Letter

Universal tripartite entanglement signature of ungappable edge states

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Gapped two-dimensional topological phases can feature ungappable edge states which are robust even in the absence of protecting symmetries. In this Letter we argue that a multipartite entanglement measure recently proposed in the context of holography, the Markov gap, provides a universal diagnostic of ungappable edge states. Defined as a difference of the reflected entropy and mutual information $h(A{:}B) = S_R(A{:}B) - I(A{:}B)$ between two parties, we argue that for A, B being adjacent subregions in the bulk $h = \frac{c_+}{3}\ln 2$, where c_+ is the minimal total central charge of the boundary theory. As evidence, we prove that $h = 0$ for string-net models and numerically verify that $h = \frac{|C|}{3}\ln 2$ for a Chern-C insulator. Our Letter establishes a unique bulk entanglement criteria for the presence of a conformal field theory on the boundary.

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Introduction. Long-range entangled topological phases are characterized by a pattern of ground-state (gs) entanglement which cannot be adiabatically transformed into a product state [1]. Among two-dimensional (2D) topological orders (TOs) we may distinguish between two types. “Ungappable” TOs, such as the integer and fractional quantum Hall effects, have irremovable gapless edge states [2–4] even in the absence of protecting symmetries. One mechanism for such behavior is a mismatch between the number of left and right movers $c_- = c_R - c_L$ of the conformal field theory (CFT) governing the edge; although certain nonchiral theories with fractionalized excitations are also ungappable [5–7]. On the other hand, “gappable” TOs, such as the toric code [8] and string-net states [9] admit a gapped boundary theory for a suitable choice of edge Hamiltonian [10]. It has been shown that a TO is gappable if *and only if* it admits a string-net representation [10–12].

Whereas topological order has traditionally been probed via its excitations (e.g., edge states and fractionalized quasiparticles), more recently quantum information measures have been used as a method for detecting the pattern of long-range entanglement in the ground-state itself. Previous work has largely focused on bipartite entanglement. In particular the topological entanglement entropy (TEE) [5,13,14], which is a linear combination of the entanglement entropy of different subregions, measures the total quantum dimension of the anyonic excitations. However, the TEE does not distinguish between ungappable and gappable TOs. Indeed the integer quantum Hall effect is an ungappable TO with vanishing TEE. In order to distinguish *chiral* ungappable TO from gappable TO, one may look at the entanglement spectrum, which has the same anomalies as the physical edge [5,15]. On a translationally invariant cylinder it was argued that the entanglement spectrum encodes c_- mod 24 [16,17]. However, this approach relies on translation symmetry and does not detect nonchiral ungappable TOs. A quantitative bulk entanglement criterion for ungappable TO is, thus, still lacking.

In this Letter we provide such a measure by going beyond bipartite entanglement and considering a multipartite entanglement measure $h(A{:}B)$ recently referred to as the “Markov gap” [18]. This quantity was first discussed in the context of 1 + 1D (one-dimensional) CFTs and holography [18–21]. After defining a procedure which eliminates the nonuniversal short-distance contribution, we argue that the remainder takes the universal value,

$$h_{\mathrm{IR}} = \frac{c_+}{3}\ln 2, \tag{1}$$

where $c_+ = c_L + c_R$ is the minimal total central charge of a *single* edge [22] of the TOs boundary CFT. To give evidence for our conjecture we first prove that $h = 0$ for string-net states, consistent with their gappable edge. Second, we numerically compute h for integer quantum Hall states and find excellent agreement with Eq. (1). Compared with previous work, our method has two merits. First, it relies only on the reduced density matrix of a subregion in the bulk and does not require translation or other symmetries. Second, it is a quantitative measure which determines the central charge of the boundary CFT with finite-size corrections which appear to converge exponentially. Our Letter, thus, establishes a quantitative bulk entanglement criteria distinguishing between gappable and ungappable TO.

Markov gap $h(A{:}B)$. We start by defining the multipartite entanglement measure h for a quantum state. Given a pure state $|\psi\rangle_{\mathrm{ABC}}$ tripartitioned into A, B, and C, the reduced density matrix on AB is given by $\rho_{AB} = \mathrm{Tr}_C|\psi\rangle\langle\psi|$. One purification of ρ_{AB}, known as the canonical purification, is

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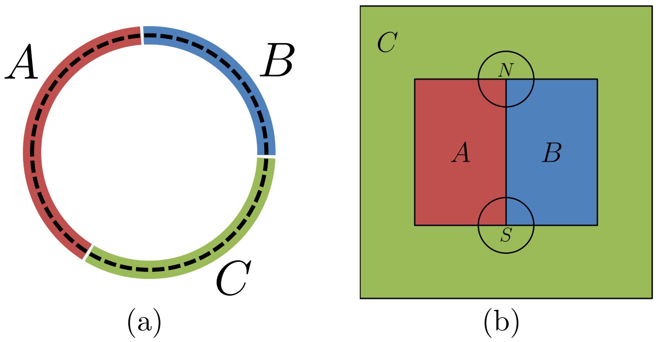

FIG. 1. (a) Tripartition of a 1D system on the circle. (b) A 2D system on a square lattice.

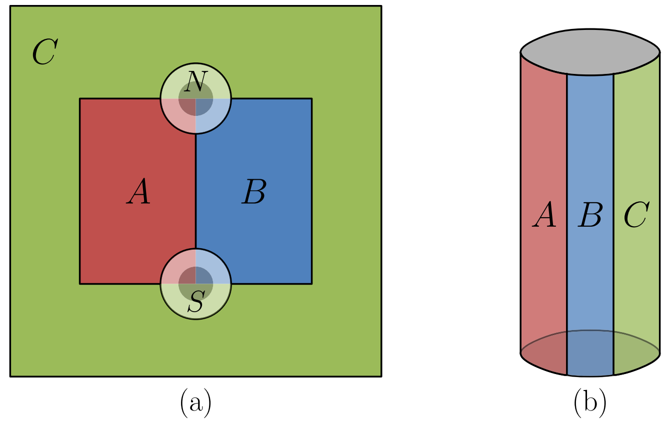

FIG. 2. (a) The tripartition of a two-dimensional system with punctures illustrated by shaded regions. There are left- and right-moving modes on the boundary of the shaded regions. Starting from the geometry in Fig. 1(b), we act with unitary disentanglers on the larger circular regions N and S (including the shaded interiors). Intuitively, in systems with a gapped bulk, such a unitary can be chosen to turn the degrees of freedom in the shaded interior into a product state. (b) An open cylinder with a tripartition where the three parties are strips that connect partitions of the two circles. Viewing the product state regions as punctures, the two geometries are topologically equivalent and can be deformed into each other using finite-depth local unitaries.

given by the square root of the density matrix taken as a state $|\sqrt{\rho}\rangle_{ABA^*B^*}$ in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_A^* \otimes \mathcal{H}_B^*$. The reflected entropy $S_R(A{:}B)$ is given by the entanglement entropy AA^* in the canonical purification, $S_R(A{:}B) = S_{AA^*}(|\sqrt{\rho}\rangle_{ABA^*B^*})$ [19]. The Markov gap $h(A : B)$ is then defined as

$$h(A{:}B) = S_R(A{:}B) - I(A{:}B) \geqslant 0, \tag{2}$$

where $I(A{:}B)$ is the mutual information between A and B. As shown in Ref. [21], h is a nonnegative quantity that vanishes if and only if the state $|\psi\rangle_{\rm ABC}$ has an algebraic form given by a sum of triangle states (SOTS).

Definition 1. A pure state $|\psi\rangle_{\rm ABC}$ is a SOTS if for each local Hilbert space $\mathcal{H}_\alpha$ ($\alpha \in \{A, B, C\}$) there exists a decomposition $\mathcal{H}_\alpha = \bigoplus_j \mathcal{H}_{\alpha_L^j} \otimes \mathcal{H}_{\alpha_R^j}$ such that

$$|\psi\rangle_{\rm ABC} = \sum_j \sqrt{p_j}\, |\psi_j\rangle_{A_R^j B_L^j}\, |\psi_j\rangle_{B_R^j C_L^j}\, |\psi_j\rangle_{C_R^j A_L^j}\,, \tag{3}$$

and $\sum_j p_j = 1$.

Roughly speaking, a SOTS only contains bipartite entanglement and Greenberger-Horne-Zeilinger (GHZ) type of entanglement. Therefore, nonvanishing $h(A{:}B)$ indicates entanglement across the three subregions beyond the GHZ type.

Tripartition for 1D and 2D systems. In Fig. 1 we show the tripartition that is considered in this Letter. For a one-dimensional system, we choose A, B, C to be adjacent intervals. It has been shown [19,21] that the ground state of a gapped system has $h = 0$ and the ground state of a gapless system has $h = h^{\rm CFT} \equiv \frac{c}{3} \ln 2$, where c is the central charge of the CFT.

Now we consider a two-dimensional lattice with the tripartition given in Fig. 1. In contrast to the one-dimensional case, there are two trisection points N and S where the three regions meet. The trisections can contribute a lattice-scale *nonuniversal* contribution to $h(A{:}B)$. Intuitively, UV physics can dress the trisection with an entangled tripartite state, which can contribute a finite h. There are two ways around this nonuniversal contribution. First, one may consider a modified geometry in which a disk is removed from each trisection so that the system becomes topologically equivalent to an open cylinder (Fig. 2). As will become clear, this approach does work, but it creates additional edges in the bulk. However, since we aim to demonstrate that h can be made a universal quantity *purely from the bulk ground state*, we instead develop a method for the disk geometry.

Instead of the ground-state $|\psi\rangle_{\rm ABC}$, we consider the space of "smoothed" states $U_N U_S |\psi\rangle_{\rm ABC}$, where $U_{N/S}$ is a unitary supported on a circle of radius R centered at N/S. We define the bulk entanglement quantity h_R at length scale R as

$$h_R \equiv \min_{U_N, U_S} h(A{:}B). \tag{4}$$

We then define $h_{\rm IR} = \lim_{R\to\infty} h_R$, where the limit is such that A, B, C must all be kept large in comparison with R. In practice, we will argue h_R converges exponentially quickly at a rate which is determined by some length scale ξ of the bulk ground state. The main result of this Letter is that $h_{\rm IR} = \frac{c_+}{3} \ln 2$ for a 2D system, where c_+ is the minimal central charge of the boundary theory. For a gapped theory $c_+ = 0$ and for an ungappable theory $c_+ \geqslant \frac{1}{2}$. Note that by construction $h \geqslant h_{\rm IR} = \frac{c_+}{3} \ln 2$, so our result may also be interpreted as a lower bound on the bare value of h.

Argument for universal $h_{\rm IR}$. Here we make an intuitive argument for the main result. Suppose $U_{N/S}$ (of radius R) are chosen to transform a subregion of radius $R' < R$ centered on each trisection into a product state. This can always be performed if we allow for a buffer of width $\xi < R - R'$, where ξ is related to a correlation length. Physically, for example, this operation can be accomplished by adiabatically turning on a topologically trivial mass term around the trisection. We then view the product state subregions as "punctures" [shaded in gray in Fig. 2(a)]. If C is one point compactified at infinity, the geometry is topologically equivalent to Fig. 2(b), where A, B, C are strips winding around an open cylinder. For an ungappable TO, there are edge modes on the top and bottom circles of the cylinder. As the bulk is gapped, the low-energy theory of the system is completely determined by the edge theory, and we can view it as a 1D theory defined on the edges of the cylinder, reducing to the geometry of Fig. 1(a). As long as the top and bottom are far enough from each other, the edge theories do not couple, and the theory on the circle is the full boundary CFT of the TO by combining left and right moving modes on the two boundaries of the cylinder. As $h(A{:}B)$ is invariant under local unitary operations, we expect that $h(A{:}B) = h^{\text{CFT}} = \frac{c_+}{3}\ln 2$ with the disentangler applied. As h_R is the minimum over U_N, U_S, we may take h^{CFT} given by the particular choice of disentangler in the thought experiment above to provide an upper bound on h_R. We expect that the disentangler gives the optimal h as the contribution from the edge modes are ungappable by local perturbations. Thus, $h_R = h^{\text{CFT}}$ if $R \gg \xi$.

If the edges are instead gapped, after reducing the cylinder along the vertical direction, the remaining state can be described by a matrix product state with finite bond dimension. For such states, it was proved in Ref. [21] that $h(A{:}B) = 0$. We may think of the smoothers on the disk as ensuring that when the trisection is mapped to the edge of a cylinder, a local nonuniversal tripartite entangled state does not become global.

Returning to the disk since the reduced density matrix of A, B plus the range of smoothers only measures local correlations, it is not affected by the topology of the whole system. The quantity h_R defined in Eq. (4) is, thus, universal given that the length scales of A, B, C much larger than R and $R \gg \xi$. Below we confirm the statement by showing that (i) $h = 0$ for string-net states, which are commonly believed to give a complete classification of topological orders with gappable edges in 2D, and that (ii) $h = \frac{c}{3}\ln 2$ for a stack of Chern insulators, where c is the minimal central charge of the boundary CFT.

Vanishing h for gappable topological order. We consider a tripartition as in Fig. 1 of the string-net liquids introduced in Ref. [9] which (can be generalized to) characterize the fixed-point (zero correlation length) wave functions of all topological orders with gappable edges in two dimensions. The string-net states have $\xi = 0$, and we are able to show explicitly that they have a SOTS structure. We review the formalism of string-net liquids and present the general proof in the Supplemental Material [23] but illustrate the argument here for the simple case of the toric code [8,24].

Consider the toric code on a trivalent lattice in Fig. 3. The degrees of freedom are located on the links of the lattice, and a two-dimensional Hilbert space is associated with each link. Let $\{|0\rangle, |1\rangle\}$ label a basis for each space. The global wave function is defined as the uniform superposition over all configurations of 0's and 1's subject to the local constraint that at any trivalent vertex, only an even number of the three adjacent links can take on value 1.

The lattice shown in Fig. 3(a) may be viewed as covering the surface of a sphere tripartitioned according to the dashed black lines. Regions A and B are adjacent, and C is complementary, and degrees of freedom on links straddling the partitions are doubled. Since this wave function has zero correlation length, we may employ such a minimal representation of the toric code. As in Ref. [13] we may reduce each region, but because each has a boundary it can only be reduced to a treelike diagram as in Fig. 3(b).

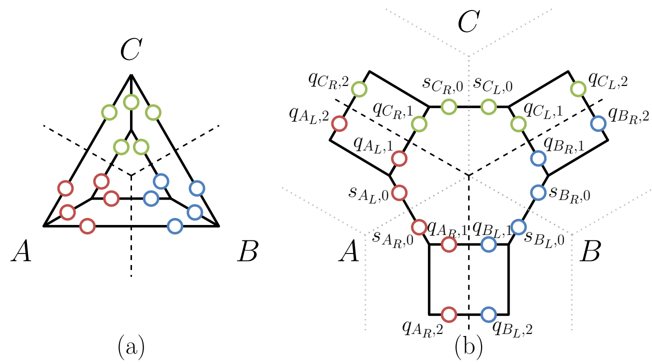

FIG. 3. (a) Toric code on a mesh covering the surface of a sphere. Degrees of freedom (0 or 1, denoted by circles) live on links, and the ground state is a uniform superposition over configurations of 0's and 1's satisfying the constraint that at each vertex an even number of 1's must meet. (b) The configuration in (a) may be reduced to this one using local unitaries acting on A, B, and C.

We now analyze the tripartite wave function in Fig. 3(b). The doubling on links implies that $q_{A_L,i} = q_{C_R,i}$, $q_{C_L,i} = q_{B_R,i}$, etc., as well as $s_{A_L,0} = s_{A_R,0}$, $s_{B_L,0} = s_{B_R,0}$, etc. Next, the graphical rules describing the relations between string configurations in Ref. [9] require that $s_{A_L,0} = s_{C_R,0} = s_{C_L,0} = s_{B_R,0} = s_{B_L,0} = s_{A_R,0}$. Let s denote the value of these central degrees of freedom. The total wave function may be organized as a sum over the central s, and the value of each $q_{\alpha,2}$ is set by the fusion of s and $q_{\alpha,1}$ for $\alpha \in \{A_L, A_R, B_L, \ldots\}$. Define

$$|AB(s)\rangle \equiv \frac{1}{\sqrt{2}}\,|s\rangle_{A_R,0}\,|s\rangle_{B_L,0} \otimes \sum_{q=\{0,1\}} |q\rangle_{A_R,1}\,|q\rangle_{B_L,1}\,|q\oplus s\rangle_{A_R,2}\,|q\oplus s\rangle_{B_L,2}\,, \quad (5)$$

and similarly define $|BC(s)\rangle$ and $|CA(s)\rangle$. For each value of s, this state in Eq. (5) is essentially a Bell pair among A_R, B_L, and $\langle AB(1)|AB(0)\rangle = 0$. The ground-state $|\psi_{\text{gs}}\rangle$ may then be written as

$$|\psi_{\text{gs}}\rangle = \sum_{s=\{0,1\}} \frac{1}{\sqrt{2}}\,|AB(s)\rangle \otimes |BC(s)\rangle \otimes |CA(s)\rangle\,. \quad (6)$$

This wave function manifestly satisfies the SOTS form in Eq. (3) with the factorization into L and R Hilbert spaces on each region as indicated by the dotted gray line in Fig. 3(b), and we can, therefore, conclude that $h(A{:}B) = 0$ for toric code. In the Supplemental Material [23], we apply this approach more generally to other string-net wave functions and find that they, too, may be written as a SOTS as in Eq. (3).

Universal h for stacked Chern insulators. The simplest chiral topological order is the Chern insulator where the minimal central charge c_+ on the boundary is given by the magnitude of the Chern number $|C|$ of the bulk [25–27]. The Chern insulator can be realized on a lattice by a tight-binding model coupled to an external magnetic-field B [28]. We consider the Hamiltonian given by

$$H(B) = -t\sum_{\vec{x},\vec{a}}(c_{\vec{x}}^{\dagger}e^{-i\vec{a}\cdot\vec{A}(\vec{x})}c_{\vec{x}+\vec{a}} + \text{H.c.}) + \mu\sum_{\vec{x}} c_{\vec{x}}^{\dagger}c_{\vec{x}}, \quad (7)$$

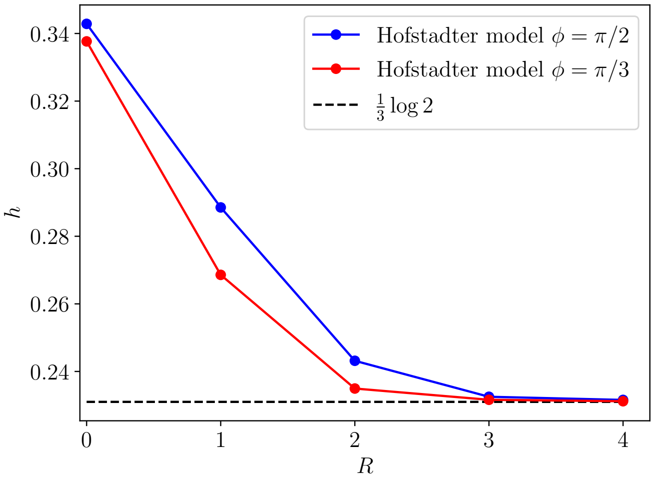

FIG. 4. h for the Hofstadter model with Chern number $C = 1$. The dashed line denotes the theoretical value $\frac{1}{3}$ ln 2.

where $\vec{a}$ runs over lattice vectors and $\vec{A}$ is the vector potential which equals $\vec{A} = (0, Bx_1)$ for the square lattice in the Landau gauge. The Chern number of the system is a sum of the Chern numbers of individual bands that are filled. We consider the lowest band for $B = \pm\pi/2$, which has $C = \pm 1$ and the lowest two bands for $B = \pi/3$, which both have $C = 1$. A topological insulator [29,30] can be constructed by stacking two layers of the Chern insulator with $C = 1$ and $C = -1$. The topological insulator is an example of symmetry-protected topological (SPT) phase where the edge modes are protected by the time-reversal (TR) symmetry [29]. The minimal central charge on (both) the boundaries is $c_+ = 0$ if TR is broken, and $c_+ = 2$ if TR is preserved.

The model is quadratic in the fermionic variables and the entanglement quantities can be computed by the standard covariance matrix techniques [31,32]. In order to obtain h_R in Eq. (4), we restrict the generators of the smoothers $U_{N/S}$ to be quadratic in the fermionic variables. When the edge modes are protected by TR symmetry, we further demand the $U_{N/S}$ are generated by a TR-invariant flow. The smoothers are optimized with a gradient optimization where the gradient can be computed from the covariance matrix. We compute the optimized h for different disentangler sizes up to $R = 6$ and different subsystem sizes up to $L_A = L_B = 24$.

We find that h converges to within 0.5% of the predicted value $h = \frac{c_+}{3}$ ln 2, where c_+ is the minimal central charge of the edge modes, once $\xi \ll R \ll L_A, L_B$. The numerical result is shown in Fig. 4 and Table I. Further details can be found in the Supplemental Material [23].

Discussion. In this Letter we have established a bulk multipartite entanglement quantity $h_{\rm IR}$ for two-dimensional topologically ordered systems. We have shown that $h_{\rm IR} = \frac{c_+}{3}$ ln 2, where c_+ is the minimal central charge of the boundary CFT. One numerically irksome feature of the definition is the use of disentanglers to remove short-distance entanglement at the trisection points. It would be interesting if instead a "subtraction scheme" as in the TEE could be devised.

TABLE I. h_R as defined in Eq. (4) for different topological orders. The second column shows the central charge for the edge modes. The third column shows the theoretical $h^{\rm CFT} \equiv \frac{c_+}{3}$ ln 2. The fourth column shows the values of h_R for different lattice models. h_R is computed analytically for string nets and optimzed numerically for other models. The smoother range is $R = 6$ for $B = \pi/3$ with the lowest two bands filled and $R = 4$ for other models.

Bulk TO	c_+	$h^{\rm CFT}$	h_R
String net	0	0	0
$B = \pi/2$, lowest band	1	0.2310	0.2316
$B = \pi/3$, lowest band	1	0.2310	0.2312
$B = \pi/3$, lowest two bands	2	0.4621	0.4641
Topological insulator (TR preserved)	2	0.4621	0.4632
Topological insulator (TR broken)	0	0	0.0014

One may naturally wonder whether the result is sensitive to the form of the disentangler. We first note that if the disentanglers were restricted entirely in AB, then we would no longer obtain $h_{\rm IR} = h^{\rm CFT}$ as we are no longer able to puncture a hole near the trisections. Next, as shown in the Supplemental Material [23], we find $h = 0$ for all models if the unitaries are allowed to entangle the degrees of freedom of the two tripartitions (i.e., when acting with a joint U_{NS}). This is expected as it allows the left-moving modes to hybridize with the right-moving modes such that they can be removed simultaneously. We also found that when the bulk is in an SPT phase, we obtain different $h_{\rm IR}$'s depending on whether the disentanglers respect or break the symmetry. Therefore, in addition to distinguishing between gappable and ungappable long-range entanglement, h may also be used to detect short-range entangled symmetry protected topological order.

Note added. Near the completion of this Letter, we became aware of independent work by Liu *et al.* [33], which also considers the reflected entropy of 2D states, and Kim *et al.* [34], which proposes a distinct entanglement measure for detecting the chiral central charge of a 2D topological state.

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[22] For a 1D nonanomalous critical chain, the total is $c_+ = 2c$, where c is the central charge (as conventionally used) of the 1D system because $c = c_L = c_R$.

[23] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevB.106.L041107 for a general proof that $h = 0$ for gappable topological orders and details regarding numerical methods employed for the ungappable topological orders examined in this paper.

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