

Seeing Algebra in Arithmetic Through Mathematical Problem Posing

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I. INTRODUCTION

Students' difficulties with learning algebraic ideas are extensively documented (e.g., Kaput, 2008; Loveless, 2008; RAND Mathematics Study Panel, 2003). These difficulties have been interpreted in different ways. Some have posited that students' difficulties result from their cognitive development. That is, students are not yet ready to learn algebra in middle and high school. Other researchers have challenged this interpretation (e.g., Stephens et al., 2017a; Cai & Knuth, 2011; Carpenter et al., 2003), suggesting that students are capable of developing their algebraic thinking given appropriate opportunities to do so. In fact, researchers have not only found that elementary school students are capable of thinking algebraically (e.g., Cai & Knuth, 2011) but have also developed ways to foster students' algebraic thinking (Carpenter et al., 2003; Cai & Moyer, 2008). These findings are consistent with the arguments of Loveless (2008) about students' lack of algebra learning opportunities.

One of the main approaches that has been explored for developing students' algebraic thinking is to help them see algebra in arithmetic (see, e.g., Russell et al., 2011). This means that important algebraic ideas and ways of thinking are surfaced while students are learning arithmetic. In other words, elements of algebra and algebraic thinking are made visible by surfacing regularities and generalizations across arithmetic situations, tasks, and solutions that students might otherwise not recognize with typical arithmetic instruction. Traditionally, most school mathematics curricula separate the study of arithmetic and algebra, making arithmetic the primary focus of elementary school mathematics and algebra the primary focus of middle and high school mathematics. This separation makes it more difficult for students to learn algebra in the later grades (Stephens et al., 2017a). The field has gradually reached a consensus that students can learn and should be exposed to algebraic ideas through arithmetic, allowing them to see algebra in arithmetic (Cai & Moyer, 2008; Ding, 2021; Russell et al., 2011). In this paper, we proceed from the same position, but we focus our discussion of helping students see algebra in arithmetic through a lens of mathematical problem posing.

This paper has three major sections. We begin by examining the idea of seeing algebra in arithmetic and how this idea has gained support in the research literature. We then focus on the use of mathematical problem posing in the classroom to help students see the algebra in arithmetic. In particular, we examine three cases of teaching mathematics with problem posing that provide perspectives on developing students' algebraic thinking. Finally, we consider how teachers might be supported in using problem posing to help their students see the algebra in arithmetic. This includes analyzing how useful problem-posing tasks may be constructed.

II. SEEING ALGEBRA IN ARITHMETIC

There is a broad consensus that students are capable of thinking algebraically even in the early grades. One approach to building algebra into the arithmetic activity of the early grades is to focus on functional thinking. Functional thinking can build elementary schools students' capacity for algebraic reasoning (Blanton & Kaput, 2011; Carpenter et al., 2003; Stephens et al., 2017b). Indeed, Blanton and Kaput (2011) have proposed ways that teachers can adapt current arithmetic curricular materials to incorporate more pattern building, representing quantities that vary in relation to each other, generalizing, and conjecturing. More specifically, they discuss three kinds of functional thinking: recursive patterning, covariational thinking, and correspondence relationships. Recursive patterning is variation that occurs within a sequence, covariational thinking involves how two quantities change simultaneously by a constant (e.g., "as x increases by one, y increases by three"), and correspondence relationships are defined as correlations between variables (e.g., " y is 3 times x plus 2").

In their 5-year study of professional development in an urban elementary school district, Blanton and Kaput (2011) found that teachers could learn to modify their current instructional resources to build these different types of functional thinking into the curriculum and progressively scaffold elementary students' algebraic thinking development. They did this by turning simple arithmetic problems with single numerical answers into problems with varying parameters that open the possibility for pattern building and recognizing mathematical relationships. As an example, the telephone problem asks:

How many telephone calls could be made among 5 friends if each person spoke with each friend exactly once on the telephone?

Varying this problem by adding some strategically chosen parameters and prompts turns it into a problem that opens up opportunities for functional thinking:

How many telephone calls would there be if there were 6 friends? Seven friends? Eight friends? Twenty friends? One hundred friends? Organize your data in a table. Describe any relationship you see between the number of phone calls and the number of friends in the group. How many phone calls would there be for n friends?

As Blanton and Kaput state, varying the parameters of a single-numerical-answer arithmetic problem allows the generation of a set of data from which students can generalize and build functional relationships.

Alternatively, Empson et al. (2011) focused on the algebraic nature of fractions as an approach to developing relational thinking in the early grades. In their view, the fundamental properties of operations

and equality used by children in problems involving fractions also form the basis for algebraic thinking. They refer to relational thinking as encompassing the thinking underlying various strategies that children use to solve fraction problems, thinking which mirrors the mathematical relationships in high school algebra. According to Empson et al.,

Relational thinking is powerful because the applicability of fundamental properties such as the associative property of addition and the distributive property of multiplication over addition cut across number domains and into the domain of Algebra where one reasons about general Quantities rather than specific numbers (p. 412).

Given that fractions are typically taught separately from whole number operations, Empson et al. argue for teaching whole number and fraction addition together to better incorporate relational thinking and thereby algebraic thinking patterns in arithmetic learning. To do this is to understand fractions as quantities that involve multiplicative relationships between pairs of numbers.

Cai et al. (2011) analyzed the earlier curricula experienced by students in China and Singapore to draw other lessons on early algebraic thinking. In these two contexts, students approach early algebraic concepts in the form of the one-problem-multiple-solutions approach in China and the Pictorial Equations approach in Singapore. Both national curricula emphasize that their main goal is to deepen students' understanding of quantitative relationships, and both incorporate algebraic models and representations to do so in both earlier and current curricula.

The one-problem-multiple-solutions approach in China exemplifies the curriculum's emphasis on examining quantitative relationships from multiple perspectives and representing relationships both algebraically and arithmetically. As Cai et al. (2011) discuss, the Chinese curriculum aims to develop three thinking habits: a) examining quantitative relationships from different perspectives, b) solving problems using both arithmetic and algebraic approaches, and c) using inverse operations to solve problems. For example, Grade 5 students are invited to solve the following problem both arithmetically and algebraically and then compare the results of the different solution approaches.

Liming elementary school has funds to buy 12 basketballs at 24 Yuan each. Before buying the basketballs, they decided to spend 144 Yuan of the funds for some soccer balls. How many basketballs can they buy?

Arithmetic solutions:

Solution 1: Begin by computing the original funding and subtract the money spent on soccer balls: $(24 \times 12 - 144) \div 24 = 144 \div 24 = 6$ basketballs.

Solution 2: Begin by computing the number of basketballs that can no longer be bought: $12 - (144 \div 24) = 6$ basketballs.

Algebraic solutions:

Solution 3: Assume that the school can still buy x basketballs: $(24 \times 12 - 144) = 24x$.

Therefore, $x = 6$ basketballs.

Solution 4: Assume that the school can still buy x basketballs: $24 \times 12 = 24x + 144$.

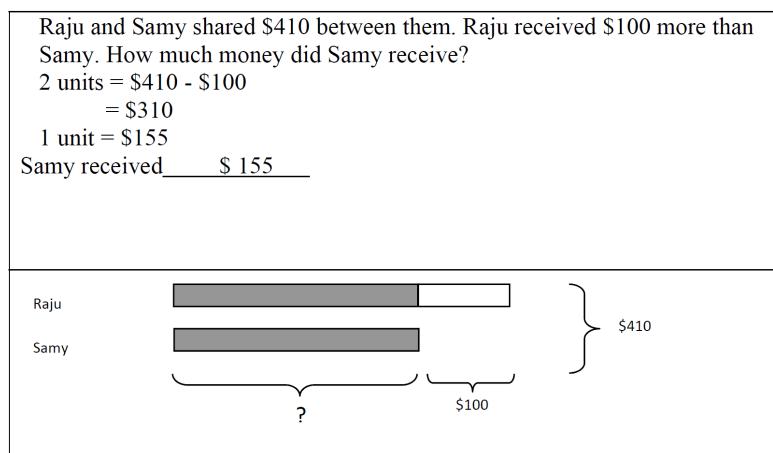
Therefore, $x = 6$ basketballs.

Solution 5: Assume that the school can still buy x basketballs. $12 = (144 \div 24) + x$.

Therefore, $x = 6$ basketballs.

Meanwhile, the Singaporean curriculum is deeply infused with solving both arithmetic and algebraic word problems, particularly using diagrams or visual components, exemplified by the Pictorial Equations approach. This approach uses number pattern activities in picture format to elicit students' generalizing and algebraic thinking abilities. An example of such a problem from the Singaporean curriculum is shown below (Figure 1).

Figure 1 Pictorial equation solving



Based on an analysis of Grades 2–6 classroom episodes, Russell et al. (2011) highlighted four mathematical activities that underlie both arithmetic and algebra, thereby serving as a bridge between the two: a) understanding the behavior of the operations, b) generalizing and justifying, c) extending the number system, and d) using notation with meaning. Through these key activities, teachers can leverage opportunities to make clear the bridge between arithmetic and algebraic ideas. First, as Russell et al. discussed, understanding the behavior of the four basic arithmetic operations—addition, subtraction, multiplication, and division—sets the basis for algebraic learning. Insufficient understanding of these operations prevents both arithmetic as well as future algebraic development. The second key activity underlying both arithmetic and algebra identified by Russell et al. is generalizing the operations and justifying those generalizations; for example, “Young students notice that when they change the order of addends, the sum does not change”

(p. 51). This also lays the basis for future algebraic reasoning. Extending the number system to include not just whole numbers but also decimal numbers, fractions, or negative numbers is a third underlying activity key to arithmetic and algebraic development: “As they consider new classes of numbers, students sort out which behaviors of the operations must remain consistent as the number system expands and which only appear to be general when considering certain classes of numbers” (Russell et al., p. 59). Finally, the fourth underlying activity of both arithmetic and algebra involves using algebraic notation with meaning—that is, building on proof and justification of generalizations in words to translate these ideas into appropriate symbolic notation. Taken together, these four key thinking patterns underlying arithmetic and algebra serve as the key points of investigation for building the bridge between both types of thinking.

From a cross-national context, Ding (2021) examined elementary Chinese and U.S. students’ experiences of learning algebra in arithmetic. For example, Ding examined the five types of inverse tasks: a) fact families or related facts, b) inverse word problems, c) using inverses to compute, d) missing numbers, and e) using inverses to check. A fact family refers to a group of four number facts involving a single operation and its inverse (e.g., $3 + 5 = 8$, $5 + 3 = 8$, $8 - 3 = 5$, $8 - 5 = 3$). Inverse word problems are a group of word problems whose solutions can form a fact family. This idea can be used to develop variations of a problem that has already been solved. The last three types of tasks require a student to solve or check a computational problem by referring to its related fact family. Finding the values of missing numbers in equations is a typical task in equation solving that does not involve letters (or pronumerals).

III. SEEING ALGEBRA IN ARITHMETIC THROUGH PROBLEM POSING

Each of the mathematical activities described above, whether the one-problem-multiple-solutions approach in China or one of the four key activities identified by Russell et al. (2011), helps students to see algebra in arithmetic by surfacing regularities and generalizations across arithmetic situations that students might otherwise not recognize with typical arithmetic instruction. In particular, the single-answer-numerical arithmetic problems so typical in arithmetic instruction do not lend themselves to seeing algebra in arithmetic.

Problem posing, in contrast, is an important mathematical activity that, by its very nature, can provide another pathway to bridge arithmetic and algebra in this way. By problem posing, we mean engaging students in generating new problems based on given situations (including mathematical expressions or diagrams) or changing (i.e., reformulating) existing problems (Cai & Hwang, 2020; Silver, 1994). We use the term “problem-posing tasks” to mean instructional tasks that engage students in generating new problems based on given situations. Each problem-posing task involves a problem situation and a prompt used to initiate students’ posing activity (Cai & Hwang, in press). When engaging in problem-posing tasks, students may generate multiple problems from a single problem-posing situation or context. Indeed, students often find it natural to explore the mathematical features of the problem-posing situation in order to formulate their own problems based on that situation. Moreover, the students’ posed problems, being rooted in a common context and mathematical structure, can open up the critical opportunity for them to see the

algebra in the arithmetic. Below, we examine two examples to illustrate this process. We begin with an overview of a lesson about using letters to represent quantitative relationships.

1. Teaching case 1: Using letters to represent quantitative relationships

In this Chinese lesson that was taught to an upper elementary grade class, the teacher's goal is to help students draw on their knowledge of arithmetic symbols and their meanings in order to generalize that knowledge to using letters to represent variable quantities in mathematical expressions and equations (Xu & Cai, 2020).

1) Episode 1: Warm-up activity

The teacher guides the students to pose problems or to describe situations that could be solved or described by the following expressions:

$$5 \times 8 \quad 120 - 30$$

The teacher also stipulates that problems or situations with the same quantitative relationship are not to be repeated. Students generate different mathematical problems and situations based on their own life experience. For example, for 5×8 , students might respond with "a box has cookies, and there are 5 boxes" or "a rectangle is 5 inches in width and 8 inches in length." This warm-up activity helps students to think through problems and situations which can be represented using an arithmetic expression. It also helps the students notice that an expression can represent multiple different situations and helps them begin to see the generalized nature of an expression such as 5×8 or $120 - 30$ (i.e., the generality of such expressions to match many different specific contexts).

2) Episode 2: Problem posing 1

The teacher shows the students " $1200 - 3x = ?$ " and asks them to pose related mathematics problems. That is, the problems the students generate should be able to be represented using $1200 - 3x = ?$ Students attempt to pose various problems and the teacher selects and discusses one of the problems posed by the students:

A large bottle of beverage has 1200 g, and 3 small cups are poured. Each small cup is x g. How many g of beverage are left in the big bottle at this time?

Other students posed similar problems, such as: "A large bottle of beverage has 1200 g; pour x number of small cups. Each small cup is 3 g. How many g of beverage are left in the big bottle then?" The teacher not only guided students to discuss whether the posed problems are appropriate to be represented by $1200 - 3x = ?$ but also guided students to think further about the range of values for x . Such discussion provides students opportunities to think through the domain of a function.

3) Episode 3: Problem posing 2

The teacher presents the figure shown below (*Figure 2*):

Figure 2 Teaching Case Episode #3



The teacher asks the students to pose related mathematical problems based on the figure and write them on the space provided. Then the teacher asks the students to present different types of problems to the whole class.

4) Episode 4: Extension

The teacher guides the students to solve one of the problems posed by a student: “How many sticks are needed to build x number of triangles and squares?” The teacher guides the students to discover that $3x + 4x$ and $(3 + 4)x$ are both answers to this problem. Then the teacher shows *Figure 3* and asks the students to compare and analyze *Figure 2* and *Figure 3*, identifying the similarities and differences. The teacher asks, “When the seventh square is placed, how many sticks are needed? If we continue to extend the squares, how many sticks do we need to have n squares?”

Figure 3 Teaching Case Episode #4



5) Episode 5: Summarizing

The teacher finishes the lesson by summarizing what the students have discussed and reiterating what they have learned in this lesson.

2. Teaching case 2: Distributive property of multiplication over addition

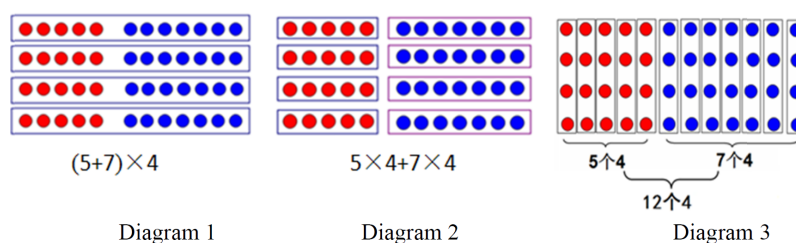
Chen and Cai (2020) compared and analyzed two lessons on the distributive property of multiplication over addition. One lesson used a problem-solving approach, and the other used a problem-posing approach by an expert elementary mathematics teacher from China. In the problem-posing lesson, the teacher divided the class into

four groups. Students in Groups 1 and 2 were asked to pose mathematical problems which could be solved using the expression $(5 + 7) \times 4$. Students in Groups 3 and 4 were asked to pose problems using the expression $5 \times 4 + 7 \times 4$. Students in Groups 1 and 2 could not see the expression given to students in Groups 3 and 4 and vice versa.

When they had generated their problems, a group leader from Groups 3 and 4 presented one of their posed problems to the entire class and asked the students in Groups 1 and 2 to determine what expression they based their posed problem on. The problem posed by Groups 3 and 4 was, "One bottle of whole milk costs 5 Yuan. One bottle of skim milk costs 7 Yuan. How much money does it cost to purchase four bottles of each type of milk?" The students in Groups 1 and 2 found that either of the two expressions could be used to solve the milk problem. The teacher then asked students in Groups 1 and 2 to present a problem and students in Groups 3 and 4 to guess what expression they used to pose their mathematical problem. Students in Groups 3 and 4 also discovered that either expression could be used to solve the problem posed by Groups 1 and 2. At this point, the teacher raised a question for the class: Why can a problem posed for one expression be solved by using another expression?

The teacher then presented Diagram 1 shown below in *Figure 4*, using the dots to represent the milk problem posed by students in Groups 3 and 4. The students then connected Diagram 1 to the other expression used by Groups 1 and 2. Similarly, the students connected Diagram 2 to the problem posed by students in Groups 1 and 2. The students were encouraged to observe and find the total number of red dots and blue dots, which can be expressed by $5 \times 4 + 7 \times 4$ or $(5 + 7) \times 4$. The students were then asked to observe Diagram 3, which depicts five fours and seven fours, the total of which is 12 fours. The teacher guided them to see that the same diagram can be viewed in different ways—in other words, both Diagrams 1 and 2 can be viewed as 12 fours, which is why the two expressions are equivalent.

Figure 4 *Teaching Case #2 diagrams*



Teaching cases 1 and 2 provide examples of how a teacher might make use of problem posing to help students make connections between their understanding of symbols in arithmetic and the use of variable notation in algebra (drawing on what Russell et al., 2011, might call “using notation with meaning”). Through their posing activity, the students generate multiple situations or problems that are connected with a single expression. This brings to the surface regularities based in their arithmetic experience and activity that the teacher can use to make algebraic ideas more visible. For example, from the students’ responses, the teacher can make further connections, such as to the distributive property. These connections are thus

rooted in the students' own posed problems that, in turn, are based in the students' own lived experiences.

IV. SUPPORTING TEACHERS TO HELP STUDENTS SEE ALGEBRA IN ARITHMETIC THROUGH PROBLEM POSING

Lessons that contain substantial opportunities for students to pose their own problems are relatively scarce in published curricula. Despite persistent calls and recommendations to make problem posing a part of the mathematics curriculum (e.g., NCTM, 1989, 2000, 2020), teachers are rarely provided with problem-posing-based lessons that align with their instructional objectives. For example, analyses of curriculum materials in the United States and elsewhere reveal a failure to incorporate problem posing in a substantial and consistent way. Only a very small proportion of problem-posing activities are included in popular elementary and middle school mathematics textbooks (Cai & Jiang, 2017; Silver, 2013). In addition to being sparse, the problem-posing activities in textbooks can be very unevenly distributed across grade levels and content areas. Of the limited number of problem-posing activities, the vast majority are related to number and operations. Very few involve algebra, geometry, measurement, or data analysis (Cai & Jiang, 2017). The unrepresentative distribution of problem-posing tasks in curricula reflects a haphazard approach to incorporating problem posing rather than a focused effort to use problem posing to achieve particular learning goals (such as making algebra visible in arithmetic).

Given the lack of existing curricular support for problem posing in general, and for problem posing as a way to see algebra in arithmetic in particular, an alternative pathway to integrating problem posing is needed. One natural way to support teachers to work around the limited support for problem posing in published curricula is to support them in reshaping existing problem-solving tasks in simple ways that transform lessons to create learning opportunities with problem posing. Teachers already translate their curriculum materials into their own instruction, engaging with their curriculum materials in a design process (e.g., Lloyd et al., 2017; Stein et al., 2007). This provides a potentially powerful point of leverage to support teachers to adapt their non-problem-posing curriculum materials to introduce more substantial problem-posing tasks and opportunities. In this way, teachers may modify familiar tasks in relatively straightforward ways to serve the purpose of helping students see the algebra in arithmetic. The teaching case below illustrates how teachers might engage in this redesign process to infuse lessons with problem posing.

1. Teaching case 3: Proportional relationships

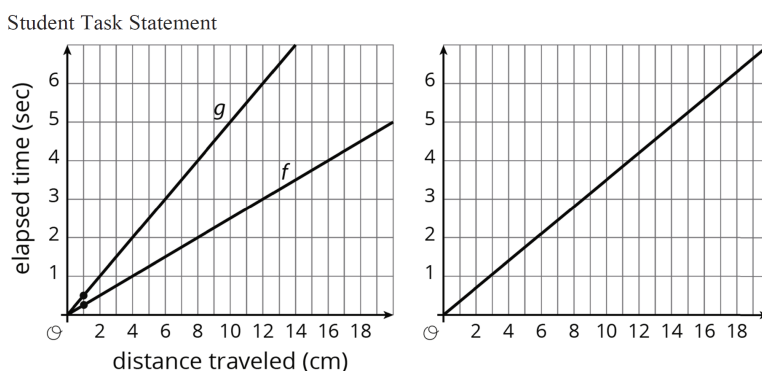
In this case, based on a lesson from the *Illustrative Mathematics* eighth-grade curriculum (Illustrative Mathematics, 2019), there are three learning goals for the students: (1) comprehend that for the equation of a proportional relationship given by $y = kx$, k represents the constant of proportionality; (2) create graphs and equations of proportional relationships in context, including an appropriate scale; and (3) interpret

diagrams or graphs of proportional relationships in context. The existing curriculum materials included several activities designed to meet these learning goals. Here, we present an example of how small modifications towards problem posing can enhance these opportunities for students to reach the goals.

1) Episode 1: Warm-up

The purpose of the warm-up is to start a conversation about what features a graph needs. Students are asked to look at a picture (Figure 5) and think of at least one thing they notice and at least one thing they wonder about regarding the picture. The teacher displays the problem for all to see and asks students to give a signal when they have noticed or wondered about something.

Figure 5 Teaching Case #3 Episode 1 diagram



This notice-and-wonder routine encourages students to pay attention to the graphs and begin to understand the situation. After examining and thinking about the graphs, students may notice:

- The second set of axes is not labeled.
- Graph g represents something going at a speed of 2 centimeters every second.
- Graph f represents something going at a pace of about 0.25 seconds per 1 centimeter.

Students may also wonder:

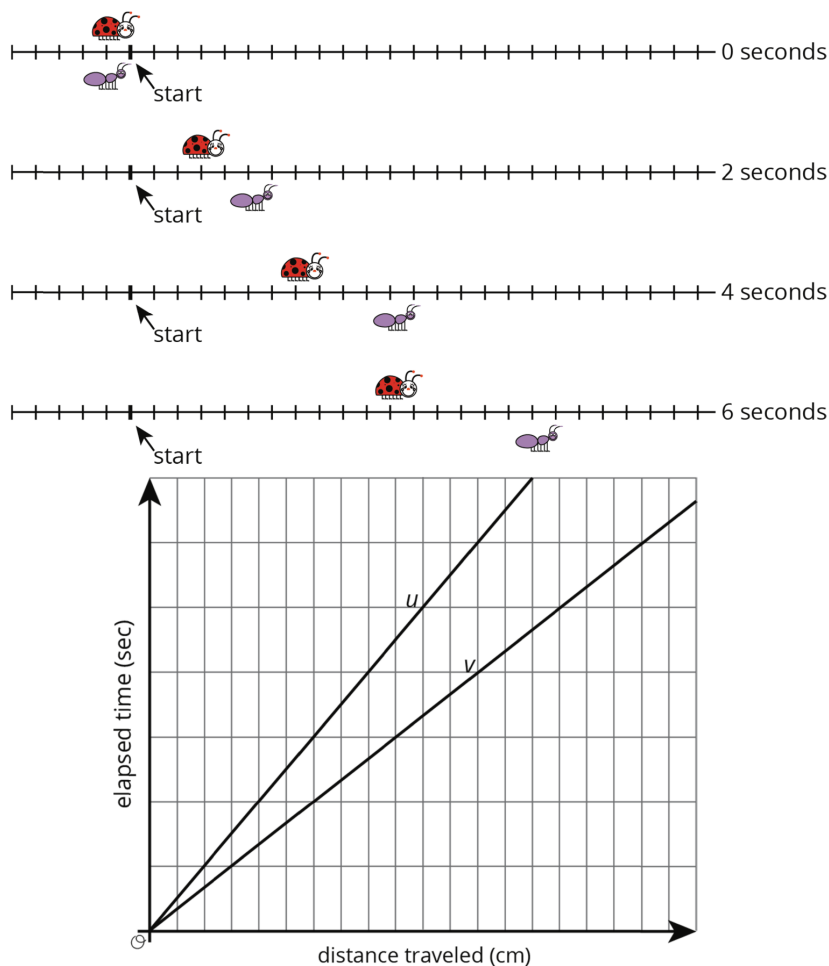
- What do the two points mean?
- Why does one graph show two lines while the other only has one?
- What does each line represent?

2) Episode 2: Paces of two different bugs

After sharing some of their responses to the notice and wonder activity, the students proceed to an investigation of the paces of two different bugs, a ladybug and an ant. Using a provided graph (see Figure 6), students answer questions about the pace of the bugs, decide on a scale for the graph axes, and mark and label the time needed to travel 1 cm for each bug (i.e., they identify a unit rate).

A ladybug and ant move at constant speeds. The diagrams with tick marks show their positions at different times. Each tick mark represents 1 centimeter.

Figure 6 Teaching Case #3 Episode 2: Ladybug and ant problem

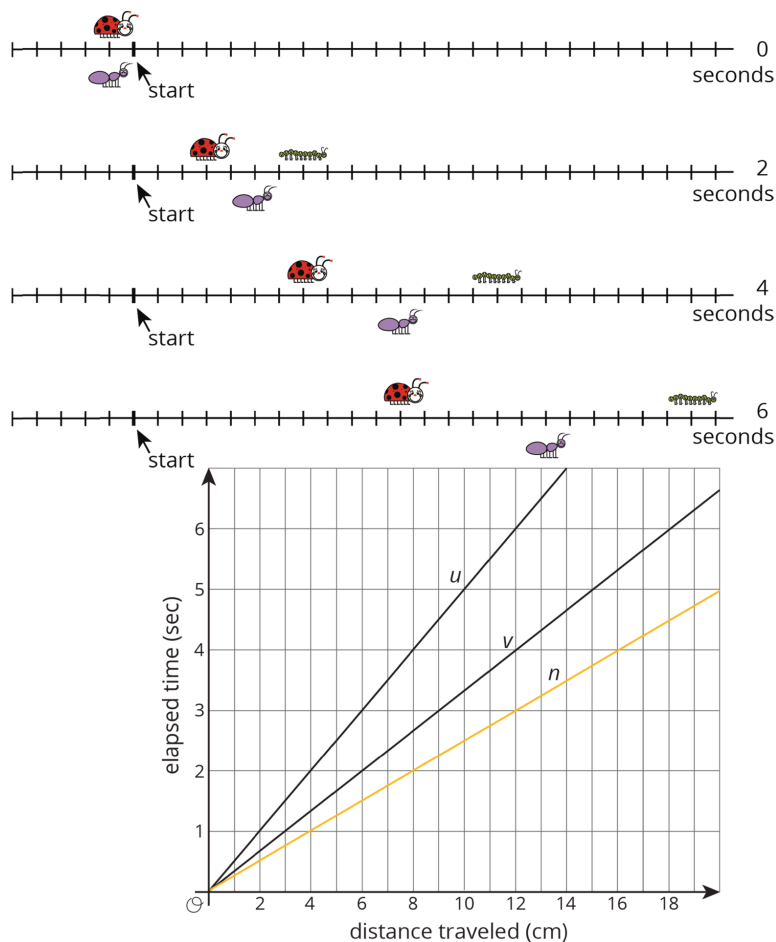


1. Lines u and v also show the positions of the two bugs. Which line shows the ladybug's? Which line shows the ant's movement? Explain your reasoning.
2. How long does it take the ladybug to travel 12 cm? The ant?
3. Scale the vertical and horizontal axes by labeling each grid line with a number. You will need to use the time and distance information shown in the tick-mark diagrams.
4. Mark and label the point on line u and the point on line v that represent the time and position of each bug after traveling 1 cm.

After answering the above four questions, students are asked to answer two additional questions in the textbook: “How fast is each bug traveling? Will there ever be a time when the ant is twice as far away from the start as the ladybug? Explain or show your reasoning.” However, with a problem-posing focus, instead of giving them these two follow-up questions, teachers can ask the students to generate questions of their own (e.g., one or two questions per student) related to the graph and the bug situation. Teachers can monitor the questions that students pose to look for potential misconceptions about the situation, the graph, or the connection between the graph and the diagrams with tick marks.

3) Episode 3. Paces of three bugs

Figure 7 Teaching Case #3 Episode 3: Paces of three bugs



In Episode 3, a third bug is introduced to the situation (see Figure 7). The textbook lists a set of tasks for students to complete: “Refer to the tick-mark diagrams and graph in the earlier activity when needed. (1)

Imagine a bug that is moving twice as fast as the ladybug. On each tick-mark diagram, mark the position of this bug. (2) Plot this bug's positions on the coordinate axes with lines u and v , and connect them with a line. (3) Write an equation for each of the three lines." However, rather than proceed directly to these tasks, the teacher modifies the activity by providing the new picture and graph and asking the students to pose problems relating the new bug to the information they already have about the other two bugs.

Some problems that the students may pose include:

- What is the speed of the new bug?
- How long will it take for the new bug to travel 12 cm?
- Is the new bug faster or slower than the other two bugs?
- How much faster is the new bug than the other bugs?
- After 5 seconds, how far will each bug have gone?
- After 1 second, how far will each bug have gone?

The students' posed problems open up an opportunity for the teacher to directly reference the constant of proportionality and the equation $y = kx$. As the students discuss and solve the problems their peers have posed, the teacher can guide the discussion and take advantage of the natural opportunities to help students with the vocabulary they will need to describe their thinking. Thus, with respect to the problem-posing approach, students begin by answering some given questions based on the graph, and this helps to scaffold their experience into posing their own problems to solve, which drives the classroom discourse to need precisely the concepts and vocabulary that are the targets of this lesson.

2. Strategies to develop problem-posing tasks

Teaching Case 3 provides an example of how teachers might effectively develop problem-posing-based lessons based on their existing curriculum materials. Because many current curricula (e.g., popular textbook series in China and the U. S., Cai & Jiang, 2017) do not incorporate significant and consistent experiences with problem posing for students, it is unreasonable to expect that problem posing will spontaneously be given much attention in classrooms. Indeed, teachers already face multiple demands on their time and attention. On any given day, they cannot devote large amounts of time preparing for significant changes in their upcoming lessons. Thus, any strategy to integrate problem posing more effectively in mathematics classrooms should ideally avoid being burdensome or perceived as a radical change in practice that would require a lot of time to adapt to. Instead, it seems reasonable to introduce problem posing through small, incremental changes that would be accessible to teachers and students but that offer the promise of rich returns in student learning, as shown in Teaching Case 3. Thus, in the spirit of Blanton and Kaput's (2011) modifications to single-numerical-answer arithmetic problems in order to surface the algebra in arithmetic, we propose three recommendations to better integrate problem posing

into existing problem-solving-focused lessons to transform them into a medium for developing algebraic thinking in existing school mathematics curriculum materials: (1) Supplement existing curriculum materials by regularly giving students the chance to pose problems or describe situations that match given mathematical equations; (2) Remove questions from existing problem-solving tasks to give students opportunities to pose problems first; and (3) Encourage students to pose problems at different levels of complexity.

Both Teaching Cases 1 and 2 portray lessons in which teachers have enhanced their existing curricula by creating new instructional tasks that ask students to pose problems or describe situations that match mathematical equations (Recommendation 1). In Teaching Case 1, the teacher transitions from asking students to generate situations for familiar arithmetic expressions to generating situations for a simple equation involving a variable, x . This gives the students the opportunity to ground a new algebraic concept (using letters to represent quantities and quantitative relationships) in both their arithmetic experience and their everyday experience. After this, the teacher engages the students in additional problem posing around a pair of patterns, opening up further opportunities for algebraic thinking.

Similarly, in Teaching Case 2, the teacher uses a problem-posing task to prompt the students to contextualize two equivalent arithmetic expressions. Again, this is an example of augmenting the curriculum with problem-posing tasks that ask students to give meaning to mathematical symbols in such a way that a generalization (algebraic thinking) can be made. Here, the generalization is the distributive property of multiplication over addition, which the students explicitly become aware of through the realization that the problems they posed could be solved by either of the two different arithmetic expressions.

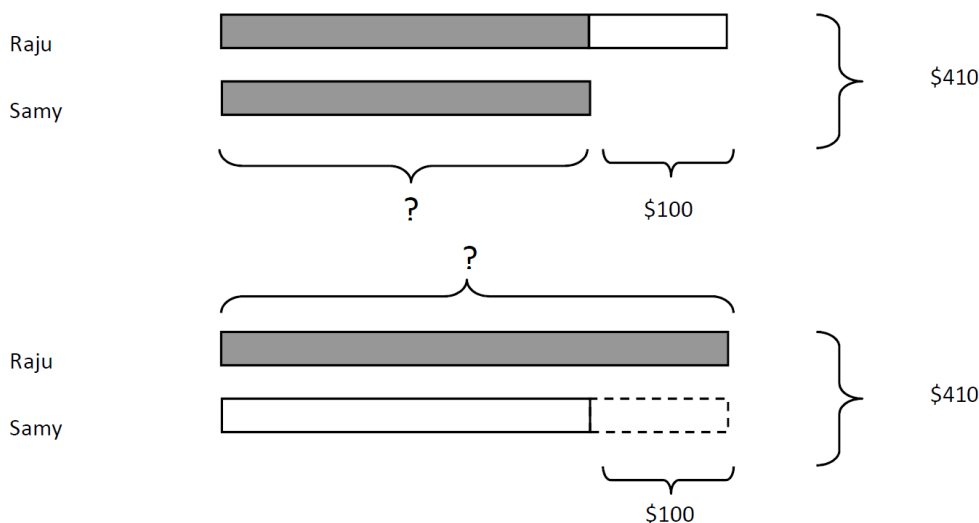
Episode 2 of Teaching Case 3 provides an example of how one might hide the given questions from a problem-solving task to give students opportunities to pose problems that will both provide the teacher with information about the students' mathematical thinking and that support the students' recognition of a generalization (Recommendation 2). To further illustrate what can be achieved by changing a problem-solving task into a problem-posing task by hiding or removing the existing question(s) from a problem situation, consider the problem from the Chinese curriculum described by Cai et al. (2011): "Liming Elementary school has funds to buy 12 basketballs at 24 Yuan each. Before buying the basketballs, they decided to spend 144 Yuan of the funds for some soccer balls. How many basketballs can they buy?" This problem can easily be changed into a problem-posing task by removing the question ("How many basketballs can they buy?") and inserting a problem-posing prompt: "Liming Elementary school has funds to buy 12 basketballs at 24 Yuan each. Before buying the basketballs, they decided to spend 144 Yuan of the funds for some soccer balls. *Pose two mathematical problems which can be solved using the information provided.*"

Now, the students will explore the parameters of the situation to generate their own problems. Some students may pose problems that require more than the given information to solve, such as, "How much does a soccer ball cost?" or "How many soccer balls can the school buy?" Other students may pose problems that include the additional information, such as, "If a soccer ball costs 48 Yuan, how many basketballs and soccer balls can the school buy in total?" As the student share these kinds of problems,

opportunities open up to analyze the situation in more detail. Because the problems they have been asked to pose should be solvable using the provided information, the discussion may involve specifying what information is not yet known and what information would need to be known in order to solve different kinds of problems. Still other students may pose problems like the original (e.g., “How many basketballs can they buy after having spent the 144 Yuan on soccer balls?”) Ultimately, the class’s posed problems provide a basis for them and the teacher to have productive discourse around the different quantitative relationships in the problem and to deepen their understanding of those relationships.

Similarly, consider the problem Cai et al. (2011) examined in the Singaporean curriculum: “Raju and Samy shared \$410 between them. Raju received \$100 more than Samy. How much money did Samy receive?” Again, this problem can easily be changed into a problem-posing task by replacing the question with a problem-posing prompt: “Raju and Samy shared \$410 between them. Raju received \$100 more than Samy. *Pose three mathematical problems which can be solved using the information provided.*” And again, the students’ responses to this problem-posing prompt can open up opportunities for discussion and discourse that highlights various relationships in the problem situation. For example, a student might pose apparently trivial yes/no problems such as, “Did Samy get as much money as Raju?” Discussing a posed problem like this immediately opens up the opportunity for the students to explicitly identify the disparity between what Raju gets and what Samy gets, thus highlighting a key mathematical condition in the problem for the whole class. Some students might ask the original question while others ask the mirror of the original question: “How much money did Raju receive?” This would create an opportunity to generate two different pictures for the same situation: one in which the unknown rectangle represents what Samy receives and one where it represents what Raju receives (see Figure 8). Comparing and discussing the two sets of diagrams corresponding to the two different posed problems would again open opportunities to make sense of the algebraic relationships in the problem situation.

Figure 8 Two pictorial equation diagrams generated by related problems for the same problem situation



Finally, our recommendation to encourage students to pose problems at different levels of complexity (Recommendation 3), is based on an observation about the importance of task variables or characteristics in shaping the opportunities to learn afforded by an instructional task. With respect to problem-solving tasks, researchers have examined the effects of various task variables on students' problem solving. For example, Goldin and McClintock (1984) considered several types of problem-solving task variables: syntax variables, content and context variables, structure variables, and heuristic behavior variables. Syntax variables are factors dealing with how problem statements are written. These factors, such as problem length and numerical and symbolic forms within the problem, may contribute to ease or difficulty in reading comprehension. Content variables refer to the semantic elements of the problem, such as the mathematical topic or the field of application, whereas context variables refer to the problem representation and the format of information in the problem. Structure variables refer to factors involved in the solution process, such as problem complexity and factors related to specific algorithms or solution strategies. Finally, heuristic process variables refer to the interactions between the mental operations of the problem solver and the task. The interaction between heuristic processes and the other task variables can have a significant impact on problem-solving ability.

Currently, much less is known about the influence of task variables on students' problem posing and, in particular, what that influence might mean for students' opportunities to see algebra in arithmetic through problem posing. As with problem solving, there are many kinds of task characteristics that could affect students' opportunities to engage with problem posing, including features of the problem situation and the problem-posing prompt. Kontorovich et al. (2012) have noted the need to investigate the effects of these characteristics. For example, Leung and Silver (1997) developed and analyzed a Test of Arithmetic Problem Posing (TAPP) which they then used to examine how the presence of numerical information affected preservice teachers' problem-posing abilities. Results from the TAPP indicated that the preservice teachers performed better on problem-posing tasks that included specific numerical information than on tasks without specific numerical information. This result provides some insight into how task variables can impact problem posing, but more research must be done on the impact of other variables. Adapting the TAPP to examine how different characteristics of problem situations affects subjects' problem posing could offer a way to study the effect of other task variables on problem posing.

With respect to Recommendation 3, early data from the Problem-Posing Based Learning (P-PBL) Project (Cai et al., 2021) suggests that asking students to generate problems at different levels of complexity (e.g., "Pose an easy problem, a somewhat difficult problem, and a difficult problem") may prompt different kinds of problem-posing activity than a more generic request (e.g., "Pose three different problems"). In interviews, P-PBL teachers raised several points about the differential effects of prompts. For example, some P-PBL teachers noted that their students would be more likely to engage with a more complex prompt like the easy/somewhat difficult/difficult prompt. They explained that their students enjoyed competition and saw the posing prompt as asking them to pose a challenging problem for their peers.

Similarly, one teacher noted that his students often wanted to make a challenge for the teacher, and thus would be highly motivated to engage with the more complex prompt in an effort to stump him. However, some other P-PBL teachers cautioned that for their students, a more complex prompt, and in particular, a prompt that includes the word “difficult,” would preemptively shut down the students’ engagement with problem posing before they even try to respond. Indeed, we found differential effects of these two prompts in the P-PBL assessment: The 6th-grade cohort posed more mathematical questions on average for the more complex prompt than the more generic prompt; however, the 7th-grade cohort exhibited the opposite behavior. These results may indeed reflect differences between classes and grades that the teachers were sensitive to. Moreover, the teachers raised the possibility of an interaction between the content of the problem situation and the prompt type, noting that problem situations embedded in certain content areas (e.g., the relatively unfamiliar topic of data) might influence how the students responded to the complex versus the generic prompt.

V. LOOKING AHEAD

In this paper, we provide some initial thoughts about the unique opportunities afforded by problem posing to help students see algebra in arithmetic. The ideas have drawn on our prior studies on problem posing involving elementary students (e.g., Cai & Hwang, 2021) and the current work of the P-PBL Project involving middle school students (Cai et al., 2021). Here, we point to some directions that the P-PBL Project is exploring that can inform future efforts to make use of problem posing as a means for students to engage with algebraic ideas through their arithmetic experiences.

Fundamentally, problem-posing based learning is a powerful approach that can open up learning opportunities for students to develop their algebraic thinking, as problem-posing tasks position students in the role of problem generator instead of the teacher or textbook. Educationally, this switching of roles is theoretically sound based on both constructivist and sociocultural aspects of learning, and it can actually increase students’ access to mathematical sense making and learning (Cobb, 1994). Rather than solving a single given problem whose parameters have been set, the task encourages students to use their conceptual understanding to make sense of and mathematize the situation and choose an aspect of it that captures their interest to problematize algebraic ideas. Students can extend from there to build connections between their existing understanding and the new context and its related mathematical ideas. The teacher can also build on the students’ posed problems to engage students with the desired mathematical learning goals, such as seeing algebra in arithmetic.

Although problem-posing activities are cognitively demanding tasks, they are adaptable to students’ abilities and thus can increase students’ access such that students with different levels of understanding can still participate and pose potentially productive problems based on their own sensemaking. Several studies have shown that problem posing is an activity with a high ceiling and low floor (Cai & Hwang, 2021; Singer

et al., 2015), offering access to all students to opportunities for mathematical sensemaking.

Although in this paper we have only provided three teaching cases about helping students see algebra in arithmetic through problem posing, it is our hope that they will provide a clearer picture for teachers and professional developers of what problem-posing-based learning can look like. Research has documented the effectiveness of using case-based instruction for teacher professional learning. This approach is effective because it situates instruction in meaningful contexts in order to learn to teach (e.g., Smith et al., 2014). Moreover, we hope that the strategies we have discussed for teachers to help students see algebra in arithmetic through problem posing will stimulate more teachers and researchers to work together to develop such lessons and to share them as new teaching cases. In turn, the growing body of teaching cases can reach ever further to support teachers as they help students see algebra in arithmetic through problem posing.

CONFLICTS OF INTEREST

No potential conflict of interest relevant to this article was reported.

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