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What Research Says About Teaching Mathematics Through Problem Posing

Jinfa Cai
University of Delaware (jcai@udel.edu)

There has been increased emphasis on integrating problem posing into curriculum and instruction with the promise of potentially providing more and higher quality opportunities for students to learn mathematics as they engage in problem-posing activities. This paper aims to provide a synthesis of what research says about teaching mathematics through problem posing. In particular, this paper addresses the following questions: (1) What does teaching mathematics through problem posing look like? (2) What is problem posing, anyway? (3) What is a problem-posing task? (4) How should teachers handle students' posed problems in classroom instruction? (5) How can teachers be supported to learn to teach through problem posing? (6) What is the effect of Problem-Posing-Based Learning (P-PBL) instruction on teachers and students? Throughout the sections, various related unanswered questions are raised, and the paper ends with a proposed P-PBL instructional model. Hopefully, the ideas presented in this paper can serve as a springboard to encourage more scholars to engage in problem-posing research so that we can provide more opportunities for students to learn mathematics through problem posing.

Keywords: problem posing, P-PBL, teaching case, classroom instruction, teacher professional learning, teaching effectiveness

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Introduction

In 2003, I synthesized a few major research findings related to teaching mathematics through problem solving (Cai, 2003). In that paper, I discussed four issues and concerns related to teaching mathematics through problem solving, which are related to four commonly asked questions: (1) Are young children really able to explore problems on their own and arrive at sensible solutions? (2) How can teachers learn to teach through problem solving? (3) What are students' beliefs about teaching through problem solving? (4) Will students sacrifice basic skills if they are taught mathematics through problem solving? I reviewed available research evidence surrounding each issue and then pointed out avenues for research that would be needed to address the issues more completely.

The current paper is a "sister paper" of Cai (2003) with a focus on teaching mathematics through problem posing. Thus, whereas the previous paper considered Problem-Based Learning (PBL), the current paper examines Problem-Posing-Based Learning (P-PBL). In recent years, there has been increased emphasis on integrating problem posing into curriculum and instruction with the promise of potentially providing more and higher quality opportunities for students to learn mathematics as they engage in problem-posing activities.

WHY PROBLEM POSING?

Problem posing has been recognized in part due to its importance in the process of scientific discovery. As the legendary Einstein put it, sometimes the posing of a problem is more important than actually solving the problem (Einstein & Infeld, 1938). At the turn of the 20th century, David Hilbert posed a set of 23 influential mathematical problems that inspired a great deal of progress in the discipline of mathematics (Hilbert, 1901-1902). In the history of science, formulating precise, answerable questions has not only advanced new discoveries but has also stimulated scientists' intellectual excitement (Mosteller, 1980). In education, problem posing has long been recognized as a critically important intellectual activity in research on reading (Rosenshine, Meister & Chapman, 1996), science education (Mestre, 2002), and mathematics education (Cai, *et al.*, 2015; Ellerton, 1986; Kilpatrick, 1987; Silver, 1994; Singer *et al.*, 2013, 2015). For this paper, my focus will be on mathematics education.

Theoretically, P-PBL is sound based on both constructivist and sociocultural perspectives on learning, and it can increase students' access to mathematical sensemaking and learning. When students have the opportunity to pose their own mathematical problems based on a situation, they must make sense of the constraints and conditions from the given information to build connections between their existing understanding and a new understanding of related mathematical ideas. Although problem-posing activities are cognitively demanding tasks, they are adaptable to students' abilities and thus can increase students' access such that students with different levels of understanding can still participate and pose potentially productive problems based on their own sensemaking. Thus, the learning opportunities provided by problem posing have a low floor and high ceiling (Cai & Hwang, 2021).

Indeed, although students traditionally find themselves positioned as simply receivers of instruction, when they formulate their own mathematical problems to investigate, they can build positive, powerful identities as mathematical creators and seekers (Silver, 1997; National Council of Teachers of Mathematics [NCTM], 1991, 2020). Problem posing shares mathematical authority in the classroom, giving students the power to create their own mathematical problems considered by the class. Moreover, because problem posing is an activity with a low floor and high ceiling (Cai & Hwang, 2021; Singer *et al.*, 2015), it offers access to all students to opportunities for mathematical sensemaking.

Researchers have begun to discuss the complex nature of teaching mathematics through problem posing. There is a need for careful analysis of practice with respect to understanding how problem posing can be productively enacted in classrooms. This kind of analysis could highlight the nature of problem-posing tasks and provide guidance to teachers about critical aspects of teaching through problem posing (e.g., problem-posing tasks and discourse patterns for handling students' posed problems). For example, Ellerton (2013) proposed an Active Learning Framework that situates the processes of problem posing in the broader processes of mathematics classrooms. Singer and Moscovici (2008) described a learning cycle in constructivist instruction that includes problem posing as an extension and application of problem solving. Meanwhile, Kontorovich *et al.* (2012) proposed a theoretical framework to help researchers handle the complexity of students' mathematical problem posing in small groups. This framework integrates five facets: task organization, students' knowledge base, problem-posing heuristics and schemes, group dynamics and interactions, and individual considerations of aptness. In addition, Matsko and Thomas (2015) had students create and solve their own problems as assignments in mathematics classes to give them experience in interacting with mathematics problems beyond the routine and mechanical. Finally, Zhang and Cai (2021) analyzed specific problem-posing teaching cases, trying to understand the nature of the problem-posing tasks the teachers used and the ways teachers handled students' posed problems.

These prior studies have provided bases for us to understand aspects of what teaching through problem posing entails and its particular features. Because classroom instruction is generally complex, with many salient features that can be investigated, researchers need to identify those features that are most relevant for problem posing and which may be most influenced by the introduction of problem-posing activities (Cai *et al.*, 2015). The purpose of this paper is to provide a synthesis of what research says about teaching mathematics through problem posing. This review is not intended to be comprehensive with a systematic literature search; instead, its purpose is to trace back through work I have engaged with to paint a picture of teaching mathematics through problem posing along with its promises and challenges. In particular, in this paper, I address the following questions: (1) What does teaching mathematics through problem posing look like? (2) What is problem posing, anyway? (3) What is a problem-posing task? (4) How should teachers handle students' posed problems in classroom instruction? (5) How can teachers be supported to learn to teach through problem posing? (6) What is the effect of P-PBL instruction on teachers and students? Although the major focus of this paper is to address the last four questions, the first two questions will provide background information and necessary explanation of the terminologies used to address the last four questions. Throughout the sections, I will also pose various related questions that the field still needs to answer.

WHAT DOES TEACHING MATHEMATICS THROUGH PROBLEM POSING LOOK LIKE?

To illustrate the practice of using problem posing to teach mathematics, I first describe a specific Grade 1 problem-posing lesson from a Chinese elementary mathematics teacher, Ms. Yang (Yang & Cai, 2016). This lesson aims to develop students' mathematical understanding related to the topic of addition and subtraction of two-digit whole numbers through problem posing. The lesson comprises four major episodes, each corresponding to an instructional task, followed by a summary.

Episode 1: Multiple Representations of a Number

T (teacher): Please write down a favorite two-digit number, and don't tell others.

[After a few minutes, the teacher asked the students to describe the numbers they wrote down, and then asked the class to guess them.]

S1 (student): The number I have is 1 less than 40. Do you know which number I like?

S2: The number you have is 39.

S1: Congratulations. Right. Tell me the number you have.

(Teacher wrote 39 on the blackboard)

S2: The number I like is the one between 26 and 28.

S3: The number you like is 27.

S2: Congratulations. You are right.

(Teacher wrote 27 on the blackboard)

T: Can you describe this number 27 in other ways?

S4: The number that is 3 less than 30 is 27.

S5: The number that is 2 more than 25 is 27.

S6: The number that is 7 more than 20 is 27.

S7: The number that is 73 less than 100 is 27.

S8: The number composed of two 10s and seven 1s is 27.

S9: The number composed of seven 1s and two 10s is 27.

S10: Two ten-stick bunches of small sticks plus seven small individual sticks mean 27.

S11: In a simplified abacus, if we have 2 beads in ones, and have 7 beads in tens, it means 27.

S12: That's wrong. We should have 2 beads in tens, and have 7 beads in ones. Then, it can be 27.

S11: Sorry, I made a mistake. Thanks for your correction.

S12: We can draw a picture that means 27. (see below)

T: Who understands his meaning?

S13: One big heart means 1 ten, and 2 big hearts means 20; one small heart means 1 unit, 7 small ones mean 7 units. So together it is 27.

In this episode, the teacher gave students the opportunity to review the content related to the composition of numbers through a game-like activity. The students described numbers and justified their descriptions. This activity involved student–student communication, and the students developed their understanding of place value by using multiple ways to represent one number.

Episode 2: Discuss the Relationship Between Numbers

T: When you see the numbers 39 and 27, which numbers can you think of that are related to 39 and 27?

S1: I associate two numbers, 72 and 93.

T: Good. Can you ask your classmates if they know how you associate 39 and 27 with the numbers 72 and 93?

S1: Dear classmates, do you know how I associate the two numbers?

S2: I think you exchanged the digit in tens with the one in units of the numbers 27 and 39.

S1: So clever. That's just what I was thinking.

T: Do the new numbers that we get by exchanging the digit in tens with units have the same meaning as the original ones?

S3: No. In 27, the number 2 is in the tens place, meaning 2 tens; the number 2 in 72 is in the unit place, meaning 2 units.

S4: 72 is bigger than 27, 27 is smaller than 72.

Teacher: Any other numbers you can think of which are associated with 39 and 27?

S5: I am thinking the numbers bigger than 27 and smaller than 39 are 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38.

S6: I am thinking the even numbers bigger than 27 and smaller than 39 are 28, 30, 32, 34, 36, 38.

S7: I am thinking the odd numbers bigger than 27 and smaller than 39 are 29, 31, 33, 35, 37.

S8: I am thinking the numbers before 27 are 26, 25, 24……

S9: I am thinking the number 40 which is after (bigger than) 27.

S10: I am thinking the number 66. Do you know how I associate it?

S11: You added the 27 to 39.

T: So good! S10 thinks of the addition of the two numbers.

(Teacher wrote "addition" and 27 + 39 = 66 on the blackboard)

S12: According to the idea of S11, I think their difference is 12.

(Teacher wrote "subtraction" and 39 – 27 = 12 on the blackboard)

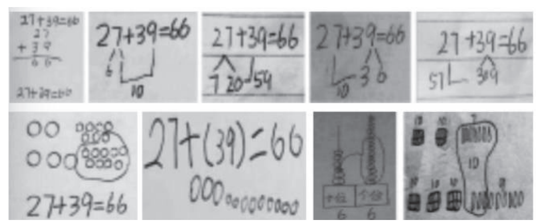
T (continually asking): Are they all correct? Can you find a way to verify their answers?

(Teacher encouraged students to use their own examination methods)

……

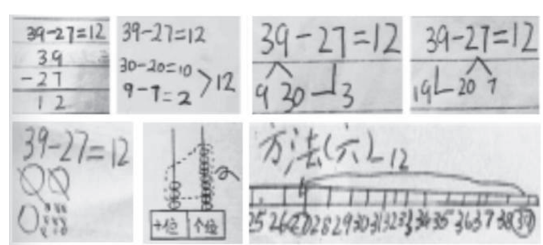
The teacher projected various solutions on the screen for the whole class to see and discuss, as shown in Figures 1 and 2 below.

Figure 1. The different methods students used to examine 27 + 39 = 66



The student solutions span multiple solution strategies, including applying the usual algorithm, grouping by place value, adding convenient numbers (e.g., 20 + 39), and drawing pictures of tens and ones followed by grouping.

Figure 2. The different methods students used to examine 39 – 27 = 12



The student solutions again span multiple solution strategies, including applying the usual subtraction algorithm, expanding by place value, subtracting convenient numbers (e.g., 30 – 27), drawing pictures of tens and ones and comparing, and using a number line.

Based on Episodes 1 and 2, we can see that students were able to observe and explain the numbers from different perspectives, construct relationships and understand the connection between two numbers, and then understand the calculations with numbers and place value.

Episode 3: Posing Addition Problems

T: How many real-life problems can you pose and solve using the addition equation 27 + 39 = 66?

In this episode, the teacher guided students to return to the meaning of addition in real life. The process of posing different real-life problems encouraged students to see the close relationship between mathematics and life, thereby deepening their understanding of the meaning of addition. When the teacher found that most students had posed different kinds of real-life problems, she asked them to present their problems and wrote them on the blackboard.

Problem 1: There are 27 red flowers and 39 yellow flowers. How many flowers are there in total?

Problem 2: We dug out 27 potatoes in the morning and 39 potatoes in the afternoon. How many potatoes did we dig out today?

Problem 3: There are 39 birds on this tree, and then another 27 birds come. How many birds are there on this tree now?

Problem 4: Xiao Ming bounced the ball 27 times, and Xiao Ping bounced it 39 times more than Xiao Ming. How many times did Xiao Ping bounce it? (A picture was drawn with this problem.)

Problem 5: There is a box of cookies. I ate 27 cookies, and 39 cookies are still left in the box. How many cookies were there in the box before?

The teacher then analyzed these posed problems and compared them in terms of the wordings used.

T: Problem 1 asked "in total." "In total" is easy to understand, as it means combining the parts of the red flowers and the yellow flowers. I understand we can use addition to calculate this total combined. But there is no "in total" in Problems 2, 3, 4, and 5; why can you all use addition to solve these problems? [The teacher used a gesture of putting two hands together]

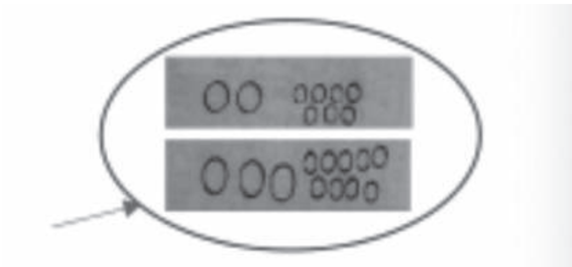
T: Who can help me? How do we explain this?

S1: Problem 2 asks "how many potatoes did we dig out today?" "Today" means we should add the potatoes we dug out in the morning with the ones we dug out in the afternoon [student imitates the teacher's gesture of "together"]. So, we should use addition.

S2: Problem 3 asks "how many birds are there on this tree now?" "Now" means combine the original birds with the ones that came later [student uses the gesture of holding two arms together], so we should use addition.

S3: Problem 4 doesn't use the words "in total," but Xiao Ping bounces 39 times more than Xiao Ming [student emphasizes the word "more"] so adding the 27 times Xiao Ming bounced with 39 times, we can get the times that Xiao Ping bounced.

T: Can you explain it with this picture?



S3: Sorry, I haven't figured it out yet.

S4: I can help him. The upper is the times Xiao Ming bounced, the lower means the times Xiao Ping bounced more than Xiao Ming. Combining the upper and the lower is the times Xiao Ping bounced.

S5: Problem 5 asks "before" which means the number of cookies before I started eating. So, we should combine the number of cookies eaten with the number of cookies still remaining in the box, using addition.

In Episode 3, after the teacher–student and student–student communication, the students reached a consensus: Although the ways of asking the question are different in these problems, the use of addition is the same. All of them require combining the two parts and calculating the total, so we can use addition. This process stands in contrast to the usual way of teaching addition word problems by focusing on key words like "in total" or "altogether." Different words were used in the problems in this episode, including "today" and "now," and these words do not ordinarily explicitly imply addition. However, the students were still able to determine the use of addition based on the meaning in the word problems they posed.

Episode 4: Posing Subtraction Problems

After discussing the meaning of addition equations, the teacher asked: "How many real-life problems can you pose and solve using the subtraction 39 – 27 = 12?" The students posed the following problems:

Problem 1: There are 39 cars in the parking lot. 27 cars drive away. How many cars are left?

Problem 2: The teachers have 39 books. They distribute 27 books to students. How many books are left?

Problem 3: There are 27 stools. 39 people are coming. How many stools do we need to add?

Problem 4: Pingping has 39 stars. Pongpong has 27 stars. How many more stars does Pingping have than Pongpong?

Problem 5: Shasha collects 27 waste batteries. Tata collects 39 waste batteries. How many fewer waste batteries does Shasha have than Tata?

Problem 6: There are 39 apples, and the number of pineapples is 27 fewer than the number of apples. How many pineapples do we have?

Problem 7: There are 39 girls in our class. The number of boys is 27 fewer than the number of girls. How many boys are in our class?

Problem 8: The white rabbit pulls 39 carrots. The brown rabbit pulls 27 carrots. How many carrots should the brown rabbit pull if she wants to have the same number of carrots as the white rabbit?

Problem 9: There are 39 building blocks in the big box and 27 building blocks in the small box. How many building blocks should we take away from the big box if we want the number of building blocks in the big box to be equal to the number in the small box?

In Episode 4, we can see that students posed many kinds of real-life problems that can be solved using subtraction. In explaining the reason for why subtraction can be used to solve these problems, students used their own language, gestures, or pictures to explain the quantity relationship, finally reaching the following consensus: Although the contexts used for these problems are different, the processes and algorithm of calculation are the same—to calculate the difference (or a part in the relationship of Whole – Part = Part). The solution can then be obtained using subtraction.

Given that the goal of education is to cultivate students' thinking ability to prepare them for life, this lesson demonstrates the benefits of using problem posing to foster students' understanding. Students were able to pose and solve real-life problems involving addition and subtraction without worrying about "key words" that explicitly suggest addition and subtraction. Thus, using problem posing to help students understand relationships benefited their understanding of subtraction and addition. One notable characteristic of this example is its novel design: Most of this lesson involved asking students to pose different types of addition and subtraction problems. Compared to the traditional method of using "key words" to help students judge whether a problem is an addition problem or a subtraction problem, this method of posing problems not only allows the teacher to assess students' understanding of addition and subtraction but also allows students to see the reasons for why we add two numbers or why we subtract two numbers, which creates better learning opportunities for them.

As indicated in Cai *et al.* (2015), it is still not entirely clear how students came up with these problems. However, it is clear that through posing problems, individual students engaged in different ways of thinking by creating situations modeled by addition or subtraction. Collectively, students discussed the posed problems, which helps *other* students in the class develop a better understanding of addition and subtraction. Moreover, the benefits of problem posing include: (1) positioning the students as the source of mathematical knowledge and insight, thus promoting positive mathematical identity formation; (2) engaging the class in a comparatively novel activity, thus promoting interest and engagement, and (3) encouraging the students to reflect on their own existing understanding of addition and subtraction situations, thus promoting social norms of understanding.

What Is Problem Posing, Anyway?

Now that we have seen a glimpse of teaching mathematics through problem posing in the above example, we can briefly describe what is meant by problem posing and problem-posing research. As stated in Cai and Hwang (2020),

> By *problem posing* in mathematics education, we refer to several related types of activity that entail or support

> teachers and students formulating (or reformulating) and expressing a problem or task based on a particular context (which we refer to as the *problem context* or *problem situation*). (p. 2)

The terms *problem* and *task* in this definitional framework broadly include any mathematical question that can be asked and any mathematical task that can be performed based on the problem situation.

Researchers have used both students and teachers as participants in problem-posing research. Thus, Cai and Hwang (2020) differentiated between how problem posing has been used with students and how it has been used with teachers. For students, problem posing has been defined as the following specific intellectual activities: (a) Students pose mathematical problems based on given problem situations which may include mathematical expressions or diagrams, and (b) students pose problems by changing (i.e., reformulating) existing problems. For teachers, problem posing has been defined as the following specific intellectual activities: (a) Teachers themselves pose mathematical problems based on given problem situations which may include mathematical expressions or diagrams, (b) teachers predict the kinds of problems that students can pose based on given problem situations, (c) teachers pose problems by changing existing problems, (d) teachers generate mathematical problem-posing situations for students to pose problems, and (e) teachers pose mathematical problems for students to solve.

The characterizations of problem posing above cover a wide range of activities because problem-posing researchers have approached problem posing from a variety of perspectives. However, Ruthven (2020) and Baumanns and Rott (2021) have raised the concern of defining problem posing too broadly. For example, Cai and Hwang (2020) found that some researchers have considered teachers posing problems by changing existing problems and teachers posing mathematical problems for students to solve as a part of problem-posing research involving teachers. One could argue that these two scenarios are related to problem solving because students are engaged in problem solving rather than problem posing in these two scenarios. For the purposes of this paper, problem posing is defined as the following specific intellectual activities: (1) Students pose mathematical problems based on given problem situations which may include mathematical expressions or diagrams; (2) students pose problems by changing (i.e., reformulating) existing problems; (3) teachers generate mathematical problem-posing situations for students to pose problems; and (4) teachers predict the kinds of problems that students can pose based on given problem situations. There are at least three advantages to defining problem posing in this way. First, this definition highlights the uniqueness of the posing aspect of problem posing. The second advantage is that it shows the major posing-related activities involved in teaching through problem posing for both students and teachers. Finally, it clearly identifies the roles of students and teachers in teaching through problem posing.

Stanic and Kilpatrick (1989) proposed three points of view in problem-solving research: problem solving as a cognitive activity, problem solving as a learning goal unto itself, and problem solving as an instructional approach. Similarly, researchers have commonly adopted perspectives of problem posing that parallel Stanic and Kilpatrick's (1989) three points of view (Cai *et al.*, in press). The first perspective deals with problem posing as a cognitive activity. Students engage in a situation or situations and then discover and pose problems based on the situations. This cognitive view is similar to scientific discovery: Students find something they really want to know. Problem-posing research from this perspective has largely focused on examining what kind of problems people can pose and the kinds of processes people use to pose problems. Also within this perspective is research using the process of problem posing to assess people's thinking and creativity (which problem solving has also been used for). That is, researchers have not only examined the capacity of students and teachers to pose mathematical problems but also the cognitive and affective processes of problem posing (Cai & Leikin, 2020).

The second perspective of problem posing is to consider it as an instructional goal. That is, through instruction and engaging in problem posing, students will be able to develop their abilities to generate problems and become better problem posers. In research from this perspective, problem posing has been used to assess people's thinking and creativity by focusing on the products—the posed problems (Leikin & Elgrably, 2020; Silver, 1997). In fact, researchers have long used posed problems as a measure of creativity (Getzels, 1979; Guilford, 1950) and have recently used them as a measure of learning outcomes (e.g., Cai *et al.*, 2013). Researchers have also investigated and confirmed that it is possible to train students and teachers to become better problem posers.

The third perspective of problem posing is to use it to teach mathematics. That is, students will understand and learn mathematics through their engagement in problem posing. Although developing problem-posing skills may be a goal of this kind of instruction, this perspective emphasizes engaging students in problem-posing tasks and activities to help them achieve both cognitive and noncognitive learning goals beyond developing their problem-posing skills. For example, in the lesson described above, a review of addition and subtraction was introduced by posing real-life mathematical problems based on addition and subtraction. In addition, problem posing might be incorporated in instruction as a way to help students develop their identities as explorers of mathematics and to foster positive dispositions towards mathematics. Below, I propose two related but unanswered research questions.

Unanswered Question 1

It is encouraging that researchers have found both teachers and students to be capable of posing mathematical problems. It is also encouraging that teachers and students can be trained to become better problem posers. However, researchers have also found that some students and teachers pose nonmathematical, unsolvable, and irrelevant problems. For example, Silver and Cai (1996) found that nearly 30% of problems posed by middle school students were either nonmathematical problems or simply nonproblem statements (even though the directions clearly asked for problems). Cai *et al.* (2015) asked: "Why do students pose non-mathematical, trivial, or otherwise suboptimal problems or statements?" This question has not been explored since 2015. Perhaps one direction for future research is to conduct "error analysis" for these undesirable responses to try to understand what leads students to provide them.

Unanswered Question 2

Related to problem-posing processes, there have been some recent advances in research on the cognitive processes of problem posing (see Cai *et al.*, 2022, for a brief review). Although research has demonstrated that students and teachers are capable of problem posing, we still know much less about their processes for posing mathematical problems, and there is not yet a definitive problem-posing framework analogous to well-established problem-solving frameworks. Although posing and solving are similar, there are enough differences to warrant a problem-posing framework that is unique from those of problem solving (Rott, Specht, & Knipping, 2021). Baumanns and Rott (2022) discussed different models of the problem-posing process and pointed out that different models served different goals. The models discussed included the model of Cruz (2006), which intended to guide teachers through the goals, formulation, and solving of the problem-posing process; that of Pelczer and Gamboa (2009), which included the five phases of *setup*, *transformation*, *formulation*, *evaluation*, and *final assessment*; that of Koichu and Kontorovich (2013), which included the four phases of warming up, searching for an interesting mathematical phenomenon, hiding the problem-posing process in the problem formulation, and reviewing with peers; and that of Baumanns and Rott (2022), which began with an initial situation analysis followed by processes of variation and generation that can feed back into one another to generate new posed problems. In addition to these models, Zhang *et al.* (2022) proposed a three-step model involving understanding the task, constructing the problem, and expressing the problem. Continuous research efforts are needed not only to propose a general problem-posing process model but also to include affective components (Cai & Leikin, 2020).

What Is a Problem-Posing Task?

The ultimate goal of instruction is to improve students' learning of mathematics. Teachers (and students) set up and implement instructional tasks to engage students to develop a deep understanding of mathematical concepts. Instructional tasks can be defined broadly as projects, questions, problems, constructions, applications, and exercises in which students engage. Doyle (1988) argued that tasks with different cognitive demands are likely to induce different kinds of learning. Tasks govern not only students' attention to particular aspects of content but also their ways of processing information, and they have the potential to provide the intellectual contexts for students' mathematical development. Such tasks can promote students' conceptual understanding, foster their ability to reason and communicate mathematically, and capture their interests and curiosity (NCTM, 1991). NCTM has recommended that students be exposed to truly problem-based tasks so that mathematical sensemaking is practiced (NCTM, 1991, 2000). Several studies have provided clear evidence supporting the connection between the nature of tasks and student learning (Cai, 2014; Hiebert & Wearne, 1993; Jacobs & Spangler, 2017; Stein & Lane, 1996). Students with the biggest gains are those from classrooms using cognitively demanding tasks.

Problem-posing tasks, then, are those instructional tasks that position students as generators or shapers of new problems based on real-life and mathematical situations (Cai & Hwang, 2020; Silver, 1994). Such tasks create opportunities for students to connect to their different experiences and backgrounds and pose very different problems, all of which are related to mathematical ideas (Cai & Leikin, 2020). Problem-posing tasks are usually cognitively demanding, but they are much more accessible than problem-solving tasks (Cai & Hwang, 2021; Silber & Cai, 2021).

Even though there are different types of problem-posing tasks (see, e.g., Baumanns & Rott, 2021; Cai & Hwang, in press), a problem-posing task usually includes two parts: a situation and a prompt (Cai & Hwang, in press; Cai *et al.*, 2022). The problem situation is what provides the context and data that the students may draw from (in addition to their own life experiences and knowledge) to craft problems. The prompt lets posers know what they are expected to do. Depending on the goal of the task, there can be many kinds of prompts for the same problem-posing situation. Take the following as an example:

> Jerome, Elliott, and Arturo took turns driving home from a trip. Arturo drove 80 miles more than Elliott. Elliott drove twice as many miles as Jerome. Jerome drove 50 miles. Pose three different mathematical problems that can be solved based on this information.

In this example, "Jerome, Elliott, and Arturo took turns driving home from a trip. Arturo drove 80 miles more than Elliott. Elliott drove twice as many miles as Jerome. Jerome drove 50 miles" is the situation; "Pose three different mathematical problems that can be solved based on this information" is the prompt. For the same situation, a different prompt could be used, such as "Pose one easy problem, one moderately difficult problem, and one difficult problem that can be solved based on this information."

Cai *et al.* (2022) specifically discussed the impact of different situations and prompts on students' problem posing (both the products and processes) at the individual, group, and classroom levels. The choice of situations and prompts can influence both the mathematical focus for the students and the level of challenge or affective engagement that the problem-posing task presents. See Cai *et al.* (2022) for more detailed discussion.

Unanswered Question 3

In mathematical problem solving, researchers have explored the effects of various task variables on students' problem solving (Goldin & McClintock, 1984), including syntax variables, content and context variables, structure variables, and heuristic behavior variables. Although some of these variables have been adopted in problem-posing research (Cai *et al.*, 2022), the question remains: Can all these variables be adapted to problem posing? Studies are needed to understand the most desirable ways to develop problem-posing tasks for classroom use, with a particular focus on problem-posing situations and prompts.

For example, returning to the 30% of responses considered undesirable in Silver and Cai (1996), the prompt used was the following: "Write three different questions that can be answered…" This prompt did not specifically require "mathematical questions." Would students still produce such a large percentage of undesirable responses if different prompts were used, such as:

- Write three different mathematical questions that can be answered based on this information;
- Write one easy, one moderately difficult, and one difficult mathematical problem that can be answered based on this information;
- Write three different mathematical problems that you can challenge your classmates to solve based on this information;

– Write three different mathematical problems that you can challenge your teacher to solve based on this information.

How Should Teachers Handle Student-Posed Problems in Classroom Instruction?

In addition to designing problem-posing tasks for classroom instruction, another important aspect of teaching mathematics through problem posing is teachers' handling of students' posed problems. An ideal consequence of using problem posing to teach mathematics is that students will pose their own problems during instruction, problems they can take ownership of. Thus, how teachers handle students' posed problems becomes a critical aspect of teaching through problem posing that can shape the effect of problem posing on the class. Understanding the ways in which teachers handle students' posed problems can help us better understand this aspect of classroom instruction in which problem-posing tasks are used.

There are at least three challenges for teachers' handling of students' posed problems. The first is that some of the students' posed problems may not be mathematical. In fact, studies have shown that students may pose various types of problems, from nonmathematical to complex mathematical problems (e.g., Cai & Hwang, 2002; Silver & Cai, 1996). It is impossible (and not beneficial) to deal with all the students' posed problems. But, will students whose posed problems are not discussed be discouraged?

The second challenge is that some of the students' posed problems may not be related to the learning goals of the lesson even though they are quite desirable mathematical problems. Moreover, some of these problems could be quite challenging mathematical problems. One of the most important aspects of handling students' posed problems is for teachers to make judgements about how the posed problems are aligned with the learning goals of the lesson. In their analysis of 22 problem-posing teaching cases, Zhang and Cai (2021) found that some of the students' posed problems were not related to the learning goals. In all 22 teaching cases, the teachers skipped the posed problems that were irrelevant to the instructional goal of the lesson, usually by saying that they wouldn't solve those problems because they were not related to the day's lesson. For posed problems relevant to the learning goals, teachers mentally classified posed problems into different difficulty levels. For easy posed problems, teachers would quickly guide students to solve them through whole-class discussion by asking students to sort out the answers. For very challenging problems, teachers would often assign them as homework or as instructional tasks to be discussed in the next class. The teachers tended to focus the discussion during the lesson on the moderately challenging problems.

Finally, the third challenge is the generative nature of posed problems. That is, problem-posing tasks are quite open ended in the sense that students can pose a variety of problems based on their own experience, and this variety complicates teachers' instantaneous decision making about how to handle these posed problems. Some studies have probed teachers' predictions of what problems their students might pose (e.g., Cai *et al.*, 2020; Xu *et al.*, 2020). The overall match between the teachers' predictions and the students' posed problems was not as consistent or accurate as one might hope. For example, Xu *et al.* (2020) found that, generally speaking, the predictions of the teachers in their study were not a good match for their students' actual problem-posing activity. Overall, the teachers predicted more complex problems that required specifying functional relationships than was borne out in the responses of the students, even those of students at higher grade levels. In addition, to overcome this challenge, teachers need to be trained to more accurately predict their students' posed problems. In particular, in the planning stage, teachers need to be equipped to consider the variety of possible problems students might pose. Anticipating their students' thinking with respect to mathematical problem posing might be a key step in using problem posing to assess students' mathematical understanding in the classroom. Thus, anticipating possible problems students might pose should be an important aspect in planning problem-posing lessons (Cai *et al.*, 2020; Koichu, 2020).

Though limited in its extent, some research has explored ways of overcoming the abovementioned challenges in handling posed problems. The teaching example presented at the beginning of this paper illustrated one way of handling students' posed problems, namely to analyze and classify the problems and then discuss the solutions based on addition and subtraction. In that teaching example, the focus was on the analysis of the problems according to their structures, with or without "key words." More generally, there is a need to develop a routine for handling students' posed problems. At the end of this paper, I propose a P-PBL instructional model. It should be indicated that the model's further development and elaboration requires the creation of more specific P-PBL teaching cases that exhibit the features of the model.

Indeed, problem-posing teaching cases provide fertile ground for exploring and addressing key questions such as how teachers should handle student-posed problems, and the sharing of such teaching cases could stimulate further discussion and exploration by teachers and researchers into these kinds of questions. A problem-posing teaching case includes major elements of a lesson and related analysis, thus capturing the instructional action of the lesson, but it is not simply a transcription of what happens during the lesson. Of course, to serve their function for both teacher education and researchers, teaching cases do include narratives describing the instructional tasks used in the lesson and the related instructional moves for those tasks (Zhang & Cai, 2021). However, teaching cases also include additional information about the underlying thinking behind major instructional decisions as well as reflections on and discussions of those decisions. The development of teaching cases is based on real, implemented lessons and the typical instructional events that arise during the lessons. Moreover, just as a teacher would ordinarily do when planning a lesson, generating a teaching case includes offering explanations about anticipated problems that students might pose. However, in implementing their lessons, teachers need to deal not only with the problems they anticipate students might pose but also the unanticipated problems that are posed. These unanticipated problems also become potential material for the teaching case.

P-PBL teaching cases such as these can serve as tangible entities to store and improve professional knowledge of teaching through problem posing (Cai *et al.*, in press). Future research should focus on analyzing video-recorded problem-posing lessons to understand the processes involved in teachers' handling of posed problems and how they determine relevant and irrelevant posed problems as well as difficulty levels of relevant problems. In particular, we need to investigate how teachers plan their lessons to facilitate their handling of posed problems.

Unanswered Question 4

Teachers not only need to handle students' posed problems but also use students' posed problems to understand students' thinking and adjust their teaching accordingly. One of the potential benefits of including problem posing in mathematics classrooms is the capacity for problem-posing tasks to reveal useful insights about students' mathematical thinking. The more information that teachers obtain about what students know and think, the more data they have to inform their efforts to create effective learning opportunities for all their students. Thus, teachers' knowledge of students' thinking has a substantial impact on their classroom instruction and, hence, on students' learning. Given that researchers have used problem-posing tasks to gain insights into students' and teachers' mathematical understanding (e.g., Cai & Hwang, 2002; Yao *et al.*, 2021), it seems reasonable to posit that teachers could also use problem posing to better understand their students' mathematical thinking. This leads to the question of how teachers can use problem posing to better understand students' thinking, especially in the process of handling students' posed problems in the classroom.

How Can Teachers Be Supported to Learn to Teach Through Problem Posing?

Problem-posing research has explored the kinds of problems that teachers can pose and has generally supported the claim that both preservice and in-service teachers are capable of posing interesting and important mathematical problems (see Cai *et al.*, 2015, for a review). Research has also shown that teachers can not only improve their problem-posing performance and change their views about problem posing through training but also learn to design problem-posing lessons (Cai *et al.*, 2020; Li *et al.*, 2020). These findings suggest that there exists a solid foundation for teachers to learn to teach through problem posing, which is quite encouraging.

However, there are also challenges that must be addressed for teachers to learn to teach through problem posing. The first challenge is the lack of problem-posing tasks in regular curricular materials. Kilpatrick (1987) observed that, in real life, problems must often be created or discovered by the solver. This suggests that, if a goal of education is to prepare students for the kinds of thinking they will need in life, problem posing must be addressed directly. Curriculum materials, and mathematics textbooks in particular, can be important resources for teachers who are teaching through problem posing (Cai & Howson, 2013). As indicated by Cai and Jiang (2017), despite the strong emphasis on problem posing in both Chinese and U.S. mathematics curriculum standards, both Chinese and U.S. elementary school curricular materials only include a very small proportion (less than 3%) of problem-posing tasks. Another challenge is teachers' buy-in to teaching through problem posing and the difficulty of implementing problem posing in classrooms even when teachers have bought in to the approach. Teacher buy-in and sense of ownership regarding the P-PBL approach is important, just as it is for other school improvement efforts (Kramer *et al.*, 2015; Redding & Viano, 2018). But, simply accepting an instructional idea does not guarantee its adequate implementation (Cai & Hwang, 2021).

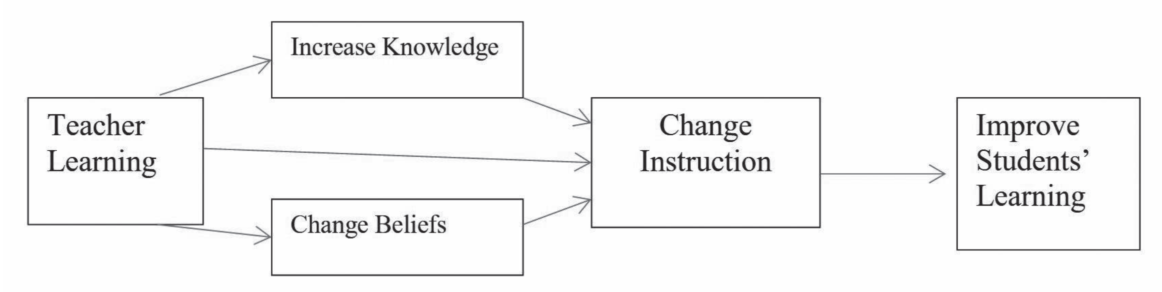
Current research on how to support teachers to learn to teach through problem posing is sparse, thus I use one problem-posing project involving elementary mathematics teachers as an example to discuss teachers' learning to teach through problem posing. The project has been designed to overcome the aforementioned challenges (Cai & Hwang, 2021; Cai *et al.*, 2020) and is based on features of effective teacher professional learning (Guskey & Yoon, 2009; Yoon *et al.*, 2007). In the project, three strategies have been developed to better integrate problem posing into the school mathematics curriculum: (a) empowering teachers to reinterpret existing curriculum materials and reshape them in simple ways to create mathematical problem-posing tasks with greater learning opportunities; (b) enhancing existing curricula with additional problem-posing tasks that include support in the form of sample posed problems; and (c) encouraging students to pose variation problems, that is, posing new problems that are based on an existing problem but that vary parameters or contextual aspects of the original problem. These strategies were identified based on a systematic search of literature on problem posing and are aimed at addressing the challenges of the dearth of problem-posing tasks in current curricular materials.

It should be indicated that even though curricular materials can be reshaped through the interactive, interpretive process that teachers already engage in to adapt what is in their textbooks to the needs of their students, there are constraints on this process. Teachers face multiple demands on their time and attention; they are not free to devote the extended amounts of time required to reinterpret large swaths of curriculum and then to incorporate significant changes to their upcoming lessons simply to increase the level of problem posing in the classroom. The three strategies identified above were also designed with this concern in mind, and they have demonstrated their feasibility in assisting teachers to pick the low-hanging fruit of developing problem-posing tasks based on the existing curriculum. Appendix A shows an example of applying these strategies to develop problem-posing tasks.

To overcome challenges related to teachers' buy-in, teacher learning should address the issue of teachers' beliefs. Understanding, studying, and working to shape teachers' beliefs is important because teachers' beliefs influence how they teach mathematics, which in turn influences students' opportunities to learn mathematics. Opportunities must be created for teacher learning with the goal of increasing teachers' knowledge and transforming their beliefs with respect to teaching through problem posing (shown in Figure 3). Through teacher learning, teachers increase their knowledge and change their beliefs, following which they change their classroom instruction with the goal of improving students' learning. As teachers learn more about problem posing and about how to design problem-posing lessons, they begin to make changes to their instruction so that they experience buy-in to the problem-posing ideas.

To address the challenge of implementation of problem-posing lessons, teachers' learning needs to closely align with practice. An important feature of effective teacher learning is its close relationship with teachers' practice (Guskey & Yoon, 2009; Yoon *et al.*, 2007). Research has found that teacher learning with direct applications of knowledge to teachers' planning and instruction has a positive influence on teaching practices, which in turn leads to gains in

Figure 3. A teacher professional learning model



student learning. Teacher learning is most effective when it focuses on the implementation of research-based instructional practices and provides teachers with opportunities to adapt the practices to their unique classroom situations (Guskey & Yoon, 2009). Teachers need to have opportunities not only to discuss actual lessons in which problem posing was used but also to design, develop, and revise lesson plans to use problem posing to teach specific topics. With such activities, teachers have opportunities to learn how to design problem-posing tasks and organize classroom discourse around problem posing.

In addition, not only do teachers need specific ideas about how they can learn to play their roles, they also need concrete examples to guide their practice. Teachers can engage in design-based research on teaching mathematics through problem posing, following which they can develop their own teaching cases based on their experiences designing and testing the new lessons. In those teaching cases, they can highlight the changes they made to produce the final design and explain the reasoning behind those changes. In developing such P-PBL teaching cases, the teachers are simultaneously learning about and doing research on teaching through problem posing (Cai & Hwang, 2021; Zhang & Cai, 2021). Moreover, the resulting P-PBL teaching cases—concrete examples of teaching through problem posing—serve to help other teachers learn what teaching through problem posing entails and what kinds of thinking and instructional decision making are involved as well as provide tested problem-posing lessons that may address the same content and pedagogical goals that they themselves have.

Unanswered Question 5

It is quite encouraging that teachers can learn to teach through problem posing (Cai *et al.*, 2020; Cai & Hwang, 2021; Li *et al.*, 2020). However, although there is a large body of literature we can draw from on teacher professional learning, we know little about the following questions: How do teachers learn to teach through problem posing, and what is the impact of teacher professional learning to teach through problem posing on classroom instruction and students' learning (see Figure 3)? Cai *et al.* (2021) are currently undertaking a longitudinal study to not only support teachers to teach mathematics through engaging their students in mathematical problem posing but also to longitudinally investigate the promise of supporting teachers to teach with P-PBL to enhance teachers' instructional practice and students' learning. In particular, Cai *et al.* (2021) are leveraging the development of P-PBL teaching cases as a key component of their project. It is quite common to use the case-based approach for teacher professional learning in various disciplines (Hillen & Hughes, 2008; Markovits & Smith, 2008; Merseth, 2003, 2016; Smith *et al.*, 2014; Stein *et al.*, 2009; Williams, 1992). As noted above, cases drawn from actual teaching can provide concrete examples that are directly connected to the content and pedagogical goals that teachers are responsible for. Moreover, for teachers who are creating their own teaching cases based on their experiences teaching through problem posing, the continuous development of P-PBL teaching cases may itself be effective for teachers' own learning. Given how effective the use of teaching cases has been, more effort is needed to accumulate teaching cases in problem posing.

With more successfully implemented teaching cases using problem posing as a resource, teachers can learn from the cases despite the paucity of problem-posing tasks in current textbooks and other curriculum materials.

What is the Effect of P-PBL Instruction on Teachers and Students?

In 2015, Cai *et al.* wrote: "Even though theoretical arguments suggest that engaging students in problem-posing activities in classrooms should have a positive impact on students' learning and problem posing, there are relatively few empirical studies that systematically document this effect" (p. 26). The good news is that over the last several years, some studies have examined the impact of problem-posing instruction on students and teachers (e.g., Akben, 2020; Bevan & Capraro, 2021; Cai & Hwang, 2021; Cai *et al.*, 2020; Li *et al.*, 2020; Klaassen & Doorman, 2015; Kopparla *et al.*, 2019; Suarsana *et al.*, 2019; Yang & Xin, 2021). Using problem posing as an instructional intervention, researchers have found positive effects of problem posing not only on teachers' problem-posing performance, beliefs, and design and teaching of problem-posing lessons (e.g., Cai *et al.*, 2020; Li *et al.*, 2020) but also on students' learning along both cognitive and noncognitive measures (e.g., Akben, 2020; Bevan & Capraro, 2021; Cai & Hwang, 2021; Yang & Xin, 2021).

Through studying a set of problem-posing professional development workshops, for example, Cai and his associates (Cai *et al.*, 2020; Cai & Hwang, 2021; Li *et al.*, 2020) have found that teachers with no prior problem-posing experience can, after attending the workshops, successfully develop their problem-posing conceptions and performance as well as their beliefs about problem posing. Li *et al.* (2020) found that, after participating in three workshops, some teachers exhibited gains in their problem-posing performance and in the scope of their beliefs about teaching using problem posing. Similarly, Cai *et al.* (2020) found that after participating in a workshop, teachers were able to pose a variety of problems and exhibited greater confidence in problem posing as well as in incorporating problem-posing in their lessons. Finally, Cai and Hwang (2021) reported that teachers participating in the professional development workshops exhibited positive changes in their beliefs about problem posing as well as their ability to pose problems and redesign existing lessons to incorporate problem-posing components.

Kopparla *et al.* (2019) conducted a quasi-experimental study in which teachers and researchers assigned elementary students into either problem-solving or problem-posing groups. The problem-posing group of students was asked to pose problems based on given information, whereas the problem-solving group of students was asked to solve problems based on given information. The results showed that students in both groups exhibited improvements in both problem posing and problem solving after the interventions. Interestingly, improvement in problem posing for the problem-solving group was stronger than it was for the problem-posing group.

Yang and Xin (2021) developed a problem-posing intervention based on the existing Conceptual Model–based Problem-Solving program (COMPS). They designed the study for three students with learning disabilities to engage with problem posing using structured problem-posing situations. The intervention was effective at improving students' problem-solving and problem-posing skills. Even though the three students had little or no experience posing problems and even had difficulty interpreting mathematical language and understanding mathematical reasoning, after the intervention phase began, students immediately began to identify the mathematical relationships provided in the given equations. Yang and Xin (2021) claimed that "the intervention appeared to help the students develop a conceptual understanding of the mathematical relationships in story problems" (p. 10).

Akben (2020) also used a quasi-experimental design to examine 61 chemistry and 40 physics students' problem-solving and metacognitive skills after engaging in problem posing in science. They found that engaging students in problem-posing activities improved these students' problem-solving skills and metacognitive awareness. Cai and Hwang (2021) also found positive effects on students' cognitive and noncognitive measures when teachers who had engaged in problem-posing training taught lessons using problem posing.

In summary, teaching through mathematical problem posing has strong theoretical and empirical support for fostering students' learning. This is an encouraging development for advances in problem-posing research in general and teaching mathematics through problem posing in particular. I conclude this paper by describing a proposed P-PBL instructional model, which not only summarizes the advances in problem-posing research but also summarizes the future directions of research.

Conclusion: A P-PBL Instructional Model

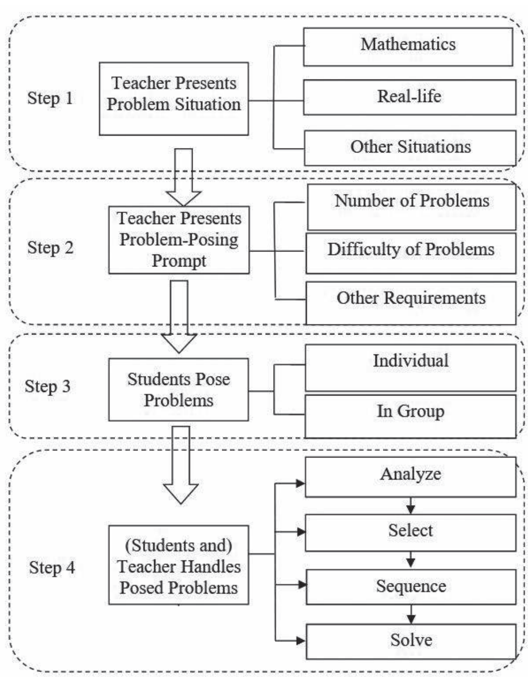
The proposed P-PBL instructional model (see Figure 4 below) treats a single instructional task in a lesson as the unit of interest. In a lesson, there might be more than one problem-posing task or a combination of problem-solving and problem-posing tasks. However, this model describes the steps for using one problem-posing task to teach mathematics—that is, implementing a problem-posing task to foster students' learning of mathematics. The model includes four steps: (a) The teacher presents a problem-posing situation, (b) the teacher provides a problem-posing prompt along with the problem situation, (c) students pose problems either individually or in a group, and (d) teachers handle the posed problems based on the learning goals for students to solve the selected problems.

The first and second steps are usually presented together. In this model, I purposefully separate them to show the importance of considering both situations and prompts when planning teaching through problem posing. Cai & Hwang (in press) have discussed various examples of problem-posing situations. Problem-posing situations could be mathematical or real life. The situations could also be related to students' out-of-school interests in topics like sports, video games, and social networking (Wilkington & Bernacki, 2015). For the second step, it should be indicated that in addition to specifying the number and difficulty levels of problems to be posed, the teacher may use prompts to motivate students' problem posing, such as "pose mathematical problems that would challenge your classmates or mathematics teachers" or "pose mathematical problems that involve percentages."

Step 3 involves students' actual posing of problems. In this step, students are provided opportunities to work on the problem-posing task either individually or in a group. Note, however, that the mathematics education field is still working to understand the cognitive and affective processes of problem posing, and there is not yet a general problem-posing analogue to well-established processes for problem-solving such as Polya's (1957) four steps (Cai *et al.*, 2015). One thing that is clear is that students need to understand the problem-posing situation and prompt before they can actually pose problems. There is also evidence that students may think about possible solutions to problems while posing them. Although various researchers have explored problem-posing processes (e.g., Rott *et al.*, 2021; Baumanns & Rott, 2022), we still need to know more about the problem-posing process (Cai *et al.*, 2022; Cai & Leikin, 2020) and problem-posing strategies.

Step 4 involves ways of handling posed problems during instruction. Posing is itself a promising activity for fostering students' learning; however, I take a strong position that it is necessary to reinforce the learning through solving some of the posed problems. That is, solving some of the posed problems creates additional learning opportunities for students. Therefore, in this step, I propose four possible instructional practices (analyze, select, sequence, and solve). Note that while students take time to work on a problem-posing task in Step 3,

Figure 4. The P-PBL Instructional Model



the teacher can monitor their progress and posed problems to facilitate their subsequent analyzing, selecting, sequencing, and solving practices.

During the *analyze* instructional, the teacher guides students to analyze and classify the posed problems into different categories. In addition to those that are not mathematical, some of the posed problems may not be clear. The teacher might ask students to clarify those posed problems in different ways. For example, the teacher could ask students to write down their posed problems on a poster and share them with other students. This writing process gives students opportunities to rephrase their posed problems and clarify their ideas. Through analysis of the posed problems, the teacher can not only correct "errors" in the posed problems but also convey criteria for desirable, "good" problems. Also through analysis, the teacher can guide the students to categorize problems into different categories and determine how relevant the posed problems are to the lesson goal as well as how difficult they are.

The analysis of posed problems lays the foundation for teachers to determine which posed problems to select to be solved together in the classroom, which forms the second instructional practice. Based on the students' thinking and the lesson goals, the teacher can select posed problems according to the level of difficulty and relevance to the learning goal. It is desirable to choose a variety of posed problems so that students can be exposed to different learning opportunities. Teachers can also get students' input to select certain problems to be solved. It should be indicated that there is a need to justify why certain posed problems are selected to be solved in the class.

During the *sequence* instructional practice, the teacher carefully thinks through the order of solving the selected problems. This sequence needs to make pedagogical sense (Stein & Smith, 2008). It is recommended that the teacher asks students or groups who posed the problems to present and discuss their posed problems in a predetermined order that makes the most pedagogical sense.

With respect to the *solve* instructional practice, teachers can draw on recommendations and practices from the literature about teaching mathematics through problem solving (e.g., Cai, 2003).

The P-PBL instructional model can provide guidance for teaching mathematics through problem posing. Existing problem-posing teaching cases conform to the P-PBL instructional model. For example, the teaching example presented at the beginning of this paper exhibits the P-PBL routine. Zhang and Cai's (2021) analysis of 22 problem-posing teaching cases also supports this instructional model. Although there is a need to develop more P-PBL teaching cases and use these teaching cases to revise the P-PBL instructional model, in the meantime, the P-PBL instructional model can guide the development of such P-PBL teaching cases.

Fundamentally, there is a need not only to further verify the model but also to specify the details of each step in a way analogous to the work that has been done for teaching through problem solving (Stein & Smith, 2008). In particular, Step 4 requires elaboration, focusing on how to analyze posed problems, how to select posed problems for class discussion, and how to sequence selected posed problems. My hope is that the ideas presented in this paper can serve as a springboard to invite more scholars to engage in problem-posing research so that we can provide more opportunities for students to learn mathematics through problem posing.

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Appendix

An Example of Developing Problem-Posing Tasks Based on Existing Curricular Materials

Original Task: During one waiter's shift he delivered 13 appetizers, 17 entrées, and 10 desserts. What percentage of the dishes he delivered were desserts? (adapted from Illustrative Mathematics, IM 6–8 Math™ V. III, illustrativemathematics.org).

Strategy 1: Empowering teachers to reinterpret existing curriculum materials and reshape them in simple ways to create mathematical problem-posing tasks with greater learning opportunities

Problem-Posing Task 1: During one waiter's shift, he delivered 13 appetizers, 17 entrées, and 10 desserts. Pose three mathematical problems that could be answered based on this situation.

With Strategy 1, teachers can simply remove the problem-solving prompt ("What percentage of the dishes he delivered were desserts?") and replace it with a problem-posing prompt ("Pose three mathematical problems that could be answered based on this situation."). Strategy 1 allows for some diversity in task design. For example, teachers may use different kinds of problem-posing prompts to specify the number of problems and types of problems to be posed, as in Problem-Posing Tasks 2 and 3 below.

Problem-Posing Task 2: During one waiter's shift, he delivered 13 appetizers, 17 entrées, and 10 desserts. Pose one easy mathematical problem, one moderately difficult mathematical problem, and one difficult mathematical problem which can be solved based on the given information.

Problem-Posing Task 3: During one waiter's shift, he delivered 13 appetizers, 17 entrées, and 10 desserts. Pose three different mathematical problems that you can challenge your classmates to solve based on this information.

Strategy 2: Enhancing existing curricula with additional problem-posing tasks that include support in the form of sample posed problems.

Problem-Posing Task 4: During one waiter's shift he delivered 13 appetizers, 17 entrées, and 10 desserts. One problem that can be asked using this information is: "What percentage of the dishes he delivered were desserts?" Pose three additional mathematical problems that can be answered based on this situation.

By providing a sample posed problem, Strategy 2 allows teachers to potentially shape the posed problems (although there is still plenty of freedom for students to pose what they wish), thus also shaping the mathematical discussion that follows the posing of problems.

Strategy 3: Encouraging students to pose variation problems.

Problem-Posing Task 5: Teacher Cai has asked his 6th grade students to solve the following problem: During one waiter's shift he delivered 13 appetizers, 17 entrées, and 10 desserts. What percentage of the dishes he delivered were desserts? Make up a similar problem so that Mr. Cai can have his students solve it.

Problem-Posing Task 6: Teacher Cai has asked his 6th grade students to solve the following problem: During one waiter's shift he delivered 13 appetizers, 17 entrées, and 10 desserts. What percentage of the dishes he delivered were desserts? Make up two problems involving percentages with different contexts so that Mr. Cai can have his students solve them.