be the fixed point set of $\bar{\sigma}$. Then there is a unique Poisson structure on the real vector space V_σ whose complexification coincides with the complexification of the Poisson structure on V.*

Proof of Lemma 3.2 The space V_σ is a real form of the complex vector space $V_{\mathbb{C}}$, so there is at most one Poisson structure on V_σ extending to the Poisson structure on $V_{\mathbb{C}}$. To prove existence, notice that we have an isomorphism of $\mathbb{R}$-algebras $\mathbb{R}[V_\sigma] \simeq \mathrm{Fix}(\bar{\sigma}^*)$, where $\bar{\sigma}^*: \mathbb{C}[V_{\mathbb{C}}] \to \mathbb{C}[V_{\mathbb{C}}]$ is an involution given by $(\bar{\sigma}^* f)(x) = \overline{f(\bar{\sigma}(x))}$. So, to obtain the desired Poisson bracket on $\mathbb{R}[V_\sigma]$, it suffices to show that the $\mathbb{R}$-subalgebra $\mathrm{Fix}(\bar{\sigma}^*)$ is closed under the Poisson bracket on $\mathbb{C}[V_{\mathbb{C}}]$. To that end, observe that $\bar{\sigma}^*$ is a composition of two commuting Poisson involutions: $f(x) \mapsto f(\sigma(x))$ and $f(x) \mapsto \overline{f(\bar{x})}$. So, $\bar{\sigma}^*$ is itself a Poisson involution and its fixed point set $\mathrm{Fix}(\bar{\sigma}^*) = \mathbb{R}[V_\sigma]$ is indeed closed under the Poisson bracket, as desired. □

Proof of Proposition 3.1 Let V be the space of real-valued functions on the vertex set of the quiver $\mathcal{Q}$. Then V carries an involution $\sigma := \tau^*$ (pull-back by τ) and a Poisson bracket defined by (4), where the coordinates $y_i: V \to \mathbb{R}$ on V are defined by $y_i(\xi) := \xi(i)$. So, by Lemma 3.2, the extension of the Poisson structure from V to $V \otimes \mathbb{C}$ restricts to the space $V_\sigma = \{\xi \in V \otimes \mathbb{C} \mid \tau^*\xi = \bar{\xi}\}$. But $Y_{\mathcal{Q}}$ is an open dense subset of $V \otimes \mathbb{C}$, while $Y^{\mathbb{R}}_{\mathcal{Q}}$ is an open dense subset of V_σ, so the Poisson structure on $Y_{\mathcal{Q}}$ restricts to $Y^{\mathbb{R}}_{\mathcal{Q}}$. □

For a quiver $\mathcal{Q}$ with an involution τ, a *real quiver mutation* is either a quiver mutation at a vertex fixed by τ, or a composition of two quiver mutations at vertices switched by τ. In what follows, we assume that no vertices of $\mathcal{Q}$ are fixed by τ. In that case, a real quiver mutation is necessarily a composition of two mutations. The order of those mutations does not matter because two vertices switched by a quiver involution are necessarily disjoint.

If a quiver $\mathcal{Q}'$ is obtained from a quiver $\mathcal{Q}$ with involution τ by means of a real quiver mutation, then τ is also an involution of $\mathcal{Q}'$. The corresponding *real Y-mutation* is the composition of two Y-mutations corresponding to quiver mutations producing $\mathcal{Q}'$ from $\mathcal{Q}$. A real Y-mutation commutes with the anti-holomorphic involution $\bar{\tau}$ and hence can be viewed a birational Poisson map $Y_{\mathcal{Q}}^{\mathbb{R}} \to Y_{\mathcal{Q}'}^{\mathbb{R}}$.

A real isomorphism $\mathcal{Q} \to \mathcal{Q}'$ of quivers with involutions is an isomorphism respecting the involutions. Such an isomorphism induces a real Poisson isomorphism $Y_{\mathcal{Q}}^{\mathbb{R}} \to Y_{\mathcal{Q}'}^{\mathbb{R}}$. A *real cluster transformation* for a quiver $\mathcal{Q}$ with involution is a composition of real Y-mutations and a map induced by a real isomorphism of the resulting quiver onto the initial one. Such a transformation is a birational Poisson map $Y_{\mathcal{Q}}^{\mathbb{R}}$ onto itself.

3.2 Recutting as a real cluster transformation

Here we apply the formalism developed in the previous section to provide a cluster description of polygon recutting on the space $\mathcal{P}_n^S/S$ of similarity classes of polygons closed up to similarity. To that end we build a quiver $\mathcal{Q}_n$ with an involution τ such that (3) is a real cluster transformation as defined in Sect. 3.1. In terminology of [6], the quiver $\mathcal{Q}_n$ is the *twist* of the affine Dynkin diagram $\tilde{A}_{n-1}$. It has $2n$ vertices which we label as $1, \dots, n, 1', \dots, n'$. There is an arrow from vertex i to vertex j if and only if $j - i \equiv 1 \mod n$, an arrow from vertex i' to vertex j' if and only if $j - i \equiv 1 \mod n$, an arrow from vertex i to vertex j' if and only if $j - i \equiv -1 \mod n$, and an arrow from vertex i' to vertex j if and only if $j - i \equiv -1 \mod n$. The involution $\tau \colon \mathcal{Q}_n \to \mathcal{Q}_n$ is given by $\tau(i) = i'$. Since on $Y_{\mathcal{Q}_n}^{\mathbb{R}}$ we have $y_{i'} = \bar{y}_i$, we denote the y-variables corresponding to i' vertices by $\bar{y}_i$. Thus, the y_i and $\bar{y}_i$ variables are independent on $Y_{\mathcal{Q}_n}$ but complex conjugate to each other on $Y_{\mathcal{Q}_n}^{\mathbb{R}}$. Figure 2 depicts the quiver $\mathcal{Q}_5$ (while the top left part of Fig. 5 shows the local structure of the general quiver $\mathcal{Q}_n$). The labels at vertices are the corresponding y-variables.

Since the space $Y_{\mathcal{Q}_n}^{\mathbb{R}}$ is parametrized by variables $y_i \in \mathbb{C}^*$, Proposition 2.5 gives a way to identify the space $Y_{\mathcal{Q}_n}^{\mathbb{R}}$ with $\mathcal{P}_n^S/S$. Namely, one takes an n-tuple $(y_1, \dots, y_n) \in Y_{\mathcal{Q}_n}^{\mathbb{R}}$ and extends it by periodicity. Under this identification recutting becomes a real Y-mutation:

Proposition 3.3 *Consider a real quiver mutation of $\mathcal{Q}_n$ given by mutating y_j and $\bar{y}_j$. Then the resulting quiver $\mathcal{Q}_n'$ is real isomorphic to $\mathcal{Q}_n$. The cluster transformation given by composition of the real Y-mutation $Y_{\mathcal{Q}_n}^{\mathbb{R}} \to Y_{\mathcal{Q}_n'}^{\mathbb{R}}$ and the mapping $Y_{\mathcal{Q}_n'}^{\mathbb{R}} \to Y_{\mathcal{Q}_n}^{\mathbb{R}}$ induced by the isomorphism $\mathcal{Q}_n' \simeq \mathcal{Q}_n$ coincides with recutting ρ_j.*

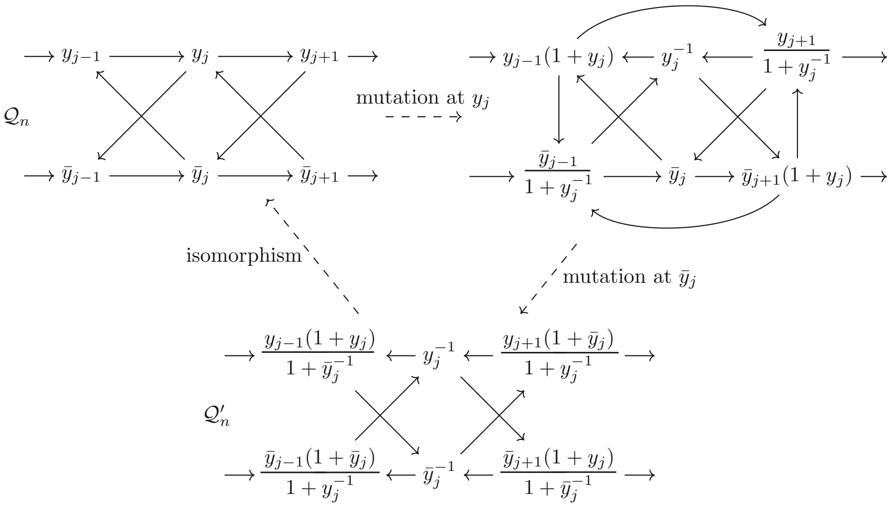

Fig. 5 Recutting as a real cluster transformation

Proof Consider Fig. 5. Observe that the mapping $\mathcal{Q}_n' \to \mathcal{Q}_n$ given by $y_j^{-1} \mapsto \bar{y}_j$ and $\bar{y}_j^{-1} \mapsto y_j$ and keeping the other vertices in place is an isomorphism. By moving the labels from $\mathcal{Q}_n'$ to $\mathcal{Q}_n$ as prescribed by the isomorphism, one gets formulas (3), as desired. □

Corollary 3.4 *Recutting on the space $\mathcal{P}_n^S/S$ preserves the following Poisson bracket:*

$$\begin{aligned} \{y_i, y_j\} &= (\delta_{i+1,j} - \delta_{i-1,j}) y_i y_j, & \{y_i, \bar{y}_j\} &= (\delta_{i-1,j} - \delta_{i+1,j}) y_i \bar{y}_j, \\ \{\bar{y}_i, \bar{y}_j\} &= (\delta_{i+1,j} - \delta_{i-1,j}) \bar{y}_i \bar{y}_j, & \{\bar{y}_i, y_j\} &= (\delta_{i-1,j} - \delta_{i+1,j}) \bar{y}_i y_j. \end{aligned} \tag{5}$$

where $\delta_{i,j} = 1$ if $i \equiv j \mod n$ and $\delta_{i,j} = 0$ if $i \not\equiv j \mod n$. (Note that the last two formulas are determined by the first two since the bracket must be real.)

Proof This is the canonical Poisson bracket on $Y_{\mathcal{Q}_n}^{\mathbb{R}}$, preserved by all real cluster transformations. □

Remark 3.5 Brackets (5) take a particular nice form when written in terms of $|y_i|$ and $\phi_i = \arg(y_i)$, i.e. ratios of lengths of consecutive sides and exterior angles of the polygon. Namely, $|y_i|$ are Casimirs, while

$$\{\phi_i, \phi_j\} = \delta_{i+1,j} - \delta_{i-1,j}. \tag{6}$$

Remark 3.6 The bracket (5) has a large number of Casimirs, namely all $|y_i|$, the product $y_1 \cdots y_n$ (equal to the coefficient $\alpha \in \mathbb{C}^*$ of the monodromy transformation $z \mapsto \alpha z + \beta$), and, for even n, the product $y_1 y_3 \cdots y_{n-1}$. Most of these Casimirs are not preserved by recutting transformations (3). The only ones that are preserved are the function $y_1 \cdots y_n$ (in particular, the angle sum $\sum \phi_i = \arg(y_1 \cdots y_n)$) and, for even n, the function $\arg(y_1 y_3 \cdots y_{n-1}) = \phi_1 + \phi_3 + \cdots + \phi_{n-1}$.

It follows from this description of Casimirs that the spaces $\mathcal{P}_n^E/S$, $\mathcal{P}_n^T/S$ of similarity classes of polygons closed up to isometry or translation are Poisson submanifolds. Indeed, the defining equation of $\mathcal{P}_n^E/S$ inside $\mathcal{P}_n^S/S$ is $|y_1 \cdots y_n| = 1$, while the defining equation of $\mathcal{P}_n^T/S$ is $y_1 \cdots y_n = 1$. So both are level sets of Casimirs and hence Poisson submanifolds. As for the submanifold $\mathcal{P}_n/S$ of similarity classes of closed polygons, it is defined by equations $y_1 \cdots y_n = 1$ and $1 + y_1 + y_1 y_2 + \cdots + y_1 \cdots y_{n-1} = 0$, and is therefore not Poisson.

Remark 3.7 The cluster transformation described in Proposition 3.3 can be regarded a particular case of a more general transformation, known as the *geometric R-matrix*. The cluster geometric R-matrix is defined in [11] for triangular grid quivers and in [5] for a more general class of *spider web quivers*. The quiver $\mathcal{Q}_n$ is not a triangular grid quiver or spider web quiver but can be seen as such if we add obsolete arrows from each y_i to $\bar{y}_i$ and from each $\bar{y}_i$ to y_i.

4 Recutting of polygons closed up to isometry: Arnold–Liouville integrability

4.1 Quaternionic polynomials

This section is a brief introduction into the theory of polynomials over quaternions. We begin by reviewing their general properties. All these results are well known but seem to be scattered in the literature, so we sketch proofs. We then move on to define what we call *special quaternionic polynomials* and present a criterion for factorization of such polynomials into linear factors. This result plays an instrumental role in our proof of integrability of recutting.

First, let us fix some terminology. Let $\mathbb{H} = \mathrm{span}_{\mathbb{R}}\langle 1, \mathbf{i}, \mathbf{j}, \mathbf{k}\rangle$ be the skew-field of quaternions. There are two different operations in $\mathbb{H}$ that are usually referred to as *conjugation*: $\alpha = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mapsto \bar{\alpha} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ and $\alpha \mapsto \beta\alpha\beta^{-1}$. To avoid confusion, we only use the term "conjugation" for the former operation. Quaternions of the form α, $\beta\alpha\beta^{-1}$ will be called *similar*.

Let $\mathbb{H}[t] := \mathbb{H} \otimes \mathbb{R}[t]$ be the $\mathbb{R}$-algebra of *unilateral* quaternionic polynomials in the indeterminate t. Those can be thought as polynomials over quaternions whose coefficients commute with the variable. For $f = \sum_{i=0}^n \alpha_i t^i \in \mathbb{H}[t]$ (where $\alpha_i \in \mathbb{H}$) and a quaternion $\beta \in \mathbb{H}$, define the *right evaluation of f at β* as $\mathrm{ev}^r_f(\beta) := \sum_{i=0}^n \alpha_i \beta^i$, and *left evaluation of f at β* as $\mathrm{ev}^l_f(\beta) := \sum_{i=0}^n \beta^i \alpha_i$ (for real β one has $\mathrm{ev}^r_f(\beta) = \mathrm{ev}^l_f(\beta)$, in which case we just write it as $f(\beta)$). Say that β is a right (left) root of f if $\mathrm{ev}^r_f(\beta) = 0$ (respectively, $\mathrm{ev}^l_f(\beta) = 0$). The following summarizes basic facts about quaternionic polynomials and their roots.

Proposition 4.1 1. *Let $f, g \in \mathbb{H}[t]$ and let $\alpha \in \mathbb{H}$ be a right root of g. Then α is a right root of the product fg.*
2. *Let $f \in \mathbb{H}[t]$ and let $\alpha \in \mathbb{H}$. Then α is a right root of f if and only if $t - \alpha \in \mathbb{H}[t]$ is a right divisor of f.*

3. *Let $f, g \in \mathbb{H}[t]$ and assume that $\alpha \in \mathbb{H}$ is not a right root of g. Then*

$$\mathrm{ev}^r_{fg}(\alpha) = \mathrm{ev}^r_f(\mathrm{ev}^r_g(\alpha) \cdot \alpha \cdot \mathrm{ev}^r_g(\alpha)^{-1})\mathrm{ev}^r_g(\alpha). \tag{7}$$

4. *Let $f \in \mathbb{H}[t]$ and let $[\alpha] := \{\beta\alpha\beta^{-1} \mid \beta \in \mathbb{H} \setminus \{0\}\} \subset \mathbb{H}$ be a similarity class of quaternions. Then one of the following is true:*
   (a) The class $[\alpha]$ contains neither right nor left roots of f.
   (b) The class $[\alpha]$ contains a unique right root of f and a unique left root of f. Such roots are called ***isolated****.*
   (c) Any element of $[\alpha]$ is both right and left root of f. Such roots are called ***spherical****.*
5. *A similarity class $[\alpha] \subset \mathbb{H}$ contains a root of $f \in \mathbb{H}[t]$ if and only if it contains the root of its* ***companion polynomial*** *$f\bar{f} = \bar{f}f \in \mathbb{R}[t]$. In particular, any non-zero quaternionic polynomial has at least one root, and hence can be factored into linear factors. (By the previous part, here we do not need to distinguish between right and left roots.)*
6. *A similarity class $[\alpha] \subset \mathbb{H}$ consists entirely of roots of f if and only if f is divisible by the* ***characteristic polynomial*** *of α, given by*

$$\chi_\alpha = t^2 - 2(\mathrm{Re}\,\alpha)t + |\alpha|^2. \tag{8}$$

   (Note that since χ_α has real coefficients, divisibility of f by χ_α on the right is equivalent to divisibility of f by χ_α on the left).
7. *For any non-zero $f \in \mathbb{H}[t]$, the total number of similarity classes $[\alpha] \subset \mathbb{H}$ containing roots of f does not exceed the degree of f.*

Proof Parts 1-3 hold for polynomials over any division ring, cf. [10, Theorem 1]. To prove part 1 we need to establish the implication $\mathrm{ev}^r_g(\alpha) = 0 \implies \mathrm{ev}^r_{fg}(\alpha) = 0$. Since the map $\mathbb{H}[t] \to \mathbb{H}$ given by $f \mapsto \mathrm{ev}^r_{fg}(\alpha)$ is a homomorphism of left $\mathbb{H}$-modules, it suffices to consider the case $f = t^m$. For such f we have $fg = t^m g = gt^m$, so $\mathrm{ev}^r_{fg}(\alpha) = \mathrm{ev}^r_g(\alpha)\alpha^m = 0$, as needed. Now that we established part 1, part 2 can be proved in the same way as for a field, using long division. To prove part 3, let $\beta := \mathrm{ev}^r_g(\alpha)$. Again, it suffices to consider the case $f = t^m$. In that case, we get $\mathrm{ev}^r_{fg}(\alpha) = \beta\alpha^m = (\beta\alpha\beta^{-1})^m\beta = \mathrm{ev}^r_f(\beta\alpha\beta^{-1})\beta$, as desired.

To prove part 4, observe that any quaternion α is a root of its characteristic polynomial (8). From this it follows that any positive power of a quaternion α can be expressed as $\alpha^m = r + s\alpha$, where r, s are polynomials with real coefficients in terms of $\mathrm{Re}\,\alpha$ and $|\alpha|^2$, and in particular only depend on the similarity class of α. This in turn implies that for any polynomial $f \in \mathbb{H}[t]$ and a similarity class $[\alpha] \subset \mathbb{H}$ there exist $\lambda, \mu \in \mathbb{H}$ such that for any $\alpha' \in [\alpha]$ we have $\mathrm{ev}^r_f(\alpha') = \lambda + \mu\alpha'$ and $\mathrm{ev}^l_f(\alpha') = \lambda + \alpha'\mu$. Now it is easy to see that (a) holds when $|\lambda| \neq |\alpha||\mu|$, (b) holds when $|\lambda| = |\alpha||\mu| \neq 0$, while possibility (c) holds when $|\lambda| = |\alpha||\mu| = 0$.

To prove part 5, notice that by part 1 any right root of f is also a right root of $\bar{f}f$. So it suffices to show that if α is a right root of $\bar{f}f$, then the similarity class of α contains a root of f. Assume that α is a right root of $\bar{f}f$. If α is also a right root of f, then we are done. If not, then by (7) we get that $\alpha' := \mathrm{ev}^r_{\bar f}(\alpha) \cdot \alpha \cdot \mathrm{ev}^r_{\bar f}(\alpha)^{-1}$ is a right root of $\bar f$, which is equivalent to saying that $\bar\alpha'$ is a left root of f. But any quaternion is similar to its conjugate (since conjugation preserves the real part and absolute value, and two quaternions $\alpha, \beta \in \mathbb{H}$ are similar if and only if $\operatorname{Re}\alpha = \operatorname{Re}\beta$ and $|\alpha| = |\beta|$), so $\bar\alpha'$ is similar to α' and hence to α. So indeed f has a root in the similarity class of α, as needed.

To prove part 6 note that any element of the class $[\alpha]$ is a root of χ_α. So the class $[\alpha]$ is annihilated by f if and only if is annihilated by the remainder of right division of f by χ_α. But that remainder is at most linear, so it can only annihilate the class $[\alpha]$ if it vanishes, which means that f is divisible by χ_α, as needed.

Part 7 is also true for any division ring, see [10, Theorem 2]. Let us sketch a quaternion-specific proof. By part 5 it suffices to show that the number of similarity classes containing roots of the companion polynomial $\bar f f$ does not exceed the degree n of f. To that end observe that since the polynomial $\bar f f$ has real coefficients, the similarity class of any of its roots consists entirely of roots. So, any similarity class containing a root of $\bar f f$ contains a complex root of $\bar f f$. Therefore it suffices to show that the number of similarity classes of complex roots of $\bar f f$ is at most n. To prove that write f as $f_1 + f_2\mathbf{i} + f_3\mathbf{j} + f_4\mathbf{k}$, where the polynomials f_i are real. Then $\bar f f = \sum f_i^2$ so all its real roots have multiplicity at least 2. As for non-real roots, any such root α has its complex conjugate counterpart $\bar\alpha$ which is similar to α. So any similarity class of complex roots of $\bar f f$ contains at least two roots (counted with multiplicity), and the total number of classes cannot exceed n, q.e.d. □

Definition 4.2 We say that a quaternionic polynomial $f \in \mathbb{H}[t]$ is *special* if it satisfies one of the following equivalent conditions:

1. $f(-t) = \mathbf{i} f(t)\mathbf{i}^{-1}$.
2. f can be written as a polynomial in $\mathbf{j}t$ with complex coefficients.
3. All even coefficients of f are complex numbers, while all odd coefficients belong to the complementary subspace $\operatorname{span}_{\mathbb{R}}\langle \mathbf{j}, \mathbf{k}\rangle$.

Special quaternionic polynomials form a subalgebra of $\mathbb{H}[t]$ which we denote by $\tilde{\mathbb{H}}[t]$. More generally, one can take an arbitrary non-zero quaternion α with zero real part and consider polynomials such that $f(-t) = \alpha f(t)\alpha^{-1}$. This always gives a subalgebra isomorphic to $\tilde{\mathbb{H}}[t]$. The following property of special quaternionic polynomials will be used to prove integrability of polygon recutting:

Proposition 4.3 *A special quaternionic polynomial $f \in \tilde{\mathbb{H}}[t]$ can be written as a product of linear special quaternionic polynomials $f_i \in \tilde{\mathbb{H}}[t]$ if and only if all complex roots of the companion polynomial $\bar f f$ of f are on the imaginary axis.*

We first prove a lemma, which will also be useful by itself.

Lemma 4.4 *Assume that the companion polynomial $\bar f f$ of a special quaternionic polynomial $f \in \tilde{\mathbb{H}}[t]$ has all its complex roots on the imaginary axis. Let α be a root of f. Then:*

1. *There exists $\beta \in \operatorname{span}_{\mathbb{R}}\langle \mathbf{j}, \mathbf{k}\rangle$ which is similar to α.*

2. *Moreover, if α is isolated, then* $\alpha \in \mathrm{span}_{\mathbb{R}}\langle \mathbf{j}, \mathbf{k}\rangle$.

Proof of Lemma 4.4 Since all roots of the companion polynomial of f are on the imaginary axis, by part 5 of Proposition 4.1 we have $\mathrm{Re}\,\alpha = 0$. So, α must be similar to some element of $\mathrm{span}_{\mathbb{R}}\langle \mathbf{j}, \mathbf{k}\rangle$, which establishes the first statement of the lemma. To prove the second statement, assume that α is isolated. Using part 2 of Proposition 4.1 write $f(t)$ as $g(t)(t-\alpha)$ where $g(t) \in \mathbb{H}[t]$. Then, using that f is special, we get

$$f(t) = \mathbf{i} f(-t)\mathbf{i}^{-1} = \mathbf{i} g(-t)(-t-\alpha)\mathbf{i}^{-1} = -\mathbf{i} g(-t)\mathbf{i}^{-1}\cdot(t + \mathbf{i}\alpha\mathbf{i}^{-1}),$$

and applying once again part 2 of Proposition 4.1 we see that $\alpha' := -\mathbf{i}\alpha\mathbf{i}^{-1}$ is also a root of f. Furthermore, since $\mathrm{Re}\,\alpha = 0$, we have $\mathrm{Re}\,\alpha' = 0$, and since $|\alpha'| = |\alpha|$, it follows that α' is similar to α. Therefore, since α is isolated, we have $\alpha' = -\mathbf{i}\alpha\mathbf{i}^{-1} = \alpha$, which is equivalent to $\alpha \in \mathrm{span}_{\mathbb{R}}\langle \mathbf{j}, \mathbf{k}\rangle$, as needed. □

Proof of Proposition 4.3 For a linear special quaternionic polynomial $a + b\mathbf{j}t$, where $a, b \in \mathbb{C}$, its companion polynomial $a\bar{a} + b\bar{b}t^2$ has roots on the imaginary axis. Furthermore, since the companion polynomial of a product is the product of companion polynomials, it follows that if f is a product of linear special quaternionic polynomial, then all complex roots of the companion polynomial of f are on the imaginary axis. Conversely, assume that f is special and all roots of $\bar{f}f$ are on the imaginary axis. Let $\alpha \in \mathbb{H}$ be an arbitrary root of f. Then α is either isolated or spherical. In the former case, by Lemma 4.4, we have $\alpha \in \mathrm{span}_{\mathbb{R}}\langle \mathbf{j}, \mathbf{k}\rangle$, so f is divisible on the right by the special quaternionic polynomial $\alpha^{-1}t - 1$. In the later case, by Lemma 4.4 we can find a root $\alpha' \in \mathrm{span}_{\mathbb{R}}\langle \mathbf{j}, \mathbf{k}\rangle$ of f similar to α, so f is divisible on the right by the special quaternionic polynomial $(\alpha')^{-1}t - 1$. So, in either case, f is divisible on the right by a linear special quaternionic polynomial, and proceeding by induction one shows that f can be written as a product of such polynomials. □

4.2 Recutting as refactorization

In this section we establish a connection between recutting and quaternionic polynomials, which is then used to derive invariants of recutting and prove its complete integrability. Let (v_i) be a polygon, and (v_i') be the result of its recutting at a vertex v_j. Consider the edge vectors $z_i = v_i - v_{i-1}$ and $z_i' = v_i' - v_{i-1}'$. Then $z_j, z_{j+1}, z_j', z_{j+1}'$ satisfy relations (1).

Proposition 4.5 *Recutting relations* (1) *are equivalent to the following relation between special quaternionic polynomials:*

$$(1 + z_j\mathbf{j}t)(1 + z_{j+1}\mathbf{j}t) = (1 + z_j'\mathbf{j}t)(1 + z_{j+1}'\mathbf{j}t). \tag{9}$$

Proof Indeed, for any $z, w \in \mathbb{C}$ one has $(1 + z\mathbf{j}t)(1 + w\mathbf{j}t) = (1 + (z+w)\mathbf{j}t - z\bar{w}t^2)$, so (9) is equivalent to (1). □

As a result, one can interpret recutting of a polygon (v_i) at a vertex v_j as refactorization of the quadratic quaternionic polynomial $g(t) := (1 + z_j\mathbf{j}t)(1 + z_{j+1}\mathbf{j}t)$.

Remark 4.6 Note that:

1. The polynomial $g(t)$ is only divisible by a real polynomial when $z_j = -z_{j+1}$ (equivalently, $v_{j-1} = v_{j+1}$), which is the case when recutting at v_j is impossible. So, as long as recutting is possible, it follows from Proposition 4.1 that the polynomial $g(t)$ has at most two right roots (both isolated) and hence at most two factorizations of the form $(1 + z\mathbf{j}t)(1 + w\mathbf{j}t)$.
2. The companion polynomial $\bar{g}(t)g(t)$ of $g(t)$ is $(1+|z_j|^2t^2)(1+|z_{j+1}|^2t^2)$. So, if $|z_j| \neq |z_{j+1}|$, the polynomial $g(t)$ has exactly two right roots (both isolated) and hence, by Lemma 4.4, exactly two factorizations of the form $(1+z\mathbf{j}t)(1+w\mathbf{j}t)$. In this case, recutting at v_j can be seen as switching between these two factorizations.
3. If $|z_j| = |z_{j+1}|$, then recutting at v_j is the identity transformation. In this case, the polynomial $g(t)$ has a unique right root and hence a unique factorization.

Summing up, unless the vertices v_{j-1} and v_{j+1} coincide, the polynomial $g(t) = (1 + z_j\mathbf{j}t)(1 + z_{j+1}\mathbf{j}t)$ has two (possibly identical) factorizations of the form $(1 + z\mathbf{j}t)(1 + w\mathbf{j}t)$, and recutting can be thought as switching from one factorization to another.

4.3 Recutting invariants for polygons closed up to translation

We begin our description of recutting invariants with the case of polygons closed up to translation. In this case, the invariants have a particularly simple form. In the next section generalize these results to polygons closed up to isometry.

Given a polygon $(v_i) \in \mathcal{P}_n^T$ closed up to translation, let z_i be its edge vectors. Consider a special quaternionic polynomial

$$f(t) := (1 + z_1\mathbf{j}t)\cdots(1 + z_n\mathbf{j}t) \in \tilde{\mathbb{H}}[t]. \tag{10}$$

Proposition 4.7 *The similarity class of the polynomial $f(t)$ in the skew-field $\tilde{\mathbb{H}}[[t]]$ is invariant under both the action of the group E of isometries, and the recutting action of $\tilde{S}_n$.*

Remark 4.8 The skew-field $\tilde{\mathbb{H}}[[t]]$ of special quaternionic power series is defined analogously to $\tilde{\mathbb{H}}[t]$: a quaternionic power series is special if and only if all its even coefficients are complex numbers, while all odd coefficients belong to the complementary subspace $\mathrm{span}_{\mathbb{R}}\langle \mathbf{j}, \mathbf{k}\rangle$. We say that $f, g \in \tilde{\mathbb{H}}[t]$ are similar if there exists an invertible formal power series $h \in \tilde{\mathbb{H}}[[t]]$ such that $hfh^{-1} = g$.

Proof of Proposition 4.7 The action of the group E of isometries amounts to multiplying all z_i by the same complex number α of absolute value 1. This is equivalent to a similarity transformation $f \mapsto \alpha^{1/2} f \alpha^{-1/2}$. So the action of E indeed preserves the similarity class of f.

To prove the invariance of the similarity class of f under recutting, observe that by Proposition 4.5 the polynomial $f_i(t) := (1 + z_i\mathbf{j}t)\dots(1 + z_{i+n-1}\mathbf{j}t)$ does not change under recutting ρ_i. Furthermore, due to n-periodicity of the sequence z_j, the polynomial $f(t)$ is similar to $f_i(t)$, so its similarity class is preserved by any recutting ρ_i and hence by the whole recutting group. □

It follows from Proposition 4.7 that any central function of $f(t)$ descends to the space $\mathcal{P}_n^T/E$ and is invariant under recutting action on that space. As such functions we take the coefficients of the real polynomials $\bar{f}(t)f(t)$ and $\operatorname{Re} f(t)$.

Proposition 4.9 *For a polygon closed up to translation, one has*

$$\bar{f}(t)f(t) = \prod_i (1 + |z_i|^2 t^2) = 1 + \sum_k E_k t^{2k},$$

$$\operatorname{Re} f(t) = 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k I_k t^{2k},$$

where

$$E_k := \sum_{i_1 < \dots < i_k} |z_{i_1}|^2 \dots |z_{i_k}|^2, \qquad I_k := \operatorname{Re} \sum_{i_1 < \dots < i_{2k}} z_{i_1} \bar{z}_{i_2} \dots z_{i_{2k-1}} \bar{z}_{i_{2k}}. \tag{11}$$

Here and in the rest of this section all summation indices run from 1 *to* n *unless otherwise specified.*

Proof The first equality follows from multiplicativity of the companion polynomial, while the second one is obtained by a straightforward computation. □

It follows that the recutting action on n-gons closed up to translation has $\lfloor 3n/2 \rfloor$ invariants, namely n elementary symmetric polynomials $E_1, \dots, E_n$ of squared side lengths $|z_i|^2$ (whose invariance is obvious from the geometric definition of recutting), and $\lfloor n/2 \rfloor$ additional invariants $I_1, \dots, I_{\lfloor n/2 \rfloor}$. The following result explains the geometric meaning of some of the invariants I_k:

Proposition 4.10
1. *For a polygon with monodromy $z \mapsto z + \beta$, the invariant I_1 is a function of squared side lengths and squared length of β: $I_1 = \frac{1}{2}(|\beta|^2 - E_1)$. In particular, a polygon is closed if and only if*

$$I_1 = -\frac{1}{2} E_1. \tag{12}$$

2. *Let n be even. Then*

$$I_{n/2} = \sqrt{E_n} \cos(\phi_1 + \phi_3 + \dots + \phi_{n-1}) = \sqrt{E_n} \cos(\phi_2 + \phi_4 + \dots + \phi_n), \tag{13}$$

where ϕ_i are exterior angles of the polygon.

3. *For closed polygons, the invariant I_2 is a function of squared side lengths and the area A of the polygon:*

$$I_2 = \frac{1}{2} E_2 - \frac{1}{8} E_1^2 - 2A^2. \tag{14}$$

Remark 4.11 Note that for $n = 3$ we have $I_2 = 0$, so relation (14) becomes $A^2 = \frac{1}{4}E_2 - \frac{1}{16}E_1^2$ which is nothing but Heron's formula for the area of a triangle. When $n = 4$, formulas (13) and (14) combined together give $A^2 = \frac{1}{4}E_2 - \frac{1}{16}E_1^2 - \frac{1}{2}\sqrt{E_n}\cos(\phi_1 + \phi_3)$ which is equivalent to Bretschneider's formula for the area of a quadrilateral.

Proof of Proposition 4.10 The first two parts are proved by a straightforward computation, so we only prove the last part. Denote by α_k be the coefficient of t^k of the polynomial $f(t)$. Then $\alpha_1 = \mathbf{j}\sum_i z_i$, $\alpha_2 = -\sum_{i<j} z_i\bar{z}_j$. For closed polygons, this gives $\alpha_1 = 0$, $\alpha_2 = -I_1 - 2A\mathbf{i}$, where

$$A = \frac{1}{2}\,\mathrm{Im}\sum_{i<j} z_i\bar{z}_j$$

is the signed area. Using also that $f \in \tilde{\mathbb{H}}[t]$ and so $\mathrm{Re}\,\alpha_k = 0$ for any odd k, we get

$$\begin{aligned}\bar{f}(t)f(t) &= 1 + 2(\mathrm{Re}\,\alpha_2)t^2 + 2(\mathrm{Re}\,\alpha_4 + \alpha_2\bar{\alpha}_2)t^4 + O(t^6)\\ &= 1 - 2I_1t^2 + (2I_2 + I_1^2 + 4A^2)t^4 + O(t^6).\end{aligned}$$

So, by definition of E_2 as the coefficient of t^4 in this expansion, we have $E_2 = 2I_2 + I_1^2 + 4A^2$. Combined with (12), this gives the desired formula. □

Remark 4.12 It follows from Proposition 4.7 that invariants I_k are well-defined on the quotient $\mathcal{P}_n^T/E$, i.e. are invariant under simultaneous rotation of all z_i. This is also easy to see from the explicit form of those invariants.

Remark 4.13 Let us show that our invariants I_k coincide with invariants c_{2k} constructed in [16, Proposition 4.3]. Consider a representation $\mathbb{H} \to GL_2(\mathbb{C})$ given by

$$\mathbf{i} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{j} \mapsto \begin{pmatrix} -\mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}, \quad \mathbf{k} \mapsto \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}.$$

The image of the polynomial $f(t)$ given by (10) under this representation is the matrix polynomial

$$F(t) = \begin{pmatrix} 1 - a_1t\cos(\psi_1)\mathbf{i} & a_1t\sin(\psi_1)\mathbf{i} \\ a_1t\sin(\psi_1)\mathbf{i} & 1 + a_1t\cos(\psi_1)\mathbf{i} \end{pmatrix} \cdots \begin{pmatrix} 1 - a_nt\cos(\psi_n)\mathbf{i} & a_nt\sin(\psi_n)\mathbf{i} \\ a_nt\sin(\psi_n)\mathbf{i} & 1 + a_nt\cos(\psi_n)\mathbf{i} \end{pmatrix},$$

where $a_i := |z_i|$ and $\psi_i := \arg(z_i)$. So

$$\mathrm{Re}\, f(t) = \frac{1}{2}\mathrm{Tr}\, F(t) = \frac{1}{2}\ell^{-n}\mathrm{Tr}\, M_1\cdots M_n, \tag{15}$$

where $\ell := (-t\mathbf{i})^{-1}$ and

$$M_i := \begin{pmatrix} \ell + a_i\cos(\psi_i) & -a_i\sin(\psi_i) \\ -a_i\sin(\psi_1) & \ell - a_i\cos(\psi_i) \end{pmatrix}.$$

The invariants c_k of [16] are defined by the relation

$$\operatorname{Tr} M_1 \cdots M_n = 2(\ell^n + c_2\ell^{n-2} + c_4\ell^{n-4} + \ldots),$$

so by (15) we have

$$\operatorname{Re} f(t) = 1 + c_2\ell^{-2} + \cdots = 1 - c_2t^2 + c_4t^4 - \ldots,$$

and hence $I_k = c_{2k}$.

4.4 Recutting invariants of polygons closed up to isometry

In the previous section we constructed recutting invariants on the space $\mathcal{P}_n^T$ of polygons closed up to translation. It turns out that those invariants do not extend to single-valued functions on the space $\mathcal{P}_n^E$ of polygons closed up to isometry. To get well-defined invariants, we consider a double covering space

$$\tilde{\mathcal{P}}_n^E := \{((v_i), \alpha) \in \mathcal{P}_n^E \times S^1 \mid (v_i) \text{ has monodromy } z \mapsto \alpha^2 z + \beta \text{ for some } \beta \in \mathbb{C}\}.$$

The projection map $\tilde{\mathcal{P}}_n^E \to \mathcal{P}_n^E$ takes a pair $((v_i), \alpha)$ to (v_i), so that elements of $\tilde{\mathcal{P}}_n^E$ can be thought of as polygons closed up to isometry with a chosen square root of the rotational part of the monodromy. Recuttings act on $\tilde{\mathcal{P}}_n^E$ by acting on the first component. Consider $((v_i), \alpha) \in \tilde{\mathcal{P}}_n^E$, and let $z_i = v_i - v_{i-1}$ be the edge vectors of the polygon (v_i). Let

$$f(t) := (1 + z_1\mathbf{j}t)(1 + z_{i+1}\mathbf{j}t) \cdots (1 + z_n\mathbf{j}t)\alpha. \tag{16}$$

Proposition 4.14 (cf. Proposition 4.7) *The similarity class of the polynomial $f(t)$ in the skew-field $\tilde{\mathbb{H}}[[t]]$ is invariant under both the action of the group E of isometries, and the recutting group action.*

We begin with a lemma, which will also be useful later on. Define the *gauge action* of $(\mathbb{C}^*)^n$ on $(\tilde{\mathbb{H}}[t])^n$ by

$$(g_i \in \tilde{\mathbb{H}}[t])_{i=1}^n \mapsto (\lambda_i g_i \lambda_{i+1}^{-1})_{i=1}^n \tag{17}$$

where $\lambda_i \in \mathbb{C}^*$, and the indices are understood cyclically, i.e. the index $n+1$ is equivalent to the index 1. Clearly, if two n-tuples $g_i(t) \in \tilde{\mathbb{H}}[t]$ and $\tilde{g}_i(t) \in \tilde{\mathbb{H}}[t]$ are gauge-equivalent, then the products $g_1(t) \cdots g_n(t)$ and $\tilde{g}_1(t) \cdots \tilde{g}_n(t)$ are similar.

Lemma 4.15 *Let $((v_i), \alpha) \in \tilde{\mathcal{P}}_n^E$, and let $z_i = v_i - v_{i-1}$ be the edge vectors of the polygon (v_i). Then the n-tuples*

$$g_1 := 1 + z_{i+1}\mathbf{j}t, \quad \ldots \quad g_{n-2} := 1 + z_{i+n-2}\mathbf{j}t, \quad g_{n-1} := (1 + z_{i+n-1}\mathbf{j}t)\alpha,$$
$$g_n := 1 + z_i\mathbf{j}t$$

and

$$\tilde{g}_1 := 1 + z_{i+1}\mathbf{j}t, \quad \ldots \quad \tilde{g}_{n-1} := 1 + z_{i+n-1}\mathbf{j}t, \quad \tilde{g}_n := (1 + z_{i+n}\mathbf{j}t)\alpha$$

are gauge-equivalent.

Proof Take $\lambda_1 = \cdots = \lambda_{n-1} = 1$, $\lambda_n = \alpha$. Then one clearly has $\lambda_j g_j \lambda_{j+1}^{-1} = \tilde{g}_j$ for $j = 1, \ldots n-1$. Furthermore,

$$\lambda_n g_n \lambda_1^{-1} = \alpha + z_i\alpha\mathbf{j}t = \alpha + z_{i+n}\alpha^{-1}\mathbf{j}t = \alpha + z_{i+n}\bar{\alpha}\mathbf{j}t = (1 + z_{i+n}\mathbf{j}t)\alpha = \tilde{g}_n,$$

where on the second step we used that $z_{i+n} = \alpha^2 z_i$, on the third step we used that $\alpha \in S^1$ and hence $\alpha^{-1} = \bar{\alpha}$, and on the second last step we used that $\bar{\alpha}\mathbf{j} = \mathbf{j}\alpha$. So we see that $\lambda_j g_j \lambda_{j+1}^{-1} = \tilde{g}_j$ for all j, as needed. □

Proof of Proposition 4.14. Let $f_i(t) := (1 + z_i\mathbf{j}t)\cdots(1 + z_{i+n-1}\mathbf{j}t)\alpha$. Then $f_i(t)$ is similar to the polynomial $\tilde{f}_i(t) := (1+z_{i+1}\mathbf{j}t)\cdots(1+z_{i+n-1}\mathbf{j}t)\alpha(1+z_i\mathbf{j}t)$, which, by Lemma 4.15, is similar to $f_{i+1}(t)$. So all f_i are similar to each other and in particular to $f_1 = f$. The rest of the proof is the same as for Proposition 4.7. □

Proposition 4.16 *For a polygon closed up to isometry, one has*

$$\begin{aligned} \bar{f}(t)f(t) &= 1 + \sum_k E_k t^{2k}, \\ \operatorname{Re} f(t) &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k I_k t^{2k}, \end{aligned} \tag{18}$$

where

$$\begin{aligned} E_k &:= \sum_{i_1 < \cdots < i_k} |z_{i_1}|^2 \ldots |z_{i_k}|^2, \\ I_k &:= \operatorname{Re}\left(\alpha \sum_{i_1 < \cdots < i_{2k}} z_{i_1}\bar{z}_{i_2}\ldots z_{i_{2k-1}}\bar{z}_{i_{2k}}\right). \end{aligned} \tag{19}$$

Proof See the proof of Proposition 4.9. □

So recutting action on n-gons closed up to isometry has has $\lfloor 3n/2 \rfloor + 1$ invariants, namely n elementary symmetric polynomials $E_1, \ldots, E_n$ of squared side lengths $|z_i|^2$ (whose invariance is obvious from the geometric definition of recutting), and $\lfloor n/2 \rfloor$ additional invariants $I_0, \ldots, I_{\lfloor n/2 \rfloor}$. Note that $I_0 = \operatorname{Re}\alpha$, which is trivially invariant since by definition of the recutting action on $\tilde{\mathcal{P}}_n^E/E$ it does not change α.

4.5 Poisson geometry of special quaternionic polynomials

In this section we show that the algebra $\tilde{\mathbb{H}}[t]$ of special quaternionic polynomials admits a multiplicative Poisson structure with nice properties. This structure can be obtained by extending the algebra $\tilde{\mathbb{H}}[t]$ to the algebra of special Laurent series in t and endowing the latter with an r-matrix of trigonometric type. Here we use a different approach based on representing special quaternionic polynomials as difference operators and then using a known Poisson structure on such operators.

Proposition 4.17 *There exists a Poisson structure on the algebra $\tilde{\mathbb{H}}[t]$ of special quaternionic polynomials with the following properties:*

1. *It is multiplicative, i.e. the multiplication map $\tilde{\mathbb{H}}[t] \times \tilde{\mathbb{H}}[t] \to \tilde{\mathbb{H}}[t]$ is Poisson.*
2. *Fixed degree polynomials form a Poisson subspace.*
3. *The Poisson structure vanishes on constant (i.e. degree 0) polynomials.*
4. *On linear polynomials $a + b\mathbf{j}t$, where $a, b \in \mathbb{C}$, the Poisson structure has the form*

$$\{a, b\} = -\tfrac{1}{2}ab, \quad \{a, \bar{b}\} = \tfrac{1}{2}a\bar{b}, \quad \{\bar{a}, \bar{b}\} = -\tfrac{1}{2}\bar{a}\bar{b},$$
$$\{\bar{a}, b\} = \frac{1}{2}\bar{a}b, \quad \{a, \bar{a}\} = 0, \quad \{b, \bar{b}\} = 0. \tag{20}$$

5. *Central functions on $\tilde{\mathbb{H}}[t]$ Poisson commute (we say that a function $\chi : \tilde{\mathbb{H}}[t] \to \mathbb{R}$ is central if $\chi(f) = \chi(g)$ for any similar $f, g \in \tilde{\mathbb{H}}[t]$).*
6. *The function $\tilde{\mathbb{H}}[t] \to \mathbb{R}$ mapping $f(t)$ to $|f(0)|$ is a Casimir.*

Remark 4.18 One can prove that $|f(t)|$ is a Casimir for any real t. That is equivalent to saying that all coefficients of the companion polynomial are Casimirs.

To prove Proposition 4.17 we recall the definition of a difference operator. Let $\mathbb{K}$ be a field, and let $\mathbb{K}^\infty = (\xi_i \in \mathbb{K})_{i \in \mathbb{Z}}$ be the vector space of bi-infinite sequences valued in $\mathbb{K}$. A degree d *difference operator* over $\mathbb{K}$ is a linear map $\mathcal{D} \colon \mathbb{K}^\infty \to \mathbb{K}^\infty$ of the form $\mathcal{D} = \sum_{i=0}^{d} a_i \mathcal{T}^i$, where $\mathcal{T} \colon \mathbb{K}^\infty \to \mathbb{K}^\infty$ is the left shift operator $(\mathcal{T}(\xi))_i := \xi_{i+1}$, while each a_i is an element of $\mathbb{K}^\infty$ acting on $\mathbb{K}^\infty$ by term-wise multiplication. A difference operator $\mathcal{D}$ is n*-periodic* if its coefficients a_i are n-periodic sequences, i.e. $(a_i)_{j+n} = (a_i)_j$.

Proposition 4.19 *As a graded associative algebra over reals, $\tilde{\mathbb{H}}[t]$ is isomorphic to the algebra of 2-periodic difference operators $\mathcal{D}$ with complex coefficients such that $\mathcal{T}\mathcal{D}\mathcal{T}^{-1} = \bar{\mathcal{D}}$.*

Proof of Proposition 4.19 The $\mathbb{R}$-algebra $\tilde{\mathbb{H}}[t]$ is generated by complex numbers $z \in \mathbb{C}$ and the polynomial $\mathbf{j}t$. The $\mathbb{R}$-algebra of 2-periodic difference operators $\mathcal{D}$ satisfying $\mathcal{T}\mathcal{D}\mathcal{T}^{-1} = \bar{\mathcal{D}}$ is generated by 2-periodic bi-infinite sequences of the form $a_z := (\dots, z, \bar{z}, \dots)$ and the operator $\mathcal{T}$. In terms of these generators, the isomorphism between these two algebras is given by $z \mapsto a_z$, $\mathbf{j}t \mapsto \mathcal{T}$. One easily checks that the relations between the generators on both sides are the same. □

Proof of Proposition 4.17 Consider the algebra of 2-periodic difference operators with real coefficients. By [13, Proposition 3.9], this algebra carries a multiplicative Poisson structure such that the map $\mathcal{D} \mapsto \mathcal{T}\mathcal{D}\mathcal{T}^{-1}$ is a Poisson automorphism. From the latter and Lemma 3.2 it follows that the extension of this structure to operators with complex coefficients restricts to operators such that $\mathcal{T}\mathcal{D}\mathcal{T}^{-1} = \bar{\mathcal{D}}$. This gives a multiplicative bracket on $\tilde{\mathbb{H}}[t]$. The desired properties of that bracket follow from properties of the bracket on difference operators. Namely, properties 1, 2, 3, 5 follow from the corresponding parts of [13, Proposition 3.9], property 4 follows from [13, Eq. (14)], while property 6 follows from [13, Proposition 3.19] combined with the fact that the determinant is a Casimir of the standard Poisson structure on GL_n. □

Remark 4.20 We changed the sign of the bracket from [13] for conformance with our cluster bracket (5).

4.6 A recutting-invariant Poisson structure

In this section we describe a Poisson bracket on the double cover $\tilde{\mathcal{P}}_n^E/E$ of the space $\mathcal{P}_n^E/E$ of polygons closed up to isometry and considered modulo isometries. This bracket is preserved by the recutting and has a property that the invariants defined in the Sect. 4.4 Poisson commute. Furthermore, this bracket descends to the space $\mathcal{P}_n^E/E$ and is taken by the natural map $\mathcal{P}_n^E/E \to \mathcal{P}_n^S/S$ to the cluster bracket (5).

Let $\tilde{\mathbb{H}}^*[t] = \{g(t) \in \tilde{\mathbb{H}}[t] \mid g(0) \neq 0\}$ be the space of special quaternionic polynomials with non-vanishing free term. Also, let $\tilde{\mathbb{H}}_k^*[t]$ be its subset consisting of polynomials of degree strictly equal to k. Then, by part 2 of Proposition 4.17, the space $\tilde{\mathbb{H}}_k^*[t]$ is a Poisson submanifold of $\tilde{\mathbb{H}}[t]$. Let $d = (d_1, \dots, d_n) \in \mathbb{Z}_+^n$. Then $Z_d := \tilde{\mathbb{H}}_{d_1}^*[t] \times \cdots \times \tilde{\mathbb{H}}_{d_n}^*[t]$ carries a product Poisson structure. Furthermore, by part 6 of Proposition 4.17, that Poisson structure restricts to $X_d := \{(g_1(t), \dots, g_n(t)) \in X_d \mid |g_1(0) \cdots g_n(0)| = 1\}$. Also note that since the Poisson structure on special quaternionic polynomials vanishes on $\tilde{\mathbb{H}}_0^*[t] = \mathbb{C}^*$, the gauge action (17) of $(\mathbb{C}^*)^n$ on X_d is Poisson, so the Poisson structure descends to $X_d/(\mathbb{C}^*)^n$.

We now show that for $d = (1, \dots, 1) \in \mathbb{Z}_+^n$, the space $X_d/(\mathbb{C}^*)^n$ can be identified with $\tilde{\mathcal{P}}_n^E/E$. The following is straightforward:

Proposition 4.21 *For any $d \in \mathbb{Z}_+^n$, every orbit of the gauge action of $(\mathbb{C}^*)^n$ on X_d has a representative $(g_i(t))_{i=1}^n$ with $g_1(0) = \cdots = g_{n-1}(0) = 1$ and $|g_n(0)| = 1$. Such a representative is unique up to a transformation of the form $(g_i(t)) \mapsto (\lambda g_i(t) \lambda^{-1})$ where $\lambda \in S^1$.*

Corollary 4.22 *Every orbit of the gauge action of $(\mathbb{C}^*)^n$ on $X_{1,\dots,1}$ has a representative of the form*

$$(1 + z_1 \mathbf{j} t, \dots, (1 + z_n \mathbf{j} t)\alpha), \tag{21}$$

where $z_i \in \mathbb{C}^*, \alpha \in S^1$. Such a representative is unique up to simultaneous rotation of all z_i.

It follows that the quotient $X_{1,\dots,1}/(\mathbb{C}^*)^n$ can be identified with the space of $\tilde{\mathcal{P}}_n^E/E$. Namely, given a gauge equivalence class in $X_{1,\dots,1}/(\mathbb{C}^*)^n$, one finds its representative of the form (21) and then maps it to a pair $((v_i), \alpha)$, where v_i is a polygon whose sequence of edge vectors z_i is obtained from numbers entering (21) by imposing quasi-periodicity condition $z_{i+n} = \alpha^2 z_i$. Since the canonical form (21) is defined up to simultaneous rotation of all z_i, this gives a well-defined map $X_{1,\dots,1}/(\mathbb{C}^*)^n \to \tilde{\mathcal{P}}_n^E/E$. Moreover, it follows from Proposition 2.6 that the so-defined map is a bijection. In particular, we get a Poisson bracket on the space $\tilde{\mathcal{P}}_n^E/E$. The following proposition summarizes its properties.

Proposition 4.23 1. *The Poisson bracket on $\tilde{\mathcal{P}}_n^E/E$ is invariant under recutting.*
2. *The invariants $E_1, \ldots, E_n, I_0, \ldots, I_{\lfloor n/2 \rfloor}$ Poisson commute.*
3. *Moreover, $E_1, \ldots, E_n, I_0$ are Casimirs.*
4. *The bracket descends to the space $\mathcal{P}_n^E/E$.*
5. *In coordinates $|z_i|, \phi_i$, the bracket on $\mathcal{P}_n^E/E$ has the following form: $|z_i|$ are Casimirs, while the bracket of ϕ_i's is given by (6). The function $\sum \phi_i$ is also a Casimir. For even n there is, in addition, a Casimir $\phi_1 + \phi_3 + \cdots + \phi_{n-1}$. Joint level sets of $|z_i|$ and these Casimirs are symplectic leaves.*
6. *The map $\mathcal{P}_n^E/E \to \mathcal{P}_n^S/S$ takes the bracket on $\mathcal{P}_n^E/E$ to the cluster bracket (5).*

We begin with a lemma. Consider a relabeling of vertices map on $\mathcal{P}_n^E$ given by $(v_i) \mapsto (\tilde{v}_i)$ where $\tilde{v}_i := v_{i+1}$. This map induces an order n map $\mathcal{S}$ on $\mathcal{P}_n^E/E$ and $\tilde{\mathcal{P}}_n^E/E$.

Lemma 4.24 *The Poisson bracket on $\tilde{\mathcal{P}}_n^E/E$ is invariant under the map $\mathcal{S}$.*

Proof Consider the map $\hat{\mathcal{S}}$ on $X_{1,\ldots,1}/(\mathbb{C}^*)^n$ induced by the cyclic shift map $(g_1, \ldots, g_n) \mapsto (g_2, \ldots, g_n, g_1)$. Clearly, this map is Poisson. So it suffices to show that the identification between the spaces $X_{1,\ldots,1}/(\mathbb{C}^*)^n$ and $\tilde{\mathcal{P}}_n^E/E$ intertwines the maps $\hat{\mathcal{S}}$ and $\mathcal{S}$. This amounts to saying that the cyclic shift of (21) to the left is gauge equivalent to $(1 + z_2\mathbf{j}t, \ldots, (1 + z_{n+1}\mathbf{j}t)\alpha)$. But this is exactly the content of Lemma 4.15 for $i = 1$. □

Proof of Proposition 4.23 We prove the parts of the proposition in a convenient order. First we prove part 1. By Lemma 4.24, the bracket on $\tilde{\mathcal{P}}_n^E/E$ is invariant under relabeling of vertices, so it suffices demonstrate the invariance of that bracket under recutting ρ_1. Let $d := (1, \ldots, 1) \in \mathbb{Z}_+^n$ and $d' := (2, 1, \ldots, 1) \in \mathbb{Z}^{n-1}$. Consider the map $\mathcal{M}\colon X_d \to X_{d'}$ given by $(g_1, \ldots, g_n) \mapsto (g_1g_2, g_3, \ldots, g_n)$. It is a Poisson map due to multiplicativity of the Poisson structure on $\tilde{\mathbb{H}}[t]$. So it descends to a Poisson map $\mathcal{M}'$ between gauge quotients $X_d/(\mathbb{C}^*)^n \to X_{d'}/(\mathbb{C}^*)^{n-1}$. Using Corollary 4.22, identify elements of $X_d/(\mathbb{C}^*)^n$ with n-tuples of the form (21) where z_n is positive real. Using Proposition 4.21, identify $X_{d'}/(\mathbb{C}^*)^{n-1}$ with the subspace of $X_{d'}$ which consists of $(g_1, \ldots, g_{n-1}) \in X_{d'}$ such that $g_i(0) = 1$ for all i, and $g_{n-1} = (1 + \beta\mathbf{j}t)\alpha$ with β positive real and $\alpha \in S^1$. With these identifications, the map $\mathcal{M}'\colon X_d/(\mathbb{C}^*)^n \to X_{d'}/(\mathbb{C}^*)^{n-1}$ is given by

$$(1+z_1\mathbf{j}t, \ldots, (1+z_n\mathbf{j}t)\alpha) \mapsto ((1+z_1\mathbf{j}t)(1+z_2\mathbf{j}t), 1+z_3\mathbf{j}t, \ldots, (1+z_n\mathbf{j}t)\alpha). \quad (22)$$

According to Proposition 4.5 and Remark 4.6, the mapping (22) is generically a 2-to-1 covering whose only non-trivial deck transformation is given by the recutting ρ_1. So ρ_1 is Poisson as a deck transformation of a Poisson covering. Thus, part 1 is proved.

We now prove part 2. Recall that the invariants $E_1, \ldots, E_n, I_0, \ldots, I_{\lfloor n/2 \rfloor}$ on the space $X_{1,\ldots,1}/(\mathbb{C}^*)^n \simeq \tilde{\mathcal{P}}_n^E/E$ are defined as central functions on $\mathbb{H}_0[t]$ applied to the polynomial (16). Therefore, the pullbacks of those invariants to $X_{1,\ldots,1}$ coincide with pullbacks of central functions $\tilde{\mathbb{H}}[t]$ by the product map $(g_1, \dots, g_n) \in X_{1,\dots,1} \mapsto g_1 \cdots g_n \in \tilde{\mathbb{H}}[t]$. Since the product map is Poisson, it follows that the invariants Poisson commute, as desired.

Next we prove part 4. Observe that the projection $\tilde{\mathcal{P}}_n^E/E \to \mathcal{P}_n^E/E$ can be viewed as quotient by the involution $\alpha \mapsto -\alpha$. That involution lifts to a Poisson involution of $X_{1,\dots,1}$ given by $(g_1, \dots, g_n) \mapsto (g_1, \dots, -g_n)$ and is, therefore, Poisson as well. So the space $\mathcal{P}_n^E/E$ inherits a Poisson structure from $\tilde{\mathcal{P}}_n^E/E$ as a quotient by Poisson involution.

Now prove parts 5 and 6. Consider functions $\tilde{y}_1, \dots, \tilde{y}_n$ on $X_{1,\dots,1}$ defined by

$$\tilde{y}_i(a_1 + b_1\mathbf{j}t, \dots, a_n + b_n\mathbf{j}t) := \frac{a_i b_{i+1}}{b_i \bar{a}_{i+1}}$$

where the index i is understood modulo n. It is easy too see that $\tilde{y}_i$ are gauge-invariant and thus descend to the quotient $X_{1,\dots,1}/(\mathbb{C}^*)^n \simeq \tilde{\mathcal{P}}_n^E/E$. To compute their pushforward to the quotient one just needs to restrict them to $X_{1,\dots,1}$ elements of the form (21). By doing so one finds that the pushforward of $\tilde{y}_i$ is the ratio y_i of consecutive edge vectors. That allows one to find Poisson brackets of y_i by computing brackets of $\tilde{y}_i$ and then pushing the result forward to the quotient. The brackets of $\tilde{y}_i$ can be found using formulas (20). As a result, one finds that the brackets of y_i's are given by (5), which precisely means that the map $\mathcal{P}_n^E/E \to \mathcal{P}_n^S/S$ (equivalently, $\tilde{\mathcal{P}}_n^E/E \to \mathcal{P}_n^S/S$) is Poisson. Thus part 6 is proved. In view of Remark 3.5, this also proves that the brackets of ϕ_i are of the form (6). So to complete the proof of part 5 it suffices to show that $|z_i|$ are Casimirs (the description of additional Casimirs and symplectic leaves easily follows using the constant form of the bracket (6)). In the same way as we showed that the pullbacks of y_i to the quotient are given by $\tilde{y}_i$, one shows that the pull-backs of $|z_i|$ are

$$\zeta_i(a_1 + b_1\mathbf{j}t, \dots, a_n + b_n\mathbf{j}t) := \frac{|b_i|}{|a_i|}.$$

These are easily seen to be Casimirs using formulas (20). Thus, part 5 is proved.

Finally, we prove part 3. The functions E_i are clearly Casimirs, since they are symmetric functions of the Casimirs $|z_i|^2$. Also notice that function α on $\tilde{\mathcal{P}}_n^E/E$ is a (complex-valued) Casimir, because the pushforward of α^2 to $\mathcal{P}_n^E/E$ is the function $\exp(\mathbf{i}(\phi_1 + \dots + \phi_n))$, which is a Casimir. Therefore, $I_0 = \operatorname{Re}\alpha$ is also a Casimir. Finally, for even n, the function $I_{n/2} = \sqrt{E_n}\operatorname{Re}(\alpha\exp(\mathbf{i}(\phi_1 + \phi_3 + \dots + \phi_{n-1})))$ is a Casimir since so are E_n, α, and $\phi_1 + \phi_3 + \dots + \phi_{n-1}$. Thus, the proposition is proved. □

Remark 4.25 (*cf. Remark* 3.6) The submanifold of $\mathcal{P}_n^E/E$ defined by $\sum \phi_i = 0 \mod 2\pi$ is the space $\mathcal{P}_n^T/E$ of isometry classes of polygons closed up to translation. So, since $\sum \phi_i$ is a Casimir, it follows that $\mathcal{P}_n^T/E$ is a Poisson submanifold of $\mathcal{P}_n^E/E$. As for the space $\mathcal{P}_n/E$ os isometry classes of closed polygons, it is not a Poisson submanifold of $\mathcal{P}_n^E/E$ because $\mathcal{P}_n/S$ is not a Poisson submanifold of $\mathcal{P}_n^S/S$.

4.7 Integrability

In this section we prove Theorem 1.1 on integrability of recutting on the space $\mathcal{P}_n^E/E$ of isometry classes of polygons closed up to isometry. Given the result of Proposition 4.23, it suffices to show that the integrals $E_1, \dots, E_n, I_0, \dots, I_{\lfloor n/2 \rfloor}$ are functionally independent. This is provided by the following:

Lemma 4.26 *The mapping $(\mathbb{C}^*)^n \times S^1 \to \mathbb{R}^{\lfloor 3n/2 \rfloor + 1}$ taking $(z_1, \dots, z_n, \alpha) \in (\mathbb{C}^*)^n \times S^1$ to the values of $E_1, \dots, E_n, I_0, \dots, I_{\lfloor n/2 \rfloor}$ defined by (19) is a submersion away from a set of measure zero.*

Proof The cases $n = 1, 2$ are straightforward so from this point on assume $n \geq 3$. Let $\mathcal{A} := (\mathbb{C}^*)^n \times S^1$ and let $\mathcal{C}$ be the set of pairs of real polynomials g, h with the following properties: both are even, the degree of g is exactly $2n$, the degree of h is at most $2[n/2]$, and $g(0) = 1$. Then the desired statement can be reformulated as follows. Consider the map $\Psi \colon \mathcal{A} \to \mathcal{C}$ that takes $(z_1, \dots, z_n, \alpha) \in (\mathbb{C}^*)^n \times S^1$ to the polynomials (18). Then Ψ is a submersion away from a set of measure zero. We prove this result by representing Ψ as a composition of two maps. Let $\mathcal{B}$ be the set of special quaternionic polynomials $f \in \tilde{\mathbb{H}}[t]$ of degree strictly n and such such that $|f(0)| = 1$. Then we have a map $\Psi_1 \colon \mathcal{A} \to \mathcal{B}$ that sends $(z_1, \dots, z_n, \alpha) \in \mathcal{A}$ to the special quaternionic polynomial $f(t)$ given by (16). Further, we have a map $\Psi_2 \colon \mathcal{B} \to \mathcal{C}$ given by $f(t) \mapsto \bar{f}(t) f(t), \operatorname{Re} f(t)$. Clearly $\Psi = \Psi_2 \circ \Psi_1$. So it suffices to show that both maps Ψ_1, Ψ_2 are submersions away from a set of measure zero. Since both maps are polynomial, that is equivalent to saying that the images of both maps have non-empty interior. First consider Ψ_1. Its image consists of those polynomials $f \in \mathcal{B}$ which can be factored into linear factors. According to Proposition 4.3, that happens if and only if all complex roots of the companion polynomials of f are on the imaginary axis. Since the companion polynomial of a special quaternionic polynomial is necessarily even, the set of f with the desired property has non-empty interior, as needed.

Now consider Ψ_2. First assume that n is odd, $n = 2k + 1$. Consider $(g, h) \in \mathcal{C}$. By definition of $\mathcal{C}$, both g, h are even, have degrees at most $4k + 2$ and $2k$ respectively (with degree of g being exactly $4k + 2$), and $g(0) = 1$. Therefore, the function $g(t) - h(t)^2 - 1 + h(0)^2$ is an even polynomial of degree at most $4k + 2$ vanishing at the origin. So, there exists a polynomial $\zeta(s)$ of degree at most $2k$ such that

$$g(t) - h(t)^2 - 1 + h(0)^2 = t^2 \zeta(t^2). \tag{23}$$

Say that $\zeta(s)$ is *strictly positive* if its coefficient of s^{2k} is positive, and $\zeta(s) > 0$ for any $s \in \mathbb{R}$. The set of strictly positive polynomials is open in the space of all real polynomials of degree at most $2k$. Now consider the subset Σ of $\mathcal{C}$ consisting of pairs $g(t), h(t)$ such that $|h(0)| < 1$ and the associated polynomial $\zeta(s)$ defined by (23) is strictly positive. That is an open subset of $\mathcal{C}$. Furthermore, $(t^{2n} + t^2 + 1, 0) \in \Sigma$, so Σ is non-empty. Let us show that Σ is contained in the image of Ψ_2. Assume $(g, h) \in \Sigma$. Consider the polynomial $\zeta(s)$ defined by (23). Since ζ is positive of degree $2k$, there exist real polynomials u, v of degree at most k such that $u^2 + v^2 = \zeta$.

Let also $a := \sqrt{1 - h(0)^2}$ (this is a real number since $|h(0)| < 1$). Consider a special quaternionic polynomial $f(t) := h(t) + \mathbf{i}a + \mathbf{j}tu(t^2) + \mathbf{k}tv(t^2)$. Then $\operatorname{Re} f(t) = h(t)$, and $\bar{f}(t)f(t) = g(t)$. The latter in particular means that the degree of f is exactly n. Furthermore, $f(0) = h(0) + \mathbf{i}a = h(0) + \mathbf{i}\sqrt{1 - h(0)^2}$ so $f \in \mathcal{B}$. Thus, $f \in \mathcal{B}$ and $\Psi_2(f) = (g, h)$, as needed.

Now assume n is even, $n = 2k$. Consider $(g, h) \in \mathcal{C}$. Then the polynomial $\zeta(s)$ defined by (23) has degree at most $2k - 1$. Consider the subset Σ of $\mathcal{C}$ consisting of pairs $g(t), h(t)$ such that $|h(0)| < 1$, the associated polynomial $\zeta(s)$ has positive coefficient b^2 of s^{2k-1}, and the polynomial

$$\xi(s) := \zeta(s) - b^2 s^{2k-1} - 2abs^{k-1} \tag{24}$$

is strictly positive of degree $2k - 2$ (here, as before, we define $a := \sqrt{1 - h(0)^2}$). The set Σ is open in $\mathcal{C}$, and non-empty since $(t^{2n} + t^{2n-2} + 2t^2 + 1, 0) \in \Sigma$. Let us show that Σ is contained in the image of Ψ_2. Let $(g, h) \in \Sigma$. Consider the polynomial $\xi(s)$ defined by (24). Since ξ is positive of degree $2k - 2$, there exist real polynomials u, v of degree at most $k - 1$ such that $u^2 + v^2 = \xi$. Let $f(t) := h(t) + \mathbf{i}(a + bt^{2k}) + \mathbf{j}tu(t^2) + \mathbf{k}tv(t^2)$. Then $f(t) \in \mathcal{B}$, and $\Psi_2(f) = (g, h)$. Thus, the lemma is proved. □

Proof of Theorem 1.1 We begin with part 1 of the theorem. Consider $E_1, \dots, E_n, I_0, \dots, I_{\lfloor n/2 \rfloor}$ as functions on the double covering of the space $\mathcal{P}_n^E/E$. The functions E_j are symmetric polynomials of squared lengths of sides and hence descend to $\mathcal{P}_n^E/E$. The functions I_j are defined on $\mathcal{P}_n^E/E$ up to sign. So $E_1, \dots, E_n, I_0^2, \dots, I_{\lfloor n/2 \rfloor}^2$ are well-defined functions on $\mathcal{P}_n^E/E$ preserved by recutting. They Poisson commute by Proposition 4.23 and are independent by Lemma 4.26. The number of those functions is $\lfloor 3n/2 \rfloor + 1$. The functions $E_1, \dots, E_n, I_0^2$ are Casimirs. For even n, the function $I_{n/2}^2$ is a Casimir too. So the total number $2\lfloor n/2 \rfloor + 2$ of Casimirs coincides with the codimension of symplectic leaves. Therefore, the remaining first integrals $I_1^2, \dots, I_{\lceil n/2 \rceil - 1}^2$ are independent on generic symplectic leaves. Their number is exactly one half of the dimension of the leaves. So, the recutting dynamics on $\mathcal{P}_n^E/E$ is indeed Arnold–Liouville integrable.

We now prove the second part. Since the squared lengths of sides considered as functions on $\mathcal{P}_n^E/E$ are Casimirs, it follows that the symplectic leaves of $\mathcal{P}_n^E/E$ are compact. Therefore, by Arnold–Liouville theorem, connected components of generic joint level sets of first integrals are tori. Furthermore, since $E_1, \dots, E_n$ are symmetric functions of squared side lengths, their joint level sets are compact too. So any joint level set of the recutting invariants $E_1, \dots, E_n, I_0^2, \dots, I_{\lfloor n/2 \rfloor}^2$ is compact and hence has finitely many connected components. For that reason, the subgroup of the recutting group preserving a given component must have finite index. That subgroup acts by translations by the discrete version of Arnold–Liouville theorem [18]. Thus, Theorem 1.1 is proved. □

5 Recutting of closed polygons: non-Hamiltonian integrability

In this section we prove Theorem 1.3 on integrability of recutting on the space $\mathcal{P}_n/E$ of isometry classes of closed polygons. Here is a version of Lemma 4.26 adapted to closed polygons:

Lemma 5.1 *Assume that $n \geq 3$. Then the mapping $\{(z_1, \dots, z_n) \in (\mathbb{C}^*)^n \mid \sum z_i = 0\} \to \mathbb{R}^{\lfloor 3n/2 \rfloor - 1}$ taking $(z_1, \dots, z_n)$ to the values of $E_1, \dots, E_n, I_2, \dots, I_{\lfloor n/2 \rfloor}$ defined by (11) is a submersion away from a set of measure zero.*

Proof The cases $n = 3, 4$ are straightforward so from this point on assume $n \geq 5$. The proof is a modification of that of Lemma 4.26. The sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and the maps Ψ_1, Ψ_2 are now defined as follows. The set $\mathcal{A}$ is $\{(z_1, \dots, z_n) \in (\mathbb{C}^*)^n \mid \sum z_i = 0\}$. The set $\mathcal{B}$ consists of special quaternionic polynomials $f \in \tilde{\mathbb{H}}[t]$ of degree strictly n and such such that $f(0) = 1$ and the coefficient of t is equal to 0. The map $\Psi_1 \colon \mathcal{A} \to \mathcal{B}$ takes $(z_1, \dots, z_n) \in \mathcal{A}$ to the special quaternionic polynomial $f(t)$ given by (10). The set $\mathcal{C}$ consists of pairs of real polynomials g, h of the form

$$g(t) = 1 + \sum_{i=1}^{n} E_i t^{2i}, \quad h(t) = 1 + \sum_{i=1}^{\lfloor n/2 \rfloor} J_i t^{2i},$$

where $E_1 = 2J_1$, and $E_n \neq 0$. The map $\Psi_2 \colon \mathcal{B} \to \mathcal{C}$ is again given by $f(t) \mapsto \bar{f}(t) f(t), \operatorname{Re} f(t)$. The statement of the lemma is equivalent to saying that $\Psi_2 \circ \Psi_1$ is a submersion away from a set of measure zero. The proof that Ψ_1 has this property is exactly the same as in Lemma 4.26. So it suffices to show that the image of Ψ_2 has non-empty interior.

First assume that n is odd, $n = 2k + 1$. Consider $(g, h) \in \mathcal{C}$. Then the polynomial $g(t) - h(t)^2$ is even, has degree at most $4k + 2$, and can be written as $(E_2 - J_1^2 - 2J_2)t^4 + O(t^6)$. Therefore, there is a polynomial $\zeta(s)$ of degree at most $2k - 2$ such that

$$g(t) - h(t)^2 - (E_2 - J_1^2 - 2J_2)t^4 = t^6 \zeta(t^2). \tag{25}$$

Consider the subset $\Sigma \subset \mathcal{C}$ consisting of pairs g, h such that $E_2 - J_1^2 - 2J_2 > 0$, and the polynomial $\zeta(s)$ is strictly positive. Then Σ is open in $\mathcal{C}$, and non-empty since $(t^{2n} + t^6 + t^4 + 1, 1) \in \Sigma$. Let us show that Σ is contained in the image of Ψ_2. Let $(g, h) \in \Sigma$. Let also $a := \sqrt{E_2 - J_1^2 - 2J_2}$. Since the polynomial ζ is strictly positive, there exist polynomials u, v of degree at most $k - 1$ such that $u^2 + v^2 = \zeta$. Let $f(t) := h(t) + \mathbf{i}at^2 + \mathbf{j}t^3 u(t^2) + \mathbf{k}t^3 v(t^2)$. Then $f \in \mathcal{B}$, and $\Psi_2(f) = (g, h)$, as needed.

Now assume n is even, $n = 2k$. Consider $(g, h) \in \mathcal{C}$. Then the polynomial $\zeta(s)$ defined by (25) has degree at most $2k - 3$ and its coefficient of s^{2k-3} is equal to $E_{2k} - J_k^2$. Consider the subset Σ of $\mathcal{C}$ consisting of pairs $g(t), h(t)$ such that $E_2 - J_1^2 - 2J_2 = a^2 > 0$, $E_{2k} - J_k^2 = b^2 > 0$, and the polynomial

$$\xi(s) := \zeta(s) - b^2 s^{2k-3} - 2abs^{k-2} \tag{26}$$

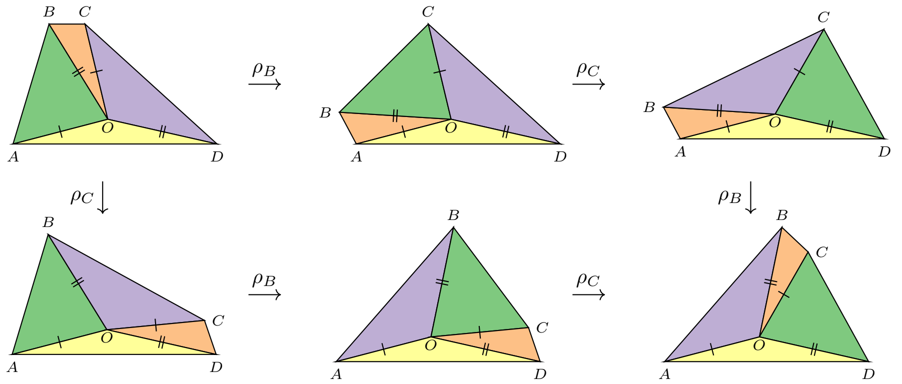

Fig. 6 The braid relation

is strictly positive of degree $2k - 4$ (note that $k > 1$ so ξ is indeed a polynomial). The set Σ is open in $\mathcal{C}$, and non-empty since $(t^{2n} + t^{2n-2} + 2t^6 + t^4 + 1, 1) \in \Sigma$. Let us show that Σ is contained in the image of Ψ_2. Let $(g, h) \in \Sigma$. Consider the polynomial $\xi(s)$ defined by (26). Since ξ is positive of degree $2k - 4$, there exist real polynomials u, v of degree at most $k - 2$ such that $u^2 + v^2 = \xi$. Let $f(t) := h(t) + \mathbf{i}(at^2 + bt^{2k}) + \mathbf{j}t^3u(t^2) + \mathbf{k}t^3v(t^2)$. Then $f \in \mathcal{B}$, and $\Psi_2(f) = (g, h)$. Thus, the lemma is proved. □

Proof of Theorem 1.3 We begin with the first part of the theorem. It is easy to see that $\mathcal{P}_n/E$ is a codimension 3, and hence dimension $2n - 3$, submanifold of $\mathcal{P}_n^E/E$. By Lemma 5.1, the restrictions of the functions $E_1, \dots, E_n, I_2^2, \dots, I_{\lfloor n/2 \rfloor}^2$ on $\mathcal{P}_n^E/E$ to $\mathcal{P}_n/E$ are functionally independent. Since on $\mathcal{P}_n/E$ we have $I_0^2 = 1$, this in particular means that the differentials of $I_0^2, E_1, \dots, E_n, I_2^2, \dots, I_{\lfloor n/2 \rfloor}^2$, considered as 1-forms on the ambient manifold $\mathcal{P}_n^E/E$, are independent at generic points of $\mathcal{P}_n/E$. From this it follows that the commuting Hamiltonian vector fields $X_2, \dots, X_{\lceil n/2 \rceil - 1}$ generated by $I_2^2, \dots, I_{\lceil n/2 \rceil - 1}^2$ are linearly independent almost everywhere on $\mathcal{P}_n/E$. Since the submanifold $\mathcal{P}_n/E$ of $\mathcal{P}_n^E/E$ is locally defined by the equations $I_0^2 = 1$ and (12), and all I_j's and E_j's Poisson commute, it follows that the vector fields $X_2, \dots, X_{\lceil n/2 \rceil - 1}$ are tangent to $\mathcal{P}_n/E$. Moreover, those vector fields preserve the invariants $E_1, \dots, E_n, I_2^2, \dots, I_{\lfloor n/2 \rfloor}^2$ (again, because I_j's and E_j's Poisson commute). So, the recutting action on $\mathcal{P}_n/E$ has $\lfloor 3n/2 \rfloor$ independent first integrals $E_1, \dots, E_n, I_2^2, \dots, I_{\lfloor n/2 \rfloor}^2$ and a complementary number $\lceil n/2 \rceil - 2$ of commuting invariant vector fields $X_2, \dots, X_{\lceil n/2 \rceil - 1}$ which are also independent and tangent to joint level sets of first integrals. Thus, the first part of Theorem 1.3 is proved. Furthermore, the second part now follows from the non-Hamiltonian version of the Arnold–Liouville theorem, see e.g. [4]. Thus, Theorem 1.3 is proved. □

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Appendix: The braid relation

Recuttings at adjacent vertices satisfy the braid relation. Adler [2] provides an algebraic argument. Here we give a geometric proof.

Proposition A.1 *Recuttings at adjacent vertices satisfy the braid relation* $\rho_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\rho_{i+1}$.

Proof It suffices to show that for any quadrilateral $ABCD$ one has $\rho_B\rho_C\rho_B = \rho_C\rho_B\rho_C$. Moreover, it is sufficient to consider the case of a convex quadrilateral. The general case follows by analytic continuation.

Let $ABCD$ be a convex quadrilateral, and let O be the intersection point of perpendicular bisectors to its diagonals. Consider the triangles AOB, BOC, COD, DOA shown in Fig. 6. Observe that recutting at any vertex is equivalent to detaching two of those triangles (namely those that are adjacent to the given vertex), and then attaching them back but switched and with opposite orientations. As a result, both transformations $\rho_B\rho_C\rho_B$ and $\rho_C\rho_B\rho_C$ boil down to cutting $ABCD$ into four triangles AOB, BOC, COD, DOA and then gluing them back interchanging AOB with COD. So, we indeed have $\rho_B\rho_C\rho_B = \rho_C\rho_B\rho_C$. □

Remark A.2 This argument shows that the intersection point of perpendicular bisectors of diagonals is invariant under recuttings of a quadrilateral. For more general polygons, a point with this property is known as the *circumcenter of mass*. Consider an arbitrarily triangulation of a polygon. Place point masses at circumcenters of the triangles, with each mass being proportional to the area of the corresponding triangle. Then their center of mass is independent of the triangulation and is called the circumcenter of mass of the polygon [3, 17]. Its invariance under recutting was observed in [1]. For quadrilaterals, the circumcenter of mass is precisely the intersection point of perpendicular bisectors of diagonals [17, Remark 3.3].

Remark A.3 Since recutting of a quadrilateral amounts to switching colored triangles in Fig. 6, there are only finitely many (isometry classes of) quads that can be obtained from a given one by means of a sequence of recuttings.

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