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The leftmost column of ordered Chinese restaurant process up-down chains: intertwining and convergence*

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Abstract

Recently there has been significant interest in constructing ordered analogues of Petrov's two-parameter extension of Ethier and Kurtz's infinitely-many-neutral-alleles diffusion model. One method for constructing these processes goes through taking an appropriate diffusive limit of Markov chains on integer compositions called ordered Chinese Restaurant Process up-down chains. The resulting processes are diffusions whose state space is the set of open subsets of the open unit interval. In this paper we begin to study nontrivial aspects of the order structure of these diffusions. In particular, for a certain choice of parameters, we take the diffusive limit of the size of the first component of ordered Chinese Restaurant Process up-down chains and describe the generator of the limiting process. We then relate this to the size of the leftmost maximal open subset of the open-set valued diffusions. This is challenging because the function taking an open set to the size of its leftmost maximal open subset is discontinuous. Our methods are based on establishing intertwining relations between the processes we study.

Keywords: ordered Chinese restaurant process; up-down Markov chains; intertwining; functional limit theorem.
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1 Introduction

The construction and analysis of ordered analogues of Petrov's [15] two-parameter extension of Ethier and Kurtz's [3] infinitely-many-neutral-alleles diffusion model has recently attracted significant interest in the literature [7, 8, 20, 21, 23]. Recall that for

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$0 \le \alpha < 1$ and $\theta > -\alpha$, [15] constructed a Feller diffusion on the closure of the Kingman simplex

$$\overline{\nabla}_\infty = \left\{ \mathbf{x} = (x_1, x_2, \dots) \ : \ x_1 \ge x_2 \ge \dots \ge 0, \sum_{i \ge 1} x_i \le 1 \right\}$$

whose generator acts on the unital algebra generated by $\phi_m(\mathbf{x}) = \sum_{i\ge 1} x_i^m$, $m \ge 2$ by

$$\mathcal{G}^{(\alpha,\theta)} = \sum_{i=1}^{\infty} x_i \frac{\partial^2}{\partial x_i^2} - \sum_{i,j=1}^{\infty} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{\infty} (\theta x_i + \alpha) \frac{\partial}{\partial x_i}.$$

We will call a diffusion with generator $\mathcal{G}^{(\alpha,\theta)}$ an $\texttt{EKP}(\alpha, \theta)$ diffusion. In [20], for each $\theta \ge 0$, $0 \le \alpha < 1$, and $\alpha + \theta > 0$, we constructed a Feller diffusion $\mathbf{X}^{(\alpha,\theta)}$ whose state space $\mathcal{U}$ is the set of open subsets of $(0, 1)$ such that the ranked sequence of lengths of maximal open intervals in $\mathbf{X}^{(\alpha,\theta)}$ is an $\texttt{EKP}(\alpha, \theta)$ diffusion. This was done by considering the scaling limit of integer composition-valued up-down chains associated to the ordered Chinese Restaurant Process.

While many interesting properties of $\mathbf{X}^{(\alpha,\theta)}$ can be obtained from the corresponding properties for $\texttt{EKP}(\alpha, \theta)$ diffusions, properties that depend on the order structure cannot be. In this paper we begin to study nontrivial aspects of the order structure of these diffusions. Motivated by [6, Theorem 2 and Theorem 19] and [5, Theorem 5], which consider similar properties in closely related tree-valued processes, we consider the evolution of the leftmost maximal open interval of $\mathbf{X}^{(\alpha,0)}$ in running in its $(\alpha, 0)$-Poisson-Dirichlet interval partition stationarity distribution. Recall that the $(\alpha, 0)$-Poisson-Dirichlet interval partition is the distribution of $\{t \in (0, 1) : V_{1-t} > 0\}$ where V_t is a $(2 - 2\alpha)$-dimensional Bessel process started from 0. We prove the following result.

Theorem 1.1. *Define* $\xi\colon \mathcal{U} \to [0, 1]$ *by* $\xi(u) = \inf\{s > 0 : s \in [0, 1] \setminus u\}$. *If* $\mathbf{X}^{(\alpha,0)}$ *is running in its* $(\alpha, 0)$*-Poisson-Dirichlet interval partition stationarity distribution, then* $\xi(\mathbf{X}^{(\alpha,0)})$ *is a Feller process*[§]*. Moreover, the generator of its semigroup* $\mathcal{L}\colon \mathcal{D} \subseteq C[0, 1] \to C[0, 1]$ *is given by*

$$\mathcal{L}f(x) = x(1 - x)f''(x) - \alpha f'(x)$$

for $x \in (0, 1)$*, where the domain* $\mathcal{D}$ *of* $\mathcal{L}$ *consists of functions* f *satisfying*

(D1) $f \in C^2(0, 1)$ *and* $x(1 - x)f''(x) - \alpha f'(x)$ *extends continuously to* $[0, 1]$*,*

(D2) $\int_0^1 (f(x) - f(0))x^{-\alpha-1}(1 - x)^{\alpha-1}\, dx = 0$*, and*

(D3) $f'(x)(1 - x)^\alpha \to 0$ *as* $x \to 1$.

We consider only the $(\alpha, 0)$ case because the known stationary distribution of $\mathbf{X}^{(\alpha,\theta)}$ is an (α, θ)-Poisson-Dirichlet interval partition and, except in the $(\alpha, 0)$ case, with probability 1 interval partitions with these distributions do not have leftmost maximal open intervals. We remark that our theorem statement could be slightly simpler if we knew that $\mathbf{X}^{(\alpha,\theta)}$ had a unique stationary distribution, but this is currently an open problem.

Our proof is based on taking the scaling limit of the leftmost coordinate in an up-down chain on compositions based on the ordered Chinese Restaurant Process, which are the same chains that were used in [20] to construct $\mathbf{X}^{(\alpha,0)}$.

Definition 1.2. *For* $n \ge 1$*, a composition of* n *is a tuple* $\sigma = (\sigma_1, ..., \sigma_k)$ *of positive integers that sum to* n*. The composition of* $n = 0$ *is the empty tuple, which we denote by* $\varnothing$*. We write* $|\sigma| = n$ *and* $\ell(\sigma) = k$ *when* σ *is a composition of* n *with* k *components. We denote the set of all compositions of* n *by* $\mathcal{C}_n$ *and their union by* $\mathcal{C} = \cup_{n\ge 0} \mathcal{C}_n$.

[§]Recall that a Feller process is a Markov process on a (locally) compact state-space E with a transition semigroup that is a strongly continuous semigroup on $C_0(E)$.

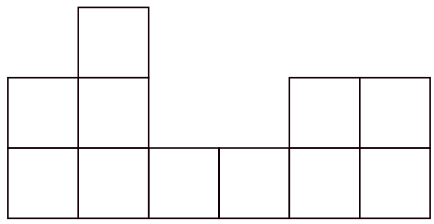


Figure 1: The diagram corresponding to the composition $\tau = (2, 3, 1, 1, 2, 2)$.

Each composition has a corresponding diagram of boxes: the diagram corresponding to σ has $|\sigma|$ boxes arranged into $\ell(\sigma)$ columns so that the i^{th} column contains σ_i boxes. See Figure 1 for an example.

An up-down chain on $\mathcal{C}_n$ is a Markov chain whose steps can be factored into two parts: 1) an up-step from $\mathcal{C}_n$ to $\mathcal{C}_{n+1}$ according to a kernel $p^{\uparrow}$ followed by 2) a down-step from $\mathcal{C}_{n+1}$ to $\mathcal{C}_n$ according to a kernel $p^{\downarrow}$. The probability $T_n(\sigma, \sigma')$ of transitioning from σ to σ' can then be written as

$$T_n(\sigma, \sigma') = \sum_{\tau \in \mathcal{C}_{n+1}} p^{\uparrow}(\sigma, \tau) p^{\downarrow}(\tau, \sigma'). \tag{1.1}$$

Up-down chains on compositions, and more generally, on graded sets, have been studied in a variety of contexts [2, 6, 9, 10, 11, 15, 16], often in connection with their nice algebraic and combinatorial properties.

In the up-down chains we considered, the up-step kernel $p^{\uparrow}_{(\alpha,\theta)}$ is given by an (α, θ)-ordered Chinese Restaurant Process growth step [18]. In the Chinese Restaurant Process analogy, we view the initial composition $\tau = (\tau_1, \ldots, \tau_k) \in \mathcal{C}_n$ as an ordered list of the number of customers at k occupied tables in a restaurant, so that τ_i is the number of customers at the i^{th} table on the list. An up-step from τ then corresponds to the entrance of a new customer to the restaurant who chooses a table to sit at according to the following rules:

- The new customer joins table i with probability $(\tau_i - \alpha)/(n + \theta)$, resulting in a step from τ to $(\tau_1, \ldots, \tau_{i-1}, \tau_i + 1, \tau_{i+1}, \ldots, \tau_k)$.
- The new customer starts a new table directly after the table i with probability $\alpha/(n + \theta)$, resulting in a step from τ to $(\tau_1, \ldots, \tau_{i-1}, \tau_i, 1, \tau_{i+1}, \ldots, \tau_k)$.
- The new customer starts a new table at the start of the list with probability $\theta/(n+\theta)$, resulting in a step from τ to $(1, \tau_1, \tau_2 \ldots, \tau_k)$.

A pictorial description of the up-step is given in Figure 2. We note that, for consistency with [7, 8], this up-step is the left-to-right reversal of the growth step in [18].

The down-step kernel $p^{\downarrow}$ we consider can also be thought of in terms of the restaurant analogy. As before, we view the initial composition $\tau \in \mathcal{C}_{n+1}$ as describing the arrangement of customers in the restaurant. A down-step from τ then corresponds to a uniformly random customer being chosen to leave the restaurant. Hence,

- the seated customer is chosen from table i with probability $\tau_i/(n + 1)$, resulting in a step from τ to

$$\begin{cases} (\tau_1, \ldots, \tau_{i-1}, \tau_i - 1, \tau_{i+1}, \ldots, \tau_k), & \text{if } \tau_i > 1, \\ (\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_k), & \text{if } \tau_i = 1. \end{cases}$$

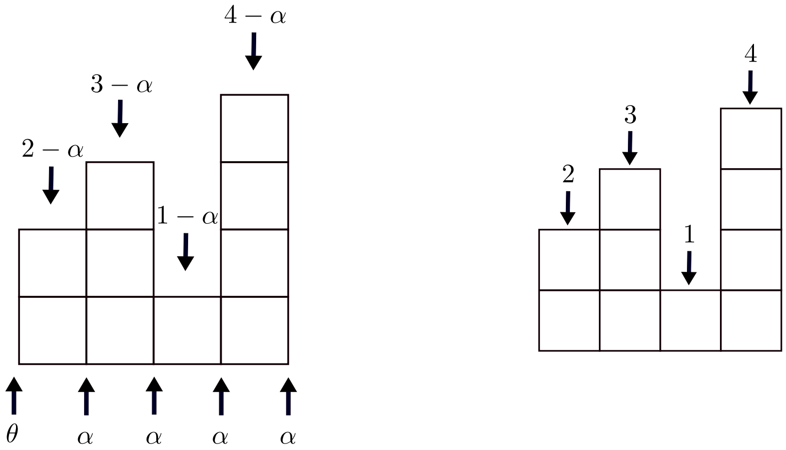


Figure 2: *Left.* An (α,θ) up-step from $\sigma=(2,3,1,4)$ inserts a box into a position above with probability proportional to the respective weight. *Right.* A down-step from σ reduces a column by one box with probability proportional to its size.

A pictorial description of the down-step is given in Figure 2. Note that, in contrast to the up-step, the down-step does not depend on (α,θ).

Let $(\mathbf{X}_n^{(\alpha,\theta)}(k))_{k\geq 0}$ be a Markov chain on $\mathcal{C}_n$ with transition kernel $T_n^{(\alpha,\theta)}$ defined as in Equation (1.1) using the $p^{\uparrow}_{(\alpha,\theta)}$ and $p^{\downarrow}$ just described. A Poissonized version of this chain was considered in [21, 23]. It can be shown that $\mathbf{X}_n^{(\alpha,\theta)}$ is an aperiodic, irreducible chain. We denote its unique stationary distribution by $M_n^{(\alpha,\theta)}$ and note that this is the left-to-right reversal of the (α,θ)-regenerative composition structures introduced in [12].

The projection $\phi(\sigma)=\sigma_1$ for $\sigma\neq\varnothing$ gives rise to the leftmost column processes, defined by $Y_n^{(\alpha,\theta)}=\phi(\mathbf{X}_n^{(\alpha,\theta)})$. Let $\nu_n^{(\alpha,\theta)}=M_n^{(\alpha,\theta)}\circ\phi^{-1}$, the distribution of the leftmost column when the up-down chain is in stationarity. The following result, interesting in its own right, is a key step in our proof of Theorem 1.1.

Theorem 1.3. *For $n\geq 1$, let μ_n be a distribution on $\{1,\dots,n\}$. Then, for all n, the up-down chain $\mathbf{X}_n^{(\alpha,0)}$ can be initialized so that $Y_n^{(\alpha,\theta)}$ is a Markov chain with initial distribution μ_n. Moreover, for any such sequence of initial conditions for $\mathbf{X}_n^{(\alpha,0)}$, if the sequence $\{n^{-1}Y_n^{(\alpha,0)}(0)\}_{n\geq 1}$ has a limiting distribution μ, then we have the convergence*

$$\left(n^{-1}Y_n^{(\alpha,0)}(\lfloor n^2 t\rfloor)\right)_{t\geq 0}\Longrightarrow (F(t))_{t\geq 0}$$

in the Skorokhod space $D([0,\infty),[0,1])$, where F is a Feller process with generator $\mathcal{L}$ (as in Theorem 1.1) and initial distribution μ.

While there are many ways to prove a result like Theorem 1.3, we take an approach based on the algebraic properties of the ordered Chinese Restaurant Process up-down chains. In particular, our proof is based on the following surprising intertwining result. For a positive integer i and composition σ, we use the notation (i,σ) as a shorthand for the composition $(i,\sigma_1,\sigma_2,\dots,\sigma_{\ell(\sigma)})$.

Theorem 1.4. *For $n\geq 1$, let Λ_n be the transition kernel from $\{1,\dots,n\}$ to $\mathcal{C}_n$ given by*

$$\Lambda_n(i,(i,\sigma))=M_{n-i}^{(\alpha,\alpha)}(\sigma),$$

and let K_n be the transition kernel from $[0,1]$ to $\{1,\dots,n\}$ given by

$$K_n(x,i)=\binom{n}{i}x^i(1-x)^{n-i}+\nu_n^{(\alpha,\alpha)}(i)(1-x)^n.$$

If the initial distribution of $\mathbf{X}_n^{(\alpha,0)}$ is of the form $\mu\Lambda_n$ for some distribution μ on $\{1,\dots,n\}$, then the process $Y_n^{(\alpha,0)}$ is Markovian. In this case, the following intertwining relations hold:

(i) $\Lambda_n T_n^{(\alpha,0)} = Q_n^{(\alpha,0)}\Lambda_n$*, where* $Q_n^{(\alpha,0)}$ *is the transition kernel of* $Y_n^{(\alpha,0)}$*, and*

(ii) $K_n e^{tn(n+1)(Q_n^{(\alpha,0)}-\mathbf{1})} = U_t K_n$ *for* $t \geq 0$*, where* U_t *is the semigroup generated by the operator* $\mathcal{L}$ *defined in Theorem 1.1 and* $\mathbf{1}$ *denotes the identity operator.*

This paper is organized as follows. In Section 2, we show that the $(\alpha, 0)$ leftmost column process is intertwined with its corresponding up-down chain and describe its transition kernel explicitly. This establishes part of Theorem 1.4. In Section 3, we state a condition under which the convergence of Markov processes can be obtained from some commutation relations involving generators. In Section 4, we analyze the generator of the limiting process. In Section 5, we show that our generators satisfy the commutation relations appearing in the result of Section 3. In Section 6, we verify the convergence condition appearing in the result in Section 3. In Section 7, we provide general conditions under which commutation relations involving generators lead to the corresponding relations for their semigroups. Finally in Section 8, we prove Theorems 1.1, 1.3, and 1.4.

The following will be used throughout this paper. For a compact topological space X, we denote by $C(X)$ the space of continuous functions from X to $\mathbb{R}$ equipped with the supremum norm. Finite topological spaces will always be equipped with the discrete topology. Any sum or product over an empty index set will be regarded as a zero or one, respectively. The set of positive integers $\{1, ..., k\}$ will be denoted by $[k]$. The falling factorial will be denoted using *factorial exponents* – that is, $x^{\downarrow b} = x(x-1)\cdot\ldots\cdot(x-b+1)$ for a real number x and nonnegative integer b, and $0^{\downarrow 0} = 1$ by convention. The rising factorial will be denoted by $(x)_b = x(x+1)\cdots(x+b-1)$. We denote the gamma function by $\Gamma(x)$. Multinomial coefficients will be denoted using the shorthand

$$\binom{|\sigma|}{\sigma} = \begin{cases} \dbinom{|\sigma|}{\sigma_1, ..., \sigma_{\ell(\sigma)}}, & \sigma \neq \varnothing, \\ 1, & \sigma = \varnothing. \end{cases}$$

2 The leftmost column process

Our study of the leftmost column process will be mainly focused on the $\theta = 0$ case. However, it will be useful to study the distribution of the (α, α) leftmost column process when the up-down chain is in stationarity. As we will see, this distribution has a role in the evolution of the $(\alpha, 0)$ process.

Proposition 2.1. *The stationary distribution of* $\mathbf{X}_n^{(\alpha,\alpha)}$ *is given by*

$$M_n^{(\alpha,\alpha)}(\sigma) = \binom{n}{\sigma}\frac{1}{(\alpha)_n}\prod_{j=1}^{\ell(\sigma)} \alpha\,(1-\alpha)_{\sigma_j - 1}, \qquad \sigma \in \mathcal{C}_n,\, n \geq 0.$$

Moreover, the following consistency conditions hold:

$$M_n^{(\alpha,\alpha)} = M_{n-1}^{(\alpha,\alpha)} p^{\uparrow}_{(\alpha,\alpha)} = M_{n+1}^{(\alpha,\alpha)} p^{\downarrow}, \qquad n \geq 1. \tag{2.1}$$

Proof. The stationary distribution of $\mathbf{X}_n^{(\alpha,\theta)}$ is identified in [20, Theorem 1.1] and the formula in the special case $\alpha = \theta$ follows from [12, Formula 48]. The consistency conditions follows from [18, Proposition 6]. ☐

Proposition 2.2. *If $\mathbf{X}_n^{(\alpha,\alpha)}$ has distribution $M_n^{(\alpha,\alpha)}$, then $Y_n^{(\alpha,\alpha)}$ has distribution*

$$\nu_n^{(\alpha,\alpha)}(i) = \binom{n}{i} \frac{\alpha\,(1-\alpha)_{i-1}}{(n-i+\alpha)_i} \mathbb{1}(1 \le i \le n), \qquad i \ge 0,\, n \ge 1.$$

Proof. Let $1 \le i \le n$ and $\sigma \in \mathcal{C}_{n-i}$. It can be verified that

$$M_n^{(\alpha,\alpha)}(i,\sigma) = \nu_n^{(\alpha,\alpha)}(i) M_{n-i}^{(\alpha,\alpha)}(\sigma). \tag{2.2}$$

Summing over σ concludes the proof. □

Let $n \ge i \ge 1$ and $\sigma \in \mathcal{C}_{n-i}$. Consider taking an $(\alpha,0)$ up-step from (i,σ) followed by a down-step. Let U be the event in which this up-step stacks a box on the first column of (i,σ), and let D be the event in which the down-step removes a box from the first column of a composition. Then, $r_{i,i+1} = \mathbb{P}(U \cap D^c)$, $r_{i,i-1} = \mathbb{P}(U^c \cap D)$, $r_{i,i}^{(1)} = \mathbb{P}(U^c \cap D^c)$, $r_{i,i}^{(2)} = \mathbb{P}(U \cap D)$, and $r_{i,i} = r_{i,i}^{(1)} + r_{i,i}^{(2)}$ do not depend on σ. Indeed, we have the formulas

$$\begin{aligned} r_{i,i-1} &= \tfrac{i(n-i+\alpha)}{n(n+1)}, & r_{i,i}^{(1)} &= \tfrac{(n-i+1)(n-i+\alpha)}{n(n+1)}, \\ r_{i,i+1} &= \tfrac{(i-\alpha)(n-i)}{n(n+1)}, & r_{i,i}^{(2)} &= \tfrac{(i-\alpha)(i+1)}{n(n+1)}. \end{aligned} \tag{2.3}$$

We use these formulas to define $r_{0,-1}$, $r_{0,1}$, $r_{0,0}^{(1)}$, $r_{0,0}^{(2)}$, and $r_{0,0} = r_{0,0}^{(1)} + r_{0,0}^{(2)}$. Moreover, we extend $r_{i,j}$ to be zero for all other integer arguments i and j.

The following is a useful identity relating the transition kernels of the $(\alpha,0)$ and (α,α) chains.

Proposition 2.3. *For $n \ge 1$ and $(i,\sigma),(j,\sigma') \in \mathcal{C}_n$, we have the identity*

$$\begin{aligned} T_n^{(\alpha,0)}\Big((i,\sigma),(j,\sigma')\Big) &= r_{i,j} p^{\uparrow}_{(\alpha,\alpha)}(\sigma,\sigma')\mathbb{1}(j=i-1) + r_{i,j} p^{\downarrow}(\sigma,\sigma')\mathbb{1}(j=i+1) \\ &\quad + (r_{i,i}^{(1)} T_{n-i}^{(\alpha,\alpha)}(\sigma,\sigma') + r_{i,i}^{(2)}\mathbb{1}(\sigma=\sigma'))\mathbb{1}(j=i) \\ &\quad + r_{1,0} p^{\uparrow}_{(\alpha,\alpha)}(\sigma,(j,\sigma'))\mathbb{1}(i=1). \end{aligned}$$

Proof. Given a composition $\tau = (\tau_1, \tau_2, \ldots, \tau_{\ell(\tau)})$, let $\tau_2^{\ell} = (\tau_2, \tau_3, \ldots, \tau_{\ell(\tau)})$ be the composition obtained by removing the first column of τ. Fix (i,σ) and (j,σ') in $\mathcal{C}_n$. Let $\mathbf{C}^{\uparrow}$ be the composition obtained by performing an $(\alpha,0)$ up-step from (i,σ) and $\mathbf{C}^{\downarrow}$ be the composition obtained by performing a down-step from $\mathbf{C}^{\uparrow}$. As before, let U be the event in which the up-step adds to the first column of a composition and D be the event in which the down-step removes from the first column of a composition. Then, we have that

$$U = \big\{\mathbf{C}^{\uparrow} = (i+1,\sigma)\big\}, \quad U^c = \{\mathbf{C}_1^{\uparrow} = i\}, \quad D^c \subseteq \{\mathbf{C}_1^{\downarrow} = \mathbf{C}_1^{\uparrow}\},$$

and

$$D \subseteq \Big\{\mathbf{C}_1^{\uparrow} > 1,\, \mathbf{C}^{\downarrow} = \Big(\mathbf{C}_1^{\uparrow} - 1, (\mathbf{C}^{\uparrow})_2^{\ell}\Big)\Big\} \cup \Big\{\mathbf{C}_1^{\uparrow} = 1,\, \mathbf{C}^{\downarrow} = (\mathbf{C}^{\uparrow})_2^{\ell}\Big\}.$$

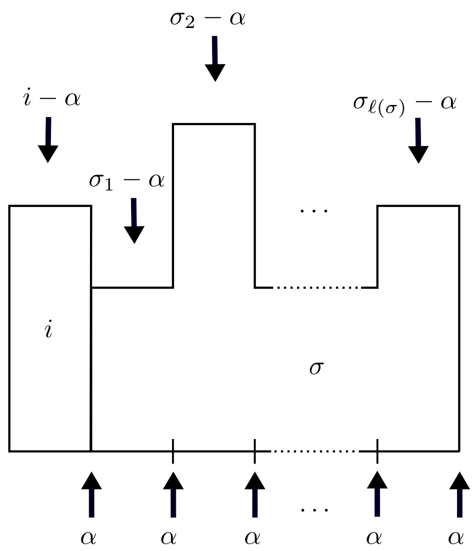
To obtain the identity, we note that

$$T_n^{(\alpha,0)}((i,\sigma),(j,\sigma')) = \mathbb{P}\{\mathbf{C}^{\downarrow} = (j,\sigma')\},$$

and rewrite this probability by conditioning on the above sets. Of particular importance will be the following observations:

(i) Conditionally given U^c, $(\mathbf{C}^{\uparrow})_2^{\ell}$ has distribution $p^{\uparrow}_{(\alpha,\alpha)}(\sigma,\cdot\,)$ and is independent of D.

(ii) Conditionally given D^c and $(\mathbf{C}^{\uparrow})_2^{\ell}$, $(\mathbf{C}^{\downarrow})_2^{\ell}$ has distribution $p^{\downarrow}\left((\mathbf{C}^{\uparrow})_2^{\ell},\,\cdot\,\right)$.



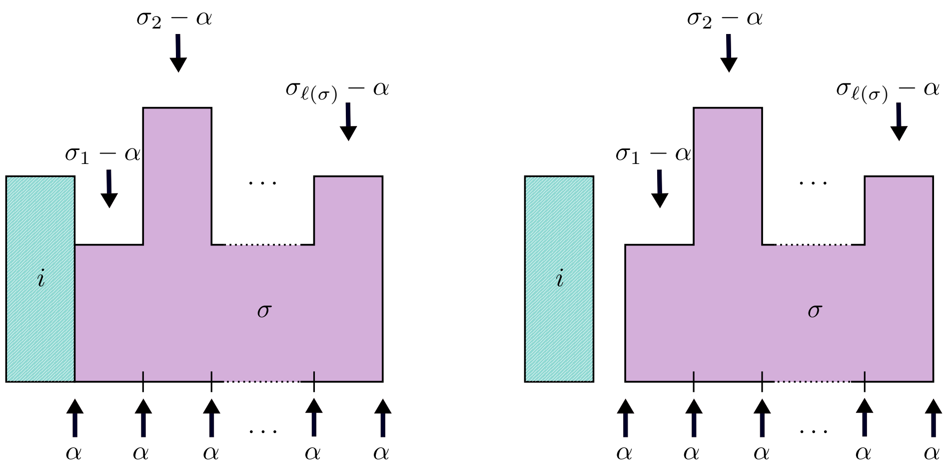


Figure 3: *Top.* An $(\alpha,0)$ up-step from (i,σ) describes $\mathbf{C}^{\uparrow}$. *Bottom-Left.* A modified up-step describes $\mathbf{C}^{\uparrow}$ conditionally given U^c. *Bottom-Right.* A column of size i and an (α,α) up-step from σ describe $\mathbf{C}_1^{\uparrow}$ and $(\mathbf{C}^{\uparrow})_2^{\ell}$, respectively, conditionally given U^c.

These facts can be established as follows:

(i) Recall that $\mathbf{C}^{\uparrow}$ is the result of an $(\alpha,0)$ up-step from (i,σ), which increases a component $i,\sigma_1,\ldots,\sigma_{\ell(\sigma)}$ by 1 with probability proportional to $i-\alpha,\sigma_1-\alpha,\ldots,\sigma_{\ell(\sigma)}-\alpha$, respectively, or inserts a component of value 1 after a given component with probability proportional to α. When we condition on U^c, $\mathbf{C}^{\uparrow}$ is the result of an up-step that uses the above rule but excludes the possibility of increasing the component i. The value of $(\mathbf{C}^{\uparrow})_2^{\ell}$ resulting from this modified up-step is described in the table below.

*weight***	*type of up-step*	*resulting value of* $(\mathbf{C}^{\uparrow})_2^{\ell}$
$\sigma_j-\alpha$	increase the component σ_j in (i,σ)	$(\sigma_1,\ldots,\sigma_{j-1},\sigma_j{+}1,\sigma_{j+1},\ldots,\sigma_{\ell(\sigma)})$
α	insert a 1 in (i,σ) after σ_j	$(\sigma_1,\ldots,\sigma_j,1,\sigma_{j+1},\ldots,\sigma_{\ell(\sigma)})$
α	insert a 1 in (i,σ) after i/before σ_1	$(1,\sigma)$

Meanwhile, an (α,α) up-step from σ results in the composition described below.

**each type of up-step occurs with probability proportional to the respective weight

weight	*type of up-step*	*resulting composition*
$\sigma_j - \alpha$	increase the component σ_j in σ	$(\sigma_1, \ldots, \sigma_{j-1}, \sigma_j + 1, \sigma_{j+1}, \ldots, \sigma_{\ell(\sigma)})$
α	insert a 1 in σ after σ_j	$(\sigma_1, \ldots, \sigma_j, 1, \sigma_{j+1}, \ldots, \sigma_{\ell(\sigma)})$
α	insert a 1 in σ before σ_1	$(1, \sigma)$

A direct comparison establishes the first claim. See also Figure 3. For the second claim, observe that the occurrence of D depends on $(\mathbf{C}^{\uparrow})_2^{\ell}$ only through its size, which is constant on U^c.

(ii) This fact can be proven similarly. A modified down-step from $\mathbf{C}^{\uparrow}$ (that does not remove from the first column) will describe $\mathbf{C}^{\downarrow}$ conditionally given D^c, and the resulting value of $(\mathbf{C}^{\downarrow})_2^{\ell}$ will be described by the composition resulting from a down-step from $(\mathbf{C}^{\uparrow})_2^{\ell}$.

We also make use of the fact that the events $\{\mathbf{C}^{\uparrow} = (n+1-|\rho|, \rho)\}$ and $\left\{(\mathbf{C}^{\uparrow})_2^{\ell} = \rho\right\}$ are identical, since the size of $\mathbf{C}^{\uparrow}$ is known to be $n+1$.

Our first conditional probability is given by

$$\begin{aligned}\mathbb{P}(\mathbf{C}^{\downarrow} = (j, \sigma')|U, D) &= \mathbb{P}(\mathbf{C}^{\downarrow} = (j, \sigma')|\mathbf{C}^{\uparrow} = (i+1, \sigma), D) \\ &= \mathbb{P}((i, \sigma) = (j, \sigma')|\mathbf{C}^{\uparrow} = (i+1, \sigma), D) \\ &= \mathbb{1}((j, \sigma') = (i, \sigma)).\end{aligned}$$

Next, we will condition on $U \cap D^c$. Notice that this is a null set when $i = n$. When $i < n$, we have

$$\begin{aligned}\mathbb{P}(\mathbf{C}^{\downarrow} = (j, \sigma')|U, D^c) &= \mathbb{P}(\mathbf{C}_1^{\uparrow} = j, (\mathbf{C}^{\downarrow})_2^{\ell} = \sigma'|\mathbf{C}^{\uparrow} = (i+1, \sigma), D^c) \\ &= \mathbb{1}(j = i+1)\mathbb{P}((\mathbf{C}^{\downarrow})_2^{\ell} = \sigma'|(\mathbf{C}^{\uparrow})_2^{\ell} = \sigma, D^c) \\ &= \mathbb{1}(j = i+1)p^{\downarrow}(\sigma, \sigma').\end{aligned}$$

Conditioning on $U^c \cap D$ will require two cases. For $i > 1$, we have

$$\begin{aligned}\mathbb{P}(\mathbf{C}^{\downarrow} = (j, \sigma')|U^c, D) &= \mathbb{P}(\mathbf{C}^{\downarrow} = (j, \sigma')|\mathbf{C}_1^{\uparrow} = i, D) \\ &= \mathbb{P}((i-1, (\mathbf{C}^{\uparrow})_2^{\ell}) = (j, \sigma')|\mathbf{C}_1^{\uparrow} = i, D) \\ &= \mathbb{1}(j = i-1)\,\mathbb{P}((\mathbf{C}^{\uparrow})_2^{\ell} = \sigma'|U^c, D) \\ &= \mathbb{1}(j = i-1)\,\mathbb{P}((\mathbf{C}^{\uparrow})_2^{\ell} = \sigma'|U^c) \\ &= \mathbb{1}(j = i-1)\,p^{\uparrow}_{(\alpha,\alpha)}(\sigma, \sigma'),\end{aligned}$$

and for $i = 1$, we have

$$\begin{aligned}\mathbb{P}(\mathbf{C}^{\downarrow} = (j, \sigma')|U^c, D) &= \mathbb{P}(\mathbf{C}^{\downarrow} = (j, \sigma')|\mathbf{C}_1^{\uparrow} = 1, D) \\ &= \mathbb{P}((\mathbf{C}^{\uparrow})_2^{\ell} = (j, \sigma')|U^c, D) \\ &= \mathbb{P}((\mathbf{C}^{\uparrow})_2^{\ell} = (j, \sigma')|U^c) \\ &= p^{\uparrow}_{(\alpha,\alpha)}(\sigma, (j, \sigma')).\end{aligned}$$

Finally, we condition on $U^c \cap D^c$. We have that

$$\begin{aligned}
\mathbb{P}\left(\mathbf{C}^{\downarrow} = (j,\sigma')|U^c, D^c\right) &= \mathbb{P}\left(\mathbf{C}_1^{\uparrow} = j, (\mathbf{C}^{\downarrow})_2^{\ell} = \sigma' | \mathbf{C}_1^{\uparrow} = i, D^c\right) \\
&= \mathbb{1}(j=i)\,\mathbb{P}\left((\mathbf{C}^{\downarrow})_2^{\ell} = \sigma' | U^c, D^c\right) \\
&= \mathbb{1}(j=i) \sum_{\tau \in \mathcal{C}_{n+1-i}} \mathbb{P}\left((\mathbf{C}^{\uparrow})_2^{\ell} = \tau | U^c, D^c\right) \mathbb{P}\left((\mathbf{C}^{\downarrow})_2^{\ell} = \sigma' | \mathbf{C}^{\uparrow} = (i,\tau), D^c\right) \\
&= \mathbb{1}(j=i) \sum_{\tau \in \mathcal{C}_{n+1-i}} \mathbb{P}\left((\mathbf{C}^{\uparrow})_2^{\ell} = \tau | U^c\right)\, \mathbb{P}\left((\mathbf{C}^{\downarrow})_2^{\ell} = \sigma' | (\mathbf{C}^{\uparrow})_2^{\ell} = \tau, D^c\right) \\
&= \mathbb{1}(j=i) \sum_{\tau \in \mathcal{C}_{n+1-i}} p^{\uparrow}_{(\alpha,\alpha)}(\sigma,\tau) p^{\downarrow}(\tau,\sigma') \\
&= \mathbb{1}(j=i)\, T^{(\alpha,\alpha)}_{n-i}(\sigma,\sigma').
\end{aligned}$$

Collecting the terms above with the appropriate terms in (2.3) establishes the result. □

Let $n \geq 1$. We define a transition kernel Λ_n from $[n]$ to $\mathcal{C}_n$ by

$$\Lambda_n(i,(i,\sigma)) = M^{(\alpha,\alpha)}_{n-i}(\sigma),$$

and a transition kernel Φ_n from $\mathcal{C}_n$ to $[n]$ by

$$\Phi_n(\sigma,i) = \mathbb{1}(\sigma_1 = i).$$

Proposition 2.4. *For $n \geq 1$, the transition kernel $Q_n^{(\alpha,0)} = \Lambda_n T_n^{(\alpha,0)} \Phi_n$ satisfies*

$$\Lambda_n T_n^{(\alpha,0)} = Q_n^{(\alpha,0)} \Lambda_n. \tag{2.4}$$

Consequently, if the initial distribution of $\mathbf{X}_n^{(\alpha,0)}$ is of the form $\mu\Lambda_n$, then $Y_n^{(\alpha,0)}$ is a time-homogeneous Markov chain with transition kernel $Q_n^{(\alpha,0)}$. Moreover, the transition kernel $Q_n^{(\alpha,0)}$ is given explicitly by

$$Q_n^{(\alpha,0)}(i,j) = r_{i,j} + r_{1,0}\nu_n^{(\alpha,\alpha)}(j)\mathbb{1}(i=1).$$

Proof. Let C_n be the kernel on $[n]$ defined by the right side of the above equation. Fix $i,j \in [n]$ and $\sigma' \in \mathcal{C}_{n-j}$. Using Proposition 2.3 and the identities (2.1) and (2.2), we compute

$$\begin{aligned}
&(\Lambda_n T_n^{(\alpha,0)})(i,(j,\sigma')) \\
&\quad = \sum_{\sigma\in\mathcal{C}_{n-i}} \Lambda_n(i,(i,\sigma)) T_n^{(\alpha,0)}((i,\sigma),(j,\sigma')) \\
&\quad = r_{i,j} \sum_{\sigma\in\mathcal{C}_{n-i}} M^{(\alpha,\alpha)}_{n-i}(\sigma) \left(p^{\uparrow}_{(\alpha,\alpha)}(\sigma,\sigma')\mathbb{1}(j=i-1) + p^{\downarrow}(\sigma,\sigma')\mathbb{1}(j=i+1)\right) \\
&\qquad + \mathbb{1}(j=i) \sum_{\sigma\in\mathcal{C}_{n-i}} M^{(\alpha,\alpha)}_{n-i}(\sigma) \left(T^{(\alpha,\alpha)}_{n-i}(\sigma,\sigma') r^{(1)}_{i,i} + \mathbb{1}(\sigma=\sigma') r^{(2)}_{i,i}\right) \\
&\qquad + r_{1,0}\mathbb{1}(i=1) \sum_{\sigma\in\mathcal{C}_{n-i}} M^{(\alpha,\alpha)}_{n-i}(\sigma) p^{\uparrow}_{(\alpha,\alpha)}(\sigma,(j,\sigma')) \\
&\quad = r_{i,j}\left(M^{(\alpha,\alpha)}_{n-j}(\sigma')\mathbb{1}(j=i-1) + M^{(\alpha,\alpha)}_{n-j}(\sigma')\mathbb{1}(j=i+1)\right) \\
&\qquad + \mathbb{1}(j=i)\left(M^{(\alpha,\alpha)}_{n-j}(\sigma') r^{(1)}_{i,i} + M^{(\alpha,\alpha)}_{n-j}(\sigma') r^{(2)}_{i,i}\right) + r_{1,0}\mathbb{1}(i=1) M^{(\alpha,\alpha)}_{n}(j,\sigma') \\
&\quad = r_{i,j} M^{(\alpha,\alpha)}_{n-j}(\sigma') + r_{1,0}\mathbb{1}(i=1)\nu^{(\alpha,\alpha)}_n(j) M^{(\alpha,\alpha)}_{n-j}(\sigma') \\
&\quad = C_n(i,j)\Lambda_n(j,(j,\sigma')) \\
&\quad = (C_n\Lambda_n)(i,(j,\sigma')).
\end{aligned}$$

The final equality follows from the fact that $\Lambda_n(j,\,\cdot\,)$ is supported on $\{\sigma \in \mathcal{C}_n : \sigma_1 = j\}$. This establishes the identity $\Lambda_n T_n^{(\alpha,0)} = C_n\Lambda_n$. Observing that $\Lambda_n\Phi_n$ is the identity kernel on $[n]$, we find that

$$Q_n^{(\alpha,0)} = \Lambda_n T_n^{(\alpha,0)}\Phi_n = C_n\Lambda_n\Phi_n = C_n,$$

from which we obtain (2.4) and the explicit description of $Q_n^{(\alpha,0)}$. The final claim follows from applying Theorem 2 in [22]. □

3 Convergence from commutation relations

In this section, we provide a condition under which commutation relations between operators implies the convergence of those operators in an appropriate sense. In the interest of generality, we first state this condition in the setting of Banach spaces, but we then reformulate it in the context of Markov processes to suit our purposes. The general setting is as follows.

Let $V, V_1, V_2, \ldots$ be Banach spaces and $\pi_1, \pi_2, \ldots$ be uniformly bounded linear operators with $\pi_n \colon V \to V_n$. These spaces will be equipped with the following mode of convergence.

Definition 3.1. *A sequence $\{f_n\}_{n\geq 1}$ with $f_n \in V_n$ converges to an element $f \in V$ (and we write $f_n \to f$) if*

$$\|f_n - \pi_n f\| \xrightarrow[n\to\infty]{} 0,$$

where for convenience, we denote every norm by the same symbol $\|\cdot\|$.

Proposition 3.2. *For $n \geq 1$, let $L_n \colon D_n \subset V \to V_n$ and $A_n \colon V_n \to V_n$ be linear operators in addition to $A \colon D \subset V \to D$. Suppose that for every $f \in D$,*

(i) $A_n L_n f = L_n A f$ for large n, and

(ii) $(L_n - \pi_n) f \longrightarrow 0$ as $n \to \infty$ (the sequence need only be defined for large n).

Then for $f \in D$, the sequence $f_n = L_n f$ (defined for large n) satisfies

$$f_n \longrightarrow f \qquad \text{and} \qquad A_n f_n \longrightarrow Af.$$

Proof. Let $f \in D$ and n be large enough so that (i) holds. In particular, we can define $f_n = L_n f$. Writing

$$\|f_n - \pi_n f\| = \|L_n f - \pi_n f\|,$$

it is clear that $f_n \to f$. Writing

$$\begin{aligned}\|A_n f_n - \pi_n A f\| &= \|A_n L_n f - \pi_n A f\| \\ &= \|L_n A f - \pi_n A f\| \\ &= \|(L_n - \pi_n) A f\|\end{aligned}$$

and noting that $Af \in D$, we obtain the other convergence. □

In the probabilistic context, the above result has some additional consequences.

Theorem 3.3. *Let E be a compact, separable metric space, A be the generator of the Feller semigroup $S(t)$ on $C(E)$, and D be a core for A that is invariant under A. For each $n \geq 1$, let E_n be a finite set endowed with the discrete topology, Z_n be a Markov chain on E_n, $\gamma_n \colon E_n \to E$ be any function, and $L_n \colon D_n \subset C(E) \to C(E_n)$ be a linear operator. Denote the transition operator of Z_n by S_n and the projection $f \mapsto f \circ \gamma_n$ by $\pi_n \colon C(E) \to C(E_n)$. Let $\{\delta_n\}_{n\geq 1}$ and $\{\varepsilon_n\}_{n\geq 1}$ be positive sequences converging to zero such that $\varepsilon_n^{-1}\delta_n \to 1$. Suppose that for $f \in D$, the following statements hold:*

(a) $\delta_n^{-1}(S_n - \mathbf{1})L_n f = L_n A f$ *for large* n*, and*

(b) $(L_n - \pi_n) f \longrightarrow 0$ *as* $n \to \infty$ *(the sequence need only be defined for large* n*).*

Then,

(i) the discrete semigroups $\{1, S_n, S_n^2, ...\}_{n\geq 1}$ *converge to* $\{S(t)\}_{t\geq 0}$ *in the following sense: for all* $f \in C(E)$ *and* $t \geq 0$*,*

$$S_n^{\lfloor t/\varepsilon_n \rfloor} \pi_n f \xrightarrow[n\to\infty]{} S(t) f$$

(ii) the above convergence is uniform in t *on bounded intervals, and*

(iii) if A *is conservative and the distributions of* $\gamma_n(Z_n(0))$ *converge, say to* μ*, then we have the convergence of paths*

$$\gamma_n(Z_n \lfloor t/\varepsilon_n \rfloor) \Longrightarrow F(t)$$

in the Skorokhod space $D([0,\infty), E)$*, where* $F(t)$ *is a Feller process with initial distribution* μ *and generator* A*.*

Proof. This is a combination of Proposition 3.2 and standard convergence results. In particular, for $f \in D$, we can define the sequence $f_n = L_n f$ for large n and obtain the convergence

$$f_n \longrightarrow f \qquad \text{and} \qquad \delta_n^{-1}(S_n - \mathbf{1}) f_n \longrightarrow Af.$$

Recalling that $\varepsilon_n^{-1}\delta_n \to 1$, we then obtain the convergence $\varepsilon_n^{-1}(S_n - \mathbf{1}) f_n \to Af$. Applying Chapter 1 Theorem 6.5 in [4] then yields the convergence of semigroups in (i) and (ii). Applying Chapter 4 Theorem 2.12 in [4] yields the path convergence in (iii). □

4 The limiting generator

In this section, we introduce the generator of a Feller process on $[0,1]$ that will be identified as the limiting process. We describe this generator both on a core of polynomials and on its full domain. However, the core description is sufficient for the analysis that will follow.

Let $\mathcal{P}$ denote the space of polynomials on $[0,1]$ equipped with the supremum norm. We will study the operator $\mathcal{B}\colon \mathcal{P} \to \mathcal{P}$ and the functional $\eta\colon \mathcal{P} \to \mathbb{R}$ given by

$$(\mathcal{B}f)(x) = x(1-x)f''(x) - \alpha f'(x), \quad x \in [0,1],$$

and

$$\begin{aligned} \eta(f) &\coloneqq \int_0^1 (f(x) - f(0)) x^{-\alpha-1}(1-x)^{\alpha-1}\, dx \\ &= \int_0^1 f'(x) x^{-\alpha}(1-x)^{\alpha}\alpha^{-1}\, dx. \end{aligned} \tag{4.1}$$

Letting $\mathbb{N} = \{0,1,2,\ldots\}$, we define a family of polynomials $\{h_n\}_{n\in\mathbb{N}\setminus\{1\}}$ by

$$h_n(x) = \sum_{s=0}^{n} x^s (-1)^{n-s} \frac{(n-1)_s}{s!} \frac{(s-\alpha)_{n-s}}{(n-s)!}, \qquad x \in [0,1].$$

Note that $h_0 \equiv 1$ and h_n has degree n. Moreover, these polynomials are related to the Jacobi polynomials $P_n^{(a,b)}$ and the shifted Jacobi polynomials $J_n^{(a,b)}$ [19, 24] by the identity

$$h_n(x) = J_n^{(\alpha-1,-\alpha-1)}(x) = P_n^{(\alpha-1,-\alpha-1)}(2x-1), \qquad x \in [0,1].$$

Proposition 4.1. *Let $\mathcal{H} = \ker \eta$ and $\omega_n = -n(n-1)$ for $n \in \mathbb{N} \setminus \{1\}$. The following statements hold:*

(i) $\mathcal{B} h_n = \omega_n h_n$ *for all* $n \in \mathbb{N} \setminus \{1\}$,

(ii) the family $\{h_n\}_{n\in\mathbb{N}\setminus\{1\}}$ *is a Hamel*[††] *basis for* $\mathcal{H}$ *and*

(iii) $\mathcal{H}$ *is a dense subspace of* $C[0,1]$.

Proof. The claim in (i) can be obtained from the classical theory of Jacobi polynomials (e.g. (4.1.3), (4.21.2), and (4.21.4) in [24]).

Noting that h_n has degree n shows that the family $\{h_n\}_{n\in\mathbb{N}\setminus\{1\}}$ is linearly independent. Since $h_0 \equiv 1$, it clearly lies in $\mathcal{H}$. To see that the other h_n also lie in $\mathcal{H}$, we use (i) to identify them as elements in the range of $\mathcal{B}$ and observe that this range lies in $\mathcal{H}$. Indeed, this can be verified using (4.1): for $f \in \mathcal{P}$, we have that

$$\begin{aligned}
\eta(\mathcal{B}f) &= \int_0^1 (x(1-x)f''(x) - \alpha f'(x) + \alpha f'(0)) x^{-\alpha-1}(1-x)^{\alpha-1}\,dx \\
&= \int_0^1 f''(x)x^{-\alpha}(1-x)^{\alpha}\,dx - \alpha \int_0^1 (f'(x) - f'(0))x^{-\alpha-1}(1-x)^{\alpha-1}\,dx \\
&= \alpha\,\eta(f') - \alpha\,\eta(f') \\
&= 0.
\end{aligned}$$

To obtain equality from the containment $\mathrm{span}\{h_n\}_{n\in\mathbb{N}\setminus\{1\}} \subset \mathcal{H}$, we observe that the former space is a maximal subspace of $\mathcal{P}$ (it has codimension one) while the latter is a proper subspace of $\mathcal{P}$.

The claim in (iii) will follow from showing that η is not continuous (see Chapter 3 Theorem 2 in [1]). To see that this holds, notice that the functions $f_j(x) = (1-x)^j$, $j \geq 1$, have norm 1 but their images under η are unbounded:

$$\begin{aligned}
\eta(f_j) &= -\int_0^1 jx^{-\alpha}(1-x)^{j-1+\alpha}\alpha^{-1}\,dx \\
&= -\frac{\Gamma(1-\alpha)\Gamma(j+\alpha)}{\alpha\,\Gamma(j)}.
\end{aligned}$$

□

Proposition 4.2. *The operator* $\mathcal{B}|_{\mathcal{H}}$ *is closable and its closure,* $\overline{\mathcal{B}|_{\mathcal{H}}}$, *is the generator of a Feller semigroup on* $C[0,1]$.

Proof. We show that $\mathcal{B}|_{\mathcal{H}}$ satisfies the conditions of the Hille-Yosida Theorem. For $\lambda > 0$, Proposition 4.1(i)-(ii) show that the range of $\lambda - \mathcal{B}|_{\mathcal{H}}$ is exactly $\mathcal{H}$. Proposition 4.1(iii) then tells us that this range, as well as the domain of $\mathcal{B}|_{\mathcal{H}}$, is dense in $C[0,1]$.

To establish the positive-maximum principle, suppose that $f \in \mathcal{H}$ has a nonnegative maximum at $y \in [0,1]$. If $y \neq 0$, the tools of differential calculus show that $(\mathcal{B}\big|_{\mathcal{H}} f)(y) \leq 0$, as desired. When $y = 0$, consider the element $F \in L^1[0,1]$ given by

$$F(x) = (f(x) - f(0))x^{-\alpha-1}(1-x)^{\alpha-1}$$

[††]Recall that a Hamel basis for a vector space V is a subset $K \subset V$ such that every element of V can be written uniquely as a finite linear combination of elements of K. In particular, although $\mathcal{H}$ is infinite dimensional, we are only considering finite linear combinations of elements of $\{h_n\}_{n\in\mathbb{N}\setminus\{1\}}$.

almost everywhere. Since $f(x) \le f(0)$ on $[0,1]$, the norm of F is given by

$$\begin{aligned}\|F\|_1 &= \int_0^1 |f(x)-f(0)| x^{-\alpha-1}(1-x)^{\alpha-1}\,dx \\ &= -\int_0^1 (f(x)-f(0)) x^{-\alpha-1}(1-x)^{\alpha-1}\,dx \\ &= -\eta(f).\end{aligned}$$

Recalling that $f \in \mathcal{H} = \ker \eta$, it follows that $F = 0$ almost everywhere. Together with the continuity of f, this implies that $f \equiv f(0)$, and consequently, $(\mathcal{B}\big|_{\mathcal{H}} f)(y) \le 0$. □

The final result in this section is the explicit description of the generator $\overline{\mathcal{B}|_{\mathcal{H}}}$ and its domain $\mathrm{Dom}(\overline{\mathcal{B}|_{\mathcal{H}}})$.

To begin, we define an operator $\hat{\mathcal{L}} \colon C[0,1] \cap C^2(0,1) \to C(0,1)$ by

$$\hat{\mathcal{L}} f(x) = x(1-x)f''(x) - \alpha f'(x).$$

We will write $\hat{\mathcal{L}} f \in C[0,1]$ whenever $\hat{\mathcal{L}} f$ can be continuously extended to $[0,1]$. Recalling the definition of $\mathcal{L}$ and $\mathcal{D}$ from Theorem 1.1, we see that $\mathcal{L}$ is the restriction of $\hat{\mathcal{L}}$ to $\mathcal{D}$. We also define functions $m \colon (0,1] \to \mathbb{R}$ and $s \colon (0,1] \to \mathbb{R}$ by

$$m(x) = \int_1^x t^{-1-\alpha}(1-t)^{\alpha-1}\,dt = -\alpha^{-1} x^{-\alpha}(1-x)^{\alpha}$$

and

$$s(x) = \int_1^x t^{\alpha}(1-t)^{-\alpha}\,dt.$$

Note that $\hat{\mathcal{L}}$ admits the factorization

$$\hat{\mathcal{L}} f = \frac{1}{m'}\left(\frac{f'}{s'}\right)',$$

from which we obtain the formula

$$f(x) - f(c) = \frac{f'(c)}{s'(c)}(s(x)-s(c)) + \int_c^x \int_c^y \hat{\mathcal{L}} f(z) m'(z) dz\, s'(y) dy, \quad x, c \in (0,1). \tag{4.2}$$

Another identity that will be useful is

$$\int_1^y m'(z) dz\, s'(y) = m(y)\, s'(y) = -\alpha^{-1}, \qquad y \in (0,1). \tag{4.3}$$

Proposition 4.3. *The identity* $\overline{\mathcal{B}|_{\mathcal{H}}} = \mathcal{L}$ *holds, where* $\mathcal{L}$ *is as defined in Theorem 1.1.*

Proof. We begin by showing that the following holds:

$$f(x) - f(1) = \int_1^x \int_1^y \mathcal{L} f(z) m'(z) dz\, s'(y) dy, \quad f \in \mathcal{D},\, x \in [0,1]. \tag{4.4}$$

To do this, we will take limits in (4.2). First we take the limit $c \to 1$. The term $\frac{f'(c)}{s'(c)}$ converges to zero due to (D3) (see Theorem 1.1). The limit of the integral is handled by the dominated convergence theorem – a suitable bound follows from the boundedness of $\mathcal{L} f$ and (4.3). This establishes the formula for $x \in (0,1)$. Taking now the limit $x \to 0$ (the dominated convergence theorem can be applied as before) establishes the $x = 0$ case. The $x = 1$ case is trivial.

Now we show that $\mathrm{Dom}(\overline{\mathcal{B}|_{\mathcal{H}}}) \subset \mathcal{D}$. Fixing $f \in \mathrm{Dom}(\overline{\mathcal{B}|_{\mathcal{H}}})$, there exists a sequence $\{f_n\}_{n\geq 1}$ of functions in $\mathcal{H}$ such that

$$f_n \longrightarrow f \qquad \text{and} \qquad \mathcal{B}f_n \longrightarrow \overline{\mathcal{B}|_{\mathcal{H}}}f. \tag{4.5}$$

Noting that $f_n \in \mathcal{D}$ for all n, we can apply (4.4). In this case, the identity $\mathcal{B}f_n = \mathcal{L}f_n$ yields

$$f_n(x) - f_n(1) = \int_1^x \int_1^y \mathcal{B}f_n(z)m'(z)dz\, s'(y)dy, \quad x \in [0,1]. \tag{4.6}$$

Using (4.5) and the dominated convergence theorem, we can take the limit $n \to \infty$ above. A suitable bound follows from the boundedness of the sequence $\{\mathcal{B}f_n\}$ and (4.3). We obtain

$$f(x) - f(1) = \int_1^x \int_1^y \overline{\mathcal{B}|_{\mathcal{H}}}f(z)m'(z)dz\, s'(y)dy, \quad x \in [0,1]. \tag{4.7}$$

Together with the fact that $\overline{\mathcal{B}|_{\mathcal{H}}}f \in C(0,1)$, $m \in C^1(0,1)$ and $s \in C^2(0,1)$, this expression implies that $f \in C^2(0,1)$. Differentiating the expression yields the identity

$$\overline{\mathcal{B}|_{\mathcal{H}}}f = \frac{1}{m'}\left(\frac{f'}{s'}\right)' = \hat{\mathcal{L}}f \quad \text{on } (0,1). \tag{4.8}$$

This shows that f satisfies (D1). To obtain (D2), we recall that

$$\int_0^1 (f_n(x) - f_n(0))x^{-\alpha-1}(1-x)^{\alpha-1}\, dx = 0$$

for all n and extend this to f by taking the limit $n \to \infty$. Once again, we apply the dominated convergence theorem. A preliminary bound can be obtained from (4.3) and (4.6):

$$\begin{aligned}\left|x^{-1}(f_n(x) - f_n(0))\right| &= x^{-1}\left|\int_0^x \int_1^y \mathcal{B}f_n(z)m'(z)dz\, s'(y)dy\right| \\ &\leq x^{-1}\|\mathcal{B}f_n\| \int_0^x \int_y^1 m'(z)dz\, s'(y)dy \\ &= \|\mathcal{B}f_n\|\alpha^{-1}.\end{aligned}$$

The boundedness of the sequence $\{\mathcal{B}f_n\}$ then provides a suitable bound.

To obtain (D3), we differentiate (4.7) and compute

$$\begin{aligned}\left|\frac{f'(x)}{s'(x)}\right| &= \left|\int_1^x \overline{\mathcal{B}|_{\mathcal{H}}}f(z)m'(z)dz\right| \\ &\leq \|\overline{\mathcal{B}|_{\mathcal{H}}}f\| \int_x^1 m'(z)dz \\ &= \|\overline{\mathcal{B}|_{\mathcal{H}}}f\|(-m(x)) \\ &\xrightarrow[x\to 1]{} 0.\end{aligned}$$

We have shown that $\mathrm{Dom}(\overline{\mathcal{B}|_{\mathcal{H}}}) \subset \mathcal{D}$ and $\overline{\mathcal{B}|_{\mathcal{H}}} = \mathcal{L}$ on $\mathrm{Dom}(\overline{\mathcal{B}|_{\mathcal{H}}})$ (see (4.8)). Therefore, it only remains to show that $\mathrm{Dom}(\overline{\mathcal{B}|_{\mathcal{H}}}) = \mathcal{D}$. From Lemma 19.12 in [13], it suffices to show that $\mathcal{L}$ satisfies the positive maximum principle. To this end, suppose that $f \in \mathcal{D}$ has a nonnegative maximum at $y \in [0,1]$. If $y \neq 1$, then the desired inequality can be obtained as in Proposition 4.2. If $y = 1$, we use (D1), L'Hôpital's rule, (D3), and (4.3) to establish the existence of limits

$$\begin{aligned}
\mathcal{L}f(1) &= \lim_{x\to 1} \mathcal{L}f(x) \\
&= \lim_{x\to 1} \frac{1}{m'(x)} \left(\frac{f'}{s'}\right)'(x) \\
&= \lim_{x\to 1} \frac{1}{m(x)} \frac{f'(x)}{s'(x)} \\
&= \lim_{x\to 1} -\alpha f'(x) \\
&= \lim_{x\to 1} -\alpha \, \frac{f(x)-f(1)}{x-1} \\
&= -\alpha f'(1).
\end{aligned}$$

Noticing that $f'(1) \geq 0$ concludes the proof. □

5 Generator relations

In this section, we show that our generators satisfy the commutation relations appearing in Theorem 3.3. Here, we rely on an alternative description of the limiting generator in terms of Bernstein polynomials.

For $k \geq 0$, let $\mathcal{P}_k$ be the subspace of $\mathcal{P}$ consisting of polynomials with degree at most k. Similarly, define

$$\mathcal{H}_k = \mathcal{H} \cap \mathcal{P}_k, \qquad k \geq 0.$$

Recall the Bernstein polynomials

$$b_{i,k}(x) = \binom{k}{i} x^i (1-x)^{k-i}, \quad i \in \mathbb{Z},\, k \geq 0.$$

Note that $b_{i,k} \equiv 0$ whenever $i < 0$ or $i > k$. For each $k \geq 0$, the collection $\{b_{i,k}\}_{i=0}^k$ forms a basis of $\mathcal{P}_k$ and a partition of unity – that is, $\sum_{i=0}^k b_{i,k} \equiv 1$. We also have the relations

$$b'_{i,k} = k(b_{i-1,k-1} - b_{i,k-1}), \tag{5.1}$$

$$b_{i,k} = \tfrac{k+1-i}{k+1}\, b_{i,k+1} + \tfrac{i+1}{k+1}\, b_{i+1,k+1}, \tag{5.2}$$

and

$$x(1-x)\, b_{i,k} = \tfrac{(i+1)(k+1-i)}{(k+1)(k+2)}\, b_{i+1,k+2}, \tag{5.3}$$

which hold whenever the relevant quantities are defined.

For $n \geq 1$, we define a transition kernel from $[0,1]$ to $[n]$ by

$$K_n(x,i) = b_{i,n}(x) + \nu_n^{(\alpha,\alpha)}(i) b_{0,n}(x).$$

Proposition 5.1. *Let $n \geq 1$. As an operator from $C([n])$ to $C[0,1]$, K_n is injective and*

$$\mathcal{H}_n = \left\{ \sum_{j=0}^n c_j b_{j,n} : c_0, \ldots, c_n \in \mathbb{R},\;\; c_0 = \sum_{j=1}^n \nu_n^{(\alpha,\alpha)}(j) c_j \right\} \tag{5.4}$$

$$= \textit{range } K_n. \tag{5.5}$$

Proof. Let $n \geq 1$. From the independence of the Bernstein polynomials and the identity

$$\text{range } K_n = \text{span}\big\{ b_{i,n}(x) + \nu_n^{(\alpha,\alpha)}(i) b_{0,n}(x) \big\}_{i=1}^n,$$

it follows that the range of K_n is an n-dimensional space. As a result, K_n is injective. Observing that the right hand side of (5.4) has dimension at most n and contains the range of K_n, it follows that these two spaces are equal. Since $\mathcal{H}_n$ also has dimension n (see Proposition 4.1(ii)), it only remains to show that the range of K_n is contained in $\mathcal{H}_n$. The containment in $\mathcal{P}_n$ is clear. For the containment in $\mathcal{H}$, we simply compute, for $i \in [n]$,

$$
\begin{aligned}
&\eta(b_{i,n}(x) + \nu_n^{(\alpha,\alpha)}(i) b_{0,n}(x)) \\
&\qquad = \binom{n}{i} \int_0^1 x^{i-\alpha-1}(1-x)^{n-i+\alpha-1}\, dx - n\alpha^{-1}\nu_n^{(\alpha,\alpha)}(i) \int_0^1 x^{-\alpha}(1-x)^{n-1+\alpha}\, dx \\
&\qquad = \binom{n}{i} \frac{\Gamma(i-\alpha)\Gamma(n-i+\alpha)}{\Gamma(n)} - n\alpha^{-1}\nu_n^{(\alpha,\alpha)}(i) \frac{\Gamma(1-\alpha)\Gamma(n+\alpha)}{\Gamma(n+1)} \\
&\qquad = 0.
\end{aligned}
$$

□

Proposition 5.2. *The action of $\mathcal{B}$ on the Bernstein polynomials is given by*

$$\mathcal{B} b_{i,n} = n(n+1) \sum_{k=0}^{n} (r_{k,i} - \mathbb{1}(k=i))\, b_{k,n}, \qquad 0 \le i \le n.$$

Proof. Let $n \ge 2$ and $0 \le i \le n$. Applying (5.1) twice, we see that

$$
\begin{aligned}
b''_{i,n} &= n(b'_{i-1,n-1} - b'_{i,n-1}) \\
&= n(n-1)(b_{i-2,n-2} - 2b_{i-1,n-2} + b_{i,n-2}).
\end{aligned}
$$

Applying now (5.3), we have that

$$
\begin{aligned}
&x(1-x)b''_{i,n}(x) \\
&\quad = n(n-1)\Big(\tfrac{(i-1)(n+1-i)}{(n-1)n}\, b_{i-1,n}(x) - \tfrac{2i(n-i)}{(n-1)n}\, b_{i,n}(x) + \tfrac{(i+1)(n-1-i)}{(n-1)n}\, b_{i+1,n}(x) \Big) \qquad (5.6) \\
&\quad = (i-1)(n+1-i)\, b_{i-1,n}(x) - 2i(n-i)\, b_{i,n}(x) + (i+1)(n-1-i)\, b_{i+1,n}(x)
\end{aligned}
$$

Using (5.1) and (5.2), we find that

$$
\begin{aligned}
b'_{i,n} &= n(b_{i-1,n-1} - b_{i,n-1}) \\
&= n\Big(\tfrac{n+1-i}{n}\, b_{i-1,n} + \tfrac{i}{n}\, b_{i,n} - \tfrac{n-i}{n}\, b_{i,n} - \tfrac{i+1}{n}\, b_{i+1,n} \Big) \qquad (5.7) \\
&= (n+1-i)\, b_{i-1,n} + (2i-n)\, b_{i,n} - (i+1)\, b_{i+1,n}.
\end{aligned}
$$

As a result,

$$
\begin{aligned}
\mathcal{B} b_{i,n} &= (i-1-\alpha)(n+1-i)\, b_{i-1,n} - (\alpha(2i-n) + 2i(n-i))\, b_{i,n} \\
&\quad + (i+1)(n-1-i+\alpha)\, b_{i+1,n} \\
&= n(n+1)\, (r_{i-1,i}\, b_{i-1,n} + (r_{i,i} - 1)\, b_{i,n} + r_{i+1,i}\, b_{i+1,n}) \\
&= n(n+1) \sum_{k=i-1}^{i+1} (r_{k,i} - \mathbb{1}(k=i))\, b_{k,n}.
\end{aligned}
$$

Recalling that $r_{k,i} - \mathbb{1}(k=i)$ is zero unless $i-1 \le k \le i+1$ and $b_{k,n} \equiv 0$ unless $0 \le k \le n$, we can change the lower and upper limits of the sum to 0 and n, respectively. This establishes the $n \ge 2$ case. When $n = 1$, we observe that (5.7) still holds and the first and last quantities of (5.6) are still equal. When $n = 0$, the claim is trivial. □

Proposition 5.3. *For $n \ge 1$, the following relation holds on $C([n])$:*

$$\mathcal{B} K_n = K_n\, n(n+1)(Q_n^{(\alpha,0)} - \mathbf{1}).$$

Proof. Let $n \geq 1$ and $i \in [n]$. Define $e_i \colon [n] \to \mathbb{R}$ by $e_i = \mathbb{1}(i = \cdot)$. From Proposition 5.2, we have that

$$\begin{aligned}
&n^{-1}(n+1)^{-1}\mathcal{B}K_n e_i \\
&\quad = n^{-1}(n+1)^{-1}\mathcal{B}(b_{i,n} + b_{0,n}\nu_n^{(\alpha,\alpha)}(i)) \\
&\quad = \sum_{k=0}^{n}(r_{k,i} - \mathbb{1}(k=i) + \nu_n^{(\alpha,\alpha)}(i)(r_{k,0} - \mathbb{1}(k=0)))\, b_{k,n} \\
&\quad = (r_{0,i} + \nu_n^{(\alpha,\alpha)}(i)(r_{0,0} - 1))\, b_{0,n} + \sum_{k=1}^{n}(r_{k,i} - \mathbb{1}(k=i) + \nu_n^{(\alpha,\alpha)}(i) r_{1,0}\mathbb{1}(k=1))\, b_{k,n}.
\end{aligned}$$

On the other hand, Proposition 2.4 gives us that

$$\begin{aligned}
K_n(Q_n^{(\alpha,0)} - \mathbf{1})e_i &= \sum_{k=1}^{n}(b_{k,n} + b_{0,n}\nu_n^{(\alpha,\alpha)}(k))((Q_n^{(\alpha,0)} - \mathbf{1})e_i)(k) \\
&= \sum_{k=1}^{n}(b_{k,n} + b_{0,n}\nu_n^{(\alpha,\alpha)}(k))(Q_n^{(\alpha,0)} - \mathbf{1})(k,i) \\
&= \sum_{k=1}^{n}(b_{k,n} + b_{0,n}\nu_n^{(\alpha,\alpha)}(k))(r_{k,i} - \mathbb{1}(i=k) + \nu_n^{(\alpha,\alpha)}(i) r_{1,0}\mathbb{1}(k=1)).
\end{aligned}$$

To show that the two expressions are equal, it will suffice to show that the coefficients of $b_{k,n}$ are the same in each. For $k \geq 1$, this is immediate. For $k = 0$, we observe that each of the above functions lies in $\mathcal{H}_n$ (see Proposition 4.1 and (5.5)) and apply (5.4). □

6 The convergence argument

In this section, we verify the convergence condition appearing in Theorem 3.3. We rely on a description of the inverse of the transition operator K_n in terms of a variant of the Bernstein polynomials.

These variants fall into the class of degenerate Bernstein polynomials [14] and are given by

$$b^*_{i,k,n}(x) = \binom{k}{i}\frac{(nx)^{\downarrow i}(n-nx)^{\downarrow(k-i)}}{n^{\downarrow k}}, \quad 0 \leq i \leq k \leq n.$$

Proposition 6.1. *For $k \geq i \geq 0$, we have the expansions*

$$b_{i,k} = \sum_{j=0}^{n} b^*_{i,k,n}\big(\tfrac{j}{n}\big) b_{j,n}, \quad n \geq k.$$

Proof. The expansions of a Bernstein polynomial in the Bernstein bases are given in Equation (2) in [19]. Let us verify that the coefficients in those expansions match the coefficients in the above expansions. Fix $n \geq k \geq i \geq 0$. The coefficient of $b_{j,n}$ in the above expansion is given by

$$b^*_{i,k,n}\left(\frac{j}{n}\right) = \binom{k}{i}\frac{j^{\downarrow i}(n-j)^{\downarrow(k-i)}}{n^{\downarrow k}}.$$

When $j < i$ or $j > n-k+i$, it is clear that this coefficient is zero. If instead

$i \leq j \leq n-k+i$, this coefficient is reduces to

$$\begin{aligned}\binom{k}{i}\frac{j^{\downarrow i}(n-j)^{\downarrow(k-i)}}{n^{\downarrow k}} &= \binom{k}{i}\frac{\frac{j!}{(j-i)!}\frac{(n-j)!}{(n-j-k+i)!}}{\frac{n!}{(n-k)!}}\\ &= \binom{k}{i}\frac{\frac{(n-k)!}{(j-i)!(n-j-k+i)!}}{\frac{n!}{j!(n-j)!}}\\ &= \binom{k}{i}\frac{\binom{n-k}{j-i}}{\binom{n}{j}}.\end{aligned}$$

In either case, this coefficient agrees with the coefficient in [19]. ☐

Let $\iota_n \colon [n] \to [0,1]$ be defined by $j \mapsto \frac{j}{n}$ and $\rho_n \colon C[0,1] \to C[n]$ be the associated projection, $f \mapsto f \circ \iota_n$.

Proposition 6.2. *For $n \geq k \geq i \geq 1$, we have the identity*

$$K_n\rho_n(b^*_{i,k,n} + \nu_k^{(\alpha,\alpha)}(i)b^*_{0,k,n}) = b_{i,k} + \nu_k^{(\alpha,\alpha)}(i)b_{0,k}.$$

Proof. It follows from definition that

$$K_n\rho_n(b^*_{i,k,n} + \nu_k^{(\alpha,\alpha)}(i)b^*_{0,k,n}) = \sum_{j=1}^{n}(b_{j,n} + \nu_n^{(\alpha,\alpha)}(j)b_{0,n})\big(b^*_{i,k,n}\big(\tfrac{j}{n}\big) + \nu_k^{(\alpha,\alpha)}(i)b^*_{0,k,n}\big(\tfrac{j}{n}\big)\big).$$

Meanwhile, Proposition 6.1 gives us the expansion

$$b_{i,k} + \nu_k^{(\alpha,\alpha)}(i)b_{0,k} = \sum_{j=0}^{n}\big(b^*_{i,k,n}\big(\tfrac{j}{n}\big) + \nu_k^{(\alpha,\alpha)}(i)b^*_{0,k,n}\big(\tfrac{j}{n}\big)\big)b_{j,n}.$$

Upon comparison, we find that the coefficient of $b_{j,n}$ is the same in both expressions whenever $j \geq 1$. Since both functions lie in $\mathcal{H}_n$, the coefficients of $b_{0,n}$ must agree as well (see (5.4)). As a result, the two functions are equal. ☐

Proposition 6.3. *For $k \geq i \geq 0$, we have the convergence*

$$b^*_{i,k,n} \xrightarrow[n\to\infty]{} b_{i,k}.$$

Proof. We write

$$\begin{aligned}b^*_{i,k,n}(x) &= \binom{k}{i}\frac{1}{n^{\downarrow k}}\prod_{r=0}^{i-1}(nx-r)\prod_{s=0}^{k-i-1}(n-nx-s)\\ &= \binom{k}{i}\frac{n^k}{n^{\downarrow k}}\prod_{r=0}^{i-1}\left(x-\frac{r}{n}\right)\prod_{s=0}^{k-i-1}\left(1-x-\frac{s}{n}\right),\end{aligned}$$

and handle each factor separately. The constants $\frac{n^k}{n^{\downarrow k}}$ converge to 1 and each factor in a product converges to either $u(x) = x$ or $v(x) = 1-x$. ☐

Proposition 6.4. *Let $f \in \mathcal{H}$ and fix $m \geq 1$ such that $f \in \mathcal{H}_m$. Then we have the convergence*

$$(K_n^{-1} - \rho_n)f \xrightarrow[n\to\infty]{n\geq m} 0$$

in the sense of Definition 3.1.

Proof. It suffices to consider the case when $f = b_{i,k} + \nu_k^{(\alpha,\alpha)}(i)b_{0,k}$ for some i and k satisfying $1 \le i \le k$. Defining $f_n = b^*_{i,k,n} + \nu_k^{(\alpha,\alpha)}(i)b^*_{0,k,n}$ for $n \ge 1$, it follows from Proposition 6.2 that

$$(K_n^{-1} - \rho_n)f = \rho_n(f_n - f).$$

Since the ρ_n are uniformly bounded, the result follows from Proposition 6.3. □

7 Semigroup relations from generator relations

In this section, we provide general conditions under which commutation relations involving generators lead to the corresponding relations for their semigroups.

Theorem 7.1. *Let A and B be the generators of the Feller semigroups V_t and W_t, respectively, and let $\mathcal{E}$ and $\mathcal{F}$ denote their respective domains. Suppose that there is a subspace $E \subset \mathcal{E}$, a linear operator $L\colon \overline{E} \to \overline{\mathcal{F}}$, and a set $I \subset (0,\infty)$ such that*

(i) L is bounded,

(ii) I is unbounded,

(iii) $E \subset (\lambda - A)E$ for $\lambda \in I$, and

(iv) $LA = BL$ on E.

Then $LV_t = W_tL$ on $\overline{E}$ for each $t \ge 0$.

Proof. Fix $\lambda \in I$ and let R^A_λ and R^B_λ be the resolvent operators corresponding to A and B respectively. It follows from (iii) that E is invariant under R^A_λ. Combining this with (iv), we obtain the following relation on E:

$$\begin{aligned} R^B_\lambda L &= R^B_\lambda L(\lambda - A)R^A_\lambda \\ &= R^B_\lambda(\lambda - B)LR^A_\lambda \\ &= LR^A_\lambda. \end{aligned}$$

It then follows easily that

$$L\lambda(\lambda R^A_\lambda - I) = \lambda(\lambda R^B_\lambda - I)L \quad \text{on } E,$$

or equivalently, $LA_\lambda = B_\lambda L$ on E, where A_λ and B_λ are the Yosida approximations of A and B respectively. Noting that E is invariant under A_λ, this extends to nonnegative integers k:

$$LA^k_\lambda = B^k_\lambda L \quad \text{on } E.$$

Applying now (i), we have for $f \in E$ and $t \ge 0$ the identity

$$\begin{aligned} Le^{tA_\lambda}f &= L\sum_{k=0}^\infty \frac{t^k}{k!}(A^k_\lambda f) \\ &= \sum_{k=0}^\infty \frac{t^k}{k!}(LA^k_\lambda f) \\ &= \sum_{k=0}^\infty \frac{t^k}{k!}(B^k_\lambda Lf) \\ &= e^{tB_\lambda}Lf. \end{aligned}$$

Letting λ become arbitrarily large (see (ii)) yields $LV_tf = W_tLf$. This establishes the result on E. The extension to $\overline{E}$ follows from the boundedness of L. □

Corollary 7.2. *Let A and B be the generators of the Feller semigroups V_t and W_t, respectively, and let $\mathcal{E}$ and $\mathcal{F}$ denote their respective domains. Suppose that there is a subspace $E \subset \mathcal{E}$, a linear operator $L\colon E \to \overline{\mathcal{F}}$, and a filtration of E by finite dimensional spaces $\{E_k\}_{k\geq 1}$ such that*

(i) $AE_k \subset E_k$ *for all k, and*

(ii) $LA = BL$ *on E.*

Then $LV_t = W_tL$ on E for each $t \geq 0$.

Proof. Let $k \geq 1$. It follows from (i) that E_k is invariant under the injective operators $\{\lambda - A\}_{\lambda>0}$. Together with the fact that E_k is finite-dimensional, this implies that

$$(\lambda - A)E_k = E_k, \quad \lambda > 0.$$

Letting $L_k\colon E_k \to \overline{\mathcal{F}}$ denote the restriction of L to E_k, it follows from (i) and (ii) that

$$L_kA = BL_k \quad \text{on } E_k.$$

Since E_k is finite-dimensional, L_k is bounded and $\overline{E_k} = E_k$. Applying Theorem 7.1, we find that $LV_t = W_tL$ on E_k for each $t \geq 0$. Taking a union over k extends the identity to E. □

8 Proofs of main results

Proof of Theorem 1.4. The first claim was proved in Proposition 2.4. For the second claim, we appeal to Corollary 7.2. We take $A = n(n+1)(Q_n^{(\alpha,0)} - \mathbf{1})$, $B = \mathcal{L}$, $L = K_n$, and $E = C([n]) = E_k$ for all k. The containment $AE_k \subset E_k$ holds trivially and the identity $LA = BL$ was established in Proposition 5.3. Applying Corollary 7.2, we obtain the desired identity in terms of transition operators, which implies the same relation in terms of transition kernels. □

Proof of Theorem 1.3. The claim about the existence of initial distributions for $\mathbf{X}_n^{(\alpha,0)}$ follows from Theorem 1.4. The second claim follows from applying Theorem 3.3 with $E = [0,1]$, $A = \mathcal{L}$, $D = \mathcal{H}$, $E_n = [n]$, $Z_n = Y_n^{(\alpha,\theta)}$, $\gamma_n(j) = \frac{j}{n}$, $D_n = \mathcal{H}_n$, $L_n = K_n^{-1}$, $\delta_n^{-1} = n(n+1)$, and $\varepsilon_n^{-1} = n^2$. To verify that A is the generator of a conservative Feller semigroup on $C[0,1]$, D is a core for A, and D is invariant under A, we appeal to Propositions 4.3, 4.2, and 4.1. Condition (a) can be obtained from the identity in Proposition 5.3 by recalling that K_n is injective (see Proposition 5.1) and that each f in $D = \mathcal{H}$ lies in $D_n = \mathcal{H}_n$ for large n. Condition (b) is exactly the result of Proposition 6.4. □

Proof of Theorem 1.1. Define $\iota : \mathcal{C} \to \mathcal{U}$ by

$$\iota(\sigma) = \left(0, \frac{\sigma_1}{|\sigma|}\right) \cup \left(\frac{\sigma_1}{|\sigma|}, \frac{\sigma_1+\sigma_2}{|\sigma|}\right) \cup \ldots \cup \left(\frac{|\sigma| - \sigma_{\ell(\sigma)}}{|\sigma|}, 1\right).$$

From [20, Theorem 1.3], we have that if

$$\iota(\mathbf{X}_n^{(\alpha,\theta)}(0)) \Longrightarrow \mathbf{X}^{(\alpha,\theta)}(0),$$

then

$$\left(\iota(\mathbf{X}_n^{(\alpha,\theta)}(\lfloor n^2t \rfloor))\right)_{t\geq 0} \Longrightarrow \left(\mathbf{X}^{(\alpha,\theta)}(t)\right)_{t\geq 0},$$

where $\lfloor a \rfloor$ is the integer part of a and the convergence is in distribution on the Skorokhod space $D([0,\infty),\mathcal{U})$, where the metric on $\mathcal{U}$ is given by the Hausdorff distance between the complements (complements being taken in $[0,1]$). If ξ were continuous, the result would follow immediately, but ξ is discontinuous. However, it is straightforward to show that if $u_n \to u$ in $\mathcal{U}$ and $\xi(u_n) \to c > 0$, then $\xi(u) = c$.

Assuming now that $\mathbf{X}_n^{(\alpha,0)}$ is running in stationarity, the fact that $\iota(\mathbf{X}_n^{(\alpha,0)}(0))$ converges in distribution to an $(\alpha,0)$ Poisson-Dirichlet interval partition distribution follows from [18] and the fact that $\phi(\mathbf{X}_n^{(\alpha,0)})$ is a Markov chain follows from Theorem 1.4. Observe that $(p^{\uparrow}_{(\alpha,0)})^{n-1}((1),\cdot)$ is the stationary distribution of $\mathbf{X}_n^{(\alpha,0)}$ and, in the $(\alpha,0)$ ordered Chinese Restaurant Process growth step, no new table is ever created at the start of the list. Thus, for every k, $\phi(\mathbf{X}_n^{(\alpha,0)}(k))$ is distributed like the size of the table containing 1 in the usual $(\alpha,0)$ Chinese Restaurant Process after n customers are seated, see [17]. Consequently, since our chain is stationary, for each t,

$$\frac{1}{n}\phi(\mathbf{X}_n^{(\alpha,0)}(\lfloor n^2t\rfloor)) = \xi(\iota(\mathbf{X}_n^{(\alpha,0)}(\lfloor n^2t\rfloor))) =_d \xi(\iota(\mathbf{X}_n^{(\alpha,0)}(0))) \Rightarrow W,$$

where W has a Beta$(1-\alpha,\alpha)$ distribution, see [17].

Therefore, from Theorem 1.3 with F as defined there and $F(0) =_d W$, passing to a subsequence if necessary, and using the Skorokhod representation theorem, we may assume that

$$\Big(\Big(\iota(\mathbf{X}_n^{(\alpha,0)}(\lfloor n^2t\rfloor)), \xi(\iota(\mathbf{X}_n^{(\alpha,0)}(\lfloor n^2s\rfloor)))\Big)\Big)_{t,s\geq 0} \xrightarrow{a.s.} \Big((\mathbf{X}^{(\alpha,0)}(t), F(s))\Big)_{t,s\geq 0}$$

in $D([0,\infty),\mathcal{U}) \times D([0,\infty),[0,1])$. Fix $t \geq 0$. Since Feller processes have no fixed discontinuities, F is almost surely continuous at t and, therefore, since convergence in $D([0,\infty),\mathcal{U})$ implies convergence at continuity points,

$$\xi(\iota(\mathbf{X}_n^{(\alpha,0)}(\lfloor n^2t\rfloor))) \xrightarrow{a.s.} F(t).$$

Since $F(t) =_d W$, $\mathbb{P}(F(t) > 0) = 1$ and, since

$$\iota(\mathbf{X}_n^{(\alpha,0)}(\lfloor n^2t\rfloor)) \xrightarrow{a.s.} \mathbf{X}^{(\alpha,0)}(t),$$

it follows that $F(t) =_{a.s.} \xi(\mathbf{X}^{(\alpha,0)}(t))$. Consequently, $F(t)$ is a modification of $\xi(\mathbf{X}^{(\alpha,0)}(t))$ and since F has a Feller semigroup, so does $\xi(\mathbf{X}^{(\alpha,0)})$. □

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