



Euler characteristic of stable envelopes

Hunter Dinkins¹ · Andrey Smirnov^{1,2}

Accepted: 26 June 2022

© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2022

Abstract

In this paper we prove a formula relating the equivariant Euler characteristic of K -theoretic stable envelopes to an object known as the index vertex for the cotangent bundle of the full flag variety. Our formula demonstrates that the index vertex is the power series expansion of a rational function. This result is a consequence of the 3d mirror self-symmetry of the variety considered here. In general, one expects an analogous result to hold for any two varieties related by 3d mirror symmetry.

Keywords Stable envelopes · 3d mirror symmetry · Quasimaps · Enumerative geometry

Mathematics Subject Classification 05E14 · 14N10 · 14N35

Contents

1	Introduction	
2	Basic properties of X	
2.1	Description as a quiver variety	
2.2	Torus action	
2.4	Dual variety	
2.5	Polarization and chamber	
2.6	Chamber	
2.7	Ordering on fixed points	
3	3d mirror symmetry	
3.1	Exchange of equivariant and Kähler parameters	
3.2	Walls and resonances	

✉ Hunter Dinkins
hdinkins@live.unc.edu

Andrey Smirnov
asmirnov@email.unc.edu

¹ Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA

² Steklov Mathematical Institute of Russian Academy of Sciences, Gubkina str. 8, Moscow, Russia 119991

4	Index vertex	
4.1	Quasimaps	
4.2	Virtual classes on QM_p	
4.3	Index vertex	
4.4	Vertex function	
4.5	Index limit	
5	Stable envelopes	
5.1	Notations	
5.2	Stable envelopes	
6	Index vertex and stable envelopes	
6.1	Big enough slopes	
6.2	Generic slopes	
6.3	Non-generic slopes	
7	Example: $T^*\mathbb{P}^1$	
7.1	Torus fixed quasimaps	
7.2	Virtual tangent space	
7.3	Vertex functions	
7.4	Index	
7.5	Stable envelopes	
	References	

1 Introduction

Let X be a smooth symplectic quasiprojective variety. We make the following assumptions:

- A torus T acts on X such that it scales the symplectic form with character $\hbar$. We denote by $A := \ker(\hbar) \subset T$ the subtorus preserving the symplectic form.
- The T -action has finitely many fixed points.
- The tangent bundle of X has a polarization $T^{1/2}X$. In other words, there exists some class $T^{1/2}X \in K_T(X)$ so that the tangent bundle of X decomposes as

$$TX = T^{1/2}X + \hbar^{-1} \left(T^{1/2}X \right)^\vee \in K_T(X)$$

- A chamber $\mathfrak{C} \subset \text{Lie}_{\mathbb{R}}(A)$ is fixed, which is a choice of connected component of the complement of

$$\bigcup_{p \in X^A} \bigcup_{w \in \text{char}_A(T_p X)} \{ \sigma \in \text{Lie}_{\mathbb{R}}(A) \mid \langle \sigma, w \rangle = 0 \}$$

where $\langle \cdot, \cdot \rangle$ is the pairing of characters and cocharacters. This chamber decomposes the tangent space at the fixed points into attracting and repelling directions.

Our main interest is the case when X is a Nakajima quiver variety of linear or affine type A . If such a variety satisfies some natural conditions which hold for quiver varieties, it is known that the cohomological, K -theoretic, and elliptic stable envelopes exist, see [1] and [20].

For a torus fixed point $p \in X^T$, the K -theoretic stable envelope provides a class

$$\text{Stab}_{\mathfrak{C}, T^{1/2}X}^{s, X, K}(p) \in K_T(X)$$

which depends on the chosen polarization and chamber, as well as a generic choice of $s \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ called the slope. For an appropriately normalized version of the stable envelope, we consider the K -theoretic equivariant Euler characteristic

$$\chi \left(\text{Stab}_{\mathfrak{e}, T^{1/2}X}^{s, X, K}(p) \right) \in K_{\mathbb{T}}(pt)_{loc} = \mathbb{C}(\mathbf{a}, \hbar) \quad (1)$$

where $\mathbf{a}$ denotes the equivariant parameters of the torus A . A natural question, and the one we study in this paper, is to ask for an explicit description of this rational function and to study its expansion as a series in $\mathbf{a}$.

In this paper, we study the problem described above for the special case when X is the cotangent bundle of the full flag variety, and we formulate our answer using 3d mirror symmetry. This variety is known to be self-dual with respect to 3d mirror symmetry, see [26] and [7]. We denote the 3d mirror dual copy of this variety by $X^!$ and denote the torus acting on this variety by $T^!$. While X and $X^!$ are isomorphic as varieties, the 3d mirror symmetry relationship requires that various tori associated to X and $X^!$ are related in non-trivial ways, see (8) below. Hence it will be necessary to distinguish them.

Using results from [7], we identify in Theorem 2 the power series expansion of (1) around a certain point with an object we call the index vertex of $X^!$. We will now briefly explain the geometric meaning of the index vertex.

The index vertex arises in the enumerative geometry of quasimaps from $\mathbb{P}^1$ to $X^!$, see Sect. 4 of [21] and Sect. 8.2 of [18]. For a $T^!$ -fixed point $p^! \in X^!$, we denote by $\text{QM}_{p^!}$ the moduli space of stable quasimaps from $\mathbb{P}^1$ to $X^!$ that evaluate to $p^!$ at ∞ . The torus $T_q^! := T^! \times \mathbb{C}_q^\times$, where $\mathbb{C}_q^\times$ acts on $\mathbb{P}^1$, acts on $\text{QM}_{p^!}$ with a discrete fixed point set. A generic choice of the slope s provides a decomposition of the virtual tangent space of $\text{QM}_{p^!}$ at a $T_q^!$ -fixed point into attracting and repelling directions. The index vertex $\text{Ind}_{p^!}^{!s}$ is defined as the generating function that counts these repelling directions, with the sum taken over all possible degrees of quasimaps, see Definition 3 below. Then, roughly speaking, our first main theorem states:

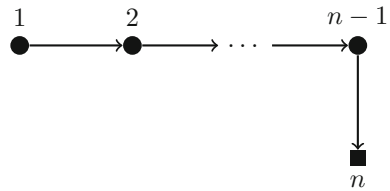
$$\chi \left(\text{Stab}_{\mathfrak{e}, T^{1/2}X}^{s, X, K}(p) \right) = \text{Ind}_{p^!}^{!s}$$

where p and $p^!$ are related under the bijection provided by 3d mirror symmetry as in Sect. 3.

As a consequence of this result, we deduce that the index vertex is the power series expansion of a rational function.

While K -theoretic stable envelopes require the slope s to be generic, the index vertex can be defined for non-generic slopes s as a limit of the so-called vertex function of $X^!$. In this case, the index vertex is still the power series expansion of a rational function, but the identification of the rational function is more complex. Using results from [12], we prove in Theorem 4 that it gives the power series expansion of a certain rational function obtained from the stable envelopes of X and stable envelopes of a subvariety $Y_s \subset X^!$.

Fig. 1 The quiver data for the cotangent bundle of the flag variety



The outline of this paper is as follows. In Sect. 2, we describe some basic notions regarding the description of the cotangent bundle of the full flag variety as a Nakajima quiver variety. In Sect. 3, we describe a few pieces of data involved in the 3d mirror self-symmetry of this variety that will be important for us. In the next two sections we discuss the main objects of study in this paper: the index vertex in Sect. 4 and stable envelopes in 5. We state and prove our main theorems in Sect. 6. In the final section, we calculate all relevant quantities directly and verify our result for generic slopes explicitly in the simplest possible example.

We conclude this introduction with a few general remarks. Although the cotangent bundle of the full flag variety is a particularly nice variety, we expect the results of this paper to hold much more generally. More specifically, we expect that for any two varieties related by 3d mirror symmetry satisfying the conditions listed at the beginning of the introduction, the index vertex and the K -theoretic stable envelopes should be related in the same way as described here.

In fact these results are an example of the program initiated by A. Okounkov in his talk [19] at the 2018 MSRI workshop “Structures in Enumerative Geometry”. Okounkov outlined an approach, based on joint work with M. Aganagic and N. Nekrasov in [1] and [18] respectively, to relating curve counts between 3d mirror dual varieties. Our main result can be understood as a degeneration of Okounkov’s more general conjecture. Indeed, the proof of Theorem 2 shows that the relationship between the K -theoretic stable envelopes and the index vertex is really a consequence of a more general conjecture regarding elliptic stable envelopes and vertex functions of 3d mirror dual varieties, see [7, 9, 14, 29], and the introduction of [1]. In the case at hand, this conjecture was proven by the first author in [9]. It would be interesting to find independent techniques to prove the results of this paper.

While a general construction of 3d mirror dual pairs is not presently known, the construction of Coloumb branches in [17] and [2] provides a large class of varieties 3d mirror dual to quiver varieties, and more generally Higgs branches of 3d $N = 4$ gauge theories. In the case of type A , which closely resembles this paper, 3d mirror dual pairs can be realized as Cherkis bow varieties, see [3, 5, 16, 24], and [4].

2 Basic properties of X

2.1 Description as a quiver variety

We construct the cotangent bundle of the full flag variety as a Nakajima quiver variety. The quiver data is given in Fig. 1.

We place a vector space V_i of dimension i at the vertex of the quiver labeled by i . We denote the n -dimensional framing vector space by V_n . We choose the stability condition given by the $G := \prod_{i=1}^{n-1} GL(V_i)$ -character

$$\theta : G \rightarrow \mathbb{C}^\times, \quad (g_i)_i \mapsto \prod_{i=1}^{n-1} (\det g_i)^{-1}$$

Let

$$\mu : T^* \left(\prod_{i=1}^{n-1} \text{Hom}(V_i, V_{i+1}) \right) \rightarrow \mathfrak{g}^*, \quad \mathfrak{g} = \text{Lie}(G)$$

be the moment map associated to the natural G -action. Points in the associated quiver variety

$$X := \mu^{-1}(0)^{\theta-ss} / G$$

are represented by tuples of maps

$$A_i : V_i \rightarrow V_{i+1}, \quad B_i : V_{i+1} \rightarrow V_i, \quad \text{for } i \in \{1, \dots, n-1\}$$

A collection of maps describes a θ -semistable if and only if the maps A_i are injective. The variety X is the cotangent bundle of the full flag variety.

2.2 Torus action

The action of $(\mathbb{C}^\times)^n$ on V_n induces an action of $(\mathbb{C}^\times)^n$ on X . The diagonal $\mathbb{C}^\times \subset (\mathbb{C}^\times)^n$ acts trivially on X , and we denote by A the quotient of $(\mathbb{C}^\times)^n$ by this subtorus. We denote the coordinates on $(\mathbb{C}^\times)^n$ by $(u_1, u_2, \dots, u_n)$ which means that coordinates on A are given by $a_i = u_i / u_{i+1}$ for $i \in \{1, \dots, n-1\}$.

An additional torus $\mathbb{C}_\hbar^\times$ acts on X by scaling the cotangent data, which is given by the maps $B_i : V_{i+1} \rightarrow V_i$ for $i \in \{1, \dots, n-1\}$, by $\hbar^{-1}$. We define

$$T := A \times \mathbb{C}_\hbar^\times$$

The torus $A \subset T$ preserves the symplectic form of X and T/A scales it with character $\hbar$.

The T -fixed points of X are indexed naturally by permutations $I = (I_1, \dots, I_n)$ of n . As remarked above, θ -semistability implies that the maps $A_i : V_i \rightarrow V_{i+1}$ are injective. We identify each of these vector spaces with a subspace of $V_n = \mathbb{C}^n$ and denote by $e_1, \dots, e_n$ the standard basis of $\mathbb{C}^n$. Then a fixed point indexed by I corresponds to a chain of vector spaces

$$V_1 \subset V_2 \subset \dots \subset V_n$$

such that $V_i/V_{i-1} = \text{Span}_{\mathbb{C}}\{e_{i_i}\}$.

2.3

The vector spaces V_i descend to bundles $\mathcal{V}_i$ on X . It is known from [15] that the tautological line bundles $\mathcal{L}_i = \det \mathcal{V}_i$ generate $\text{Pic}(X)$.

We define the Kähler torus of X by

$$K = \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}^{\times}$$

and write $\mathbf{z} = (z_1, \dots, z_{n-1})$ for the coordinates on it induced by the tautological line bundles. The variables z_i are usually referred to as Kähler parameters.

The real Lie algebra of K is

$$\text{Lie}_{\mathbb{R}}(K) := \text{cochar}(K) \otimes_{\mathbb{Z}} \mathbb{R} = \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

We denote elements of $\text{Lie}_{\mathbb{R}}(K)$ by s and call them slopes. We identify $\text{Lie}_{\mathbb{R}}(K)$ with $\mathbb{R}^{n-1}$ by the map

$$\sum_{i=1}^{n-1} \mathcal{L}_i \otimes r_i \mapsto (-r_1, \dots, -r_{n-1}) \in \mathbb{R}^{n-1} \quad (2)$$

The minus signs are chosen to match later notation.

2.4 Dual variety

The cotangent bundle of the full flag variety is known to be self dual with respect to 3d mirror symmetry, see [7, 26], and [10]. We denote by $X^!$ the same variety, constructed as a quiver variety in the same way. For $X^!$ we likewise have tori $T^!$, $A^!$, and $K^!$.

2.5 Polarization and chamber

Stable envelopes depend on a choice of polarization and chamber. We explain the meaning of these here.

A polarization of X is the choice of a K -theory class $T^{1/2}X$ so that the tangent bundle decomposes as

$$TX = T^{1/2}X + \hbar^{-1}(T^{1/2}X)^{\vee} \in K_{\mathbb{T}}(X)$$

We fix the polarization of X , given in terms of the tautological bundles $\mathcal{V}_i, i = 1, \dots, n$ by

$$T^{1/2}X = \sum_{i=1}^{n-1} \mathcal{V}_i^{\vee} \otimes \mathcal{V}_{i+1} - \sum_{i=1}^{n-1} \mathcal{V}_i^{\vee} \otimes \mathcal{V}_i \in K_{\mathbb{T}}(X) \quad (3)$$

In terms of the Chern roots $x_1^{(i)}, \dots, x_i^{(i)}$ of $\mathcal{V}_i$, this is given by

$$T^{1/2}X = \sum_{i=1}^{n-1} \left(\sum_{j=1}^i \frac{1}{x_j^{(i)}} \right) \left(\sum_{k=1}^{i+1} x_k^{(i+1)} \right) - \sum_{i=1}^{n-1} \left(\sum_{j=1}^i \frac{1}{x_j^{(i)}} \right) \left(\sum_{k=1}^i x_k^{(i)} \right) \in K_T(X)$$

At the fixed point given by a permutation I of $\{1, 2, \dots, n\}$, the tangent space can be calculated by substituting $x_j^{(i)} = u_{I_j}$. So

$$\begin{aligned} T_I^{1/2}X &= \sum_{i=1}^{n-1} \sum_{j=1}^i \sum_{k=1}^{i+1} \frac{u_{I_k}}{u_{I_j}} - \sum_{i=1}^{n-1} \sum_{j=1}^i \sum_{k=1}^i \frac{u_{I_k}}{u_{I_j}} \\ &= \sum_{1 \leq j < k \leq n} \frac{u_{I_k}}{u_{I_j}} \in K_T(pt) \end{aligned}$$

and

$$T_I X = \sum_{1 \leq j < k \leq n} \frac{u_{I_k}}{u_{I_j}} + \hbar^{-1} \sum_{1 \leq j < k \leq n} \frac{u_{I_j}}{u_{I_k}} \in K_T(pt)$$

2.6 Chamber

A chamber is a choice of connected component of

$$\mathrm{Lie}_{\mathbb{R}} \mathbf{A} - \bigcup_w \{ \sigma \in \mathrm{Lie}_{\mathbb{R}} \mathbf{A} \mid \langle \sigma, w \rangle = 0 \} \quad (4)$$

where the union is taken over all $\mathbf{A}$ -weights of the tangent spaces at the fixed points and $\langle \cdot, \cdot \rangle$ is induced by the natural pairing on characters and cocharacters. A choice of generic cocharacter σ of $\mathbf{A}$ gives a chamber $\mathfrak{C}$, and the dependence of the chamber on the cocharacter is locally constant. The tangent space at a fixed point decomposes into a direct sum of T -weight spaces

$$T_I X = \bigoplus_{w \in \mathrm{Hom}(T, \mathbb{C}^\times)} V_I(w)$$

A choice of chamber given by a cocharacter σ decomposes the tangent space at a fixed point I into attracting and repelling directions:

$$T_I X = N_I^+ + N_I^-$$

where

$$N_I^+ = \bigoplus_{\substack{w \\ \langle \sigma, w \rangle > 0}} V_I(w)$$

$$N_I^- = \bigoplus_{\substack{w \\ \langle \sigma, w \rangle < 0}} V_I(w)$$

Here, σ is viewed as a cocharacter of T via the inclusion $A \subset T$. Since the fixed point set is finite and σ is chosen to lie in (4), every direction is either attracting or repelling.

In the case of the cotangent bundle of the full flag variety, we fix once and for all the cocharacter

$$\sigma : u \mapsto (u^{-1}, u^{-2}, \dots, u^{-n}), \quad u \in \mathbb{C}^\times \quad (5)$$

and denote the corresponding chamber as $\mathcal{C}$. With respect to this chamber, attracting weights look like u_i/u_j where $i < j$, or equivalently, like monomials with positive powers in $a_i := u_i/u_{i+1}$. Explicitly, we have

$$T_I X = N_I^+ + N_I^-$$

where

$$N_I^- = \sum_{\substack{1 \leq j < k \leq n \\ I_k > I_j}} \frac{u_{I_k}}{u_{I_j}} + \hbar^{-1} \sum_{\substack{1 \leq j < k \leq n \\ I_k < I_j}} \frac{u_{I_j}}{u_{I_k}}$$

$$N_I^+ = \sum_{\substack{1 \leq j < k \leq n \\ I_k < I_j}} \frac{u_{I_k}}{u_{I_j}} + \hbar^{-1} \sum_{\substack{1 \leq j < k \leq n \\ I_k > I_j}} \frac{u_{I_j}}{u_{I_k}} \quad (6)$$

2.7 Ordering on fixed points

Given a permutation $I = (I_1, \dots, I_n)$ of n , we define the ordered indices $i_1^{(k)}, \dots, i_k^{(k)}$ so that

$$\{i_1^{(k)} < \dots < i_k^{(k)}\} = \{I_1, \dots, I_k\}$$

Definition 1 For permutations I and J with ordered indices $i_m^{(k)}$ and $j_m^{(k)}$, we define

$$I < J \iff i_m^{(k)} < j_m^{(k)} \text{ for all } k = 1, \dots, n-1 \text{ and } m = 1, \dots, k \quad (7)$$

The partial order $<$ coincides with the partial order on fixed points given by attraction in Sect. 3.1.2 of [1]. In what follows, we will also denote by $<$ an arbitrary refinement of this partial order to a total order.

3 3d mirror symmetry

3.1 Exchange of equivariant and Kähler parameters

One property of 3d mirror symmetry, see [12] and [28], is the existence of a bijection

$$X^T \longleftrightarrow X^{\dagger T^!}$$

and an isomorphism of tori

$$\kappa : T \times K \times \mathbb{C}_q^\times \rightarrow T^! \times K^! \times \mathbb{C}_q^\times$$

that induces isomorphisms $A \cong K^!$ and $K \cong A^!$. The torus $\mathbb{C}_q^\times$ acts on the domain of quasimaps to X and $X^!$, as will be explained in Sect. 4.1 below.

Following [7], we define the bijection on fixed points by

$$I \longleftrightarrow I^{-1} =: I^!$$

and the map κ by

$$\begin{aligned} z_i &\mapsto \hbar^! a_i^! \\ a_i &\mapsto \frac{\hbar^!}{q} z_i^! \\ \hbar &\mapsto \frac{q}{\hbar^!} \\ q &\mapsto q \end{aligned} \tag{8}$$

The differential $d\kappa$ induces isomorphisms

$$\mathrm{Lie}_{\mathbb{R}}(A) \cong \mathrm{Lie}_{\mathbb{R}}(K^!), \quad \mathrm{Lie}_{\mathbb{R}}(K) \cong \mathrm{Lie}_{\mathbb{R}}(A^!) \tag{9}$$

We obtain an induced map

$$\mathbb{C}(\hbar, q)((\mathbf{a}))((z)) \rightarrow \mathbb{C}(\hbar^!, q)((z^!))((\mathbf{a}^!)) = \mathbb{C}(\hbar^!, q)((\mathbf{a}^!))((z^!)) \tag{10}$$

We abuse notation and also denote (10) by κ . We note that taking the Laurent series expansion of a rational function in $\mathbf{a}$ around the point $a_i = 0$ identifies $K_{T \times \mathbb{C}_q^\times}(pt)_{loc}[[z]]$ as a subset of the domain of (10).

3.2 Walls and resonances

The K -theoretic stable envelopes depend on a choice of slope $s \in \mathrm{Lie}_{\mathbb{R}}(K)$. The dependence is locally constant, and the K -theoretic stable envelopes change only

when s crosses the walls of a certain hyperplane arrangement, which we denote by $\text{Wall}(X)$.

From (2) and (9) we have

$$s = (s_1, \dots, s_{n-1}) \in \mathbb{R}^{n-1} \cong \text{Lie}_{\mathbb{R}}(\mathbf{K}) \cong \text{Lie}_{\mathbb{R}}(\mathbf{A}^!)^{\dagger}$$

and we write

$$\begin{aligned} zq^s &= (z_1q^{s_1}, \dots, z_{n-1}q^{s_{n-1}}) \\ a^!q^s &= (a_1^!q^{s_1}, \dots, a_{n-1}^!q^{s_{n-1}}) \end{aligned}$$

To each element $w \in \text{Lie}_{\mathbb{R}}(\mathbf{A})$ with $w = (w_1, \dots, w_{n-1})$, we let v_w be the cyclic subgroup of $\mathbf{A}$ generated by

$$(e^{2\pi i w_1}, \dots, e^{2\pi i w_{n-1}}) \in \mathbf{A} \quad (11)$$

We define

$$\text{Res}(X) = \{w \in \text{Lie}_{\mathbb{R}}(\mathbf{A}) \mid X^{v_w} \neq X^{\mathbf{A}}\}$$

In [11], it is shown that

$$\text{Res}(X) = \{w \mid \langle \alpha, w \rangle + m = 0 \text{ for some } m \in \mathbb{Z}, I \in X^{\mathbf{A}}, \alpha \in \text{char}_{\mathbf{A}}(T_I X)\}$$

Under the identification (9), the walls and resonances are exchanged ([12] Theorem 2):

$$\text{Wall}(X) \cong \text{Res}(X^!), \quad \text{Res}(X) \cong \text{Wall}(X^!)$$

A simple calculation shows that under (2), the walls are given by

$$\text{Wall}(X) = \{(s_1, \dots, s_{n-1}) \mid \exists I \subset \{1, \dots, n-1\}, \sum_{i \in I} s_i \in \mathbb{Z}\}$$

We use the term generic slopes to refer to elements of $\text{Lie}_{\mathbb{R}}(\mathbf{K}) \setminus \text{Wall}(X)$.

4 Index vertex

In this section, we define one of the objects involved in our main theorem: the index vertex. We are interested in it for $X^!$.

4.1 Quasimaps

Let $[x : y]$ denote homogeneous coordinates on $\mathbb{P}^1$. We denote

$$0 = [0 : 1], \quad \infty = [1 : 0]$$

The torus $\mathbb{C}_q^\times$ acts on $\mathbb{P}^1$ by

$$q \cdot [x_0 : x_1] = [x_0 q : x_1], \quad q \in \mathbb{C}^\times \quad (12)$$

For $p \in (X^!)^{\mathsf{T}^!}$, let QM_p be the moduli space of stable quasimaps from $\mathbb{P}^1$ to $X^!$ that take the value p at ∞ , see [6] and [21] Sect. 4. The data of a stable quasimap to $X^!$ provides:

- Vector bundles $\mathcal{V}_i$ for $i \in \{1, \dots, n-1\}$ over $\mathbb{P}^1$ such that $\mathrm{rank}(\mathcal{V}_i) = i$.
- A trivial vector bundle $\mathcal{V}_n$ over $\mathbb{P}^1$ of rank n .
- A section

$$f \in H^0(\mathbb{P}^1, \mathcal{M} \oplus (\hbar^!)^{-1}(\mathcal{M})^\vee), \quad f(\infty) = p$$

where

$$\mathcal{M} = \bigoplus_{i=1}^{n-1} \mathrm{Hom}(\mathcal{V}_i, \mathcal{V}_{i+1})$$

such that the section f lands in the GIT stable locus for all but finitely many points of $\mathbb{P}^1$. We abuse notation by writing a quasimap as f , with the understanding that the data of the vector bundles are included.

The degree of a quasimap is given by

$$\deg f = (\deg \mathcal{V}_1, \dots, \deg \mathcal{V}_{n-1})$$

which gives a decomposition

$$\mathrm{QM}_p = \bigsqcup_d \mathrm{QM}_p^d$$

of the quasimap moduli space into components corresponding to quasimaps with fixed degree d . It is known that the degrees for which QM_p^d is nonempty lie inside a certain cone, see [21] Sect. 7.2.

The actions of $\mathsf{T}^!$ on $X^!$ and of $\mathbb{C}_q^\times$ on $\mathbb{P}^1$ induce an action on quasimaps. We denote $\mathsf{T}_q^! = \mathsf{T}^! \times \mathbb{C}_q^\times$.

4.2 Virtual classes on $\mathbf{QM}_p$

It is known that $\mathbf{QM}_p$ has a perfect obstruction theory, which allows one to define the symmetrized virtual structure sheaf $\widehat{\mathcal{O}}_{\text{vir}}$ and virtual tangent space $\mathcal{T}_{\text{vir}}$, see [6] Theorem 7.2.2 and [21] Sect. 6.

Fix $f \in (\mathbf{QM}_p)^{\mathbb{T}_q^1}$. As a $\mathbb{T}_q^1$ -module, the virtual tangent space of $\mathbf{QM}_p$ at f is

$$\mathcal{T}_{\text{vir},f} = H^*\left(\mathbb{P}^1, \mathcal{T}^{1/2} + \hbar^{-1}(\mathcal{T}^{1/2})^\vee\right)$$

where

$$\mathcal{T}^{1/2} = \bigoplus_{i=1}^{n-1} \text{Hom}(\mathcal{V}_i, \mathcal{V}_{i+1}) - \bigoplus_{i=1}^{n-1} \text{End}(\mathcal{V}_i) \quad (13)$$

The reduced virtual tangent space at f is defined to be

$$\mathcal{T}_{\text{vir},f}^{\text{red}} = \mathcal{T}_{\text{vir},f} - T_p X^! \quad (14)$$

The class $\mathcal{T}^{1/2}$ induces a polarization $\mathcal{T}^{1/2}$ of the reduced virtual tangent space at a fixed quasimap f :

$$\mathcal{T}_{\text{vir},f}^{\text{red}} = \mathcal{T}^{1/2} + \hbar^{-1}(\mathcal{T}^{1/2})^\vee \quad \text{where} \quad \mathcal{T}^{1/2} = H^*(\mathbb{P}^1, \mathcal{T}^{1/2}) - T_p^{1/2} X^! \quad (15)$$

4.3 Index vertex

Let $s \in \text{Lie}_{\mathbb{R}}(A^!) \setminus \text{Res}(X^!)$ and denote the substitution $a_i^! = q^{s_i}$ by $\mathbf{a}^! = q^s$. Such an s induces a decomposition

$$\mathcal{T}_{\text{vir},f}^{\text{red}}|_{\mathbf{a}^!=q^s, \hbar^!=q} = \mathcal{T}_{s,+} + \mathcal{T}_{s,-}$$

where $\mathcal{T}_{s,+}$ (resp. $\mathcal{T}_{s,-}$) consists of the terms containing positive (resp. negative) powers of q . The subtraction of the term $T_p X^!$ in (14) and the substitution $\hbar^! = q$ together ensure that $\mathcal{T}_{\text{vir},f}^{\text{red}}|_{\hbar^!=q}$ contains no monomials of power 0 in q . By definition of $\text{Res}(X^!)$, the same will thus be true for $\mathcal{T}_{\text{vir},f}^{\text{red}}|_{\mathbf{a}^!=q^s, \hbar^!=q}$. Hence $\mathcal{T}_{s,\pm}$ are well-defined.

Definition 2 Let $s \in \text{Lie}_{\mathbb{R}}(A^!) \setminus \text{Res}(X^!)$. The s -index of $f \in (\mathbf{QM}_p)^{\mathbb{T}_q^1}$ is

$$\mathcal{I}_s(f) = \text{rank}(\mathcal{T}_{s,-})$$

where we understand the rank of a virtual $\mathbb{T}_q^1$ -bundle to be counted with sign.¹

¹ In other words, if A and B are $\mathbb{T}_q^1$ -modules, then $\text{rk}(A - B) = \text{rank}(A) - \text{rank}(B)$.

Definition 3 ([21] Sect. 7.3) The index vertex of $X^\dagger$ at $p \in (X^\dagger)^\dagger$ with respect to $s \in \text{Lie}_{\mathbb{R}}(A^\dagger) \setminus \text{Res}(X^\dagger)$ is the generating function

$$\text{Ind}_p^{!s} = \sum_{\substack{\mathbf{d} \\ \text{QM}_p^{\mathbf{d}} \neq \emptyset}} (z^\dagger)^{-\mathbf{d}} \sum_{f \in (\text{QM}_p^{\mathbf{d}})^{\dagger q}} \left(\frac{\hbar^\dagger}{q} \right)^{\frac{1}{2} \mathcal{I}_s(f) - |\mathbf{d}|}$$

where $(z^\dagger)^{-\mathbf{d}} = \prod_{i=1}^{n-1} (z_i^\dagger)^{-d_i}$ and $|\mathbf{d}| = \sum_{i=1}^{n-1} d_i$.

The choice of $-\mathbf{d}$ in the definition of the index vertex is made for the sake of consistency with our other conventions from [7], which will be used below.

4.4 Vertex function

There is an alternative perspective on the index vertex which describes it as a limit of the so-called vertex function for $X^\dagger$. Since we will need the vertex function for both X and $X^\dagger$, we briefly switch our variables back to those of X . The vertex function for $X^\dagger$ is given by the same formula as below after the trivial change of variables

$$\hbar \rightarrow \hbar^\dagger, \quad u_i \rightarrow u_i^\dagger, \quad \text{and} \quad z_i \rightarrow z_i^\dagger$$

Definition 4 We define $C \subset \mathbb{Z} \times \mathbb{Z}^2 \times \dots \times \mathbb{Z}^{n-1}$ as the collection of integers $d_{i,j}$ where $i \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, i\}$ such that

- $d_{i,j} \geq 0$ for all i, j .
- For each $i \in \{1, \dots, n-2\}$, there exists $\{j_1, \dots, j_i\} \subset \{1, \dots, i+1\}$ so that $d_{i,k} \geq d_{i+1,j_k}$ for all k .

The vertex function of X is defined by an equivariant count of quasimaps from $\mathbb{P}^1$ to X , see Theorem 1 below. To streamline our presentation here, we define the vertex function through a formula.

Definition 5 The vertex function of the cotangent bundle of the full flag variety restricted to a fixed point I is given by the following power series:

$$V_I(\mathbf{a}, \mathbf{z}) = \sum_{d_{i,j} \in C} \mathbf{z}^{\mathbf{d}} \prod_{i=1}^{n-2} \prod_{j=1}^i \prod_{k=1}^{i+1} \frac{\left(\hbar \frac{u_{I_k}}{u_{I_j}} \right)_{d_{i,j}-d_{i+1,k}}}{\left(q \frac{u_{I_k}}{u_{I_j}} \right)_{d_{i,j}-d_{i+1,k}}} \\ \prod_{i=1}^{n-1} \prod_{j,k=1}^i \frac{\left(q \frac{u_{I_k}}{u_{I_j}} \right)_{d_{i,j}-d_{i,k}}}{\left(\hbar \frac{u_{I_k}}{u_{I_j}} \right)_{d_{i,j}-d_{i,k}}} \prod_{i=1}^n \prod_{j=1}^{n-1} \frac{\left(\hbar \frac{u_i}{u_{I_j}} \right)_{d_{n-1,j}}}{\left(q \frac{u_i}{u_{I_j}} \right)_{d_{n-1,j}}}$$

where $z^d = \prod_{i=1}^{n-1} \prod_{j=1}^i z_i^{d_{i,j}}$ and $(x)_d$ denotes the q -Pochhammer symbol

$$(x)_d := \frac{\varphi(x)}{\varphi(xq^d)}, \quad \varphi(x) = \prod_{i=0}^{\infty} (1 - xq^i)$$

In what follows, we will need the roof function, which is defined by

$$\hat{a}(t) = \frac{1}{t^{1/2} - t^{-1/2}}, \quad \hat{a}(t_1 + t_2 - t_3) = \frac{\hat{a}(t_1)\hat{a}(t_2)}{\hat{a}(t_3)} \quad (16)$$

The geometric meaning of the vertex function is given by the following theorem. For precise definitions, see [21] Sect. 7.

Theorem 1 ([13] Theorem 3.1) *Let $\widehat{\mathcal{O}}_{\text{vir}}^d$ be the symmetrized virtual structure sheaf on QM_I^d . The vertex function $V_I(\mathbf{a}, \mathbf{z})$ in Definition 5 is equal to*

$$\frac{1}{\hat{a}(T_I X)} \sum_{\substack{d \\ QM_I^d \neq \emptyset}} (z^\#)^{-d} \chi(\widehat{\mathcal{O}}_{\text{vir}}^d) \in K_{T_q}(pt)_{\text{loc}}[[z]] \quad (17)$$

where χ denotes the K -theoretic equivariant Euler characteristic and

$$(z^\#)^{-d} = \prod_{i=1}^{n-1} (z_i^\#)^{-d_i} \quad \text{where} \quad z_i^\# = \hbar^{a_i/2} q^{b_i/2} z_i$$

for some $a_i, b_i \in \mathbb{Z}$.

Remark 1 The $\hat{a}$ prefactor is equivalent to replacing the virtual tangent space by the reduced virtual tangent space in localization formulas, which causes the series to start with 1.

Remark 2 The precise form of the shift $z_i^\# = \hbar^{a_i/2} q^{b_i/2} z_i$ is described in the proof of Proposition 2 below.

4.5 Index limit

In this subsection, we switch back to the variables for $X^!$.

Definition 6 The index limit of an element of $F(\mathbf{a}^!, \mathbf{z}^!) \in K_{T_q^!}(pt)[[z^!]]$ with respect to $s \in \text{Lie}_{\mathbb{R}}(\mathbf{A}^!)$ is defined to be

$$\kappa \left(\lim_{q \rightarrow 0} \kappa^{-1} \left(F(\mathbf{a}^!, \mathbf{z}^!) \right) \Big|_{z=zq^{s+1}} \right)$$

provided this limit exists. Here, $zq^{s+1} = (z_1 q^{s_1+1}, \dots, z_{n-1} q^{s_{n-1}+1})$.

Remark 3 The reason we shift by q^{s+1} instead of q^s is for consistency with the index vertex, see Proposition 2 below.

Remark 4 It may appear more natural to consider the limit

$$\lim_{q \rightarrow 0} V_{I^!}^!(\mathbf{a}^! q^s, \mathbf{z}^!)$$

However, this limit will not exist for vertex functions. For example, the vertex function of one of the fixed points of $X^! = T^*\mathbb{P}^1$ is

$$V(\mathbf{a}^!, \mathbf{z}^!) = \sum_{d=0}^{\infty} \frac{(\hbar^!)_d (\hbar^! u_1^! / u_2^!)_d}{(q)_d (q u_1^! / u_2^!)_d} z_1^{!d}$$

and one can easily check that the limit

$$\lim_{q \rightarrow 0} V(\mathbf{a}^! q^s, \mathbf{z}^!)$$

does not exist. On the other hand, the index limit does exist. By Proposition 1 below, this is also the case in general.

Remark 5 The index limit with respect to s could also be defined by formally substituting $\hbar^! = q/t$, shifting $\mathbf{a}^! \rightarrow \mathbf{a}^! q^s$, taking the limit $q \rightarrow 0$, and substituting back $t = q/\hbar^!$.

Proposition 1 *The index limit of the vertex function restricted to any fixed point exists for all s .*

Proof The coefficients of the vertex function consist of terms of the form

$$\left(\frac{(\hbar^! x)_d}{(q x)_d} \right)^{\pm 1}$$

where x is a character of $A^!$ and $d \in \mathbb{Z}$. Using Remark 5, one can see that the limit of all such terms exists. $\square$

Proposition 2 *For $s \in \text{Lie}_{\mathbb{R}}(A^!) \setminus \text{Res}(X^!)$, the index limit with respect to s of the vertex function of $X^!$ restricted to $I^!$ is equal to the index vertex at $I^!$:*

$$\kappa \left(\lim_{q \rightarrow 0} \kappa^{-1} \left(V_{I^!}^!(\mathbf{a}^!, \mathbf{z}^!) \right) \Big|_{\mathbf{z} \rightarrow \mathbf{z} q^{s+1}} \right) = \text{Ind}_{I^!}^{!s}$$

Proof Up to normalization, the vertex function of $X^!$ is equal to the generating function of the equivariant Euler characteristic of $\text{QM}_{I^!}^d$, as in Theorem 1. We will compute this here using equivariant localization.

Table 1 Contributions of $\mathcal{T}^{1/2}$ to $\mathcal{F}_{\text{vir},f}^{\text{red}}$

Virtual sub-bundle of $\mathcal{T}^{1/2}$	Contribution to $\mathcal{F}_{\text{vir},f}^{\text{red}}$
$\pm x q^d \mathcal{O}(d) \pm (\hbar^!)^{-1} x^{-1} q^{-d} \mathcal{O}(-d)$	$\pm x q (1 + q + \dots + q^{d-1}) \mp \frac{1}{x \hbar^!} (1 + q^{-1} + \dots + q^{-d+1})$
$\pm x q^{-d} \mathcal{O}(-d) \pm (\hbar^!)^{-1} x^{-1} q^d \mathcal{O}(d)$	$\pm \frac{q}{x \hbar^!} (1 + q + \dots + q^{d-1}) \mp x (1 + q^{-1} + \dots + q^{-d+1})$

At a fixed point $f \in (\text{QM}_{T^!}^d)^{T^!}_q$, the vector bundles $\mathcal{V}_i$ over $\mathbb{P}^1$ split into the sum of equivariant line bundles:

$$\mathcal{V}_i = \bigoplus_{j=1}^i w_{i,j} q^{d_{i,j}} \mathcal{O}(d_{i,j})$$

where $w_{i,j}$ stands for a character of $A^!$, see Sect. 4.5 of [23]. The injectivity of the maps $V_{i-1} \rightarrow V_i$ constrains the possible degrees of the bundles $\mathcal{V}_i$. In particular, the collection $\{-d_{i,j}\}$ of negatives of the degrees must lie in C . Hence we have

$$\mathcal{T}^{1/2} = \bigoplus_i x_i q^{a_i} \mathcal{O}(a_i) - \bigoplus_j y_j q^{b_j} \mathcal{O}(b_j)$$

where x_i and y_j are characters of $A^!$, a_i and b_j are integers, and the sums are taken over some indexing sets.

The possible virtual sub-bundles of $\mathcal{T}^{1/2}$ come in pairs:

$$\pm x q^{\pm d} \mathcal{O}(\pm d) \quad \text{and} \quad \pm (\hbar^!)^{-1} x^{-1} q^{\mp d} \mathcal{O}(\mp d)$$

which together contribute to the reduced virtual tangent space as in Table 1 (see [23] Lemma 1). We assume in Table 1 without loss of generality that $d \geq 0$.

By definition of the symmetrized virtual structure sheaf, these terms contribute to localization formula via the roof function (16). The contributions of each of these terms is given in Table 2.

The terms $(-q^{1/2} \hbar^{-1/2})^{\pm d}$ account for precisely the difference between z_i and $z_i^\#$ in Theorem 1.

Applying Remark 5, we see that the index limit can be interpreted as a count of certain monomials in

$$\mathcal{F}_{\text{vir},T^!}^{\text{red}}|_{a^!=q^s, \hbar^!=q}$$

Putting this together with Tables 1 and 2, we see that the index limit of the vertex function is equal to the index vertex. $\square$

We can now extend the definition of the index vertex to include non-generic slopes.

Table 2 Contributions in the localization formula

Virtual sub-bundle of $\mathcal{T}^{1/2}$	Contribution to (17)
$\pm x q^d \mathcal{O}(d) \pm (h^!)^{-1} x^{-1} q^{-d} \mathcal{O}(-d)$	$\left((-q^{1/2} (h^!)^{-1/2})^d \frac{(h^! x)_d}{(qx)_d} \right)^{\pm 1}$
$\pm x q^{-d} \mathcal{O}(-d) \pm (h^!)^{-1} x^{-1} q^d \mathcal{O}(d)$	$\left((-q^{1/2} (h^!)^{-1/2})^{-d} \frac{(h^! x)_{-d}}{(qx)_{-d}} \right)^{\pm 1}$

Definition 7 The index vertex of $X^!$ at $I^!$ with respect to $s \in \text{Lie}_{\mathbb{R}}(A^!)$ is defined to be the index limit of $V_{I^!}^!(a^!, z^!)$ with respect to s .

Thanks to Proposition 2, this definition agrees with Definition 3 for $s \in \text{Lie}_{\mathbb{R}}(A^!) \setminus \text{Res}(X^!)$ and extends it for $s \in \text{Res}(X^!)$.

5 Stable envelopes

In this section, we explain our conventions used for stable envelopes. The main references for stable envelopes are [1] and Sect. 9 of [21].

5.1 Notations

Given a T-module expressed in weights as $V = w_1 + \dots + w_r \in K_{\text{T}}(pt)$ with $w_i \neq 1$ for all i , we define the symmetric and exterior powers as

$$S^{\bullet}(V) := \prod_{i=1}^r (1 - w_i)^{-1}, \quad \bigwedge^{\bullet} V := \prod_{i=1}^r (1 - w_i)$$

We extend these to all of $K_{\text{T}}(pt)$ by

$$S^{\bullet}(-V) := \bigwedge^{\bullet} V, \quad \bigwedge^{\bullet}(-V) := S^{\bullet}(V)$$

Similarly, we define

$$\Theta(V) = \prod_{i=1}^r \vartheta(w_i) \quad \text{and} \quad \Phi(V) = \prod_{i=1}^r \varphi(w_i)$$

where

$$\varphi(x) = \prod_{i=0}^{\infty} (1 - xq^i) \quad \text{and} \quad \vartheta(x) = (x^{1/2} - x^{-1/2})\varphi(qx)\varphi(q/x)$$

We similarly extend these to $K_{\text{T}}(pt)$ by multiplicativity.

5.2 Stable envelopes

Stable envelopes depend on a choice of polarization and chamber. In all that follows, we assume that these are given by (3) and (5).

For $I \in X^A$, let $\text{Stab}_{-\mathfrak{C}}^{X, Ell}(I)$ be the elliptic stable envelopes of I for X corresponding to the polarization $T^{1/2}X$ and chamber $-\mathfrak{C}$.

The elliptic stable envelope of a fixed point gives a section of a line bundle over the extended elliptic cohomology scheme of X . This scheme can be described as

$$\text{Ell}_T(X) = \left(\bigsqcup_{I \in X^T} \widehat{\mathcal{O}}_I \right) / \Delta$$

where $\widehat{\mathcal{O}}_I$ is a product of elliptic curves isomorphic to $\mathbb{C}^\times / q^{\mathbb{Z}}$ for fixed q with $|q| < 1$ and Δ denotes a certain gluing of these abelian varieties, see [27] Sect. 2.13. Restricting this section to a component $\widehat{\mathcal{O}}_J$, one obtains the matrix of restrictions of the elliptic stable envelope:

$$T_{I,J}^X := \text{Stab}_{-\mathfrak{C}}^{X, Ell}(I)|_{\widehat{\mathcal{O}}_J}$$

We use the normalization of the elliptic stable envelope determined by

$$T_{I,I}^X = \Theta(N_I^+)$$

which differs from the normalization of [1] by a sign. We assume the fixed points are ordered from highest to lowest with respect to $\prec$ from Definition 1, which means that the matrix of restrictions is upper triangular. Explicit formulas for the elliptic stable envelopes in terms of the theta function ϑ can be written using the so-called elliptic weight functions, see [25] and Sect. 3 of [7]. More generally, explicit formulas for the elliptic stable envelopes of any type A quiver variety were written in [8].

We also define a normalized matrix of restrictions of the elliptic stable envelope by

$$\widetilde{T}_{I,J}^X = \frac{\text{Stab}_{-\mathfrak{C}}^{X, Ell}(I)|_{\widehat{\mathcal{O}}_J}}{\text{Stab}_{-\mathfrak{C}}^{X, Ell}(J)|_{\widehat{\mathcal{O}}_J}}$$

Similarly, we have K -theoretic stable envelopes $\text{Stab}_{-\mathfrak{C}}^{s, X, K}(I)$ that depend further on a choice of slope $s \in \text{Lie}_{\mathbb{R}}(K)$, see [21] Sect. 9 and [22] Sect. 2. We normalize them by requiring that the diagonal terms of the matrix of fixed point restrictions are given by

$$\text{Stab}_{-\mathfrak{C}}^{s, X, K}(I)|_I = \sqrt{\frac{\det N_I^+}{\det T_I^{1/2}X}} \bigwedge^\bullet (N_I^+)^{\vee}$$

We denote the matrix of restrictions of the K -theoretic stable envelopes by

$$A_{I,J}^{s,X} = \text{Stab}_{-\mathfrak{e}}^{s,X,K}(I)|_J$$

and a renormalized restriction matrix by

$$\tilde{A}_{I,J}^{s,X} = \frac{\text{Stab}_{-\mathfrak{e}}^{s,X,K}(I)|_J}{\text{Stab}_{-\mathfrak{e}}^{s,X,K}(J)|_J}$$

6 Index vertex and stable envelopes

This section contains our main theorems, which relate the index vertex of $X^!$ to the K -theoretic stable envelopes of X .

6.1 Big enough slopes

In what follows, we will also be interested in the index limits of $\kappa(V_I(\mathbf{a}, \mathbf{z}))$ and $\Phi((q - \hbar^!)N_{I^!}^{!+})$ in the sense of Definition 6. It will be convenient to have a notion of slopes for which these limits are trivial.

Definition 8 A slope $s \in \text{Lie}_{\mathbb{R}}(A^!)$ is said to be big enough if $s_i > 0$ for all i and

$$s_i + s_{i+1} + \cdots + s_j > j - i - 1$$

for all $j \geq i$.

Lemma 1 If $s \in \text{Lie}_{\mathbb{R}}(A^!)$ big enough, then

$$\lim_{q \rightarrow 0} V_I(\mathbf{a}, \mathbf{z}q^{s+1}) = 1$$

and

$$\kappa \left(\lim_{q \rightarrow 0} \kappa^{-1} \left(\Phi((q - \hbar^!)N_{I^!}^{!+}) \right) \Big|_{\mathbf{z} \rightarrow \mathbf{z}q^{s+1}} \right) = 1$$

for all permutations I .

Proof By (8), $\kappa^{-1}(a_i^!) = \frac{\hbar}{q} z_i$. By definition, $a_i^! = u_i^! / u_{i+1}^!$, and so $\kappa^{-1} \left(\frac{u_{I_j}^!}{u_{I_k}^!} \right) = \left(\frac{\hbar}{q} \right)^{I_k - I_j} z_{I_j} \cdots z_{I_{k-1}}$ if $I_j < I_k$. Now the result follows from Definition 5 and (6) by a straightforward computation. $\square$

6.2 Generic slopes

Let D be the diagonal matrix given by

$$D = \text{diag} \left(\sqrt{\frac{\det T_I^{1/2}}{\det N_I^+}} \right)_{I \in X^\tau} \quad (18)$$

Let $S^\bullet(\hbar \otimes N^+)$ be the column vector

$$S^\bullet(\hbar \otimes N^+) = (S^\bullet(\hbar \otimes N_I^+))_{I \in X^\tau}$$

and Ind^{ls} be the column vector

$$\text{Ind}^{ls} = (\text{Ind}_{I^!}^{ls})_{I \in X^\tau}$$

of index vertices for $X^!$. We remind the reader that $I^!$ is the inverse permutation of I .

Our main theorem describes the index vertex of $X^!$ for generic slopes in terms of the K -theoretic stable envelopes of X .

Theorem 2 *If $s \in \text{Lie}_{\mathbb{R}}(K) \setminus \text{Wall}(X)$ is big enough, then*

$$\text{Ind}^{ls} = \kappa \left(D \cdot \tilde{A}^{s+1, X} \cdot D^{-1} \cdot S^\bullet(\hbar \otimes N^+) \right)$$

Equivalently,

$$\text{Ind}_{I^!}^{ls} = \kappa \left(\sum_{J \in X^\tau} \sqrt{\frac{\det T_I^{1/2} X}{\det N_I^+}} \tilde{A}_{I, J}^{s+1, X} \sqrt{\frac{\det N_J^+}{\det T_J^{1/2} X}} S^\bullet(\hbar \otimes N_J^+) \right)$$

Proof In [7], we proved that

$$\begin{aligned} & \Phi((q - \hbar^!) N_{I^!}^{!+}) V_{I^!}^!(\mathbf{a}^!, \mathbf{z}^!) \\ &= \kappa \left(\sum_{J \in X^\tau} \sqrt{\frac{\det T_I^{1/2} X}{\det N_I^+}} \tilde{T}_{I, J}^X \sqrt{\frac{\det N_J^+}{\det T_J^{1/2} X}} \Phi((q - \hbar) N_I^+) V_J(\mathbf{a}, \mathbf{z}) \right) \end{aligned} \quad (19)$$

To deduce Theorem 2, we take the index limit of both sides with respect to s .

Lemma 1 implies that $\Phi((q - \hbar^!) N_{I^!}^{!+})$ and $V_J(\mathbf{a}, \mathbf{z})$ contribute a factor of 1 to the limit. The term

$$\sqrt{\frac{\det T_I^{1/2} X}{\det N_I^+}} \sqrt{\frac{\det N_J^+}{\det T_J^{1/2} X}}$$

does not depend on z or q . The limit of the normalized stable envelope $\tilde{T}_{I,J}^X$ was calculated in [12]. For $s \notin \text{Wall}(X)$, we have

$$\lim_{q \rightarrow 0} \tilde{T}_{I,J}^X|_{z=zq^{s+1}} = \tilde{A}_{I,J}^{s+1,X}$$

The contribution from $\Phi((q - \hbar)N_J^+)$ is given by

$$\prod_{w \in \text{char}_T(N_J^+)} (1 - \hbar w)^{-1} = S^\bullet(\hbar \otimes N_J^+)$$

Putting all this together gives the result. $\square$

The normalized K -theoretic stable envelopes are related to the usual ones by

$$A_{I,J}^{s+1,X} = \sqrt{\frac{\det N_J^+}{\det T_J^{1/2} X}} \bigwedge^\bullet (N_J^+)^{\vee} \tilde{A}_{I,J}^{s+1,X}$$

Also,

$$N_J^+ = \hbar^{-1} (N_J^-)^{\vee} \implies S^\bullet(\hbar \otimes N_J^+) = S^\bullet((N_J^-)^{\vee})$$

So Theorem 2 is equivalent to

$$\text{Ind}_{I!}^{ls} = \kappa \left(\sum_{J \in X^T} \sqrt{\frac{\det T_I^{1/2} X}{\det N_I^+}} A_{I,J}^{s+1,X} S^\bullet(T_J X^{\vee}) \right)$$

By equivariant localization, the right side of this equation is the K -theoretic equivariant Euler characteristic of the K -theoretic stable envelope of I twisted by a line bundle:

Theorem 3

$$\text{Ind}_{I!}^{ls} = \kappa \left(\chi \left(\sqrt{\frac{\det T_I^{1/2} X}{\det N_I^+}} \text{Stab}_{-\mathfrak{e}}^{s+1,X,K}(I) \right) \right)$$

As a consequence, we obtain:

Corollary 1 *For big enough generic slopes, the index vertex $\text{Ind}_{I!}^{ls}(\mathbf{a}, \mathbf{z})$ is a rational function of $\mathbf{a}^!$, $\mathbf{z}^!$, and $\sqrt{\hbar^!}$.*

6.3 Non-generic slopes

The limits of the elliptic stable envelopes for non-generic slopes is one of the main results of [12]. Under the identification given by κ , a slope $s \in \text{Lie}_{\mathbb{R}}(K)$ can be viewed as an element of $\text{Lie}_{\mathbb{R}}(A^!)$, which has an associated cyclic subgroup $v_s \subset A^!$ as in (11). In particular, if $s \in \text{Wall}(X) = \text{Res}(X^!)$, then we obtain a subvariety

$$Y_s := (X^!)^{v_s} \neq (X^!)^{A^!}$$

The limits of the elliptic stable envelopes for non-generic slopes is expressed in terms of the K -theoretic stable envelopes of Y_s . We need to explain the choice of chamber, polarization, and slope used in the latter.

We first explain the choice of slope. Let $\mathcal{U}_0$ denote an open analytic neighborhood of 0 in $\text{Lie}_{\mathbb{R}}(K^!)$. Let $\text{Wall}_0(X^!)$ be the set of walls passing through 0. Then $\mathcal{U}_0 \setminus \text{Wall}_0(X^!)$ is the disjoint union of connected components. Let $\mathfrak{D}_+(X^!)$ denote the connected component containing ample line bundles on $X^!$. Explicitly, $\mathfrak{D}_+(X^!)$ is generated by small positive real multiples of the Chern classes of the tautological line bundles $\mathcal{L}_i^!$ for $i \in \{1, \dots, n-1\}$.

The inclusion

$$\iota : Y_s \rightarrow X^!$$

induces a map

$$\iota^* : \text{Pic}(X^!) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \text{Pic}(Y_s) \otimes_{\mathbb{Z}} \mathbb{R}$$

Define $\mathfrak{D}_+(Y_s) = \iota^*(\mathfrak{D}_+(X^!))$. Slopes in $\mathfrak{D}_+(Y_s)$ will be the right choice for the stable envelopes of Y_s .

Since the torus $A^!$ acts on Y_s , the chamber $-\mathfrak{C}$ automatically gives a chamber for the stable envelopes of Y_s .

The v_s -invariant part of the polarization for $X^!$ gives a polarization $(T^{1/2}X^!)^{v_s}$ of Y_s .

With respect to these choices, we can now talk about the K -theoretic stable envelopes of Y_s .

Theorem 4 *Let $s \in \text{Wall}(X)$ be big enough and let $\epsilon \in \mathfrak{D}_+(X)$ be a small ample slope of X such that $s' = s + \epsilon$ is a generic slope. Then*

$$\text{Ind}^{!s} = \kappa \left(D \cdot H \cdot \tilde{A}^{\mathfrak{D}_+(Y_{s+1}), Y_{s+1}} \cdot H^{-1} \cdot \tilde{A}^{s', X} \cdot D^{-1} \cdot S^{\bullet}(\hbar \otimes N^+) \right)$$

where H is the diagonal matrix given by

$$H = \text{diag} \left((-1)^{\gamma_I(s)} \hbar^{m_I(s)/2} \prod_{i=1}^{n-1} \mathcal{L}_i|_I \right)_{I \in X^T}$$

where $\gamma_I(s)$ and $m_I(s)$ are integers.

Fig. 2 The quiver data for the variety $T^*\mathbb{P}^1$ 

Proof From Theorem 3 and Theorem 5 in [12], the twisted limit of the restriction matrix of the elliptic stable envelope is

$$\lim_{q \rightarrow 0} \tilde{T}^X|_{z=zq^{s+1}} = \tilde{Z}' \cdot \tilde{A}^{s'+1, X}$$

Here, the matrix $\tilde{Z}'$ is given by

$$\tilde{Z}' = H \cdot \tilde{A}^{\mathfrak{D}+(Y_{s+1}), Y_{s+1}} \cdot H^{-1}$$

where H is the diagonal matrix

$$H = \text{diag} \left((-1)^{\gamma_I(s)} \hbar^{m_I(s)/2} \left(\prod_{i=1}^{n-1} \mathcal{L}_i|_I \right) \right)_{I \in X^\Gamma}$$

for integers $\gamma_I(s)$ and $m_I(s)$. For the precise description of these integers, see Theorem 2 in [11] and Theorem 4 in [12].

The limits of the rest of the terms in (19) are the same as in Theorem 2, which finishes the proof. $\square$

Contrary to the case of generic s , the right hand side of Theorem 4 does not admit a nice interpretation as an equivariant Euler characteristic of a simple twist of the K -theoretic stable envelope of X . However, we still have:

Corollary 2 *For any big enough $s \in \text{Lie}_{\mathbb{R}}(K)$, the index vertex $\text{Ind}_I^{!s}$ is a rational function of $\mathbf{a}^!$, $\mathbf{z}^!$, and $\sqrt{\hbar^!}$.*

7 Example: $T^*\mathbb{P}^1$

We work out our main result for generic slopes explicitly here in the simplest example: $X = T^*\mathbb{P}^1$. This corresponds to the quiver shown in Fig. 2.

Let $V = \mathbb{C}$ and $W = \mathbb{C}^2$. Points in X are represented by maps

$$(I, J) \in \text{Hom}(W, V) \oplus \text{Hom}(V, W)$$

such that I is injective and $I \circ J = 0$, modulo the action of $GL(1)$ on V . The torus $(\mathbb{C}^\times)^2$ acts on $\mathbb{C}^2$ by

$$(u_1, u_2) \cdot (x_1, x_2) = (u_1^{-1}x_1, u_2^{-1}x_2)$$

which induces an action of $A = \mathbb{C}^\times$, where the coordinate on this torus is $a = u_1/u_2$.

The torus $\mathbb{C}_\hbar^\times$ acts by

$$\hbar \cdot (I, J) = (\hbar^{-1}I, J)$$

We denote $T = A \times \mathbb{C}_\hbar^\times$. There are two T -fixed points p_1 and p_2 , which are labeled by the identity permutation and the simple transposition, respectively. The vector space V descends to a line bundle $\mathcal{V}$ on X and the T -character of this tautological bundle at the fixed point p_i is u_i .

We denote the dual copy of $T^*\mathbb{P}^1$ as $X^!$ and denote the torus as $T^! = A^! \times \mathbb{C}_{\hbar^!}^\times$. The exchange of equivariant and Kähler parameters is given by (8) and the bijection on fixed points is the identity map.

7.1 Torus fixed quasimaps

A quasimap $f \in \mathcal{QM}_{p_i}^d$ to $X^!$ consists of a degree d line bundle $\mathcal{V}$ and a trivial rank 2 vector bundle $\mathcal{W}$ over $\mathbb{P}^1$, along with a section

$$f \in H^0(\mathbb{P}^1, \mathcal{M} \oplus (\hbar^!)^{-1}(\mathcal{M})^\vee)$$

where

$$\mathcal{M} = \text{Hom}(\mathcal{V}, \mathcal{W})$$

Assume the quasimap is $T_q^!$ -fixed. Equivariantly with respect to the torus $T^!$, we have

$$\mathcal{W} = u_1^! \mathcal{O}_{\mathbb{P}^1} \oplus u_2^! \mathcal{O}_{\mathbb{P}^1}$$

where $w\mathcal{O}_{\mathbb{P}^1}$ denotes a twist of the structure sheaf of $\mathbb{P}^1$ by a trivial line bundle with weight w . Since the section f takes the value p_i at ∞ , in order f to be $T^!$ -fixed, $\mathcal{V}$ must have $T^!$ -weight $u_i^!$. Furthermore, the component of the section in

$$\text{Hom}(\mathcal{V}, u_i^! \mathcal{O}_{\mathbb{P}^1}) \cong \mathcal{O}(-d)$$

must be nonzero at ∞ . So $d \leq 0$ and there is only one such section, which is given in homogeneous coordinates by x_0^{-d} . To be invariant under $\mathbb{C}_q^\times$, the bundle $\mathcal{V}$ must be further twisted by the trivial line bundle with $\mathbb{C}_q^\times$ -weight of q^{-d} . This completely determines the quasimap.

To summarize, there is a unique $T_q^!$ -fixed quasimap in $\mathcal{QM}_{p_i}^d$ for each $d \leq 0$, and the data is as follows:

- $\mathcal{W} = u_1^! \mathcal{O}_{\mathbb{P}^1} \oplus u_2^! \mathcal{O}_{\mathbb{P}^1}$
- $\mathcal{V} = u_i^! q^{-d} \mathcal{O}(-d)$
- $f = x_0^{-d}$

7.2 Virtual tangent space

Fix $d \leq 0$ and let $\mathcal{W}$, $\mathcal{V}$, and f_d be the unique T_q^1 -fixed quasimap of degree d as above. We have the induced virtual bundle from (13):

$$T^{1/2} = \mathcal{V}^\vee \otimes \mathcal{W} - \mathcal{V}^\vee \otimes \mathcal{V} = q^{-d} \mathcal{O}(-d) + u^! q^{-d} \mathcal{O}(-d) - \mathcal{O}_{\mathbb{P}^1} \quad (20)$$

where

$$u^! = \frac{u_j}{u_i}, \quad j \in \{1, 2\} \setminus \{i\}$$

The reduced virtual tangent space at f_d is

$$\mathcal{T}_{\text{vir}, f_d}^{\text{red}} = H^*(T^{1/2} \oplus \hbar^{-1}(T^{1/2})^\vee) - T_{p_i} X$$

The action of $\mathbb{C}_q^\times$ in (12) is such that the global sections of $\mathcal{O}(m)$ for $m \geq 0$ have character $1 + q^{-1} + \dots + q^{-m}$. Thus, any given term $xq^{-d} \mathcal{O}(-d)$ in $\mathcal{T}^{1/2}$ of (20) leads to contributions of the form

$$\begin{aligned} & H^* \left(xq^{-d} \mathcal{O}(-d) + (\hbar^!)^{-1} x^{-1} q^d \mathcal{O}(d) \right) - x - (\hbar^!)^{-1} x^{-1} \\ &= xq \left(1 + q + \dots + q^{-d-1} \right) - \frac{1}{\hbar^! x} \left(1 + q^{-1} + \dots + q^{d+1} \right) \end{aligned}$$

So the T_q^1 -character of the reduced virtual tangent space at f_d is

$$\begin{aligned} \mathcal{T}_{\text{vir}, f_d}^{\text{red}} &= q \left(1 + q + \dots + q^{-d-1} \right) - \frac{1}{\hbar^!} \left(1 + q^{-1} + \dots + q^{d+1} \right) \\ &\quad + qu^! \left(1 + q + \dots + q^{-d-1} \right) - \frac{1}{\hbar^! u^!} \left(1 + q^{-1} + \dots + q^{d+1} \right) \end{aligned} \quad (21)$$

7.3 Vertex functions

The twist in [21] Sect. 6.1.8 that transforms the virtual structure sheaf to the symmetrized virtual structure sheaf means that the vertex function from Theorem 1 is

$$\sum_{d \leq 0} (z^!)^{-d} q^{\deg T^{1/2}/2} \hat{a} \left(\mathcal{T}_{\text{vir}, f_d}^{\text{red}} \right)$$

Writing this out explicitly, we find that for the fixed point p_i with $u^!$ as above, we have

$$\begin{aligned} \sum_{d \leq 0} (z^!)^{-d} q^{\deg T^{1/2/2} \hat{a}} \left(\mathcal{T}_{\text{vir}, f_d}^{\text{red}} \right) &= \sum_{d=0}^{\infty} (z^!)^d q^d \frac{(\hbar^!)_d}{(q)_d} \left(-\frac{q}{\hbar^!} \right)^{d/2} \frac{(\hbar^! u^!)_d}{(q u^!)_d} \left(-\frac{q}{\hbar^!} \right)^{d/2} \\ &= \sum_{d \geq 0} \left(\frac{q^2}{\hbar^!} z^! \right)^d \frac{(\hbar^!)_d}{(q)_d} \frac{(\hbar^! u^!)_d}{(q u^!)_d} \end{aligned}$$

The vertex functions from Definition 5 are

$$\begin{aligned} V_{p_1}^! (a^!, z^!) &= \sum_{d=0}^{\infty} \frac{(\hbar^!)_d (\hbar^! u_2^! / u_1^!)_d}{(q)_d (q u_2^! / u_1^!)_d} (z^!)^d \\ V_{p_2}^! (a^!, z^!) &= \sum_{d=0}^{\infty} \frac{(\hbar^!)_d (\hbar^! u_1^! / u_2^!)_d}{(q)_d (q u_1^! / u_2^!)_d} (z^!)^d \end{aligned}$$

which clearly agree with the previously ones after renormalizing the Kähler parameter.

7.4 Index

With respect to the choice of chamber (5), the attracting weight is $a^! := u_1^! / u_2^!$. Write $f_d^!$ for the unique $T_q^!$ -fixed quasimap of degree d on $\text{QM}_{p_i}^d$. The reduced virtual tangent space (21) for $\text{QM}_{p_1}^d$ at $f_d^!$ is

$$\begin{aligned} \mathcal{T}_{\text{vir}, f_d^!}^{\text{red}} &= q \left(1 + q + \dots + q^{-d-1} \right) - \frac{1}{\hbar^!} \left(1 + q^{-1} + \dots + q^{d+1} \right) \\ &\quad + \frac{q}{a^!} \left(1 + q + \dots + q^{-d-1} \right) - \frac{a^!}{\hbar^!} \left(1 + q^{-1} + \dots + q^{d+1} \right) \end{aligned}$$

Thus the s -index from Definition 2 is

$$\mathcal{I}_s(f) = \begin{cases} 0 & -d < s \\ 2\lfloor s \rfloor + 2d & 1 < s < -d \\ 2d & s < 1 \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Hence

$$\text{Ind}_{p_1}^{!s} = \begin{cases} \frac{1 - (z^! \hbar^! / q)^{\lfloor s \rfloor + 1}}{1 - z^! \hbar^! / q} + \frac{(\hbar^! / q)^{\lfloor s \rfloor} z^{\lfloor s \rfloor + 1}}{1 - z^!} & s > 1 \\ \frac{1}{1 - z^!} & s < 1 \end{cases}$$

Similarly, the virtual tangent space (21) for $\mathrm{QM}_{p_2}^d$ at f_d^2 is

$$\begin{aligned} \mathcal{T}_{\mathrm{vir}, f_d^2}^{\mathrm{red}} &= q \left(1 + q + \dots + q^{-d-1} \right) - \frac{1}{\hbar!} \left(1 + q^{-1} + \dots + q^{d+1} \right) \\ &\quad + qa! \left(1 + q + \dots + q^{-d-1} \right) - \frac{1}{\hbar! a!} \left(1 + q^{-1} + \dots + q^{d+1} \right) \end{aligned}$$

So

$$\mathcal{I}_s(f) = \begin{cases} 2d & s > -1 \\ 2\lfloor |s| \rfloor + 2d & d < s < -1 \\ 0 & s < d \end{cases}$$

and

$$\mathrm{Ind}_{p_2}^{!s} = \begin{cases} \frac{1 - (z! \hbar! / q)^{\lfloor |s| \rfloor + 1}}{1 - z! \hbar! / q} + \frac{(\hbar! / q)^{\lfloor |s| \rfloor} z!^{\lfloor |s| \rfloor + 1}}{1 - z!} & s < -1 \\ \frac{1}{1 - z!} & s > -1 \end{cases}$$

7.5 Stable envelopes

Now we switch from $X^!$ to X . We need to compute the K -theoretic stable envelopes for X . With respect to the chamber $\mathfrak{C}$ from (5), $a := u_1/u_2$ is an attracting weight. The choice of polarization is assumed to be

$$T^{1/2}X = \mathrm{Hom}(V, W) - \mathrm{Hom}(V, V)$$

Under (2), big enough slopes are given by $s > 0$.

Analogous to the argument given in Sect. 7.1 of [22], one can use the defining properties of stable envelopes to calculate the matrix of the K -theoretic stable envelope in the basis of fixed points ordered as $[p_1, p_2]$ to be:

$$A^{s,X} = \left(\mathrm{Stab}_{-\mathfrak{C}}^{s,X,K}(I)|_J \right)_{I,J \in X^\Gamma} = \begin{pmatrix} \left(\frac{a}{\hbar} - 1 \right) \sqrt{\hbar} (\hbar^{-1} - 1) \sqrt{\hbar} a^{\lfloor s \rfloor} & \\ 0 & (1 - a^{-1}) \end{pmatrix}$$

Alternatively, one could use the formulas [8] and their implementation in Maple described there. Normalizing by dividing each column by the diagonal entry gives

$$\tilde{A}^{s,X} = \begin{pmatrix} 1 - \frac{(1 - \hbar) a^{\lfloor s \rfloor + 1}}{(1 - a) \sqrt{\hbar}} & \\ 0 & 1 \end{pmatrix}$$

The polarization at the fixed points is

$$\begin{aligned} T^{1/2}X|_{p_1} &= a^{-1} \\ T^{1/2}X|_{p_2} &= a \end{aligned}$$

So conjugating by D in (18) and multiplying on the right by $S^\bullet(\hbar \otimes N^+)$ gives

$$D \cdot \tilde{A}^{s,X} \cdot D^{-1} \cdot S^\bullet(\hbar \otimes N^+) = \begin{pmatrix} \frac{1}{1-a} - \frac{(1-\hbar)a^{\lfloor s \rfloor}}{(1-a)(1-\hbar a)} \\ \frac{1}{1-\hbar a} \end{pmatrix}$$

Applying κ and summing the two components of this vector, we see that Theorem 2 for $s > 1$ reads

$$\frac{1 - (z^! \hbar^! / q)^{\lfloor s \rfloor + 1}}{1 - z^! \hbar^! / q} + \frac{(\hbar^! / q)^{\lfloor s \rfloor} z^{\lfloor s \rfloor + 1}}{1 - z^!} = \frac{1}{1 - z^! \hbar^! / q} - \frac{(1 - q / \hbar^!) (z^! \hbar^! / q)^{\lfloor s + 1 \rfloor}}{(1 - z^! \hbar^! / q) (1 - z^!)}$$

$$\frac{1}{1 - z^!} = \frac{1}{1 - q / \hbar^! (z^! \hbar^! / q)}$$

which is easily checked to be true. For $0 < s < 1$, Theorem 2 reads

$$\frac{1}{1 - z^!} = \frac{1}{1 - z^! \hbar^! / q} - \frac{(1 - q / \hbar^!) (z^! \hbar^! / q)}{(1 - z^! \hbar^! / q) (1 - z^!)}$$

$$\frac{1}{1 - z^!} = \frac{1}{1 - q / \hbar^! (z^! \hbar^! / q)}$$

which is also true.

Acknowledgements This work was partially supported by NSF grant DMS-2054527.

References

1. Aganagic, M., Okounkov, A.: Elliptic stable envelopes. *J. Amer. Math. Soc.* **34**, 79–133 (2021)
2. Braverman, A., Finkelberg, M., Nakajima, H.: Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II. *Adv. Theor. Math. Phys.* **22**, 1071–1147 (2018). <https://doi.org/10.4310/ATMP.2018.v22.n5.a1>
3. Cherkis, S.: Instantons on the Taub-NUT space. *Adv. Theor. Math. Phys.* **14**(2), 609–642 (2010). <https://doi.org/10.4310/ATMP.2010.v14.n2.a7>
4. Cherkis, S.A.: Instantons on Gravitons. *Commun. Math. Phys.* **306**(2), 449–483 (2011). <https://doi.org/10.1007/s00220-011-1293-y>
5. Cherkis, S.A.: Moduli Spaces of Instantons on the Taub-NUT Space. *Commun. Math. Phys.* **290**(2), 719–736 (2009). <https://doi.org/10.1007/s00220-009-0863-8>
6. Ciocan-Fontanine, I., Kim, B., Maulik, D.: Stable quasimaps to GIT quotients. *J. Geom. Phys.* **75**, 17–47 (2014)
7. Dinkins, H.: 3d mirror symmetry of the cotangent bundle of the full ag variety. (Nov. 2020). [arXiv:2011.08603](https://arxiv.org/abs/2011.08603) [math.AG]
8. Dinkins, H.: Elliptic stable envelopes of affine type A quiver varieties. (2021). [arXiv:2107.09569](https://arxiv.org/abs/2107.09569) [math.AG]
9. Dinkins, H.: “Symplectic Duality of $T^*Gr(k, n)$ ”. *Math. Res. Lett.* (2021), to appear
10. Gaiotto, D., Koroteev, P.: On Three Dimensional Quiver Gauge Theories and Integrability. *J. High Energy Phys.* **126**, 1–59 (2013). [https://doi.org/10.1007/JHEP05\(2013\)126](https://doi.org/10.1007/JHEP05(2013)126)
11. Kononov, Y., Smirnov, A.: Pursuing quantum difference equations I: stable envelopes of subvarieties. (Apr. 2020). [arXiv:2004.07862](https://arxiv.org/abs/2004.07862) [math.RT]

12. Kononov, Y., Smirnov, A.: Pursuing quantum difference equations II: 3D-mirror symmetry. (Aug. 2020). [arXiv:2008.06309](https://arxiv.org/abs/2008.06309) [math.AG]
13. Koroteev, P., Zeitlin, A.M.: qKZ/tRS Duality via Quantum K-Theoretic Counts. *Math. Res. Lett.* **28**(2), 435–470 (2021). <https://doi.org/10.4310/MRL.2021.v28.n2.a5>
14. Liu, H.: Quasimaps and stable pairs. *Forum Math. Sigma* **9**, e32 (2021). <https://doi.org/10.1017/fms.2021.25>
15. McGerty, K., Nevins, T.: Kirwan surjectivity for quiver varieties. *Invent. Math.* **212**(1), 161–187 (2018). <https://doi.org/10.1007/s00222-017-0765-x>
16. Nakajima, H., Takayama, Y.: Cherkis bow varieties and Coulomb branches of quiver gauge theories of affine type A. *Sel. Math. New Ser.* **23**, 2553–633 (2017). <https://doi.org/10.1007/s00029-017-0341-7>
17. Nakajima, H.: Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, I. *Adv. Theor. Math. Phys.* **20**(3), 595–669 (2016)
18. Nekrasov, N., Okounkov, A.: “Membranes and Sheaves”. *Algebraic Geometry*, 3 (Apr. 2014). <https://doi.org/10.14231/AG-2016-015>
19. Okounkov, A.: Enumerative symplectic duality. Lecture given at the MSRI Workshop “Structures in Enumerative Geometry”. (2018). <https://www.msri.org/workshops/816/schedules/23898>
20. Okounkov, A.: Inductive construction of stable envelopes. *Lett. Math. Phys.* **111**(6), 1–56 (2020). [arXiv:2007.09094](https://arxiv.org/abs/2007.09094) [math.AG]
21. Okounkov, A.: Lectures on K-theoretic computations in enumerative geometry. In: *Lectures on K-theoretic computations in enumerative geometry*, vol. 24. IAS/Park City Mathematics Series. American Mathematical Society, Utah (2017)
22. Okounkov, A., Smirnov, A.: Quantum difference equation for Nakajima varieties. (2016). [arXiv:1602.09007](https://arxiv.org/abs/1602.09007) [math-ph]
23. Pushkar, P., Smirnov, A., Zeitlin, A.: Baxter Q-operator from quantum K-theory. *Adv. Math.* **360**, 106919 (2016). <https://doi.org/10.1016/j.aim.2019.106919>
24. Rimányi, R., Shou, Y.: Bow varieties—geometry, combinatorics, characteristic classes. (Dec. 2020). [arXiv:2012.07814](https://arxiv.org/abs/2012.07814) [math.AG]
25. Rimányi, R., Tarasov, V., Varchenko, A.: Elliptic and K-theoretic stable envelopes and Newton polytopes. *Sel. Math. New Ser.* **25**, 16 (2019). <https://doi.org/10.1007/s00029-019-0451-5>
26. Rimányi, R., et al.: Three-Dimensional Mirror Self-Symmetry of the Cotangent Bundle of the Full Flag Variety. *SIGMA* **15**, 093 (2019). <https://doi.org/10.3842/SIGMA.2019.093>
27. Smirnov, A.: Elliptic stable envelope for Hilbert scheme of points in the plane. *Sel. Math. New Ser.* **26**, 3 (2020). <https://doi.org/10.1007/s00029-019-0527-2>
28. Smirnov, A., Dinkins, H.: Characters of tangent spaces at torus fixed points and 3d-mirror symmetry. *Lett. Math. Phys.* **110**, 2337–2352 (2020). <https://doi.org/10.1007/s11005-020-01292-y>
29. Smirnov, A., Zhou, Z.: 3d Mirror Symmetry and Quantum K-theory of Hypertoric Varieties. (May 2020). [arXiv:2006.00118](https://arxiv.org/abs/2006.00118) [math.AG]