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RESEARCH ARTICLE

Ricci fall-off in static and stationary non-singular spacetimes, revisited: the null geodesic method

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Abstract

We revisit an older concept in singularity theory: that in the presence of the strong energy condition (SEC), a static or stationary spacetime must have a quadratic fall-off in a characteristic Ricci quantity, in order for the spacetime to be without singularities (or, at least, to be both globally hyperbolic and timelike or null geodesically complete). We replace SEC with the null energy condition (NEC), and apply the methods used previously on timelike geodesics, to null geodesics instead. The results are noticeably weaker for the NEC case than for SEC: using a somewhat different characteristic measure of Ricci curvature, we obtain a fall-off which is quadratic only if there is not much asymptotic change in the size of the Killing field: we employ a ratio of maximum size to square of minimum size of the Killing field—within a ball of given radius r—in addition to $1/r^2$.

Keywords Singularity theorem · Stationary · Static · Ricci fall-off · Null energy condition

Contents

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1 Introduction

Singularity theorems fall into two classes: (1) those that use the behavior of null geodesics (like the Penrose singularity theorem [8]) and thus must assume the null convergence condition, and (2) those that use the behavior of timelike geodesics (like the Hawking singularity theorem [6]) and thus must assume the timelike convergence condition. Here the null convergence condition is the statement that $\mathrm{Ric}(k, k) \geq 0$ for all null vectors k, while the timelike convergence condition is the statement that $\mathrm{Ric}(t, t) \geq 0$ for all timelike vectors t. (By continuity the timelike convergence condition implies the null convergence condition).

These convergence conditions are more commonly stated as energy conditions: that is, one assumes the Einstein field equation and writes the convergence condition as a condition on the stress-energy tensor T. The null convergence condition then becomes the Null Energy Condition (NEC) which states that $T(k, k) \geq 0$ for all null k, while the timelike convergence condition becomes the strong energy condition (SEC) which states that $T(t, t) - \frac{1}{2}\langle t, t\rangle \mathrm{tr}(T) \geq 0$ for all timelike t.

Even in the early days of the singularity theorems, it was realized that the Strong Energy Condition was problematic from the point of view of physics; for example, Hawking and Ellis [7] note that a massive scalar field violates the Strong Energy Condition. However, over the years the Strong Energy Condition has become considerably more problematic due to three developments: (1) the theory of inflationary cosmology, (2) the discovery of the Higgs boson, and (3) the discovery of the acceleration of the expansion of the universe:

It's not only a massive scalar field that violates the Strong Energy Condition, but also any scalar field with a positive potential. Such a field plays an essential role in inflationary theories of the early universe, which are widely considered to be our best explanation for the detailed properties of both the Cosmic Microwave Background and the distribution of galaxies and galaxy clusters.

At the time [7] was written, it was not known whether nature contained a fundamental scalar field with a positive potential. However, such a scalar field was theorized: the Higgs field that both gives mass to other particles and whose quantum excitations are the Higgs boson. Now that the Higgs boson has been discovered, we know that nature contains a fundamental matter field that violates the Strong Energy Condition.

Even more embarassing for the Strong Energy Condition was the discovery of the acceleration of the expansion of the universe. In a Friedmann–Lemaitre–Robertson–Walker spacetime an accelerated expansion can only happen through a violation of the Strong Energy Condition. Thus we can now say definitively that not only *can* nature violate the Strong Energy Condition, but also at the present time most regions of space *are* places where the Strong Energy Condition is violated.

Thus, from the physical point of view, it is better to have singularity theorems that don't depend on the Strong Energy Condition. Based on this consideration, your authors are motivated to revisit their own "non-singularity" result of 1997 [1]. That paper considered a particular class of spacetimes: those that are static or stationary, globally hyperbolic, and non-singular. We then used the methods of the singularity theorems (assuming the Strong Energy Condition and treating the behavior of timelike geodesics) to obtain bounds on the behavior of the Ricci curvature in such spacetimes.

We found that a characteristic eigenvalue of the Ricci tensor $R^a{}_b$ must fall off quadratically in terms of a geometrically important conformal metric on the leaf-space of the foliation $\mathcal{F}$ of integral curves of the Killing field. Put another way: if the Ricci curvature does not exhibit this quadratic fall-off condition, yet we assume SEC and global hyperbolicity, then timelike-geodesic completeness must fail: the classic rendering of the singularity theorems. Along the way we obtained some general results on the topology and geometric properties of this class of spacetimes.

The current paper is an attempt to obtain analogous results to those of [1] but not assuming SEC and using the properties of null geodesics rather than timelike geodesics. In Sect. 2 we will summarize the results of [1], including results that are independent of energy conditions and can therefore also be used in the present work. In Sect 3 we treat static spacetimes, while Sect. 4 treats stationary spaccetimes (with somewhat less general applicability). Discussion is presented in Sect. 5. In general: the results are not as simply expressed as when using SEC and timelike geodesics (in particular, asymtoptic behavior of the size of the Killing field has a strong effect), and they can be unsatisfying in specific situations (making the results vacuous); but, then, the universe is not as we supposed it a quarter century gone.

2 Earlier results summarized

We recall here the chief results from [1], including the notation there.

We let M be a chronological stationary spacetime with U its Killing field. Let $\mathcal{F}$ be the foliation of integral curves of U (all of which must be lines, not circles, as M is chronological), with $Q = M/\mathcal{F}$ the leaf-space of the foliation (also called the stationary orbit-space), i.e., equivalence relation is "lying on the same foliate"; $\pi : M \to Q$ is the projection.

Theorem 2.1 *If M is timelike or null geodesically complete, then U is complete and Q is a (Hausdorff) manifold; also, M is diffeormorphic to $\mathbb{R} \times Q$.*

Proof Lemma 1, Theorem 2, and Theorem 3 in [1]. □

Although, when M is globally hyperbolic, this yields Q as having the topology of a Cauchy surface, it can be non-trivial to locate Cauchy surfaces in the spacetime, even in the static case: if M is not simply connected, then the rest-spaces to the Killing field are not diffeormorphic to Q.

But we can invest Q with a great deal of important geometry. With U^a (or U) being the Killing field, let $\Omega = |U|^2 = -U_a U^a$, a function on Q, and define the 1-form α on M by $\alpha(U) = 1$ and $\mathrm{Ker}(\alpha) = U^\perp$ (i.e., $\alpha_a = -(1/\Omega)U_a$). Then, following [2], we derive a Riemannian metric $h = h_{ab}$ on Q such that we can express the spacetime metric g on M via

$$g = -(\Omega \circ \pi)\alpha^2 + \pi^* h$$

or

$$g_{ab} = -\Omega \alpha_a \alpha_b + h_{ab}$$

Then M is static iff $d\alpha = 0$.

The geometrically important metric on Q, however, is not h, but what we may call the *geometric conformal metric* $\bar{h} = (1/\Omega)h$ (to distinguish this from another conformal metric we will also be using). If, similarly, we let $\bar{g} = (1/(\Omega \circ \pi))g$, then we have, on M,

$$\bar{g} = -\alpha^2 + \bar{h}$$

Theorem 2.2 *If M—timelike or null geodesically complete and stationary—is globally hyperbolic, then Q is complete in the geometric conformal metric.*

Proof Theorems 4 and 8 in [1]. □

Another general result on stationary spacetimes and covering spaces:

Lemma 2.3 *Let $\tilde{M}$ be the universal cover of a spacetime M. Then $\tilde{M}$ (with metric induced from M) inherits these properties from M:*

(1) *static or stationary,*
(2) *energy conditions such as SEC or NEC,*
(3) *geodesic completeness of any type,*
(4) *global hyperbolicity.*

Furthermore, if M is stationary and chronological, with complete Killing field:

Let $\tilde{Q}$ be the universal cover of Q; let $p_M : \tilde{M} \to M$ and $p_Q : \tilde{Q} \to Q$ be the corresponding projections; and let $\tilde{\pi} : \tilde{M} \to \tilde{Q}$ be defined by $p_Q \circ \tilde{\pi} = \pi \circ p_M$. Then $\tilde{\pi}$ is the projection of $\tilde{M}$ to its stationary orbit-space.

Proof Lemmas 4.1 and 4.2 of [1]. □

The crucial role of SEC is this: SEC—more precisely, the timelike convergence condition—gives us that for timelike vector t, $\text{Ric}(t, t) \geq 0$ ($R_{ab}t^a t^b \geq 0$). A result of Hall [3] then yields $R^a{}_b$ (also denoted $\text{Ric}^\sharp$, using $\sharp/\flat$ notations for raising and lowering indices) has exactly one non-spacelike eigenspace at each point. We denote the corresponding eigenvalue by $-S$. Then we have

Theorem 2.4 *Let M be a spacetime satisfying SEC. Then $S \geq 0$, and for any unit timelike vector t, $R_{ab}t^a t^b \geq S$.*

Proof Proposition 5 of [1]. □

In case M is static, then U is itself an eigenvector of $R^a{}_b$, and we obtain $S = R_{ab}U^a U^b/|U|^2$.

That is all the ingredients needed for the static case; but for non-static stationary, we need a bit more. First note that the projection $\pi : M \to Q$ is a line-bundle: M has an $\mathbb{R}$-action given by $s \cdot x = \gamma_x(s)$, where γ_x is the U-integral curve through x with $\gamma_x(0) = x$. This also means π has smooth cross-sections $z : Q \to M$, i.e., so that $\pi \circ z = \text{id}_Q$; indeed, there is gauge-freedom in choice of such cross-sections, in that for any such cross-section z, for any smooth function $\eta : Q \to \mathbb{R}$, $z^\eta : Q \to M$

is also a smooth cross-section, defined by $z^\eta(q) = \eta(q) \cdot z(q)$—and for any fixed cross-section z, this construction encompasses all possible cross-sections.

Such cross-sections define global time-functions τ on M (also smooth), obeying $(d\tau)U = 1$; for a given cross-section z, this works by defining $\tau_z : M \to \mathbb{R}$ so that $\tau_z(x) \cdot z(\pi(x)) = x$, i.e., $\tau_z(x)$ moves $z(\pi(x))$ back to x. Conversely, any smooth time-function τ on M obeying $(d\tau)U = 1$ defines a smooth cross-section z_τ via $z_\tau(q)$ is that element $x \in \pi^{-1}(q)$ such that $\tau(x) = 0$; and the operations $z \mapsto z_\tau$ and $\tau \mapsto \tau_z$ are inverses. If we start with τ, derive z_τ, then modify that by $\eta : Q \to \mathbb{R}$ yielding $(z_\tau)^\eta$, then the corresponding time-function is given by $\tau^\eta(x) = \tau_{(z_\tau)^\eta}(x) = \tau(x) + \eta(\pi(x))$; and for fixed τ, that encompasses all possible such time-functions.

Any such time-function τ then defines a smooth 1-form ω, the “drift form”, defined on Q by

$$\alpha - d\tau = \pi^*\omega$$

giving us

$$\begin{aligned} g &= -(\Omega \circ \pi)(d\tau + \pi^*\omega)^2 + \pi^* h \\ \bar{g} &= -(d\tau + \pi^*\omega)^2 + \pi^*\bar{h} \end{aligned}$$

Notice that if we change τ to $\tau^\eta = \tau + \eta \circ \pi$, we have $d(\tau^\eta) = d\tau + \pi^* d\eta$, so ω is well-defined up to an exact 1-form on Q (ω shifting to $\omega - d\eta$). We have M is static iff $d\omega = 0$, in which case ω defines a unique element of the first de Rham cohomology of Q. But things are more complicated in the general stationary case.

Pick a basepoint $q_0 \in Q$. Define $Z(Q)$ (the *cycles* of Z) to be the Abelian group generated by all the loops in Q base-pointed at q_0, with concatenation being sum, reparametrization in the same direction being irrelevant, and reversed orientation being the negative. Change in basepoint is an isomorphism of groups, and we can ignore it. Any cycle z has a length $\bar{L}(z)$ defined using the geometic conformal metric $\bar{h}$ on Q. For any basepointed loop c in Q, let $[c]$ be the cycle defined by c (i.e., if c includes basepointed sub-loops, $[c]$ is independent of rearranging the order in which one traverses them—and eliminating any pair of sub-loops which are the same execept for having reverse orientations); and any cycle can be represented as coming from a loop in such a manner—in fact, by a loop which doesn’t contain any pairs of reverse-orientation sub-loops.

Define the *cocycles* $Z^*(Q)$ of Q by $Z^*(Q) = \mathrm{Hom}(Z(Q), \mathbb{R})$. Any 1-form θ on Q defines a cocycle $\{\theta\}$ via $\{\theta\}([c]) = \int_c \theta$. We define the *weight* $\mathrm{wt}(\beta)$ of any cocycle β by

$$\mathrm{wt}(\beta) = \sup_{z \in Z(Q)} \frac{\beta(z)}{\bar{L}(z)}$$

Define the *fundamental cocycle* of M by $\beta_M = \{\omega\}$, for ω the drift-form on Q. Then a great deal of the global causal structure of M is bound up in $\mathrm{wt}(\beta_M)$; see, for example, [4]. In particular, M is chronological iff $\mathrm{wt}(\beta_M) \leq 1$, and $\mathrm{wt}(\beta_M) < 1$

implies M is causal. For a static spacetime, the fundamental cocycle is an element of $H^1_{\mathrm{dR}}(M)$, the first de Rham cohomology, thus: consider any loop of static observers, say, from q_0 to q_0. If we constrain a photon to move along this path—or any path homotopic to it—then the fundamental cocycle returns the difference between the q_0-observed time between photon emission and its return to that observer, and the length of the photon's path in Q (measured in the geometric conformal metric $\bar{h}$). We naively may expect this difference to be 0, but it's easy to construct examples in which it is not (take a product spacetime $\mathbb{R}^1 \times Q$ for Q not simply connected, pass to the universal cover, then take a quotient by $\pi_1(Q)$, but with non-trivial action on the $\mathbb{R}^1$-factor). Note the fundamental cocycle is constant within a homotopy class of paths.

For stationary spacetimes, the only difference is that the fundamental cocycle can vary continuously among paths, instead of being discretely defined on homotopy classes. In either case, the weight of the fundamental cocycle tells us the extent to which the spacetime is subject to a sort of chronological decoherence among the stationary observers.

The main result in the SEC case is this:

Theorem 2.5 *Let M be a stationary, globally hyperbolic, timelike or null geodesically complete spacetime satisfying SEC and having $wt(\beta_M) = w < 1$; or, if M is static, take $w = 0$ instead. For any point $q_0 \in Q$, for any $r > 0$, define $S_r(q_0) = \inf_{\bar{B}_r(q_0)} S$, where $\bar{B}_r(q_0)$ is the ball of radius r in Q, using the geometric conformal metric $\bar{h}$. Then for all $r > 0$,*

$$S_r(q_0) \leq K/r^2$$

where $K = 3\pi^2/(4|U|^2_{q_0}(1-w)^2)$.

Proof Theorems 6 and 10 of [1]. □

3 Static case

The approach followed here is based on that in [1], but we are forced to be somewhat more general in our approach. For the static case, we again will make use of a timelike eigenvalue for $R^a{}_b$—meaning, the eigenvalue for the (necessarily existing) timelike eigenspace—but we will also have to consider the other eigenvalues as well. The important point is that the Killing vector U is itself an eigenvector for $R^a{}_b$, which implies that there is a basis of eigenvectors for the tangent space at each point (since $R^a{}_b$, being self-adjoint, preserves the perp-space to any eigenvector; then, with U being a timelike eigenvector, $R^a{}_b$ is self-adjoint in the spacelike space $U^\perp$, which then splits into orthogonal eigenspaces as per usual behavior with a positive-defininte metric).

This result was referred to in [1], but was never used. As it is crucial to the static case, we call specific attention to it now.

Proposition 3.1 *In any static spacetime, the Killing vector U is an eigenvector of the Ricci tensor $R^a{}_b$.*

Proof sketch A stationary spacetime M has metric

$$g = -(\Omega \circ \pi)(d\tau + \pi^*\omega)^2 + \pi^* h$$

for U the Killing field, $\pi : M \to Q$ the projection to stationary orbit-space, $\Omega = |U|$, τ any function on M with $(d\tau)U = 1$, ω the corresponding drift-form on Q, and h the induced Riemannian metric on Q. In abstract-index notation (with $\pi_a{}^b = \pi^*$, $\pi^a{}_b = \pi_*$):

$$g_{ab} = -\Omega(d_a\tau + \pi_a{}^c\omega_c)(d_b\tau + \pi_b{}^d\omega_d) + \pi_a{}^c\pi_b{}^d h_{cd}$$

Let ∇^h, Δ^h, Hess^h, and $(R^h)_{ab}$ denote, respectively, the covariant derivative, Laplacian, Hessian operator, and Ricci tensor in (Q, h). For any $q \in Q$ and $x \in \pi^{-1}(q)$, let $\beta : T_q Q \to T_x M$ be such that $\pi_* \circ \beta = \mathrm{id}$ and image of β is in $U^\perp$. Then we have, for any X^a, Y^a in Q,

$$\begin{aligned}
R_{ab}U^aU^b &= -\frac{1}{4}\frac{1}{\Omega}|(\nabla^h)^a\Omega|^2 - \frac{1}{2}\Delta^h\Omega \\
R_{ab}U^a\beta^b{}_dY^d &= \frac{1}{2}(d_a\omega_b)(\nabla^h\Omega)^aY^b \\
R_{ab}\beta^a{}_cX^c\beta^b{}_dY^d &= \left(\frac{1}{2}(\mathrm{Hess}^h_\Omega)_{ab} - \frac{3}{4}\frac{1}{\Omega^2}(d_a\Omega)(d_b\Omega) + (R^h)_{ab}\right)X^aY^b
\end{aligned}$$

or

$$\begin{aligned}
\mathrm{Ric}(U,U) &= -\frac{1}{4}\frac{1}{\Omega}|\nabla^h\Omega|^2 - \frac{1}{2}\Delta^h\Omega \\
\mathrm{Ric}(U,\beta Y) &= \frac{1}{2}(d\omega)(\nabla^h\Omega, Y) \\
\mathrm{Ric}(\beta X,\beta Y) &= \left(\frac{1}{2}\mathrm{Hess}^h_\Omega - \frac{3}{4}\frac{1}{\Omega^2}(d\Omega)^2 + \mathrm{Ric}^h\right)(X,Y)
\end{aligned}$$

The important point here is that for M static, $d\omega = 0$, producing U^a as an eigenvector of $R^a{}_b$. (If M is not static, then U is not an eigenvector, unless $(d\omega)(\nabla^h\Omega, -) = 0$.) □

So in a static spacetime U is a timelike eigenvector; call its eigenvalue $-S$. Since $\mathrm{Ric}^\sharp = R^a{}_b$ is self-adjoint, it follows that $U^\perp$, spacelike subspace, is preserved by $R^a{}_b$, hence, has an orthonormal basis of (spacelike) eigenvectors; let p be the smallest (possibly, most negative) of those eigenvalues.

Proposition 3.2 *Let M be a static spacetime obeying NEC. Then*

$$S + p \geq 0$$

Proof Any null vector k can be expressed as $k = aU + k^\perp$ for non-0 a, where $k^\perp \in U^\perp$, hence, is spacelike, and $|a||U| = |k^\perp|$. Then $\mathrm{Ric}^\sharp k = \mathrm{Ric}^\sharp(aU + k^\perp) =$

$a(-S)U + \mathrm{Ric}^\sharp(k^\perp)$, and we know $\mathrm{Ric}^\sharp k$ lies in $U^\perp$. By NEC (more properly, the null convergence condition), we get

$$\begin{aligned}0 &\leq \mathrm{Ric}(k,k)\\ &= \langle \mathrm{Ric}^\sharp k, k\rangle\\ &= \langle -aSU + \mathrm{Ric}^\sharp(k^\perp), aU + k^\perp\rangle\\ &= a^2 S|U|^2 + \langle \mathrm{Ric}^\sharp(k^\perp), k^\perp\rangle\end{aligned}$$

Select k so that $k^\perp$ is in the eigenspace of p; then $\langle \mathrm{Ric}^\sharp(k^\perp), k^\perp\rangle = \langle pk^\perp, k^\perp\rangle = pa^2|U|^2$. Combining with the inequality above yields $0 \leq a^2|U|^2(S+p)$. □

It is this sum, $S + p$, that we will use as a characteristic value for the Ricci curvature. (While SEC allowed us to deduce that $S \geq 0$, NEC doesn't give us that; instead we have the somewhat less physically compelling quantity $S + p \geq 0$. In the static case—which implies timelike Ricci eigenvector—we can employ the interpretations of section 4.3 of [7]: S is the static-observer energy density, and the other eigenvalues are pressures in the static rest-frame, with p being the smallest of those: $S + p$ is static energy density plus minimum of all static pressures.) Define $A = S + p$. As in [1], we will be concerned with how this quantity behaves with respect to the geometric conformal metric $\bar{h}$ in the observer space Q. Recall that Theorem 2.2 guarantees for us (in a timelike or null geodesically complete stationary spacetime) that Q is complete in $\bar{h}$; thus, any closed metric ball $\bar{B}_r(q_0) = \{q \in Q \mid \bar{d}(q, q_0) \leq r\}$ (with $\bar{d}$ denoting distance in $\bar{h}$) is compact. We define, for $r > 0$,

$$\bar{A}_r^- = \inf_{q \in \bar{B}_r(q_0)} A(q)$$

(the bar reminding us we are using $\bar{h}$ in the definition). By Proposition 3.2, we know $\bar{A}_r^- > 0$ unless there is some point q in $\bar{B}_r(q_0)$ with $\bar{A}_r^- = 0$ (in which case $\bar{A}_s^- = 0$ for all $s \geq r$).

It turns out that for null geodesics, the geometric conformal metric is not the one that has direct application. Rather, we will have need to consider what we'll call the *null-specific conformal metric* $\hat{h} = \Omega h$. So we have $\hat{h} = \Omega^2 \bar{h} = |U|^4 \bar{h}$. We will have occassion to use

$$\bar{\Omega}_r^- = \inf_{q \in \bar{B}_r(q_0)} \Omega(q)$$
$$\bar{\Omega}_r^+ = \sup_{q \in \bar{B}_r(q_0)} \Omega(q)$$

Note that in the treatment of [1], the results on Ricci fall-off really make sense only when Q is non-compact (although the statements of the theorems aren't false in case Q is compact: merely not very interesting). We will specifically restrict to the non-compact case in this paper. In light of Theorem 2.1, we see this is equivalent to saying that the Cauchy surface topology is non-compact.

Theorem 3.3 *Let M be a static, globally hyperbolic (with non-compact Cauchy surface), null geodesically complete spacetime obeying NEC; take $\pi : M \to Q$ to be the projection to the static-observer space and the spacetime metric to be $g = -(\Omega \circ \pi)\alpha^2 + \pi^* h$ with U^a the Killing field, $\Omega = |U|^2$, $\alpha_a = -(1/\Omega \circ \pi)U_a$, h the induced Riemannian metric on Q, $A = S + p$ the sum of Ricci eigenvalues, and $\bar{A}_r^-$, $\bar{\Omega}_r^+$, $\bar{\Omega}_r^-$ defined as above using the geometric conformal metric $\bar{h} = (1/\Omega)h$ with respect to any base-point $q_0 \in Q$. Then for all $r > 0$,*

$$\bar{A}_r^- \leq 2\pi^2 \frac{\bar{\Omega}_r^+}{(\bar{\Omega}_r^-)^2} \frac{1}{r^2}$$

Proof (A) First we consider the case for M simply connected.

We can write the spacetime metric as

$$g = -(\Omega \circ \pi)(d\tau + \pi^* \omega)^2 + \pi^* h$$

where $\tau : M \to \mathbb{R}$ derives from any cross-section of $\pi : M \to Q$ and the 1-form ω on Q is thereby derived from $\alpha = d\tau + \pi^* \omega$ represesenting an element $\{\omega\}$ of the first de Rham cohomology $H^1_{\mathrm{dR}}(Q)$ (i.e., since M is static, $d\alpha = 0$, hence $d\omega = 0$). Since M is simply connected, so is Q (Theorem 2.1). It follows that $H^1_{\mathrm{dR}}(Q) = 0$, so $\{\omega\} = 0$, so there is some $\eta : Q \to \mathbb{R}$ with $\omega = d\eta$. Replacing τ by $\tau + \eta \circ \pi$, we can rewrite g as

$$g = -(\Omega \circ \pi)(d\tau)^2 + \pi^* h$$

that is to say, we have presented M as *standard static*. (This always can be done locally, but in the simply connected case we have it globally.)

Consider any null geodesic $\beta : [0, b] \to M$ with $\pi(\beta(0)) = q_0$. First note that $E = \langle \dot{\beta}, U \rangle$ is constant, simply due to β being a geodesic: Let $c = \pi \circ \beta : [0, b] \to Q$. We can extend $\dot{\beta}$ to a vector field k on $\pi^{-1}[c]$ via the $\mathbb{R}$-action, $k_{t \cdot \beta(\lambda)} = t \cdot \dot{\beta}(\lambda)$, i.e., $\mathcal{L}_U(k) = 0$; then, since this action is an isometry, the curves $t \cdot \beta : [0, b] \to M$ (t constant) are also geodesics. It follows (note here that $k = k^a$ is a vector) that $\dot{\beta}\langle \dot{\beta}, U \rangle = k\langle k, U \rangle = \langle k, \nabla_U k \rangle = 0$. Let us take β and U as both future-directed, so $E < 0$.

Since β is null, we have $\Omega((d\tau)\dot{\beta})^2 = h(\dot{c}, \dot{c})$. Rewrite $d\tau = \alpha$ and recall $\alpha_a = -(1/\Omega \circ \pi)U_a$. Thus we have $|\dot{c}|_h = \sqrt{\Omega(-E/\Omega)^2} = -E/\sqrt{\Omega}$ or $-E = \sqrt{\hat{h}(\dot{c}, \dot{c})}$, where $\hat{h} = \Omega h$ is the null-specific conformal metric. It follows that if we let $\hat{L}$ denote the length-functional from $\hat{h}$, then $\hat{L}(c) = -Eb$.

But $\hat{h}$ is not known to be a complete metric on Q; it is $\bar{h} = (1/\Omega)^2 \hat{h}$ that has that property from M being timelike/null geodesically complete. So we will be obliged to relate these two metrics via $\bar{\Omega}^+$ and $\bar{\Omega}^-$.

As in the proof of Proposition 3.2, for any null vector k, we can write $k = aU + k^\perp$ and obtain $\mathrm{Ric}(k, k) = a^2 S\Omega + \langle \mathrm{Ric}^\sharp(k^\perp), k^\perp \rangle$; and we also have $\langle \mathrm{Ric}^\sharp(k^\perp), k^\perp \rangle \geq p|k^\perp|^2 = pa^2\Omega$ (with p the smallest eigenvalue of $\mathrm{Ric}^\sharp$ restricted to $U^\perp$). Taking $k = \dot{\beta}$, we have $E = -a\Omega$. This gives us $\mathrm{Ric}(\dot{\beta}, \dot{\beta}) \geq (S + p)a^2\Omega = AE^2/\Omega$. If

$\pi \circ \beta$ is contained within $\bar{B}_r(q_0)$, then $\mathrm{Ric}(\dot{\beta}, \dot{\beta}) \geq \bar{A}_r^- E^2/\bar{\Omega}_r^+$. (Note that as β has compact image, there must exist such r: c cannot escape all $\bar{B}_r(q_0)$.)

But what do we take for r? First note that $\dot{\beta}\tau = (d\tau)\dot{\beta} = -\langle U, \dot{\beta}\rangle/\Omega = -E/\Omega$. The r we want is $r = \bar{L}(c) = \int_0^b \frac{1}{\sqrt{\Omega}}|\dot{c}|_h = \int_0^b \dot{\beta}\tau = \Delta(\tau) = \tau(\beta(b)) - \tau(\beta(0)$. So let us define, for any $\lambda \geq 0$, $\rho(\lambda) = \tau(\beta(\lambda)) - \tau(\beta(0))$; we will be taking $r = \rho(b)$. We have

$$\begin{aligned} \frac{d}{d\lambda}\rho(\lambda) &= \frac{d}{d\lambda}\tau(\beta(\lambda)) \\ &= \dot{\beta}\tau \\ &= -\frac{E}{\Omega} \end{aligned}$$

which tells us that $\rho(b) = \int_0^b \frac{-E}{\Omega} \leq \int_0^b \frac{-E}{\bar{\Omega}^-_{\rho(b)}} = \frac{-Eb}{\bar{\Omega}^-_{\rho(b)}}$ or $\rho(b)\bar{\Omega}^-_{\rho(b)} \leq -Eb$. Conclusion: any r satisfying $c \subset \bar{B}_r(q_0)$—i.e., $\beta \subset \pi^{-1}[\bar{B}_r(q_0)]$—must satisfy $r\bar{\Omega}_r^- \leq -Eb$.

We can, of course, always choose to parametrize our null geodesic so that $E = -1$; let us do so from this point onwards. Thus, any r yielding c as contained within $\bar{B}_r(q_0)$ satisfies $r\bar{\Omega}_r^- \leq b$.

The Null Myers Theorem (e.g., Proposition 2.6 [5]) asserts that if $\mathrm{Ric}(\dot{\beta}, \dot{\beta}) \geq (n-2)q^2$, then β is not maximizing on $[0, b]$ if $b > \pi/q$. We have, from above, $\mathrm{Ric}(\dot{\beta}, \dot{\beta}) \geq \bar{A}_r^-/\bar{\Omega}_r^+$ for the r considered above.

Thus, on the one hand, $\beta : [0, b] \to M$, null geodesic with $\langle\dot{\beta}, U\rangle = -1$, cannot be maximizing if $b > \pi\sqrt{2}\sqrt{\frac{\bar{\Omega}_r^+}{\bar{A}_r^-}}$ for $\pi \circ \beta$ contained within $\bar{B}_r(q_0)$, which implies $r\bar{\Omega}_r^- \leq b$. On the other hand, consider any x_0 with $\pi(x_0) = q_0$ and any x on the boundary of $I^+(x_0)$. Since M is globally hyperbolic, there is a maximizing null geodesic $\beta : [0, b] \to M$ from x_0 to x, with $\langle\dot{\beta}(0), U\rangle = -1$. A given null geodesic could simply leave the boundary of $I^+(x_0)$. However, since we are assuming a non-compact Cauchy surface, it follows from the result of [8] that not all the null geodesics emanating from x_0 can leave this boundary. We now consider only those geodesics that remain on the boundary (i.e., are maximizing). This gives us $\frac{\bar{\Omega}_r^+}{\bar{A}_r^-} \geq b^2/(2\pi^2)$, so $\bar{A}_r^- \leq \frac{2\pi^2\bar{\Omega}_r^+}{b^2}$. Then we use $r\bar{\Omega}_r^- \leq b$ (from $\beta[0, b] \subset \bar{B}_r(q_0)$), and we have the promised result:

$$\bar{A}_r^- \leq 2\pi^2 \frac{\bar{\Omega}_r^+}{(\bar{\Omega}_r^-)^2}\frac{1}{r^2} \tag{*}$$

(B) Now consider the general case. We use Lemma 2.3 and examine the static spacetime $\tilde{M}$, the universal cover of M, with $p_M : \tilde{M} \to M$ the universal covering space projection; the induced Killing field $\tilde{U}$ on $\tilde{M}$ and the projection $\tilde{\pi} : \tilde{M} \to \tilde{Q}$; and the universal covering space projection $p_Q : \tilde{Q} \to Q$, with $p_Q \circ \tilde{\pi} = \pi \circ p_M$. We know $\tilde{M}$ is globally hyperbolic, is null complete, and satisfies NEC, so by (A), we have (*), as applied to $\tilde{M}$ with respect to some $\tilde{q}_0$ with $p_Q(\tilde{q}_0) = q_0$. But $p_Q : (\tilde{Q}, \bar{\tilde{h}}) \to (Q, \bar{h})$ is distance-decreasing, and all quantities are otherwise identical (p_Q being a local isometry). It follows that (*) applies equally well to M and q_0. □

4 Stationary case

If we do not know M is static, then we gain no simplicity by looking at $\tilde{M}$. So instead we must deal directly the given metric on M. However, we can take some advantage of knowing we can write the metric as $g = (\Omega \circ \pi)(-(d\tau + \pi^*\omega)^2 + \pi^*\bar{h})$, where we have total gauge freedom to replace ω by $\omega^\eta = \omega + d\eta$ for arbitrary $\eta : Q \to \mathbb{R}$. We are concerned with maximizing null geodesics in M. These project to curves in Q, but not necessarily to loops (as was the case with the SEC exploration in [1], where we looked at maximizing timelike geodesics between elements of a single stationary-observer orbit); thus we cannot make use of the integration of ω around closed curves (which is invariant under change to any ω^η). We will, instead, simply control the size of ω directly:

Whereas in [1] we required $|\int_c \omega| \leq w\bar{L}(c)$ for any base-pointed loop c in Q, for some $w < 1$, we will here instead require $|\omega|_{\bar{h}} \leq w$ at all points, for some constant $w < 1$ (i.e., using the geometric conformal metric $\bar{h}$, $|\omega|_{\bar{h}} = (\bar{h}_{ab}\omega^a\omega^b)^{\frac{1}{2}}$)—or at least, that this be true for some presentation of the metric, i.e., choice of cross-section of π. This gives us, for any curve $c : [0, b] \to Q$, $|\int_c \omega| = \int_0^b \omega(\dot{c}) \leq \int_0^b |\omega|_{\bar{h}}|\dot{c}|_{\bar{h}} \leq w \int_0^b |\dot{c}|_{\bar{h}} = w\bar{L}(c)$.

Now, for any null curve $\beta : [0, b] \to M$, with $c = \pi \circ \beta$, recall from the proof of Theorem 3.3 we have $(d\tau + \pi^*\omega)\dot{\beta} = |\dot{c}|_{\bar{h}}$. Then with $|\omega| \leq w$ we have

$$\begin{aligned} \dot{\beta}\tau &= |\dot{c}|_{\bar{h}} - \omega\dot{c} \\ &\geq |\dot{c}|_{\bar{h}} - |\omega_{\bar{h}}||\dot{c}|_{\bar{h}} \\ &\geq (1 - w)|\dot{c}|_{\bar{h}} \end{aligned}$$

and we conclude $\Delta\tau = \tau(\beta(b)) - \tau(\beta(0)) \geq (1 - w)\bar{L}(c)$. (This constrasts with the standard static situation—Theorem 3.3, section (A)—in which $\Delta\tau = \bar{L}(c)$; although the non-standard static case is more complex in terms of using $\bar{d}$, we were able to elide that complication by means of direct comparison with the universal cover.)

Another issue is that in the static case we have a timelike eigenvector of Ric and therefore a basis of eigenvectors of Ric. In [1] we used the strong energy condition to reduce to two cases: one where there is a timelike eigenvector and one where there is a null eigenvector. However, that strategy will not work in this case, because we need to consider all the eigenvalues of Ric, not just the timelike (or null) one. Therefore, for the next theorem we will add the assumption that Ric has a timelike eigenvector.

Theorem 4.1 *Let M be a stationary, globally hyperbolic (with a non-compact Cauchy surface), null geodesically complete spacetime obeying NEC, with various quanti-*

ties as specified in Theorem 3.3 (save that Q is the space of stationary, not static, observers); and let us also assume that Ric *has a timelike eigenvector. Suppose for some choice of cross-section of $\pi : M \to Q$ (yielding $\alpha = d\tau + \pi^*\omega$), the corresponding drift form ω has, at all points, geometric-conformal norm $|\omega|_{\bar{h}} \le w$ for some constant $w < 1$. Then for any basepoint $q_0 \in Q$, for all $r > 0$,*

$$\bar{A}_r^- \le \frac{2\pi^2}{(1-w)^2} \frac{\bar{\Omega}_r^+}{(\bar{\Omega}_r^-)^2} \frac{1}{r^2}$$

Proof Pick a point x_0 with $\pi(x_0) = q_0$. Consider any point $q \in Q$ with $\bar{d}(q, q_0) = r$ (recall Q is complete in $\bar{d}$). We know the stationary orbit γ_q enters the future of x_0 (pick any curve $\sigma : [0, L] \to Q$ from q_0 to q, unit-speed in $\bar{h}$; then $P = \pi^{-1}[\sigma]$ is conformal to $(\mathbb{R} \times [0, 1], -(d\tau + f(x)\,dx)^2 + (dx)^2)$ for $f(x) = \omega(\dot{\sigma})$, so $|f(x)| \le |\omega|_{\bar{h}} |\dot{\sigma}|_{\bar{h}} \le w$; it follows that the lightcones in P are no wider than those from the metric $-(d\tau)^2 + (\frac{dx}{1-w})^2$, so each $\pi^{-1}(x)$ enters the future of every point in P). Thus, from global hyperbolicity, there must be a maximizing null geodesic $\beta : [0, b] \to M$ from x_0 to some point x_1 on γ_q, parametrized with $\langle \dot{\beta}, U \rangle = -1$; let $c = \pi \circ \beta$.

From the argument above, we know $\tau(x_1) - \tau(x_0) \ge (1 - w)\bar{L}(c)$. The desired r for which $c \subset \bar{B}_r(q_0)$ is, again, $\bar{L}(c)$; so we have $(1 - w)r \le \Delta\tau$. This effectively puts us back into the same situation as in the proof of Theorem 3.3, section (A), with $(1 - w)r$ substituted for r. And that yields the result. □

But what if we have the non-generic situation, with $R^a{}_b$ having no timelike eigenvector at some points? What can we say then? We just work with what we have instead of with eigenvectors:

At each point we define a quantity

$$B = \inf\{\mathrm{Ric}(k, k) \mid k \text{ is a null vector with } \langle k, U/|U| \rangle = -1\}$$

This is plainly a measure of how “big” Ricci is (and is constant along the stationary orbits, hence, it descends to Q), though it’s not one that we’re used to dealing with; but it’s the one that works for the general algebraic form of Ric. With NEC in place we know $B \ge 0$. As before we define $\bar{B}_r^- = \inf\{B(q) \mid \bar{d}(q, q_0) \le r\}$ for any basepoint q_0.

Theorem 4.2 *Let M be a stationary, globally hyperbolic, null geodesically complete spacetime obeying NEC, with various quantities as specified in Theorem 3.3 (save that Q is the space of stationary, not static, observers). Suppose for some choice of cross-section of $\pi : M \to Q$ (yielding $\alpha = d\tau + \pi^*\omega$), the corresponding drift form ω has, at all points, geometric-conformal norm $|\omega|_{\bar{h}} \le w$ for some constant $w < 1$. Then for any basepoint $q_0 \in Q$, for all $r > 0$,*

$$\bar{B}_r^- \le \frac{2\pi^2}{(1-w)^2} \frac{\bar{\Omega}_r^+}{(\bar{\Omega}_r^-)^2} \frac{1}{r^2}$$

Proof We just everywhere use B in place of A. □

5 Discussion

We first examine some general issues relevant to the current exploration.

To understand the role of all the conditions we use, it is helpful to consider the fate of "would-be counter-examples" to our theorems: that is, spacetimes that appear to violate our conclusions but that do so by failing to satisfy one or more of our conditions.

To begin, consider de Sitter spacetime and anti-de Sitter spacetime. Both spacetimes are geodesically complete. However each of these spacetimes is "spacetime homogeneous" (i.e. the same at all points of spacetime) and therefore neither the Ricci curvature nor anything else can "fall off" in either spacetime. However, each of these spacetimes has $A = 0$, as indeed does any spacetime whose only stress-energy is a cosmological constant. Furthermore, de Sitter spacetime is not globally static, and its Cauchy surfaces are compact; whereas anti-de Sitter spacetime is not globally hyperbolic.

(Another class of would-be counter-examples are low-regularity spaces. For instance, we could marry two copies of anti-de Sitter space together, along their common causal boundary; this produces a smooth manifold that bears a covariant derivative which extends smoothly across the marriage-boundary as well as a smooth sense of causality, but there is no global spacetime metric which extends even continuously across the boundary. Doubled anti-de Sitter space is arguably globally hyperbolic, in having Cauchy hypersurfaces and compact causal diamonds; but it is a low-regularity variety of global hyperbolicity, so does not come under our theorem. This is a subject area that may deserve exploration.)

Two would-be counter-examples that we considered in [1] are the Einstein static universe and Melvin's magnetic universe. The Einstein static universe is homogeneous (the same at all points in space) and thus again no quantity can "fall off." This spacetime fails to be a counter-example to the theorem of [1] because $S = 0$. In the present case A is a nonzero constant. However, the Einstein static universe fails to be a counter-example to our theorem here because its Cauchy surfaces are compact.

Melvin's magnetic universe is a static solution of the Einstein-Maxwell equations representing an infinitely long tube of magnetic flux held together by its own self-gravity. Its curvature "fails to fall off" in the sense that it does not change at all as one moves in the direction parallel to the axis of the flux tube. However, this is not a counter-example to the theorem of [1] because that theorem only states that curvature must fall off in at least one direction, not in all directions. One might expect a similar resolution here; however things are even simpler: $A = 0$ at all points of Melvin's magnetic universe. Furthermore the $A = 0$ property is not some peculiarity of Melvin's magnetic universe, but rather is a property of any static solution of the Einstein-Maxwell equations that is either purely magnetic (like Melvin's magnetic universe) or purely electric. Thus we cannot look to the Einstein–Maxwell equations for interesting would-be counter-examples to our theorems.

We next examine some of the specifics involved in the theorems here, that are different from those using the SEC condition.

One point to be explored is the difference between the static and stationary cases. Unlike in [1], in the stationary case we have not made use of an essential quantity (the weight of the fundamental cocycle), but have simply said, "for some choice of cross-section, find some $w < 1$ for which at all points $|\omega|_{\bar{h}} \leq w$." This is a much less transparent process. Let us look at a specific example in illustration:

We take as spacetime manifold $M = \mathbb{R}^3 = \mathbb{R}^1 \times \mathbb{R}^2$ (coordinates (t, x, y)) with metric (for fixed constant a, $|a| < 1$)

$$g = -(dt + ax \cos y \, dy)^2 + (dx)^2 + (dy)^2$$

so that we have timelike killing field $U = \frac{\partial}{\partial t}$, stationary orbit space $Q = \mathbb{R}^2$ with projection $\pi : M \to Q$ given by $\pi(t, x, y) = (x, y)$ and orbit-space metric $h = (dx)^2 + (dy)^2$; we have length of Killing field $\Omega = 1$. With the obvious cross-section $z : Q \to M$ given by $z(x, y) = (0, x, y)$, we have the corresponding time function $\tau : M \to \mathbb{R}$ given by $\tau(t, x, y) = t$ and drift-form $\omega = ax \cos y \, dy$ on Q. Note that $|\omega|_{\bar{h}} = |ax \cos y|$ is unbounded. As Ω is constant and h is flat, we have $\mathrm{Ric} = 0$ for M (see proof of Proposition 3.1).

We cannot apply Theorem 4.1 to this spacetime with the given presentation, as we do not have ω sufficiently bounded. But we can fix this with an alternative presentation: we will select $\eta : Q \to \mathbb{R}$ so that $|\omega^\eta|_{\bar{h}}$ is bounded appropriately. Specifically: choose $\eta(x, y) = ax \sin y$. Then $\omega^\eta = \omega - d\eta = ax \cos y \, dy - (a \sin y \, dx + ax \cos y \, dy) = -a \sin y \, dx$, and $|\omega^\eta|_{\bar{h}} = |a \sin y|$, and so we can apply Theorem 4.1 with $w = |a|$, so long as $|a| < 1$. The presentation of the metric now is with $\tau^\eta = \tau + \eta \circ \pi$ given by $\tau^\eta(t, x, y) = t + ax \sin y$; that is to say, the metric can be expressed as

$$g = -(d\tau^\eta - a \sin y \, dx)^2 + (dx)^2 + (dy)^2$$

on the same manifold with coordinates (τ^η, x, y) and with cross-section $z^\eta = \eta \cdot z$ given by $z^\eta(x, y) = (ax \sin y, x, y)$ in (t, x, y) coordinates. (However, the application of Theorem 4.1 is trivial: $A_q = 0$ for all points $q \in Q$.)

Another point to consider is just what sort of Ricci fall-off Theorem 3.3 (for instance) gives us. The complication here, as opposed to [1], is the max and min values of Ω over a geometric-conformal ball of radius r; while the former can only grow and the latter only shrink, the ratio can do anything—and, in particular, $\bar{\Omega}_r^+ / (\bar{\Omega}_r^-)^2$ need not decrease with r. Let's explore this with a spherically symmetric standard-static example.

We take $M = \mathbb{R}^1 \times \mathbb{R}^3$ with $\mathbb{R}^3$ realized as $((0, \infty) \times \mathbb{S}^2) \cup \{\mathbf{0}\}$; use coordinates (t, ρ) for $\mathbb{R}^1 \times (0, \infty)$, and for metric use

$$g = -\Omega(\rho)(dt)^2 + (d\rho)^2 + \rho^2 k_{\mathbb{S}^2}$$

where $k_{\mathbb{S}^2}$ is the metric on the round unit $\mathbb{S}^2$ and Ω is some positive function on $[0, \infty)$, sufficiently differentiable at 0 to make for a differentiable metric on M. Then we have $\partial/\partial t$ as Killing field with static-orbit space $Q = \mathbb{R}^3$ and projection $\pi(t, \rho, x) = (\rho, x)$. What sort of behavior can we get in $\dfrac{\bar{\Omega}_r^+}{(\bar{\Omega}_r^-)^2} \dfrac{1}{r^2}$? (Note we are using base-point $\mathbf{0} \in Q$ and $r = \bar{d}(\mathbf{0}, (\rho, x))$.) It can vary widely.

For instance, let's restrict ourselves to $\Omega(\rho) = \rho^q$ for any constant q (more precisely: for some $a > 0$, use that only for $\rho \geq a$, and take Ω constant on $[0, a]$). Then we get $F(r) = \frac{\bar{\Omega}_r^+}{(\bar{\Omega}_r^-)^2} \frac{1}{r^2}$ decreasing in r iff $-2 < q < 1$. For $0 \leq q < 1$, we have

$$F(r) \sim \frac{1}{r^{2\left(\frac{1-q}{1-\frac{1}{2}q}\right)}}$$

while for $-2 < q \leq 0$, we have

$$F(r) \sim \frac{1}{r^{2\left(\frac{2+q}{2-q}\right)}}$$

(where $\sim$ indicates asymptotic behavior). For $q = 1$, $F(r)$ is constant, and for $q = -2$, $F(r)$ approaches a finite number; for all other q, $F(r)$ increases unboundedly in r.

Are all such q viable candidates for Theorem 4.1? If we calculate the Ricci curvature for the metric g above, we get

$$\mathrm{Ric} = -\left(\frac{1}{4}\frac{\Omega'^2}{\Omega} - \frac{1}{2}\Omega'' - \frac{1}{\rho}\Omega'\right)(dt)^2 + \frac{1}{\Omega}\left(\frac{1}{4}\frac{\Omega'^2}{\Omega} - \frac{1}{2}\Omega''\right)(d\rho)^2 - \frac{\rho}{2}\frac{\Omega'}{\Omega}k_{\mathbb{S}^2}$$

This shows us NEC is obeyed precisely when both

$$\Omega' \geq 0$$
$$\Omega'' - \frac{1}{2}\frac{\Omega'^2}{\Omega} + \frac{1}{\rho}\Omega' \geq 0$$

So NEC is fulfilled for $\Omega(\rho) = \rho^q$ precisely when $q \geq 0$, and the only cases in which Theorem 3.3 gives us a Ricci fall-off are for $0 \leq q < 1$.

In sum: there are considerable challenges to achieving anything like the classical singularity results, using only NEC. But these results show that something like the SEC results can be achieved in the presence of a timelike Killing field, so long as the size of that field does not vary strongly. What this might mean for a more general consideration of singularity theorems is far from clear.

Data availibility Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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