Slope of Orderable Dehn Filling of Two-Bridge Knots

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ABSTRACT

In this paper, we study the Riley polynomial of double twist knots with higher genus. Using the root of the Riley polynomial, we compute the range of rational slope r such that r-filling of the knot complement has left-orderable fundamental group. Further more, we make a conjecture about left-orderable surgery slopes of two-bridge knots.

Keywords: left-orderability, two-bridge knot, character variety

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1. Introduction

The study of Riley polynomial of two-bridge knots dates back to [1]. In a series of papers by Ryoto Hakamata and Masakazu Teragaito [2,3], they studied a special class of two-bridge knot, the double twisted knot $J(k, l)$ (see Figure 1). Under their convention, the half-twists in the upper box are left-handed (resp. right-handed) if $k > 0$ (resp. $k < 0$), while those in the lower box are right-handed (resp. left-handed) if $l > 0$ (resp. $l < 0$). By symmetry, $J(k, l)$ is isotopic to $J(-l, -k)$. For example, $J(3, 2)$ is the knot 5_2, while $J(-3, 2)$ is the figure-eight knot 4_1. We will follow this convention in the paper.

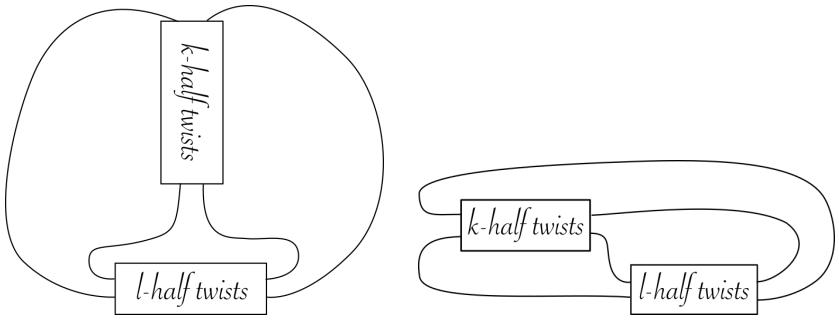

Fig. 1: The double twist knot $J(k, l)$

In [3], the special case of the double twisted knot $K = J(k, l)$ with $k = 2m$ and $l = 2n$ half twists is studied. And a property of the fundamental group of the Dehn filling of K called left-orderability is discussed. A non-trivial group G is called left-orderable if there is a strict total ordering invariant under left multiplication. That is to say, if $f < g$ then $hf < hg$ for any $f, g, h \in G$. We call a 3-manifold orderable if its fundamental group is left-orderable. The main result of [3] states:

Theorem 1.1. *[3, Theorem 1.1] Let $K = J(2m, 2n)$ be the hyperbolic genus one 2-bridge knot in the 3-sphere S^3 as illustrated in Figure 1. Let I be the interval defined by*

$$I = \begin{cases} (-4n, 4m) & \text{if } m > 0 \text{ and } n > 0, \\ (4m, -4n) & \text{if } m < 0 \text{ and } n < 0, \\ [0, \max\{4m, -4n\}) & \text{if } m > 0 \text{ and } n < 0, \\ (\min\{4m, -4n\}, 0] & \text{if } m < 0 \text{ and } n > 0. \end{cases}$$

Then any slope in I is left-orderable. That is, $\pi_1(K(r))$ is left-orderable.

As Tran informed the author, the above result is also proved independently by him in [4] using a different method. And in Tran's recent paper [5], he improved the above result by showing slopes in $(-\infty, 0)$ also give rise to orderable Dehn fillings.

While in the papers [3], the authors mostly focused on the double twist knot with $2m$ and $2n$ half twists, which has genus 1. In this paper, I computed the Riley polynomial of the double twist knot with $2m + 1$ and $2n$ half twists $J(2m + 1, 2n)$, which has higher genus. By studying the root of the Riley polynomial, I proved the following theorem of double twist knots.

Theorem 1.2. *For rational $r \in I$, the Dehn filling of $J(2m + 1, 2n)$ of slope r is orderable, where*

$$I = \begin{cases} (-\infty, 1) & \text{if } m > 0 \text{ and } n > 0, \text{ or } m < -1 \text{ and } n > 1, \\ (-1, \infty) & \text{if } m < -1 \text{ and } n < 0, \text{ or } m > 0 \text{ and } n < -1, \\ [0, 4n) & \text{if } m < -1 \text{ and } n > 0, \\ (4n, 0] & \text{if } m > 0 \text{ and } n < 0. \end{cases}$$

Remark 1.3. When $m = -1, 0$ or $n = 0$, $J(-1, 2n)$, $J(1, 2n)$ and $J(2m + 1, 0)$ become torus knots. So these three cases are excluded. All two-bridge links except the torus links are hyperbolic (See for example [6, Corollary 2]). Therefore as a subset of two-bridge links, all the double twist knots in our theorem are hyperbolic.

Remark 1.4. After the author posted this paper on arxiv, Tran informed the author that in version 3 of his paper [5], he independently computed the interval in the first two cases of Theorem 1.2.

Section 2-7 of this paper are devoted to proving the theorem. In Section 8, we use computer program to check the theorem for some examples. In Section 9, we make connections with general two-bridge knots.

2. $\mathbf{SL_2(\mathbb{R})}$ Representations of the Fundamental Group of $J(k, 2n)$

We will mostly follow the notations in [3].

Let us look at the link diagram of $J(k, l)$ in Figure 1. The link diagram $J(k, l)$ is a knot (called the double twist knot) if and only if either k or l is even. Then one may apply symmetries of the knot to assume that $l = 2n$ is even. Moreover, since $J(k, 2n)$ and $J(-k, -2n)$ differ by mirror, then in our computation, we can assume $n > 0$ without loss of generality.

The fundamental group of $J(k, 2n)$ has the following presentation $\pi_1(S^3 - J(k, 2n)) = \langle x, y | w^n x = y w^n \rangle$, where

$$w = \begin{cases} (xy^{-1})^m (x^{-1}y)^m, & \text{if } k = 2m, \\ (xy^{-1})^m xy (x^{-1}y)^m, & \text{if } k = 2m+1. \end{cases} \tag{2.1}$$

In the first case $J(2m, 2n)$ corresponds to $K(m, n)$ as in [3]. We are interested in the second case $J(2m + 1, 2n)$, double twist knot with $2m + 1$ half twists and $2n$ half twists, which has genus n when $m \neq -1$ (see e.g. [7, Corollary 8.7.5]). Under the above presentation of the fundamental group, the meridian $\mathcal{M} = x$ and the longitude $\mathcal{L} = w_*^n w^n x^{-4n}$, where $w_* = (yx^{-1})^m yx(y^{-1}x)^m$ is obtained from w by reversing the order of letters. See Section 4 of [8] for more details about the presentation of $\pi_1(S^3 - J(k, 2n))$ and $\mathcal{L}$. But remark that there is a slight difference in the notation and choice of framing.

In this paper, we will study the case when $k = 2m + 1$ in $J(k, l)$.

Let ρ be a $\mathrm{SL}_2(\mathbb{R})$ representation of $\pi_1(S^3 - J(2m + 1, 2n))$. Then by taking conjugation, we can assume that

$$\rho(x) = \begin{pmatrix} \sqrt{t} & 1/\sqrt{t} \\ 0 & 1/\sqrt{t} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \sqrt{t} & 0 \\ -s\sqrt{t} & 1/\sqrt{t} \end{pmatrix}, \tag{2.2}$$

where s and t are real numbers. To compute the image of $\mathcal{M}$ and $\mathcal{L}$ under ρ, we will need some computational trick. Let f_m and g_m be two sequences of polynomials in the variable s defined by inductive relations $f_{m+2} - (s + 2)f_{m+1} + f_m = 0$ and $g_{m+2} - (s + 2)g_{m+1} + g_m = 0$, with initial conditions $f_0 = g_0 = 1$, $f_1 = s + 1$ and $g_1 = s + 2$. Set $f_{-m} = f_{m-1}$ for $m \geq 1$. Set $g_{-1} = 0$ and $g_{-m} = -g_{m-2}$ for $m \geq 2$. So both f_m and g_m are defined for any $m \in \mathbb{Z}$.

Lemma 2.1. *[3, Lemma 2.2] The closed formulas for f_m and g_m are*

$$f_m = \sum_{i=0}^{m} \binom{m+i}{m-i} s^i, \quad g_m = \sum_{i=0}^{m} \binom{m+1+i}{m-i} s^i.$$

In particular, all coefficients of f_m and g_m are positive integers, and the degree of both f_m and g_m are m. Also, f_m and g_m are monic.

Then it is easy to check that f_m and g_m satisfy the following properties.

Lemma 2.2. *[3, Lemma 2.3]*

(1) $f_m + g_{m-1} = g_m$

(2) $f_m + sg_m = f_{m+1}$

(3) $f_m^2 = sg_m g_{m-1} + 1$

Now we can compute the image of w under ρ.

Proposition 2.3. *Let $W = \rho(w)$, then*

$$W = \begin{pmatrix} tf_m^2 - sg_m^2 & \frac{f_m g_m}{t} - f_m g_{m-1} \\ stf_m g_{m-1} - sf_m g_m & \frac{f_m^2}{t} - sg_{m-1}^2 \end{pmatrix}. \tag{2.3}$$

Proof. We compute directly,

$$\rho(xy^{-1}) = \begin{pmatrix} s+1 & 1 \\ s & 1 \end{pmatrix},$$

$$\rho(x^{-1}y) = \begin{pmatrix} s+1 & -1/t \\ -st & 1 \end{pmatrix},$$

$$\rho(xy) = \begin{pmatrix} t-s & 1/t \\ -s & 1/t \end{pmatrix}.$$

Then

$$\rho(xy^{-1})^m = \begin{pmatrix} f_m & g_{m-1} \\ f_m - f_{m-1} & g_{m-1} - g_{m-2} \end{pmatrix} = \begin{pmatrix} f_m & g_{m-1} \\ sg_{m-1} & f_{m-1} \end{pmatrix},$$

$$\rho(x^{-1}y)^m = \begin{pmatrix} f_m & -\frac{f_{m+1} - (s+1)f_m}{st} \\ -stg_{m-1} & g_m - (s+1)g_{m-1} \end{pmatrix} = \begin{pmatrix} f_m & -\frac{g_{m-1}}{t} \\ -stg_{m-1} & f_{m-1} \end{pmatrix}.$$

So

$$W = \rho(w) = \rho(xy^{-1})^m \rho(xy)\rho(x^{-1}y)^m = \begin{pmatrix} tf_m^2 - sg_m^2 & \frac{f_m g_m}{t} - f_m g_{m-1} \\ stf_m g_{m-1} - sf_m g_m & \frac{f_m^2}{t} - sg_{m-1}^2 \end{pmatrix}. \square$$

Denote

$$W = \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix}.$$

Let z and $1/z$ be the eigenvalues of W. Set $\tau = \mathrm{tr}W = z + 1/z$ and $\tau_k = z^{k-1} + z^{k-3} + \cdots + z^{3-k} + z^{1-k}$. Then τ_k satisfies $\tau_{k+2} - \tau\tau_{k+1} + \tau_k = 0$.

Lemma 2.4. *[3, Lemma 3.5] For* $W^n = \begin{pmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{pmatrix} = \rho(w^n)$*, we have*

$$W^n = \begin{pmatrix} w_{1,1}\tau_n - \tau_{n-1} & w_{1,2}\tau_n \\ w_{2,1}\tau_n & \tau_{n+1} - w_{1,1}\tau_n \end{pmatrix}.$$

The proof of [3, Lemma 3.5] does not depend on the explicit expression of W. So this lemma still holds in our case.

3. Longitude

In this section, we follow the procedures in Section 5 of [3] to compute the image of the longitude $\mathcal{L}$ under certain representation ρ_s, which will be defined below.

Conjugating ρ by $Q = \begin{pmatrix} t-1 & 1 \\ 0 & \sqrt{t} - 1/\sqrt{t} \end{pmatrix}$, we can diagonalize $\rho(x)$. Call this new representation ρ_s. So

$$\rho_s(x) = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}, \text{ and } \rho_s(y) = \begin{pmatrix} \frac{t-s-1}{\sqrt{t}-\frac{1}{\sqrt{t}}} & \frac{s}{(\sqrt{t}-\frac{1}{\sqrt{t}})^2} - 1 \\ -s & \frac{s+1-1/t}{\sqrt{t}-\frac{1}{\sqrt{t}}} \end{pmatrix}.$$

We will compute $\rho_s(\mathcal{L})$. Let

$$\rho_s(w^n) = \begin{pmatrix} v_{1,1} & v_{2,1} \\ v_{1,2} & v_{2,2} \end{pmatrix}, \text{ and } \sigma = \frac{s(\sqrt{t} - \frac{1}{\sqrt{t}})^2}{(\sqrt{t} - \frac{1}{\sqrt{t}})^2 - s}.$$

We will first compute $\rho_s(w_*^n)$.

Lemma 3.1.

$$\rho_s(w_*^n) = \begin{pmatrix} v_{1,1} & \frac{v_{2,1}}{\sigma} \\ v_{1,2}\sigma & v_{2,2} \end{pmatrix}.$$

Proof. It's easy to compute that

$$\rho_s(xy) = \begin{pmatrix} \frac{t^2-st-t}{t-1} & \frac{s\sqrt{t}}{(t-1/t)^2} - \sqrt{t} \\ -\frac{s}{\sqrt{t}} & \frac{st+t-1}{t(t-1)} \end{pmatrix} = \begin{pmatrix} \frac{t^2-st-t}{t-1} & -\frac{s\sqrt{t}}{\sigma} \\ -\frac{s}{\sqrt{t}} & \frac{st+t-1}{t(t-1)} \end{pmatrix},$$

$$\rho_s(yx) = \begin{pmatrix} \frac{t^2-st-t}{t-1} & \frac{s}{\sqrt{t}(t-1/t)^2} - \frac{1}{\sqrt{t}} \\ -s\sqrt{t} & \frac{st+t-1}{t(t-1)} \end{pmatrix} = \begin{pmatrix} \frac{t^2-st-t}{t-1} & -\frac{s}{\sqrt{t}\sigma} \\ -s\sqrt{t} & \frac{st+t-1}{t(t-1)} \end{pmatrix}.$$

We can observe that the $(1,2)$-th entry of $\rho_s(yx)$ is exactly the $(2,1)$-th entry of $\rho_s(xy)$ divided by σ, and the $(2,1)$-th entry of $\rho_s(yx)$ is exactly the $(1,2)$-th entry of $\rho_s(xy)$ multiplied by σ. And the other two entries of $\rho_s(yx)$ and $\rho_s(xy)$ coincide.

The authors of [3] proved in Lemma 5.1 that similar relation holds between $\rho_s(y^{-1}x)$ and $\rho_s(xy^{-1})$, and also between $\rho_s(yx^{-1})$ and $\rho_s(x^{-1}y)$. Moreover, they showed that this relation is preserved under matrix multiplication;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix},$$

$$\begin{pmatrix} p & \frac{r}{\sigma} \\ q\sigma & s \end{pmatrix} \begin{pmatrix} a & \frac{c}{\sigma} \\ b\sigma & d \end{pmatrix} = \begin{pmatrix} ap+br & \frac{cp+dr}{\sigma} \\ (aq+bs)\sigma & cq+ds \end{pmatrix}.$$

Therefore the same relation must also hold for $\rho_s(w^n)$ and $\rho_s(w_*^n)$. □

Lemma 3.2. *[3, Lemma 5.4] For* $U = \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} = \rho_s(w)$,

$$u_{1,1} = w_{1,1} + \frac{w_{2,1}}{t-1}, \qquad u_{1,2} = \sqrt{t}\left(w_{1,2} - \frac{w_{1,1}}{t-1}\right) + \frac{\sqrt{t}}{t-1}\left(w_{2,2} - \frac{w_{2,1}}{t-1}\right),$$
$$u_{2,1} = \frac{w_{2,1}}{\sqrt{t}}, \qquad u_{2,2} = w_{2,2} - \frac{w_{2,1}}{t-1}.$$

Lemma 3.3. *[3, Lemma 5.5] For* $U^n = \begin{pmatrix} v_{1,1} & v_{2,1} \\ v_{1,2} & v_{2,2} \end{pmatrix} = \rho_s(w)^n$, *we have*

$$v_{1,1} = u_{1,1}\tau_n - \tau_{n-1}, \qquad v_{1,2} = u_{1,2}\tau_n,$$
$$v_{2,1} = u_{2,1}\tau_n, \qquad v_{2,2} = \tau_{n+1} - u_{1,1}\tau_n.$$

Again, the above two lemmas still hold because no explicit expression of entries of W is involved.

Let B_s be the $(1,1)$-entry of $\rho_s(\mathcal{L})$. In our case, B_s obtains the form in the following lemma, which is different from [3, Lemma 5.6].

Lemma 3.4. $B_s = -\dfrac{u_{2,1}}{u_{1,2}\sigma} t^{-2n}$.

Proof. We compute $\rho_s(\mathcal{L}) = \rho_s(w_*^n w^n x^{-4n})$ first.

$$\rho_s(x)^{-4n} = Q\rho(x)^{-4n}Q^{-1} = Q\begin{pmatrix} 1/t & -1-1/t \\ 0 & t \end{pmatrix}^{2n} Q^{-1}$$
$$= Q\begin{pmatrix} 1/t^{2n} & * \\ 0 & t^{2n} \end{pmatrix}^{2n} Q^{-1} = \begin{pmatrix} 1/t^{2n} & * \\ 0 & t^{2n} \end{pmatrix}.$$

Then

$$\rho_s(\mathcal{L}) = \rho_s(w_*^n)\rho_s(w^n)\rho_s(x)^{-4n} = \begin{pmatrix} v_{1,1} & \frac{v_{2,1}}{\sigma} \\ v_{1,2}\sigma & v_{2,2} \end{pmatrix} \begin{pmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{pmatrix} \begin{pmatrix} 1/t^{2n} & * \\ 0 & t^{2n} \end{pmatrix}$$
$$= \begin{pmatrix} (v_{1,1}^2 + \frac{v_{2,1}^2}{\sigma})t^{-2n} & * \\ (v_{1,1}v_{1,2}\sigma + v_{2,1}v_{2,2})t^{-2n} & * \end{pmatrix}.$$

So $B_s = (v_{1,1}^2 + \dfrac{v_{2,1}^2}{\sigma})t^{-2n}$. To simplify B_s, we need to use the relation $\det U^n = v_{1,1}v_{2,2} - v_{1,2}v_{2,1} = 1$. Moreover $v_{1,1}v_{1,2}\sigma + v_{2,1}v_{2,2} = 0$, because $\rho_s(\mathcal{L})$ is diagonal. Then

$$\begin{aligned} B_s &= (v_{1,1}^2 + \frac{v_{2,1}^2}{\sigma})t^{-2n} = (-\frac{v_{1,1}v_{2,2}}{v_{1,2}} + v_{2,1})\frac{v_{2,1}}{\sigma}t^{-2n} \\ &= (-\frac{v_{1,2}v_{2,1}+1}{v_{1,2}} + v_{2,1})\frac{v_{2,1}}{\sigma}t^{-2n} = -\frac{v_{2,1}}{v_{1,2}\sigma}t^{-2n} \\ &\overset{Lemma\ 3.3}{=\!=} -\frac{u_{2,1}\tau_n}{u_{1,2}\tau_n\sigma}t^{-2n} = -\frac{u_{2,1}}{u_{1,2}\sigma}t^{-2n}. \end{aligned}$$

□

Proposition 3.5. *Let B_s be the $(1,1)$-entry of $\rho_s(\mathcal{L})$, where $\mathcal{L}$ is the longitude of $J(2m+1,2n)$. Then*

$$B_s = \frac{g_m - tg_{m-1}}{g_{m-1} - tg_m}t^{-2n}.$$

Proof. We compute directly using Lemma 3.2 and Proposition 2.3.

$$u_{2,1} = \frac{w_{2,1}}{\sqrt{t}} = \frac{s}{\sqrt{t}}f_m(tg_{m-1} - g_m),$$

$$\begin{aligned} u_{1,2} &= \sqrt{t}\left(w_{1,2} - \frac{w_{1,1}}{t-1}\right) + \frac{\sqrt{t}}{t-1}\left(w_{2,2} - \frac{w_{2,1}}{t-1}\right) \\ &= \frac{\sqrt{t}}{t(t-1)}\left((t-1)f_mg_m - (t^2-t)f_mg_{m-1} - t^2f_m^2 + stg_m^2\right) \\ &+ \frac{\sqrt{t}}{t(t-1)}\left(-stg_{m-1}^2 + f_m^2 - \frac{st^2}{t-1}f_mg_{m-1} + \frac{st}{t-1}f_mg_m\right) \\ &= \frac{\sqrt{t}}{t(t-1)}\left((t-1+\frac{st}{t-1})f_mg_m - (t^2-t+\frac{st^2}{t-1})f_mg_{m-1} + (1-t^2)f_m^2 + st(g_m^2 - g_{m-1}^2)\right) \\ &= \frac{\sqrt{t}}{t(t-1)}f_m\left((t-1+\frac{st}{t-1})g_m - (t^2-t+\frac{st^2}{t-1})g_{m-1} + (1-t^2)(g_m - g_{m-1}) + st(g_m + g_{m-1})\right) \\ &= \frac{\sqrt{t}}{t(t-1)}f_m\left((t+\frac{st}{t-1} - t^2 + st)g_m - (-t+\frac{st^2}{t-1} + 1 - st)g_{m-1}\right) \\ &= \frac{\sqrt{t}(st-(t-1)^2)}{t(t-1)^2}f_m\left(tg_m - g_{m-1}\right) = -\frac{s}{\sqrt{t}\sigma}f_m\left(tg_m - g_{m-1}\right). \end{aligned}$$

Applying Lemma 3.4, we have

$$B_s = -\frac{u_{2,1}}{u_{1,2}\sigma} = \frac{g_m - tg_{m-1}}{g_{m-1} - tg_m}t^{-2n}.$$

□

4. Root of the Riley Polynomial

The Riley polynomial of two-bridge knot was first studied by Robert Riley in his paper [1]. The root of the Riley polynomial of a two-bridge knot K describes all the nonabelian $\mathrm{PSL}_2(\mathbb{C})$ representations of $\pi_1(S^3 - K)$. By examining real roots of

the Riley polynomial of K, we will be able to construct $\mathrm{PSL}_2(\mathbb{R})$ representations that we need.

Proposition 4.1. *The Riley polynomial of $K = J(2m+1, 2n)$ is*

$$\phi_K(s,t) = (\tau_{n+1} - \tau_n) + (t + 1/t - s - 2) f_m g_{m-1} \tau_n, \tag{4.1}$$

where $\tau = (t + 1/t - s - 2) f_m^2 + 2$.

Proof. By [1, Theorem 1], the Riley polynomial for a two-bridge knot K is $\phi_K(s,t) = z_{1,1} + (1-t) z_{1,2}$. So for the double twist knot $J(2m+1, 2n)$,

$$\begin{aligned}\phi_K(s,t) &= (w_{1,1}\tau_n - \tau_{n-1}) + (1-t) w_{1,2} \tau_n \\ &= (\tau_{n+1} - \tau_n) + ((1-t) w_{1,2} + 1 - w_{2,2}) \tau_n.\end{aligned}$$

$$\begin{aligned}(1-t) w_{1,2} + 1 - w_{2,2} &= (1-t)(\frac{f_m g_m}{t} - f_m g_{m-1}) + f_m^2 - s g_m g_{m-1} - (\frac{f_m^2}{t} - s g_{m-1}^2) \\ &= (t + 1/t - s - 2) f_m g_{m-1} \tau_n.\end{aligned}$$

So $\phi_K(s,t) = (\tau_{n+1} - \tau_n) + (t + 1/t - s - 2) f_m g_{m-1} \tau_n$.

Direct computation using Proposition 2.3 and Lemma 2.2 shows

$$\tau = \mathrm{tr} W = t f_m^2 - s g_m^2 + \frac{f_m^2}{t} - s g_{m-1}^2 = (t + 1/t - s - 2) f_m^2 + 2. \tag{4.2}$$

□

Lemma 4.2. *When $m > 0$, both f_m and g_m have m simple negative roots.*

Proof. First, we show both f_m and g_m have at least one sign change when $m \neq 0$. Notice that both f_m and g_m are monic polynomials in terms of s of degree m. In particular $g_m(2x-2)$ is the Chebyshev Polynomial of the second kind. Therefore g_m has exactly m different negative simple roots $2\cos\left(\frac{k\pi}{m+1}\right) - 2$, $k = 1, 2, \cdots m$.

When $s \to \infty$, both f_m and g_m approach $+\infty$. So $g_m < 0$ on $(2\cos\left(\frac{2j\pi}{m+1}\right) - 2, 2\cos\left(\frac{(2j-1)\pi}{m+1}\right) - 2)$, while $g_m > 0$ on $(2\cos\left(\frac{(2j+1)\pi}{m+1}\right) - 2, 2\cos\left(\frac{2j\pi}{m+1}\right) - 2)$ and $(2\cos\left(\frac{\pi}{m+1}\right) - 2, \infty)$, $j = 1, \ldots, \lfloor \frac{m}{2} \rfloor$. Since $g_{m-1}\left(2\cos\left(\frac{(2j-1)\pi}{m}\right) - 2\right) = 0$ and $2\cos\left(\frac{(2j-1)\pi}{m}\right) - 2 \in \left(2\cos\left(\frac{2j\pi}{m+1}\right) - 2, 2\cos\left(\frac{(2j-1)\pi}{m+1}\right) - 2\right)$ for $j = 1, \ldots, \lfloor \frac{m}{2} \rfloor$, then $f_m\left(2\cos\left(\frac{(2j-1)\pi}{m}\right) - 2\right) < 0$ by Lemma 2.2 (1). Similarly, we have $2\cos\left(\frac{2j\pi}{m}\right) - 2 \in \left(2\cos\left(\frac{(2j+1)\pi}{m+1}\right) - 2, 2\cos\left(\frac{2j\pi}{m+1}\right) - 2\right)$ and $g_{m-1}\left(2\cos\left(\frac{2j\pi}{m}\right) - 2\right) = 0$ for $j = 1, \ldots, \lceil \frac{m}{2} \rceil - 1$. Then it follows from Lemma 2.2 (1), that $f_m\left(2\cos\left(\frac{2j\pi}{m}\right) - 2\right) > 0$. And moreover $f_m(0) = 1 > 0$. Then the degree m polynomial f_m has exactly m sign changes, which implies that f_m should have at exactly m simple roots. □

Let $r_{f_m} < 0$ be the largest root of f_m, i.e. r_{f_m} has the smallest absolute value among all roots of f_m. Then for $m > 0$, both g_m and g_{m-1} are positive on $(r_{f_m}, 0)$.

Setting $t = 1$ in the Riley polynomial $\phi_K(s,t) = (\tau_{n+1} - \tau_n) + (t + 1/t - s - 2)f_m g_{m-1}\tau_n = 0$, we get $\phi_K(s,1) = (\tau_{n+1} - \tau_n) - sf_m g_{m-1}\tau_n = 0$, with $\tau = -sf_m^2 + 2$. Then we can choose $s_0 > 0$ to be the smallest positive solution of $\phi_K(s,1) = 0$ with respect to s.

Proposition 4.3. *Set $T = t + \frac{1}{t}$.*

(1) When $n > 1$ or $n = 1$ $m > 0$, the Riley polynomial (4.1) has a root $\frac{-C_2}{f_m^2} < T - s - 2 < \frac{-C_1}{f_m^2}$ for $s > 0$, where C_1 and C_2 are positive constants. Moreover, $0 < T < 2$ for $s \in (0, s_0)$.

(2) When $m < -1$ and $n > 0$, the Riley polynomial has a root $\frac{C_3}{f_m} < T - s - 2 < \frac{C_4}{f_m}$ with $r_{f_m} < s < 0$, where C_3 and C_4 are positive constants. Moreover, $T > 2$ for $s \in (r_{f_m}, 0)$.

Proof.

(1) Suppose $n > 1$, we set $\tau_{n+1} = \tau_n$ as in [3, Lemma 4.1]. Then we can find two roots $z = e^{\frac{\pi}{2n+1}i}$ and $z = e^{\frac{3\pi}{2n+1}i}$, with

$$\tau_n(e^{\frac{\pi}{2n+1}i}) = \tau_{n+1}(e^{\frac{\pi}{2n+1}i}) > 0, \quad \tau_n(e^{\frac{3\pi}{2n+1}i}) = \tau_{n+1}(e^{\frac{3\pi}{2n+1}i}) < 0.$$

Let $C_1 = 2 - 2\cos(\frac{\pi}{2n+1}) > 0$, $C_2 = 2 - 2\cos(\frac{3\pi}{2n+1}) > 0$. Then

$$\phi_K(s, T = s + 2 - \frac{C_1}{f_m^2}) = -\frac{C_1 g_{m-1}}{f_m}\tau(e^{\frac{\pi}{2n+1}i}) < 0,$$
$$\phi_K(s, T = s + 2 - \frac{C_2}{f_m^2}) = -\frac{C_2 g_{m-1}}{f_m}\tau(e^{\frac{3\pi}{2n+1}i}) > 0.$$

Since $\phi_K(s,T)$ is a polynomial function of T, it is continuous. So it has a root $\frac{-C_2}{f_m^2} < T - s - 2 < \frac{-C_1}{f_m^2}$ by Intermediate Value Theorem, with $C_1, C_2 \in (0,4)$.

When $m > 0$ and $n = 1$, $\phi_K(s,t) = (\tau - 1) + (t + 1/t - s - 2)f_m g_{m-1}$, where $\tau = (t+1/t-s-2)f_m^2 + 2$. So $\phi_K(s,t) = (T-s-2)f_m g_m + 1$, which has the solution $T = s+2+\frac{-1}{f_m g_m}$. Since $\frac{f_m}{g_m}$ is continuous on $[0, s_0]$, then it must be bounded below by C_1 and above by C_2. And it is not hard to see that $0 < C_1 < C_2 < 1$, as $g_m > f_m \geq 1$ when $s \geq 0$. Therefore when $n = 1$, we still have $\frac{-C_2}{f_m^2} < T - s - 2 < \frac{-C_1}{f_m^2}$.

To see $0 < T < 2$ for $s \in (0, s_0)$, notice that by assumption s_0 is the smallest positive value for s such that $T = 2$.

(2) When $m < -2$, let $m' = -m$, then $f_m = f_{m'-1}$ and $g_{m-1} = -g_{m'-1}$. So the Riley polynomial becomes $\tau_{n+1} - \tau_n - (T - s - 2)f_{m'-1}g_{m'-1}\tau_n = 0$. Since $2\cos\left(\frac{\pi}{m'-1}\right) - 2$, the largest root of $g_{m'-2}$, is smaller than r_{f_m}, then $g_{m'-2}$ is positive and increasing on $s \in (r_{f_m}, 0)$. As a result, $g_{m'-1} = f_{m'-1} + g_{m'-2}$ must also be positive and increasing on $(r_{f_m}, 0)$. Setting $s = r_{f_m}$ and $f_{m'-1} = 0$ in Lemma 2.2 (1) and (3), we get $g_{m'-1}(0) = \sqrt{\frac{1}{-r_{f_m}}}$. So $g_{m'-1} \in (\sqrt{\frac{1}{-r_{f_m}}}, m')$ and

$f_{m'-1} \in (0,1)$, when $s \in (r_{f_m}, 0)$. Since $-1 < r_{f_m} < 0$, $\sqrt{\frac{1}{-r_{f_m}}} > 1$, claim that we can choose $C_3, C_4 > 0$ such that $2 + C_3 f_{m'-1} < \tau < 2 + C_4 f_{m'-1}$. In fact, we can choose $C_4 > 0$ large enough such that $1 + C_4\sqrt{\frac{1}{-r_{f_m}}} > 2 + C_4 > 2 + C_4 f_{m'-1}$. When $\tau > 2$, both τ_n and τ_{n+1} are positive. So

$$\begin{aligned}\phi_K(s, \tau = 2 + C_4 f_{m'-1}) = \tau_{n+1} - \tau_n - C_4 g_{m'-1}\tau_n &< \tau_{n+1} - (1 + C_4\sqrt{\frac{1}{-r_{f_m}}})\tau_n \\ &< \tau_{n+1} - \tau\tau_n = -\tau_{n-1} < 0.\end{aligned}$$

When $m = -2$, $m' = 2$. So $f_{m'-1} = s + 1$, $g_{m'-1} = s + 2$. Let $C_4 \geq 1$, then

$$\begin{aligned}\phi_K(s, \tau = 2 + C_4 f_{m'-1}) =& \phi_K(s, \tau = 2 + C_4(s+1)) = \tau_{n+1} - \tau_n - C_4(s+2)\tau_n \\ =& \tau_{n+1} - (2 + C_4(s+1) + C_4 - 1)\tau_n \\ =& \tau_{n+1} - \tau\tau_n + (1 - C_4)\tau_n \\ =& -\tau_{n-1} + (1 - C_4)\tau_n \leq -\tau_{n-1} < 0.\end{aligned}$$

For $\tau > 2$, it is easy to verify that $\frac{\tau_{n+1}}{\tau_n}$ is an increasing function (for example using first derivative test), so $\frac{\tau_{n+1}}{\tau_n} > \frac{n+1}{n}$. So we can choose $C_3 > 0$ such that $1 + C_3 m' < \frac{n+1}{n}$ and it follows that

$$\phi_K(s, \tau = 2 + C_3 f_{m'-1}) = \tau_{n+1} - \tau_n - C_3 g_{m'-1}\tau_n > \tau_{n+1} - (1 + C_3 m')\tau_n > 0.$$

Apparently $T > 2$ when $s \to r_{f_m}+$. To see $T = t + 1/t > 2$ for all $s \in (r_{f_m}, 0)$, we will need to look at B_s. In the proof of Lemma 3.4, notice that $B_s = (v_{1,1}^2 + \frac{v_{2,1}^2}{\sigma})t^{-2n}$. So B_s is a polynomial in terms of $s, \frac{1}{s}, t, \frac{1}{t}$ and $\frac{1}{t-1}$. Since by Proposition 3.5 $B_s = -1$ when t approaches and takes the value 1, then $t = 1$ is a removable singularity of B_s. Therefore B_s is continuous for $s \in (r_{f_m}, 0)$. Now assume $T = 2$ when s equals some $s_1 \in (r_{f_m}, 0)$ and choose s_1 such that $T > 2$ for $s \in (r_{f_m}, s_1)$. In fact, we can assume $t > 1$ for $s \in (r_{f_m}, s_1)$, because similar computations with $1/t > 1$ instead of $0 < t < 1$ could be carried out. Then $g_m - tg_{m-1} = tg_{m'-1} - g_{m'-2} = (t-1)g_{m'-1} + g_{m'-1} - g_{m'-2} = (t-1)g_{m'-1} + f_{m'-1} > 0$ for $s \in (r_{f_m}, s_1)$. Moreover, $g_{m-1} - tg_m = tg_{m'-2} - g_{m'-1} > (t-1)g_{m'-2} > 0$. This implies $B_s = \dfrac{g_m - tg_{m-1}}{g_{m-1} - tg_m} t^{-2n} > 0$ for $s \in (r_{f_m}, s_1)$, and it follows by continuity that $B_s \geq 0$ for $s \in (r_{f_m}, s_1]$, which contradicts $B_s(s = s_1) = -1$. □

5. Slopes

In this section, we will compute slopes of asymptotes of the graph of the root of the Riley polynomial under logarithmic scale.

Lemma 5.1.

(1) $\lim_{s \to r_{f_m}+} t = \infty$ *when* $m < -1$ *and* $n > 0$.

Proof. When $m < -1$ and $n > 0$, by Proposition 4.3 (2) we have $s + 2 + \frac{C_3}{f_m} < T < s + 2 + \frac{C_4}{f_m}$. So $\lim_{s \to r_{f_m}+} T = \infty$, and we can choose t such that $\lim_{s \to r_{f_m}+} t = \infty$□

Lemma 5.2.

(1) $\lim_{s \to r_{f_m}+} B_s t^{2n} = 1$ *when* $m < -1$.

(2) $\lim_{s \to 0+} B_s = 1$ *when* $m \neq -1$.

(3) $\lim_{s \to s_0-} B_s = -1$ *and* $\lim_{s \to s_0-} Ln(B_s) = -\pi + 2d\pi$ *for some* $d \in \mathbb{Z}$.

Proof. (1) When $f_m = 0$, from Lemma 2.2(1) we know $g_m = g_{m-1}$. So

$$\lim_{s \to r_{f_m}+} B_s t^{2n} = \lim_{s \to r_{f_m}+} \frac{g_m - t g_{m-1}}{g_{m-1} - t g_m} = \lim_{s \to r_{f_m}+} \frac{g_m(1-t)}{g_m(1-t)} = 1.$$

(2) To prove the second limit, set $s = 0$ in τ, then $\tau = t + 1/t$. When $m \neq -1$, set $s = 0$ in $\phi_K(s,t) = 0$, then we have $(\tau_{n+1} - \tau_n) + m(t + 1/t - 2)\tau_n = 0$. So $t^{n+1} - t^{-n-1} - t^n + t^{-n} + m(t + 1/t - 2)(t^n - t^{-n}) = 0$, which simplifies to $t^{2n}[m - (m+1)t] - [(m+1) - mt] = 0$. So

$$\lim_{s \to 0+} B_s = \frac{m + 1 - mt}{m - (m+1)t} t^{-2n} = 1.$$

(3) When $s = s_0$, we have $t = 1$ and

$$\lim_{s \to s_0-} B_s = \frac{g_m - g_{m-1}}{g_{m-1} - g_m} = -1.$$

So $\lim_{s \to s_0-} \mathrm{Ln}(B_s) = -\pi + 2d\pi$ for some integer d. □

It is hard in general to give a formula to compute d, but we know $|2d - 1| \geq 1$, and we can choose proper framing such that $2d - 1 > 0$.

Let A_s be the $(1,1)$-entry of $\rho_s(\mathcal{M}) = \rho_s(x)$, which equals $\sqrt{t}$. We define $g : (,) \to \mathbb{R}$ to be $g(s) = -\dfrac{\ln(B_s)}{\ln(A_s)}$ and examine the image of g.

Proposition 5.3. *For points corresponding to the root of the Riley polynomial, the image of* g *contains every number in* I*, with*

$$I = \begin{cases} (-\infty, 0) & \text{if } m \neq -1 \text{ and } n > 1, \text{ or } m > 0 \text{ and } n = 1, \\ (0, 4n) & \text{if } m < -1 \text{ and } n > 0, \end{cases}$$

Proof.

$$\lim_{s \to s_0-} g(s) = -\lim_{s \to s_0-} \frac{\ln(B_s)}{\ln(A_s)} = -\frac{\ln(-1)}{\ln(1^+)} = -\frac{\pi}{0^+} = -\infty.$$

Applying Lemma 5.2, when $m \neq -1$ we have

$$\lim_{s\to 0+} g(s) = -\lim_{s\to 0+} \frac{\ln(B_s)}{\ln(A_s)} = -\lim_{s\to 0+} \frac{2\ln(B_s)}{\ln(t)} = 0.$$

When $m < -1$ and $n > 0$,

$$\lim_{s\to r_{f+}} g(s) = -\lim_{s\to r_{f+}} \frac{\ln(B_s)}{\ln(A_s)} = -\lim_{s\to r_{f+}} \frac{2\left(\ln(B_s t^{2n}) - 2n\ln(t)\right)}{\ln(t)} = 4n.$$

Then the lemma follows from Proposition 4.3. □

6. Root of the Alexander Polynomial

Setting $s = 0$ in (4.2), then $\tau = T = t + 1/t$ and $\phi_K(s = 0, t) = \tau_{n+1} - \tau_n + m(\tau - 2)\tau_n = 0$, which simplifies to

$$0 = \phi_K(0, t) = (m + 1)(t^n + t^{-n}) + (2m + 1)\sum_{i=0}^{n-1}(t^i + t^{-i})(-1)^{n-i}. \tag{6.1}$$

Next, we compute the Alexander polynomial $\Delta_{J(2m+1,2n)}(a)$ of $J(2m + 1, 2n)$ and show that it is the same as (6.1).

Let $\Delta_{T(p,q)}$ be the Alexander polynomial of the torus knot $T(p, q)$.

Lemma 6.1. *(see for example [9, Example 9.15])*

$$\Delta_{T(p,q)}(a) = \frac{(a^{pq} - 1)(a - 1)}{(a^p - 1)(a^q - 1)} \times a^{-\frac{(p-1)(q-1)}{2}}.$$

The extra factor $a^{-\frac{(p-1)(q-1)}{2}}$ is multiplied to normalize $\Delta_{T(p,q)}(a)$ so that it is symmetric, i.e. $\Delta_{T(p,q)}(a^{-1}) = \Delta_{T(p,q)}(a)$.

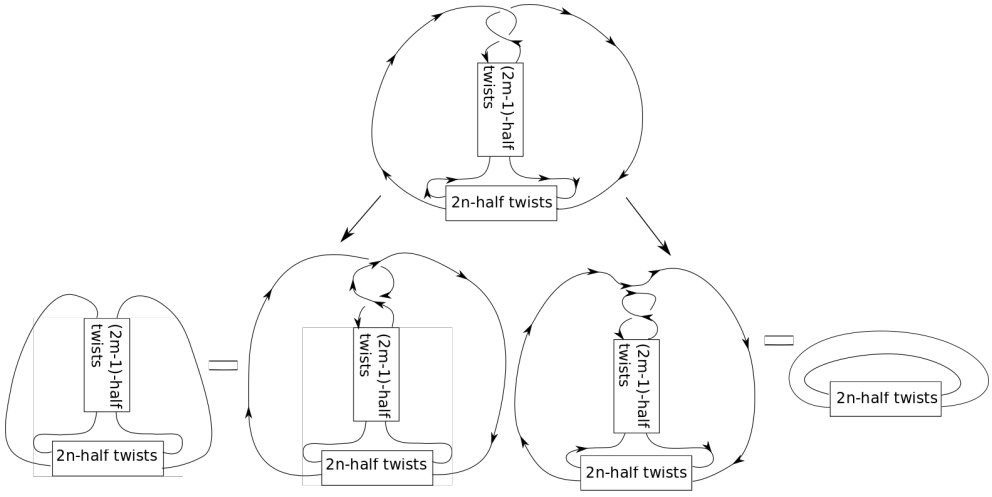

Fig. 2: The Skein relations

Proposition 6.2. *The Alexander polynomial of the double twist knot* $J(2m+1,2n)$ *is* $\Delta_{J(2m+1,2n)}(a) = (m+1)(a^n + a^{-n}) + (2m+1)\sum_{i=0}^{n-1}(a^i + a^{-i})(-1)^{n-i}$.

Proof.

We will use the fact that the Alexander polynomial satisfies the Skein relations (see for example [10, Chapter 6]) and prove by induction.

Let $\Delta_{J(2m+1,2n)}(a)$ be the Alexander polynomial of the double twist knot $J(2m+1,2n)$. Normalize $\Delta_{J(2m+1,2n)}(a)$ so that $\Delta_{J(2m+1,2n)}(a) = \Delta_{J(2m+1,2n)}(a^{-1})$ and the coefficient of its highest degree term is positive. From the Skein relations as shown in Figure 2, we have

$$\begin{aligned}
\Delta_{J(2m+1,2n)} - \Delta_{J(2m-1,2n)} &= (\sqrt{t} - 1/\sqrt{t})\Delta_{J(0,2n)},\\
\Delta_{J(2m-1,2n)} - \Delta_{J(2m-3,2n)} &= (\sqrt{t} - 1/\sqrt{t})\Delta_{J(0,2n)},\\
&\cdots\\
\Delta_{J(3,2n)} - \Delta_{J(1,2n)} &= (\sqrt{t} - 1/\sqrt{t})\Delta_{J(0,2n)}.
\end{aligned}$$

Adding the above equations all together, we have $\Delta_{J(2m+1,2n)} - \Delta_{J(1,2n)} = m(\sqrt{t} - 1/\sqrt{t})\Delta_{J(0,2n)}$. Notice that $J(0,2n)$ is the torus link $T(2,2n)$, and $J(1,2n)$ is the torus knot $T(2,2n+1)$. Let $\Delta_{T(p,q)}$ be the Alexander polynomial of the torus knot or link $T(p,q)$. Then $\Delta_{J(0,2n)} = \Delta_{T(2,2n)}$ and $\Delta_{J(1,2n)} = \Delta_{T(2,2n+1)}$. Applying the Skein relations of torus links, we have $(\sqrt{t} - 1/\sqrt{t})\Delta_{J(0,2n)} = (\sqrt{t} - 1/\sqrt{t})\Delta_{T(2,2n)} =$

$\Delta_{T(2,2n+1)} - \Delta_{T(2,2n-1)}$. So $\Delta_{J(2m+1,2n)} - \Delta_{T(2,2n+1)} = \Delta_{J(2m+1,2n)} - \Delta_{J(1,2n)} = m(\Delta_{T(2,2n+1)} - \Delta_{T(2,2n-1)})$. Thus

$$\begin{aligned}
\Delta_{J(2m+1,2n)} &= (m+1)\Delta_{T(2,2n+1)} - m\Delta_{T(2,2n-1)} \\
&= (m+1)\frac{(a^{2(2n+1)}-1)(a-1)}{(a^{2n+1}-1)(a^2-1)a^n} - m\frac{(a^{2(2n-1)}-1)(a-1)}{(a^{2n-1}-1)(a^2-1)a^{n-1}} \\
&= (m+1)\frac{(a^{2n+1}+1)}{(a+1)a^n} - m\frac{a^{2n-1}+1}{(a+1)a^{n-1}} \\
&= (m+1)\sum_{i=-n}^{n}(a^i)(-1)^{n+i} - m\sum_{i=-n+1}^{n-1}(a^i)(-1)^{n-1+i} \\
&= (m+1)(a^n+a^{-n}) + (2m+1)\sum_{i=0}^{n-1}(a^i+a^{-i})(-1)^{n-i}.
\end{aligned}$$

□

We can see that the Alexander polynomial of $J(2m+1,2n)$ is exactly the same as its Riley polynomial when $s=0$ (as shown in (6.1)). So when $s=0$, t takes the value of a root ξ of the Alexander polynomial.

Proposition 6.3.

(1) When $m \neq -1$ and $n > 1$ or $m > 0$ and $n = 1$, $\lim_{s\to 0+} t = \xi$ is a unit complex root of the Alexander polynomial $\Delta_{J(2m+1,2n)}$.

(2) When $m < -1$ and $n > 0$, $\lim_{s\to 0-} t = \xi$ is a positive real root of $\Delta_{J(2m+1,2n)}$.

Proof. As we see in (6.1), setting $s=0$ in the Riley polynomial, we get

$$\phi_K(0,t) = (m+1)(t^n+t^{-n}) + (2m+1)\sum_{i=0}^{n-1}(t^i+t^{-i})(-1)^{n-i},$$

which is the same as the Alexander polynomial $\Delta_{J(2m+1,2n)}$. Notice that it is a palindrome (symmetric) polynomial of even degree $2n$ after multiplying by t^n. So by [11, Theorem 1], it has a complex root on the unit circle when $m \neq -1$ and $n > 1$. When $n = 1$, $\phi_K(0,t) = (m+1)t + (2m+1) + (m+1)t^{-1}$. If $m > 0$, then $m+1 > \frac{1}{2}(2m+1)$. So again by [11, Theorem 1], $\phi_K(0,t)$ has a unit complex root.

When $m < -1$, there is a positive real root different from 1. To see this, let $m' = -m > 1$, then $\phi_K(0,t) = t^{2n}(m' - (m'-1)t) + m't - (m'-1) = 0$. Set $h(t) = t^{2n}(m' - (m'-1)t) + m't - (m'-1)$. When $t = 0$, $h(t) = 1 - m' < 0$; when $t = 1$, $h(t) = 2 > 0$. So by Intermediate Value Theorem, $h(t)$ must have a real root between 0 and 1, and also a root> 1 by symmetry of $h(t)$.

This proposition could also be proved by taking $s = 0$ in $0 < 2 + \frac{-C_1}{f_m^2} < T < 2 + \frac{-C_2}{f_m^2} < 2$, and in $2 < 2 + \frac{C_3}{f_m^2} < T < 2 + \frac{C_4}{f_m^2}$ when $m < -1$. □

7. Left-Orderability

Boyer, Rolfsen and Wiest proved the following theorem about left-orderability.

Theorem 7.1. *[12, Theorem 1.1] Suppose that M is a compact, connected and P^2-irreducible 3-manifold. A necessary and sufficient condition that $\pi_1(M)$ be left-orderable is that either $\pi_1(M)$ is trivial or there exists a non-trivial homomorphism from $\pi_1(M)$ to a left-orderable group.*

Consider the Lie group $SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \right\}$. So we can parameterize $SU(1,1)$ by (γ, ω) where $\gamma = -\overline{\beta}/\alpha \in \mathbb{C}$ and $\omega = \arg\alpha$ is defined modulo 2π. Then $\mathrm{SL}_2\mathbb{R} \simeq \mathrm{SU}(1,1)$ can be described as $\{(\gamma,\omega) \mid |\gamma| < 1, -\pi \leq \omega < \pi\}$. The nonlinear Lie group $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ is defined to be the universal cover of $\mathrm{SL}_2\mathbb{R}$ and $\mathrm{PSL}_2\mathbb{R}$. In particular, it can be described as $\{(\gamma,\omega) \in \mathbb{C}\times\mathbb{R} \mid |\gamma| < 1, -\infty < \omega < \infty\}$ with group operation given by:

$$(\gamma,\omega)(\gamma',\omega') = \left((\gamma + \gamma' e^{-2i\omega})(1+\bar{\gamma}\gamma' e^{-2i\omega})^{-1}, \omega + \omega' + \frac{1}{2i}\ln\left(1+\bar{\gamma}\gamma' e^{-2i\omega}\right)(1+\gamma\bar{\gamma}' e^{2i\omega})^{-1}\right).$$

As a subgroup of $\mathrm{Homeo}^+(\mathbb{R})$, $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ is left-orderable. We follow the notation in [13]. Denote $\mathrm{PSL}_2\mathbb{R}$ by G, and $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ by $\widetilde{G}$. We call an element $\widetilde{g}$ of $\widetilde{G}$ elliptic, parabolic or hyperbolic if it covers an element g of the corresponding type in $\mathrm{PSL}_2\mathbb{R}$. In particular, if $\widetilde{g}$ covers $\pm I$, then $\widetilde{g}$ is called central.

Suppose M is a knot complement in a rational homology 3-sphere. Let $R_{\widetilde{G}}(M) = \mathrm{Hom}(\pi_1(M), \widetilde{G})$ be the variety of $\widetilde{G}$ representations of $\pi_1(M)$. Similarly define $R_{\widetilde{G}}(\partial M) = \mathrm{Hom}(\pi_1(\partial M), \widetilde{G})$. For a precise definition of the representation variety, see for example [14]. We call a $\widetilde{G}$ representation $\widetilde{\rho} \in R_{\widetilde{G}}(\partial M)$ elliptic, parabolic, hyperbolic or central if $\widetilde{\rho}(\pi_1(\partial M))$ contains the corresponding elements.

So in the case of double twist knot $J(2m+1, 2n)$, all the $\widetilde{G}$ representations corresponding to the root of the Riley polynomial described in Proposition 4.3 (1) are elliptic, parabolic or central when restricted to the boundary. All the $\widetilde{G}$ representations corresponding to the solution described in Proposition 4.3 (2) are hyperbolic, parabolic or central when restricted to the boundary.

7.1. *Translation Extension Locus*

[13, Section 4]

The name translation extension locus comes from the fact that we need to use translation number in the definition. For an elements $\widetilde{g}$ in $\widetilde{G}$, define the translation number of $\widetilde{g}$ to be

$$\mathrm{trans}(\widetilde{g}) = \lim_{n\to\infty} \frac{\widetilde{g}^n(x) - x}{n} \text{ for some } x \in \mathbb{R}.$$

Then trans: $R_{\widetilde{G}}(\partial M) \to H^1(\partial M;\mathbb{R})$ can be defined by taking $\widetilde{\rho}$ to trans$\circ\widetilde{\rho}$.

Let M be a knot complement in a rational homology 3-sphere. To study $\widetilde{G}$ representations of M whose restrictions to $\pi_1(\partial M)$ are elliptic, Culler and Dunfield gave the following definition of translation extension locus.

Definition 7.2. [13, Section 4] Let $PE_{\widetilde{G}}(M)$ be the subset of representations in $R_{\widetilde{G}}(M)$ whose restriction to $\pi_1(\partial M)$ are either elliptic, parabolic, or central. Consider composition

$$PE_{\widetilde{G}}(M) \subset R_{\widetilde{G}}(M) \xrightarrow{\iota^*} R_{\widetilde{G}}(\partial M) \xrightarrow{\text{trans}} H^1(\partial M;\mathbb{R}).$$

The closure in $H^1(\partial M;\mathbb{R})$ of the image of $PE_{\widetilde{G}}(M)$ under trans $\circ\ \iota^*$ is called translation extension locus and denoted $EL_{\widetilde{G}}(M)$.

In particular, $EL_{\widetilde{G}}(M)$ contains the x-axis, which corresponds to abelian G representations of $\pi_1(M)$ that are elliptic, parabolic, or central when restricted to $\pi_1(\partial M)$.

Let $D_\infty(M)$ be the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. They showed that the translation extension locus $EL_{\widetilde{G}}(M)$ satisfies the following properties.

Theorem 7.3. *[13, Theorem 4.3] The extension locus $EL_{\widetilde{G}}(M)$ is a locally finite union of analytic arcs and isolated points. It is invariant under $D_\infty(M)$ with quotient homeomorphic to a finite graph. The quotient contains finitely many points which are ideal or parabolic in the sense defined above. The locus $EL_{\widetilde{G}}(M)$ contains the horizontal axis L_0, which comes from representations to $\widetilde{G}$ with abelian image.*

In the notation of this paper, a $\widetilde{G}$ representation corresponds to point with coordinates $(\frac{1}{2\pi}Ln(t), \frac{1}{\pi}Ln(B_s))$ in $EL_{\widetilde{G}}(M)$. Once we build up the translation extension locus, we will use the following lemma to prove left-orderability.

Lemma 7.4. *[13, Lemma 4.4] Suppose M is a compact orientable irreducible 3-manifold with ∂M a torus, and assume the Dehn filling $M(r)$ is irreducible. If L_r meets $EL_{\widetilde{G}}(M)$ at a nonzero point which is not parabolic or ideal, then $M(r)$ is orderable.*

7.2. *Holonomy Extension Locus*

In [15], I constructed the holonomy extension locus which is an analog of the translation extension locus and has similar properties to translations extension locus.

Let M be the complement of a knot in a rational homology 3-sphere.

Definition 7.5. [15, Definition 3.3], Let $PH_{\widetilde{G}}(M)$ be the subset of representations whose restriction to $\pi_1(\partial M)$ are either hyperbolic, parabolic, or central. Consider the composition

$$PH_{\widetilde{G}}(M) \subset R^{\text{aug}}_{\widetilde{G}}(M) \xrightarrow{\iota^*} R^{\text{aug}}_{\widetilde{G}}(\partial M) \xrightarrow{\text{EV}} H^1(\partial M;\mathbb{R}) \times H^1(\partial M;\mathbb{Z})$$

The closure of EV $\circ\, \iota^*(PH_{\widetilde{G}}(M))$ in $H^1(\partial M;\mathbb{R})$ is called the holonomy extension locus and denoted $HL_{\widetilde{G}}(M)$.

In particular, $HL_{\widetilde{G}}(M)$ contains the x-axis, which corresponds to abelian G representations of $\pi_1(M)$ that are hyperbolic, parabolic, or central when restricted to $\pi_1(\partial M)$.

The holonomy extension locus $HL_{\widetilde{G}}(M)$ satisfies the following properties.

Theorem 7.6. *[15, Theorem 3.1] The holonomy extension locus $HL_{\widetilde{G}}(M) = \bigsqcup_{i,j\in\mathbb{Z}} H_{i,j}(M)$, $-\frac{k_M}{k} \le j \le \frac{k_M}{k}$ is a locally finite union of analytic arcs and isolated points. It is invariant under the affine group $D_\infty(M)$ with quotient homeomorphic to a finite graph with finitely many points removed. Each component $H_{i,j}(M)$ contains at most one parabolic point and has finitely many ideal points locally.*

The locus $H_{0,0}(M)$ contains the horizontal axis L_0, which comes from representations to $\widetilde{G}$ with abelian image.

Under the notation of this paper, a $\widetilde{G}$ representation corresponds to a point which obtains coordinates $(\frac{1}{2}\ln(t), \ln(B_s), \mathrm{trans}(\widetilde{\rho_s}(\mathcal{M})), \mathrm{trans}(\widetilde{\rho_s}(\mathcal{L})))$ in $HL_{\widetilde{G}}(M)$, or equivalently $(\frac{1}{2}\ln(t), \ln(B_s)) \in H_{i,j}(M)$, with $i = \mathrm{trans}(\widetilde{\rho_s}(\mathcal{M}))$ and $j = \mathrm{trans}(\widetilde{\rho_s}(\mathcal{L}))$. We are using ln instead of Ln in the coordinates now because both t and B_s are real numbers under this setting. Once we build up the holonomy extension locus, we will use the following lemma to prove left-orderability.

Lemma 7.7. *If L_r intersects $H_{0,0}(M)$ component of $HL_{\widetilde{G}}(M)$ at non parabolic or ideal points, and assume $M(r)$ is irreducible, then $\pi_1(M(r))$ is left-orderable.*

Remark 7.8. A point in $EL_{\widetilde{G}}(M)$ (or $HL_{\widetilde{G}}(M)$) is called an ideal point if it does not come from an actual $\widetilde{G}$ representation of $\pi_1(M)$ but only lives in the closure. But in this paper, since we build translation extension locus/holonomy extension locus from the root of the Riley polynomial, every pair (s,t) corresponds to a $\widetilde{G}$ representation of $\pi_1(S^3 - J(2m+1,2n))$. So we do not encounter ideal points.

7.3. *Proof of the Main Theorem*

Now we can prove our main theorem.

Theorem 1.2. *For any rational $r \in I$, the Dehn filling of $J(2m+1,2n)$ of slope r is orderable, where*

$$I = \begin{cases} (-\infty, 1) & \text{if } m > 0 \text{ and } n > 0, \text{ or } m < -1 \text{ and } n > 1, \\ (-1, \infty) & \text{if } m < -1 \text{ and } n < 0, \text{ or } m > 0 \text{ and } n < -1, \\ [0, 4n) & \text{if } m < -1 \text{ and } n > 0, \\ (4n, 0] & \text{if } m > 0 \text{ and } n < 0. \end{cases}$$

Proof. First of all, notice that Dehn filling of $J(2m+1,2n)$ of slope r is irreducible as long as $n \neq 0$ and $m \neq 0, -1$ by [16, Theorem 2 (a)]. In particular, the 0 Dehn

filling of $J(2m+1,2n)$ is irreducible and has first betti number equal to 1. Therefore by [12, Corollary 3.4], $\pi_1(S^3 - J(2m+1,2n)(0))$ is left-orderable.

Case 1: $m > 0$ and $n > 0$ or $m < -1$ and $n > 1$.

From Proposition 4.3 (1) and Lemma 5.3, we know there is an arc in $EL_{\widetilde{G}}(J(2m+1,2n))$ going from a point $(\frac{1}{2\pi}\ln\xi, 0)$ on the positive half of the x-axis to a point $(0, 2d-1)$ (not included in the arc) on the positive half of the y-axis, where ξ is a unit complex root of the Alexander polynomial. Here d is defined by $\lim_{s\to s_0-} \mathrm{Ln}(B_s) = -\pi + 2d\pi$. By the discussion in 5.3, we can choose proper framing such that $2d-1 > 0$. So by Lemma 7.4, Dehn filling of rational slope $r \in (-\infty, 0]$ is orderable. By symmetry of translation extension locus as described in Theorem 7.3, there is also an arc going from $(1 - \frac{1}{2\pi}\ln\xi, 0)$ to $(1, -(2d-1))$ (excluded from the arc). Therefore Dehn filling of rational slope $r \in [0, 2d-1)$ is also orderable. So in total, we know the interval of left-orderable Dehn fillings should at least contain $(-\infty, 1)$.

Case 2: $m < -1$ and $n > 0$.

When $s = 0$, the representation ρ_s is reducible, so $\mathrm{trans}(\widetilde{\rho_s}(\mathcal{L})) = 0$. By continuity of translation number, all representations corresponding to the same continuous root of the Riley polynomial (i.e. for all $s \in (r_{f_m}, 0)$) should satisfy $\mathrm{trans}(\widetilde{\rho_s}(\mathcal{L})) = 0$. From Proposition 4.3 (2) and Lemma 5.3, we know there is an arc in $H_{0,0}(J(2m+1,2n))$ going from a point $(\frac{1}{2}\ln\xi, 0)$ on the positive half of the x-axis to infinity with asymptote of slope $-4n$, where ξ is a positive real root of the Alexander polynomial. Applying Lemma 7.7, then we see immediately that Dehn filling of rational slope $r \in [0, 4n)$ is orderable.

When $n < 0$, all computations could be carried out similarly. □

8. Examples

In this section, I will demonstrate the translation extension locus and holonomy extension locus of some double twist knots and some general two-bridge knots. All the figures are produced by the program PE [17] written by Marc Culler and Nathan Dunfield.

8.1. *Double Twist Knots*

Let us first look at some double twist knots. Our first example (figure 3) is $J(5,4) = 9_4$, with $m = 2$ and $n = 2$. Our second example (figure 4) is $J(-3,4) = 6_2$, with $m = -2$ and $n = 2$.

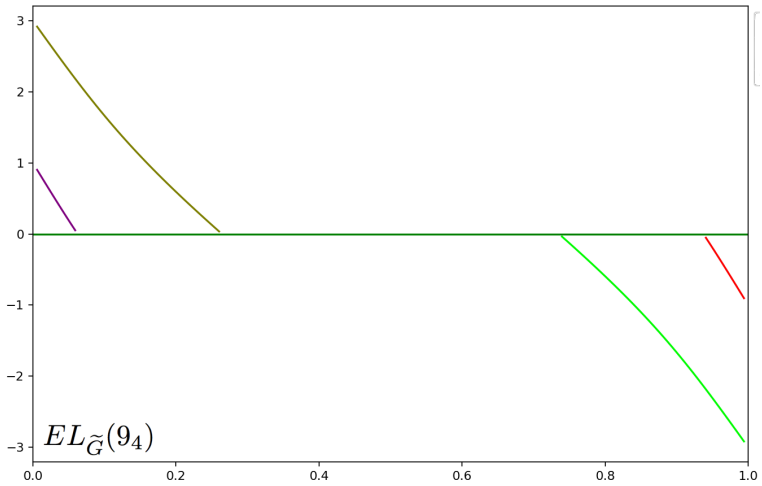

Fig. 3: Translation Extension Locus $EL_{\widetilde{G}}(9_4)$

This figure demonstrates the quotient of translation extension locus of 9_4 under the action of integral translation in the x direction. The x coordinate of a point in $EL_{\widetilde{G}}(9_4)$ is $\frac{1}{2\pi}Ln(t)$, the y coordinate is $\frac{1}{\pi}Ln(B_s)$. From this figure we know that Dehn filling of $J(5,4)$ of rational slope from $-\infty$ to 3 is orderable. The number 3 comes from the translation number of the parabolic element $\widetilde{\rho_{s_0}}(\mathcal{L})$ (or equivalently $\frac{1}{\pi}\mathrm{Ln}(B_{s_0})$). It is in general difficult to compute this number. But we know it has to be at least 1.

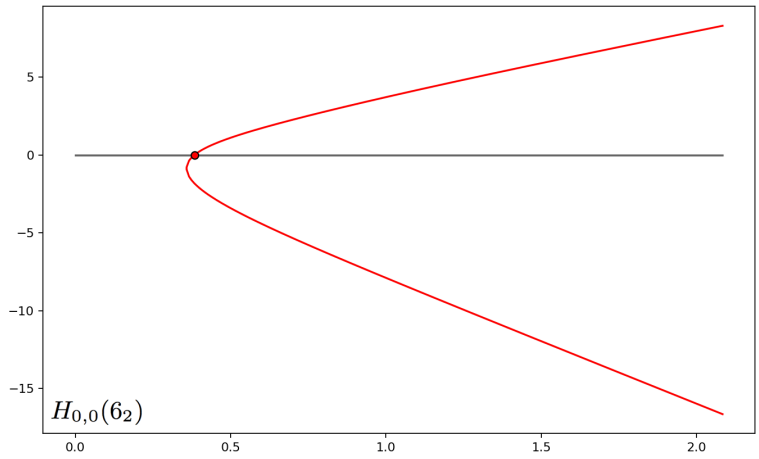

Fig. 4: Holonomy Extension Locus $HL_{\widetilde{G}}(6_2)$.

This figure is the quotient of the $H_{0,0}(6_2)$ component of the holonomy extension locus of 6_2 under the $\mathbb{Z}/2\mathbb{Z}$ action of reflection about the origin. The x coordinate of a point in $H_{0,0}(6_2)$ is $\frac{1}{2}\ln(t)$, the y coordinate is $\ln(B_s)$. The asymptotes have slope -8 and 4. So we know Dehn filling of $J(-3,4)$ of rational slope $r \in (-8,4)$ is orderable. Unfortunately, our Theorem 1.2 could only predict the slope $-8 = -4n$, but not the other slope 4.

8.2. *Two-Bridge Knots*

Now let us look at more general examples of two-bridge knots.

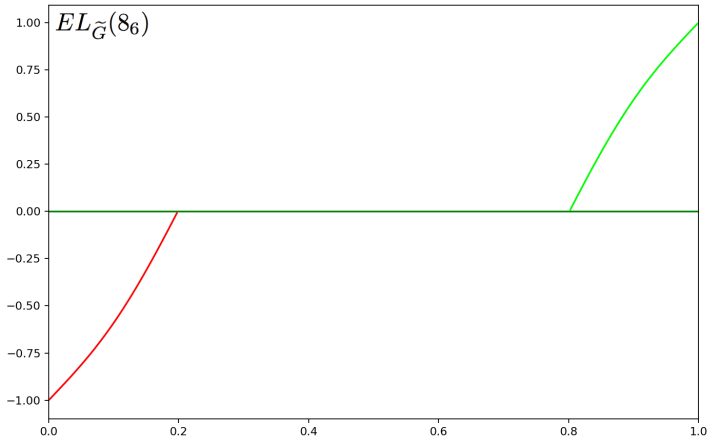

Fig. 5: Translation Extension Locus $EL_{\widetilde{G}}(8_6)$.

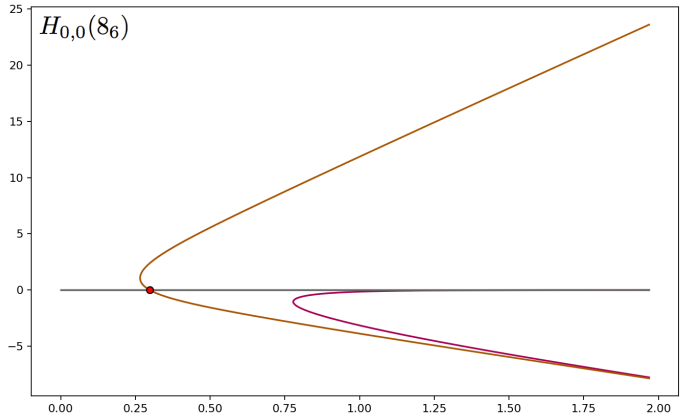

Fig. 6: Holonomy Extension Locus $HL_{\widetilde{G}}(8_6)$.

Figure 5 is the quotient of translation extension locus of 8_6 under the $\mathbb{Z}/2\mathbb{Z}$ action of integral translation in the x direction. Figure 6 is the quotient of the $H_{0,0}(8_6)$ component of the holonomy extension locus of 8_6 under the action of reflection about the origin.
The two-bridge knot 8_6 is not a double twist knot. Its Alexander polynomial is $2a^4-6a^3+7a^2-6a+2$, which has a pair (reciprocal to each other) of positive real roots and a pair of unit complex roots.

We can see from the above example that the holonomy extension locus and translation extension locus of 8_6 have similar patterns as those of double twists knots shown in Figure 3 and Figure 4.

When the Alexander polynomial of a two-bridge knot has no positive real roots or unit complex roots, in most cases we are still able to find intervals of left-orderable surgery slopes by constructing $\widetilde{G}$ representations and studying the holonomy extension loci. However, in other cases, it is possible that we may not be able to obtain such intervals of left-orderable surgery slopes using $\widetilde{G}$ representations. The two-bridge knot 6_3 is such an example.

The translation extension locus $EL_{\widetilde{G}}(6_3)$ contains only abelian boundary elliptic $\widetilde{G}$ representations, so we look at the holonomy extension locus $HL_{\widetilde{G}}(6_3)$. Let $\widetilde{\rho}$ be a nonabelian boundary hyperbolic $\widetilde{G}$ representation of $\pi_1(6_3)$ in $HL_{\widetilde{G}}(6_3)$. Then the translation number of $\widetilde{\rho}(\mathcal{L})$ must be ± 1. So we only see the $H_{0,\pm 1}(6_3)$ components in the holonomy extension locus of 6_3, except abelian boundary hyperbolic $\widetilde{G}$ representations which constitute $H_{0,0}(6_3)$.

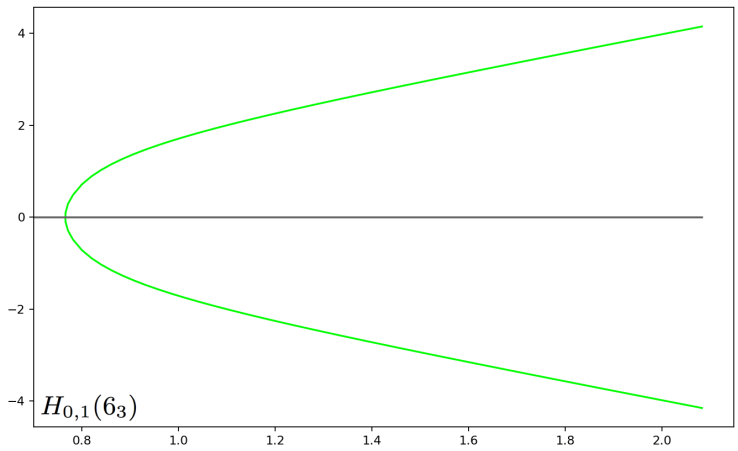

Fig. 7: Holonomy Extension Locus $HL_{\widetilde{G}}(6_3)$.

This figure is the $H_{0,1}(6_3)$ component of the holonomy extension locus of 6_3. (The component $H_{0,-1}(6_3)$ only differ from $H_{0,1}(6_3)$ by reflection about the origin. So we will not show it here.) There are two asymptotes of slope -2 and 2. So for every rational $r \in (-2, 2)$, $\pi_1(M(r))$ has a nonabelian G representation. However, such a representation does not lift to $\widetilde{G}$ because the translation number of $\widetilde{\rho}(\mathcal{L})$ does not vanish, which means we cannot obtain an interval of left-orderable surgery slopes by constructing $\widetilde{G}$ representations.

9. Conjecture

9.1. *Improving the Results*

As observed in Figure 4, the arc in the holonomy extension locus $H_{0,0}(6_2)$ has another asymptote of slope $4 = -(4m+4)$, in addition to the slope $-8 = -4n$ which is confirmed by Proposition 5.3. This phenomenon is also observed in other examples of $J(2m+1, 2n)$ with $m < -1$ and $n > 0$. Actually, according to our computational data, the other slope should be $-(4m+4)$. Unfortunately the author does not know how prove this.

But as we will see next, $-4n$ is the boundary slope of some incompressible surface in $S^3 - J(2m+1, 2n)$. We follow the notation in [16]. For rational number $0 < \frac{p}{q} < 1$, there is a continued fraction expansion

$$\frac{p}{q} = [a_1, -a_2, a_3, -a_4, \ldots, \pm a_k] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_k}}}, \quad |a_k| > 1\,.$$

Consider the case when $m < -1$. The two-bridge slope of $J(2m+1, 2n)$ is $\cfrac{1}{2m+1+\cfrac{1}{2n}} = [2m+1, -2n]$. Let n^+ and n^- be the number of positive and negative numbers in $[2m+1, -2n]$, then $n^+ = 0$ and $n^- = 2$.

Rewrite this slope in the unique continued fraction expansion with each number even, then

$$\frac{1}{2m+1+\cfrac{1}{2n}} = \frac{1}{2m+2+\cfrac{1-2n}{2n}} = [2m+2, \underbrace{2, \ldots, 2}_{2n-1}].$$

Let n_0^+ and n_0^- be the number of positive and negative numbers in $[2m+2, \underbrace{2, \ldots, 2}_{2n-1}]$, then $n_0^+ = 2n-1$ and $n_0^- = 1$.

So $m(2, \ldots, 2) = 2[(n^+ - n^-) - (n_0^+ - n_0^-)] = -4n$. By [16, Proposition 2], $-4n$ is the boundary slope of some incompressible surface in $S^3 - J(2m+1, 2n)$.

9.2. *General Case for Two-Bridge Knots*

In [15, Lemma 3.6], the author noticed that an arc in the holonomy extension locus approaches the line through the origin of slope equal to the slope of the incompressible surface associated to an ideal point of the character variety. So Dunfield made the following conjecture.

Conjecture 9.1. **Let K be a two-bridge knot. Suppose the Alexander polynomial of K has a root p_0. For any rational $r \in I$ as defined below, Dehn filling of K of slope r is orderable, with**

$$I = \begin{cases} (-S_1, -S_2) & \text{if } p_0 > 0, \\ (-\infty, k) & \text{if } p_0 \text{ is a unit complex number}, \end{cases}$$

where S_1 and S_2 are slopes of incompressible surfaces associated to ideal points of the $\mathbf{PSL}_2\mathbb{C}$ character variety of the complement of K, and $0 < k \leq 2g(K) - 1$ is some odd integer.

The interval is optimal in the sense that any interval I' such that $\pi_1(M(r))$ has a nontrivial $\widetilde{PSL_2(\mathbb{R})}$ representation for any rational $r \in I'$, is contained in I. This conjecture is also the motivation of this paper.

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