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ON THE QUOTIENT OF THE HOMOLOGY COBORDISM GROUP BY SEIFERT SPACES

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Abstract. We prove that the quotient of the integer homology cobordism group by the subgroup generated by the Seifert fibered spaces is infinitely generated.

1. Introduction

The homology cobordism group $\Theta^3_{\mathbb{Z}}$ consists of integer homology 3-spheres modulo integer homology cobordism and is a fundamental structure in geometric topology. For example, $\Theta^3_{\mathbb{Z}}$ played a central role in Manolescu's [Man16] disproof of the triangulation conjecture in high dimensions.

A natural question to ask is which types of manifolds can represent a given class $[Y] \in \Theta^3_{\mathbb{Z}}$. The first answers to this question were in the positive direction. Livingston [Liv81] showed that every class in $\Theta^3_{\mathbb{Z}}$ can be represented by an irreducible integer homology sphere, and Myers [Mye83] improved this to show that every class admits a hyperbolic representative. More recently, Mukherjee [Muk20, Theorem 1.18] showed that every class admits a Stein fillable representative.

In the negative direction, Frøyshov (in unpublished work), Lin [Lin17], and Stoffregen [Sto17] showed that there are classes in $\Theta^3_{\mathbb{Z}}$ that do not admit a Seifert fibered representative. Nozaki, Sato, and Taniguchi [NST19, Corollaries 1.6 and 1.7] improved this result to show that there are classes that admit neither a Seifert fibered representative nor a representative that is surgery on a knot in S^3. The Frøyshov, Stoffregen, and Nozaki-Sato-Taniguchi examples are all connected sums of Seifert fibered spaces, and Lin's example has Floer homology consistent with it being representable by a Seifert fibered space. In particular, these results are insufficient to show $\Theta^3_{\mathbb{Z}}$ is not generated by Seifert fibered spaces.

Using the involutive Heegaard Floer homology of Hendricks and Manolescu [HM17], we proved in [HHSZ20, Theorem 1.9] that Seifert fibered spaces do not generate $\Theta^3_{\mathbb{Z}}$. More precisely, let Θ_{SF} denote the subgroup of $\Theta^3_{\mathbb{Z}}$ generated by Seifert fibered spaces. We showed that the quotient $\Theta^3_{\mathbb{Z}}/\Theta_{SF}$ contains a subgroup isomorphic to $\mathbb{Z}$, generated by $Y = S^3_{+1}(-2T_{6,7}\#T_{6,13}\#T_{-2,3;2,5})$. The main result of this paper is that the quotient $\Theta^3_{\mathbb{Z}}/\Theta_{SF}$ is in fact infinitely generated:

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Theorem 1.1. *The quotient $\Theta^3_{\mathbb{Z}}/\Theta_{SF}$ contains a subgroup isomorphic to $\mathbb{Z}^\infty$, spanned by*

$$Y_n = S^3_{+1}(T_{2,3}\# - 2T_{2n,2n+1}\#T_{2n,4n+1}), \quad n \geq 3, \ n \text{ odd}.$$

Involutive Heegaard Floer homology associates to an integer homology sphere Y (or more generally a spin rational homology sphere) an algebraic object called an iota-complex. The local equivalence class of this iota-complex is an invariant of the homology cobordism class of Y, and the set of iota-complexes modulo local equivalence forms a group under tensor product. For technical reasons, it is often convenient to consider a slightly weaker notion of equivalence, called almost local equivalence, and the associated group $\widehat{\mathfrak{I}}$ of almost iota-complexes modulo almost local-equivalence, as in [DHST18]. There is a group homomorphism

$$\hat{h}\colon \Theta^3_{\mathbb{Z}} \to \widehat{\mathfrak{I}}$$

induced by sending $[Y]$ to the almost local equivalence class of its iota-complex.

The proof of Theorem 1.1 relies on the following steps:

(1) A computation of the almost local equivalence class of the iota-complex associated to Y_n using the involutive surgery formula of [HHSZ20, Theorem 1.6]. We call this complex $C(n-1)$.
(2) A computation of the almost local equivalence class of linear combinations of $C(n-1)$, for $n \geq 2$, following the strategy of [DHST18, Section 8.1].
(3) A comparison of the results from step (2) with the computation of $\hat{h}(\Theta_{SF})$ in [DHST18, Theorem 8.1].

Remark 1.2. Let Θ_{AR} denote the subgroup of $\Theta^3_{\mathbb{Z}}$ spanned by almost-rationally plumbed 3-manifolds; see [Ném05] for the precise definition of an almost-rational plumbing. By [DS19, Theorem 1.1], $\hat{h}(\Theta_{AR}) = \hat{h}(\Theta_{SF})$, so the proof of Theorem 1.1 actually shows that the quotient $\Theta^3_{\mathbb{Z}}/\Theta_{AR}$ contains subgroup isomorphic to $\mathbb{Z}^\infty$.

Recall that a *graph manifold* is a prime 3-manifold whose JSJ decomposition contains only Seifert fibered pieces. The manifolds Y_n in Theorem 1.1 are all graph manifolds, since they are surgery along connected sums of torus knots. Similarly, the manifold Y in [HHSZ20, Theorem 1.9] is a graph manifold, since it is surgery along a connected sum of iterated torus knots. A natural question to ask is whether every homology sphere is homology cobordant to a graph manifold, or more generally, whether graph manifolds generate $\Theta^3_{\mathbb{Z}}$. As far as the authors know, both of these questions remain open; we expect that the answer to both is no. Note that if [NST19, Conjecture 1.22] is true, then graph manifolds do not generate $\Theta^3_{\mathbb{Z}}$, as pointed out in [NST19, Proposition 1.23]. Another natural question to ask is whether surgeries on knots in S^3 generate $\Theta^3_{\mathbb{Z}}$.

Organization. This paper is organized as follows. In Section 2 we recall some background on involutive Heegaard Floer homology. In Section 3 we prove that the almost iota-complex of the manifolds Y_n in Theorem 1.1 is $C(n-1)$. In Section 4 we compute the almost local equivalence classes of linear combinations of $C(n)$, and use it to complete the proof of Theorem 1.1.

2. Background on involutive Heegaard Floer homology

We will assume the reader is familiar with the basics of knot Floer homology [OS04] [Ras03], and confine ourselves to listing some definitions necessary for studying involutive Heegaard Floer homology [HM17]. In fact, in the present paper we will only need a few properties of this theory, which we summarize here. For more details, see [HHSZ20, Section 3].

Definition 2.1. An *iota-complex* (or *ι-complex*) (C, ι) is a chain complex C, which is free and finitely generated over $\mathbb{F}[U]$, equipped with an endomorphism ι. Here $\mathbb{F}$ is the field of 2 elements, and U is a formal variable with grading -2. Furthermore, the following hold:

(1) C is equipped with a $\mathbb{Z}$-grading, compatible with the action of U. We call this grading the *Maslov* or *homological* grading.
(2) There is a grading-preserving isomorphism $U^{-1}H_*(C) \cong \mathbb{F}[U, U^{-1}]$.
(3) ι is a grading-preserving chain map and $\iota^2 \simeq \mathrm{id}$.

Given two iota-complexes (C_1, ι_1) and (C_2, ι_2), a homogeneously graded $\mathbb{F}[U]$-chain map $f\colon C_1 \to C_2$ is said to be an *ι-homomorphism* if $\iota_2 \circ f + f \circ \iota_1 \simeq 0$. Two iota-complexes (C_1, ι_1) and (C_2, ι_2) are called *ι-equivalent* if there is a homotopy equivalence $\Phi\colon C_1 \to C_2$ which is an ι-homomorphism.

For any closed oriented 3-manifold Y equipped with self-conjugate spin^c structure $\mathfrak{s}$, Hendricks–Manolescu [HM17] prove that the $\mathbb{F}[U]$-chain complex with homotopy involution $(CF^-(Y, \mathfrak{s}), \iota)$ is well defined up to homotopy-equivalence. In the case that Y is a rational homology 3-sphere, $(CF^-(Y, \mathfrak{s}), \iota)$ is an iota-complex.

The tensor product of iota-complexes (C_1, ι_1) and (C_2, ι_2) is given by

$$(C_1, \iota_1) \otimes (C_2, \iota_2) := (C_1 \otimes_{\mathbb{F}[U]} C_2, \iota_1 \otimes \iota_2). \tag{2.1}$$

Moreover, Hendricks–Manolescu–Zemke [HMZ18] establish that

$$(CF^-(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2), \iota) \simeq (CF^-(Y_1, \mathfrak{s}_1), \iota_1) \otimes (CF^-(Y_2, \mathfrak{s}_2), \iota_2),$$

where $\simeq$ denotes homotopy-equivalence of iota-complexes.

Definition 2.2. Suppose (C, ι) and (C', ι') are two iota-complexes.

(1) A *local map* from (C, ι) to (C', ι') is a grading-preserving ι-homomorphism $F\colon C \to C'$, which induces an isomorphism from $U^{-1}H_*(C)$ to $U^{-1}H_*(C')$.
(2) We say that (C, ι) are (C', ι') are *locally equivalent* if there is a local map from (C, ι) to (C', ι'), as well as a local map from (C', ι') to (C, ι).

The set of local equivalence classes forms an abelian group, denoted $\mathfrak{I}$, with product given by the operation $\otimes$ in equation (2.1). See [HMZ18, Section 8]. Inverses are given by dualizing both the chain complex C and the map ι with respect to $\mathbb{F}[U]$; we write $-(C, \iota)$ for this dual iota-complex. According to [HMZ18, Theorem 1.8], the map

$$Y \mapsto [(CF^-(Y), \iota)]$$

determines a homomorphism from $\Theta^3_{\mathbb{Z}}$ to $\mathfrak{I}$.

There is an additional, weaker, equivalence relation between iota-complexes, introduced in [DHST18] (see also [HHSZ20, Section 3.3]).

Definition 2.3 ([DHST18, Definition 3.1])**.** Let C_1 and C_2 be free, finitely generated chain complexes over $\mathbb{F}[U]$, such that each C_i has an absolute $\mathbb{Q}$-grading and a

relative $\mathbb{Z}$-grading with respect to which U has grading -2. Two grading-preserving $\mathbb{F}[U]$-module homomorphisms

$$f, g \colon C_1 \to C_2$$

are *homotopic mod* U, denoted $f \simeq g \mod U$, if there exists an $\mathbb{F}[U]$-module homomorphism $H \colon C_1 \to C_2$ such that H increases grading by one and

$$f + g + H \circ \partial + \partial \circ H \in \operatorname{im} U.$$

Definition 2.4 ([DHST18, Definition 3.2])**.** An *almost iota-complex* (or *almost ι-complex*) $\mathcal{C} = (C, \overline{\iota})$ consists of the following data:

- A free, finitely-generated, $\mathbb{Z}$-graded chain complex C over $\mathbb{F}[U]$, with
$$U^{-1} H_*(C) \cong \mathbb{F}[U, U^{-1}].$$
Here U has degree -2 and $U^{-1}H_*(C)$ is supported in even gradings.
- A grading-preserving $\mathbb{F}[U]$-module homomorphism $\overline{\iota} \colon C \to C$ such that
$$\overline{\iota} \circ \partial + \partial \circ \overline{\iota} \in \operatorname{im} U \qquad \text{and} \qquad \overline{\iota}^2 \simeq \mathrm{id} \mod U.$$

Of course, any iota-complex induces an almost iota-complex. The definition of tensor product of almost iota-complexes is the same as equation (2.1).

In analogy with the terminology above, an *almost ι-homomorphism* from $(C_1, \overline{\iota}_1)$ to $(C_2, \overline{\iota}_2)$ is a homogeneously-graded, $\mathbb{F}[U]$-equivariant chain map $f \colon C_1 \to C_2$ such that $f \circ \overline{\iota} \simeq \overline{\iota} \circ f \mod U$. We then have the following new relation between almost ι-complexes.

Definition 2.5 ([DHST18, Definition 3.5])**.** Suppose $(C_1, \overline{\iota}_1)$ and $(C_2, \overline{\iota}_2)$ are almost ι-complexes.

(1) An *almost local map* from $(C_1, \overline{\iota}_1)$ to $(C_2, \overline{\iota}_2)$ is a grading-preserving almost ι-homomorphism $F \colon C_1 \to C_2$, which induces an isomorphism from $U^{-1}H_*(C)$ to $U^{-1}H_*(C')$.
(2) We say that $(C_1, \overline{\iota}_1)$ are $(C_2, \overline{\iota}_2)$ are *almost locally equivalent* if there is an almost local map from $(C_1, \overline{\iota}_1)$ to $(C_2, \overline{\iota}_2)$, as well as an almost local map from $(C_2, \overline{\iota}_2)$ to $(C_1, \overline{\iota}_1)$.

One special case of this definition will be especially useful to us: if $\overline{\iota}$ and $\overline{\iota}'$ are maps on the same complex C such that $(C, \overline{\iota})$ and $(C, \overline{\iota}')$ are each almost iota-complexes, and the difference $\overline{\iota} - \overline{\iota}' \in \operatorname{im}(U)$, then the identity map from C to itself is an almost local equivalence between $(C, \overline{\iota})$ and $(C, \overline{\iota}')$.

Using the definitions above, one may construct an almost local equivalence group $\widehat{\mathfrak{I}}$ of almost iota-complexes. It is a non-trivial result that $\widehat{\mathfrak{I}}$ can be parametrized explicitly [DHST18, Theorem 6.2], as we now describe. To a sequence $(a_1, b_2, a_3, b_4, \dots, a_{2m-1}, b_{2m})$, where $a_i \in \{\pm\}$ and $b_i \in \mathbb{Z} \setminus \{0\}$, we may associate an almost iota-complex

$$C(a_1, b_2, a_3, b_4, \dots, a_{2m-1}, b_{2m}),$$

called the *standard complex of type* $(a_1, b_2, a_3, b_4, \dots, a_{2m-1}, b_{2m})$, as follows. The standard complex is freely generated over $\mathbb{F}[U]$ by $t_0, t_1, \dots, t_{2m}$. For each symbol a_i, we introduce an $\omega = (1 + \iota)$-arrow between t_{i-1} and t_i as follows:

- If $a_i = +$, then $\omega t_i = t_{i-1}$.
- If $a_i = -$, then $\omega t_{i-1} = t_i$.

For each symbol b_i, we introduce a ∂-arrow between t_{i-1} and t_i as follows:

- If $b_i > 0$, then $\partial t_i = U^{|b_i|} t_{i-1}$.
- If $b_i < 0$, then $\partial t_{i-1} = U^{|b_i|} t_i$.

In computations with standard complexes, it will frequently be convenient to represent the group operation with $+$ instead of $\otimes$. The dual of the standard complex $C(a_1, b_2, a_3, b_4, \dots, a_{2m-1}, b_{2m})$ is the standard complex

$$-C(a_1, b_2, a_3, b_4, \dots, a_{2m-1}, b_{2m}) = C(-a_1, -b_2, -a_3, -b_4, \dots, -a_{2m-1}, -b_{2m}),$$

where if a_i is $+$ then $-a_i$ is $-$ and vice versa.

Every element of $\widehat{\mathfrak{I}}$ is locally equivalent to a unique standard complex [DHST18, Theorem 6.2]. Thus, in spite of $\widehat{\mathfrak{I}}$ being infinitely-generated, its elements are easy to describe. We write $\hat{h}$ for the composite

$$\Theta^3_{\mathbb{Z}} \to \mathfrak{I} \to \widehat{\mathfrak{I}}.$$

Note that there is not a simple formula for the group operation in terms of standard complexes. Nevertheless, the image $\hat{h}(\Theta_{SF}) \subseteq \widehat{\mathfrak{I}}$ has a simple description; see [DHST18, Section 8]. Indeed,

$$\hat{h}(\Theta_{SF}) = \{C(a_1, b_1, \dots, a_k, b_k) \mid |b_i| \leq |b_{i-1}| \text{ and } \operatorname{sgn}(b_i) = -\operatorname{sgn}(a_i)\} \subseteq \widehat{\mathfrak{I}}.$$

2.1. **Involutive knot Floer homology.** Hendricks-Manolescu also constructed an involutive knot Floer homology $\mathcal{CFK}(Y, K)$ for $K \subseteq Y$ a null-homologous knot, which, from our viewpoint, is a finitely-generated $\mathbb{F}[\mathscr{U}, \mathscr{V}]$-complex with an endomorphism ι_K, with properties as follows.

Suppose that (C_K, ∂) is a free, finitely generated complex over the ring $\mathbb{F}[\mathscr{U}, \mathscr{V}]$. There are two naturally associated maps

$$\Phi, \Psi \colon C_K \to C_K,$$

as follows. We write ∂ as a matrix with respect to a free $\mathbb{F}[\mathscr{U}, \mathscr{V}]$-basis of C_K. We define Φ to be the endomorphism obtained differentiating each entry of this matrix with respect to $\mathscr{U}$. We define Ψ to be the endomorphism obtained by differentiating each entry with respect to $\mathscr{V}$. These maps naturally appear in the context of knot Floer homology, see [Sar11, Zem17, Zem19c]. The maps Φ and Ψ are independent of the choice of basis, up to $\mathbb{F}[\mathscr{U}, \mathscr{V}]$-equivariant chain homotopy [Zem19a, Corollary 2.9].

We say an $\mathbb{F}$-linear map $F \colon C_K \to C'_K$ is *skew-*$\mathbb{F}[\mathscr{U}, \mathscr{V}]$*-equivariant* if

$$F \circ \mathscr{V} = \mathscr{U} \circ F \quad \text{and} \quad F \circ \mathscr{U} = \mathscr{V} \circ F.$$

We may view a free complex over $\mathbb{F}[\mathscr{U}, \mathscr{V}]$ also as an infinitely generated complex over $\mathbb{F}[U]$, where U acts diagonally via $U = \mathscr{U}\mathscr{V}$. Concretely, if $B = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is an $\mathbb{F}[\mathscr{U}, \mathscr{V}]$-basis, then an $\mathbb{F}[U]$-basis is given by the elements $\mathscr{U}^i \cdot \mathbf{x}_k$ and $\mathscr{V}^j \cdot \mathbf{x}_k$, ranging over all $i \geq 0$, $j \geq 0$ and $k \in \{1, \dots, n\}$.

Definition 2.6.

(1) An ι_K*-complex* (C_K, ι_K) is a finitely generated, free chain complex C_K over $\mathbb{F}[\mathscr{U}, \mathscr{V}]$, equipped with a skew-equivariant endomorphism ι_K satisfying

$$\iota_K^2 \simeq \mathrm{id} + \Phi\Psi.$$

(2) We say an ι_K-complex (C_K, ι_K) is of $\mathbb{Z}HS^3$*-type* if there are two $\mathbb{Z}$ valued gradings, $\mathrm{gr}_{\mathbf{w}}$ and $\mathrm{gr}_{\mathbf{z}}$, such that $\mathscr{U}$ and $\mathscr{V}$ have $(\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$-bigrading $(-2, 0)$ and $(0, -2)$, respectively. We assume ∂ has $(\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$-bigrading $(-1, -1)$, and that ι_K switches $\mathrm{gr}_{\mathbf{w}}$ and $\mathrm{gr}_{\mathbf{z}}$. Furthermore, we assume that

$A := \frac{1}{2}(\mathrm{gr}_{\mathbf{w}} - \mathrm{gr}_{\mathbf{z}})$ is integer valued. We call A the *Alexander* grading, and we call $\mathrm{gr}_{\mathbf{w}}$ and $\mathrm{gr}_{\mathbf{z}}$ the *Maslov* gradings. Writing $\mathcal{A}_s \subseteq C_K$ for the subspace in Alexander grading s, we assume that there is a grading-preserving isomorphism $U^{-1}H_*(\mathcal{A}_s) \cong \mathbb{F}[U, U^{-1}]$ for all $s \in \mathbb{Z}$.

In Definition 2.6, an ι_K-complex of $\mathbb{Z}HS^3$-type is equipped with two Maslov gradings, $\mathrm{gr}_{\mathbf{w}}$ and $\mathrm{gr}_{\mathbf{z}}$. We note that in the literature, usually one considers just the $\mathrm{gr}_{\mathbf{w}}$-grading, which is referred to as the homological grading. All ι_K-complexes in this paper will be of $\mathbb{Z}HS^3$-type (as they arise as the complexes associated to knots in S^3). For further details on the translation to other versions of knot Floer homology, see [Zem19b, Section 1].

The tensor product of ι_K-complexes has a slightly subtle definition:

$$(C_K, \iota_K) \otimes (C'_K, \iota'_K) = (C_K \otimes_{\mathbb{F}[\mathscr{U},\mathscr{V}]} C'_K, \iota_K \otimes \iota'_K + \iota_K \Psi \otimes \iota'_K \Phi).$$

Local equivalence of ι_K-complexes can defined much as for iota-complexes. See [Zem19a, Section 2]. For the present paper however, it is helpful to work with the (equivalent) definition that local equivalence of iota-complexes is the equivalence relation generated by declaring two ι_K-complexes of $\mathbb{Z}HS^3$-type C_1 and C_2 locally equivalent if C_1 is an ι_K-equivariant summand of C_2 (cf. [HH19]). With respect to this definition one may form a local equivalence group of ι_K-complexes; inverses are given by dualizing over $\mathbb{F}[\mathscr{U}, \mathscr{V}]$. As previously, we let $-(C_K, \iota_K)$ denote the dual ι_K-complex of (C_K, ι_K).

At present, it is very difficult to compute the ι_K-complexes associated to most knots. However, for L-space knots $K \subseteq S^3$, Hendricks-Manolescu [HM17] observed that there is a unique choice of ι_K such that the knot Floer complex $\mathcal{CFK}(K)$ is an ι_K-complex. In particular, the involutive knot Floer complex of an L-space knot K is determined by the Alexander polynomial $\Delta_K(t)$ of K.

2.2. **The surgery formula.** Our main tool is the surgery formula from [HHSZ20], which gives an expression for the involutive Heegaard Floer complex $(CF^-(S^3_{+1}(K)), \iota)$ in terms of the involutive knot Floer complex of K.

We will only need a small part of the surgery formula. For $K \subseteq S^3$, let $A_0(K)$ denote the $\mathbb{F}[U]$-subcomplex of $(\mathscr{U}, \mathscr{V})^{-1}\mathcal{CFK}(K)$, generated over $\mathbb{F}$ by the monomials $\mathscr{U}^i\mathscr{V}^j \cdot \mathbf{x}$ satisfying $A(\mathbf{x}) + j - i = 0$ with $i \geq 0$ and $j \geq 0$. The U-action on $A_0(K)$ is given by $U = \mathscr{U}\mathscr{V}$. Moreover, we can define a chain map $\iota\colon A_0(K) \to A_0(K)$ by $\iota(\mathbf{x}) = \iota_K(\mathbf{x})$, since ι_K preserves $A_0(K)$. It turns out (but is not obvious) that $(A_0(K), \iota_K)$ is an iota-complex (cf. [HM17, Theorem 1.5] and [HHSZ20, Lemma 3.16]). A consequence of the full surgery formula is:

Proposition 2.7 ([HHSZ20, Theorem 1.6])**.** *The local equivalence class of $(CF^-(S^3_{+1}(K)), \iota)$ is that of $(A_0(K), \iota_K)$. In particular, the ι-local equivalence class of $(CF^-(S^3_{+1}(K)), \iota)$ depends only on the ι_K-local class of $(\mathcal{CFK}(K), \iota_K)$.*

3. Computation of the almost iota-complex of Y_n

In this section we give a computation of the almost iota-complex associated to the manifold $Y_n = S^3_{+1}(T_{2,3}\# - 2T_{2n,2n+1}\#T_{2n,4n+1})$ for $n \geq 3$ odd. We start by describing the knot Floer homology of the two torus knots $T_{2n,2n+1}$ and $T_{2n,4n+1}$, followed by computing and simplifying several tensor products.

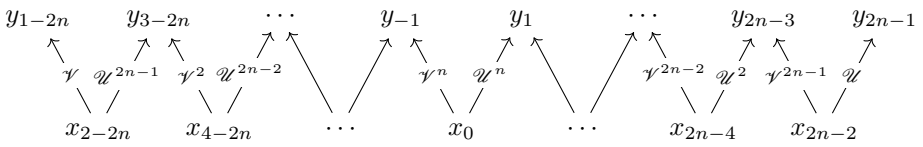


FIGURE 3.1. The complex $\mathcal{C}_n$

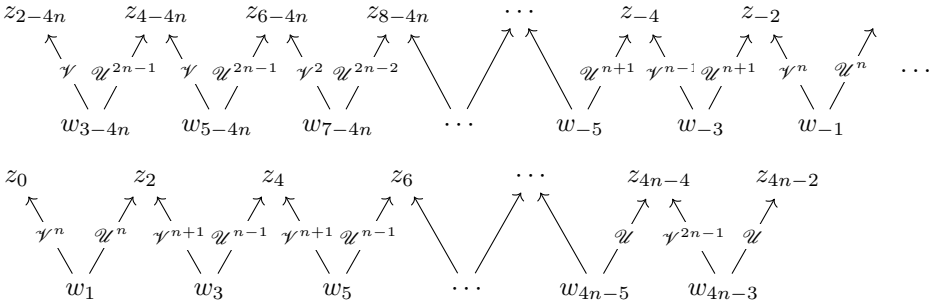


FIGURE 3.2. The complex $\mathcal{D}_n$. Note that the staircase continues onto the second row of the figure, such that $\partial(w_{-1}) = \mathscr{V}^n z_{-2} + \mathscr{U}^n z_0$.

3.1. **The knot Floer homology of two families of torus knots.** In this subsection we compute the ι_K-complexes associated to the torus knots $T_{2n,2n+1}$ and $T_{2n,4n+1}$.

Let $\mathcal{C}_n$ denote the $\mathbb{F}[\mathscr{U},\mathscr{V}]$ complex in Figure 3.1 generated by elements x_k such that $-2n+2 \le k \le 2n-2$ with k even and y_ℓ such that $-2n+1 \le \ell \le 2n-1$ with ℓ odd, with nonzero differentials given by

$$\partial(x_k) = \mathscr{V}^{c_{2n-1+k}} y_{k-1} + \mathscr{U}^{c_{2n+k}} y_{k+1}$$

determined by the symmetric sequence of positive integers $(c_1, c_2, \dots, c_{4n-3}, c_{4n-2})$ $= (1, 2n-1, 2, 2n-2, \dots, 2n-2, 2, 2n-1, 1)$.

Likewise, let $\mathcal{D}_n$ denote the complex defined similarly using the symmetric string of positive integers $(c_1, \cdots, c_{8n-4})$ given by

$$(1, 2n-1, 1, 2n-1, 2, 2n-2, 2, 2n-2, \cdots, 2n-2, 2, 2n-2, 2, 2n-1, 1, 2n-1, 1)$$

with generators w_k such that $3-4n \le k \le 4n-3$ with k odd and z_ℓ such that $2-4n \le \ell \le 4n-2$ with ℓ even, and nonzero differentials given by

$$\partial(w_k) = \mathscr{V}^{c_{4n-2+k}} z_{k-1} + \mathscr{U}^{c_{4n-1+k}} z_{k+1}.$$

See Figure 3.2 for a depiction of this staircase.

Proposition 3.1. *For n odd, the knot Floer homology $\mathcal{CFK}(T_{2n,2n+1})$ is chain homotopy equivalent to the complex $\mathcal{C}_n$, and the knot Floer homology $\mathcal{CFK}(T_{2n,4n+1})$ is chain homotopy equivalent to the complex $\mathcal{D}_n$. In both cases the involution ι_K is given by the natural reflection which interchanges the bigradings.*

By [OS05] and [HM17], the ι_K-complex associated to a torus knot is determined by its Alexander polynomial, so it suffices to compute the Alexander polynomials of

$T_{2n,2n+1}$ and $T_{2n,4n+1}$ for n odd. Suppose that $(c_1, c_2, \dots, c_{2k-1}, c_{2k})$ is a symmetric sequence of positive integers. We will define

$$\Delta(c_1, c_2, \dots, c_{2k-1}, c_{2k}) := 1 - t^{c_1} + t^{c_1+c_2} - t^{c_1+c_2+c_3} + \cdots - t^{c_1+\cdots+c_{2k-1}} + t^{c_1+\cdots+c_{2k}}.$$

Lemma 3.2. *The Alexander polynomial of $T_{2n,2n+1}$ is given by the formula*

(3.1)
$$\begin{aligned}\Delta_{T_{2n,2n+1}}(t) &= \Delta(1, 2n-1, 2, 2n-2, \dots, 2n-1, 1)\\ &= 1 - t + t^{2n} - t^{2n+2} + t^{4n} - \cdots + t^{2n(2n-2)} - t^{2n(2n-1)-1} + t^{2n(2n-1)}.\end{aligned}$$

Proof. Write $\Delta = \Delta(1, 2n-1, 2, 2n-2, \dots, 2n-1, 1)$. The Alexander polynomial is given by

$$\Delta_{T_{2n,2n+1}} = \frac{(t^{2n(2n+1)} - 1)(t-1)}{(t^{2n+1} - 1)(t^{2n} - 1)}.$$

Rearranging, it becomes sufficient to show that

$$\frac{(t^{2n})^{2n+1} - 1}{t^{2n} - 1} = \Delta \cdot \frac{t^{2n+1} - 1}{t - 1}.$$

Expanding this out, our desired relation becomes

$$\sum_{i=0}^{2n} t^{2ni} = \Delta \cdot \sum_{i=0}^{2n} t^i. \tag{3.2}$$

It is helpful to state a simple algebraic fact. Note that if N and M are positive integers, and $\{a_j\}_{j\in\mathbb{Z}}$ is a sequence which is zero for $j \notin \{0, \dots, N\}$, then

$$\left(\sum_{j=0}^{N} a_j t^j\right)\left(\sum_{i=0}^{M} t^i\right) = \sum_{j=0}^{N+M} (a_{j-M} + a_{j-M+1} + \cdots a_j) t^j.$$

In particular, if we write $a_0, \dots, a_{2n(2n-1)}$ for the coefficients of Δ (and set $a_j = 0$ for $j \notin \{0, \dots, 2n(2n-1)\}$), then the t^j coefficient of the right hand side of equation (3.2) is

$$a_{j-2n} + a_{j-2n+1} + \cdots + a_j. \tag{3.3}$$

However, by examining the description of Δ given in equation (3.1) it is easy to verify that equation (3.3) is 1 if $j = 2nk$ for some $k \in \{0, \dots, 2n\}$, and is 0 otherwise. This verifies equation (3.2), and completes the proof. □

Lemma 3.3. *The Alexander polynomial of $T_{2n,4n+1}$ satisfies*

$$\Delta_{T_{2n,4n+1}}(t) = \Delta(1, 2n-1, 1, 2n-1, 2, 2n-2, 2, 2n-2, \dots, 2n-1, 1, 2n-1, 1).$$

Proof. The proof is in much the same spirit as the proof of Lemma 3.2. Let Δ denote $\Delta(1, 2n-1, 1, 2n-1, 2, 2n-2, 2, 2n-2, \dots, 2n-1, 1, 2n-1, 1)$. Using the definition of the Alexander polynomial and rearranging terms, as in Lemma 3.2, it is sufficient to show that

$$\frac{(t^{2n})^{4n+1} - 1}{t^{2n} - 1} = \Delta \cdot \frac{t^{4n+1} - 1}{t - 1},$$

which we expand to

$$\sum_{i=0}^{4n} t^{2ni} = \Delta \cdot \sum_{i=0}^{4n} t^i.$$

Following the argument of Lemma 3.2, it is sufficient to show that if a_j denote the coefficients of Δ, then $a_{j-4n} + \cdots + a_j$ is 1 if $j = 2nk$, for some $k \in \{0, \dots 4n\}$, and is 0 otherwise. This is straightforward to verify. □

Proof of Proposition 3.1. By [OS05] and [HM17], the ι_K-complex of a torus knot is determined by its Alexander polynomial. The Alexander polynomials computed in Lemmas 3.2 and 3.3 correspond to the staircases $\mathcal{C}_n$ and $\mathcal{D}_n$ respectively, with ι_K given by the natural involution in each case. □

3.2. **The ι_K-complex associated to $-2T_{2n,2n+1}\#T_{2n,4n+1}$.** In this subsection we compute the ι_K-complex associated to the connect sum of torus knots $-2T_{2n,2n+1}\#T_{2n,4n+1}$ up to ι_K-local equivalence.

3.2.1. *The ι_K-local equivalence class of $T_{2n,2n+1}\#T_{2n,2n+1}$.* As in Section 3.1, let $\mathcal{C}_n$ denote the complex of $T_{2n,2n+1}$ for n odd which appears in Figure 3.1, and let $\mathcal{D}_n$ denote the complex associated to $T_{2n,4n+1}$ which appears in Figure 3.2. We first consider $\mathcal{X}_n := \mathcal{C}_n \otimes \mathcal{C}_n$. We will choose a new basis for $\mathcal{X}_n$ with respect to which our complex decomposes as a direct sum of $\mathcal{Y}_n \oplus \mathcal{Z}_n$, as follows. The subset $\mathcal{Y}_n$ is generated by the basis elements appearing in Figure 3.3.

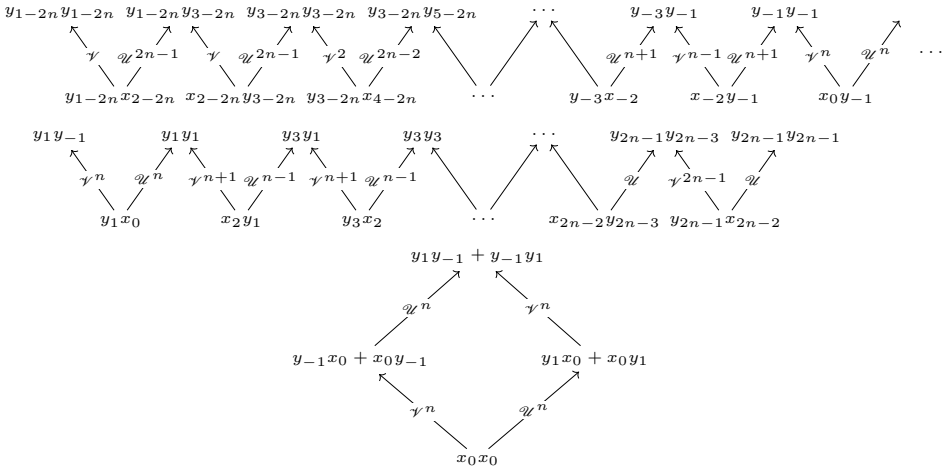


FIGURE 3.3. The subcomplex $\mathcal{Y}_n \subseteq \mathcal{X}_n$. Note that the top two rows form a staircase complex, such that $\partial(x_0y_{-1}) = \mathscr{V}^n y_{-1}y_{-1} + \mathscr{U}^n y_1 y_{-1}$.

Observe that in the staircase summand of the subcomplex $\mathcal{Y}_n$, the pattern of the construction changes at the basis element y_1y_1. Namely, traveling left to right in Figure 3.3 along the top row, we increase the second index of the generators y_iy_j, followed by the first. Along the second row, we increase the first index of the generators y_iy_j, followed by the second. This complex is equipped with the

involution ι_K arising from the tensor product, which in particular sends

$$\begin{aligned}
\iota_K(x_0x_0) &= x_0x_0 + \mathscr{U}^{n-1}\mathscr{V}^{n-1}y_1y_{-1}\\
\iota_K(y_1y_{-1}) &= y_1y_{-1} + (y_1y_{-1} + y_{-1}y_1)\\
\iota_K(y_1y_{-1} + y_{-1}y_1) &= y_1y_{-1} + y_{-1}y_1\\
\iota_K(y_{-1}x_0 + x_0y_{-1}) &= y_1x_0 + x_0y_1\\
\iota_K(y_1x_0 + x_0y_1) &= y_{-1}x_0 + x_0y_{-1}\\
\iota_K(x_0y_{-1}) &= y_1x_0 + (y_1x_0 + x_0y_1)\\
\iota_K(y_1x_0) &= x_0y_{-1} + (y_{-1}x_0 + x_0y_{-1})
\end{aligned}$$

and is otherwise a reflection.

Before defining the summand $\mathcal{Z}_n$, we make a few preliminary observations about gradings. Firstly, we note that

$$A(y_{i+2}) = A(y_i) + 2n,$$

for all odd i. As a consequence, if i and j are odd, then

$$A(y_iy_j) = A(y_{i+2}y_{j-2}).$$

In particular, if i, j are odd, then there is an $\gamma_{i,j} \in \mathbb{Z}$ such that

$$y_iy_j + (\mathscr{U}\mathscr{V})^{\gamma_{i,j}}y_{i+2}y_{j-2}$$

has homogeneous $(\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$-bigrading. It is not hard to compute that if $i < j$, then $\gamma_{i,j} \geq 0$.

Suppose that i and j are even and $i < j$. By considering the differential applied to x_ix_j and using the fact that the $\mathscr{U}$ powers in ∂x_i decrease as we increase the index of x_i, we see that if $i < j$, then there is an $\alpha_{i,j} \geq 0$ such that

$$\mathscr{U}^{\alpha_{i,j}}y_{i+1}x_j + x_iy_{j+1}$$

has homogeneous $(\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$-bigrading. Entirely analogously, if $i < j$, then there is a $\beta_{i,j} \geq 0$ so that

$$y_{i-1}x_j + \mathscr{V}^{\beta_{i,j}}x_iy_{j-1}$$

has homogeneous $(\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$-bigrading.

We now describe the summand $\mathcal{Z}_n$. The generators have the following form:

(Z-1) If i and j are both odd and $i \neq \pm j$, then $y_iy_j + y_jy_i$ is a generator of $\mathcal{Z}_n$.

(Z-2) If i is odd, j is even, and $j \neq -i \pm 1$, then $y_ix_j + x_jy_i$ is a generator of $\mathcal{Z}_n$.

(Z-3) If $i > 0$ is even and nonzero, write $i = 2n - 2k$ for some $n > k \geq 1$. Then $\mathcal{Z}_n$ has a generator

$$x_ix_i + k(2n-k)\mathscr{U}^{k-1}\mathscr{V}^{2n-k-1}y_{i-1}y_{i+1}.$$

(Z-4) If $i < 0$ is even then $\mathcal{Z}_n$ has a generator x_ix_i.

(Z-5) If i and j are even with $i < j$, then

$$x_ix_j + k_i(2n-k_j)\mathscr{U}^{k_i-1}\mathscr{V}^{2n-k_j-1}y_{i-1}y_{j+1}$$

is a generator of $\mathcal{Z}_n$, where $i = 2n - 2k_i$ and $j = 2n - 2k_j$.

(Z-6) If i and j are even and $i > j$, then x_ix_j is a generator of $\mathcal{Z}_n$.

(Z-7) If i and j are odd and $i < j - 2$, then

$$y_iy_j + (\mathscr{U}\mathscr{V})^{\gamma_{i,j}}y_{i+2}y_{j-2}$$

is a generator of $\mathcal{Z}_n$.

(Z-8) If $i > 2$ is odd, then

$$y_i y_{-i} + (\mathscr{U}\mathscr{V})^{\gamma_{-i,i}} y_{i-2} y_{-i+2}$$

is a generator.

(Z-9) If i and j are even and $i < j$, then following are generators of $\mathcal{Z}_n$:
- (a) $x_i y_{j+1} + \mathscr{U}^{\alpha_{i,j}} y_{i+1} x_j$;
- (b) $x_{-i} y_{-j-1} + \mathscr{V}^{\beta_{-i,-j}} y_{-i-1} x_{-j}$.

(Z-10) For even $j > 0$, the following are generators of $\mathcal{Z}_n$:
- (a) $y_{j+1} x_{-j} + \mathscr{V}^{\beta_{j,-j}} x_j y_{-j+1}$;
- (b) $y_{-j-1} x_j + \mathscr{U}^{\alpha_{-j,j}} x_{-j} y_{j-1}$.

(Z-11) For even $j > 0$, the following are generators of $\mathcal{Z}_n$:
- (a) $y_{j-1} x_{-j} + \mathscr{U}^{\alpha_{-j,j-2}} x_{j-2} y_{-j+1}$;
- (b) $y_{-j+1} x_j + \mathscr{V}^{\beta_{-j+2,j}} x_{-j+2} y_{j-1}$.

In the following, we use

$$(\iota_K \otimes \iota_K) \circ (\mathrm{id} \otimes \mathrm{id} + \Psi \otimes \Phi)$$

as our model of the involution.

Lemma 3.4. *$\mathcal{Z}_n$ and $\mathcal{Y}_n$ satisfy the following:*

1. *$\mathcal{Z}_n$ and $\mathcal{Y}_n$ are free.*
2. *$\mathcal{X}_n \cong \mathcal{Y}_n \oplus \mathcal{Z}_n$.*
3. *$\partial \mathcal{Z}_n \subseteq \mathcal{Z}_n$ and $\partial \mathcal{Y}_n \subseteq \mathcal{Y}_n$*
4. *$\iota_K \mathcal{Z}_n \subseteq \mathcal{Z}_n$ and $\iota_K \mathcal{Y}_n \subseteq \mathcal{Y}_n$.*

In particular, $\mathcal{X}_n$ is ι_K-locally equivalent to $\mathcal{Y}_n$.

Proof. To prove (1) and (2), we will first show that $\mathcal{X}_n = \mathcal{Y}_n + \mathcal{Z}_n$, and then we will show that the generating set obtained by concatenating the obvious basis of $\mathcal{Y}_n$ with the basis for $\mathcal{Z}_n$ above gives a generating set of $\mathcal{X}_n$ of the correct number of elements. In particular, this will imply that $\mathcal{Z}_n$ is free since it has a generating set with no linear relations.

We first address $\mathcal{X}_n = \mathcal{Y}_n + \mathcal{Z}_n$. Suppose i and j are both odd. Note $y_i y_i$ is in $\mathcal{Y}_n$ so we may assume that $i \neq j$. Consider the case $i \neq -j$. By adding (Z-1), it is sufficient to consider $i < j$. By adding (Z-7), we reduce to the case of $y_i y_i$ or $y_i y_{i+2}$ which are either in $\mathcal{Y}_n$, or are a sum of an element in $\mathcal{Y}_n$ with an element of (Z-1). Now consider $y_i y_{-i}$. By adding elements (Z-8), we reduce to the case of $y_1 y_{-1}$ and $y_{-1} y_1$, which are both in $\mathcal{Y}_n$. We now consider elements $x_i y_j$ and $y_j x_i$. Note that if $|i - j| = 1$, then either $x_i y_j$ is in $\mathcal{Y}_n$, or $x_i y_j$ plus an element (Z-2) is in $\mathcal{Y}_n$. The same holds for $y_j x_i$. Next, we consider an arbitrary $x_i y_j$. Using (Z-9), we may relate $x_i y_j$ with sums of $x_n y_m$ and $y_m x_n$ with $|m - n| < |i - j|$. Hence, by induction, it suffices to show that we can do the same to $y_j x_i$. If $j \neq -i \pm 1$, then we use (Z-2) to relate $y_j x_i$ with $x_i y_j$, and apply the previous argument. If $j = -i \pm 1$, then we use (Z-10) or (Z-11). This shows that all $x_i y_j$ and $y_j x_i$ are in the span. Finally, each $x_i x_j$ is a sum of generators (Z-4), (Z-5) and (Z-6), as well as the above terms. Hence $\mathcal{X}_n = \mathcal{Y}_n + \mathcal{Z}_n$.

We now show that the generating set obtained by concatenating $\mathcal{Y}_n$ and $\mathcal{Z}_n$ has the same cardinality as the rank of $\mathcal{X}_n$, which implies that $\mathcal{Z}_n$ is free and $\mathcal{X}_n \cong \mathcal{Y}_n \oplus \mathcal{Z}_n$. Firstly,

$$\mathrm{rank}(\mathcal{X}_n) = 16n^2 - 8n + 1 \quad \text{and} \quad \mathrm{rank}(\mathcal{Y}_n) = 8n + 1.$$

Similarly, $\mathcal{Z}_n$ has $2n^2-2n$ generators of type (Z-1), $4n^2-6n+2$ generators of type (Z-2), $4n^2-4n$ generators of types (Z-3), (Z-4), (Z-5) or (Z-6), $2n^2-2n$ generators of type (Z-7) and (Z-8), and $4n^2-6n+2$ generators of type (Z-9), and $4n-4$ generators of type (Z-10) or (Z-11). Hence, we have a generating set of $\mathcal{Z}_n$ with $16n^2-16n$ generators. Concatenating these generating sets gives a generating set of $\mathcal{C}_n \otimes \mathcal{C}_n$ with rank $16n^2-8n+1$, which must be a basis.

We now prove (3). Clearly $\partial\mathcal{Y}_n \subseteq \mathcal{Y}_n$, so we focus on $\mathcal{Z}_n$. On (Z-1), ∂ vanishes. The map ∂ sends elements of type (Z-2) to a sum of two elements of (Z-1). Elements (Z-3) and (Z-4) are mapped to sums of (Z-2). Basis elements in (Z-5) are mapped to a sum of (Z-9a) and (Z-9b). Basis elements (Z-6) are as follows. If $|j+i-1|>1$, they are mapped to a sum of (Z-9a) and (Z-9b). If $i+j=2$, they are mapped to a sum of (Z-9a), (Z-9b) and (Z-2). If $i+j=-2$, they are mapped to a sum of (Z-9b) and (Z-11a). The differential vanishes on (Z-7) and (Z-8). Elements (Z-9a) are mapped to elements (Z-7). Elements (Z-10) are mapped to a sum of (Z-1) and (Z-7) if $i \neq -j$, or (Z-8) if $i=-j$. Elements (Z-10a) are mapped to (Z-8). Elements (Z-10b) are mapped to (Z-7). Elements (Z-11a) are mapped to a sum of (Z-1) and (Z-7). Finally (Z-11b) is mapped to (Z-7).

We now prove (4). Clearly $\iota_K\mathcal{Y}_n \subseteq \mathcal{Y}_n$, so we focus on $\mathcal{Z}_n$. The map ι_K sends elements (Z-1) to elements (Z-1). Similarly elements (Z-2) are sent to elements (Z-2). Elements (Z-3) are sent to elements (Z-4). Elements (Z-4) are sent to the sum of an element (Z-3) and an element (Z-1). Similarly elements (Z-5) are sent to elements (Z-6), while elements (Z-6) and sent to sums of (Z-5) and (Z-7). Generators (Z-7) with $i \neq -j$ are sent to a sum (Z-7) and two elements of (Z-1). Generators (Z-7) with $i=-j$ are interchanged with generators (Z-8). Elements (Z-9a) and (Z-9b) are interchanged. Elements (Z-10a) and (Z-10b) are interchanged. Similarly elements (Z-11a) and (Z-11b) are interchanged. □

3.2.2. *The ι_K-local equivalence class of $-2T_{2n,2n+1}\#T_{2n,4n+1}$.* In this section, we compute the ι_K-local equivalence class of $\mathcal{CFK}(-2T_{2n,2n+1}\#T_{2n,4n+1})$.

We begin by introducing a new complex, called the *box complex.* Let $\mathcal{B}_n$ denote the knot-like complex in Figure 3.4 with five generators v, u, s_1, s_{-1}, s_0, with differential

$$\partial v = 0, \quad \partial s_0 = \mathscr{V}^n s_{-1} + \mathscr{U}^n s_1, \quad \partial s_{-1} = \mathscr{U}^n u, \quad \partial s_1 = \mathscr{V}^n u, \quad \text{and} \quad \partial u = 0.$$

The $\mathrm{gr} = (\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$-bigradings are as follows:

$$\begin{aligned} \mathrm{gr}(v) &= (0,0), \\ \mathrm{gr}(s_0) &= (2-2n, 2-2n), \\ \mathrm{gr}(s_{-1}) &= (1-2n, 1) \\ \mathrm{gr}(s_1) &= (1, 1-2n) \quad \text{and} \\ \mathrm{gr}(u) &= (0,0). \end{aligned}$$

The involution on $\mathcal{B}_n$ is as follows:

$$\begin{aligned} \iota_K(v) &= v+u \\ \iota_K(s_0) &= s_0 + \mathscr{U}^{n-1}\mathscr{V}^{n-1}v \\ \iota_K(s_{-1}) &= s_1 \\ \iota_K(s_1) &= s_{-1} \\ \iota_K(u) &= u. \end{aligned}$$

$$\begin{array}{ccc} & s_{-1} & \xleftarrow{\mathscr{U}^n} & u \\ v & \uparrow{\scriptstyle \mathscr{V}^n} & & \uparrow{\scriptstyle \mathscr{V}^n} \\ & s_0 & \xrightarrow{\mathscr{U}^n} & s_1 \end{array} \qquad \begin{array}{ccc} & s_{-1}^\vee & \xleftarrow{\mathscr{U}^n} & u^\vee \\ v^\vee & \downarrow{\scriptstyle \mathscr{V}^n} & & \downarrow{\scriptstyle \mathscr{V}^n} \\ & s_0^\vee & \xleftarrow{\mathscr{U}^n} & s_1^\vee \end{array}$$

FIGURE 3.4. The box complex $\mathcal{B}_n$ and its dual $\mathcal{B}_n^\vee$

We will also be interested in the dual complex $\mathcal{B}_n^\vee$, which is generated by $v^\vee$, $s_0^\vee$, $s_{-1}^\vee$, $s_1^\vee$, $u^\vee$ with gradings

$$\begin{aligned} \mathrm{gr}(v^\vee) &= (0,0), \\ \mathrm{gr}(s_0^\vee) &= (2n-2, 2n-2), \\ \mathrm{gr}(s_{-1}^\vee) &= (-1, 2n-1) \\ \mathrm{gr}(s_1^\vee) &= (2n-1, -1) \quad \text{and} \\ \mathrm{gr}(u^\vee) &= (0,0), \end{aligned}$$

and involution

$$\begin{aligned} \iota_K(v^\vee) &= v^\vee + \mathscr{U}^{n-1}\mathscr{V}^{n-1}s_0^\vee \\ \iota_K(s_0^\vee) &= s_0^\vee \\ \iota_K(s_{-1}^\vee) &= s_1^\vee \\ \iota_K(s_1^\vee) &= s_{-1}^\vee \\ \iota_K(u^\vee) &= u^\vee + v^\vee. \end{aligned}$$

Proposition 3.5. *The ι_K-complex of $(\mathcal{CFK}(-2T_{2n,2n+1}\#T_{2n,4n+1}), \iota_K)$ is ι_K-locally equivalent to the complex $\mathcal{B}_n^\vee$ with the involution described above.*

Recall that by Lemma 3.4, $\mathcal{CFK}(2T_{2n,2n+1})$ is ι_K-locally equivalent to the complex $\mathcal{Y}_n$ of Figure 3.3. Moreover, by Proposition 3.1, $\mathcal{CFK}(T_{2n,4n+1})$ is ι_K-locally equivalent to the complex $\mathcal{D}_n$ of Figure 3.2. Our proof of Proposition 3.5 proceeds by demonstrating that the ι_K-complex $\mathcal{Y}_n$ is ι_K-locally equivalent to $\mathcal{D}_n \otimes \mathcal{B}_n$.

Indeed, we prove a general lemma about the tensor product of (positive) staircase complexes with an even number of steps and box complexes. Let k be an even number, and let $\mathcal{S}$ be a staircase complex with generators x_j such that $-k+1 \le j \le k-1$ with j odd, and y_i such that $-k \le i \le k$ with i even. Let the differentials

$$\partial(x_j) = \mathscr{V}^{c_{k+j}} y_{j-1} + \mathscr{U}^{c_{k+j+1}} y_{j+1}$$

be specified by a symmetric sequence of positive integers $(c_1, c_2, \dots, c_{2k-1}, c_{2k})$ with the property that $c_k = c_{k+1} = n$. Most importantly, $\mathcal{S}$ has an even number of steps and the central arrows with target y_0 are both weighted by n, so that

$$\partial x_{-1} = \mathscr{V}^{c_{k-1}} y_{-2} + \mathscr{U}^n y_0 \quad \text{and} \quad \partial x_1 = \mathscr{U}^{c_{k-1}} y_2 + \mathscr{V}^n y_0.$$

(Recall that $c_{k-1} = c_{k+2}$.) We will compute the ι_K-local equivalance class of $\mathcal{S} \otimes \mathcal{B}_n$ for any staircase of this form. Similarly to the methods of the previous subsection, we construct an ι_K-equivariant splitting

$$\mathcal{S} \otimes \mathcal{B}_n \cong \mathcal{Y} \oplus \mathcal{W}$$

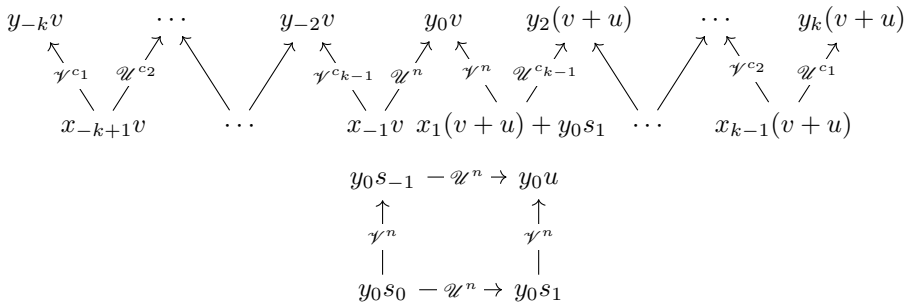


FIGURE 3.5. The complex $\mathcal{Y}$. Note that on the bottom row of the staircase complex, the terms to the right of $x_1(v+u)+y_0s_1$ are all of the form $x_i(v+u)$.

into two summands $\mathcal{Y}$ and $\mathcal{W}$, which we now describe. The complex $\mathcal{Y}$, which is the simpler of the two, appears in Figure 3.5. The complex $\mathcal{W}$ has the following generators:

(W-1) For even $i \neq 0$, the element y_iu.
(W-2) For odd $i \notin \{1,-1\}$, the element x_iu.
(W-3) The elements

$$x_{-1}u + y_0s_{-1} \quad \text{and} \quad x_1u + y_0s_1.$$

(W-4) For $i > 0$ even, the elements

$$y_is_{-1}, \quad y_is_1 \quad \text{and} \quad y_i(s_0 + \mathscr{U}^{n-1}\mathscr{V}^{n-1}v).$$

(W-5) For $i < 0$ even, the elements

$$y_is_{-1}, \quad y_is_1 \quad \text{and} \quad y_is_0.$$

(W-6) For $i > 1$ odd, then

$$x_is_{-1}, \quad x_is_1 \quad \text{and} \quad x_i(s_0 + \mathscr{U}^{n-1}\mathscr{V}^{n-1}v).$$

(W-7) For $i < -1$ odd, then

$$x_is_{-1}, \quad x_is_1 \quad \text{and} \quad x_is_0.$$

(W-8) The elements x_1s_1 and $x_{-1}s_{-1}$.
(W-9) The elements

$$x_1s_{-1} + y_0(s_0 + \mathscr{U}^{n-1}\mathscr{V}^{n-1}v), \quad \text{and} \quad x_{-1}s_1 + y_0s_0.$$

(W-10) The elements

$$x_1(s_0 + \mathscr{U}^{n-1}\mathscr{V}^{n-1}v) \quad \text{and} \quad x_{-1}s_0.$$

As in the previous example, we are using the model of the involution

$$(\iota_K \otimes \iota_K) \circ (\mathrm{id} \otimes \mathrm{id} + \Psi \otimes \Phi).$$

Note in particular that we have

$$\begin{aligned}
\iota_K(y_0v) &= y_0v + y_0u\\
\iota_K(y_0s_0) &= y_0s_0 + \mathscr{U}^{n-1}\mathscr{V}^{n-1}y_0v\\
\iota_K(y_0u) &= y_0u\\
\iota_K(y_0s_1) &= y_0s_{-1}\\
\iota_K(y_0s_{-1}) &= y_0s_1\\
\iota_K(x_{-1}v) &= (x_1(v+u) + y_0s_1) + y_0s_1\\
\iota_K(x_1(v+u) + y_0s_1) &= x_{-1}v + y_0s_{-1}.
\end{aligned}$$

Lemma 3.6. *The ι_K-complex $\mathcal{S}\otimes\mathcal{B}_n$ decomposes as the direct sum of ι_K-complexes $\mathcal{Y}\oplus\mathcal{W}$.*

Proof. Confirming that ∂ and ι_K both preserve $\mathcal{Y}$ and $\mathcal{W}$, and furthermore that $\mathcal{Y}\oplus\mathcal{W}\cong\mathcal{S}\otimes\mathcal{B}_n$, proceeds straightforwardly and similarly to Lemma 3.4. □

Proof of Proposition 3.5. We recall that by Lemma 3.4, $\mathcal{CFK}(2T_{2n,2n+1})$ is ι_K-locally equivalent to the complex $\mathcal{Y}_n$ of Figure 3.3. Moreover, by Proposition 3.1, $\mathcal{CFK}(T_{2n,4n+1})$ is ι_K-locally equivalent to the staircase complex $\mathcal{D}_n$ of Figure 3.2. Applying Lemma 3.6 to $\mathcal{D}_n\otimes\mathcal{B}_n$ shows that $\mathcal{D}_n\otimes\mathcal{B}_n$ is ι_K-locally equivalent to $\mathcal{Y}_n$. Therefore $\mathcal{Y}_n^\vee\otimes\mathcal{D}_n$ is ι_K-locally equivalent to $\mathcal{B}_n^\vee$. The statement of the proposition follows immediately. □

3.3. **The almost iota-complex associated to $S^3_{+1}(T_{2,3}\#-2T_{2n,2n+1}\#T_{2n,4n+1})$.** We now consider the tensor product of $\mathcal{B}_n^\vee$ with the complex of the trefoil $T_{2,3}$, again for n odd. Recall that $\mathcal{CFK}(T_{2,3})$ is the staircase complex generated by three elements r_0, s_1, s_{-1} with $\partial(r_0) = \mathscr{V}s_{-1} + \mathscr{U}s_1$ and other differentials trivial. We are interested in the iota-complex $(E_n,\iota) = A_0(\mathcal{B}_n^\vee\otimes\mathcal{CFK}(T_{2,3}))$ obtained from the ι_K-complex $\mathcal{B}_n^\vee\otimes\mathcal{CFK}(T_{2,3})$ by restricting to monomials $\mathscr{U}^i\mathscr{V}^j\mathbf{x}$ in $(\mathscr{U},\mathscr{V})^{-1}(\mathcal{B}_n^\vee\otimes\mathcal{CFK}(T_{2,3}))$ for which $A(\mathbf{x})+j-i=0$ and i and j are non-negative.

Proposition 3.7. *For $n\geq 3$ odd, the iota-complex (E_n,ι) is almost-locally equivalent to the standard complex $C(n-1) = C(+,-1,+,-n+1)$.*

Proof. The chain complex $(E_n,\iota) = A_0(\mathcal{B}_n^\vee\otimes\mathcal{CFK}(T_{2,3}))$ has fifteen generators and differentials as shown in Figure 3.6. (Recall that the action of U is generated by the action of $\mathscr{U}\mathscr{V}$.)

Using the usual model

$$\iota = (\iota_K\otimes\iota_K)\circ(\mathrm{id}\otimes\mathrm{id}+\Psi\otimes\Phi),$$

the involution takes the following form on E_n:

$$\begin{aligned}
\iota(a) &= c + U^{n-1}k & \iota(i) &= n\\
\iota(b) &= b + U^{n-1}j & \iota(j) &= j\\
\iota(c) &= a + U^{n-1}l & \iota(k) &= l\\
\iota(d) &= d + b + p & \iota(l) &= k\\
\iota(e) &= c + f & \iota(m) &= g + U^{n-1}l\\
\iota(f) &= a + e & \iota(n) &= i\\
\iota(g) &= m & \iota(p) &= h\\
\iota(h) &= p
\end{aligned}$$

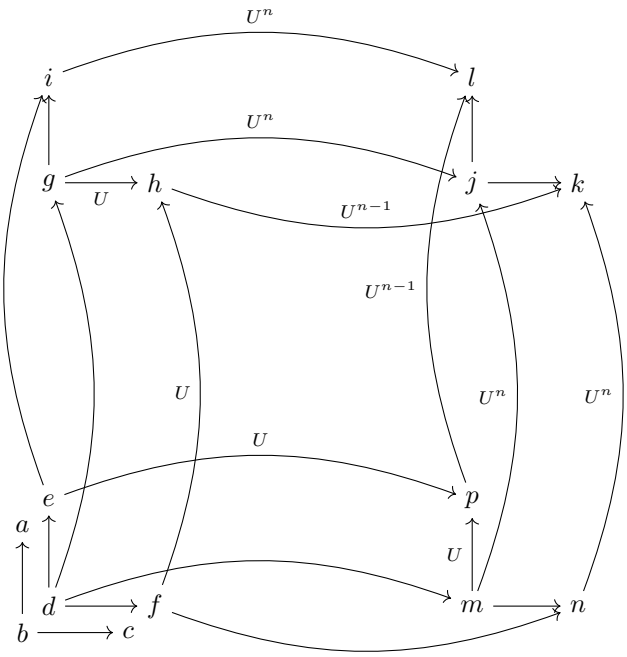


FIGURE 3.6. The complex $A_0(\mathcal{B}_n^\vee \otimes \mathcal{CFK}(T_{2,3}))$

We now do a change of basis to E_n to obtain the presentation of E_n shown in Figure 3.7.

$$a \qquad e + g \xrightarrow{U} h + U^{n-1}j + p \qquad p \xrightarrow{U^{n-1}} l$$

$$b \longrightarrow a + c \qquad j \longrightarrow l + k$$

$$e \longrightarrow i + Up \qquad m \longrightarrow n + Up + U^n j$$

$$d \longrightarrow e + f + g + m$$

FIGURE 3.7. A new basis of E_n. Arrows denote the differential.

Let F_n denote the top line of Figure 3.7. There are projection and inclusion maps

$$\Pi\colon E_n \to F_n \quad \text{and} \quad I\colon F_n \to E_n,$$

which are obviously homotopy equivalences. In particular, (F_n, ι') is ι-equivalent to (E_n, ι), where

$$\iota' = \Pi \circ \iota \circ I.$$

We compute

$$\begin{aligned} \iota'(a) &= \Pi(c + U^{n-1}k) = a + U^{n-1}l \\ \iota'(e+g) &= \Pi(c+f+m) = a+e+g \\ \iota'(h + U^{n-1}j + p) &= \Pi(h + U^{n-1}j + p) = h + U^{n-1}j + p \\ \iota'(p) &= \Pi(h) = h + U^{n-1}j \\ \iota'(l) &= \Pi(k) = l. \end{aligned} \tag{3.4}$$

We briefly remark how Π is computed in (3.4). The procedure is to write an element in terms of the basis in Figure 3.7, and then project to the top row. As an example

$$\Pi(c + U^{n-1}k) = \Pi(a + (a+c) + U^{n-1}l + U^{n-1}(l+k)) = a + U^{n-1}l.$$

We now consider the induced almost iota-complex. We claim that (F_n, ι') is ι-homotopy equivalent equivalent to the complex (F_n, ι'') where ι'' is the following map

$$\begin{aligned} \iota''(a) &= a \\ \iota''(e+g) &= a + (e+g) \\ \iota''(h + U^{n-1}j + p) &= h + U^{n-1}j + p \\ \iota''(p) &= (h + U^{n-1}j + p) + p \\ \iota''(l) &= l. \end{aligned}$$

The equivalence of (F_n, ι') and (F_n, ι'') is seen as follows. The map $\iota' + \iota''$ sends a to $U^{n-1}l$ and vanishes on all other generators of F_n. In particular, $\iota' + \iota'' = [\partial, H]$ on F_n, where H is the $\mathbb{F}[U]$-equivariant map which satisfies $H(a) = p$ and vanishes on all other generators.

However, (F_n, ι'') is the iota-complex

$$a \dashleftarrow e+g \xrightarrow{U} h + U^{n-1}j + p \dashleftarrow p \xrightarrow{U^{n-1}} l$$

where dashed arrows denote $\omega := \iota'' + \mathrm{id}$. This clearly reduces to the almost iota-complex $C(+, -1, +, -n+1) = C(n-1)$. □

4. Tensor products of almost iota-complexes

4.1. **The subgroup of the group of almost iota-complexes spanned by $C(n)$.** We now compute the subgroup of the group of almost iota-complexes spanned by linear combinations of the almost iota-complexes $C(n) = (+, -1, +, -n)$ for varying $n > 1$. The results of this section are similar to [DHST18, Section 8.1]. In this section we use the $+$ symbol instead of $\otimes$ to represent the tensor product of almost iota complexes. Observe that $-C(n)$ is parametrized by $(-, 1, -, n)$. We will consider sums of the form

$$C = \pm C(n_1) \pm C(n_2) \pm \cdots \pm C(n_m),$$

where each $n_k > 0$. Without loss of generality, we assume that the n_k are non-increasing, that is, $n_1 \geq n_2 \geq \cdots \geq n_m$. Furthermore, we assume that C is *fully simplified*, meaning that if $n_i = n_{i+1}$, the complexes $C(n_i)$ and $C(n_{i+1})$ occur with the same sign. Theorem 4.1 and its proof are analogous to [DHST18, Theorem 8.1].

Theorem 4.1. *Let*

$$C = \pm C(n_1) \pm C(n_2) \pm \cdots \pm C(n_m)$$

be fully simplified with $n_1 \geq n_2 \geq \cdots \geq n_m > 1$. Then the standard representative of C is obtained by concatenating the parameters of the above terms in the order that they appear.

Example 4.2. The standard representative of $C(n_1) + C(n_2) + \cdots + C(n_m)$ is

$$\begin{aligned}(+,-1,+,-n_1) + \cdots + (+,-1,+,-n_m)& \\ = (+,-1,+,-n_1,+,-1,+,-n_2,\dots,+,-1,+,-n_m).&\end{aligned}$$

Example 4.3. The standard representative of $-C(n_1) - C(n_2) - \cdots - C(n_m)$ is

$$(-,1,-,n_1) + \cdots + (-,1,-,n_m) = (-,1,-,n_1,-,1,-,n_2,\dots,-,1,-,n_m).$$

Example 4.4. The standard representative of $C(n_1) - C(n_2)$ is

$$(+,-1,+,-n_1) + (-,1,-,n_2) = (+,-1,+,-n_1,-,1,-,n_2).$$

Proof of Theorem 4.1. This proof closely follows the proof of [DHST18, Theorem 8.1]. We begin with a model calculation in the case $m = 2$. Let N and M be positive integers and consider $C(N) = (+,-1,+,-N)$ and $C(M) = (+,-1,+,-M)$. We consider the following two cases:

(1) $C_1 = -C(N) - C(M)$ with $N \geq M$,
(2) $C_2 = C(N) - C(M)$ with $N > M$,

and show that we have the following almost local equivalences

$$C_1 \sim (-,1,-,N,-,1,-,M) \qquad \text{and} \qquad C_2 \sim (+,-1,+,-N,-,1,-,M).$$

The other two cases $-C(N) + C(M)$ and $C(N) + C(M)$ follow by dualizing. For both C_1 and C_2, the obvious tensor product basis consists of 25 generators. These bases are displayed in the left of Figures 4.1 and 4.2, where they are labeled a through y. The dashed red arrows represent the action of ω and the solid black arrows represent ∂, with the label over the arrow denoting the associated power of U; for example, in C_1, we have that $\partial o = Uj + U^M n$ and that $\omega(m) = n + r + s$.

On the right of Figure 4.1, we have performed the change of basis

$$\begin{aligned}f' &= f + b + g\\ k' &= k + c + \ell\\ p' &= p + d + i + h + \ell\\ u' &= u + U^{N-M}e + U^{N-1}m + U^{N-1}n\\ v' &= v + U^{N-M}j + U^{N-1}n + U^{N-1}r\\ w' &= w + U^{N-M}o\\ x' &= x + U^{N-M}t,\end{aligned}$$

keeping the other basis elements the same. The reader should verify that this results in the diagram in the right of Figure 4.1. It is then evident from the right of Figure 4.1 that $C(N) + C(M)$ is almost locally equivalent to

$$(-,1,-,N,-,1,-,M),$$

as desired.

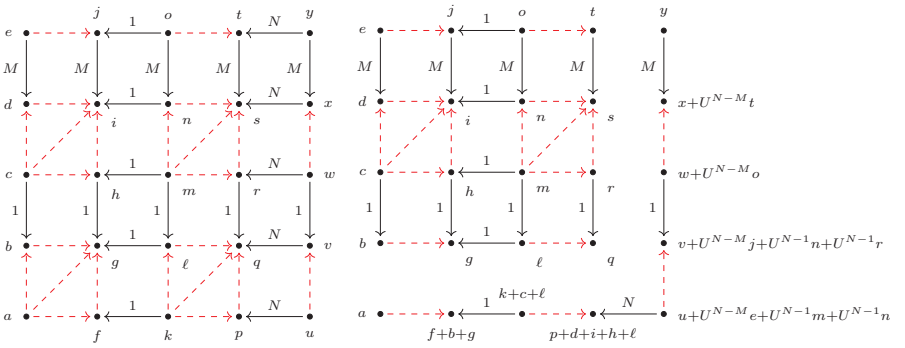


FIGURE 4.1. Left, the obvious tensor product basis for C_1. Right, after a change of basis. Recall that $N \geq M$.

The computation of $C(N) - C(M)$ is similar. On the right of Figure 4.2, we have performed a change of basis

$$\begin{aligned} a' &= a + b + g + m + n + s + U^{N-M}y \\ f' &= f + r \\ k' &= k + q + U^{N-1}w \\ p' &= p + k \\ q' &= q + U^{N-1}w \\ s' &= s + U^{N-M}y, \end{aligned}$$

keeping the other basis elements the same (e.g., $b' = b$, etc). The reader should verify that this results in the diagram on the right of Figure 4.2, where we consider ω modulo U. For example,

$$\begin{aligned} \omega(f + r) &= a + b + g + m + n + s \\ &\equiv a + b + g + m + n + s + U^{N-M}y \mod U. \end{aligned}$$

We have marked the dashed red arrows that are congruence modulo U (rather than equality) with congruence symbols to emphasize this point. (Here is where we first use the notion of almost local equivalence; in the computations of Section 3, all of the maps were local equivalences.) Note that since $N > M$, we have that $N - M > 0$. It is then evident from the right of Figure 4.2 that $C(N) - C(M)$ is almost locally equivalent to the standard complex $(+, -1, +, -N, -, 1, -, M)$, as desired.

We now consider the general case, by induction on m. Suppose we have established the claim for

$$C = \pm C(n_1) \pm C(n_2) \pm \cdots \pm C(n_m)$$

as in the statement of the theorem. Let M be a positive integer such that $M \leq n_m$. Now consider

$$C' = C - C(M).$$

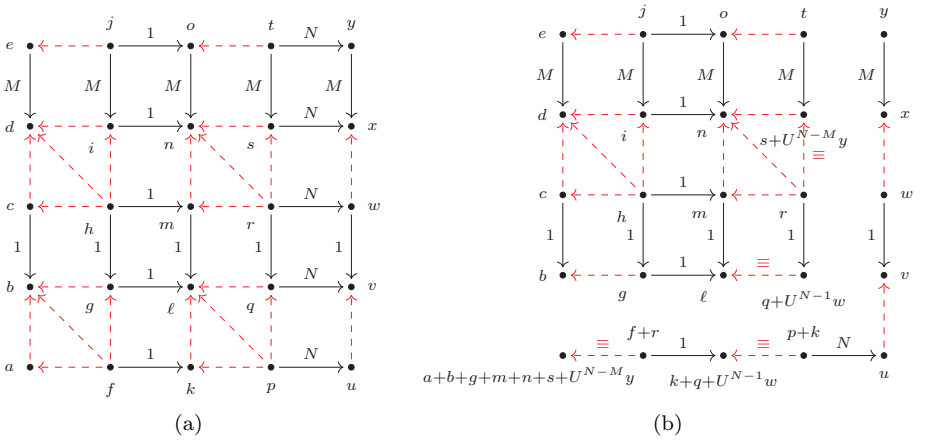


FIGURE 4.2. Left, the obvious tensor product basis for C_2. Right, after a change of basis. Recall that $N > M$.

The case $C + C(M)$ where we add rather than subtract $C(M)$ follows by dualizing. The obvious tensor product basis for $C - C(M)$ is schematically depicted in Figure 4.3 (where we have arbitrarily chosen signs in front of each $C(n_k)$). Using the inductive hypothesis applied to C, this complex has $5(4m+1)$ generators.

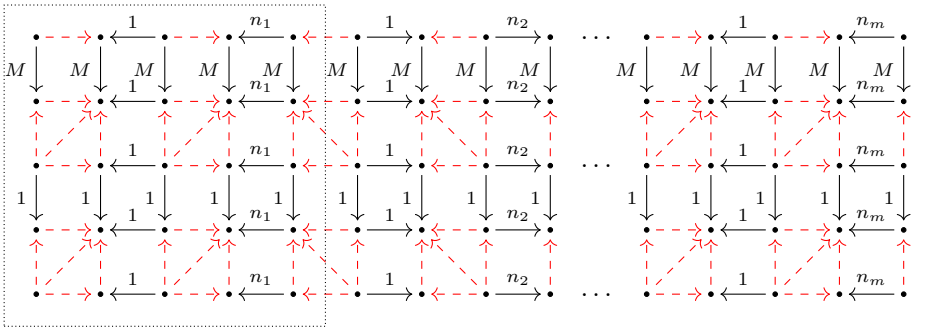
Our strategy will be to split off subcomplexes by change-of-basis moves paralleling those defined for C_1 and C_2. We begin by comparing the leftmost 25 generators of C'. Label these a through y, as usual. We begin by letting n_1 assume the role of N from the previous argument, so that applying the appropriate change of basis based as in Figure 4.1 if the coefficient of $C(n_1)$ is negative and as in Figure 4.2 if the coefficient of $C(n_1)$ is positive results in the second row of Figure 4.3. Note that in the first case, there is an additional subtlety: since we replace u, v, w, x with u', v', w', x' respectively, we are in danger of changing the dashed red arrows entering/exiting u, v, w, and x on the right. To check that this does not happen, we consider two cases:

(1) Suppose that there are dashed red arrows entering u, v, w, x from the right. We claim that in order for this to happen, we must have $n_1 > M$. Indeed, because C is fully simplified, if $n_1 = M$, then all subsequent terms in our sum are $-C(M)$, in which case u, v, w, x would have dashed red arrows exiting them, rather than entering. Hence $n_1 > M$. But this shows that

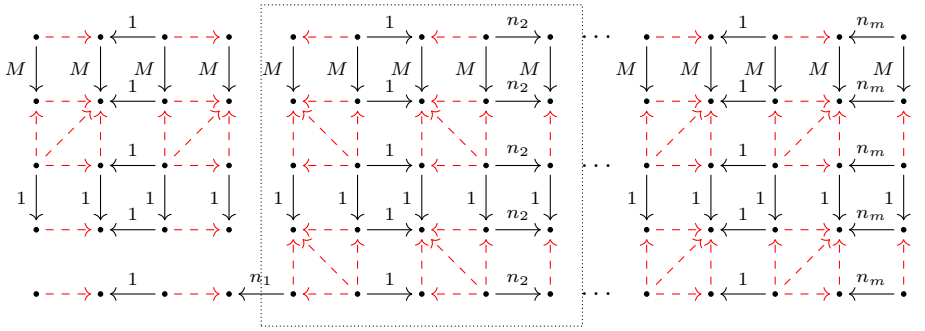
$$\begin{aligned} u' &\equiv u \mod U \\ v' &\equiv v \mod U \\ w' &\equiv w \mod U \\ x' &\equiv x \mod U, \end{aligned}$$

which means that the original dashed red arrows hold modulo U.

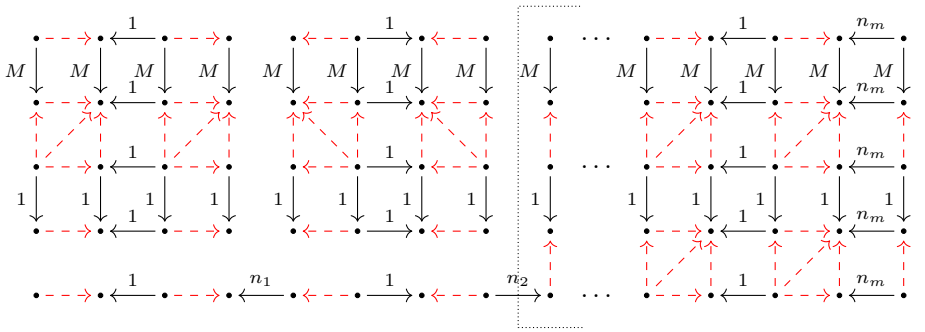
(2) Suppose that there are dashed red arrows exiting u, v, w, x to the right. Then we can explicitly check that the dashed red arrows exiting u', v', w', x'



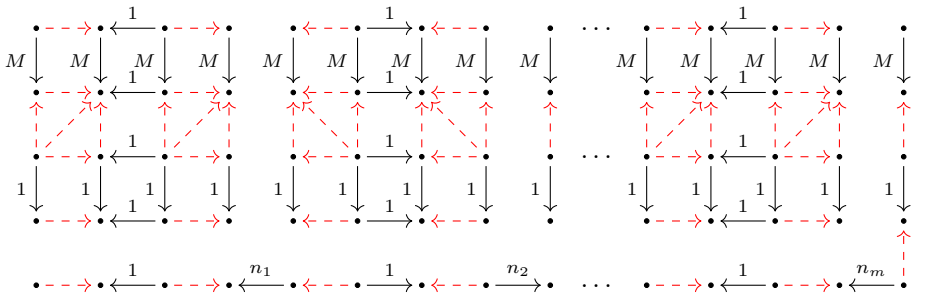
(a)



(b)



(c)



(d)

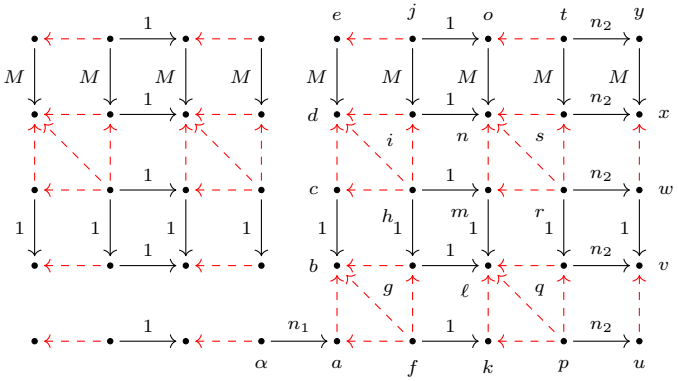
FIGURE 4.3

are unchanged:

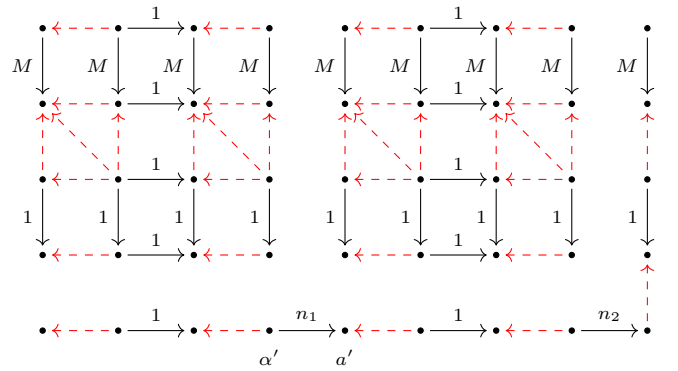
$$\begin{aligned}\omega(u') &= \omega(u) + U^{n_1-M} j + U^{n_1-1}(n+r) = \omega(u) + v' + v\\ \omega(v') &= \omega(v)\\ \omega(w') &= \omega(w) + U^{n_1-M} t = \omega(w) + x' + x\\ \omega(x') &= (x + U^{n_1-M} t) = \omega(x).\end{aligned}$$

In particular, we see that in either case, our change of basis does not change the form of the diagram lying to the right of u, v, w, x, and y.

We now consider the 25 generators lying inside the dashed box in the second row of Figure 4.3, relabeling them a through y as usual. Again, we attempt to perform a change of basis as in Figure 4.1 or 4.2, now with n_2 taking the role of N from the initial argument, as follows.



(a)



$\alpha' = \alpha + U^{n_1-1}(c+h+i) + U^{n_1-M} t$

$a' = a + b + g + m + n + s + U^{n_1-M} y$

(b)

FIGURE 4.4

If the second term $C(n_2)$ appears with negative sign in C, then we use the change of basis in Figure 4.1.

If the second term $C(n_2)$ appears with positive sign in C, then we attempt to use the change of basis in Figure 4.2. However, there is an additional subtlety, as depicted in Figure 4.4. Namely, we have a black arrow entering/exiting a from the left, so when we set

$$a' = a + b + g + m + n + s + U^{n_1 - M} y,$$

we must ensure that we don't change the diagram to the left of the dashed box. If the black arrow to the left of a is exiting a, then this follows from the fact that $\partial a' = \partial a$. However, if the arrow is instead entering a (representing the relation $\partial \alpha = U^{n_1} a$), then the diagram is no longer accurate, since evidently $\partial \alpha \neq U^{n_1} a'$. In this situation, we carry out the additional (retroactive) basis change

$$\alpha' = \alpha + U^{n_1 - 1}(c + h + i) + U^{n_1 - M} t$$

as in Figure 4.4, so that $\partial \alpha = U^{\zeta} a'$. Note that $n_1 > 1$ by hypothesis. Furthermore, note that $n_1 > M$, since C is fully simplified. Hence $\alpha' \equiv \alpha \mod U$, so modulo U, our basis change does not change the dashed red arrow leaving α. In any case, we see that performing the appropriate change-of-basis splits off another subcomplex and leads to a diagram as in the third row of Figure 4.3. Iterating this procedure results in the complex depicted in the bottom row of Figure 4.3, as desired. □

4.2. **Proof of Theorem 1.1.** We are now ready to complete the proof of our main theorem.

Proof of Theorem 1.1. By Proposition 2.7, the iota-complex $CF^-(S_{+1}(T_{2,3}\# - 2T_{2n,2n+1}\#T_{2n,4n+1}), \iota)$ is locally equivalent to $(A_0(T_{2,3}\#-2T_{2n,2n+1}\#T_{2n,4n+1}), \iota_K)$. By Proposition 3.5 and Proposition 3.7, for $n \geq 3$ odd, the almost local equivalence class of $(A_0(T_{2,3}\# - 2T_{2n,2n+1}\#T_{2n,4n+1}), \iota_K)$ is $C(n-1)$. Theorem 4.1 implies that the complexes $C(n)$ span a $\mathbb{Z}^\infty$ subgroup in $\widehat{\mathfrak{I}}$; in particular, elements in this subgroup of $\widehat{\mathfrak{I}}$ are of the form

$$(a_1, b_1, \ldots, a_{2m}, b_{2m}),$$

where

$$|b_1| = |b_3| = \ldots = |b_{2m-1}| = 1$$

and

$$|b_2| \geq |b_4| \geq \cdots \geq |b_{2m}|.$$

By [DHST18, Theorem 8.1], elements in $\hat{h}(\Theta^3_{SF})$ are of the form

$$(a_1, b_1, \ldots, a_m, b_m),$$

where

$$|b_1| \geq |b_2| \geq \cdots \geq |b_m|$$

and so the span of the $C(n)$ intersects $\hat{h}(\Theta^3_{SF})$ trivially. Therefore, we conclude that the classes

$$[S_{+1}(T_{2,3}\# - 2T_{2n,2n+1}\#T_{2n,4n+1})]$$

span a $\mathbb{Z}^\infty$ subgroup of $\Theta^3_{\mathbb{Z}}/\Theta^3_{SF}$.

□

Acknowledgments

We are grateful to Irving Dai and Linh Truong for helpful conversations. The search for the examples in this paper was inspired by the examples of knot-like complexes computed by Wenzhao Chen in [Che20] (although our ultimate examples and methods are significantly different from his). We also thank the anonymous referees for helpful comments and feedback.

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