CONCENTRATION OF THE NUMBER OF INTERSECTIONS OF RANDOM EIGENFUNCTIONS ON FLAT TORI

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Abstract. We show that in two dimensional flat torus the number of intersections between random eigenfunctions of general eigenvalues and a given smooth curve is almost exponentially concentrated around its mean, even when the randomness is not gaussian.

1. Introduction

Let $\mathbf{T}^2$ be the two dimensional flat tori $\mathbf{R}^2/\mathbf{Z}^2$. Let F be a real-valued eigenfunction of the Laplacian on $\mathbf{T}^2$ with eigenvalue λ^2,

$$-\Delta F = \lambda^2 F.$$

It is known that all eigenvalues λ^2 have the form $4\pi^2 m$ where $m = a^2 + b^2$ for some $a, b \in \mathbf{Z}$. Let $\mathcal{E}_\lambda$ be the collection of $\mu = (\mu_1, \mu_2) \in \mathbf{Z}^2$ such that

$$\mu_1^2 + \mu_2^2 = m.$$

Denote $N = \#\mathcal{E}_\lambda$. Note that if we express m in the form $m = m_1^2 m_2$ with $m_1 = 2^r \prod_{q_k \equiv 3 \mod 4} q_k^{b_k}$ and $m_2 = 2^c \prod_{p_j \equiv 1 \mod 4} p_j^{a_j}$ $(c = 0, 1)$, where p_k, q_k are primes, and $a_k, b_k \in \mathbf{N}$, then

$$N = \prod_j (a_j + 1). \tag{1}$$

Notice that for any $\varepsilon > 0$ we have $N = O(\lambda^\varepsilon)$.

The toral eigenfunctions $F(x) = e^{2\pi i \langle \mu, x \rangle}, \mu \in \mathcal{E}_\lambda$ form an orthonormal basis in the eigenspace corresponding to λ^2. For a given toral eigenfunction F the nodal set N_F is defined to be the zero set of F,

$$N_F := \left\{ x \in \mathbf{T}^2 : F(x) = 0 \right\}.$$

The nodal set N_F has been studied intensively in analysis and differential geometry. In this note we will be focusing on the intersection between N_F and a given smooth reference curve $\mathcal{C} \subset \mathbf{T}^2$ parametrized by $\gamma : [0, 1] \to \mathbf{T}^2$ with the following properties.

Condition 1 (Assumption on γ). *$\mathcal{C}$ has unit length and $\gamma(t)$ is real analytic with positive curvature. More specifically, there exists a positive constant c such that*

$$\|\gamma'(t)\|_2 = 1 \text{ and } \|\gamma''(t)\|_2 > c \text{ for all } t.$$

The number of nodal intersections $\mathcal{Z}(F)$ between F and $\mathcal{C}$ is defined to be the cardinality of the intersection $N_F \cap \mathcal{C}$

$$\mathcal{Z}(F) := \#\{x \in \mathbf{T}^2 : x \in \mathcal{C} \wedge F(x) = 0\}.$$

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1.1. **Deterministic results.** About ten years ago Bourgain and Rudnick provided uniform upper and lower bounds for the L^2-norm of the restriction of F to $\mathcal{C}$ as follows.

Theorem 1.2. [7, Main Theorem, Theorem 1.1] *Assume that $\mathcal{C}$ is as in Condition 1. We have*

$$\int_{\mathcal{C}} |F|^2 d\gamma \gg \left(\int_{\mathbf{T}^2} |F(x)|^2 dx \right). \tag{2}$$

Also, for any $\varepsilon > 0$,

$$\lambda^{1-\varepsilon} \ll \mathcal{Z}(F) \ll \lambda,$$

where the implicit constants depend only on $\mathcal{C}$ and ε but not on λ.

Here we say that $f = O(g)$, or $f \ll g$, if there exists a positive constant C such that $|f| \leq C|g|$.

It was then conjectured by Bourgain and Rudnick that the lower bound is of order λ.

Conjecture 1.3. [7] *We have*

$$\mathcal{Z}(F) \gg \lambda.$$

In a subsequent paper, to support this conjecture they showed

Theorem 1.4. [8, Theorem 1.1] *Assume that $\mathcal{C}$ is as in Condition 1, then*

$$\mathcal{Z}(F) \gg \frac{\lambda}{B_\lambda^{5/2}}$$

where B_λ denote the maximal number of lattice points which lie on an arc of size $\sqrt{\lambda}$ on the circle $\|x\|_2 = \lambda$,

$$B_\lambda := \max_{\|x\|_2 = \lambda} \# \left\{ \mu \in \mathcal{E}_\lambda : \|x - \mu\|_2 \leq \sqrt{\lambda} \right\}.$$

In particular, as one can show that $B_\lambda \ll \log \lambda$ (see [8]), we have

$$\mathcal{Z}(F) \gg \lambda / \log^{5/2} \lambda.$$

The link in Theorem 1.4 between $\mathcal{Z}(F)$ and B_λ yields another interesting relationship between Bourgain-Rudnick conjecture 1.3 and Cilleruelo-Granville conjecture [13] which predicts that $B_\lambda = O(1)$ uniformly. This is known to hold for almost all λ^2, see for instance [6, Lemma 5]. It is worth noting that when the curvature of $\mathcal{C}$ is zero, it could happen that $\liminf_\lambda \mathcal{Z}(F) = 0$ (see for instance the construction in [8].)

Notations. We consider λ as an asymptotic parameter going to infinity and allow all other quantities to depend on λ unless they are explicitly declared to be fixed or constant. As mentioned earlier, we write $X = O(Y)$, $Y = \Omega(X)$, $X \ll Y$, or $Y \gg X$ if $|X| \leq CY$ for some fixed C; this C can depend on other fixed quantities such as the the parameter C_0 in the condition of ξ and the curve γ. If $X \ll Y$ and $Y \ll X$, we say that $Y = \Theta(X)$ or $X \asymp Y$. Also, for sequences of positive numbers $(X_k), (Y_k)$, we write $X_k = o(Y_k)$ if $Y_k / X_k \to \infty$ as $k \to \infty$.

Throughout the note, if not specified otherwise, a property $p(m)$ holds for *almost all* m if the set of m up to T that $p(m)$ does not hold has cardinality much smaller than that of the set of m for which $p(m)$ holds, i.e. $|\{m \leq T, \bar{p}(m)\}| = o(|\{m \leq T, p(m)\}|)$ as $T \to \infty$. Finally, the norm $\|.\|_2$ (or $d_2(.)$) in this note, if not specified otherwise, will be the usual L_2-norm.

1.5. **Arithmetic random wave model.** Recall that $N = \#\mathcal{E}_\lambda$ is the dimension of the eigenspace corresponding to the eigenvalue λ^2. A probabilistic approach to the study of $\mathcal{Z}(F)$ was introduced in the pioneer paper of Rudnick and Wigman [32]. Consider the random eigenfunction

$$F(x) = \frac{1}{\sqrt{N}} \sum_{\mu \in \mathcal{E}_\lambda} \varepsilon_\mu e^{2\pi i \langle \mu, x \rangle}, \tag{3}$$

for all $x \in \mathbf{T}^2$, where ε_μ are iid standard complex Gaussian with a saving

$$\varepsilon_{-\mu} = \bar{\varepsilon}_\mu.$$

This saving ensures that F is real-valued. The random function F (called *arithmetic random wave* [2]) is a centered Gaussian field over $\mathbf{T}^2$ which is stationary because the correlation $\mathbb{E}(F(x)F(y))$ is invariant under translation. As we can also see, the law of this model is independent of the choice of the orthonormal basis of the eigenspaces.

Rudnick and Wigman showed that for all eigenvalues, "almost all" [1] eigenfunctions satisfy Conjecture 1.3. More specifically, they showed the following.

Theorem 1.6. [32, Theorems 1.1, 1.2] *Let $\mathcal{C} \subset \mathbf{T}^2$ be a smooth curve on the torus, with nowhere vanishing curvature and of total length one. Then*

(1) *The expected number of nodal intersections is precisely*

$$\mathbb{E}_{\mathbf{g}} \mathcal{Z}(F) = \sqrt{2m}.$$

(2) *The variance is bounded from above as follows*

$$\mathrm{Var}_{\mathbf{g}}(\mathcal{Z}(F)) \ll \frac{m}{N}.$$

(3) *Furthermore, let $\{m\}$ be a sequence such that $N_m \to \infty$ and the Fourier coefficient $\{\widehat{\tau_m}(4)\}$ do not accumulate at ± 1, then*

$$\mathrm{Var}_{\mathbf{g}}(\mathcal{Z}(F)) = \frac{m}{N} \int_{\mathcal{C}} \int_{\mathcal{C}} \left(\sum_{\mu \in \mathcal{E}_\lambda} 4 \frac{1}{N} \left\langle \frac{\mu}{|\mu|}, \dot{\gamma}(t_1) \right\rangle^2 \left\langle \frac{\mu}{|\mu|}, \dot{\gamma}(t_2) \right\rangle^2 - 1 \right) dt_1 dt_2 + O\left(\frac{m}{N^{3/2}} \right).$$

Here the subscript $\mathbf{g}$ is used to emphasize standard Gaussian randomness, and τ_m is the probability measure on the unit circle $S^1 \subset \mathbf{R}^2$ associated with $\mathcal{E}_\lambda$,

$$\tau_m := \frac{1}{N} \sum_{\mu \in \mathcal{E}_\lambda} \delta_{\mu/\sqrt{m}},$$

where δ is the Diract delta measure.

We also refer the reader to [34, Proposition 2.2] by Rudnick et.al. where general estimates were given when the condition on $\{\widehat{\tau_m}(4)\}$ is lifted, and to [29, Theorem 1.3] by Rossi and Wigman for further extension when the first term in $\mathrm{Var}_{\mathbf{g}}(\mathcal{Z}(F))$ vanishes. We also refer the reader to [22] by Maffucci where nodal intersections with segments were considered.

1.7. **Our main results.** The magnitude m/N of the variance in Theorem 1.6 suggests that $\mathcal{Z}(F)$ is concentrated around its mean. Indeed, by Markov's bound, for any $\varepsilon > 0$ we have that

$$\mathbb{P}_{\mathbf{g}}(|\mathcal{Z}(F) - \mathbb{E}\mathcal{Z}(F)| \geq \varepsilon\lambda) \ll \frac{1}{N\varepsilon^2}. \quad (4)$$

Furthermore, the aforementioned work [29] showed that the fluctuation of $\mathcal{Z}(F)$ satisfies Central Limit Theorem (as long as γ is non-static, see [29, Theorem 1.1] for the definition). Perhaps it is natural to ask

Question 1.8. *How well is $\mathcal{Z}(F)$ concentrated around its mean for the gaussian model?*

As far as we are concerned, despite of significant breakthroughs regarding the statistics of $\mathcal{Z}(F)$ for the gaussian model mentioned above, there has been no attempt to study this simple question. Relatedly, there has been a few results in the literature to study concentration for various models, including [1, 23, 24, 28, 30], but unfortunately none of those works seem to be applicable here. With this note we hope to provide a robust method for these types of questions. In the first step we show

[1]As $(\varepsilon_\mu/\sqrt{\sum_\mu^2})_{\mu \in \mathcal{E}_\lambda}$ is random uniform over the sphere $S^{|\mathcal{E}_\lambda|}$, we can interpret $F(x)$ as a random uniform eigenfunction.

Theorem 1.9 (Concentration of the gaussian case). *Assume that γ satisfies Conditions 1. Then there exist positive constants c, c' such that for $N^{-c'} \le \varepsilon \le c'/\log N$ we have*

$$\mathbb{P}_{\mathbf{g}}(|\mathcal{Z}(F) - \mathbb{E}\mathcal{Z}(F)| \ge \varepsilon\lambda) = O(e^{-c\varepsilon^9 N}).$$

Our next focus is to show that $\mathcal{Z}(F)$ is very well concentrated even for *non-gaussian* distributions. Here

$$F(x) = \frac{1}{\sqrt{N}} \sum_{\mu \in \mathcal{E}_\lambda} \varepsilon_\mu e^{2\pi i \langle \mu, x\rangle}, \tag{5}$$

where $\varepsilon_\mu = \varepsilon_{1,\mu} + i\varepsilon_{2,\mu}$ and $\varepsilon_{1,\mu}, \varepsilon_{2,\mu}, \mu \in \mathcal{E}_\lambda$ are iid copies of a common random variable ξ of mean zero and variance one, and $\varepsilon_{-\mu} = \bar{\varepsilon}_\mu$. We will denote by $\mathbb{P}_{\varepsilon_\mu}, \mathbb{E}_{\varepsilon_\mu}$, and $\mathrm{Var}_{\varepsilon_\mu}$ the probability, expectation, and variance with respect to the random variables $(\varepsilon_\mu)_{\mu \in \mathcal{E}_\lambda}$. We will focus on two extreme families of randomness: either on bounded ξ, or on ξ satisfying the log-Sobolev inequality: that is there is a positive constant C_0 such that for any smooth, bounded, compactly supported functions f we have

$$\mathrm{Ent}_\xi(f^2) \le C_0 \mathbb{E}|\nabla f(\xi)|^2, \tag{6}$$

where $\mathrm{Ent}_\xi(f) = \mathbb{E}(f(\xi)\log f(\xi)) - \mathbb{E}f(\xi)\log \mathbb{E}(f(\xi))$. The general model (5) of random function was first considered in [11] by the current author with Chang, O. Nguyen and Vu where it was shown that the moments of $\mathcal{Z}(F)$ are asymptotically universal.

Theorem 1.10 (Universality of moment statistics). *Let C_0 be a given positive constant, and suppose that either*

- *$1/C_0 < |\xi| < C_0$ with probability one,*
- *or ξ is continuous with density bounded by C_0 and satisfies the logarithmic Sobolev inequality with parameter C_0 in (6).*

Assume that γ satisfies Condition 1. Then for almost all m we have

- *$\mathbb{E}_{\varepsilon_\mu}\mathcal{Z}(F) = \mathbb{E}_{\mathbf{g}}\mathcal{Z}(F) + O\left(\lambda/N^{c'}\right)$;*
- *More generally, for any fixed $k \in \mathbf{N}$, $\mathbb{E}_{\varepsilon_\mu}\mathcal{Z}(F)^k = \mathbb{E}_{\mathbf{g}}\mathcal{Z}(F)^k + O\left(\lambda^k/N^{c'}\right)$,*

where c' depends on the implicit constants in Conditions 1 and C_0. Furthermore, if ξ is continuous and have bounded density function, then the above holds for all m. In particular, we have

$$\mathbb{E}_{\varepsilon_\mu}\mathcal{Z}(F) = \sqrt{2m} + O\left(\lambda/N^{c'}\right) \quad \textit{and} \quad \mathrm{Var}_{\varepsilon_\mu}(\mathcal{Z}(F)) \ll \frac{\lambda^2}{N^{c'}}.$$

One crucial corollary of this result is that $\mathcal{Z}(F)$ is already concentrated around its mean via Markov's bound

$$\mathbb{P}_{\varepsilon_\mu}(|\mathcal{Z}(F) - \mathbb{E}\mathcal{Z}(F)| \ge \varepsilon\lambda) \ll \frac{1}{N^{c'}\varepsilon^2}. \tag{7}$$

In this note we upgrade this polynomial concentration to exponential.

Theorem 1.11 (Concentration of the non-gaussian case). *With the same conditions on ξ and γ as in Theorem 1.10, then for almost all m there exist positive constants $c, c' > 0$ such that for $N^{-c'} \le \varepsilon \le c'/\log N$ we have*

$$\mathbb{P}_{\varepsilon_\mu}(|\mathcal{Z}(F) - \mathbb{E}\mathcal{Z}(F)| \ge \varepsilon\lambda) \le e^{-c\varepsilon^9 N}.$$

Furthermore, the above is true for all m when ξ is continuous.

For $\varepsilon > c'/\log N$, we can certainly bound

$$\mathbb{P}_{\varepsilon_\mu}(|\mathcal{Z}(F) - \mathbb{E}\mathcal{Z}(F)| \ge \varepsilon\lambda) \le \mathbb{P}_{\varepsilon_\mu}(|\mathcal{Z}(F) - \mathbb{E}\mathcal{Z}(F)| \ge (c'/\log N)\lambda) = O(e^{-cN/(\log N)^9}),$$

which also seems non-trivial. However, we suspect that the logarithmic power can be removed when ε has order 1. Our bound on c' (where $\varepsilon \geq N^{-c'}$) and the growth of ε^9 in the exponent are far from being optimal, however the problem to obtain optimal bounds for these parameters seems to be highly non-trivial, even in the gaussian case. Relatedly, in the gaussian setting is seems interesting to obtain moderate deviation principle for $(\mathcal{Z}(F) - \mathbb{E}\mathcal{Z}(F))/\sqrt{\mathrm{Var}(\mathcal{Z}(F))}$.

We notice that in the Bernoulli case (i.e. Rademacher case, when ξ takes value ± 1 with probability 1/2) one cannot obtain anything better than $\exp(-\Theta(N))$. The main technical reason preventing us from covering for all m is that in general we cannot rely on Theorem 1.6. We will use Theorem 1.10 instead, which in turn is known only for almost all m for general ensembles.

We remark that Theorem 1.11 can also be extended for almost all m to other types of ξ not necessarily bounded nor satisfying the logarithmic Sobolev inequality. For instance our result also covers the following cases.

- When $|\xi| > 1/C_0$ with probability one and $|\xi|$ has sub-exponential tail. Then our method, by taking $C_0 = N^{\delta'}$ in Theorem 2.5 with an appropriate δ', yields a sub-exponential concentration of type $\mathbb{P}_{\varepsilon_\mu}(|\mathcal{Z}(F) - \mathbb{E}\mathcal{Z}(F)| \geq \varepsilon N) = O(e^{-(\varepsilon N)^\delta})$ for some constant $0 < \delta < 1$.
- Additionally, by the same argument, when $|\xi| > 1/C_0$ with probability one for given $C_0 > 0$ and when $\mathbb{E}(|\xi|^{C'}) < \infty$ for some sufficiently large C', then $\mathbb{P}_{\varepsilon_\mu}(|\mathcal{Z}(F) - \mathbb{E}\mathcal{Z}(F)| \geq \varepsilon n) = O((\varepsilon N)^{-C})$ as long as $N^{-c'} \leq \varepsilon \leq 1/\log N$.

Finally, our result can be seen as a continuation of [28] where exponential concentration of the number of real roots of random trigonometric polynomials was shown. Although our general approach is similar to that of [28], the technical details are very different. More specifically we have to incorporate various non-trivial results such as Theorem 1.2, Theorem 2.1, Theorem 2.3, Theorem 1.10, Theorem 3.3, Proposition 5.1 for the current model, all seem to be of their own interest.

2. Supporting lemmas and proof method

We first cite here one of the key technical ingredients by Bourgain and Rudnick in their proof of Theorem 1.4 above.

Theorem 2.1. [8, Lemma 4.1] *For each $\mu \in \mathcal{E}$ let $h_\mu(t) \in C^1[0,1]$ and $\varepsilon_\mu \in \mathbf{C}$ with $\sum_\mu |\varepsilon_\mu|^2 = 1$. Let*

$$H(t) = \sum_{\mu \in \mathcal{E}_\lambda} \varepsilon_\mu h_\mu(t) e^{i\langle \mu, \gamma(t)\rangle}.$$

Then there exists a constant C_0 depending on $\mathcal{C}$ such that

$$\int_0^1 |H(t)|^2 dt \leq 2 \max_\mu \int_0^1 |h_\mu(t)|^2 dt + C_0 \frac{N}{\lambda^{1/6}} \Big(\max_\mu \max_t |h_\mu(t)|^2 + \max_\mu \max_t |h_\mu(t)| \max_\mu \int_0^1 |h'_\mu(t)| dt \Big).$$

We also refer the reader to [3, 9] for related results for deterministic eigenfunctions, which were obtained by passing to randomized ones. Next, for $t \in [0,1]$, consider the following deterministic function

$$F(t) = \sum_{\mu \in \mathcal{E}_\lambda} \varepsilon_\mu e^{i\langle \mu, \gamma(t)\rangle} = \sum_\mu \varepsilon_\mu e^{i\lambda\langle \mu/\lambda, \gamma(t)\rangle} \tag{8}$$

with $\varepsilon_\mu = \bar{\varepsilon}_{-\mu}$ and $\sum_\mu |\varepsilon_\mu|^2 = 1$.

For each positive integer $d = 1, 2, 3$ let

$$H_d(t) = \frac{\partial^d}{\partial^d t} F(t) =: \sum_\mu \varepsilon_\mu \lambda^d h_{d,\mu}(t) e^{i\lambda\langle \mu/\lambda, \gamma(t)\rangle}.$$

We first show that

Claim 2.2. *There exists a constant C depending on γ such that*

$$|h_{d,\mu}(t)| \le C, d = 1, 2, 3.$$

Proof. When $d = 1$, $|h_{1,\mu}(t)| = |\langle \mu/\lambda, \gamma'(t)\rangle| \le \|\mu/\lambda\|_2 \|\gamma'(t)\|_2 \le 1$. When $d = 2$,

$$|h_{2,\mu}(t)| = |\lambda^{-1}\langle \mu/\lambda, \gamma''(t)\rangle + \langle \mu/\lambda, \gamma'(t)\rangle^2| \le |\lambda^{-1}\langle \mu/\lambda, \gamma''(t)\rangle| + |\langle \mu/\lambda, \gamma'(t)\rangle^2| \le C.$$

The case $d = 3$ is similar, noting that γ is real analytic. □

Theorem 2.3 (Restricted large sieve inequality)**.** *Assume that $F(t)$ and $H_d(t)$ are as above, where $d = 1, 2$. Then for any $M \in \mathbf{N}$ and any $0 \le t_1 < t_2 < \cdots < t_M \le 1$, with δ being the minimum of the gaps between t_i, t_{i+1}, we have*

$$\sum_{i=1}^{M} |H_d(t_i)|^2 \le C_1^d \lambda^{2d}(\lambda + \delta^{-1}),$$

where C_1 depends on γ.

Proof. (of Theorem 2.3) It suffices to assume that $\delta \le t_1$ and $t_M \le 1 - \delta$. We follow the classical approach by Gallagher [16] with the important input of Theorem 2.1.

Claim 2.4. *Let g be a differentiable function on $I = [a - h, a + h]$. Then*

$$g(a) \le \frac{1}{2h}\int_I |g(t)|dt + \frac{1}{2}\int_I |g'(t)|dt.$$

Proof. Let $\rho(t) = t - (a - h)$ if $t \in (a - h, a)$ and $\rho(t) = t - (a + h)$ if $t \in (a, a + h)$. Partal intergrals (over $(a - h, a)$ and $(a, a + h)$) give

$$\int_I \rho(t)g'(t)dt = 2hg(a) - \int_I g(t)dt.$$

Note that $|\rho(.)| \le h$, so the claim follows by triangle inequality. □

By this claim,

$$\sum_{i=1}^{M} |H_d(t_i)|^2 \le \frac{1}{\delta}\sum_i \int_{t_i-\delta/2}^{t_i+\delta/2} |H_d(t)|^2 dt + \sum_i \int_{t_i-\delta/2}^{t_i+\delta/2} |H_d(t)H_d'(t)|dt \le \frac{1}{\delta}\int_0^1 |H_d(t)|^2 dt + \int_0^1 |H_d(t)H_d'(t)|dt.$$

Note that by Cauchy-Schwarz, $\int_0^1 |H_d(t)H_d'(t)|dt \le \sqrt{\int_0^1 |H_d(t)|^2 dt}\sqrt{\int_0^1 |H_d'(t)|^2 dt}$. Recall that $H_d(t) = \sum_\mu \varepsilon_\mu \lambda^d h_{d,\mu}(t) e^{i\lambda\langle \mu/\lambda, \gamma(t)\rangle}$, where the $h_{d,\mu}$ can be bounded as in Claim 2.2. The L_2-bound from Theorem 2.1 yields (where we use the fact from the formula of N from (1) that $N \ll \lambda^\varepsilon$ for any ε)

$$\int_0^1 |H_d(t)|^2 dt = O(\lambda^{2d}).$$

Similarly, as $H_d(t)' = H_{d+1}(t) = \sum_\mu \varepsilon_\mu \lambda^d h_{d+1,\mu}(t) e^{i\lambda\langle \mu/\lambda, \gamma(t)\rangle}$, Theorem 2.1 also yields

$$\int_0^1 |H_d'(t)|^2 dt = O(\lambda^{2d+2}).$$

Putting the bounds together, we thus obtain $\sum_{i=1}^{M} |H_d(t_i)|^2 O(\lambda^{2d}(\lambda + \delta^{-1})), d = 1, 2$.

□

On the probability side, for bounded random variables we will rely on the following consequence of McDiarmid's inequality.

Theorem 2.5. *Assume that* $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$*, where* ξ_i *are iid copies of* ξ *of mean zero, variance one, taking values in* Ω*, a subset of* $[-C_0, C_0]$*. Let* $\mathcal{A}$ *be a set of* Ω^n*. Then for any* $t > 0$ *we have*

$$\mathbb{P}(\boldsymbol{\xi} \in \mathcal{A})\mathbb{P}(d_2(\boldsymbol{\xi}, \mathcal{A}) \geq t\sqrt{n}) \leq 4\exp(-t^4 n/16C_0^4).$$

For random variables ξ satisfying the log-Sobolev inequality (6), we use the following.

Theorem 2.6. *Assume that* $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$*, where* ξ_i *are iid copies of* ξ *satisfying* (6) *with a given* C_0*. Let* $\mathcal{A}$ *be a set in* $\mathbf{R}^n$*. Then for any* $t > 0$ *we have*

$$\mathbb{P}\big(d_2(\boldsymbol{\xi}, A) \geq t\sqrt{n}\big) \leq 2\exp\big(-\mathbb{P}^2(\boldsymbol{\xi} \in \mathcal{A})t^2 n/4C_0\big).$$

In particularly, if $\mathbb{P}(\boldsymbol{\xi} \in \mathcal{A}) \geq 1/2$ *then* $\mathbb{P}(d_2(\boldsymbol{\xi}, A) \geq t\sqrt{n}) \leq 2\exp(-t^2 n/16C_0)$*. Similarly if* $\mathbb{P}(d_2(\boldsymbol{\xi}, A) \geq t\sqrt{n}) \geq 1/2$ *then* $\mathbb{P}(\boldsymbol{\xi} \in \mathcal{A}) \leq 2\exp(-t^2 n/16C_0)$*.*

These results are standard, whose proof can be found for instance in [28, Appendix B].

Now we discuss the proof method for Theorem 1.11. Broadly speaking, the approach follows a perturbation framework (see also [23, 28, 30] for recent adaptions) with detailed steps as follows:

(1) Our starting point is an input from [11] which shows that $\mathbb{E}\mathcal{Z}(F)$ is close to $\mathbb{E}_{\mathbf{g}}\mathcal{Z}(F)$ and $\mathcal{Z}(F)$ is moderately concentrated around its mean.

(2) We then show that it is highly unlikely that there is a small set of intervals where both $|F|$ and $|F'|$ are small. We justify this by relying on a strong *repulsion estimate* (Theorem 3.3) and on a variant of *large sieve inequality* (Theorem 2.3). This step is carried out in Section 4.

(3) Furthermore, we will show in Section 5 via Jensen's bound that the number of roots over these intervals (called *unstable*, where $|F|$ and $|F'|$ are small simultaneously) is small.

(4) In addition to these results, we will show that the number of roots of $F + g$ is close to the number of roots of F over the stable intervals as long as $\|g\|_2$ is small. Basing on these results, geometric tools such as Theorem 2.5 and 2.6 can be invoked to show that indeed $\mathcal{Z}(F)$ satisfies exponential concentration.

3. Proof of Theorem 1.11: preparation

Here and later, to ease our presentation, instead of $\mathbb{P}_{\varepsilon_\mu}$ and $\mathbb{E}_{\varepsilon_\mu}$ we will only write $\mathbb{P}, \mathbb{E}$, assuming that all of these statistics are with respect to the underlying iid random variables $\varepsilon_\mu, \mu \in \mathcal{E}_\lambda$.

One of our key ingredients, Theorem 3.3 below, is a *repulsion-type* estimate which shows that at any point it is unlikely that the function and its derivative vanish simultaneously.

First, for $t \in [0, \lambda]$, we consider the rescaled function

$$J(t) := F(t/\lambda) = \frac{1}{\sqrt{N}} \sum_{\mu \in \mathcal{E}_\lambda} \varepsilon_{1,\mu} \cos(2\pi\langle \mu, \gamma(t/\lambda)\rangle) + \varepsilon_{2,\mu} \sin(2\pi\langle \mu, \gamma(t/\lambda)\rangle) \tag{9}$$

and

$$J'(t) = \frac{1}{\sqrt{N}} \sum_{\mu \in \mathcal{E}_\lambda} -\varepsilon_{1,\mu} 2\pi\langle \mu/\lambda, \gamma'(t/\lambda)\rangle \sin(2\pi\langle \mu, \gamma(t/\lambda)\rangle) + \varepsilon_{2,\mu} 2\pi\langle \mu/\lambda, \gamma'(t/\lambda)\rangle \cos(2\pi\langle \mu, \gamma(t/\lambda)\rangle), \tag{10}$$

where $\mu, -\mu \in \mathcal{E}_\lambda$ and $\bar{\varepsilon}_\mu = \varepsilon_{-\mu}$.

We prove our repulsion result via the study of small ball probability of the random walk $\frac{1}{\sqrt{N}} \sum_\mu \varepsilon_{1,\mu} \mathbf{u}_\mu + \varepsilon_{2,\mu} \mathbf{v}_\mu$ where

$$\mathbf{u}_\mu(t) = \Big(\cos(2\pi\langle \mu, \gamma(t/\lambda)\rangle), -2\pi\langle \mu/\lambda, \gamma'(t/\lambda)\rangle \sin(2\pi\langle \mu, \gamma(t/\lambda)\rangle)\Big)$$

and

$$\mathbf{v}_\mu(t) = \Big(\sin(2\pi\langle \mu, \gamma(t/\lambda)\rangle), 2\pi\langle \mu/\lambda, \gamma'(t/\lambda)\rangle \cos(2\pi\langle \mu, \gamma(t/\lambda)\rangle)\Big).$$

We first show that these vectors are asymptotically isotropic.

Claim 3.1. *For each $t \in [0, \lambda]$, and for all $(a, b) \in \mathbf{S}^1$ we have*

$$\sum_\mu \langle \mathbf{u}_\mu(t), (a, b)\rangle^2 + \langle \mathbf{v}_\mu(t), (a, b)\rangle^2 \asymp N.$$

Proof. We have

$$\begin{aligned}\sum_\mu \langle \mathbf{u}_\mu(t), (a, b)\rangle^2 + \langle \mathbf{v}_\mu(t), (a, b)\rangle^2 &= \sum_\mu \left[a \cos(2\pi\langle \mu, \gamma(t/\lambda)\rangle) - b\langle \mu/\lambda, \gamma'(t/\lambda)\rangle \sin(2\pi\langle \mu, \gamma(t/\lambda)\rangle)\right]^2 \\ &\quad + \left[a \sin(2\pi\langle \mu, \gamma(t/\lambda)\rangle) + b\langle \mu/\lambda, \gamma'(t/\lambda)\rangle \cos(2\pi\langle \mu, \gamma(t/\lambda)\rangle)\right]^2 \\ &= Na^2 + b^2 \sum_\mu \langle \mu/\lambda, \gamma'(t/\lambda)\rangle^2 \asymp N,\end{aligned}$$

where we used the fact that if $\mu = (\mu_1, \mu_2) \in \mathcal{E}_\lambda$ then $(\pm\mu_1, \pm\mu_2), (\pm\mu_2, \pm\mu_1) \in \mathcal{E}_\lambda$, noting that

$$\langle (\mu_1, \mu_2)/\lambda, \gamma'(t/\lambda)\rangle^2 + \langle (-\mu_2, \mu_1)/\lambda, \gamma'(t/\lambda)\rangle^2 = \|\gamma'(t/\lambda)\|_2^2 = 1.$$

□

Notice that $\|\mathbf{u}_\mu(t)\|_2, \|\mathbf{v}_\mu\|_2(t) \ll 1$. The above claim implies that a positive portion of the $\{|\langle \mathbf{u}_\mu, (a, b)\rangle|, |\langle \mathbf{v}_\mu, (a, b)\rangle|\}$ are of order 1. Using this information we obtain the following key bound.

Lemma 3.2. *Assume that $\varepsilon_{1,\mu}, \varepsilon_{2,\mu}$ are iid copies of ξ as in Theorem 1.11. For any $r \geq 1/\sqrt{N}$ we have*

$$\sup_{a \in \mathbf{R}^2} \mathbb{P}\left(\frac{1}{\sqrt{N}} \sum_\mu \varepsilon_{1,\mu}\mathbf{u}_\mu + \varepsilon_{2,\mu}\mathbf{v}_\mu \in B(a, r)\right) = O(r^2),$$

where $B(a, r)$ is the open ball of center a and radius r, and where the implied constant is allowed to depend on C_0.

Proof. This is [18, Theorem 1] where we cover a ball of radius r by Nr^2 balls of radius $1/\sqrt{N}$. □

We also refer the reader to [27, 33] for further developments of similar anti-concentration estimates. We deduce from Lemma 3.2 the following corollary.

Theorem 3.3 (Repulsion estimate)**.** *Assume that the coefficients $\varepsilon_{1,\mu}, \varepsilon_{2,\mu}$ of $J(t)$ are iid copies of ξ as in Theorem 1.11. Then as long as $\alpha > 1/\sqrt{N}$, $\beta > 1/\sqrt{N}$, for every $t \in [-\lambda, \lambda]$ we have*

$$\mathbb{P}\big(|J(t)| \leq \alpha \wedge |J'(t)| \leq \beta\big) = O(\alpha\beta).$$

In application we just choose α, β to be at least N^{-c} for some small constant c.

4. Exceptional eigenfunctions

This current section is motivated by the treatment in [23, Section 4.2] and [28, Section 4]. Let $C > 4$ be a sufficiently large number and choose

$$R = C \log N. \tag{11}$$

Cover $[0, 1]$ by $\lfloor \frac{\lambda}{R} \rfloor$ open intervals I_i of length (approximately) R/λ each. Let $3I_i$ be the interval of length $3R/\lambda$ having the same midpoint with I_i. Given some parameters α, β, we call an interval I_i *stable* for a function f if there is no point in $x \in 3I_i$ such that $|f(x)| \leq \alpha$ and $|f'(x)| \leq \beta\lambda$. In other words, there is no

$x \in 3I_i$ where $|f(x)|$ and $|f'(x)|/\lambda$ are both small. Let δ be another small parameter (so that $\delta R < 1/4$), we call f *exceptional* if the number of unstable intervals is at least $\delta\lambda$. We call f not exceptional otherwise.

For convenience, for each $F(x) = \frac{1}{\sqrt{N}}\sum_{\mu\in\mathcal{E}_\lambda} a_\mu \cos(2\pi\langle\mu,\gamma(x)\rangle) + b_\mu \sin(2\pi\langle\mu,\gamma(x)\rangle)$ we assign a unique (unscaled) vector $\mathbf{v}_F = (a_\mu, b_\mu)_{\mu\in\mathcal{E}_\lambda}$ in $\mathbf{R}^{2N}$. Note that when F is random, that is when a_μ, b_μ (playing the role of $\varepsilon_{1,\mu}, \varepsilon_{2,\mu}$) are iid copies of ξ as in Theorem 1.11 then the components of F are iid copies of ξ. Let $\mathcal{E}_e = \mathcal{E}_e(R,\alpha,\beta;\delta)$ denote the set of vectors $\mathbf{v}_F$ associated to exceptional functions F. Our goal in this section is the following.

Theorem 4.1. *Assume that α,β,δ satisfy $\delta \le 1/4R$ and*

$$\alpha \asymp \delta^{3/2}, \beta \asymp \delta^{3/4}, \delta > N^{-2/5}. \tag{12}$$

Let $\mathbf{v}_F = (a_\mu, b_\mu)_{\mu\in\mathcal{E}_\lambda}$ be a random vector, where $\varepsilon_{i\mu}$ are iid copies of ξ as in Theorem 1.11. Then we have

$$\mathbb{P}\Big(\mathbf{v}_F \in \mathcal{E}_e\Big) \le e^{-c\delta^8 N},$$

where c is absolute.

We now discuss the proof. First assume that f (playing the role of F) is exceptional, then there are $K = \lfloor \delta\lambda/3 \rfloor$ unstable intervals that are R/λ-separated (and hence $4/\lambda$-separated as N is sufficiently large). Now for each unstable interval in this separated family we choose $x_j \in 3I_j$ where $|f(x_j)| \le \alpha$ and $|f'(x_j)| \le \beta\lambda$ and consider the interval $B(x_j, \gamma/\lambda)$ for some $\gamma < 1$ chosen sufficiently small (given δ, see (14)). Let

$$M_j := \max_{x\in B(x_j,\gamma/\lambda)} |f''(x)|.$$

By Theorem 2.3 we have

$$\sum_{j=1}^K M_j^2 \le \frac{2\lambda + (4/\lambda)^{-1}}{2\pi}\int_{x\in[0,1]} f''(x)^2 dx \le \lambda^5 \frac{\sum_\mu \varepsilon_{1,\mu}^2 + \varepsilon_{2,\mu}^2}{N}.$$

On the other hand, in both the boundedness and the log-Sobolev cases of ξ in Theorem 1.11 we have $\frac{\sum_\mu a_\mu^2 + b_\mu^2}{N} \ge 4$ with exponentially small probability, so without loss of generality it suffices to focus on the event

$$\frac{\sum_\mu a_\mu^2 + b_\mu^2}{N} \le 4.$$

We thus infer from the above that the number of j for which $M_j \ge C_2\delta^{-1/2}\lambda^2$ is at most $2C_2^{-2}\delta\lambda$. Hence for at least $(1/3 - 2C_2^{-2})\delta\lambda$ indices j we must have $M_j < C_2\delta^{-1/2}\lambda^2$.

Consider our function over $B(x_j,\gamma/\lambda)$ where $M_j < C_2\delta^{-1/2}\lambda^2$, then by Taylor expansion of order two around x_j, we obtain for any x in this interval

$$|f(x)| \le \alpha + \beta\gamma + C_2\delta^{-1/2}\gamma^2/2 \text{ and } |f'(x)| \le (\beta + C_2\delta^{-1/2}\gamma)\lambda.$$

Now consider a function g of the form $g(x) = \frac{1}{\sqrt{N}}\sum_{\mu\in\mathcal{E}_\lambda} a'_\mu \cos(2\pi\langle\mu,\gamma(x)\rangle) + b'_\mu \sin(2\pi\langle\mu,\gamma(x)\rangle)$, for which $\|g\|_2 = (\sum_\mu {a'}_\mu^2 + {b'}_\mu^2)/N \le \tau$, where τ is another parameter to be chosen (such as it satisfies (14)). Then as the intervals $B(x_j,\gamma/\lambda)$ are $4/\lambda$-separated, by Theorem 2.3 we have

$$\sum_j \max_{x\in B(x_j,\gamma/\lambda)} g(x)^2 \le 8\lambda\frac{\sum_\mu {a'}_\mu^2 + {b'}_\mu^2}{N} \le 8\lambda\tau^2$$

and

$$\sum_j \max_{x\in B(x_j,\gamma/\lambda)} g'(x)^2 \le 8\lambda\frac{\sum_\mu {a'}_\mu^2 + {b'}_\mu^2}{N} \le 8\lambda^3\tau^2.$$

Hence, again by an averaging argument, the number of intervals where either $\max_{x\in B(x_j,\gamma/\lambda)}|g(x)| \geq C_3\delta^{-1/2}\tau$ or $\max_{x\in B(x_j,\gamma/\lambda)}|g'(x)| \geq C_3\delta^{-1/2}\tau\lambda$ is bounded from above by $(1/3-2C_2^{-2})\delta\lambda/2$ if C_3 is sufficiently large. On the remaining at least $(1/3-2C_2^{-2})\delta\lambda/2$ intervals, with $h=f+g$, we have simultaneously that

$$|h(x)| \leq \alpha+\beta\gamma+C_2\delta^{-1}\gamma^2/2+C_3\delta^{-1/2}\tau \text{ and } |h'(x)| \leq (\beta+C_2\delta^{-1}\gamma+C_3\delta^{-1/2}\tau)\lambda.$$

For short, let

$$\alpha' = \alpha+\beta\gamma+C_2\delta^{-1}\gamma^2/2+C_3\delta^{-1/2}\tau \text{ and } \beta' = \beta+C_2\delta^{-1/2}\gamma+C_3\delta^{-1/2}\tau.$$

It follows that $\mathbf{v}_h$ belongs to the set $\mathcal{U}=\mathcal{U}(\alpha,\beta,\gamma,\delta,\tau,C_1,C_2,C_3)$ in $\mathbf{R}^{2N}$ of the vectors corresponding to h, for which the measure of x with $|h(x)|\leq\alpha'$ and $|h'(x)|\leq\beta'\lambda$ is at least $(1/3-2C_2^{-2})\delta\gamma$ (because this set of x contains $(1/3-2C_2^{-2})\delta\lambda/2$ intervals of length $2\gamma/\lambda$). Putting together we have obtained the following claim.

Claim 4.2. *Assume that $\mathbf{v}_f \in \mathcal{E}_e$. Then for any g with $\|g\|_2 \leq \tau$ we have $\mathbf{v}_{f+g}\in\mathcal{U}$. In other words,*

$$\Big\{\mathbf{v}\in\mathbf{R}^{2N}, d_2(\mathcal{E}_e,\mathbf{v}) \leq \tau\sqrt{N}\Big\} \subset \mathcal{U}.$$

We next show that $\mathbb{P}(\mathbf{v}_f\in\mathcal{U})$ is smaller than $1/2$. Indeed, for each F, let $B(f)$ be the measurable set of $x\in\mathbf{T}$ such that $\{|f(x)|\leq\alpha'\}\wedge\{f'(x)|\leq\beta'\lambda\}$. Then the Lebesgue measure of $B(f)$, $\mu(B(f))$, is bounded by

$$\mathbb{E}\mu(B(f)) = \int_{x\in\mathbf{T}} \mathbb{P}(\{|f(x)|\leq\alpha'\}\wedge\{|f'(x)|\leq\lambda\beta'\})dx = O(\alpha'\beta'),$$

where we used Theorem 3.3 for each x. It thus follows that $\mathbb{E}\mu(B(f))=O(\alpha'\beta')$. So by Markov inequality,

$$\mathbb{P}(\mathbf{v}_f\in\mathcal{U}) \leq \mathbb{P}\big(\mu(B(f)) \geq (1/3-2C_2^{-2})\delta\gamma\big) = O(\alpha'\beta'/\delta\gamma) < 1/2 \tag{13}$$

if α,β are as in (12) and then γ,τ are chosen appropriately, for instance as

$$\gamma \asymp \delta^{5/4}, \tau \asymp \delta^2. \tag{14}$$

Proof. (of Theorem 4.1) By Theorems 2.5 and 2.6, and by Claim 4.2 and (13) we have

$$\mathbb{P}(\mathbf{v}\in\mathcal{E}_e) \leq e^{-c\tau^4 N},$$

completing the proof with τ from (14). □

5. Roots over unstable intervals

To start with, consider a (deterministic) function $F(t)$ of type

$$F(t) = \sum_\mu a_\mu e^{i\langle\mu,\gamma(t)\rangle}.$$

This is a realization of our random eigenfunction. One of the main goals in this section is the following lemma.

Proposition 5.1. *Let ε be given as in Theorem 1.11. Assume that the parameters R,α,β,τ are chosen as in (11), (12) and (14). Assume that there are $\delta\lambda$ disjoint intervals I of length R/λ over which $F(t)$ has at least $\varepsilon\lambda/2$ roots, then there exists a measurable set $A\subset[0,1]$ of measure at least $c\varepsilon/4$ over which*

$$\max_{t\in A}|F(t)| \leq \alpha \text{ and } \max_{t\in A}|F'(t)| \leq \beta\lambda.$$

Before proving this result, we deduce that non-exceptional polynomials cannot have too many roots over the unstable intervals.

Corollary 5.2. *Let the parameters $R, \varepsilon, \alpha, \beta, \tau$ and δ be as in Proposition 5.1. Then a non-exceptional F cannot have more than $\varepsilon\lambda/2$ roots over any $\delta\lambda$ intervals I_i from Section 4. In particularly, F cannot have more than $\varepsilon\lambda/2$ roots over the unstable intervals.*

Proof. (of Corollary 5.2) If F has more than $\varepsilon\lambda/2$ roots over some $\delta\lambda$ intervals I_i, then Proposition 5.1 implies the existence of a set $A = A(F)$ that intersects with the set of stable intervals (because the total size of the unstable intervals is at most $\delta\lambda R/\lambda = \delta R < c\varepsilon/8$), so that $\max_{x\in A}|F(x)| \le \alpha$ and $\max_{x\in A}|F'(x)| \le \beta\lambda$. However, this is impossible because for any x in the union of the stable intervals we have either $|F(x)| > \alpha$ or $|F'(x)| > \beta\lambda$. □

We now discuss the proof of Proposition 5.1. We first recall the following Jensen's bound (see for instance [31] and [26, Appendix A1]) on the number of roots of an analytic function $\psi(w)$ over the closed ball of center z and radius R (denoted by $\bar{B}(z,R)$) in $\mathbf{C}$

$$\#\{w \in B(z,r) : \psi(w) = 0\} \le \frac{\log \frac{M}{m}}{\log \frac{R^2+r^2}{2Rr}}$$

where $0 < r < R$ and $M = \max_{w\in \bar{B}(z,R)} |\psi(w)|, m = \max_{w\in\bar{B}(z,r)} |\psi(w)|$. Next, by Condition 1, the curve γ has an analytic continuation to $[0,1] + B(0,\varepsilon_\gamma) \subset \mathbf{C}$ for a sufficiently small ε_γ. In what follows $F(z)$ is $\sum_\mu a_\mu e^{i\langle \mu,\gamma(z)\rangle}$, where $\gamma(z)$ is the extension of $\gamma(t)$.

Lemma 5.3. *Let I be any interval in $[0,1]$ with length $\delta = |I| \le \varepsilon_\gamma/2$. Assume furthermore that $\sum_\mu |a_\mu| \le N$. Then there exists a constant c (depending on γ) such that*

$$\#\{z \in I + B(0,\delta) : F(z) = 0\} \le c\lambda\delta + \log N - \log\max_{t\in I}|F(t)|$$

and

$$\#\{z \in I + B(0,\delta) : F'(z) = 0\} \le c\lambda\delta + \log(2N) - \log\max_{t\in I}|\frac{1}{\lambda}F'(t)|.$$

Proof of Lemma 5.3. We first work with roots of $F(z)$. For $z \in I + B(0,2\delta), \exists t \in \mathbf{R}$ such that $|z - t| < 2\delta$, and by analyticity

$$|\gamma(z) - \gamma(t)| \le c\delta,$$

for some constant c depending on γ. Hence for $\mu \in \mathcal{E}_\lambda$,

$$\left|e^{i\langle \mu,\gamma(z)\rangle}\right| = \left|e^{i\langle\mu,\gamma(z)-\gamma(t)\rangle}\right| \le e^{c\lambda\delta}.$$

Therefore by the triangle inequality

$$|F(z)| \le (\sum_{\mu\in\mathcal{E}_\lambda}|a_\mu|)e^{c\lambda\delta} \le Ne^{c\lambda\delta}.$$

Jensen's inequality (applied to $B(z_I, 2\delta), B(z_I,\delta)$, where z_I is the midpoint of I) implies

$$\begin{aligned}\#\{z \in I + B(0,\delta), F(z) = 0\} &\le \log\left(Ne^{c\lambda\delta}\right) - \log\max_{t\in I}|F(t)| \\ &\le c\lambda\delta + \log N - \log\max_{t\in I}|F(t)|.\end{aligned}$$

We next work with roots of $F'(z)$, where the argument is similar. For $z \in I + B(0,2\delta), \exists t \in \mathbf{R}$ such that $|z - t| < 2\delta$, and hence by analyticity $|\gamma(z) - \gamma(t)| \le c\delta$ and also $|\gamma'(z) - \gamma'(t)| \le c\delta$ for some constant c depending on γ. Hence for $\mu \in \mathcal{E}_\lambda$, as before we have $\left|e^{i\langle\mu,\gamma(z)\rangle}\right| \le e^{c\lambda|I|}$, as well as $|\gamma'(z)| \le |\gamma'(t)| + c'\delta = 1 + c'\delta$. This implies that

$$|\frac{1}{\lambda}F'(z)| = |\sum_\mu a_\mu\langle\mu/\lambda, \gamma'(z)\rangle e^{i\langle\mu,\gamma(z)\rangle}| \le (1 + c'\delta)(\sum_{\mu\in\mathcal{E}_\lambda}|a_\mu|)e^{c\lambda\delta} \le 2Ne^{c\lambda\delta}.$$

By Jensen's inequality (again applied to $B(z_I, 2\delta), B(z_I, \delta)$),

$$\#\left\{z \in I + B(0,\delta), \frac{1}{\lambda}F'(z) = 0\right\} \le \log\left(2Ne^{c\lambda\delta}\right) - \log \max_{t\in I} |\frac{1}{\lambda}F'(t)|.$$

□

As a consequence we obtain the following

Corollary 5.4. *Assume that I is any interval in $[0,1]$ with length*

$$\frac{2\log(2N)}{c\lambda} \le |I| \le \varepsilon_\gamma/2.$$

Assume furthermore that $\sum_\mu |a_\mu| \le N$ and one of the following holds,

- $\max_{t\in I} |F(t)| \ge \exp(-c\lambda|I|/2)$,
- $\max_{t\in I} |F'(t)| \ge \lambda \exp(-c\lambda|I|/2)$.

Then we have

$$\#\{t \in I, F(t) = 0\} \le 2c|I|\lambda.$$

Proof. (of Corollary 5.4)It is clear that if $\max_{t\in I} |F(t)| \ge \exp(-c\lambda|I|/2)$ then the first part of Lemma 5.3 implies the claim. In the second case that $\max_{t\in I} |F'(t)| \ge \lambda\exp(-c\lambda|I|/2)$, by the mean value theorem one has $\#\{t \in I, F(t) = 0\} \le \#\{t \in I, F'(t) = 0\} + 1$, and we can bound the latter by the second part of Lemma 5.3. □

We next provide an overview of the proof of Proposition 5.1. By Corollary 5.4, if there is an interval I over which F has many roots, then over the entire I both $|F(t)|$ and $|F'(t)|$ are small. As such, if there are many intervals over which F has many roots, the measure of t for which $|F(t)|$ and $|F'(t)|$ are both small is non-negligible.

Proof. (of Proposition 5.1) Among the $\delta\lambda$ intervals we first throw away those of less than $\varepsilon\delta^{-1}/4$ roots, hence there are at least $\varepsilon\lambda/4$ roots left from the original set of $\varepsilon\lambda/2$ roots. For convenience we denote the remaining intervals by $J_1, \dots, J_M$, where $M \le \delta\lambda$, and let $m_1, \dots, m_M$ denote the number of roots over each of these intervals respectively.

In the next step we expand each interval J_j by consecutively adding nearby intervals of length R/λ (at the beginning of Section 4) of J_j to form a larger interval $\bar{J}_j$ of length $\lceil cm_j/R\rceil \times (R/\lambda)$ for some small constant c (and we recall from (11) that $R = C\log n$). Furthermore, if the expanded intervals $\bar{J}'_{i_1}, \dots, \bar{J}'_{i_k}$ of $\bar{J}_{i_1}, \dots, \bar{J}_{i_k}$ form an intersecting chain, then we create a longer interval $\bar{J}'$ of length $\lceil c(m_{i_1} + \dots + m_{i_k})/R\rceil \times (R/\lambda)$, which contains them and therefore contains at least $m_{i_1} + \dots + m_{i_k}$ roots.

After the merging process, we obtain a collection $\bar{J}'_1, \dots, \bar{J}'_{M'}$ with the number of roots $m'_1, \dots, m'_{M'}$ respectively, so that $\sum m'_i \ge \varepsilon\lambda/2$. Note that now $\bar{J}'_i$ has length $\lceil cm'_i/R\rceil \times (R/\lambda) \approx cm'_i/\lambda$ (because $\varepsilon\delta^{-1}$ is sufficiently large compared to R) and these intervals are R/λ-separated. Now over each J'_i of length $cm'_i/\lambda \ge c(C\log N)/\lambda$ there are m'_i roots, by Corollary 5.4 we must have

$$\max_{t\in J'_i} |F(t)| \le \exp(-cm'_i/2) \text{ and } \max_{t\in J'_i} |F'(t)| \le \lambda\exp(-cm'_i/2). \quad (15)$$

As α, β from (12) are of order at least $N^{-O(1)}$, while C from (11) is sufficiently large, so we automatically have in this case that $\max_{t\in J'_i} |F(t)| \le \alpha$ and $\max_{t\in J'_i} |F'(t)| \le \beta\lambda$.

Letting A denote the union of all such intervals J_i'. Then we have $\max_{x\in A}|F(x)| \le \alpha$ and $\max_{x\in A}|F'(x)| \le \beta\lambda$ and its Lebesgue measure satisfies

$$\lambda_{Leb}(A) \ge \sum_i cm_i'/\lambda \ge c\varepsilon/4.$$

□

We conclude the section by a quick consequence of our lemma. For each F that is not exceptional we let $S(F)$ be the collection of intervals over which F is stable. Let $N_s(F)$ denote the number of roots of F over the set $S(F)$ of stable intervals.

Corollary 5.5. *With the same parameters as in Corollary 5.2, we have*

$$\mathbb{P}\Big(N_s(F)1_{F\in\mathcal{E}_e^c} \le \mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda\Big) = o(1)$$

and

$$\mathbb{E}\Big(N_s(F)1_{F\in\mathcal{E}_e^c}\Big) \ge \mathbb{E}\mathcal{Z}(F) - 2\varepsilon\lambda/3.$$

Proof. (of Corollary 5.5) For the first bound, by Corollary 5.2, if $N_s(F)1_{F\in\mathcal{E}_e^c} \le \mathbb{E}\mathcal{Z}(F)-\varepsilon\lambda$ then $\mathcal{Z}(F)1_{F\in\mathcal{E}_e^c} \le \mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda/2$. Thus

$$\begin{aligned}\mathbb{P}\big(N_s(F)1_{F\in\mathcal{E}_e^c} \le \mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda\big) &\le \mathbb{P}\big(\mathcal{Z}(F)(F)1_{F\in\mathcal{E}_e^c} \le \mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda/2\big)\\ &\le \mathbb{P}\big(\mathcal{E}_e^c \wedge \mathcal{Z}(F) \le \mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda/2\big) + \mathbb{P}(\mathcal{E}_e) = o(1),\end{aligned}$$

where we used (4) and Theorem 4.1. For the second bound regarding $\mathbb{E}(\mathcal{Z}(F)1_{F\in\mathcal{E}_e^c})$, let $N_{us}(F)$ denote the number of roots of F over the set of unstable intervals. By Corollary 5.2, for non-exceptional F we have that $N_{us}(F) \le \varepsilon\lambda/2$, and hence trivially $\mathbb{E}(N_{us}(F)1_{F\in\mathcal{E}_e^c}) \le \varepsilon\lambda/2$. Because each F has $O(\lambda)$ roots by Theorem 1.2, we then obtain

$$\begin{aligned}\mathbb{E}(N_s(F)1_{F\in\mathcal{E}_e^c}) &\ge \mathbb{E}\mathcal{Z}(F) - \mathbb{E}(N_{us}(F)1_{F\in\mathcal{E}_e^c}) - \mathbb{E}(\mathcal{Z}(F)1_{F\in\mathcal{E}_e})\\ &\ge \mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda/2 - O(\lambda\times e^{-c\tau^4\lambda}) \ge \mathbb{E}\mathcal{Z}(F) - 2\varepsilon\lambda/3.\end{aligned}$$

□

6. PROOF OF THEOREM 1.11: COMPLETION

We first give a few deterministic results to control the number of roots under perturbation.

Lemma 6.1. *Fix strictly positive numbers μ and ν. Let $I = (a, b)$ be an interval of length greater than $2\kappa/\nu$, and let f be a C^1-function on I such that at each point $x \in I$ we have either $|f(x)| > \kappa$ or $|f'(x)| > \nu$. Then for each root $x_i \in I$ with $x_i - a > \kappa/\nu$ and $b - x_i > \kappa/\nu$ there exists an interval $I(x_i) = (a', b')$ where $f(a')f(b') < 0$ and $|f(a')| = |f(b')| = \kappa$, such that $x_i \in I(x_i) \subset (x_i - \kappa/\nu, x_i + \kappa/\nu)$ and the intervals $I(x_i)$ over the roots are disjoint.*

As a consequence we obtain

Corollary 6.2. *Fix positive κ and ν. Let $I = (a, b)$ be an interval of length at least $2\kappa/\nu$, and let f be a C^1-function on I such that at each point $x \in I$ we have either $|f(x)| > \kappa$ or $|f'(x)| > \nu$. Let g be a function such that $|g(x)| < \kappa$ over I. Then for each root $x_i \in I$ of f with $x_i - a > \kappa/\nu$ and $b - x_i > \kappa/\nu$ we can find a root x_i' of $f + g$ such that $x_i' \in (x_i - \kappa/\nu, x_i + \kappa/\nu)$, and also the x_i' are distinct.*

The proof of Lemma 6.1 above is elementary, we refer the reader to [23, Claim 4.2] and more specifically to [28, Lemma 6.1] for a complete proof.

Now we prove Theorem 1.11 by considering the two tails separately.

6.3. **The lower tail.** We need to show that

$$\mathbb{P}(\mathcal{Z}(F) \leq \mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda) \leq e^{-c\varepsilon^9\lambda}. \tag{16}$$

With the parameters $\alpha, \beta, \delta, \tau, R$ chosen as in Corollary 5.2, consider a non-exceptional eigenfunction F. Let g be an eigenfunction with $\|g\|_2 \leq \tau$, where τ is chosen as in (14). Consider a stable interval I_j with respect to F (there are at least $(\frac{2\pi}{R} - \delta)\lambda$ such intervals). We first notice that the number of stable intervals I_j over which $\max_{x\in 3I_j} |g(x)| > \alpha$ is at most $O(\delta\lambda)$. Indeed, assume that there are M such intervals $3I_j$. Then we can choose $M/6$ such intervals that are R/λ-separated. By Theorem 2.3 we have $(M/6)\alpha^2 \leq \lambda\tau^2$, which implies $M \leq 6\lambda(\tau\alpha^{-1})^2 = O(\delta\lambda)$. From now on we will focus on the stable intervals with respect to F on which $|g|$ is smaller than α.

By Corollary 6.2 (applied to $I = 3I_j$ with $\mu = \alpha$ and $\nu = \beta n$, note that $\alpha/\beta \asymp \delta^{3/4} < R$), because $\max_{x\in 3I_j} |g(x)| < \alpha$, the number of roots of $F + g$ over each interval I_j is at least as that of F. Hence if F is such that $\mathcal{Z}(F) \geq \mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda/2$ and also F has at least $\mathbb{E}\mathcal{Z}(F) - 2\varepsilon\lambda/3$ roots over the stable intervals, then by Corollary 5.2, with appropriate choice of the parameters, F has at least $\mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda$ roots over the stable intervals I_j above where $|g| \leq \alpha$, and hence Corollary 6.2 implies that $F + g$ has at least $\mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda$ roots over these stable intervals I_j. In particularly $F + g$ has at least $\mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda$ roots over $\mathbf{T}$. Let $\mathcal{U}^{lower}$ be the collection of $\mathbf{v}_F$ from such F (where $\mathcal{Z}(F) \geq \mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda/2$ and F has at least $\mathbb{E}\mathcal{Z}(F) - 2\varepsilon\lambda/3$ roots over the stable intervals). Then by Corollary 5.5 and (7)

$$\mathbb{P}(\mathbf{v}_F \in \mathcal{U}^{lower}) \geq 1 - \mathbb{P}\big(\mathcal{Z}(F) \leq \mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda/2\big) - \mathbb{P}\big(N_s(F)1_{F\in\mathcal{E}_e^c} \leq \mathbb{E}\mathcal{Z}(F) - 2\varepsilon\lambda/3\big) \geq 1/2. \tag{17}$$

Proof. (of Equation (16)) By our application of Corollary 6.2 above, the set $\{\mathbf{v}, d_2(\mathbf{v}, \mathcal{U}^{lower}) \leq \tau\sqrt{2\lambda}\}$ is contained in the set of having at least $\mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda$ roots. Furthermore, (17) says that $\mathbb{P}(\mathbf{v}_F \in \mathcal{U}^{lower}) \geq 1/2$. Hence by Theorems 2.5 and 2.6

$$\mathbb{P}(\mathcal{Z}(F) \geq \mathbb{E}\mathcal{Z}(F) - \varepsilon\lambda) \geq \mathbb{P}\Big(\mathbf{v}_F \in \big\{\mathbf{v}, d_2(\mathbf{v}, \mathcal{U}^{lower}) \leq \tau\sqrt{\lambda}\big\}\Big) \geq 1 - \exp(-c\varepsilon^9\lambda),$$

where we used the fact that $\tau \asymp \delta^2$ from (14). □

6.4. **The upper tail.** Our goal here is to justify the upper tail

$$\mathbb{P}(\mathcal{Z}(F) \geq \mathbb{E}\mathcal{Z}(F) + \varepsilon\lambda) \leq e^{-c\varepsilon^9\lambda}. \tag{18}$$

Let $\mathcal{U}^{upper}$ denote the set of $\mathbf{v}_F$ for which $\mathcal{Z}(F) \geq E\mathcal{Z}(F) + \varepsilon\lambda$. By Theorem 4.1 it suffices to assume that F is non-exceptional.

Proof. (of Equation (18)) Assume that for a non-exceptional F we have $\mathcal{Z}(F) \geq \mathbb{E}\mathcal{Z}(F) + \varepsilon\lambda$. Then by Corollary 5.2 the number of roots of F over the stable intervals is at least $\mathbb{E}\mathcal{Z}(F) + 2\varepsilon\lambda/3$. Let us call the collection of $\mathbf{v}_F$ of these eigenfunctions by $\mathcal{S}^{upper}$. Then argue as in the previous subsection (with the same parameters of $\alpha, \beta, \tau, \delta$), Corollary 5.2 and Corollary 6.2 imply that any $h = F + g$ with $\|g\|_2 \leq \tau$ has at least $\mathbb{E}\mathcal{Z}(F) + \varepsilon\lambda/2$ roots. On the other hand, we know by (4) that the probability that F belongs to this set of functions is smaller than $1/2$. It thus follows by Theorems 2.5 and 2.6 that

$$\mathbb{P}(\mathbf{v}_F \in \mathcal{U}^{upper}) \leq e^{-c\varepsilon^9\lambda},$$

where we again used that $\tau \asymp \delta^2$. □

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