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Vector Line Integrals in Mathematics and Physics

Tevian Dray[1] · **Corinne A. Manogue**[2]

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Abstract

Representational transformation diagrams are used to compare and contrast standard textbook presentations of vector line integrals in undergraduate courses in both mathematics and physics. These presentations are taken as the lower anchor in a learning trajectory. Two principal approaches in the lower division are identified, roughly but not entirely correlated with these two disciplines. These textbook approaches are compared to existing characterizations for (single-variable) integration in the theory literature, notably *adding up pieces* (or *multiplicatively-based summation*), which is generalized here to *chop, multiply, add*; *accumulation from rate*; *quantitatively-based summation*; and a new characterization, *parametric integrals*. A review of upper-division textbooks establishes key features of an upper anchor in the learning trajectory. In conclusion, a hypothetical learning trajectory is presented, designed to scaffold student acquisition of rich concept images for vector line integrals.

Introduction and Research Context

Calculus is a gateway to physics and physics is a gateway to engineering. It is well known that students struggle to apply mathematics to science (e.g. Hammer et al., 2005). Building robust but basic understanding in early mathematics classes opens the gateway for poorly prepared but bright/hard-working students who are initially struggling in their mathematics classes, but only if students are able to effectively transfer their knowledge to applied disciplines. Recognizing the need to align undergraduate mathematics programs with the needs of the partner disciplines, the Mathematical Association of America sponsored the Curriculum Foundations project (MAA, 2004), whose reports include both general and discipline-specific

✉ Tevian Dray
tevian@math.oregonstate.edu

Corinne A. Manogue
corinne@physics.oregonstate.edu

[1] Department of Mathematics, Oregon State University, Corvallis, OR 97331, USA

[2] Department of Physics, Oregon State University, Corvallis, OR 97331, USA

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recommendations, which a successor project is currently attempting to implement (SUMMIT-P, 2018).

We are a curriculum development team with over 30 years experience (each) teaching vector calculus (first author; TD) and upper-division physics (second author; CAM). We are developing a learning trajectory for integration that extends from lower-division mathematics and physics all the way through vector line (and eventually flux) integrals in physics.

Single-variable calculus is typically covered in the first year of university studies, with multivariable and vector calculus being covered in the second year. The introductory, calculus-based physics sequence often begins after the first semester of calculus has been completed, but often introduces concepts from second-year calculus in parallel with, or even slightly earlier than, they are covered in a calculus class. A good example of such a topic is line integrals, especially those involving vector fields. Such line integrals are then used extensively in middle-division physics courses covering classical mechanics and electromagnetism. This pattern of requiring physics students to generalize their formal mathematics knowledge at an extremely rapid pace is typical of the transition of physics majors from the lower to the upper division. The need to begin preparing physics majors in lower-division mathematics and physics courses for the advanced topics of the upper division is analogous to (and, we acknowledge, in tension with) the need to prepare mathematics majors in lower-division mathematics courses for upper-division, proof-based courses.

Theoretical Perspective

Both mathematicians and physicists would recognize the expression $\int_C \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ as a generic vector line integral, expressing the work done by a force $\vec{\boldsymbol{F}}$ along a path C. This expression is sometimes derived in the form of the line integral $\int_C \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{T}}\, ds$. The choice of which of these expressions to treat as fundamental during instruction naturally affects student understanding of line integrals. This paper delineates how textbooks in lower-division mathematics, lower-division physics, and upper-division physics implement this choice and then discusses the consequences of these choices for student learning, both in the moment and in future coursework.

In our previous studies of derivatives, we have been strongly influenced by Zandieh's "concept image" of derivatives (Zandieh, 2000a). Tall and Vinner (1981) (see also Vinner, 1983; Zandieh, 2000b; Thompson, 2013), define a *concept image* as the set of properties associated with a concept together with mental pictures of the concept. To us, concept images have also come to include information such as physical context, relevant physical laws, dimensions, understanding of how to interpret equations geometrically and physically, understanding of special cases, and more. Our work (Roundy et al., 2015; Emigh & Manogue, 2022a, b) has generalized and extended Zandieh's concept image for ordinary derivatives to multivariable and vector calculus examples in physics contexts.

Faculty have *rich* concept images, which students must accumulate gradually. This work forms part of a long-term project to identify an idealized concept image for integrals and use it to construct a learning trajectory to inform the work of curriculum developers, textbook authors, and teachers. Our learning trajectory and concept map for partial derivatives can be found online (Roundy, 2018, under development).

A *learning trajectory* (LT) in mathematics, known as a *learning progression* in physics, is a possible sequence of increasingly sophisticated understandings of a topic. Key features of learning progressions described by the National Research Council (Duschl et al., 2007) and the National Assessment Governing Board (Lemke & Gonzales, 2006) include:

- Learning progressions are hypotheses about learning in a given domain;
- Learning progressions include upper and lower anchors, with the upper anchor grounded in societal goals for learning core knowledge and practices in science, and the lower anchor grounded in the ideas that students bring to the classroom; and
- Learning progressions describe ways students may develop more sophisticated ways of thinking in a domain, often with support of specific instructional strategies.

A major goal of developing learning progressions is to deepen the focus of science and mathematics education on central concepts rather than on inconsequential topics (Plummer, 2012).

Overview

This paper describes our early empirical efforts to flesh out the broad lower and upper anchors of part of the learning trajectory; to incorporate, and contribute to, the theoretical education research discussion on integration; and to suggest the outline of a learning trajectory for vector line integrals based on both the empirical and theoretical work.

The research question that underpins the empirical part of this investigation is:

- What does an analysis of textbook treatments of vector line integrals reveal about the learning objectives and (abbreviated) learning trajectories of the associated courses?

The question that underpins the theoretical part of this investigation is:

- How do the revealed learning objectives and learning trajectories harmonize across courses? Where are there possible disconnects and tensions that may trip up students?

If the upper anchor of a learning trajectory is intended to be "grounded in societal goals for learning core knowledge" and the lower anchor "grounded in the ideas students bring to the classroom," then how, in practice, can we identify these anchors?

In "Methods", we introduce our empirical analysis method, Representational Transformation Diagrams, and the language we will use to describe the features of the anchors. We then use this analysis strategy in "The Lower Anchor in Mathematics and Physics" to establish the lower anchor as instantiated in the treatment of vector line integrals in lower-division mathematics and physics texts. Of course, this lower anchor will need to be tested empirically in a later phase of the investigation to identify how much of this lower anchor students actually access. We follow this empirical work with a theoretical discussion in "Theoretical Discussion" of the existing literature on student understanding of integration. In this section we also introduce new concepts into the theoretical discussion, motivated both by our earlier empirical analysis and by our own experiences in an applied field. In "The Upper Anchor", we introduce our choice of upper anchor as empirically instantiated in middle-division physics textbooks. In "Discussion", we first propose a (hypothetical) learning trajectory for vector line integrals that suggests ways teaching approaches might be improved, including some comments from the viewpoint of applied disciplines.

Methods

To represent and analyze the rich concept images, as enacted in the textual part of the textbooks, we turned to an analysis strategy developed by Bajracharya et al. (2019) called a *representational transformation diagram* (RTD). These diagrams are a type of flowchart introduced to document the transformations from one representation to another made by students while solving problems.

Bajracharya et al. (2019) identified three common transformation phenomena, which they called *translation*, *consolidation*, and *dissociation* depending on whether a box labeling a representation had no more than one incoming and/or outgoing arrow, more than one incoming arrow, or more than one outgoing arrow, respectively. Further, they found that their student interview participants had most trouble with *consolidation*, and then with *dissociation*, rather than with *translation*. We use the length of the RTD together with the presence or absence of consolidation and dissociation as a proxy for cognitive load.

How are we to understand the content of such a diagram, when using it to analyze the concept image of a vector line integral? We chose to look for three things: an *iconic expression or equation*, how the iconic expression is *unpacked*, and what expression is provided to students as the *starting point for calculation*. We introduce the phrase *iconic expression* to describe the symbolic representation of a fundamental concept in its simplest, most compact form. Such expressions should be *geometric*, that is, independent of origin, coordinate system, and parameterization, and should also be *easy to remember*. If one understands the symbols, an iconic expression should contain instructions for *unpacking* the expression in cases with different origins, coordinate systems, and parameterizations. Elements of an "unpacked" representation can be inferred from "consolidation" where the several arrows entering the box describe how the origin, coordinate system, and parameterization are identified or calculated. The process of "unpacking" in problem solving can be inferred from "dissociation," the set of lines emerging from an iconic equation.

In his review of two supplemental books on calculus, McCallum (2001) points out that calculus textbooks are written for two distinct audiences, students and instructors, with conflicting needs. Students read the book "backward from the homework problems," then look at worked examples, but read the text itself only as a "last resort." Instructors instead read textbooks "forward," starting with the table of contents, checking that the desired topics are covered, with the right level of rigor, and that there are enough problems. In short, these two "zones" barely overlap. In our textbook analysis, we make special note of the representations that students are expected to use in examples and homework problems, which we call the *starting point for calculation*.

Notation and Language

Mathematics and physics are two disciplines separated by a common language. Although superficially speaking the same language, experts attempting to communicate across this disciplinary boundary can easily be misunderstood due to subtle – and some not-so-subtle – usage differences. We have previously pointed out some of these differences (Dray & Manogue, 1999, 2002, 2003, 2004, 2005; Dray et al., 2008; Dray, 2016), mostly in the context of differentiation. Here, we describe the notational differences relevant to vector line integrals that appear in this paper.

Mathematicians typically write vectors as n-tuples, for instance writing $\vec{\boldsymbol{r}} = \langle x, y, z \rangle$, whereas physicists almost always insert explicit basis vectors, writing $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$. [1] The physics use of hats rather than arrows is a useful mnemonic to denote unit vectors. Both disciplines will often omit the arrows in print in favor of a bold-faced font. But the usage $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{a}}$, with the assumptions that $a = |\vec{\boldsymbol{a}}|$ and $\hat{\boldsymbol{a}} = \vec{\boldsymbol{a}}/a$, is unique to physics.

The use of explicit basis vectors encourages a more geometric understanding of vectors, emphasizing that they are coordinate independent arrows in space, not merely three separate objects in a preferred (rectangular) basis and coordinate system. This more geometric interpretation of vectors occasionally shows up in mathematics in the notation $\overrightarrow{OA}$, denoting the vector from point O to point A, a representation that is common in high school mathematics but not used in physics. However, physicists often denote *points* in three dimensions using vector notation such as $\vec{\boldsymbol{b}}$, as special instances of the position vector $\vec{\boldsymbol{r}}$.

The examples above involve different notations for the same underlying concept. There are also instances of the same notation being used for different concepts. Foremost among these is the physics usage of subscripts to denote components, as in $\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{x}} + F_y\hat{\boldsymbol{y}} + F_z\hat{\boldsymbol{z}}$, rather than the mathematics usage to denote partial differentiation, as in $f_x = \frac{\partial f}{\partial x}$. The physics notation again has a mnemonic affordance, it is easy to keep track of which components come from which vectors. Interestingly, general relativity, with practitioners from both mathematics and physics, combines

[1] The modern trend in physics to use $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ rather than the (quaternionic!) names $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$ is a minor perturbation of this notation.

the notations as in $F_{x,y}$, where the subscript before the comma indicates a component and the subscript after the comma denotes partial differentiation.

In the RTDs in this paper, algebraic representations of the vector line integral are shown in bold boxes. Representations for other mathematical objects (typically parts of the integral) are in unbold boxes. We are careful in each RTD below to use the same notation as the textbook being analyzed. However, in the text of this article, we have chosen for clarity to use a consistent notation throughout, translating implicitly when necessary.

We feel there are strong pedagogical reasons at the lower- and middle-division levels for always putting arrows and hats on vectors. Physics texts are about equally divided whether they denote the vector differential as $d\vec{\boldsymbol{\ell}}$ or $d\vec{\boldsymbol{r}}$; we prefer $d\vec{\boldsymbol{r}}$.

The Lower Anchor in Mathematics and Physics

We analyzed a wide range of textbooks in both disciplines, covering an equally wide range of viewpoints. Because of space limitations, we deliberately chose to discuss here only two representative textbooks, one each in mathematics (Briggs et al., 2019) and physics (Giancoli, 2009), that are both widely used and generally viewed as traditional. We acknowledge that these criteria led us to textbooks that are as different as possible from each other. Our goal is to understand and demonstrate the full spectrum of possibilities, not to claim that all mathematics textbooks take the approaches of Briggs et al. nor that all physics textbooks take the approaches of Giancoli .

Traditional Vector Calculus Textbooks

We have chosen to discuss Briggs et al. (2019) in detail as a representative example of a traditional calculus textbook, as well as having the most detailed concept image among the textbooks we examined.

The treatment of line integrals presented in Briggs et al. (2019) is given by the representational transformation diagram in Fig. 1. This diagram is very rich! In this concept image, we identify

$$\int_C \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{T}}\, ds \tag{1}$$

as the iconic expression for vector line integrals. This iconic equation is then carefully unpacked into the expression $\int_C \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{r}}\,'(t)\, dt$. (Note the consolidation evident in the three new incoming arrows.) This latter expression is then rapidly unpacked into three equivalent versions, marked by bold-faced boxes at the bottom of Fig. 1. The equivalence of these four "unpacked" expressions is in fact presented as a boxed theorem, summarizing this book's treatment of vector line integrals.

Having made it this far, the careful reader of this textbook might be somewhat surprised to discover that very little of this rich concept image is used in the examples and problems. None of f, g, h, $d\vec{\boldsymbol{r}}$, dx, dy, dz make a further appearance.

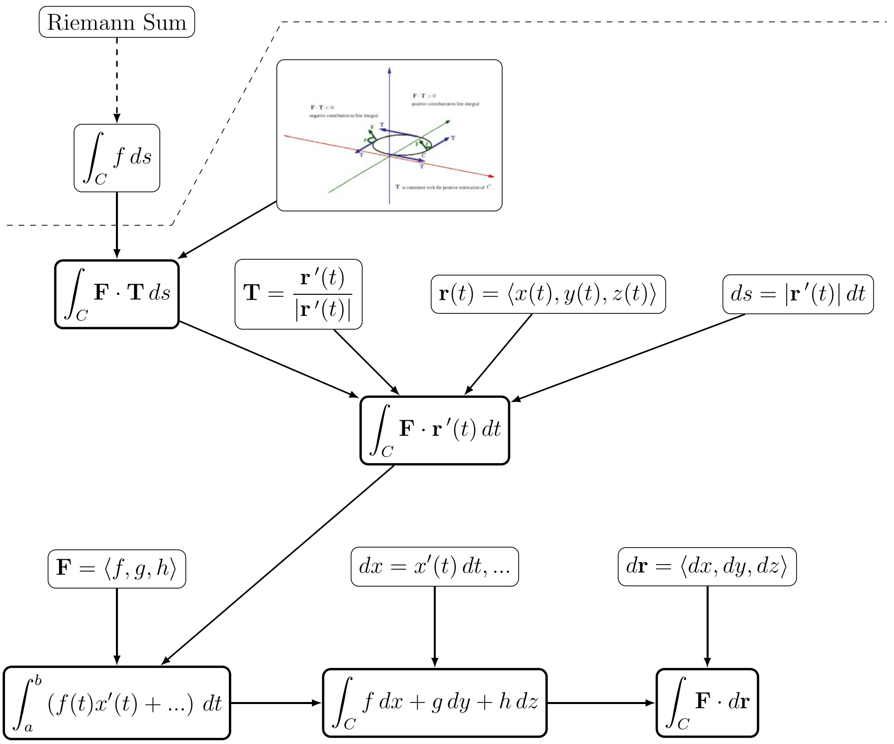

Fig. 1 The representational transformation diagram for the treatment of line integrals in Briggs et al. (2019). The dashed line represents a division between information contained in prerequisite sections and the section on vector line integrals

Rather, the starting point for computation in all examples is $\int_C \vec{F}\cdot\vec{r}\,'(t)\,dt$, with $\vec{r}(t)$ either given explicitly as a vector function of t, or chosen from a small collection of elementary examples, such as circles and lines, for which students are expected to be able to determine $\vec{r}(t)$ by themselves. In either case, $\vec{r}(t)$ must be differentiated and then inserted into the dot product. The only part of the rich, expert concept image summarized in Fig. 1 that appears to be elevated to a learning outcome in Briggs et al. (2019) is the ability to evaluate this last integral for a given vector field $\vec{F}$ along a given, explicitly parameterized path $\vec{r}(t)$, all starting from the single expression $\int_C \vec{F}\cdot\vec{r}\,'(t)\,dt$. Students can be successful in completing these problems using only algebraic reasoning.

The dichotomy described by McCallum (2001) is apparent in Fig. 1. A careful derivation of the expression $\int_C \vec{F}\cdot\vec{r}\,'(t)\,dt$ is given in Briggs et al. (2019), using only the top half of the concept image shown in Fig. 1, whereas the examples and problems start with this expression, using only the bottom half. Furthermore, the final expression, $\int_C \vec{F}\cdot d\vec{r}$, is hardly mentioned. The overall length of this RTD is a clear signal that if students were to actually try to follow the derivation, the cognitive load would be very high.

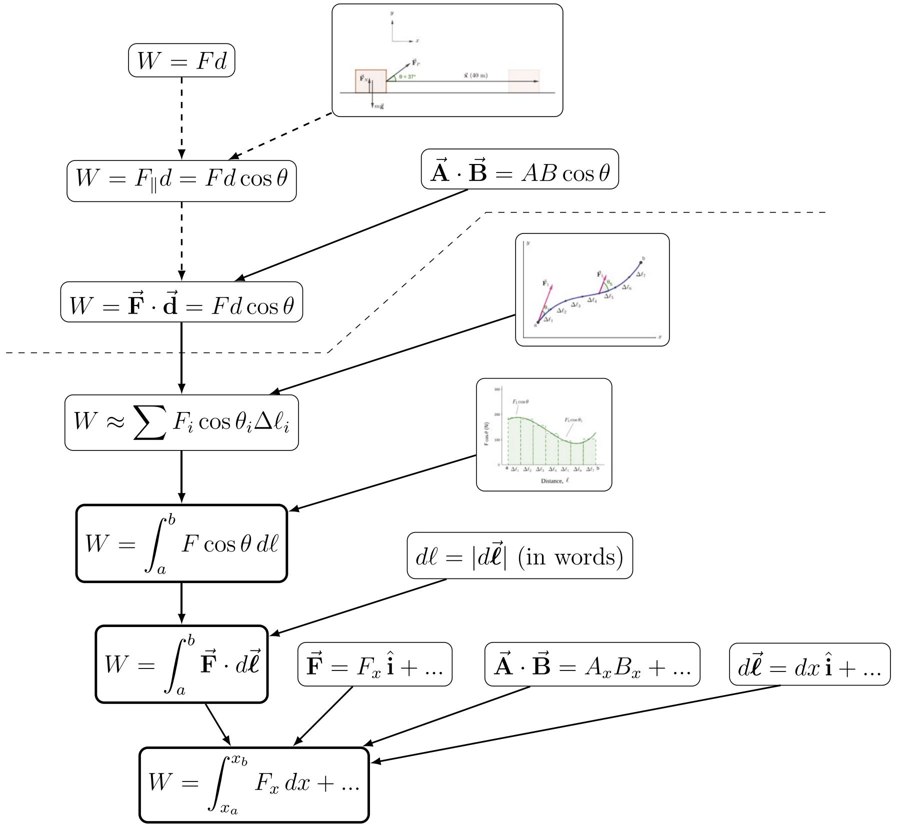

Fig. 2 The representational transformation diagram for the treatment of line integrals in Giancoli (2009). The dashed line represents a division between information contained in prerequisite sections and the section on vector line integrals

There are two possible interpretations of $\int_C \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{r}}\,'(t)\,dt$. One possibility is to put implicit parentheses around $\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{r}}\,'$, in which case this integral becomes an explicit representation of $\vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{T}}\,ds$ in terms of a parameterization, thus reducing vector line integrals to the previously-considered case of scalar line integrals. Alternatively, the parentheses can be put around $\vec{\boldsymbol{r}}\,'\,dt$, which then becomes an explicit representation of $d\vec{\boldsymbol{r}}$. Unfortunately, the notation $d\vec{\boldsymbol{r}}$ never appears in what students are expected to do.

Traditional Introductory Physics Textbooks

We have chosen Giancoli (2009) as a representative example of a traditional introductory physics textbook. We construct a representational transformation diagram for the presentation of line integrals in this textbook, which is given in Fig. 2. This concept image is also very rich, but differs considerably from the one shown in Fig. 1.

Here we are able to identify

$$W = \int_a^b \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} \tag{2}$$

as the iconic equation, which appears near the bottom of the figure (using $d\vec{\boldsymbol{\ell}}$ rather than $d\vec{\boldsymbol{r}}$).

Most of the heavy lifting in the chapter on work (see the parts of Fig. 2 above the dotted line) comes in the first two sections, which build a physical understanding of work in situations where the force is constant, but not necessarily in the direction of motion. These sections include the first formal algebraic presentation of the dot product anywhere in the textbook, although the geometric process of finding the component of a vector in a particular direction (not necessarily x) is a common feature of the previous four chapters. The only figures in these two sections show people pulling blocks or lifting backpacks, including vector force diagrams with angles marked.

In the third section of the chapter, the line integral for work is finally introduced (see the parts of Fig. 2 below the dotted line). This section includes three figures: for a chopped path with sample arrows for $\vec{\boldsymbol{F}}$ and sample scalar intervals $d\ell$, for a flat Riemann sum graph, and for an area under a curve (not shown). In this part of the RTD, most instances of consolidation consist of an algebraic representation and a figure that serves as a geometric/physical elaboration of that algebraic expression together consolidated into the next algebraic representation.

It is not until the second to the last line at the very end of the diagram that the iconic equation is presented and then immediately unpacked in a flurry of consolidation arrows.

What are students asked to do? End-of-chapter problems closely follow the order of examples presented in all three sections of the book. Most are word problems. Students are expected to interpret diagrams and/or words to find $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{d}}$ as vectors and then evaluate the dot product. Most, but not all, problems are one-dimensional. A few problems stand out, notably one in which the one-dimensional force is presented as a piecewise constant graph, clearly invoking "area under the curve." Other problems involve circular motion in two-dimensions.

Interestingly, several examples and many end-of-chapter problems deal with chopping a path (the word curve is never used) into two or three finite chunks, on each of which $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{d}}$ are constant, so that the integration is trivial on each separate chunk. The idea of *chopping* is scaffolded in this way immediately before the formal line integral is introduced.

Because the problems that students are asked to do involve so many different representations, we would argue that the starting point for calculation is in fact the iconic equation itself! Students' ability to unpack this equation for themselves is a clear learning goal and is being explicitly supported.

Other Texts

Among other widely-used calculus textbooks, Stewart (2003) has a much more abbreviated derivation of the several equivalent representations of vector line integrals, but the learning outcomes inferred from the examples and problems are essentially the same as those in Briggs et al. (2019), as were those in the OpenStax textbook by Strang et al. (2016). On the physics side Halliday et al. (1993) is quite similar to Giancoli (2009).

Several other textbooks land somewhere in the middle. The textbook by Hughes Hallett and coauthors (McCallum et al., 2012) represents the most successful "reform" calculus text, emphasizing conceptual reasoning, multiple representations, and open-ended problems. Geometric reasoning is incorporated throughout, resulting in several instances where the presentation is more closely aligned with the usage in physics textbooks such as Giancoli (2009) than with mathematics textbooks such as Briggs et al. (2019). The reform textbook by Ostebee and Zorn (1997) is also in this category, again emphasizing multiple representations.

Alternate treatments on the physics side include the introductory textbook by Knight (2008), designed using evidence-based pedagogy. It is an "energy first" textbook, and therefore motivates the concept of work as a change in (kinetic) energy, leading to a "chain rule" treatment of vector line integrals that is quite different from the one in Giancoli (2009). The textbook *Matter and Interactions* (Chabay & Sherwood, 2011) takes a modeling approach, emphasizing student computations using `vpython` to study applications. Again, work is introduced as the transfer of energy.

Each of these alternative textbooks, in both mathematics and physics, makes significant efforts to cross the divide between the two extremes represented by our choice of more traditional textbooks.

Theoretical Discussion

Adding Up Pieces and Definite Integrals

Integration in the context of single-variable calculus has been studied extensively. We are interested here in the subset of that literature which characterizes approaches to integration; a more complete bibliography can be found in Jones (2020).

The first part of Jones (2020) provides a theoretical characterization of five different "ways of thinking" about definite integrals, based on the research literature, which we partially reproduce in Table 1. This characterization is the culmination of nearly a decade of work by Jones and others, which we summarize briefly in the next two paragraphs.

In earlier work, Jones (2013) studied *cognitive resources* that students have when thinking about integrals, summarizing this description by introducing the symbolic forms "adding up pieces," "perimeter and area," and "function matching" (antiderivatives), as well as a few special cases. In Jones (2015a), he considers area, antiderivatives, and adding up pieces in both pure mathematics and applied science,

Table 1 Theoretical ways of thinking about definite integrals from Jones (2020) and Pina and Loverude (2019)

Way of thinking	The main characteristic is that integrals are interpreted as ...
Space underneath a graph	... the amount of space underneath the graph of the integrand
Antiderivative	... an instruction to compute an antiderivative
Adding up pieces	... the summation of infinitesimal quantities
Accumulation from rate	... the accumulation from a rate function
Averaging	... an averaging out of the integrand across the domain
Procedural	... the integral acting as an operator to further a derivation

showing that the latter form is the most helpful; however, in separate work (Jones, 2015b), he points out the prevalence of the other two forms. Earlier work (Sealey, 2006) has shown that an overreliance on "area under the curve" can reduce the integral's applicability. More recently, several studies expand on the importance of the "adding up pieces" concept image. Both Bajracharya et al. (2023) and Kontorovich (2023) discuss student difficulties when integrating quantities that are not everywhere positive. Stevens (2019) proposes a hypothetical learning trajectory based on adding up pieces and Stevens and Jones (2023) describe a detailed instructional sequence intended to foster student development of this concept image across a full unit on integration and, in an empirical study, describe how students responded to this sequence in a series of four interview questions.

Pina and Loverude (2019) did a coded, quantitative analysis of integration in lower-division physics texts using an early version of Jones's codes. They identified an additional category, which they call "procedural," which they describe as "the integral acting as an operator to further a derivation." We have added this category as a separate line in Table 1. Their analysis showed that the "adding up pieces" and "procedural" categories were by far the dominant uses of integration in these physics texts. They also point out that "perimeter and area" is the category least used in introductory physics textbooks, and argue that it is also the least productive for physics students.

Chop, Multiply, Add

Wagner (2015, 2018) argues that the "Riemann sum-based interpretation" of integration is the "most valuable" in applied contexts such as physics, emphasizing the importance of multiplication of height and width while constructing the summands that represent the area of the rectangles under the graph. In a similar vein and at about the same time, Jones (2015a, b) renamed his "adding up pieces" category to "multiplicatively-based summation." Simmons and Oehrtman (2019) then express concern that the concept of multiplicatively-based summation is not sufficiently general. Their analysis of student work, developed more fully in Simmons (2021) and Oehrtman and Simmons (2023), shows that the integration measure (dr) is an essential component of their local model ($dA = 2\pi r\,dr$ for a thin ring), rather than a small piece of the domain (Δr) in a local Riemann product. Their study demonstrates that the distinction between integrand and measure commonly taught in calculus courses does not capture the reasoning actually used by their student interviewees, which they name *quantitatively-based summation*.[2]

There is a representation in words of the process of integration which we like to describe to students as "*chop, multiply, add*" (CMA). This representation emphasizes three fundamental steps that we believe are common to all uses of definite integration. First, the region of integration must be *chopped* into small pieces. Second, the integrand (such as mass density) must be *multiplied* by some small measure of the piece (such as its length) in order to obtain the contribution of that piece to the total. Finally, the contributions from each piece must be *added* together.[3] This language becomes a "pedagogical chant" to guide students through the steps of problem solving. The CMA language is intended to apply equally well to Simmons and Oehrtman's example where the chopped piece is all of $2\pi r\,dr$ which is multiplied by the constant function 1.

To us, a more interesting question than what physical/geometric quantities get multiplied together is the question of where the chopping happens conceptually. To a mathematician (e.g. TD), it is natural to map the physical situation onto the graph of a function and then think about a Riemann sum interpretation of CMA. To a physicist (e.g. CAM), it is more natural to chop the physical situation directly and to think of the multiplication and addition as taking place floating in space. This distinction can be hidden by the fact that mathematical physicists (e.g. both of us), like all bilingual people, are capable of code-switching and can and do switch languages mid-discussion. A good example of this code switching can be seen in Fig. 2 where the physicist's chopping in physical space is clearly shown in the second inserted figure, but the third inserted figure code-switches to a mathematics graph of a Riemann sum. To the best of our knowledge, this distinction has not been explored in the research literature, although it comes up frequently at our dinner table.[4]

Accumulation From Rate and Half-Definite Integrals

The FTC is really about *half-definite* (or semidefinite) integrals, in which one limit is constant, but the other is a variable. Thompson et al. (2013) (see also Thompson & Carlson, 2017) have built a calculus course based on the interpretation of such integrals as the *accumulation* of *rate-of-change functions*, that is, using AR.

Jones and Ely (2022, 2023) compare the treatment of integration as *accumulation from rate* (AR), in which half-definite integrals are fundamental, with *adding up pieces* (AUP), which more naturally yields definite integrals. As they point out, AR requires *all* integrals to be viewed as accumulation functions, whereas AUP allows more general quantitative relationships. One could argue that this distinction is minor

[2] In a similar vein, Nilsen and Knutsen (2023) document the use of infinitesimal reasoning by some Norwegian calculus students, despite instruction emphasizing the rigorous use of limits.

[3] We had been using the phrase "chop and add" to describe integration for more than a decade when Wagner's work, reinforced by Jones's, persuaded us to add the "multiply" in the middle.

[4] The coauthors are a married couple.

in the context of single-variable integration, where it is relatively straightforward to convert in either direction between definite and half-definite integrals, although the latter are more closely tied to the Fundamental Theorem of Calculus (FTC). However, only AUP readily generalizes to higher dimensions – including the vector line integrals considered here, contexts in which the FTC plays only a secondary role.[5]

Furthermore, the interplay of variables, constants, and parameters used in half-definite integrals is quite challenging for students.[6] The need for *dummy variables* such as u in $\int_a^x f(u)\,du$ is not obvious to students, many of whom are still using x for every unknown. To further complicate matters, physicists try not to switch letters in such expressions because of mnemonic value, writing x' instead of u, resulting in an obvious conflict with the use of prime to denote derivatives that students may be more familiar with from their mathematics classes.

Parametric Integrals

As noted above, it is not obvious how to generalize AR to higher dimensions. We propose here an alternative characterization that could be viewed as filling the gap.

Physics applications often involve definite integrals with limits that can be considered parameters. For instance, when computing the electrostatic potential corresponding to the electric field of a point charge, one integrates from a fixed location where the potential is defined to be zero (typically infinity!) to the desired radius. (See also "The Electric Field ($\vec{\boldsymbol{E}}$)".) This radius is viewed as a constant while performing the integral, but is actually a parameter; the resulting potential is a function of r. This interplay between parameters and variables is reminiscent of the discussion in "Accumulation From Rate and Half-Definite Integrals", although the conceptual approach is quite different. In AR, the integral is regarded as fundamentally an accumulation of a rate, rather than as CMA; here, the integral is fundamentally CMA. *After* integration, the bounds are reinterpreted as parameters, resulting in what we call a *parametric integral* (PI). These two interpretations are of course equivalent, thanks to the FTC. But only the latter generalizes naturally to higher dimensions. The transition from definite integral to parametric integral is then analogous to the transition from derivative at a point to derivative at a parametric point, that is, as a function.

The Upper Anchor

To understand the physics learning trajectory after the lower division, we examined textbooks in classical mechanics (Marion & Thornton, 1995; Taylor, 2005), electromagnetism (Griffiths, 1999; Reitz et al., 1993), and mathematical methods (Boas,

[5] Jones and Ely (2023) discuss the use of both AUP and AR to find arclength, which involves a *scalar* line integral, pointing out that the reasoning required to apply AR in this context is quite intricate.

[6] The difference between these quantities is sometimes described as depending on whether their names occur at the end, beginning, or middle of the alphabet!

2006; Riley et al., 2006; Arfken & Weber, 1995). These subjects occur in middle-division physics and are the place where vector line integrals are first used in their full geometric glory. For the purposes of this paper, we will let these textbooks define the upper anchor of our learning trajectory. We identified three physical contexts, described in more detail below – work, electrostatic potential, and Ampère's law – involving quantities represented by a vector line integral. Each takes the same iconic form, namely

$$P = \int \vec{\boldsymbol{G}} \cdot d\vec{\boldsymbol{r}}. \tag{3}$$

None of the textbooks we examined used the iconic form

$$P = \int \vec{\boldsymbol{G}} \cdot \hat{\boldsymbol{T}}\, ds. \tag{4}$$

Although each of the three contexts below involves the same iconic expression, the three concept images are quite different. There are fundamental differences in the mathematical and physical contexts that affect the type of path (arbitrary, closed, open) and strategies used to evaluate the integral. In particular, whether or not $\vec{\boldsymbol{G}}$ is conservative affects most other considerations. We summarize here the mathematical features of these three physical contexts and in each case analyze integration strategies using the three different descriptions of integration above, namely *accumulation from rate* (AR), *chop, multiply, add* (CMA), and *parametric integration* (PI). Although we use physics language throughout, we emphasize that it is the mathematical differences between these contexts that is relevant to our analysis.

Work ($\vec{\boldsymbol{F}}$)

Work is the first physical context in which vector line integrals typically arise in undergraduate-level physics. The work W done by a force $\vec{\boldsymbol{F}}$ as an object moves along a path C is given by

$$W = \int_C \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}. \tag{5}$$

This formula is both the iconic equation and the starting point for calculation for the physics lower-division anchor described in "Traditional Introductory Physics Textbooks". The approach in all of the upper-division texts to evaluating this integral is pure CMA, with chopping defined by little pieces of $d\vec{\boldsymbol{r}}$ and multiplication given by the dot product. The relevant mathematical features of work integrals are that the force $\vec{\boldsymbol{F}}$ can be conservative (e.g. gravity) or non-conservative (e.g. friction), and the path C can be either open or closed. Furthermore, there is not necessarily any underlying symmetry. In this sense, line integrals representing work are "generic."

There is one further feature of work: Not all forces are represented by vector fields defined in space independently of the path.[7] The simplest friction forces depend on the trajectory, not merely on an object's location in space, and are directed oppositely to the motion. More complicated dissipative forces may depend, for example, on the velocity of the object.

The Electric Field ($\vec{E}$)

The electrostatic field $\vec{E}$ is always conservative; it is the gradient[8] of the scalar potential V

$$\vec{E} = -\vec{\nabla} V. \tag{6}$$

In this context, the iconic equation represents a mathematical relationship between a vector field and its potential. The fundamental theorem for gradients tells us that

$$-\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{r} = V(\vec{b}) - V(\vec{a}), \tag{7}$$

where the line integral is evaluated on *any* path from point $\vec{a}$ to point $\vec{b}$. Since this line integral is independent of path, an important evaluation strategy is to *choose* a path along which the antidifferentiation is easy, such as piecewise curves along which the tangential component of $\vec{E}$ is constant or the dot product is zero. The conceptual understanding of the integral is CMA, where the chopping step may need to be flexible in the sense of Simmons and Oehrtman (2019). Independence of path also implies that the line integral (7) around a closed path, that is, with $\vec{b} = \vec{a}$, would be zero.

If $\vec{a}$ is fixed and $V(\vec{a})$ set equal to zero, the line integral yields (minus) the value of the potential at $\vec{b}$. In the notation of Griffiths (where $\mathcal{O}$ denotes the special point $\vec{a}$)

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E} \cdot d\vec{r}, \tag{8}$$

where the clear intent is that the upper limit $\vec{r}$ is a variable, i.e. this is an example of CMA followed by PI. To interpret this integral using AR, it would be necessary to first rewrite the integral in the form (1) and then reinterpret $\vec{E} \cdot \hat{T}$ as a rate of change, which strikes this physicist (CAM) as a strange interpretation both on dimensional grounds and because the word "rate" invokes time dependence in this physical context which is static (i.e. time independent).

[7] Friction forces are an example of a "vector field over a curve", although that terminology is rarely used in physics.

[8] The conventional minus sign captures the mismatch between the gradient pointing "uphill", and physical forces pointing "downhill."

The Magnetic Field ($\vec{\boldsymbol{B}}$)

For a magnetostatic field $\vec{\boldsymbol{B}}$, Ampère's law states that

$$\oint_C \vec{\boldsymbol{B}} \cdot d\vec{\boldsymbol{r}} = \mu_0 I_{\text{enc}} \tag{9}$$

where μ_0 is a constant and I_{enc} denotes the current enclosed by the closed path C.[9] (The circle added to the integral sign indicates that the path is closed.) Rather than expressing a mathematical relationship, as in the previous case, Ampère's law is the expression of an empirically verifiable physical law. In homework problems, students are asked to evaluate the two different physical quantities represented by the two sides of the equation and show that they are equal. Because the vector line integral is around a closed loop, the interpretation is CMA without PI.

A common use of Ampère's law is to determine[10] the magnetic field $\vec{\boldsymbol{B}}$ if the current density $\vec{\boldsymbol{J}}$ is known. In the presence of symmetry – the only situation in which this strategy works – an important aspect of the strategy is to *choose* appropriate paths that respect the symmetry of the situation over which to evaluate the line integral. The simplest, idealized example is the magnetic field $\vec{\boldsymbol{B}} = B_\phi \,\hat{\boldsymbol{\phi}}$ of an infinite straight line of current which has cylindrical symmetry, so students are immediately led outside the realm of rectangular coordinates.

Discussion

Suggested Learning Trajectory

We propose in Fig. 3 a partial learning trajectory based on our own teaching that provides a framework for discussing both the results of our textbook analysis in "The Lower Anchor in Mathematics and Physics", "Theoretical Discussion", "The Upper Anchor" and some additional aspects of integration from the point of view of applied sciences and engineering that may be of benefit to mathematics education researchers and/or mathematics teachers.

Concept images are fundamental not only to our theoretical framework for research, as discussed in "Introduction and Research Context", but also to our pedagogical commitments. An explicit pedagogical goal of our own teaching is to help our students expand and enrich their concept images. One of the key aspects of an expert concept image of line integrals, as indeed for many related concepts, is the way in which they incorporate multiple representations and the connections amongst them. For this reason, we have deliberately given each box in our

9 I_{enc} is the flux of the current density $\vec{\boldsymbol{J}}$ through *any* surface for which C is the boundary. This surface independence is a deep consequence of current conservation, described mathematically as the property that magnetic fields are divergence free.

10 From *inside* the integrand!

learning trajectory a descriptive label in words and/or symbols and a geometric image, emphasizing the need to include the use of multiple representations of a single concept. In a proper RTD, which Fig. 3 is not, the relationship between these representations would also need to be shown.

The overall framing in Fig. 3 reflects our commitment to CMA as part of a concept image: the column headings emphasize the CMA characterization introduced in "Chop, Multiply, Add". Additionally, the first heading (Object) acknowledges that the concept of integration generalizes from functions of one variable to objects with different mathematical and physical properties. The first two rows of the LT represent the lower anchor, the prerequisite knowledge needed about integration, with the first row representing the elementary integration of functions, and the second the generalization to scalar line integrals, expressed in terms of linear densities λ.

Scalar vs. Vector Line Integrals at the Lower Anchor

In "Introduction and Research Context", we noted two expressions for line integrals, namely $\int_C \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ and $\int_C \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{T}}\, ds$. As shown in Fig. 1, both of these expressions indeed occur in mathematics textbooks; we have identified the latter as iconic. In physics textbooks, however, only the first of these expressions occurs, albeit in the slightly different form (2), which we have also identified as iconic. We do not wish to pigeonhole one of these expressions as "mathematics" and the other as "physics." Instead, they represent two different routes to understanding line integrals, each with its own affordances. Furthermore, the reform calculus texts McCallum et al. (2012) and Ostebee and Zorn (1997) use the first expression above for the iconic equation, and the education research-based physics textbook Knight (2008) uses the second, exactly reversing any attempt at categorization.

Nonetheless, there is a dramatic shift in emphasis between the two representational transformation diagrams in Figs. 1 and 2, and thus between the two iconic expressions for line integrals. In Fig. 1, the fundamental concept is the scalar line integral; the iconic expression (1) suggests that the curve is being *chopped* into small pieces, on each of which the scalar quantity $\vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{T}}$ is evaluated, then *multiplied* by the small distance ds, then finally *added* up.

In Fig. 2, on the other hand, the fundamental concept is work, and the fundamental mathematical idea is the dot product. The iconic Eq. (2) suggests that the *chopping* is into small *vector* pieces $d\vec{\boldsymbol{r}}$ (which is written as $d\vec{\boldsymbol{\ell}}$ in Giancoli (2009)), the *multiplication* is in fact the dot product of $\vec{\boldsymbol{F}}$ with $d\vec{\boldsymbol{r}}$, after which the contributions are *added* up.

As the preceding discussion makes clear, both of these approaches fit naturally into the CMA characterization. Although it is straightforward to provide an AUP interpretation for scalar line integrals, it is less obvious how to handle "$\vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$" in this characterization. The infinitesimal vector $d\vec{\boldsymbol{r}}$ is a composite object, not only combining part of the integrand with the measure (as in $ds = r\, d\theta$), but also combining vector and scalar aspects (Dray & Manogue, 2009–2022c). CMA allows the

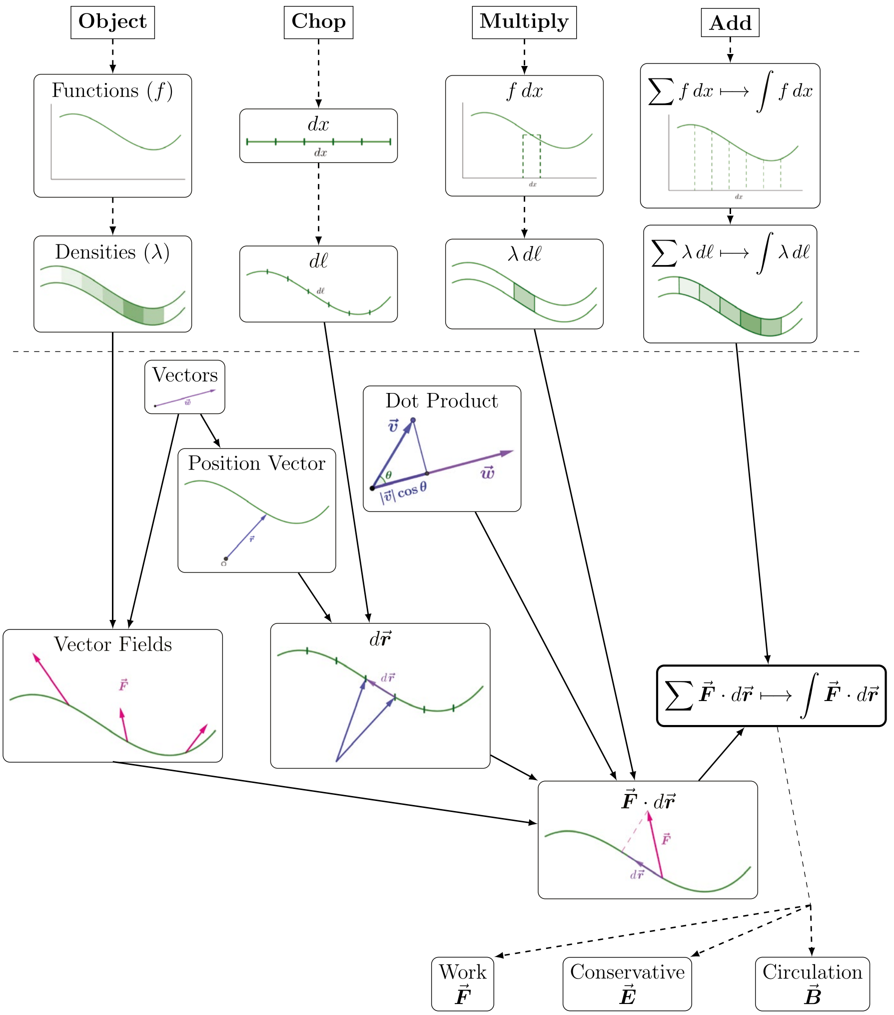

Fig. 3 A suggested partial learning trajectory for an introduction to line integrals. Prerequisite content from single-variable integration appears above the dotted line, representing the lower anchor; the upper anchor is indicated by the dashed arrows at the bottom

geometry to be fundamental, since the chopping can be vectorial, whereas the pieces being added in AUP must be scalars.

As Fig. 1 makes clear, mathematicians discuss scalar integrals first, then reduce vector line integrals to the scalar case. Figure 2 suggests, however, that this order may not represent the best pedagogical strategy in physics. We note here two further advantages of making $d\vec{r}$ fundamental, rather than $\hat{T}\,ds$. First, it is far easier computationally to go from $d\vec{r}$ to $ds = |d\vec{r}|$ than it is to go from $\hat{T}\,ds$, with its two competing square roots, to $d\vec{r}$. Second, using the relationship $ds = |d\vec{r}|$ naturally shows that vector line integrals depend on the direction in which the path is traversed, whereas the absolute value in ds ensures that scalar line integrals do not (Dray & Manogue, 2009–2022d).

The second part of Jones (2020) reports on an interview-based study with ten students. One of the interview prompts is:

> What does $\int_C \vec{\boldsymbol{V}} \cdot d\vec{\boldsymbol{r}}$ mean, where $\vec{\boldsymbol{V}}$ represents a vector field and C represents a curve?

Student responses were coded according to Jones's five ways of thinking. All of the students were able to make sense of the vector $\vec{\boldsymbol{V}}$, either as literal arrows ($N = 5$), physical phenomena ($N = 4$), and/or an algebraic function ($N = 3$). But the most striking result was the large variety of struggles students had trying to make sense of $d\vec{\boldsymbol{r}}$. One student, in particular, went through seven different interpretations while trying to choose one they were satisfied with. These interpretations of $d\vec{\boldsymbol{r}}$ included "infinitesimal vector," "arc length of an infinitesimal piece" of the curve, and "distance from the origin" to a point on the curve.

Perhaps the trouble students have making sense of $d\vec{\boldsymbol{r}}$ can be attributed to the lack of a strong pedagogical connection between $d\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{r}}\,'\,dt$ in a learning trajectory such as the one in Fig. 1.

Returning to our proposed LT, the transition from ordinary integration to vector line integrals goes through scalar line integrals, to honor the mathematicians' learning objective, but with an added emphasis on physical interpretation, to honor the physicists' learning objective. Although the first two rows of Fig. 3 are formally identical in terms of symbol manipulation, both their mathematical and physical content differ. Expressing scalar line integrals in terms of densities builds on the concept of mass density, which, in our experience, most students can understand regardless of their background in science. The LT also emphasizes the need to chop up the domain, multiply by the density, and add up the pieces. Finally, the use of densities provides a natural place to discuss physical dimensions and the fact that the dimensions of the final integral must be those of the integrand λ multiplied by those of the measure $d\ell$.

Unpacking

Our examination of middle-division physics books showed that the iconic expression $\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ is universally fundamental, with an interpretation either as CMA or as CMA followed by PI, depending on the context. What is also clear from the middle-division texts, although the data is too spread out for us to cite briefly, is that theoretical parts of the textbooks use the coordinate-independent iconic expression $\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$ in derivations, but applied parts of the texts expect students to unpack these vector line integrals in many contexts including using curvilinear coordinates, i.e. along (parts of) circles, cylinders, and spheres. Which coordinate system is best to use can be dictated either by the path or the given form of the vector field or both.

The introduction of vector line integrals can be set up both to facilitate the transition to the coordinate-independent iconic expression and to begin to develop skills that would allow its unpacking in these curvilinear contexts. We signal this pedagogical commitment by "ending" our LT with the iconic expression $\int \vec{F} \cdot d\vec{r}$ as a *starting point for calculation*. The upper anchor is represented in Fig. 3 by the three boxes at the bottom of the figure, representing the three applications discussed in "The Upper Anchor". Our intent is that an instructor would choose classroom examples and homework problems that would help students develop a toolbox of flexible evaluation strategies to "unpack" the iconic equation, appropriate for the content of the course and the time available. For us, this is a key learning goal and pedagogical commitment. Our work outside the scope of this paper includes designing and testing a variety of suitable examples (Paradigms Team at Oregon State University, 2019–2022b).

The trickiest part of unpacking the iconic expression $\int \vec{F} \cdot d\vec{r}$ involves the notion of curves (mathematics) or paths (physics). Both disciplines describe curves and paths using parameterization, although physicists often describe paths with words and figures and expect students to do the rest. In an effort to capture the many ways of arriving at a parameterization for a given curve, we introduced a strategy we call "*use what you know*" (Dray & Manogue, 2003). This flexible strategy encourages students to use geometric reasoning when available, saving formal parameterization for a last resort. For example, the line segment from (1, 0) to (0, 1) has slope −1, and lies along the line $y = -x + 1$. Therefore, $dy = -dx$, which can be plugged directly into $d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$, yielding $d\vec{r} = dx(\hat{x} - \hat{y})$ (or $d\vec{r} = dy(-\hat{x} + \hat{y})$). Depending on the integrand, no further substitution may be necessary; the curve has effectively been parameterized with respect to either x or y, depending on which way the substitution was done.[11]

The "use what you know" strategy generalizes directly to vector surface (flux) integrals, which are even more common than vector line integrals in physics applications. These integrals involve finding the component of a vector field perpendicular to a surface. But how are we to chop up a surface? As for vector line integrals, we regard $d\vec{r}$ as key. A small, directed piece of the surface $d\vec{A}$ is the cross product of two small, directed sides; $d\vec{A} = d\vec{r}_1 \times d\vec{r}_2$. Scalar surface integrals can now be evaluated by taking the magnitude $dA = |d\vec{A}|$. This approach, which naturally extends "chop, multiply, add" to higher dimensions, is further described in Dray and Manogue (2003, 2009–2022a), This additional application of $d\vec{r}$ helps to justify the class time necessary to give the vector differential a comprehensive treatment.

The Dot Product

The dot product plays an essential role in vector line integrals, namely to pick out the component ($F_{\|}$) of a vector field ($\vec{F}$) in the direction ($\hat{T}$) along a curve (C).

In the introductory physics textbooks, the dot product is often introduced in the context of work; the geometry of projections comes first, with algebraic

[11] Some students, accustomed to determining $ds = \sqrt{2}\,dx$ geometrically in scalar line integrals, fail to realize that this geometric factor is contained in these expressions for $d\vec{r}$, and attempt to insert an extra factor of $\sqrt{2}$ before integrating (Dray & Manogue, 2009–2022d).

computation following. In calculus, the dot product is usually introduced immediately after vectors themselves, with the algebraic expression getting at least equal footing, and often coming first.[12] Teachers should be aware of the high cognitive load of combining a first formal treatment of the dot product with the transition from single-variable integration to vector line integrals, as we saw in the physics RTD in "Traditional Introductory Physics Textbooks". If students have an initial concept image of the dot product primarily as an algebraic calculation rather than a projection, then the cognitive load is even higher.

In an attempt to ease this cognitive load, both mathematics and physics lower-division textbooks introduce a *starting point for calculation* in which the dot product is evaluated algebraically in rectangular coordinates. Calculus textbooks write this expression in the form $\int P\,dx + ...$, and physics textbooks write this expression as $\int F_x\,dx + ...$, illustrating physicists' desire to avoid switching letters, and resulting in an obvious notational conflict, in this case with the use of "F_x" in mathematics to denote differentiation. However, there is a price to be paid for this attempt at simplification. Reducing the integrand to an expression involving dx, dy, and dz may encourage students to evaluate these three integrals separately – and may prevent students from realizing that they are still line integrals, requiring parameterization.

The heart of the learning trajectory in Fig. 3 lies below the dashed line. New inputs appear here for vectors and dot products, both of which are challenging for students, as evidenced by their emphasis in the physics RTD in Fig. 2. We further separate the concepts of *position vectors* (anchored at the origin) from *vector fields* (anchored at points in some domain), which are quite different mathematical objects, but often conflated (Dray & Manogue, 2009–2022b).

Antidifferentiation and Limits

A major success of real analysis was the formalization by Riemann (in his 1854 Habilitationsschift) of the domain-chopping process we now know as the Riemann sum, leading to a rigorous definition of the definite integral as a formal limit of such sums. If the limit exists, the function is (Riemann) *integrable*, and the limiting value is, by definition, the result of evaluating the integral.

However, for physical scientists, the details of the limit process just don't matter, and for a very fundamental reason: It is impossible to measure a physical quantity at a 0-dimensional point in space. Any measuring device, be it a physical probe, a beam of particles in a particle accelerator, or even a beam of laser light has a fundamental length associated with it. So, there is no way to know if the underlying physics is continuous. Furthermore, we do know that, for example, a table top is *not* smooth at the molecular level. Physical scientists model physical situations as continuous when they are confident that this model gives a reasonable approximation to the quantity they are trying to calculate. For these reasons, we have minimized the mechanics of Riemann sums and limits and confined this part of the concept image to the right-hand column of Fig. 3.

[12] It is much easier to derive the algebraic formula for the dot product from the geometric definition than vice versa (Dray & Manogue, 2006).

These concepts can and should be reinforced, in terms of what it means to "Add" in CMA, but these concepts are not new for students at this stage and, we believe, not where the main cognitive difficulties lie when learning about line integrals.

Similarly, the evaluation of these definite integrals, involving antidifferentiation and inserting the given limits into the result, is also minimized and confined to the right-hand column. Of course, students must be able to antidifferentiate quickly and accurately, but, again, this step is not where the cognitive difficulties lie.

Furthermore, in practice, very few integrals can be evaluated analytically in closed form to obtain a primitive. Physical scientists often choose to look at models and limiting cases for which the necessary integrals can be evaluated, knowing that the actual situation is only being approximated. This art of choosing illustrative idealizations is a strategy that students must learn to implement for themselves. When more accuracy is needed, physicists compute definite integrals numerically, a process that requires its own strategies for limiting error.

Finally, we point out that many complex physical systems cannot be described by known/named functions at all. The only information known about them comes from measurements of discrete data. In this case, integration must be numerical and is inherently CMA.

Components and Bases

Remarkable by its absence from Fig. 3 is the use of vector components. We regard the use of vector components as part of the process of "using what you know" to unpack the iconic equation of a line integral, which sometimes – but not always – will involve expressing the components of the given vectors with respect to some basis. We expect to have more to say about this issue when we add detail to the suggested LT.

Furthermore, as noted in "Unpacking", physics applications often involve high symmetry, leading not only to the use of curvilinear coordinates, but also to the corresponding basis vectors. Even though such vectors are rarely used in lower-division mathematics courses, writing the (rectangular) basis vectors explicitly, rather than as n-tuples of components, allows students to make the transition between disciplines more easily. Deciding which basis to use is, of course, also part of "using what you know."

Many professional mathematicians also use explicit basis elements and care deeply about changes of basis. We suggest to mathematics educators and mathematics education researchers that it would be helpful in these subfields of mathematics as well as in client disciplines for lower-division courses to set the groundwork by making the basis explicit. It is difficult to talk about, let alone algebraically manipulate, a mathematical structure for which one does not have an algebraic symbol.

Conclusion

Many mathematics departments offer only a single semester course in multivariable calculus, with a week or two at the end devoted to vector calculus concepts. In these settings, the mathematics RTD in Fig. 3 can be a reasonable choice, treating both the position vector and the vector field from the integrand as triples of numbers, and giving a simplified purely algebraic formula for the integrand in rectangular coordinates (such as $f(t)x'(t)\,dt + ...$). After all, this is the context in which this RTD has been developed.

We acknowledge that it takes time to develop a solid understanding of $d\vec{r}$, or the ability to move fluidly between multiple representations. Nor are these concepts easy to convey through traditional lectures or homework assignments. In our own teaching of both second-year courses in vector calculus and third-year courses in electromagnetism, we spend considerable time modeling the use of $d\vec{r}$, which we regard as the single concept that unifies vector calculus (Dray & Manogue, 2003, 2009–2021). An effective way of teaching these crucial skills is through the use of in-class activities that can be supported by the teaching team; see Kustusch et al. (2020) for a description of how to choose when to do activities vs. lecture vs. homework. Whether the rewards (increased understanding of the geometry of both line and surface integrals) as represented by our suggested LT in "Suggested Learning Trajectory" is worth this extra time commitment is a matter for individual programs to decide. Whether students can develop richer concept images as suggested by this LT should be the subject of lots of future curriculum development and research.

Since the topic of vector line integrals crosses disciplinary boundaries between mathematics and physics, a shared commitment between departments to a chosen learning trajectory is important. Such a shared commitment is not always easy to achieve; one successful example is the *Paradigms in Physics* project (Paradigms Team at Oregon State University, 2019–2022a) at our own institution; see van Zee and Manogue (2018) for a history.

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