Distribution-Level Markets under High Renewable Energy Penetration

Peizhong Ju
School of Electrical & Computer Engineering, Purdue University
West Lafayette, Indiana, USA
jup@purdue.edu

Xiaojun Lin
School of Electrical & Computer Engineering, Purdue University
West Lafayette, Indiana, USA
linx@purdue.edu

Jianwei Huang
School of Science & Engineering, The Chinese University of Hong Kong
Shenzhen, China
Shenzhen Institute of Artificial Intelligence and Robotics for Society
Shenzhen, China
jianweihuang@cuhk.edu.cn

ABSTRACT

We study the market structure for emerging distribution-level energy markets with high renewable energy penetration. Renewable generation is known to be uncertain and has a close-to-zero marginal cost. In this paper, we use solar energy as an example of such zero-marginal-cost resources for our focused study. We first show that, under high penetration of solar generation, the classical real-time market mechanism can either exhibit significant price-volatility (when each firm is not allowed to vary the supply quantity), or induce price-fixing (when each firm is allowed to vary the supply quantity), the latter of which leads to extreme unfairness of surplus division. To overcome these issues, we propose a new rental-market mechanism that trades the usage-right of solar panels instead of real-time solar energy. We show that the rental market produces a stable and unique price (therefore eliminating price-volatility), maintains positive surplus for both consumers and firms (therefore eliminating price-fixing), and achieves the same social welfare as the traditional real-time market. A key insight is that rental markets turn uncertainty of renewable generation from a detrimental factor (that leads to price-volatility in real-time markets) to a beneficial factor (that increases demand elasticity and contributes to the desirable rental-market outcomes).

CCS CONCEPTS

• **Hardware → Smart grid**.

KEYWORDS

Power distribution, renewable energy, market, rental market

ACM Reference Format:
Peizhong Ju, Xiaojun Lin, and Jianwei Huang. 2022. Distribution-Level Markets under High Renewable Energy Penetration. In *The Thirteenth ACM International Conference on Future Energy Systems (e-Energy '22), June 28-July 1, 2022, Virtual Event, USA.* ACM, New York, NY, USA, 30 pages. https://doi.org/10.1145/3538637.3538846

e-Energy '22, June 28-July 1, 2022, Virtual Event, USA

ACM ISBN 978-1-4503-9397-3/22/06.
https://doi.org/10.1145/3538637.3538846

1 INTRODUCTION

1.1 Motivation

As the deployment of renewable generation at the distribution level continues to rise [9, 38], there are significant interests in developing a new distribution-level electricity market that allows renewable-energy producers and electricity consumers to directly trade renewable energy [29]. In today's power systems, renewable generation at the distribution level has to be either consumed locally, or sold to the utility according to a combination of net-metering/feed-in-tariff, connection charges, and/or peak-based demand charges. At the same time, consumers are charged by the utility via a separate set of retail prices. These prices and tariffs are often decided by a consortium of utility, government regulators, and/or consumer interest groups. However, since their financial interests are often conflicting with each other, determining the "right" price/charges has already become a fierce political fight [14, 27]. With the introduction of new distribution-level markets, the renewable producers and electricity consumers will be able to directly trade energy with each other [10, 17, 24, 26, 36, 39]. The hope is that, by its "invisible hand", a well-designed market may be more effective in discovering the "right" valuation for renewable energy based on the market condition.

Most studies of distributed-level markets (including the recent pilot program in [24]) have focused on real-time markets [10, 17, 26, 36, 39], which replicate the real-time markets at the transmission-level. Real-time markets set prices based on discovering the lowest "marginal cost" of generation to meet demand. Such prices can be shown to maximize the total social surplus of the system, and is thus "efficient" [5]. However, from a market designer's point of view, efficiency is usually not the only consideration. In fact, as we will show shortly, due to the much higher penetration of zero-marginal-cost renewable generation, real-time markets at the distribution level tend to produce multiple equilibrium prices, all of which are efficient. Thus, it is no longer clear why pricing based on the marginal cost is the most appealing option.

In addition to efficiency, there are a few other equally important considerations, including: (i) Will the social surplus be fairly distributed between consumers and producers [19]? (ii) Is the market outcome unique and predictable [21]? (iii) Is the market price stable and easy to predict [12]? To the best of our knowledge, there does not exist a comprehensive study of the distribution-level markets under these more comprehensive lenses. This motivates our study

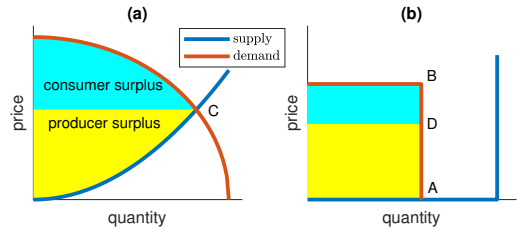

Figure 1: (a) Relatively elastic supply and demand functions assumed in classical market theory lead to a unique market price that maximizes the social surplus. (b) With zero-marginal-cost supply and highly-inelastic demand, any point D between A and B maximizes the social surplus.

in this paper, which hopes to provide some preliminary understandings of these important considerations.

1.2 Key results

Throughout the rest of the paper, we will use solar energy as an example of such zero-marginal-cost resources for our focused study (while many of the insights could also be extended to other types of renewable energy). Motivated by the above questions, we first study the traditional real-time markets, and reveal their deficiencies when facing the zero marginal cost of solar generation and inelastic demand. This then motivates us to propose the new rental market with more desirable features. Table 1 summarizes our key results.

Real-time markets: We focus on the setting where solar generation can be predicted quite accurately at the beginning of each time-interval of real-time markets [13, 35], the marginal cost of solar producers is zero [7] (which reflects that fact that solar generation incurs high investment cost but low production cost), and the consumer (if she wishes) can always buy electricity from the utility at a fixed retail price π_g. Prior studies have shown that, due to the uncertainty and variability of solar generation, the market prices will fluctuate wildly between 0 and π_g, depending on whether the total solar generation is higher or lower than the total electricity demand [20, 32]. We advance this line of study by further studying how such price volatility will incentivize solar producers to withhold supply. In particular, when the total supply is higher than the demand, the solar producers will earn zero revenue since the price becomes 0. As a result, they will have a strong incentive to withhold their supply and raise the price. Indeed, in this paper we show that, once the solar producers can withhold supply, significantly different outcomes will arise, which can be quite unpredictable and unfair (see Table 1).

The role of zero marginal cost and inelastic demand: The underlying reason for the above highly-undesirable outcome is the combination of (i) zero marginal cost of solar producers, and (ii) inelasic demand. In classical market design theory, one would expect a supply curve with an upward slope and a demand curve with a downward slope (see Fig. 1(a)). As a result, there is only one point where the demand meets the supply, which maximizes the social surplus. However, when the marginal cost of the supply is zero and the demand is fixed, both the supply and demand curves become highly-inelastic (see Fig. 1(b)). Note that although the intersection point A of the supply and demand curves still maximizes the social surplus, any points between A and B will also do! Further, the suppliers have every incentive to drive the equilibrum to B, so that they earn the entire social surplus and drive the consumer surplus to zero.

Rental markets: In view of the above issues of real-time markets, we then propose an alternate form of distribution-level markets, which we refer to as rental markets. In such a rental market, consumers rent a certain amount of solar panels from solar producers in advance. Then, in real time, the consumer gets to use the electricity generated from the rented panels at no addition costs. We then study the strategic behavior of solar producers and the resulting rental price, and show the following desirable results. First, the rental price is naturally *stable* (i.e., non-volatile) as it is not immediately affected by real-time conditions. Second, as long as the solar generation is variable, the rental demand function becomes relatively elastic (i.e., having a downward slope as in Fig. 1(a)). Thus, once the number of producers is larger than a threshold (which is function of the elasticity of the rental demand function), a *unique* outcome arises, which is always equal to the outcome of perfect competition (i.e, the point C in Fig. 1(a)) [34]. Third, at this unique market outcome, both the consumer surplus and producer surplus are not zero. Last but not least, this unique outcome also maximizes the social welfare.

The role of solar variability: A key insight revealed from our analysis is how the variability of solar generation will affect the market outcome under different market rules. Recall that in real-time markets, since we assume that solar generation can be accurately predicted for the immediately next time-interval, there is no solar variability *within* each time-interval of real-time markets. Instead, this variability manifests as high price-volatility *across* time-intervals [20, 32]. In contrast, rental markets operate over a longer time horizon of many time-intervals. Thus, the variability directly enters into the strategic consideration of the market participants. Our results suggest that, while the variability and uncertainty of solar generation is often considered a *detrimental* factor for real-time markets (e.g., it may lead to price-volatility [32]), it becomes a *beneficial* factor for rental markets (e.g., it produces demand elasticity and lowers the bar for perfectly-competitive outcome to arise). Due to this reason, we expect that rental markets may serve as a more favorable alternative for distribution-level markets under high renewable penetration.

2 RELATED WORK

There is significant interest in understanding how providers of uncertain renewable energy should participate in the energy market. Existing work can be broadly divided into two categories, according to the assumption on the market price.

The first category assumes that prices are exogenously given, and studies optimal bidding strategies when market participants are price-takers (see, e.g., [8, 10, 23]). Specifically, [8] and [23] study how renewable providers (e.g., wind farms) bid in a day-ahead electricity market, based on future scenarios of generation and prices; while [10] further studies how the availability of local generation and market recourse will impact their bidding strategy. However, this

Table 1: Features comparison of different markets.

features / markets	Social welfare	Price volatility	Fairness of surplus division	Predictability of outcome
Real-time markets (no supply withholding)	Maximized	**High**	Relatively fair	Unique outcome (for most of the time)
Real-time markets (with supply-withholding)	Maximized	Zero	**Extremely unfair (consumer surplus is 0)**	**Multiple Nash equilibria and outcomes**
Rental markets	Maximized	Zero	Relatively fair	Unique outcome (with modest assumptions)

line of work fails to capture the impact of the market participants' bidding strategies on the price.

The second category explicitly consider how the market price is formed from the agents' bidding. A significant body of related work assume traditional generators at the transmission level with significant production costs [16, 31]. In contrast, renewable generators (such as solar panels) at the distribution level have nearly-zero marginal cost. There are only limited studies on the market equilibrium with zero-marginal-cost generation, for the settings of two producers [36], with storage [20], or along with the investment game [22]. However, these results all assume that the suppliers can only change the bidding prices but cannot withhold the supply quantity, which is not realistic when firms have the flexibility to curtail their generation (especially when the price is not favorable). In contrast, our work is the first to study the setting where renewable energy suppliers can vary both price and quantity in their bids, and reveal the emergence of price-fixing behavior in real-time energy markets. Further, all of these studies assume a real-time energy market, while our work is the first to study alternative market designs in the form of rental markets.

The rental market is also inspired by the structure of other markets with zero marginal costs, in particular markets for telecommunication services [18]. For example, for mobile wireless service providers, while base stations and backbone networks are very expensive to build, the cost of providing service for one phone call or one kilobyte (KB) of data is often negligible. Although dynamic pricing based on real-time demand levels was discussed at various times, what prevails today are often fixed-price monthly contracts (e.g., unlimited voice and data for 50 dollars per month). Such a fixed-price contract has been found to reduce the financial uncertainty to the suppliers, and provide them with the guaranteed revenue for future network expansion. In addition to these potential benefits, our study of rental markets further investigates how rental markets eliminates market manipulation and supply withholding. Therefore, our study provides new insights why alternative forms of markets can be more favorable than real-time energy markets.

3 SYSTEM MODEL

We consider a distribution-level power system with one utility, N consumers, and M firms (i.e., suppliers). We will characterize different market players, based on which we will introduce the market mechanisms in Section 4 and Section 5.

The *utility* not only manages the physical power distribution grid, but also operates a distribution-level energy market. Further, it provides a reliable source of energy external to the distribution-level market at a fixed retail price π_g (see further discussions on consumer model below).

Firms own solar panels and sell solar energy to the distribution-level market. Note that a key attribute of solar generation is that it incurs high investment cost but low operation cost. Thus, we assume that the marginal cost of each firm is zero. For simplicity, we assume that a firm does not receive revenue for any solar energy that is not sold in the distribution-level market. This assumption corresponds to the case where utility either does not buy back electricity from the distribution level or imposes a feed-in tariff of zero. (See remark below for possible extensions.) Let C_i denote the size (in m^2) of firm i's solar panel. Let $C = \sum_{i=1}^{M} C_i$ denote the total size of panels of all firms. Let $G(t)$ denote the energy generated by unit size of solar panels at time t, which is assumed to be a random variable. We assume that all solar panels have the same efficiency $G(t)$ (see further discussion at the end of this section). Therefore, the amount of solar energy available from firm i is $C_i G(t)$.

Consumers (e.g., households) consume but do not generate electricity. We let $L_n(t)$ denote the real-time electricity demand of consumer n at time t, which is a continuous random variable. Let $L(t)$ denote the total electricity demand of all consumers at time t, i.e., $L(t) = \sum_{n=1}^{N} L_n(t)$. We assume that the real-time demand is inelastic, i.e., no demand-response, which reflects the practical setting where demand elasticity is low [4]. We assume that consumers can always buy electricity from the utility at the fixed retail price π_g. On the other hand, consumers would be interested in buying the solar energy from the distribution-level market if the price is lower.

The objective of the distribution-level market is to determine the price and quantity with which firms and consumers can directly trade solar energy, based on the bids submitted by them. We are particularly interested in how the equilibirum market price is formed due to the strategic behaviors of the participants. As we discussed in the introduction, we will study not only the efficiency (i.e., whether the social surplus is maximized), but also the questions of (i) fairness of surplus division between consumers and firms; (ii) the uniqueness and predictability of the market outcome; and (iii) price volatility.

Remark: We briefly comment on some of the simplifying assumption made earlier. We assume that firms and consumers are separate, i.e., one market participant cannot be both a firm and a consumer at the same time. In reality, a firm may consume energy by herself. In that case, it is common for the firm to first use the solar energy for her own demand, and then sell the remaining solar energy to

the market. Note that such firms can be equivalently viewed as having a lower generation efficiency $G(t)$. It is possible to extend our analysis to the setting with varying generation efficiency $G(t)$ across firms. Indeed, our results for real-time markets will still hold (since the equilibrium at each time only depends on the exact value of the random generation at the time, regardless of its distribution), and readers can refer to Section 5.7 on how the rental market can work with heterogeneous $G(t)$. It is also possible to generalize some of our results to the setting where utility offers non-zero feed-in tariffs, without changing the main conclusions qualitatively. For example, price volatility and price-fixing in real-time markets will still hold, with the only difference being that the price will vary between π_g and the feed-in-tariff (instead of between π_g and 0).

4 REAL-TIME MARKETS

In this section, we will consider real-time distribution-level energy markets, which operate in a way similar to how renewable suppliers bid in the existing transmission-level real-time energy markets [25]. We fix a time-instant t, and consider an instance of the real-time market at time t. Thus, we will drop the index t when there is no source of confusion. Let $q_i^0 = C_i G(t)$ be the actual amount of renewable electricity generated by firm i at time t, which is determined exogeneously (by solar irradiance). We assume that each firm knows the value of q_i^0 when she submits her bid to the real-time market, which is reasonable as short-term prediction of solar generation can be quite accurate [6]. Next, we consider the four features listed in Table 1 for two types of real-time markets. We will show that, depending on whether the suppliers can vary the bidding quantities, the real-time markets can lead to price-volatility or price-fixing behaviors.

4.1 Price-volatility in a single-price-bid market

We first consider a single-price-bid system where each firm i can only vary the bidding price p_i, while the supply quantity q_i is fixed at q_i^0 and cannot be varied. Some earlier studies [22, 36] have shown that this type of real-time markets with single-price-bids will produce high price volatility. Here, we report a similar result but for uniform prices, under a mechanism that is closer to how the current transmission-level energy markets operate.

Market clearing mechanism: After the market receives the bids of all firms, i.e., their bidding price p_i and actual generation amount q_i^0, the market stacks all the bids together to compute the supply curve, i.e., the total available quantity from all bids at or below each price point p. Then, the market clearing price π_{eq} is given by the lowest price such that the total available quantity *exceeds* the total demand $L(t)$.

The sold amount s_i of each firm i is then determined as follows. All bids with price lower than π_{eq} clear their entire quantity q_i^0, i.e., $s_i = q_i^0$. All bids with price higher than π_{eq} clear zero quantity, i.e., $s_i = 0$. For those bids with price exactly equal to π_{eq}, we assume that the sold/cleared amount is assigned proportionally to q_i^0 to split the left-over demand, i.e.,

$$s_i = \min\left\{ \frac{q_i^0 \left(L(t) - \sum_{\{j:\, p_j < \pi_{\text{eq}}\}} q_j^0\right)}{\sum_{\{j:\, p_j = \pi_{\text{eq}}\}} q_j^0},\ q_i^0 \right\}. \tag{1}$$

Under such a market rule, the whole system can be viewed as a game where each firm chooses her bidding price. The market outcome can be analyzed by the Nash equilibrium of the firms' bidding strategies, which is summarized as follows.

Proposition 4.1. *Assume* $\sum_{i\in S} q_i^0 \neq L(t)$ *for all* $S \subseteq \{1, \cdots, M\}$[1]. *Then, at each time-instant* t*, the equilibrium market outcome has three cases:*

Case 1. *(Limited Supply) When* $\sum_{i=1}^M q_i^0 \leq L(t)$*, Nash equilibrium exists, and the market clearing price must be* $\pi_{eq} = \pi_g$*.*

Case 2. *(Abundant Supply) When* $\sum_{i\neq j} q_i^0 > L(t)$ *for all* j*, Nash equilibrium exists, and at any equilibrium the market clearing price must be* $\pi_{eq} = 0$*.*

Case 3. *(Borderline Supply) Consider the situation where* $\sum_{i=1}^M q_i^0 > L(t)$*,* $\sum_{i\neq j} q_i^0 < L(t)$ *for some* j *(let* $\mathcal{I}$ *denote the set of those* j*'s). Then Nash equilibrium exists, and at any equilibrium the market clearing price must be* $\pi_{eq} = \pi_g$*.*

(Detailed characterization of these Nash equilibria can be found in the proof in Appendix A.1.)

Proposition 4.1 indicates that real-time markets with single-price bids will experience significant price-volatility: as the real-time solar generation fluctuates above or below the demand, the market price will jump between 0 and π_g (corresponding to point A and point B of Fig. 1, respectively). Specifically, when the total solar generation is limited (Case 1), the consumers have to buy electricity from the utility, which drives the market price to π_g. When there is too much solar generation in the market (Case 2), the demand shortage leads to fierce competition among firms, and eventually drives the market price to 0. In Case 3, the firms in $\mathcal{I}$ have a high market power. Specifically, without any firm in $\mathcal{I}$, the system changes from "excessive supply" to "insufficient supply". Thus, firms in $\mathcal{I}$ will be the ones that set the market price to π_g.

Cases 1 and 2 of Proposition 4.1 are similar to results reported in [1, 36], but the result for Case 3 is different. In [1, 36], there exists no pure Nash equilibrium for Case 3. The reason for the existence of Nash equilibrium here is that we assume uniform price for all firms, while [1, 36] considered differentiated prices.

4.2 Price-fixing in markets allowing price-quantity bids

In this paper, we advance this line of study of real-time markets by furthering considering the impact of price-volatility on firms' strategic behavior. Specifically, whenever the total solar generation exceeds the demand (Case 2 in Proposition 4.1), firms will receive zero revenue because the market price is driven to zero. Intuitively, there will then be a strong incentive for firms to withhold supply, which will likely lead to very different equilibrium dynamics compared to Section 4.1.

We note that existing studies in the literature [22, 36] assume that renewable suppliers cannot vary their bidding quantity, partly because renewable generation (unlike fossil-fuel generation) is usually considered uncontrollable. However, there are several reasons

[1]This assumption holds almost surely, e.g., when all q_i^0's follows a continuous distribution with a finite probability density function (i.e., with no atoms).

that we believe it is important to study this possibility of withholding. First, in practice it is actually quite easy for a solar-energy supplier to adjust her supply. For example, a firm can shut down part of her solar panels. Second, even though the utility may build additional mechanisms to detect such withholding (e.g., by monitoring irradiance and cloud covers), such detection mechanisms would incur extra costs. Further, if each firm's solar panel is small (e.g., for rooftop solar panels of a small house), even a small amount of local change in cloud conditions can impact its generation amount significantly, making fine-grade monitoring based on cloud conditions difficult. Due to this inherently uncertainty, it would be difficult for the utility to tell whether the lower supply is due to withholding or due to lack of solar irradiance. Finally, when a utility penalizes a withholding firm, it limits the freedom of the bidders, which may also be undesirable. (Ideally, we want a free market that can lead to desirable outcomes even without such limitations.) Due to these reasons, we believe that it is practically important to consider the possibility of withholding supply.

Supply withholding leads to the following model for a real-time market with price-quantity bids. Recall that $q_i^0 = C_i G(t)$ is the real solar generation available to firm i at time t. Now, each firm can submit a bid that varies both the price $p_i \in [0, \pi_g]$ and the quantity $q_i \in [0, q_i^0]$. The market clearing mechanism remains the same as in Section 4.1, except that the supply curve is computed based on the declared quantity q_i instead of q_i^0. The following result shows that the equilibrium market outcome changes drastically once firms can submit price-quantity bids.

Proposition 4.2. *With price-quantity bids, in additional to the Nash equilibria characterized in Proposition 4.1, a Nash equilibrium with $\pi_{eq} = \pi_g$ always exists, regardless of the demand or the generation. Specifically, any bidding strategy that satisfies the following conditions is a Nash equilibrium with $\pi_{eq} = \pi_g$:*

$$\begin{cases} \sum_{i=1}^{M} q_i = \min\left\{L(t), \sum_{i=1}^{M} q_i^0\right\} \text{ (firms may withhold supply),} \\ \text{and } p_i = 0 \text{ for all } i \text{ (every firm bids zero price).} \end{cases} \tag{2}$$

In sharp contrast to Proposition 4.1, Proposition 4.2 shows that real-time markets with price-quantity bids always have a Nash equilibrium where the market price[2] is π_g. Such an equilibrium is highly unfair to the consumers because the consumer surplus (which corresponds to the difference between the consumer's cost of buying from the market and that of directly purchasing electricity from the utility) will always be zero. In the other words, the entire social surplus is earned by the firms as the producer surplus (corresponding to Fig. 1(b) where point D is moved to B). As a result, consumers will be disincentivized to participate in the market. Compared to Case 2 of Proposition 4.1, a key difference in Proposition 4.2 is that, when the total solar generation exceeds the supply (i.e., $L(t) < \sum_{i=1}^{M} q_i^0$), some firms do understate their generation at the new Nash equilibrium (2), which then drives the price back up to π_g. We note that in practice it is also easy for each firm to reach such an equilibrium strategy. To see this, consider two successive time-instants such that the solar generation amount does not vary much. If the market price was zero in the previous time-instant due to over-supply, each firm can simply withhold her supply at the next time-instant to be slightly below her sold/cleared-amount from the previous time-instant.

Another consequence of the above analysis is that there exist multiple equilibrium market outcomes. In particular, it is not hard to verify that, in addition to the equilibrium stated in Proposition 4.2, the outcome predicted in Proposition 4.1 is also a Nash equilibrium for the setting with price-quantity bids. Note that the equilibrium stated in Proposition 4.2 is Pareto optimal, i.e., no firms can gain higher payoff without hurting other firms, and thus may be more favorable. However, because (2) has multiple solutions, there will be multiple such Pareto-optimal equilibria too. The existence of multiple equilibria means that it is difficult to predict the outcome of the market.

4.2.1 When there exists demand elasticity. Note that our earlier discussions have assumed that the energy demand $L(t)$ has no elasticity, i.e., it does not change with price. This is a reasonable approximation due to limited demand elasticity in the current grid. Even if this assumption is relaxed, we expect that our qualitative conclusions on the real-time energy markets will still be valid. For example, we can show that even when there is a small amount of demand elasticity, the problem of price-fixing may still exist. In this case, the demand function $L(t)$ should be a non-increasing function of the price π, i.e., $L(t, \pi)$. As the price π decreases from π_g, we can define the demand elasticity as the absolute ratio between the relative change of demand and the relative change of price, i.e., $\eta_t(\pi) := \frac{L(t,\pi) - L(t,\pi_g)}{L(t,\pi_g)} \frac{\pi_g}{\pi_g - \pi} > 0$. The following result shows that the price-fixing problem still exists when the demand elasticity and the number of firms are small.

Proposition 4.3. *With price-quantity bids, if $\max_{\pi \in [0,\pi_g]} \eta_t(\pi) \le \min_{i \in \{1,2,\cdots,M\}} \frac{q_i}{L(t,\pi_g)}$, then Eq. (2) (after replacing $L(t)$ in Eq. (2) by $L(t, \pi_g)$) still corresponds to a Nash equilibrium with $\pi_{eq} = \pi_g$.*

We prove Proposition 4.3 in Appendix A.3. Notice that $\frac{q_i}{L(t,\pi_g)}$ is the fraction of the sold amount of firm i. Its value is likely to be large when the number of firms is small (in which case the condition in Proposition 4.3 is easier to hold).

5 SOLAR-PANEL RENTAL MARKETS

In view of the issues of price-volatility and pricing-fixing in real-time markets, we propose an alternative form of distribution markets that avoid these issues and lead to desirable outcomes in terms of all four considerations in Table 1. The key ideas of this new market are two-fold. First, instead of trading in real-time, this market trades once over a time-period of length T (e.g., T could correspond to a month). Second, instead of trading energy, this market trades the usage right for a certain size of solar panels. Specifically, consumers lease a certain size of solar panels from the firms T time ahead and can then use all the electricity generated by the rented solar panels in real-time. Therefore, we refer to this type of markets as *rental markets*. Note that if the real-time demand of a consumer exceeds the generation of her rented panels, she still has to buy the deficit part from the utility at the grid price π_g.

[2]Note that under Eq. (2) the total bidding quantity is no greater than the demand. Thus, even though every firm bids zero price, the market clearing price is set by the grid price π_g because we define π_{eq} as the lowest price such that the total available quantity *exceeds* the supply.

We will show that rental markets can eliminate both price-volatility and price-fixing. First, the price in the rental market is determined once over the time-period of length T. Thus, real-time price-volatility is eliminated by design. Second and more importantly, we will show soon that solar variability naturally induces elasticity of the solar panel demand function in such a rental market. Recall that the inelasticity of the demand was one of the main reasons for price-fixing in real-time markets. In contrast, we will show that this variability-induced demand elasticity will help the rental market avoid price-fixing. We note that the work in [2] also proposed to trade usage rights in an electricity market. However, [2] focused on studying the existence of a feasible contract without addressing the price-fixing behavior in such markets.

Readers familiar with transmission-level energy markets will recognize rental markets as a form of *forward* markets. Indeed, one may also use a form of forward energy market similar to the design of day-ahead markets in current transmission-level energy markets [3]. A key difference, however, is that, in traditional day-ahead markets, due to their uncertainty in generation, solar producers has to bid more conservatively to avoid the penalty of under-generation. In contrast, firms in rental markets do not need to bid conservatively because the forward commitment is in the *usage rights* of panels, not energy. We will discuss this point further in Section 5.6.

5.1 Panel demand function and its elasticity in rental markets

We first derive the panel demand function in rental markets, which corresponds to the size (in m^2) of the solar panel that consumer n wishes to rent at a unit rental price of π (in \$/m^2, normalized to one time-instant). Recall that the real-time electricity demand of consumer n is $L_n(t)$.

Suppose that the consumer n has rented c_n unit of solar panels. Then, the generation at time t is $G(t)c_n$. If $L_n(t) \le G(t)c_n$, the consumer does not need to buy any electricity from the utility. Otherwise, she needs to buy the deficit $L_n(t) - G(t)c_n$ at the price π_g. Thus, when the market price of renting unit-size solar panel for a unit-time is π, the time-average expected cost of the consumer n is given by

$$J_n(\pi, c_n) = \pi c_n + \frac{\pi_g}{T}\int_0^T \mathbb{E}[L_n(t) - G(t)c_n]^+ dt. \tag{3}$$

We take the panel demand function $d_n(\pi)$ of customer n as the largest c_n^* that minimizes $J_n(\pi, c_n)$ over c_n, i.e.,

$$d_n(\pi) = \sup\left\{c_n^* \middle| J_n(\pi, c_n^*) \le J_n(\pi, c_n), \text{ for all } c_n \ge 0\right\}. \tag{4}$$

When there are multiple global cost minimizers for $J_n(\pi, \cdot)$, Eq. (4) chooses $d_n(\pi)$ to be the maximum global minimizer. This will ensure the existence of the Nash equilibrium, especially when the clearing price is at one of the discontinuity points of panel demand function. We will discuss this point further after we introduce Theorem 5.2.

For any demand functions we define elasticity as the absolute ratio between the relative change of demand and the relative change of price, i.e., $\eta(\pi) := |\frac{\pi}{D(\pi)}\frac{dD(\pi)}{d\pi}|$. Due to this definition, the elasticity is always a non-negative value. We call a demand function perfectly inelastic when its elasticity is zero. Otherwise, we call it elastic.

It is easy to see that, if $L_n(t)$ and $G(t)$ were both constant over t, i.e., $L_n(t) = L$ and $G(t) = G$, then the demand function $d_n(\pi)$ would be equal to $\frac{L}{G}$ when $\pi \le \pi_g$, and 0, otherwise. In other words, the demand function would still be inelastic. In contrast, the following result shows that, if the solar generation is sufficiently random, then the demand function will become elastic. Towards this end, we introduce the following assumption on the randomness of solar generation.

Assumption 1. *Suppose that for any $t \in [0, T]$, $L_n(t)$ and $G(t)$ are non-negative continuous random variables with a joint probability density function (pdf) $f^t_{G,L_n}(x, y) = f^t_{G|L_n}(x|y) \cdot f^t_{L_n}(y)$, where $f^t_{G|L_n}(x|y)$ denotes the conditional pdf of $G(t)$ conditioned on $L_n(t)$, and $f^t_{L_n}(y)$ denotes the marginal pdf of $L_n(t)$. We assume that there is a finite $\gamma > 0$ such that*

$$x^2 f^t_{G|L_n}(x|y) \le \gamma \text{ for all } x, y, \text{ and } t. \tag{5}$$

Intuitively, Assumption 1 states that the probability distribution of solar generation should not be too concentrated (e.g., there should not be any atom in the distribution) and its tail should not be too heavy. Assumption 1 holds for many common continuous random distributions, such as Gaussian distribution and any sub-Gaussian distributions [11] whose tail probability decays at least as fast as that of Gaussian distribution.

We now define the total demand function $D(\pi)$ as the sum of all consumers' demand functions, i.e., $D(\pi) = \sum_{n=1}^N d_n(\pi)$. We then have the following result, which relates the elasticity of $D(\pi)$ to the parameter γ in Assumption 1.

Proposition 5.1. *At any π such that $D(\pi) > 0$, the elasticity of the total demand function $D(\pi)$ is bounded by*

$$\eta(\pi) \ge \frac{\pi}{\pi_g \gamma}. \tag{6}$$

(Note that since $D(\pi)$ may be discontinuous, $\frac{\partial D(\pi)}{D(\pi)}$ may be $+\infty$. Nonetheless, Eq. (6) still holds.)

Proposition 5.1 shows that the demand elasticity in rental markets is bounded from below by a function of γ. Note that according to (5), as γ decreases, the distribution of random solar generation is even less concentrated (i.e., more random). Proposition 5.1 then shows that the demand elasticity will also be higher. In this sense, while the variability and uncertainty of solar generation is often regarded as a detrimental factor for energy systems, in our proposed rental markets it becomes a beneficial factor in contributing to demand elasticity.

5.2 Mechanism of rental markets

Recall from Section 4.2 that price-fixing in real-time markets arises when each firm is allowed to vary both her price and quantity. Later in this section, we wish to show that rental markets can eliminate such price-fixing under price-quantity bids. However, we also note that in today's transmission-level energy markets, a generator can even bid multiple blocks of price-quantity pairs, which is more general than a single price-quantity bid. It is conceivable that future

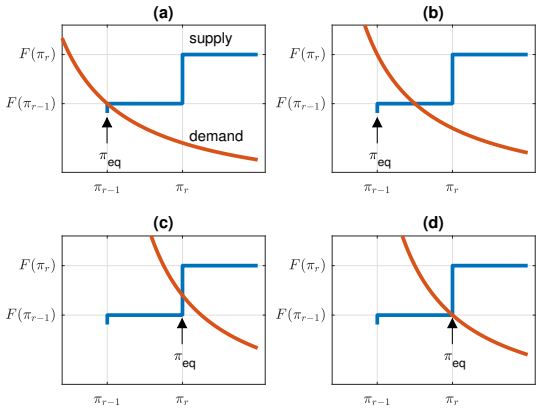

Figure 2: Determination of the market clearing price π_{eq} in rental markets, depending on the different ways that the supply curve intersects the demand curve. The supply curve $F(\pi)$ is given by as $F(\pi) = \sum_{\{i,k:\, p_{i,k} \le \pi\}} q_{i,k}$. We use π_{r-1} and π_r to denote the bidding prices of two bids in this figure. Here we assume that the bids are ordered by the bidding prices, and thus π_{r-1} and π_r are the prices of two consecutive bids.

distribution-level markets may also allow such flexibility of multi-block bids. Further, when a few firms form a coalition, they can coordinate their single price-quantity bids in a way that effectively simulates multi-block bids. In this paper, we wish to show that our positive result on rental markets will hold even for this more general setting. Hence, below we will introduce the rental market mechanism directly for multi-block bids, so that we can state a more general and stronger result. (For real-time markets, we can also show that our conclusion in Section 4.2 about price-fixing also holds under multi-block bids.)

We now describe the mechanism of rental markets under multi-block bids. The key difference from the mechanism in Section 4.2 is that the traded product is solar panel size instead of electrical energy. Suppose that the firm i makes a multi-block bid containing K_i sub-bids. Denote the price and the quantity (i.e., solar panel size) of her k-th sub-bid as $p_{i,k}$ and $q_{i,k}$, respectively. Without loss of generality, assume that $p_{i,1} < p_{i,2} < \cdots < p_{i,K_i}$, $q_{i,k} > 0$ for all $k \in \{1, \cdots, K_i\}$, and $\sum_{k=1}^{K_i} q_{i,k} \le C_i$. Firm i's bid can then be described by $\vec{p}_i = [p_{i,1}\ p_{i,2}\ \cdots\ p_{i,K_i}]$ and $\vec{q}_i = [q_{i,1}\ q_{i,2}\ \cdots\ q_{i,K_i}]$. Let $\mathbf{p}$ and $\mathbf{q}$ denote the collection of $\vec{p}_i$ and $\vec{q}_i$, respectively, for all i.

The market collects the bids from all firms, as well as the panel demand functions $d_n(\pi)$ from all consumers.[3] The market then stacks all the sub-bids together to compute the supply curve (similar to real-time markets in Section 4) for the total available solar-panel size at each price point. Similarly, the demand function $d_n(\pi)$ is added together to form the total demand curve $D(\pi)$. The market clearing price is then determined by the intersection of the supply curve and the demand curve. Algorithm 1 describes the detailed process to determine the market price π_{eq}. Denote $s_{i,k}$ as the cleared/sold amount of the k-th sub-bid of firm i. In Algorithm 2, we specify the process of determining the cleared amount. Specifically, for those sub-bids with prices lower than (or higher than, correspondingly) π_{eq}, the sold amount is equal to the bidding quantity (or zero, correspondingly). For those sub-bids with price equal to π_{eq}, the sold amount is assigned proportionally to the bidding quantity (similar to (1)).

[3]Note that we assume that the demand functions are truthful, which may be reasonable when the number of consumers is large and each consumer cannot influence the market price.

Algorithm 1: Compute market price π_{eq}.

1 Sort elements in $\{p_{i,k} :$ for all $i,k\}$ in an ascending order as $\pi_1 < \pi_2 < \cdots < \pi_R$ (note that all π_r's are distinct);
2 $Q \leftarrow 0$;
3 **for** $r = 1 : R$ **do**
4 **if** $D(\pi_r) < Q$ **then**
5 $\pi_{eq} \leftarrow \pi_{r-1}$; (cases (a) and (b) in Fig. 2)
6 exit;
7 $Q \leftarrow Q + \sum_{\{i,k:\, p_{i,k}=\pi_r\}} q_{i,k}$;
8 **if** $D(\pi_r) < Q$ **then**
9 $\pi_{eq} \leftarrow \pi_r$; (cases (c) and (d) in Fig. 2)
10 exit;
11 $\pi_{eq} \leftarrow \pi_R$;

Algorithm 2: Decide the sold amount $s_{i,k}$.

1 $Q \leftarrow 0$
2 $s_{i,k} \leftarrow 0$, for all i,k;
3 **for** $i = 1 : M$ **do**
4 **for** $k = 1 : K_i$ **do**
5 **if** $p_{i,k} < \pi_{eq}$ **then**
6 $s_{i,k} \leftarrow q_{i,k}$;
7 $Q \leftarrow Q + q_{i,k}$;
8 Denote those bids with the price π_{eq} as $q_{a_1,b_1}, q_{a_2,b_2}, \cdots, q_{a_h,b_h}$.
9 **for** $i = 1 : h$ **do**
10 $$s_{a_i,b_i} \leftarrow \min\left\{\frac{q_{a_i,b_i}\left(D(\pi_{eq}) - Q\right)}{\sum_{l=1}^{h} q_{a_l,b_l}},\ q_{a_i,b_i}\right\}. \quad (7)$$

Comparing the former real-time market mechanism in Section 4.2 with the rental market mechanism, the main difference is that the former assumes a fixed demand while the latter assumes a demand function $D(\pi)$ that decreases with the price π. As a result, in Algorithm 1 there are multiple cases (in Line 4, Line 8 and some other corner cases) where the market price is calculated. That is because, when the demand is not fixed, there are multiple ways that the supply curve and the demand curve intersect as illustrated in Fig. 2.

Remark: The above market mechanism is described for the more general multi-block bids. It is easy to see that, even if each firm is only allowed to bid a single price-quantity block, the mechanism would be exactly the same (we just need to restrict k to be equal to 1 for all firms).

5.3 Outcomes of rental markets

Next, we show that, thanks to the demand elasticity reported in Proposition 5.1, rental markets will be able to eliminate price-fixing and produce much more desirable outcomes than real-time markets.

The following Theorem 5.2 shows that, under a condition of $\frac{\max_i C_i}{C}$, i.e., the solar panel size of the largest firm compared to the total panel size in the market, a Nash equilibrium always exists in such a rental market with desirable features.

Theorem 5.2. *Consider the rental market allowing multi-block bids. If* $\frac{\max_i C_i}{C} \le \frac{D^{-1}(C)}{\pi_g \gamma}$, *then* ***(i) (existence of Nash equilibrium)*** *there exists at least one Nash equilibrium where each firm i bids a single price-quantity pair (of quantity C_i and price $D^{-1}(C)$)*[4]*;* ***(ii) (features of any Nash equilibrium)*** *at any Nash equilibrium, all solar panels from all firms are cleared, i.e., $\sum_{k=1}^{K_i} s_{i,k} = \sum_{k=1}^{K_i} q_{i,k} = C_i$ for all i, and the market clearing price must be $\pi_{eq} = D^{-1}(C)$.*

We comment on the highly-desirable features of the market outcome predicted by Theorem 5.2. Note that $D^{-1}(C)$ can be viewed as the price under perfect competition [30]. Theorem 5.2 thus states that the market clearing price π_{eq} is always identical to perfect competition. Further, since all solar panels from all firms are cleared, there is no price-fixing or supply-withholding. Finally, since one of the equilibria corresponds to each firm bidding a single price-quantity pair, the same conclusion will hold if we allow each firm to always bid a single price-quantity pair only. We provide a proof sketch in Section 5.4 and a complete proof in Appendix C.

The condition $\frac{\max_i C_i}{C} \le \frac{D^{-1}(C)}{\pi_g \gamma}$ in Theorem 5.2 can be interpreted as follows. Since $D^{-1}(C) = \pi_{\text{eq}}$, according to Proposition 5.1, the right-hand-side of the condition is simply a lower bound on the demand elasticity. Thus, Theorem 5.2 captures precisely the importance of demand elasticity, which is induced by the randomness of solar generation via Proposition 5.1. Specifically, price-fixing is eliminated as long as no single firm dominates, i.e., the supply of each firm is smaller than a corresponding fraction of the total supply. This fraction is exactly equal to the demand elasticity. Thus, the more random the solar generation is (i.e., the smaller γ is), the higher the demand elasticity, and the larger each firm can be without the worry of price-fixing.

Why supremum in Eq. (4) *is necessary:* Consider the case that $D(\cdot)$ has a discontinuity at the clearing price $\pi_{\text{eq}} = D^{-1}(C)$ predicted by Theorem 5.2. If consumers do not define the demand function as the supremum in Eq. (4), we may have $D(\pi_{\text{eq}}) < C$. Then, there must exist a firm that cannot clear all her panels at price π_{eq}. This firm can then undercut the price π_{eq} to gain more profit, which means that $\pi_{\text{eq}} = D^{-1}(C)$ no longer corresponds to a Nash equilibrium as stated in Theorem 5.2. In contrast, Eq. (4) ensures that $D(\pi_{\text{eq}}) \ge C$, and thus the above situation will not occur.

The outcomes of the rental market for price volatility, fairness, and uniqueness of outcome follows directly from Theorem 5.2 and are concluded in Table 1. It remains to show the social welfare of the rental market. Here, for both real-time and rental markets, we define the social welfare SW as the total surplus of all firms, consumers and the utility, i.e.,

$$\text{SW} = \text{Util}_c - \text{Cost}_c + \text{Rev}_f - \text{Cost}_f + \text{Rev}_u - \text{Cost}_u,$$

where Util_c and Cost_c are the consumers' utility value and cost, respectively, Rev_f and Cost_f are firms' revenue and cost, respectively, and Rev_u and Cost_u are utility company's revenue and cost, respectively, for meeting the consumers' remaining electricity needs not met by the distribution-level market. We also need the following additional assumption.

Assumption 2. *When a consumer's demand $L_n(t)$ is below the solar generation $G(t)c_n$ of the rented panels, the surplus generation $G(t)c_n - L_n(t)$ will be fed back to the grid.*

Assumption 2 is reasonable because, after solar panels are already traded in the rental market, neither the firm nor the consumer has the incentive to curtail the surplus generation. (Note that this assumption is only needed here and is not needed for the earlier Nash equilibrium result.)

The following Proposition 5.3 shows that both markets attain the same social welfare. We provide the proof in Section 5.5.

Proposition 5.3. *Under Assumption 2, the rental market always attains the same social welfare as the real-time market.*

5.4 Proof sketch of Theorem 5.2

We will focus on the second part of the theorem, i.e., at all Nash equilibria (NE) all firms will clear all of their capacity and the resulting market price is $\pi_{\text{eq}} = D^{-1}(C)$. (The first part of theorem for verifying the existence of NE is easier. See Part 1 of Appendix C.5.) To begin with, it is not hard to show that, if at an NE all firms clear all of their capacity, the resulting market price must be $\pi_{\text{eq}} = D^{-1}(C)$ (see Lemma C.12 of the supplemental material). Thus, next we focus on proving that all firms must clear all of their capacity. We prove by contradiction. Suppose on the contrary that, at a Nash equilibrium $(\mathbf{p}, \mathbf{q})$, there exists at least one firm i that does not clear her capacity, i.e., $\sum_{k=1}^{K_i} s_{i,k} < C_i$. We can show that the corresponding market price must satisfy ① $\pi_{\text{eq}} > D^{-1}(C)$, i.e., it must be higher than the market price when all capacity is cleared (see Part 2, Step 5 of Appendix C.5. We then proceed as follows.

First, among all firms with unsold capacity, we can always find a certain firm j such that ② $\frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})} \le \frac{C_j}{C}$, i.e., the proportion of firm j's sold amount to the cleared demand is less than or equal to the proportion of her panel size to the total panel size. To see why, consider the simpler case where all firms have uncleared capacity. Then, the sum of the left-hand-side of ② over all firms j is less than or equal to 1, because the cleared capacity cannot be higher than the cleared demand. However, the corresponding sum of the right-hand-side equals to 1, which leads to a contradiction. Therefore, at least one firm j must satisfy ②. For the more general case where not every firm has uncleared capacity, we can still show ② by first eliminating the contribution of those firms that clear their capacity (see Part 2, Step 1 of Appendix C.5).

Second, we let this firm j deviate to a different bidding strategy. Below, we will use $(\cdot)'$ to denote the new values (for bidding or market outcome) after the deviation. Specifically, the new bidding strategy of firm j is $K_j' = 1$, $p_{j,1}' = \pi^* < \pi_{\text{eq}}$, and $q_{j,1}' = C_j$, i.e.,

[4] In fact, only one firm needs to bid at the price $D^{-1}(C)$ and other firms can bid at a lower price, which still results into a Nash equilibrium.

firm j bids all of her capacity at a lower price $\pi^* < \pi_{\text{eq}}$. Using the property that the original bidding strategy is an NE, we can show that, for some $\delta > 0$, as long as $\pi_{\text{eq}} - \delta \le \pi^* < \pi_{\text{eq}}$, the new market clearing price must be $\pi'_{\text{eq}} = \pi^*$. In other words, the new undercutting bid of firm j will set the market price (see Part 2, Step 3 of Appendix C.5).

Third, we can show that the new cleared quantity of firm j must satisfy ③ $s'_{j,1} - \sum_{k=1}^{K_j} s_{j,k} \ge D(\pi^*) - D(\pi_{\text{eq}})$, i.e, the firm j should be able to clear more capacity, and the increase should be at least equal to the increase in the cleared demand. (Firm j may clear more because it can "rob" other firms' cleared capacity too.)

However, because $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium, the firm j should not earn more profit by deviating to the new bidding strategy. We thus have

$$\begin{aligned}
&\pi^* s'_{j,1} \le \pi_{\text{eq}} \sum\nolimits_{k=1}^{K_j} s_{j,k} \\
\Longrightarrow\ &\pi^* \big(s'_{j,1} - \sum\nolimits_{k=1}^{K_j} s_{j,k}\big) \le \big(\pi_{\text{eq}} - \pi^*\big) \sum\nolimits_{k=1}^{K_j} s_{j,k} \\
\Longrightarrow\ &\frac{D(\pi^*) - D(\pi_{\text{eq}})}{D(\pi_{\text{eq}})} \le \frac{\pi_{\text{eq}} - \pi^*}{\pi^*} \cdot \frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})} \quad \text{(by ③)} \\
\Longrightarrow\ &\left|\frac{\partial D(\pi_{\text{eq}})}{D(\pi_{\text{eq}})}\right| \le \left|\frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}}\right| \frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})} \quad \text{(letting } \pi^* \to \pi_{\text{eq}}\text{)}.
\end{aligned}$$

Using ② and $\frac{C_j}{C} \le \frac{D^{-1}(C)}{\pi_g \gamma}$ (the condition of this theorem), we then have $\left|\frac{\partial D(\pi_{\text{eq}})}{D(\pi_{\text{eq}})}\right| \le \left|\frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}}\right| \frac{C_j}{C} \le \left|\frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}}\right| \frac{D^{-1}(C)}{\pi_g \gamma}$. Applying ①, we thus have $\left|\frac{\partial D(\pi_{\text{eq}})}{D(\pi_{\text{eq}})}\right| < \left|\frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}}\right| \frac{\pi_{\text{eq}}}{\pi_g \gamma}$. This contradicts the elasticity of $D(\cdot)$ stated in Proposition 5.1. The second part of the theorem thus follows.

5.5 Proof of Proposition 5.3 (social welfare comparison)

Proof. The insensitivity reported in Proposition 5.3 is again due to the inelasticity of consumers' real-time demand and the zero marginal-costs of the firms. First, the consumer's utility value Util_c is a constant since the real-time energy demand is inelastic. Second, the firms' cost Cost_f is always zero due to zero marginal costs. Third, the payment of the consumers must equal to the revenue of the firms plus the revenue of the utility company, i.e., $\text{Cost}_c = \text{Rev}_f + \text{Rev}_u$. Thus, for both rental and real-time markets, the only term that may change the social welfare is the utility company's cost. However, under Assumption 2 for rental markets, the amount of electricity that the utility needs to procure from the transmission level is fixed at $[L(t) - G(t)C]^+$. The same is also true for the real-time markets in Section 4. Therefore, the total social welfare is independent of the market outcome, and thus rental markets are as efficient as real-time markets in terms of social welfare. □

5.6 Comparison with forward energy markets

As we discussed at the beginning of Section 5, instead of using rental markets, an alternative form of forward markets would be a forward energy market, similar to the day-ahead market in the transmission level. In this subsection, we briefly discuss about the possible implementation of such a forward energy market at the distribution level, and compare it with the rental market.

Consider a day-ahead energy market where the market price π_{eq} and the sold amount s_i for each firm i are determined before actual generation. In real-time, firm i's actual generation quantity $C_i G(t)$ can be smaller or larger than s_i. We assume that, when under-generation occurs, i.e., $C_i G(t) < s_i$, firm i must pay a penalty for the under-generation amount at the grid price π_g, Thus, firm i's profit is equal to $s_i \pi_{\text{eq}} - \pi_g \mathbb{E}[s_i - C_i G(t)]^+$. Similar to the rental market, we can show that the firms' supply function will also possess elasticity thanks to the uncertainty of generation $G(t)$ (similar to Proposition 5.1). As a result, we can expect that such a forward energy market can also avoid price-fixing thanks to the uncertainty of future generation (similar to Theorem 5.2).

However, in such a forward energy market, firm i usually has to bid (and clear) a lower amount s_i day-ahead, due to the penalty described above. As a result, part of the generation when $C_i G(t) > s_i$, i.e., when the actual generation is larger than the sold amount, is wasted in the day-ahead market. This can also be viewed as a form of withholding supply, and will lead to a lower social welfare compared to the rental market. We provide numerical results in Section 6.4 to illustrate this effect.

One may argue that there may be ways to make this residual amount $[C_i G(t) - s_i]^+$ available to consumers in real-time. For example, the firms may feed-in this residual amount at zero or low feed-in prices, which however may be deemed as unfair to the firms. Alternatively, another real-time energy market could be established to trade this residual amount of energy. Unfortunately, the latter approach will not only increase the complexity of the market design but also re-introduce the price-volatility or price-fixing behaviors (that we reported earlier for real-time energy markets). In contrast, as illustrated by Theorem 5.2 and Proposition 5.3, the proposed rental market sells out all panels, and thus no withholding of supply occurs at the seller side. Furthermore, there is no need to establish two markets at different time-scales. Therefore, we argue that the proposed rental market may be more suitable than the traditional forward energy market.

5.7 Heterogeneity of generation efficiency

As we discussed in Section 3, we assume that all firms have the same generation efficiency $G(t)$. We now briefly discuss how heterogeneous generation efficiency can be handled in our proposed rental market by a normalization procedure (see more detailed discussions in Appendix D). At a high level, the idea is that the utility can pool the generations of all panels together, and calculate a common efficiency $G(t)$ equal to the total generation divided by the sum of declared panel sizes. Then, the utility can calculate an "effective panel size" C_i for each firm based on how her historical generation contributes to this common pool of the normalized generation efficiency $G(t)$. Intuitively, if a firm's contribution to the common pool is higher (correspondingly, lower), its "effective panel size" will also be larger (correspondingly, smaller). Then, our rental markets can operate based on the effective panel size C_i and the common normalized efficiency $G(t)$ only. Note that this normalization procedure does not introduce any surplus/deficit of

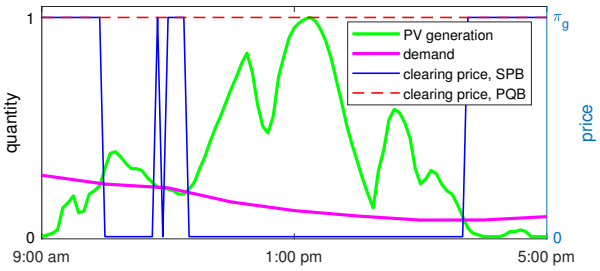

Figure 3: Real-time single-price-bid market experiences high price volatility, while real-time price-quantity-bid market experiences price fixing.

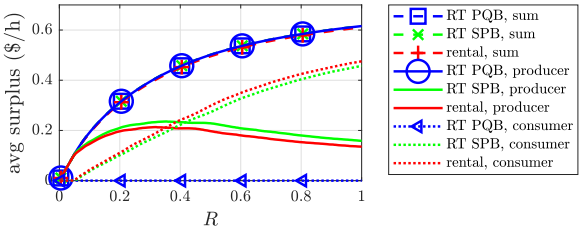

Figure 4: Producer surplus, consumer surplus, and social surplus for different markets under different amount of solar panels.

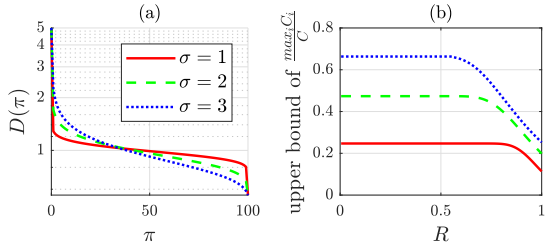

Figure 5: (a) Demand function with different amount of uncertainty for PV generation. (b) The requirement firms' size in Theorem 5.2 (i.e., $\frac{\max_i C_i}{C} \le \frac{D^{-1}(C)}{\pi_g \gamma}$) with different amount of solar panels.

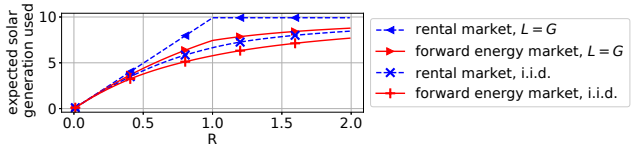

Figure 6: Comparison between the forward energy market and the rental market in terms of the solar generation amount that is actually used by consumers under different penetration ratio R.

solar energy to the market operator, and hence it will not affect the calculation of social surplus.

This normalization procedure has a number of benefits. First, the consumer does not need to be concerned about from which firm she rents the solar panel, because the amount of solar generation "credited" to her rented panel is based on the common $G(t)$ that represents the generation of the whole rental market. Second, this procedure does allow firms with low generation to be penalized, either directly by their lower "effective panel sizes," or by introducing additional penalty based on their actual generation. Finally, other real-time operational issues that may limit actual solar generation, such as malfunctioning of solar panels or physical constraints (e.g., line or voltage limits), can also be accounted for by the effective panel sizes.

6 SIMULATIONS

In this section, we verify the earlier analytical results by simulating the outcomes of different markets in Table 1 using real traces of solar generation and household energy consumption. The solar generation data are from a PV farm located near Purdue University (latitude: 40.45°, longitude: 86.85°). The data were taken every five minutes during the whole year 2006 (provided by NREL[37]). The load data are from two residential houses, one at Purdue University, the other in Indianapolis. The data were taken hourly during a typical meteorological year (provided by EERE[28]). We scale the generation and load data to simulate different solar energy penetration ratio R, which is defined as the ratio between the total solar generation and the total load in the system (averaged across the whole year). The retail price of electricity at Purdue University is equal to $\pi_g = 0.1029$\$/kWh.

6.1 Price volatility and price fixing in real-time markets

As we discussed in Section 4, real-time markets suffer from either price-volatility (with single-price-bid (SPB)) or price-fixing (with price-quantity-bid (PQB)). Indeed, in Fig. 3, for real-time markets with SPB, the clearing price (the blue curve) fluctuates between 0 and π_g as the demand (the pink curve) goes below or above the solar generation (the green curve), suggesting severe price volatility. On the other hand, for real-time markets with PQB, the market clearing price stays at π_g (the dashed curve), suggesting price-fixing.

6.2 Social welfare and fairness of surplus division

According to Proposition 5.3, the rental markets are as efficient as real-time markets in terms of social welfare. We find that this is also true in terms of social surplus (i.e., the sum of only the consumer surplus and the producer surplus, excluding utility's profit). Specifically, in Fig. 4, we plot the consumer surplus, producer surplus, and their sum, for different markets with different solar penetration level R. We can see that the social *surplus* (over consumers and firms only) is also almost the same across the three markets (the top curve). We can thus conclude that the rental market is also quite efficient in terms of social surplus.

However, Fig. 4 shows drastic difference in terms of how the social surplus is split between consumers and firms. For real-time markets with price-quantity bids (RT PQB), the producer surplus

almost overlaps with the social surplus (the top curve), while the corresponding consumer surplus equals to zero (the bottom ◂ curve). In other words, the firms takes all of the social surplus, which is highly unfair. In contrast, both rental markets (red curves) and real-time markets with SPB (green curves) have similar and positive consumer-surplus and producer-surplus. Further, the consumer surplus increases with the solar penetration level R, suggesting that consumers gain more benefit as more solar is invested in these two markets. In contrast, the producer surplus first increases with R but eventually decreases due to the increased competition as more solar is available.

6.3 Impact of solar generation uncertainty

One important insight from our analysis (Proposition 5.1 and Theorem 5.2) is that, in rental markets,, the uncertainty of solar generation becomes a beneficial factor that contributes to the elasticity of the demand function and the desirable market outcome. In order to illustrate this effect, we next turn to a synthetic setting so that we can vary the level of uncertainty easily. We consider a total time duration of unit length, i.e., $T = 1$. The generation $G(t)$ for all $t \in [0, 1]$ is equal to a common random variable G that follows a truncated normal distribution $\mathcal{N}_{\text{truncated}}(\mu, \sigma^2, \underline{G}, \overline{G})$, where a larger σ means higher uncertainty of solar generation. We let the load profile for each consumer be flat, i.e., $L_n(t) = L_n$, for all $t \in [0, 1]$. Thus, the total load $L \triangleq \sum_{n=1}^{N} L_n$ is a constant. In the following numerical simulation, we let $\underline{G} = 2$ and $\overline{G} = 18$, $\mu = \frac{\underline{G}+\overline{G}}{2} = 10$, $L = 10$, and $\pi_g = 10$.

In Fig. 5(a), we plot several demand functions $D(\cdot)$ corresponding to different values of σ. Clearly, as σ increases, the demand function becomes more elastic. For example, the demand function of $\sigma = 1$ (the red curve) is the least elastic one because it is the flattest curve. Then, in Fig. 5(b) we plot the value of $D^{-1}(C)/\pi_g\gamma$ at different solar penetration levels R. Recall from Theorem 5.2 that this quantity is an upper bound on $\max C_i/C$, i.e., the ratio between the maximum panel size of any firm i and the total panel size, so that the desirable outcome predicted by Theorem 5.2 will occur. Clearly, a larger σ (i.e., higher solar uncertainty) makes this quantity bigger, suggesting each firm can be bigger, without losing the guarantee of Theorem 5.2.

6.4 Comparison with forward energy market

We now present simulation results in Fig. 6 to illustrate our discussion in Section 5.6 that the rental market can utilize more solar generation than the forward energy market. We let both $L(t)$ and $G(t)$ be uniformly distributed in $[0, 20]$ for all t (and hence the penetration ratio R is equal to C). For the correlation between generation and consumption, we consider two cases: ① $L(t) = G(t)$, i.e., fully correlated, and ② $L(t)$ and $G(t)$ are independent. Details of other system setup can be found in Appendix E.

In Fig. 6, we plot the expected amount of solar generation used by consumers, where the two solid red curves are for the forward energy market ('►' for case ① and '+' for case ②) and two dashed blue curves are for the rental market ('◄' for case ① and '×' for case ②). We can see that both red curves are lower than the corresponding blue curves when the penetration ratio R is around 1, while their difference becomes smaller when R is away from 1. This behavior is because the used generation amount for the forward energy market equals to the minimum among the solar generation amount, the load amount, and the amount cleared by the forward energy market. When R is very low or very high (i.e., either too much consumption or too much generation), either the solar generation amount or the load amount becomes the bounding term. In contrast, when the total generation amount and the total consumption amount are comparable, the cleared amount becomes the bounding term, in which case the wasting in forward energy market becomes more significant.

7 CONCLUSION AND FUTURE WORK

In this paper, we study different forms of markets for distribution-level energy markets under high penetration of uncertain and zero-marginal-cost solar generation. While real-time markets exhibit either price-volatility or price-fixing, our proposed rental markets eliminate both price-volatility and price-fixing and achieve outcomes such that the market price is stable and uniquely determined. Further, the rental-market is as efficient as real-time markets in terms of social welfare, and maintain positive surplus to both producers and consumers. Although we focus on the case of exactly zero marginal cost in this paper (as in [7]), our conclusion that the rental market performs better than real-time markets (in terms of eliminating price-volatility and price-fixing) still holds even when the marginal cost is close to zero. Finally, our analysis of rental markets reveals the important contribution of the uncertainty of solar generation to the desirable market outcomes. Thus, rental markets could potentially be a highly-desirable alternative to real-time markets in such settings with high penetration of uncertain and zero-marginal-cost resources.

There are several interesting directions for future work. First, in this paper we have focused on the strategic bidding of the firms, but assumed that the consumers' demand functions are always truthful. It would be interesting to study the setting where both sides are strategic. Second, energy storage has been considered an important player in energy markets with uncertain renewable. It would be useful to understand how the addition of storage will change the market operation and market outcomes. Third, this paper assumes that the probability distribution of each consumer's consumption $L_n(t)$ is given. Thus, each consumer can compute her panel demand to minimize the expected cost Eq. (3). In practice, this distribution itself may be uncertain due to the long time-horizon of the rental market (e.g., the consumer may be uncertain whether the next season is hotter or colder than before). In that case, the consumer may bid more aggressively or conservatively than minimizing her expected cost, depending on how she weighs the different possibilities. Studying the rental market under such an imprecise distribution of consumption can be an interesting and valuable future direction.

ACKNOWLEDGMENT

This work has been partially supported by NSF through grants ECCS-1509536 and ECCS-2129631, the Shenzhen Science and Technology Program (Project JCYJ20210324120011032), Guangdong Basic and Applied Basic Research Foundation (Project 2021B1515120008), and the Shenzhen Institute of Artificial Intelligence and Robotics for Society.

REFERENCES

[1] Daron Acemoglu, Kostas Bimpikis, and Asuman Ozdaglar. 2009. Price and capacity competition. *Games and Economic Behavior* 66, 1 (2009), 1–26.

[2] N. Aguiar and V. Gupta. 2021. An Insurance Contract Design to Boost Storage Participation in the Electricity Market. *IEEE Transactions on Sustainable Energy* 12, 1 (2021), 543–552. https://doi.org/10.1109/TSTE.2020.3011052

[3] N. Aguiar, V. Gupta, and P. P. Khargonekar. 2020. A Real Options Market-Based Approach to Increase Penetration of Renewables. *IEEE Transactions on Smart Grid* 11, 2 (2020), 1691–1701. https://doi.org/10.1109/TSG.2019.2942258

[4] Mohamed H Albadi and Ehab F El-Saadany. 2008. A summary of demand response in electricity markets. *Electric power systems research* 78, 11 (2008), 1989–1996.

[5] Simon P Anderson and Régis Renault. 2003. Efficiency and surplus bounds in Cournot competition. *Journal of Economic Theory* 113, 2 (2003), 253–264.

[6] Peder Bacher, Henrik Madsen, and Henrik Aalborg Nielsen. 2009. Online short-term solar power forecasting. *Solar energy* 83, 10 (2009), 1772–1783.

[7] Luiz Barroso, Francisco D Munoz, Bernardo Bezerra, Hugh Rudnick, and Gabriel Cunha. 2021. Zero-Marginal-Cost Electricity Market Designs: Lessons Learned From Hydro Systems in Latin America Might Be Applicable for Decarbonization. *IEEE Power and Energy Magazine* 19, 1 (2021), 64–73.

[8] Graeme N Bathurst, Jennie Weatherill, and Goran Strbac. 2002. Trading wind generation in short term energy markets. *IEEE Transactions on Power Systems* 17, 3 (2002), 782–789.

[9] Morgan Bazilian, Ijeoma Onyeji, Michael Liebreich, Ian MacGill, Jennifer Chase, Jigar Shah, Dolf Gielen, Doug Arent, Doug Landfear, and Shi Zhengrong. 2013. Re-considering the economics of photovoltaic power. *Renewable Energy* 53 (2013), 329–338.

[10] Eilyan Y Bitar, Ram Rajagopal, Pramod P Khargonekar, Kameshwar Poolla, and Pravin Varaiya. 2012. Bringing wind energy to market. *IEEE Transactions on Power Systems* 27, 3 (2012), 1225–1235.

[11] Valerii V Buldygin and Yu V Kozachenko. 1980. Sub-Gaussian random variables. *Ukrainian Mathematical Journal* 32, 6 (1980), 483–489.

[12] Eric Cardella, Bradley T Ewing, and Ryan B Williams. 2017. Price volatility and residential electricity decisions: Experimental evidence on the convergence of energy generating source. *Energy Economics* 62 (2017), 428–437.

[13] Stanley KH Chow, Eric WM Lee, and Danny HW Li. 2012. Short-term prediction of photovoltaic energy generation by intelligent approach. *Energy and Buildings* 55 (2012), 660–667.

[14] Samuel R Dastrup, Joshua Graff Zivin, Dora L Costa, and Matthew E Kahn. 2012. Understanding the Solar Home price premium: Electricity generation and "Green" social status. *European Economic Review* 56, 5 (2012), 961–973.

[15] Allan P. Donsig and Kenneth R. Davidson. 2002. *Real Analysis with Real Applications.* Prentice Hall.

[16] Lei Fan, Jianhui Wang, Ruiwei Jiang, and Yongpei Guan. 2014. Min-max regret bidding strategy for thermal generator considering price uncertainty. *IEEE Transactions on Power Systems* 29, 5 (2014), 2169–2179.

[17] Bahman Gharesifard, Tamer Başar, and Alejandro D Domínguez-García. 2015. Price-based coordinated aggregation of networked distributed energy resources. *IEEE Trans. Automat. Control* 61, 10 (2015), 2936–2946.

[18] Sergio Luis Franklin Júnior and Nelio Domingues Pizzolato. 2013. Pricing and Costing Practices in Telecommunications: the Brazilian Experience and Challenges. *Brazilian Journal of Operations & Production Management* 10, 1 (2013), 7–24.

[19] Daniel Kahneman, Jack L Knetsch, and Richard Thaler. 1986. Fairness as a constraint on profit seeking: Entitlements in the market. *The American economic review* (1986), 728–741.

[20] Dileep Kalathil, Chenye Wu, Kameshwar Poolla, and Pravin Varaiya. 2017. The sharing economy for the electricity storage. *IEEE Transactions on Smart Grid* (2017).

[21] Vanessa Krebs, Lars Schewe, and Martin Schmidt. 2018. Uniqueness and multiplicity of market equilibria on DC power flow networks. *European Journal of Operational Research* 271, 1 (2018), 165–178.

[22] Pan Li, Shreyas Sekar, and Baosen Zhang. 2018. A capacity-price game for uncertain renewables resources. *arXiv preprint arXiv:1804.06496* (2018).

[23] Julija Matevosyan and Lennart Soder. 2006. Minimization of imbalance cost trading wind power on the short-term power market. *IEEE Transactions on Power Systems* 21, 3 (2006), 1396–1404.

[24] Esther Mengelkamp, Johannes Gärttner, Kerstin Rock, Scott Kessler, Lawrence Orsini, and Christof Weinhardt. 2018. Designing microgrid energy markets: A case study: The Brooklyn Microgrid. *Applied Energy* 210 (2018), 870–880.

[25] Midcontinent Independent System Operator, Inc. accessed May 9, 2019. *MISO Business Practices Manuals.* Midcontinent Independent System Operator, Inc. https://www.misoenergy.org/legal/business-practice-manuals/

[26] Juan M Morales, Antonio J Conejo, Henrik Madsen, Pierre Pinson, and Marco Zugno. 2013. *Integrating renewables in electricity markets: operational problems.* Vol. 205. Springer Science & Business Media.

[27] Steven Mufson. 2018. The hottest fight in American politics? Arizona's smackdown over solar power. *The Washington Post* (2018). https://www.washingtonpost.com/business/economy/the-hottest-fight-in-american-politics-arizonas-smackdown-over-solar-power/2018/11/02/9f761030-d3dc-11e8-8c22-fa2ef74bd6d6_story.html?noredirect=on&utm_term=.7b2b922f2004

[28] Office of Energy Efficiency & Renewable Energy (EERE). [n.d.]. Commercial and Residential Hourly Load Profiles for all TMY3 Locations in the United States. https://openei.org/doe-opendata/dataset/commercial-and-residential-hourly-load-profiles-for-all-tmy3-locations-in-the-united-states. Accessed: 2019-07-28.

[29] M. E. Peck and D. Wagman. 2017. Energy trading for fun and profit buy your neighbor's rooftop solar power or sell your own-it'll all be on a blockchain. *IEEE Spectrum* 54, 10 (October 2017), 56–61. https://doi.org/10.1109/MSPEC.2017.8048842

[30] Joan Robinson. 1969. *The economics of imperfect competition.* Springer.

[31] Claudia P Rodriguez and George J Anders. 2004. Bidding strategy design for different types of electric power market participants. *IEEE Transactions on Power Systems* 19, 2 (2004), 964–971.

[32] Mardavij Roozbehani, Munther A Dahleh, and Sanjoy K Mitter. 2012. Volatility of power grids under real-time pricing. *IEEE Transactions on Power Systems* 27, 4 (2012), 1926–1940.

[33] Elias M Stein and Rami Shakarchi. 2009. *Real analysis: measure theory, integration, and Hilbert spaces.* Princeton University Press.

[34] George J Stigler. 1957. Perfect competition, historically contemplated. *Journal of political economy* 65, 1 (1957), 1–17.

[35] Yan Su, Lai-Cheong Chan, Lianjie Shu, and Kwok-Leung Tsui. 2012. Real-time prediction models for output power and efficiency of grid-connected solar photovoltaic systems. *Applied energy* 93 (2012), 319–326.

[36] Joshua A Taylor and Johanna L Mathieu. 2016. Strategic bidding in electricity markets with only renewables. In *2016 American Control Conference (ACC).* IEEE, 5885–5890.

[37] The National Renewable Energy Laboratory (NREL). [n.d.]. Solar Power Data for Integration Studies. https://www.nrel.gov/grid/solar-power-data.html. Accessed: 2019-07-28.

[38] Govinda R Timilsina, Lado Kurdgelashvili, and Patrick A Narbel. 2012. Solar energy: Markets, economics and policies. *Renewable and sustainable energy reviews* 16, 1 (2012), 449–465.

[39] Wayes Tushar, Chau Yuen, Hamed Mohsenian-Rad, Tapan Saha, H Vincent Poor, and Kristin L Wood. 2018. Transforming energy networks via peer-to-peer energy trading: the potential of game-theoretic approaches. *IEEE Signal Processing Magazine* 35, 4 (2018), 90–111.

A PROOFS IN SECTION 4

We provide the following two algorithms to describe how real-time markets compute market price and sold amount according to bidding and demand. Notice that those algorithms work for both the single-price-bid market and markets allowing price-quantity bids.

Algorithm 3: Compute market price π_{eq}.

```
Sort elements in {p_i : for all i} as π_1 < π_2 < ··· < π_R (note
  that all π_r's are distinct);
Q ← 0;
π_eq ← π_g;
for r = 1 : R do
    Q ← Q + Σ_{i: p_i=π_r} q_i;
    if L(t) < Q and π_r < π_g then
        π_eq ← π_r;
        break;
```

Remark: In Eq. (8) of Algorithm 4, bids at the market price π_{eq} split the sold amount in proportion to their generation amount. Here is an example. After clearing the supply below the price π_{eq}, the remaining demand is 10, firm 1 has a bid at the price π_{eq} with the amount 60, firm 2 has a bid at the price π_{eq} with the amount 40. There is no other bid at the price π_{eq}. Note that in this case

Algorithm 4: Decide the sold amount s_i.

$Q \leftarrow 0$;
$s_i \leftarrow 0$, for all i;
for $i = 1 : M$ **do**
  if $p_i < \pi_{eq}$ **then**
    $s_i \leftarrow q_i$;
    $Q \leftarrow Q + q_i$;
Denote those bids with the price π_{eq} as $q_{a_1}, q_{a_2}, \cdots, q_{a_h}$.
for $i = 1 : h$ **do**

$$s_{a_i} \leftarrow \min\left\{\frac{q_{a_i}(L(t)-Q)}{\sum_{l=1}^{h} q_{a_l}},\ q_{a_i}\right\}. \tag{8}$$

the market price will indeed equal to π_{eq} according to Algorithm 3. Then, firm 1 sells the amount 6 and the firm 2 sells the amount 4, i.e., they split the remaining demand 10 in proportion to their bidding amount.

We immediately get the following lemma that reveals useful properties of those two algorithms.

Lemma A.1. *(a) If $\sum_{i=1}^{M} q_i > L(t)$, then we must have $\pi_{eq} = p_j$ for all j that satisfies*

$$\sum_{i:\ p_i \le p_j} q_i > L(t),$$
$$\sum_{i:\ p_i < p_j} q_i \le L(t),$$

and this kind of j must exists.
(b) If $s_i = 0$, then we must have $p_i \ge \pi_{eq}$.
(c) If $p_i > \pi_{eq}$, then $s_i = 0$. If $p_i < \pi_{eq}$, then $s_i = q_i$.
(d) If $0 < s_i < q_i$, then we must have $p_i = \pi_{eq}$. Further, if $0 < s_i < q_i$ and $s_j = q_j$, then we must have $p_j < p_i = \pi_{eq}$.
(e) When $L(t) \le \sum_i q_i$, we have

$$L(t) = \sum_{i:\ p_i < \pi_{eq}} q_i + \sum_{i:\ p_i = \pi_{eq}} s_i.$$

Remark: These results are intuitive. Part (a) states that at π_{eq}, supply and demand are close. Part (b) states that if one bid gets no sell, then this bid must have the price higher than π_{eq}. Part (c) states that bidding above π_{eq} gets no sell, while bidding below π_{eq} sells all. Part (d) states that a partly sold bid must have the price π_{eq}. Part (e) states that when the supply is enough, the total sell amount equals to the demand.

Proof. (a) Because $\sum_{i=1}^{M} q_i > L(t)$, the condition stated in Line 6 of Algorithm 3 must be met at some iteration. Thus, we can directly get the result of this Lemma by Line 5~7 of Algorithm 3.

(b) By Algorithm 4, we know that if $p_i < \pi_{\text{eq}}$, then $s_i = p_i > 0$. Thus, if $s_i = 0$, then we must have $p_i \ge \pi_{\text{eq}}$.

(c) This result is directly derived from Algorithm 4.

(d) Due to Line 5 of Algorithm 4, we get the first statement that $p_i = \pi_{\text{eq}}$ if $0 < s_i < q_i$. Next, we prove the second statement. Because $s_j = q_j > 0$, we know that $p_j \le \pi_{\text{eq}}$ (otherwise $s_j = 0$ from part (c)). Thus we only need to prove that $p_j = \pi_{\text{eq}}$ is impossible. We prove by contradiction. Suppose in contrary that $p_j = \pi_{\text{eq}}$. Since $0 < s_i < p_i$, from Eq. (8) we know that $L(t) - Q > \sum_{l=1}^{h} q_{a_l}$. Thus, we must have $s_j < q_j$ by Eq. (8). This contradicts with $s_j = q_j$. Thus, we conclude that $p_j \ne \pi_{\text{eq}}$. As a result, we must have $p_j < \pi_{\text{eq}}$. Finally, using the result of the first statement, we can get the result of the second statement that $p_j < p_i = \pi_{\text{eq}}$.

(e) This result directly follows from Algorithm 4. □

A.1 Proof of Proposition 4.1

We first give the following on the statement about any Nash equilibrium in the single-price-bid market, where every firm bids all generation amount, i.e., $q_i = q_i^0$ for all $i \in \{1, 2, \cdots, M\}$.

Lemma A.2. *Suppose $\sum_{i\in S} q_i^0 \ne L(t)$ for all $S \subseteq \{1, 2, \cdots, M\}$ and $\sum_{i=1}^{M} q_i^0 > L(t)$. At a Nash equilibrium, if $\pi_{eq} > 0$, then there must exist one and only one firm j that satisfies $0 < s_j < q_j^0$, and $s_i = q_i^0$ for all other firms $i \ne j$.*

Proof. We split the proof by several steps as below.

Step 1: we prove that, for any bidding strategy (not necessarily a Nash equilibrium), we must have $s_i > 0$ for all $p_i = \pi_{\text{eq}}$. To see this, by Line 6 of Algorithm 4, we know that Q must equal to the sum of the elements in some subset of $\{q_1^0, q_2^0, \cdots, q_M^0\}$. In other words, we can write $Q = \sum_{i\in S_0} q_i^0 \ne L(t)$, where S_0 consists of the indices of firms that bid prices less than π_{eq}. Obviously, we have $S_0 \subseteq \{1, 2, \cdots, M\}$. Because $\sum_{i\in S} q_i^0 \ne L(t)$ for all $S \subseteq \{1, 2, \cdots, M\}$, in Eq. (8) we must have $L(t) \ne Q$. Thus, by Eq. (8), we must have $s_i > 0$ for all $p_i = \pi_{\text{eq}}$.

Step 2: we prove that $s_i > 0$ for all i at any Nash equilibrium. We prove by contradiction. Suppose on the contrary that there exists a firm i^* such that $s_{i^*} = 0$. Thus, the payoff of the firm i^* equals to $s_{i^*}\pi_{\text{eq}} = 0$. Because $s_{i^*} = 0$, by Lemma A.1(b), we have $p_{i^*} \ge \pi_{\text{eq}}$. Further, considering the result in step 1, we must have $p_{i^*} > \pi_{\text{eq}}$. By Lemma A.1(a), we have

$$\sum_{i:\ p_i \le \pi_{\text{eq}}} q_i^0 > L(t), \tag{9}$$
$$\sum_{i:\ p_i < \pi_{\text{eq}}} q_i^0 \le L(t). \tag{10}$$

Now, let the firm i^* deviate to another bidding strategy with $p'_{i^*} = \pi_{\text{eq}}$. Then, Eq. (9) and Eq. (10) still hold. By Lemma A.1(a), we know that the new market price will not change, i.e., $\pi'_{\text{eq}} = \pi_{\text{eq}}$. By the result of step 1, we have $s'_{i^*} > 0$. Thus, the new payoff of the firm i^* equals to $s'_{i^*}\pi'_{\text{eq}} = s'_{i^*}\pi_{\text{eq}} > 0$, which is larger than the previous payoff of zero. This contradicts the assumption that the original bidding strategy is a Nash equilibrium. Thus, we have proven that $s_i > 0$ for all i.

Step 3: we prove that there exists one and only one firm j that satisfies $s_j < q_j^0$ at any Nash equilibrium. Because $\sum_{i\in S} q_i^0 \ne L(t)$ for all $S \subseteq \{1, 2, \cdots, M\}$, Lemma A.1(e) implies that at least one firm j satisfies $0 < s_j < q_j^0$. Now, we prove by contradiction that no other firms satisfy $s_j < q_j^0$. Suppose on the contrary that there exists another firm $k \ne j$ such that $0 < s_k < q_k^0$. By Lemma A.1(d), we have $p_j = p_k = \pi_{\text{eq}}$. The payoff of the firm j equals to $s_j\pi_{\text{eq}}$. By

Lemma A.1(a) we have

$$\sum_{i:\ p_i<\pi_{\text{eq}}} q_i^0 \le L(t).$$

By Lemma A.1(e) we have

$$L(t) = \sum_{i:\ p_i<\pi_{\text{eq}}} q_i^0 + \sum_{i:\ p_i=\pi_{\text{eq}}} s_i \ge \left(\sum_{i:\ p_i<\pi_{\text{eq}}} q_i^0\right) + s_j + s_k. \quad (11)$$

Now, let the firm j deviate to another bidding strategy that $p_j' = \pi_{\text{eq}} - \epsilon$ where

$$\epsilon = \frac{\pi_{\text{eq}} \min\{s_k,\ q_j^0 - s_j\}}{2\min\{s_j + s_k,\ q_j^0\}}, \quad (12)$$

and all other firms' bidding prices do not change, i.e., $p_i' = p_i$, for all $i \neq j$. (Note that $0 < \epsilon < \frac{\pi_{\text{eq}}}{2}$ because $\min\{s_k,\ q_j^0 - s_j\} < \min\{s_k,\ q_j^0 - s_j\} + s_j = \min\{s_j + s_k,\ q_j^0\}$.) Then, we must have $\{i:\ p_i < \pi_{\text{eq}} - \epsilon\} = \{i:\ p_i' < \pi_{\text{eq}} - \epsilon\}$, $\{i:\ p_i = \pi_{\text{eq}} - \epsilon\} = \{i:\ p_i' = \pi_{\text{eq}} - \epsilon,\ i \neq j\}$, and

$$\sum_{i:\ p_i'<\pi_{\text{eq}}-\epsilon} q_i^0 = \sum_{i:\ p_i<\pi_{\text{eq}}-\epsilon} q_i^0 \le \sum_{i:\ p_i<\pi_{\text{eq}}} q_i^0 \le L(t),$$

where the last inequality follows from Eq. (11). By Lemma A.1(a), it implies that the new market price must satisfy $\pi_{\text{eq}}' \ge \pi_{\text{eq}} - \epsilon$. There exist two possible cases. Case 1: $\pi_{\text{eq}}' > \pi_{\text{eq}} - \epsilon$. By Lemma A.1(c), we have $s_j' = q_j^0$. Case 2: $\pi_{\text{eq}}' = \pi_{\text{eq}} - \epsilon$. We have

$$\begin{aligned}
&\sum_{i:\ p_i<\pi_{\text{eq}}-\epsilon} q_i^0 + \left(\sum_{i:\ p_i=\pi_{\text{eq}}-\epsilon} s_i'\right) + s_j' \\
&= \sum_{i:\ p_i'<\pi_{\text{eq}}-\epsilon} q_i^0 + \sum_{i:\ p_i'=\pi_{\text{eq}}-\epsilon} s_i' \\
&\qquad \text{(because only the firm } j \text{ deviates)} \\
&= L(t) \quad \text{(by Lemma A.1(e))} \\
&\ge \left(\sum_{i:\ p_i<\pi_{\text{eq}}} q_i^0\right) + s_j + s_k \text{ (by Eq. (11)).}
\end{aligned}$$

Moving the first two terms of the left-hand side to the right-hand side, we have

$$\begin{aligned}
s_j' &\ge s_j + s_k + \left(\sum_{i:\ p_i<\pi_{\text{eq}}} q_i^0 - \sum_{i:\ p_i<\pi_{\text{eq}}-\epsilon} q_i^0 - \sum_{i:\ p_i=\pi_{\text{eq}}-\epsilon} s_i'\right) \\
&\ge s_j + s_k + \left(\sum_{i:\ p_i<\pi_{\text{eq}}} q_i^0 - \sum_{i:\ p_i<\pi_{\text{eq}}-\epsilon} q_i^0 - \sum_{i:\ p_i=\pi_{\text{eq}}-\epsilon} q_i^0\right) \\
&= s_j + s_k + \left(\sum_{i:\ p_i<\pi_{\text{eq}}} q_i^0 - \sum_{i:\ p_i\le\pi_{\text{eq}}-\epsilon} q_i^0\right) \\
&\ge s_j + s_k.
\end{aligned}$$

In summary, in both cases, we always have $s_j' \ge \min\{s_j + s_k,\ q_j^0\}$. Thus, the new payoff $\pi_{\text{eq}}' s_j'$ of the firm j satisfies

$$\begin{aligned}
\pi_{\text{eq}}' s_j' &\ge (\pi_{\text{eq}} - \epsilon)\min\{s_j + s_k,\ q_j^0\} \\
&= \pi_{\text{eq}}\min\{s_j + s_k,\ q_j^0\} - \epsilon\min\{s_j + s_k,\ q_j^0\} \\
&= \pi_{\text{eq}}\min\{s_j + s_k,\ q_j^0\} - \frac{\pi_{\text{eq}}\min\{s_k,\ q_j^0 - s_j\}}{2} \\
&\qquad \text{(by Eq. (12))} \\
&> \pi_{\text{eq}}\big(\min\{s_j + s_k,\ q_j^0\} - \min\{s_k,\ q_j^0 - s_j\}\big) \\
&= \pi_{\text{eq}} s_j.
\end{aligned}$$

That means that the firm j gets more payoff after deviating, which contradicts the assumption that the original bidding strategy is a Nash equilibrium. Thus, we have proven that there exists one and only one firm j that satisfies $0 < s_j < q_j^0$.

Finally, recall from step 2 that for all firm i must satisfy $s_i > 0$. Thus, the result of step 3 also implies that $s_i = q_i^0$ for all $i \neq j$ (where the firm j is the one in step 3 that is the only firm that satisfies $0 < s_j < q_j^0$). □

Now we are ready to prove Proposition 4.1.

A.1.1 Proof of Case 1.

Proof. When $\sum_{i=1}^M q_i^0 \le L(t)$, Line 6 of Algorithm 3 will never be reached. Thus, the market price must be $\pi_{\text{eq}} = \pi_g$. Next, we show that every firm bids at π_g is a Nash equilibrium. First, when every firm bids at π_g, because $\pi_{\text{eq}} = \pi_g$, we have $a_i = i$ and $Q = 0$ in Eq. (8). Thus, we have $s_i = q_i^0$ in Eq. (8) since $L(t) - Q = L(t) > \sum_{i=1}^M q_i^0$. As a result, the profit of the firm i equals $q_i^0 \pi_g$. Since at any circumstance the market price cannot exceed π_g and the firm i cannot sell more than q_i^0, the firm i cannot earn more profit than $q_i^0 \pi_g$ by choosing any other bidding strategy. Thus, the situation that every firm bids at π_g is Nash equilibrium. □

A.1.2 Proof of Case 2.

Proof. First, we prove that the strategy that all firms bid zero price is a Nash equilibrium. When every firm i bids zero price $p_i = 0$, because $\sum_{i=1}^M q_i^0 \ge \sum_{i\neq j} q_i^0 > L(t)$ for all j, the condition in Line 6 of Algorithm 3 must be met at some time. Thus, we must have $\pi_{\text{eq}} = 0$, and the payoff for every firm i equals to $q_i^0 \pi_{\text{eq}} = 0$. If one firm j^* bids differently (i.e., $p_{j^*}' > 0$), since $\sum_{i\neq j^*} q_i^0 > L(t)$, we still have $\pi_{\text{eq}}' = 0$ by Algorithm 3. Because $p_{j^*}' > 0 = \pi_{\text{eq}}'$, by Algorithm 4, we have $s_{j^*}' = 0$. As a result, the new payoff of the firm j^* equals to $\pi_{\text{eq}}' s_{j^*}' = 0$, i.e., the firm j^* cannot get more benefits. Thus, we have proven that the strategy that all firms bid zero price is a Nash equilibrium.

Then, we prove the second statement of this proposition by contradiction. Suppose on the contrary that, at a Nash equilibrium, we have $\pi_{\text{eq}} > 0$. By Lemma A.2, we have one firm j^* such that $0 < s_{j^*} < q_{j^*}$ and $s_i = q_i^0$ for all other firms $i \neq j^*$. By Lemma A.1(d), we know $p_{j^*} = \pi_{\text{eq}}$ and $p_i < \pi_{\text{eq}}$ for all $i \neq j^*$. Thus, by

Lemma A.1(e), we have

$$L(t) = \left(\sum_{i\neq j^*} q_i^0\right) + s_{j^*} > \sum_{i\neq j^*} q_i^0.$$

This contradicts the assumption that $\sum_{i\neq j} q_i^0 > L(t)$ for all j. Thus, we have proven that at any Nash equilibrium, we must have $\pi_{\text{eq}} = 0$. □

A.1.3 Proof of Case 3.

Proof. First, we illustrate that the following bidding strategy is a Nash equilibrium: one firm $j \in \mathcal{I}$ bids the price π_g, and all other firms bid the price zero. Since $\sum_{i\neq j} q_i^0 < L(t)$, we have $\pi_{\text{eq}} = \pi_g$ because Line 6 of Algorithm 3 will never be met. By Algorithm 4, any firm $i \neq j$ sells all, i.e., $s_i = q_i^0$. Thus, any firm $i \neq j$ has already achieved its maximum possible payoff $q_i^0\pi_g$ and none of them has an incentive to deviate. Consider the firm j. Since all firms except the firm j bid zero price, the firm j cannot sell more unless bidding the price zero. Thus, if the firm j bids any positive price less than π_g, its payoff will be lower. If the firm j bids zero price, by Algorithm 3 we know the market price will be zero and thus its payoff will be zero because $\sum_{i=1}^{M} q_i^0 > L(t)$. In summary, we conclude that the firm j cannot get more payoff by deviating. Thus, we have shown the strategy that $p_j = \pi_{\text{eq}},\ p_i\Big|_{i\neq j} = 0$ is a Nash equilibrium.

Then, we prove that at any Nash equilibrium we must have $\pi_{\text{eq}} = \pi_g$ and only one firm in $\mathcal{I}$ bids at π_g. We split the proof by several steps as follows.

Step 1: we show that if any firm $j \in \mathcal{I}$ bids the price $p_j = \pi_g$, then $\pi_{\text{eq}} = \pi_g$ (regardless of other firms' bids). At the beginning of this subsection, we have already made the assumption that a legitimate bid should satisfy $p_i \in [0, \pi_g]$. Thus, we have

$$\sum_{i:\ p_i\leq\pi_g} q_i^0 = \sum_{i=1}^{M} q_i^0 > L(t).$$

Further, if any firm $j \in \mathcal{I}$ bids the price $p_j = \pi_g$, we then have

$$\sum_{i:\ p_i<\pi_g} q_i^0 \leq \sum_{i\neq j} q_i^0 < L(t) \text{ (by the definition of } \mathcal{I}).$$

By Lemma A.1(a), we know $\pi_{\text{eq}} = \pi_g$. Thus, we have proven that if $p_j = \pi_g$ for some $j \in \mathcal{I}$, then $\pi_{\text{eq}} = \pi_g$.

Based on the result of step 1, to complete the rest of the proof, we only need to show that, at any Nash equilibrium, there must exist one firm $j \in \mathcal{I}$ that bids $p_j = \pi_g$. Towards this end, we will first show that $\pi_{\text{eq}} > 0$ at any Nash equilibrium in step 2, based on which we then apply Lemma A.2 in step 3.

Step 2: We prove $\pi_{\text{eq}} > 0$ at any Nash equilibrium by contradiction. Suppose on the contrary that $\pi_{\text{eq}} = 0$ at a Nash equilibrium. Consider any firm $j \in \mathcal{I}$. By step 1, we know $p_j \neq \pi_g$. Note that since $\pi_{\text{eq}} = 0$, the payoff of the firm j equals to 0. Let the firm j deviate its bidding strategy to $p_j' = \pi_g$. By step 1, we know that the new market price equals to $\pi_{\text{eq}}' = \pi_g$. Because $\sum_{i\in S} q_i^0 \neq L(t)$ for all $S \subseteq \{1, 2, \cdots, M\}$, by Eq. (8) we must have $s_j' > 0$. Thus, the new payoff of the firm j equals to $s_j'\pi_{\text{eq}}' > 0$, which is larger than the payoff of its previous bidding strategy. This contradicts the assumption that the previous bidding strategy is a Nash equilibrium. Thus, we have proven that $\pi_{\text{eq}} > 0$ at any Nash equilibrium.

Step 3: we prove $\pi_{\text{eq}} = \pi_g$ at any Nash equilibrium. We prove by contradiction. Suppose on the contrary that $\pi_{\text{eq}} < \pi_g$. By the result of step 2, we must have $\pi_{\text{eq}} > 0$. By Lemma A.2, there exists one and only one firm j such that $0 < s_j < q_j^0$ and $s_i = q_i^0$ for all $i \neq j$. Thus, by Lemma A.1(e), we have

$$L(t) = \left(\sum_{i\neq j} q_i^0\right) + s_j,$$

which implies that $\sum_{i\neq j} q_i^0 < L(t)$, i.e., $j \in \mathcal{I}$. By Lemma A.1(d), we have $p_j = \pi_{\text{eq}} < \pi_g$ and $p_i < p_j$ for all $i \neq j$. The payoff of the firm j thus equals to $\pi_{\text{eq}} s_j$. Now, let the firm j deviate to another bidding strategy that $p_j' = \pi_g$. By step 1, we know that the new market price equals to $\pi_{\text{eq}}' = \pi_g$. Because $p_i < p_j = \pi_{\text{eq}} < \pi_g = \pi_{\text{eq}}'$ for all $i \neq j$, by Lemma A.1(e), we have $s_j' = L(t) - \sum_{i\neq j} q_j^0 = s_j$. Thus, the new payoff of the firm j equals $\pi_{\text{eq}}' s_j' = \pi_g s_j' > \pi_{\text{eq}} s_j$. This contradicts the assumption that the previous bidding strategy is a Nash equilibrium. Thus, we have proven that $\pi_{\text{eq}} = \pi_g$.

In summary, we conclude that, at any Nash equilibrium, we must have $\pi_{\text{eq}} = \pi_g$ and only one firm in $\mathcal{I}$ bids at the price π_g while other firms bid the price below π_g. □

A.2 Proof of Proposition 4.2

Proof. We first verify that all Nash equilibria in Proposition 4.1 are still Nash equilibria in the price-quantity-bid mechanism. To that end, we will check whether there exists any firm that can earn more profit by withholding supply (who may also change the corresponding bidding price at the same time). For Case 1 of Proposition 4.1, the price is already at the highest possible price π_g. Further, at the Nash equilibrium (where every firm bids at π_g) every firm has already sold all generation. Thus, withholding supply and/or changing the bidding price will not increase the payoff of any firm. Therefore, all firms bidding their full generation at price π_g is still a Nash equilibrium. For Case 2 of Proposition 4.1, no firm is large enough to change the situation from abundant supply to limited supply because $\sum_{i\neq j} q_i^0 > L(t)$ for all j, which implies that the market price will still always be zero and thus the profit of the firm that withholding supply will still be zero. Therefore, the Nash equilibrium of Case 2 (every firm bidding at zero price) is still a Nash equilibrium. For the Nash equilibrium of Case 3 (one firm $j \in \mathcal{I}$ bids at π_g, all other firms $i \neq j$ bid at zero price), first note that those firms with zero bidding price already achieve their maximum payoff (selling all generation at π_g). Hence, they have no incentive to withhold supply and/or change the bidding price. It remains to consider the firm j who bids at π_g. We consider three sub-cases. ① If the firm j bids at another price $p_j' \in (0,\ \pi_g]$, then by Algorithm 3 and Algorithm 4 we know that, after deviation from π_g to p_j', the market price becomes $\pi_{\text{eq}}' = p_j'$, and the firm j's sold amount becomes $s_j' = \min\{L(t) - \sum_{i\neq j} q_i^0, q_j'\}$ (where $q_j' \in [0, q_j^0]$ denotes the new bidding quantity of firm j after deviation which allows supply withholding). Let $s_j = L(t) - \sum_{i\neq j} q_i^0$ be the sold amount of firm j before deviation. Thus, the firm j's payoff after

the deviation is

$$\begin{aligned} s_j'\pi_{\text{eq}}' &= \min\left\{L(t) - \sum_{i\neq j} q_i^0,\ q_j'\right\} \cdot p_j' = \min\{s_j, q_j'\} \cdot p_j' \\ &\leq s_j \pi_{\text{eq}}, \end{aligned}$$

which implies that the firm j cannot increase her payoff after deviation if $p_j' \in (0, \pi_g]$. ② If $p_j' = 0$ and $q_j' > L(t) - \sum_{i\neq j} q_i^0$, then $\pi_{\text{eq}}' = 0$ and thus the firm j' payoff becomes zero, which also implies that her payoff does not increase. ③ If $p_j' = 0$ and $q_j' \leq L(t) - \sum_{i\neq j} q_i^0 = s_j$, then $s_j'\pi_{\text{eq}}' \leq q_j'\pi_g \leq s_j\pi_{\text{eq}}$, which implies that the firm j cannot increase her payoff after deviation. Therefore, in all possible sub-cases ①②③ of Case 3, we have shown that firm j cannot increase her payoff by changing her bidding. We can then conclude that the original Nash equilibrium of Case 3 is still a Nash equilibrium when withholding supply is allowed. To sum up, we have shown that all Nash equilibria in all three cases of Proposition 4.1 are still Nash equilibria under price-quantity-bids.

We then focus on proving the new Nash equilibrium described in (2). First, we prove that, if $L(t) > \sum_{i=1}^M q_i^0$, then there must exist a Pareto-optimal Nash equilibrium with $\pi_{\text{eq}} = \pi_g$. Specifically, we want to show the bidding strategy that $q_i = q_i^0, p_i = \pi_{\text{eq}}$ for all i is a Pareto-optimal Nash equilibrium with $\pi_{\text{eq}} = \pi_g$. Because $L(t) > \sum_{i=1}^M q_i^0 = \sum_{i=1}^M q_i$, by Algorithm 3, we have $\pi_{\text{eq}} = \pi_g$. By Algorithm 4, we have $s_i = q_i$. Because $q_i^0 = q_i$, any firm i sells all of its generation, i.e., $s_i = q_i^0$. Thus, each firm i has already achieved its maximum possible profit $\pi_g q_i^0$. In other words, no bidding strategy can make any firm get more profit. Thus, the bidding strategy $q_i = q_i^0,\ p_i = \pi_{\text{eq}}$ for all i is a Pareto-optimal Nash equilibrium.

Then, we prove that if $L(t) \leq \sum_{i=1}^M q_i^0$, then there must exist a Pareto-optimal Nash equilibrium with $\pi_{\text{eq}} = \pi_g$. Specifically, we will show that any bidding strategy that satisfies the following conditions is such a Nash equilibrium:

$$\begin{cases} L(t) = \sum_{i=1}^M q_i, \\ p_i = 0 \text{ for all } i. \end{cases} \tag{13}$$

Because $L(t) \leq \sum_{i=1}^M q_i^0$, we can always find such $(q_0, q_1, \cdots)$ that satisfies Eq. (13). By Algorithm 3 (especially Line 6), we know that $\pi_{\text{eq}} = \pi_g$ in this situation[5]. By Algorithm 4, we know $s_i = p_i$ for all i. Thus, the profit of any firm j equals $p_j\pi_g$. Now, suppose that the firm j deviates to an arbitrary bidding strategy $(q_j',\ p_j')$, while the bids of the other firms remain the same, i.e., $p_i' = p_i,\ q_i' = q_i$ for all $i \neq j$. There are three different cases.

Case 1: $p_j' > 0$. By Algorithm 4, we know that $s_i' = q_i' = q_i$ for all $i \neq j$, because all other firms $i \neq j$ bid lower than the firm j and $\sum_{i\neq j} q_i' = \sum_{i\neq j} q_i = L(t) - q_j < L(t)$. Thus, $s_j' \leq L(t) - \sum_{i\neq j} q_i = q_j$. As a result, the payoff of the firm j becomes $s_j'\pi_{\text{eq}} \leq q_j\pi_g$, which is not greater than the original payoff.

[5]Readers may be surprised why the market price is π_g even though every firm bids at zero price. Note that by Algorithm 4, the market price is the marginal cost for one *additional* unit of demand. With the bids in Eq. (13), the demand is equal to the total bidding quantity. Thus, the marginal price that consumers have to pay for one *additional* unit of electricity is π_g. See the remark at the end of this subsection on what happens if the market price is defined as the marginal cost for the *last* unit of demand.

Case 2: $p_j' = 0$ and $q_j' \leq q_j$. The payoff of the firm j equals to $s_j'\pi_{\text{eq}} \leq q_j'\pi_g \leq q_j\pi_g$, which is also not greater than the original payoff.

Case 3: $p_j' = 0$ and $q_j' > q_j$. Thus, we have

$$\sum_{i=1}^M q_i' = q_j' + \sum_{i\neq j} q_i > q_j + \sum_{i\neq j} q_i = \sum_{i=1}^M q_i = L(t).$$

By Algorithm 3, we have that the new market price now equals to $\pi_{\text{eq}}' = 0$. Thus, the payoff of the firm j becomes zero, which cannot be greater than its original payoff.

In all three cases, the firm j cannot get more payoff by deviating to another bidding strategy. As a result, we conclude that the original bidding strategy is a Nash equilibrium. Now, we prove that this Nash equilibrium is Pareto optimal. The total payoff of all firms equals to $\pi_{\text{eq}} \sum_{i=1}^M s_i = \pi_g \sum_{i=1}^M q_i = \pi_g L(t)$. By Algorithm 3, we know that the market price cannot exceed π_g. By Algorithm 4, we know that the total sold amount cannot exceed $L(t)$. Thus, $\pi_g L(t)$ is the maximum total payoff to the firms as a whole. It implies that this Nash equilibrium is Pareto optimal. The result of this proposition thus holds. □

Remark on how the marginal price is determined: In Algorithm 3, the marginal price (i.e., market price π_{eq}) is defined as the cost for one *additional* unit of demand. There is an alternative way of defining the market price as the cost of the *last* unit of demand. Specifically, we may change the condition in Line 6 in Algorithm 3 as "if $L(t) \leq Q$ and $\pi_r < \pi_g$". For markets allowing only single-price bids (which we study in subsection 4.1), since we assume

$$\mathbf{Pr}\left\{\sum_{i\in S} q_i \neq L(t) \text{ for all } S \subseteq \{1, 2, \cdots, M\}\right\} = 1,$$

the above change makes no difference to the calculation of $\mathbb{E}[\pi_{\text{eq}}]$ in subsection 4.1. For markets allowing price-quantity bids (which we study in this subsection), as we explain below, similar outcomes as Proposition 4.2 will arise even when the market price is defined as the cost of the last unit of demand. In this case, we made the additional assumption that the quantity of a bid must be a multiple of some smallest-possible unit δ (e.g., 1 kW). Then, we can change the bidding strategy in Eq. (2) to

$$\begin{cases} \sum_{i=1}^M q_i = \min\left\{\left\lfloor \frac{L(t)}{\delta} \right\rfloor \cdot \delta,\ \sum_{i=1}^M q_i^0\right\}, \\ p_i = 0,\ \text{both } q_i \text{ and } q_i^0 \text{ are a multiple of } \delta,\ \text{for all } i. \end{cases} \tag{14}$$

We can use similar methods as in the proof of Proposition 4.2 to show that Eq. (14) is a Pareto-optimal Nash equilibrium with $\pi_{\text{eq}} = \pi_g$. Detailed proof is omitted here.

A.3 Proof of Proposition 4.3

Proof. When $\sum_{i=1}^M q_i^0 < L(t, \pi_g)$, according to Eq. (2), every firm sells all generation amount at the highest price π_g. Thus, she cannot earn more profit by changing her bid, which implies that Eq. (2) is a Nash equilibrium. It remains to consider the situation where $\sum_{i=1}^M q_i^0 \geq L(t, \pi_g)$. By Eq. (2), we have $\sum_{i=0}^M q_i = L(t, \pi_g)$. We now prove by contradiction. Suppose on the contrary that one firm i can earn more profit by deviating her bid away from what described in Eq. (2). Since the new equilibrium price π_{eq} can never

be higher than π_g, this can only happen if firm i's new sold amount satisfies $s_i' > q_i$, with the new market price $\pi_{\text{eq}}' > 0$. For all other firms $j \neq i$, since they bid zero price, it implies that $s_j' = s_j = q_j$ for all $j \neq i$ (by Lemma A.1(d) in Appendix A). Thus, we have $s_i' - q_i = L(t, \pi_{\text{eq}}') - L(t, \pi_g)$. In other words, the increased sold amount of firm i all comes from the total demand elasticity and does not come from other firms' sold amount. The profit of firm i after the deviation is $s_i'\pi_{\text{eq}}'$. The change of firm i's profit is then

$$
\begin{aligned}
&\pi_{\text{eq}}' s_i' - \pi_g q_i \\
=&\pi_g(s_i' - q_i) - (\pi_g - \pi_{\text{eq}}')s_i' \\
=&(\pi_g - \pi_{\text{eq}}')L(t,\pi_g)\left(\frac{s_i' - q_i}{L(t,\pi_g)}\frac{\pi_g}{\pi_g - \pi_{\text{eq}}'} - \frac{s_i'}{L(t,\pi_g)}\right) \\
=&(\pi_g - \pi_{\text{eq}}')L(t,\pi_g)\left(\frac{L(t,\pi_{\text{eq}}') - L(t,\pi_g)}{L(t,\pi_g)}\frac{\pi_g}{\pi_g - \pi_{\text{eq}}'} - \frac{s_i'}{L(t,\pi_g)}\right) \\
\leq&(\pi_g - \pi_{\text{eq}}')L(t,\pi_g)\left(\max_{\pi\in[0,\pi_g]}\eta_t(\pi) - \frac{q_i}{L(t,\pi_g)}\right) \\
\leq&0.
\end{aligned}
$$

(The last inequality is from the condition of this proposition.) In other words, firm i cannot earn more profit by deviating to another bid. This contradicts our previous assumption that firm i can earn more profit. The result of this proposition thus follows. □

B ABOUT THE PANEL DEMAND FUNCTION

In this part, we will show more detailed properties of the panel demand function defined in Section 5.1. From Appendix B.1 to Appendix B.4, we will first study properties of the panel demand function such as existence and monotonicity. Then we will prove Proposition 5.1 in Appendix B.5. In Appendix B.6, we will also provide a corollary of Proposition 5.1, which will be useful in the proof of Theorem 5.2. We will need some extra notations as follows. Define $\mathbf{x}_t = (L_n(t),\ G(t)) \in \mathbb{R}^2$, and $H_n(\mathbf{x}_t, c_n) = L_n(t) - G(t)c_n$. Then, $[H_n(\mathbf{x}_t, c_n)]^+$ is the amount of extra electricity that the consumer n needs to buy from the grid at time t, where $[a]^+ = \max\{0, a\}$. Let $[0, T]$ denote the range of time[6].

B.1 Existence of $d_n(\pi)$

Lemma B.1. *The function $J_n(\pi, c_n)$ is continuous with respect to both π and c_n.*

Proof. To verify the continuity of J_n, note that

$$
\begin{aligned}
&\lim_{\Delta\pi\to0,\Delta c_n\to0}\left(J_n(\pi+\Delta\pi, c_n+\Delta c_n) - J_n(\pi, c_n)\right) \\
=&\lim_{\Delta\pi\to0,\Delta c_n\to0}\left((\pi+\Delta\pi)(c_n+\Delta c_n) - \pi c_n\right) \\
&+\frac{\pi_g}{T}\lim_{\Delta c_n\to0}\int_0^T\mathbb{E}\Big[[L_n(t) - G(t)(c_n+\Delta c_n)]^+ \\
&-[L_n(t) - G(t)c_n]^+\Big]dt \text{ (by Eq. (3))} \\
=&0 + \frac{\pi_g}{T}\int_0^T\mathbb{E}\lim_{\Delta c_n\to0}\Big[[L_n(t) - G(t)(c_n+\Delta c_n)]^+ \\
&-[L_n(t) - G(t)c_n]^+\Big]dt \\
=&0 + \frac{\pi_g}{T}\int_0^T\mathbb{E}\ 0\ dt = 0.
\end{aligned}
$$

Thus, J_n is continuous with respect to both π and c_n. Note that in the second equality, we exchange the order between the limit and the expectation. This exchange of order is valid by the Dominated Convergence Theorem [33], since $\left|\int_0^T\mathbb{E}[L_n(t)]dt\right| < \infty$ and $\left|[L_n(t) - G(t)(c_n+\Delta c_n)]^+\right| \leq L_n(t)$ for all Δc_n. □

It is obvious that $d_n(0) = +\infty$ by the definition of $d_n(\pi)$. The following lemma indicates that $d_n(\pi)$ also exists for all $\pi > 0$.

Lemma B.2. *For any $\pi > 0$, the minimizer of $J_n(\pi, c_n)$ over c_n must exist, i.e., there exists c_n^* such that*

$$J_n(\pi, c_n^*) \leq J_n(\pi, c_n) \text{ for all } c_n \geq 0.$$

Proof. We first prove that, for any $\pi \geq 0$, there exists $\bar{c} > 0$ such that

$$J_n(\pi, 1) \leq J_n(\pi, c_n) \text{ for all } c_n \geq \bar{c}. \tag{15}$$

To see this, note that because $H_n(\mathbf{x}_t, c_n) \leq L(t)$, we have

$$
\begin{aligned}
J_n(\pi, 1) &\leq \pi + \frac{\pi_g}{T}\int_0^T L(t)dt \\
&= \pi\left(1 + \frac{\pi_g}{\pi T}\int_0^T L(t)dt\right).
\end{aligned}
$$

We let $\bar{c} = 1 + \frac{\pi_g}{\pi T}\int_0^T L(t)dt$. Thus, we have

$$
\begin{aligned}
J_n(\pi, 1) &\leq \pi\left(1 + \frac{\pi_g}{\pi T}\int_0^T L(t)dt\right) \\
&= \pi\bar{c} \\
&\leq J_n(\pi, c_n), \text{ for all } c_n \geq \bar{c}.
\end{aligned}
$$

Further, by the Extreme Value Theorem [15], over the closed and bounded interval $[0, \bar{c}]$, the continuous function $J_n(\pi, c_n)$ must have a minimum c_n^*, i.e.,

$$J_n(\pi, c_n^*) \leq J_n(\pi, c_n) \text{ for all } c_n \leq \bar{c}.$$

Combining with Eq. (15), the result of the lemma then follows. □

B.2 Calculation of the threshold price $\overline{\pi}_n$

Intuitively, when the price is too high, the consumer n will choose to rent zero amount of PV panels. Let $\overline{\pi}_n$ be the threshold, above which the customer n will rent zero amount of PV panels. In the following lemma, we give the exact value of $\overline{\pi}_n$ and prove that $d_n(\pi) = 0$ when $\pi > \overline{\pi}_n$.

[6] Although we adopt a continuous-time model here, the result also applies to the discrete-time model.

Lemma B.3. *If $\pi > \overline{\pi}_n \triangleq \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\{L_n(t)>0\}}]dt$, then $J_n(\pi, c_n) > J_n(\pi, 0)$ for all $c_n > 0$, and thus $d_n(\pi) = 0$.*

Proof.

$$\begin{aligned}
&\pi > \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\{L_n(t)>0\}}]dt \text{ and } c_n > 0\\
\Longrightarrow\ &\pi c_n > \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)c_n\mathbb{1}_{\{L_n(t)>0\}}]dt\\
\Longrightarrow\ &\pi c_n > \frac{\pi_g}{T}\int_0^T \mathbb{E}\Big[\left(L_n(t) - [L_n(t) - G(t)c_n]^+\right)\\
&\qquad\qquad \cdot \mathbb{1}_{\{L_n(t)>0\}}\Big]dt\\
\Longrightarrow\ &\pi c_n + \frac{\pi_g}{T}\int_0^T \mathbb{E}\left[[L_n(t) - G(t)c_n]^+\right]dt\\
&\quad > \frac{\pi_g}{T}\int_0^T \mathbb{E}[L_n(t)]dt\\
\Longrightarrow\ &J_n(\pi, c_n) > J_n(\pi, 0).
\end{aligned}$$

□

In Lemma B.3, the condition $\pi > \overline{\pi}_n \triangleq \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\{L_n(t)>0\}}]dt$ can be rewritten as

$$\frac{\pi}{\frac{1}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\{L_n(t)>0\}}]dt} > \pi_g.$$

Notice that the left hand side represents the equivalent price of one unit amount of solar energy that is useful for the consumer n (i.e., when her load is positive). Lemma B.3 thus states that, when this equivalent price is higher than the retail price, the consumer n will not rent any PV panels, i.e., $d_n(\pi) = 0$ if $\pi > \overline{\pi}_n$.

B.3 Monotonicity of the panel demand function

Lemma B.4. *Define $h(c_n) = \int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(c_n)}]dt$ where $\mathcal{A}(c_n) = \{(t,\ \mathbf{x}_t) :\ H_n(\mathbf{x}_t, c_n) > 0\}$. Then, under Assumption 1, $h(c_n)$ is continuous when $c_n > 0$. Specifically, for all $c_{i,2} > c_{i,1} > 0$, we have $0 \le h(c_{i,1}) - h(c_{i,2}) \le \gamma T \ln\left(\frac{c_{i,2}}{c_{i,1}}\right) \le \gamma T \frac{c_{i,2}-c_{i,1}}{c_{i,1}}$.*

Proof. Suppose that $c_{i,2} > c_{i,1} > 0$. Because $H_n(\mathbf{x}_t, c_{i,2}) \le H_n(\mathbf{x}_t, c_{i,1})$, we have $\mathcal{A}(c_{i,2}) \subseteq \mathcal{A}(c_{i,1})$. Thus, we have $h(c_{i,1}) - h(c_{i,2}) \ge 0$. By the definition of $h(\cdot)$, we have

$$\begin{aligned}
&h(c_{i,1}) - h(c_{i,2})\\
=&\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(c_{i,1})}]dt - \int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(c_{i,2})}]dt\\
=&\int_0^T\int_0^\infty\int_0^{\frac{y}{c_{i,1}}} x f^t_{G,L_n}(x,y)dxdydt\\
&-\int_0^T\int_0^\infty\int_0^{\frac{y}{c_{i,2}}} x f^t_{G,L_n}(x,y)dxdydt\\
=&\int_0^T\int_0^\infty\int_{\frac{y}{c_{i,2}}}^{\frac{y}{c_{i,1}}} x f^t_{G|L_n}(x|y)\cdot f^t_{L_n}(y)dxdydt\\
\le&\int_0^T\int_0^\infty\int_{\frac{y}{c_{i,2}}}^{\frac{y}{c_{i,1}}} \frac{\gamma}{x}\cdot f^t_{L_n}(y)dxdydt \quad \text{(by Assumption 1)}\\
=&\int_0^T\int_0^\infty \gamma\ln\left(\frac{c_{i,2}}{c_{i,1}}\right)\cdot f^t_{L_n}(y)dydt\\
=&\gamma T\ln\left(\frac{c_{i,2}}{c_{i,1}}\right)\\
\le&\gamma T\frac{c_{i,2}-c_{i,1}}{c_{i,1}} \text{ (because } \ln(x)\le x-1 \text{ for } x\ge 1).
\end{aligned}$$

It also implies that

$$\lim_{c_{i,2}\to c_{i,1}}\left(h(c_{i,2}) - h(c_{i,1})\right) = 0.$$

As a result, we conclude that $h(c_n)$ is continuous when $c_n > 0$. □

Lemma B.5. *Under Assumption 1, for a fixed π, the function $J_n(\pi, c_n)$ is differentiable, and*

$$\frac{\partial J_n(\pi, c_n)}{\partial c_n} = \pi - \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(c_n)}]dt, \tag{16}$$

where $\mathcal{A}(c_n) = \{(t,\ \mathbf{x}_t) :\ H_n(\mathbf{x}_t, c_n) > 0\}$.

Proof. Since πc_n (the first term of $J_n(\pi, c_n)$) is differentiable, we only need to prove that $\int_0^T \mathbb{E}\left[[H_n(\mathbf{x}_t, c_n)]^+\right]dt$ is differentiable. Towards this end, we have

$$\begin{aligned}
&\lim_{\Delta c\to 0}\frac{1}{\Delta c}\Bigg(\int_0^T \mathbb{E}\left[[H_n(\mathbf{x}_t, c_n+\Delta c)]^+\right]dt\\
&\qquad -\int_0^T \mathbb{E}\left[[H_n(\mathbf{x}_t, c_n)]^+\right]dt\Bigg)\\
=&\lim_{\Delta c\to 0}\frac{1}{\Delta c}\Bigg(\int_0^T \mathbb{E}\left[H_n(\mathbf{x}_t, c_n+\Delta c)\,\mathbb{1}_{\mathcal{A}(c_n+\Delta c)}\right]dt\\
&\qquad -\int_0^T \mathbb{E}\left[H_n(\mathbf{x}_t, c_n)\,\mathbb{1}_{\mathcal{A}(c_n)}\right]dt\Bigg)\\
=&\lim_{\Delta c\to 0}\frac{1}{\Delta c}\Big(\int_0^T \mathbb{E}[H_n(\mathbf{x}_t, c_n+\Delta c)\,(\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)})]\\
&dt + \int_0^T \mathbb{E}[(H_n(\mathbf{x}_t, c_n+\Delta c) - H_n(\mathbf{x}_t, c_n))\cdot\mathbb{1}_{\mathcal{A}(c_n)}]dt\Big)\\
=&\lim_{\Delta c\to 0}\frac{\int_0^T \mathbb{E}[H_n(\mathbf{x}_t, c_n+\Delta c)\left(\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)}\right)]dt}{\Delta c}\\
&+\lim_{\Delta c\to 0}\frac{-\int_0^T \mathbb{E}[G(t)\Delta c\cdot\mathbb{1}_{\mathcal{A}(c_n)}]dt}{\Delta c}.
\end{aligned}$$

For the first limit, we have

$$\begin{aligned}
&\left|\lim_{\Delta c\to 0}\frac{\int_0^T \mathbb{E}[H_n(\mathbf{x}_t, c_n+\Delta c)\left(\mathbb{1}_{\mathcal{A}(c_n+\Delta c)}-\mathbb{1}_{\mathcal{A}(c_n)}\right)]dt}{\Delta c}\right| \\
\le &\lim_{\Delta c\to 0}\frac{\int_0^T \mathbb{E}[|H_n(\mathbf{x}_t, c_n+\Delta c)|\cdot\left|\mathbb{1}_{\mathcal{A}(c_n+\Delta c)}-\mathbb{1}_{\mathcal{A}(c_n)}\right|]dt}{|\Delta c|} \\
\le &\lim_{\Delta c\to 0}\frac{\int_0^T \mathbb{E}\left[|G(t)\Delta c|\cdot\left|\mathbb{1}_{\mathcal{A}(c_n+\Delta c)}-\mathbb{1}_{\mathcal{A}(c_n)}\right|\right]dt}{|\Delta c|} \qquad (*) \\
= &\lim_{\Delta c\to 0}\int_0^T \mathbb{E}\left[G(t)\left|\mathbb{1}_{\mathcal{A}(c_n+\Delta c)}-\mathbb{1}_{\mathcal{A}(c_n)}\right|\right]dt \\
= &\int_0^T \mathbb{E}\left[G(t)\lim_{\Delta c\to 0}\left|\mathbb{1}_{\mathcal{A}(c_n+\Delta c)}-\mathbb{1}_{\mathcal{A}(c_n)}\right|\right]dt \\
&\text{(by the Dominated Convergence Theorem)} \\
=&0.
\end{aligned}$$

The reason of the inequality $(*)$ is as follows. If $\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)} = 1$, then $H_n(\mathbf{x}_t, c_n+\Delta c) > 0$ and $H_n(\mathbf{x}_t, c_n) \le 0$. Thus, we have $0 < H_n(\mathbf{x}_t, c_n+\Delta c) = H_n(\mathbf{x}_t, c_n) + G(t)\Delta c \le G(t)\Delta c$. If $\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)} = -1$, then $H_n(\mathbf{x}_t, c_n+\Delta c) \le 0$ and $H_n(\mathbf{x}_t, c_n) > 0$. Thus, we have $0 \ge H_n(\mathbf{x}_t, c_n+\Delta c) = H_n(\mathbf{x}_t, c_n) + G(t)\Delta c > G(t)\Delta c$. In conclusion, when $\left|\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)}\right| = 1$, we must have $|H_n(\mathbf{x}_t, c_n+\Delta c)| \le |G(t)\Delta c|$.

Thus, we have

$$\begin{aligned}
&\lim_{\Delta c\to 0}\frac{1}{\Delta c}\left(\int_0^T \mathbb{E}\left[[H_n(\mathbf{x}_t, c_n+\Delta c)]^+\right]dt - \int_0^T \mathbb{E}\left[[H_n(\mathbf{x}_t, c_n)]^+\right]dt\right) \\
=&0+\lim_{\Delta c\to 0}\frac{-\int_0^T \mathbb{E}[G(t)\Delta c\cdot \mathbb{1}_{\mathcal{A}(c_n)}]dt}{\Delta c} \\
=&-\int_0^T \mathbb{E}\left[G(t)\mathbb{1}_{\mathcal{A}(c_n)}\right]dt.
\end{aligned}$$

Thus, we have proven the differentiability of $J_n(\pi, c_n)$, and it follows that

$$\frac{\partial J_n(\pi, c_n)}{\partial c_n} = \pi - \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(c_n)}]dt.$$

□

Lemma B.6. *The demand function $d_n(\pi)$ is monotone decreasing with respect to π.*

Proof. It suffices to prove that any $c > d_n(\pi_0)$ cannot be the minimizer of $J_n(\pi_1, c)$ for all $\pi_1 > \pi_0$. In other words, it suffices to show that

$$J_n(\pi_1, c) > J_n(\pi_1, d_n(\pi_0)), \quad \text{for all } c > d_n(\pi_0).$$

To prove this, we have

$$\begin{aligned}
&J_n(\pi_1, c) - J_n(\pi_1, d_n(\pi_0)) \\
=&\pi_1(c - d_n(\pi_0)) + \frac{\pi_g}{T}\int_0^T \Big(\mathbb{E}\left[[L_n(t) - G(t)c]^+\right] \\
&- \mathbb{E}\left[[L_n(t) - G(t)d_n(\pi_0)]^+\right]\Big)dt \\
=&(\pi_1 - \pi_0)(c - d_n(\pi_0)) + \pi_0(c - d_n(\pi_0)) \\
&+ \frac{\pi_g}{T}\int_0^T \Big(\mathbb{E}\left[[L_n(t) - G(t)c]^+\right] \\
&\qquad - \mathbb{E}\left[[L_n(t) - G(t)d_n(\pi_0)]^+\right]\Big)dt \\
=&(\pi_1 - \pi_0)(c - d_n(\pi_0)) + J_n(\pi_0, c) - J_n(\pi_0, d_n(\pi_0)) \\
\ge&(\pi_1 - \pi_0)(c - d_n(\pi_0)) \\
&\text{(because } J_n(\pi_0, c) - J_n(\pi_0, d_n(\pi_0)) \ge 0 \\
&\text{by the definition of } d_n(\pi_0)) \\
>&0.
\end{aligned}$$

The result of the lemma thus holds. □

Lemma B.7. *Recall the definition of $\overline{\pi}_n$ in Appendix B.2. Under Assumption 1, the demand function $d_n(\pi)$ is strictly monotone decreasing with respect to π when $\pi \in (0, \overline{\pi}_n)$.*

Proof. Consider two prices π_0 and π_1 that $0 < \pi_1 < \pi_0 < \overline{\pi}_n$. By Lemma B.6, we already have $d_n(\pi_0) \ge d_n(\pi_1)$. Thus, we only need to prove that $d_n(\pi_0) \ne d_n(\pi_1)$. Towards this end, it is sufficient to prove that such $d_n(\pi_1)$ must not be the minimizer of $J_n(\pi_0, c_n)$.

Because $\pi_1 < \overline{\pi}_n$, we have

$$\pi_1 < \frac{\pi_g}{T}\int_0^T \mathbb{E}\left[G(t)\mathbb{1}_{\{L_n(t)>0\}}\right]dt.$$

Applying Lemma B.5, we thus have

$$\left.\frac{\partial J_n(\pi_1, c_n)}{\partial c_n}\right|_{c_n=0} = \pi_1 - \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\{L_n(t)>0\}}]dt < 0.$$

This implies that $d_n(\pi_1) > 0$. By the first-order condition at the minimizer $d_n(\pi_1)$ of $J_n(\pi_1, c_n)$ with respect to c_n, we have

$$\left.\frac{\partial J_n(\pi_1, c_n)}{\partial c_n}\right|_{c_n=d_n(\pi_1)} = 0.$$

Similarly, we have

$$\left.\frac{\partial J_n(\pi_0, c_n)}{\partial c_n}\right|_{c_n=d_n(\pi_0)} = 0.$$

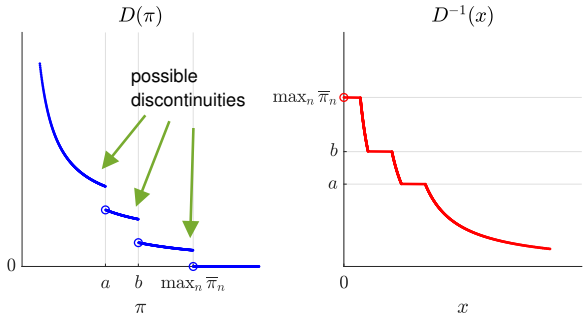

Figure 7: Demand function $D(\pi)$ and the corresponding inverse function $D^{-1}(x)$.

As a result, we have

$$\begin{aligned}
&\left.\frac{\partial J_n(\pi_0, c_n)}{\partial c_n}\right|_{c_n=d_n(\pi_1)} \\
=&\pi_0 - \frac{\pi_g}{T}\int_0^T \mathbb{E}\left[G(t)\mathbb{1}_{\mathcal{A}(d_n(\pi_1))}\right]dt \\
&\text{(where } \mathcal{A}(\cdot) \text{ is defined in Lemma B.5)} \\
=&(\pi_0 - \pi_1) + \left(\pi_1 - \frac{\pi_g}{T}\int_0^T \mathbb{E}\left[G(t)\mathbb{1}_{\mathcal{A}(d_n(\pi_1))}\right]dt\right) \\
=&(\pi_0 - \pi_1) + \left.\frac{\partial J_n(\pi_1, c_n)}{\partial c_n}\right|_{c_n=d_n(\pi_1)} \\
=&\pi_0 - \pi_1 \neq 0.
\end{aligned}$$

Therefore, we conclude that $d_n(\pi_1)$ is not the minimizer of $J_n(\pi_0, c_n)$. The result of this lemma then holds. □

B.4 Inverse function of $D(\cdot)$

Based on the properties of $d_n(\pi)$, we know that $D(\pi)$ is strictly monotone decreasing with respect to π when $\pi \in [0, \max_n \overline{\pi}_n]$, and $D(\pi)$ equals to zero when $\pi > \max_n \overline{\pi}_n$. We depict $D(\pi)$ in Fig. 7. Then, we can define the inverse function $D^{-1}(x)$ on $x > 0$. Notice that even though $d_n(\cdot)$ and $D(\cdot)$ may have discontinuities, $D^{-1}(x)$ is well-defined for all $x > 0$ due to the strict monotonicity of $D(\cdot)$. Further, we can verify that $D^{-1}(x) := \max\{\pi : D(\pi) \geq x\}$. Actually, because $D(\cdot)$ is strictly monotone decreasing, we still have $D^{-1}(D(\pi)) = \pi$. However, $D(D^{-1}(x)) = x$ is not always true. Instead, we have $D(D^{-1}(x)) \geq x$. Fig. 7 shows what $D(\pi)$ and corresponding $D^{-1}(x)$ look like.

B.5 Proof of Proposition 5.1

Proof. Let $\pi_a > \pi_b$. According to Lemma B.5 and the first-order condition, we have

$$\begin{aligned}
\frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(d_n(\pi_a))}]dt = \pi_a, \\
\frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(d_n(\pi_b))}]dt = \pi_b.
\end{aligned}$$

Thus, by Lemma B.4, we have

$$\begin{aligned}
\pi_a - \pi_b =& \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(d_n(\pi_a))}]dt \\
&- \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(d_n(\pi_b))}]dt \\
\leq& \pi_g\gamma\frac{d_n(\pi_b) - d_n(\pi_a)}{d_n(\pi_a)} \\
\implies d_n(\pi_a) \leq& \pi_g\gamma\frac{d_n(\pi_b) - d_n(\pi_a)}{\pi_a - \pi_b}.
\end{aligned}$$

Fix π_a and let π_b approach π_a. We have

$$d_n(\pi) \leq \pi_g\gamma\left|\frac{\partial d_n(\pi)}{\partial \pi}\right|.$$

(Notice that $\frac{\partial d_n(\pi)}{\partial \pi} \leq 0$ by Lemma B.6.) Summing this inequality over all consumers, we then have

$$\begin{aligned}
D(\pi) \leq& \pi_g\gamma\left|\frac{\partial D(\pi)}{\partial \pi}\right| \\
\implies& \frac{\left|\frac{\partial D(\pi)}{D(\pi)}\right|}{\left|\frac{\partial \pi}{\pi}\right|} \geq \frac{\pi}{\pi_g\gamma}.
\end{aligned}$$

□

B.6 Corollaries of Proposition 5.1

Solving the differential inequality in Proposition 5.1, we have the following corollary.

Corollary B.8. *Let π and π_0 be two arbitrary prices such that $\pi > \pi_0$. We then have*

$$D(\pi) \leq D(\pi_0)e^{-\frac{\pi-\pi_0}{\pi_g\gamma}}.$$

Proof. Notice that $D(\pi)$ is strictly monotone decreasing. We have

$$\begin{aligned}
&\left|\frac{\partial D(\pi)}{D(\pi)}\right| \geq \frac{\pi}{\pi_g\gamma}\left|\frac{\partial \pi}{\pi}\right| \\
\implies& \left|\int_{D(\pi_0)}^{D(\pi)} \frac{1}{D(a)}\partial D(a)\right| \geq \left|\int_{\pi_0}^{\pi} \frac{1}{\pi_g\gamma}\partial\pi\right| \\
\implies& \left|\ln\frac{D(\pi)}{D(\pi_0)}\right| \geq \left|\frac{\pi - \pi_0}{\pi_g\gamma}\right| \\
\implies& \ln\frac{D(\pi_0)}{D(\pi)} \geq \frac{\pi - \pi_0}{\pi_g\gamma} \\
&\text{(since } D(\pi_0) \geq D(\pi) \text{ by the monotonicity of } D(\cdot)) \\
\implies& \ln\frac{D(\pi)}{D(\pi_0)} \leq -\frac{\pi - \pi_0}{\pi_g\gamma} \\
\implies& D(\pi) \leq D(\pi_0)e^{-\frac{\pi-\pi_0}{\pi_g\gamma}}.
\end{aligned}$$

□

Replacing π_0 by $D^{-1}(x)$ and replacing $D(\pi_0)$ by x, we have a more general conclusion stated in the following corollary.

Corollary B.9. *Let $\pi > D^{-1}(x)$. We then have*

$$D(\pi) \leq xe^{-\frac{\pi-D^{-1}(x)}{\pi_g\gamma}}.$$

Proof. If $x = D(D^{-1}(x))$, then the result of this corollary is obviously true by applying Corollary B.8. If $x \neq D(D^{-1}(x))$, then $D(\cdot)$ must be discontinuous at $D^{-1}(x)$, which implies that $D(y) < x < D(D^{-1}(x))$ for all $y > D^{-1}(x)$. Thus, for all y such that $D^{-1}(x) < y < \pi$, we have

$$D(\pi) \leq D(y)e^{-\frac{\pi-y}{\pi g \gamma}} < xe^{-\frac{\pi-y}{\pi g \gamma}}.$$

Let y approach $D^{-1}(x)$, we have

$$D(\pi) \leq \lim_{y \to D^{-1}(x)} xe^{-\frac{\pi-y}{\pi g \gamma}} = xe^{-\frac{\pi - D^{-1}(x)}{\pi g \gamma}}.$$

□

C NASH EQUILIBRIA OF THE RENTAL MARKET

In this part, we provide details of how to analyze the Nash equilibrium of the rental market (i.e., the outcome shown by Theorem 5.2). In Appendix C.1, we will show a theorem that reveals the equivalence between the multi-block-bid mechanism and the two-block-bid mechanism for the rental market. Specifically, Theorem C.2 states that, for any Nash equilibrium of the rental market with multi-block bids, there must exist an "equivalent" Nash equilibrium with only two-block bids. The significance of this result is that, in our later analysis of the outcome of rental markets, we only need to focus on equilibria with two-block bids, which significantly simplifies our analysis. We then prove Theorem C.2 (with some other useful lemmas) in Appendix C.2 and Appendix C.3. In Appendix C.4, we show some results that help us to focus on what is necessary for Theorem 5.2 to hold. After that, we prove Theorem 5.2 in Appendix C.5.

C.1 Equivalence between the multi-block-bid mechanism and the two-block-bid mechanism

Directly analyzing Nash equilibria in a multi-block-bid market described by Algorithm 1 and Algorithm 2 is relatively difficult, as the action space of the players is large. Fortunately, we can simplify such analysis by introducing the concept of *outcome-equivalent* Nash equilibria (Definition C.1). With this definition, we only need to consider the situation that each firm only makes two bids at any Nash equilibrium (Theorem C.2).

Remark on Algorithm 1 and 2: We have assumed that the multiple bids of each firm have different prices, i.e., $p_{i,k_1} \neq p_{i,k_2}$ when $k_1 \neq k_2$. This assumption is without loss of generality because, if a firm has two or more bids at the same price, we can merge them into on bid, and both the market price price and the firm's profit remains the same under Algorithms 1 and 2. That is because we adopt the uniform price policy (i.e., all sold part gets paid at the common market price π_{eq}), and the proportional assignment Eq. (7) (that all bids at the market price π_{eq} are assigned sales in proportion to the bidding quantity). The above property also implies that, even if a firm divides its equity into two firms that bid cooperatively in the market, the outcome of the market would be the same as the firm bids as a single entity.

Definition C.1. Two Nash equilibriums $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{p}', \mathbf{q}')$ are **outcome-equivalent** when $\sum_{k=1}^{K_i'} s_{i,k}' = \sum_{k=1}^{K_i} s_{i,k}$ for all i, and $\pi_{\text{eq}}' = \pi_{\text{eq}}$.

Theorem C.2. *For any Nash equilibrium $(\mathbf{p}, \mathbf{q})$ in the multi-block-bid system, there must exist an outcome-equivalent Nash equilibrium $(\mathbf{p}', \mathbf{q}')$ such that each firm only makes two bids, and the price of any bid is either 0 or π_{eq}.*

C.2 Preparation for the proof of Theorem C.2

We first provide some useful definitions, lemmas, and corollaries.

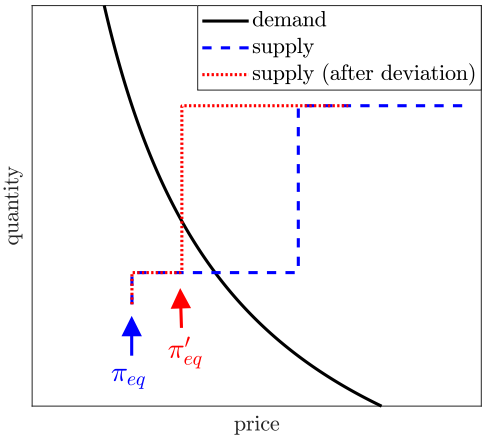

Figure 8: An example of $\pi'_{\text{eq}} > \pi_{\text{eq}}$ when $\{p'_{i,k} : \text{for all } k\} \nsubseteq \{p_{j,k} : \text{for all } j,k\} \cup \{0\}$.

Definition C.3. In the multi-bid system, suppose a firm i changes her bid from $(\vec{p}_i, \vec{q}_i)$ to $(\vec{p}_i{}', \vec{q}_i{}')$. we say that the firm i **bids more aggressively** (in the new bid $(\vec{p}_i{}', \vec{q}_i{}')$) if

$$\begin{cases} \sum_{\{k:\, p'_{i,k} < \pi\}} q'_{i,k} \geq \sum_{\{k:\, p_{i,k} < \pi\}} q_{i,k}, & \text{for all } \pi \in \mathbb{R}, \\ \{p'_{i,k} : \text{ for all } k\} \subseteq \{p_{j,k} : \text{ for all } j,k\} \cup \{0\}. \end{cases}$$

Remark on Definition C.3: The first condition states that, when a firm bids more aggressively, her total bidding quantity below any price π becomes larger. Later, we will show that, when a firm bids more aggressively, the market price should not increase (see Lemma C.5(b)). However, for this to be true, the second condition in this definition becomes necessary, i.e., the new prices must be from the set of prices in the original bids (possibly by another firm j). Fig. 8 shows an counter-example where market price actually increases after one firm increases her bidding quantity at certain prices without this constraint.

Lemma C.4. *For real numbers a, b, x, y that $a \geq 0$, $b > 0$, $x \geq 0$, $y \geq 0$, $a - x \geq 0$, $b - x + y > 0$, we must have*

$$\min\left\{\frac{a-x}{b-x+y}, 1\right\} \leq \min\left\{\frac{a}{b}, 1\right\}.$$

Proof. If $a \geq b$, then we have

$$\min\left\{\frac{a-x}{b-x+y}, 1\right\} \leq 1 = \min\left\{\frac{a}{b}, 1\right\}.$$

If $a < b$, then we only need to prove

$$\frac{a-x}{b-x+y} \le \frac{a}{b},$$

which is true because

$$\frac{a-x}{b-x+y} \le \frac{a-x}{b-x} \le \frac{a}{b}.$$

This lemma thus holds. □

Lemma C.5. *(a)*

$$\pi_{eq} = \max\left\{ p_{j,l} : \sum_{\{i,k:\ p_{i,k}<p_{j,l}\}} q_{i,k} \le D(p_{j,l}) \right\}.$$

(b) If any firm i^* bids more aggressively and other firms do not change, then $\pi'_{eq} \le \pi_{eq}$.

(c) $D(\pi_{eq}) \ge \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}$. Further, if $D(\pi_{eq}) > \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}$, we have $s_{i,k} = q_{i,k}$ for all i,k that $p_{i,k} \le \pi_{eq}$.

(d) Consider two different bidding strategies $(\mathbf{p},\mathbf{q})$ and $(\mathbf{p}',\mathbf{q}')$. If

$$\sum_{\{i,k:\ p'_{i,k}<\pi_{eq}\}} q'_{i,k} = \sum_{\{i,k:\ p_{i,k}<\pi_{eq}\}} q_{i,k}, \text{ and}$$

$$\sum_{\{i,k:\ p'_{i,k}=\pi_{eq}\}} q'_{i,k} \ge \sum_{\{i,k:\ p_{i,k}=\pi_{eq}\}} q_{i,k},$$

then $\pi'_{eq} \ge \pi_{eq}$.

(e) If $s_{i^*,k^*} = 0$ and $p_{i^*,k^*} = \pi_{eq}$, then we have $s_{i,k} = 0$ for all i,k that $p_{i,k} = \pi_{eq}$, and we also have $D(\pi_{eq}) = \sum_{\{i,k:\ p_{i,k}<\pi_{eq}\}} q_{i,k}$.

Remark on Lemma C.5: These results are intuitive. Part (a) states that π_{eq} is roughly the point where the demand curve and the supply curve intersect. Part (b) states that bidding more aggressively only makes the market price lower. Part (c) states that the total sold amount is always less than or equal to the demand. Further, if the total sold amount is less than the demand, then there are no partly sold bids. Part (d) states that, if the bidding amount below the original market price π_{eq} is the same, but the bidding amount at the original market price π_{eq} is larger (and thus the bidding amount above the original market price π_{eq} is smaller), then the new market price π'_{eq} cannot decrease. Part (e) states that if a bid with the price π_{eq} sells zero amount, then any bid with the price π_{eq} must also sell zero amount.

Proof. (a) We examine the outcome of Algorithm 1 in all possible situations. Define $F(\pi)$ as $F(\pi) = \sum_{\{i,k:\ p_{i,k}\le\pi\}} q_{i,k}$. Define $S(\pi)$ as $S(\pi) = \sum_{\{i,k:\ p_{i,k}<\pi\}} q_{i,k}$. Obviously, $F(\pi)$ and $S(\pi)$ is monotone increasing. Recall that in Algorithm 1 all different bidding prices are ranked as $\pi_1 < \pi_2 < \cdots < \pi_R$. We have $S(\pi_{a+1}) = F(\pi_a)$ for any pair of adjacent prices $(\pi_a,\ \pi_{a+1})$. We also have $S(\pi) \le F(\pi)$ for all π. What we need to prove can be written as

$$\pi_{\text{eq}} = \max_i\{\pi_i :\ S(\pi_i) \le D(\pi_i)\}. \tag{17}$$

We consider three cases (i.e., Case 1 to 3 below).

Case 1: $D(\pi_1) < F(\pi_1)$. Then $\pi_{\text{eq}} = \pi_1$ as Algorithm 1 exits on the branch of Line 8. We have $S(\pi_1) = 0 \le D(\pi_1)$, and $S(\pi_2) = F(\pi_1) > D(\pi_1)$. Eq. (17) thus follows.

Case 2: $D(\pi_R) \ge F(\pi_R)$. Then $\pi_{\text{eq}} = \pi_R$ as Algorithm 1 does not exit on the branch of Line 4 or Line 8, i.e., π_{eq} is determined by Line 11. We have $S(\pi_{\text{eq}}) \le F(\pi_{\text{eq}}) \le D(\pi_{\text{eq}})$. Eq. (17) thus follows.

Case 3: $D(\pi_1) \ge F(\pi_1)$ and $D(\pi_R) < F(\pi_R)$. Then, we can always find π_r such that $D(\pi_{r-1}) \ge F(\pi_{r-1})$ and $D(\pi_r) < F(\pi_r)$ (notice that $D(\pi) - F(\pi)$ is monotone decreasing). We consider two sub-cases (i.e., Case 3.1 and 3.2) below.

Case 3.1: $D(\pi_{r-1}) = F(\pi_{r-1})$. Then $\pi_{\text{eq}} = \pi_{r-1}$ as Algorithm 1 exits on the branch of Line 4. See Fig. 2(a). We have $S(\pi_{r-1}) \le F(\pi_{r-1}) = D(\pi_{r-1})$, and $S(\pi_r) = F(\pi_{r-1}) = D(\pi_{r-1}) > D(\pi_r)$. Eq. (17) thus follows.

Case 3.2: $D(\pi_{r-1}) \ne F(\pi_{r-1})$. Because $D(\pi_{r-1}) \ge F(\pi_{r-1})$ in Case 3, we now have $D(\pi_{r-1}) > F(\pi_{r-1})$. We then consider three further sub-cases (i.e., Case 3.2.1 to 3.2.3).

Case 3.2.1: $D(\pi_r) < F(\pi_{r-1})$. Then $\pi_{\text{eq}} = \pi_{r-1}$ as Algorithm 1 exits on the branch of Line 4. See Fig. 2(b). We have $S(\pi_{r-1}) \le F(\pi_{r-1}) \le D(\pi_{r-1})$, and $S(\pi_r) = F(\pi_{r-1}) > D(\pi_r)$. Eq. (17) thus follows.

Case 3.2.2: $D(\pi_r) > F(\pi_{r-1})$. Then $\pi_{\text{eq}} = \pi_r$ as Algorithm 1 exits on the branch of Line 8. See Fig. 2(c). We have $S(\pi_r) = F(\pi_{r-1}) < D(\pi_r)$, and $S(\pi_{r+1}) = F(\pi_r) > D(\pi_r)$. Eq. (17) thus follows.

Case 3.2.3: $D(\pi_r) = F(\pi_{r-1})$. Then $\pi_{\text{eq}} = \pi_r$ as Algorithm 1 exits on the branch of Line 8. See Fig. 2(d). We have $S(\pi_r) = F(\pi_{r-1}) = D(\pi_r)$, and $S(\pi_{r+1}) = F(\pi_r) > D(\pi_r)$. Eq. (17) thus follows.

To sum up, Eq. (17) holds for all cases. Therefore, the result of (a) thus follows.

(b) Because only one firm deviates, by the definition of bidding more aggressively, we have

$$\sum_{\{i,k:\ p'_{i,k}<\pi\}} q'_{i,k} \ge \sum_{\{i,k:\ p_{i,k}<\pi\}} q_{i,k}, \text{ for all } \pi \in \mathbb{R}, \text{ and}$$

$$\{p'_{i,k} :\ \text{for all } i,k\} \subseteq \{p_{i,k} :\ \text{for all } i,k\} \cup \{0\}.$$

As a result, we have

$$\left\{ p'_{j,l} : \sum_{\{i,k:\ p'_{i,k}<p'_{j,l}\}} q'_{i,k} \le D(p'_{j,l}) \right\}$$

$$\subseteq \left\{ p_{j,l} : \sum_{\{i,k:\ p_{i,k}<p_{j,l}\}} q_{i,k} \le D(p_{j,l}) \right\} \cup \{0\}.$$

By (a), we have $\pi'_{\text{eq}} \le \pi_{\text{eq}}$.

(c) Obviously, $s_{i,k} = q_{i,k}$ for all i,k such that $p_{i,k} < \pi_{\text{eq}}$. We now consider two cases. (i) If $D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k} \le \sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}} q_{i,k}$, by Eq. (7), we have

$$s_{j,l} = \frac{q_{j,l}\left(D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}\right)}{\sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}} q_{i,k}}$$

for all j,l such that $p_{j,l} = \pi_{\text{eq}}$.

Summing this equation over all such j and l, we have

$$\sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}} s_{i,k} = D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}$$

$$\Longrightarrow D(\pi_{\text{eq}}) = \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}.$$

(ii) If $D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k} > \sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}} q_{i,k}$, by Eq. (7), we have $s_{i,k} = q_{i,k}$ for all i,k such that $p_{i,k} = \pi_{\text{eq}}$. Thus, $s_{i,k} =$

$p_{i,k}$ for all i,k such that $p_{i,k} \le \pi_{\text{eq}}$. As a result, we have

$$D(\pi_{\text{eq}}) > \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k} + \sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}} q_{i,k} = \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}.$$

The result of part (c) then follows.

(d) By (a), we have $\sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k} \le D(\pi_{\text{eq}})$. By the given relationship between $(\mathbf{p},\mathbf{q})$ and $(\mathbf{p}',\mathbf{q}')$, we have $\sum_{\{i,k:\ p'_{i,k}<\pi_{\text{eq}}\}} q'_{i,k} = \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k} \le D(\pi_{\text{eq}})$. Thus, we have

$$\begin{aligned}
&\pi_{\text{eq}} \in \left\{ p'_{j,l} : \sum_{\{i,k:\ p'_{i,k}<p'_{j,l}\}} q'_{i,k} \le D(p'_{j,l}) \right\} \\
\implies &\max\left\{ p'_{j,l} : \sum_{\{i,k:\ p'_{i,k}<p'_{j,l}\}} q'_{i,k} \le D(p'_{j,l}) \right\} \ge \pi_{\text{eq}} \\
\implies &\pi'_{\text{eq}} \ge \pi_{\text{eq}} \text{ (applying (a)).}
\end{aligned}$$

(e) Because $s_{i^*,k^*} = 0$, by Eq. (7), we have $D(\pi_{\text{eq}}) - Q = 0$, which implies $s_{i,k} = 0$ for all i,k such that $p_{i,k} = \pi_{\text{eq}}$. Therefore, we must have $D(\pi_{\text{eq}}) = \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}$. □

Lemma C.6. *Consider any bidding strategy* $(\mathbf{p},\ \mathbf{q})$ *(not necessarily a Nash equilibrium). Suppose that a firm* i^* *bids more aggressively and other firms do not change their bids. Then, we must have* $\pi'_{eq} \le \pi_{eq}$ *and* $s'_{i,k} \le s_{i,k}$, *for all* $i \ne i^*$, *and for all* k. *Consequently, any other firm* $i \ne i^*$ *will not earn more profit.*

Proof. By Lemma C.5(b), we have $\pi'_{\text{eq}} \le \pi_{\text{eq}}$. It only remains to show that $s'_{i,k} \le s_{i,k}$, for all $i \ne i^*$ and for all k. There are two possible cases, $\pi'_{\text{eq}} < \pi_{\text{eq}}$ or $\pi'_{\text{eq}} = \pi_{\text{eq}}$. We discuss them separately as follows.

Case 1: $\pi'_{\text{eq}} < \pi_{\text{eq}}$. By Algorithm 2, we have

$$\begin{aligned}
&s'_{i,k} = s_{i,k} = q_{i,k}, \text{ for all } i,k \text{ that } p_{i,k} < \pi'_{\text{eq}},\ i \ne i^*, \\
&s'_{i,k} \le q_{i,k} = s_{i,k}, \text{ for all } i,k \text{ that } p_{i,k} = \pi'_{\text{eq}},\ i \ne i^*, \\
&\text{and} \\
&s'_{i,k} = 0 \le s_{i,k}, \text{ for all } i,k \text{ that } p_{i,k} > \pi'_{\text{eq}},\ i \ne i^*.
\end{aligned}$$

Thus, we have shown that $s'_{i,k} \le s_{i,k}$, for all $i \ne i^*$ and for all k in this case.

Case 2: $\pi'_{\text{eq}} = \pi_{\text{eq}}$. We have

$$s'_{i,k} = q_{i,k} = s_{i,k}, \text{ for all } i,k \text{ such that } p_{i,k} < \pi_{\text{eq}},\ i \ne i^*,$$

and

$$s'_{i,k} = 0 = s_{i,k}, \text{ for all } i,k \text{ such that } p_{i,k} > \pi_{\text{eq}},\ i \ne i^*.$$

Further, let

$$\begin{aligned}
x &= \sum_{\{j,l:\ p'_{j,l}<\pi_{\text{eq}}\}} q'_{j,l} - \sum_{\{j,l:\ p_{j,l}<\pi_{\text{eq}}\}} q_{j,l} \ge 0, \\
y &= \sum_{\{j,l:\ p'_{j,l}\le\pi_{\text{eq}}\}} q'_{j,l} - \sum_{\{j,l:\ p_{j,l}\le\pi_{\text{eq}}\}} q_{j,l} \ge 0.
\end{aligned}$$

(Notice that x and y are positive because of Definition C.3.) Then, for all i,k that $p_{i,k} = \pi_{\text{eq}},\ i \ne i^*$, we have

$$\begin{aligned}
s'_{i,k} &= \min\left\{ \frac{q_{i,k}\left(D(\pi_{\text{eq}}) - \sum_{\{j,l:\ p'_{j,l}<\pi_{\text{eq}}\}} q'_{j,l}\right)}{\sum_{\{j,l:\ p'_{j,l}=\pi_{\text{eq}}\}} q'_{j,l}},\ q_{i,k} \right\} \\
&= \min\left\{ \frac{q_{i,k}\left(D(\pi_{\text{eq}}) - \sum_{\{j,l:\ p_{j,l}<\pi_{\text{eq}}\}} q_{j,l} - x\right)}{\sum_{\{j,l:\ p_{j,l}=\pi_{\text{eq}}\}} q_{j,l} - x + y}, q_{i,k} \right\} \\
&\le \min\left\{ \frac{q_{i,k}\left(D(\pi_{\text{eq}}) - \sum_{\{j,l:\ p_{j,l}<\pi_{\text{eq}}\}} q_{j,l}\right)}{\sum_{\{j,l:\ p_{j,l}=\pi_{\text{eq}}\}} q_{j,l}},\ q_{i,k} \right\} \\
&\quad \text{(applying Lemma C.4)} \\
&= s_{i,k}.
\end{aligned}$$

Thus, we have also shown that $s'_{i,k} \le s_{i,k}$, for all $i \ne i^*$ and for all k. The conclusion of this lemma thus follows. □

Lemma C.7. *Suppose that the original bidding strategy* $(\mathbf{p},\mathbf{q})$ *is a Nash equilibrium. Consider a new bidding strategy where an arbitrary firm* i^* *deviates from this Nash equilibrium and bids more aggressively, and another firm* $j^* \ne i^*$ *also deviates from* $(\vec{p_{j^*}},\vec{q_{j^*}})$ *in an arbitrary way. Assume that no other firm* $k \ne i^*, j^*$ *changes her bid. Then, no matter how the firm* j^* *changes her bid, her payoff under* $(\mathbf{p}',\mathbf{q}')$ *cannot increase compared to her payoff at the original Nash equilibrium* $(\mathbf{p},\mathbf{q})$.

Proof. Because $(\mathbf{p},\mathbf{q})$ is a Nash equilibrium, no matter how the firm j^* changes its strategy to $(\vec{p_{j^*}}',\vec{q_{j^*}}')$, her payoff cannot increase. After the firm j^* deviates, suppose now the firm i^* bids more aggressively. By Lemma C.6, this change of the firm i^* cannot make the payoff of the firm j^* higher. Thus, firm j^* still cannot make more payoff. The conclusion of this lemma thus follows. □

Corollary C.8. *Assume that* $(\mathbf{p},\mathbf{q})$ *is a Nash equilibrium. Consider a new bidding strategy* $(\mathbf{p}',\mathbf{q}')$ *where the firm* i^* *bids more aggressively and other firms do not deviate. If the market outcome under* $(\mathbf{p}',\mathbf{q}')$ *satisfies* $\pi'_{eq} = \pi_{eq}$ *and* $\sum_{k=1}^{K'_i} s'_{i,k} = \sum_{k=1}^{K_i} s_{i,k}$ *for all* i, *then the new bidding strategy* $(\mathbf{p}',\mathbf{q}')$ *must also be a Nash equilibrium.*

Proof. Because $\pi'_{\text{eq}} = \pi_{\text{eq}}$ and $\sum_{k=1}^{K'_i} s'_{i,k} = \sum_{k=1}^{K_i} s_{i,k}$ for all i, we know every firm's payoff does not change under $(\mathbf{p}',\mathbf{q}')$ compared with that under $(\mathbf{p},\mathbf{q})$. Then, we check whether any firm can get more payoff by deviating to $(\mathbf{p}'',\mathbf{q}'')$ from $(\mathbf{p}',\mathbf{q}')$.

First, we consider the case where the firm i^* deviates. In this case, the firm i^* is the only firm that changes her bid from the Nash equilibrium $(\mathbf{p},\mathbf{q})$ to $(\mathbf{p}'',\mathbf{q}'')$. By the definition of the Nash equilibrium $(\mathbf{p},\mathbf{q})$, the firm i^* cannot get more payoff under $(\mathbf{p}'',\mathbf{q}'')$ than that under $(\mathbf{p},\mathbf{q})$, which is also equal to her payoff under $(\mathbf{p}',\mathbf{q}')$. Hence, we conclude that the firm i^* cannot get more payoff by deviating from $(\mathbf{p}',\mathbf{q}')$.

Second, we consider the case where another firm $i \ne i^*$ deviates. The whole deviation process from $(\mathbf{p},\mathbf{q})$ to $(\mathbf{p}'',\mathbf{q}'')$ is the same as what described in Lemma C.7. As a result, the payoff of the firm i under $(\mathbf{p}'',\mathbf{q}'')$ is not more than that under $(\mathbf{p},\mathbf{q})$, which is also equal to her payoff under $(\mathbf{p}',\mathbf{q}')$. This means that any firm $i \ne i^*$

cannot get more payoff by deviating from $(\mathbf{p}', \mathbf{q}')$. The conclusion of this corollary thus follows. □

Lemma C.9. *For any real numbers a, b, x, y that $b \geq a > 0,\ x \geq 0$, and $y > 0$, we must have*

$$\min\left\{\frac{ay}{b},\ a\right\} \leq \min\left\{\frac{(a+x)y}{b+x},\ a+x\right\},$$

where equality holds if and only if $x = 0$ or $a = b$ and $y \leq b$.

Proof. Since $a \leq a + x$ (equality holds when $x = 0$), it only remains to show that $\frac{ay}{b} \leq \frac{(a+x)y}{b+x}$. We have

$$\begin{aligned}
&\frac{ay}{b} \leq \frac{(a+x)y}{b+x}\\
\iff &\frac{a}{b} \leq \frac{a+x}{b+x}\\
\iff &a(b+x) \leq b(a+x)\\
\iff &(a-b)x \leq 0.
\end{aligned}$$

Because $a \leq b$ and $x \geq 0$, we do have $(a-b)x \leq 0$ (equality holds if and only if $a = 0$ or $x = 0$).

Now, we check the condition of $\min\left\{\frac{ay}{b},\ a\right\} = \min\left\{\frac{(a+x)y}{b+x},\ a+x\right\}$. When $x = 0$, equality obviously holds. When $x \neq 0$, equality holds if and only if $\frac{ay}{b} = \frac{(a+x)y}{b+x}$ and $\frac{ay}{b} \leq a$, which implies $a = b$ and $y \leq b$. This lemma thus follows. □

Proposition C.10. *Assume that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium. Suppose that a firm i^* has at least one bid with the bidding price higher or equal than π_{eq}. Let $k^* = \min\{k :\ p_{i^*,k} \geq \pi_{eq}\}$. Consider the new bidding strategy $(\mathbf{p}', \mathbf{q}')$ when the firm i^* bids as follows*

$$\begin{cases}
K'_{i^*} = k^*,\\
p'_{i^*,k^*} = \pi_{eq}, \quad q'_{i^*,k^*} = \sum_{k \geq k^*} q_{i^*,k},\\
p'_{i^*,k} = p_{i^*,k}, \quad q'_{i^*,k} = q_{i^*,k}, \quad \textit{for all } k < k^*,
\end{cases}$$

and other firms' bids do not change. Then, the bidding strategy $(\mathbf{p}', \mathbf{q}')$ is an outcome-equivalent Nash equilibrium.

Proof. From the assumptions, the firm i^* bids more aggressively. Note that π_{eq} must belong to $\{p_{j,k} :\ \text{for all } j,k\} \cup \{0\}$, and thus the new bid satisfies Definition C.3. Specifically, we have

$$\begin{aligned}
\sum_{\{i,k:\ p'_{i,k} < \pi_{\text{eq}}\}} q'_{i,k} &= \sum_{\{i,k:\ p_{i,k} < \pi_{\text{eq}}\}} q_{i,k},\\
\sum_{\{i,k:\ p'_{i,k} = \pi_{\text{eq}}\}} q'_{i,k} &\geq \sum_{\{i,k:\ p_{i,k} = \pi_{\text{eq}}\}} q_{i,k}.
\end{aligned}$$

By Lemma C.5(b) and C.5(d), we have $\pi'_{\text{eq}} = \pi_{\text{eq}}$. It only remains to show that $\sum_{k=1}^{K'_i} s'_{i,k} = \sum_{k=1}^{K_i} s_{i,k}$, for all i. First, we have $s'_{i,k} = q_{i,k} = s_{i,k}$ for all (i,k) that $p'_{i,k} < \pi_{\text{eq}}$. Second, we prove $\sum_{\{k:\ p_{i,k} = \pi_{\text{eq}}\}} s_{i,k} = \sum_{\{k:\ p'_{i,k} = \pi'_{\text{eq}}\}} s'_{i,k}$ for all i by three steps[7].

Step 1: We prove that, if there exists no l such that $p_{i^*,l} = \pi_{\text{eq}}$, then $s'_{i^*,k^*} = 0$ and $s'_{i,k} = s_{i,k} = 0$ for all i,k such that $p_{i,k} = \pi_{\text{eq}}$. Towards this end, note that because $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium, the firm i^*'s payoff under $(\mathbf{p}', \mathbf{q}')$ should be less than or equal to her payoff under $(\mathbf{p}, \mathbf{q})$. In other words, we have

$$\begin{aligned}
&\pi'_{\text{eq}}\left(s'_{i^*,k^*} + \sum_{\{k:\ p'_{i^*,k} < \pi'_{\text{eq}}\}} s'_{i^*,k}\right)\\
&\leq \pi_{\text{eq}} \sum_{\{k:\ p_{i^*,k} < \pi_{\text{eq}}\}} s_{i^*,k}\\
\implies &s'_{i^*,k^*} = 0 \text{ (because } \pi'_{\text{eq}} = \pi_{\text{eq}})\\
\implies &s'_{i,k} = 0 \text{ for all } i,k \text{ such that } p'_{i,k} = \pi'_{\text{eq}},\\
&\text{and thus } D(\pi'_{\text{eq}}) = \sum_{i,k:\ p'_{i,k} < \pi'_{\text{eq}}} q'_{i,k}\\
&\text{(because of Lemma C.5(e))}\\
\implies &D(\pi_{\text{eq}}) = \sum_{i,k:\ p_{i,k} < \pi_{\text{eq}}} q_{i,k}\\
&\text{(because } \pi_{\text{eq}} = \pi'_{\text{eq}} \text{ and } q_{i,k} = q'_{i,k} \text{ for all } i,k\\
&\text{that } p_{i,k} < \pi_{\text{eq}})\\
\implies &\sum_{i,k:\ p_{i,k} < \pi_{\text{eq}}} s_{i,k} = \sum_{i,k:\ p_{i,k} < \pi_{\text{eq}}} q_{i,k} = D(\pi_{\text{eq}})\\
&\geq \sum_{i=1}^{M} \sum_{k=1}^{K_i} s_{i,k} \text{ (by Lemma C.5(c))}\\
\implies &s_{i,k} = 0 \text{ for all } i,k \text{ such that } p_{i,k} = \pi_{\text{eq}}.
\end{aligned}$$

In conclusion, we have $s'_{i^*,k^*} = 0$ and $s'_{i,k} = s_{i,k} = 0$ for all i,k such that $p_{i,k} = \pi_{\text{eq}}$.

Step 2: We prove that, if there exists l such that $p_{i^*,l} = \pi_{\text{eq}}$, then $s'_{i^*,k^*} = s_{i^*,l}$. Since there exists at most one bid of user i^* at π_{eq}, we must have $l = k^*$. Thus, it is equivalent to prove $s'_{i^*,k^*} = s_{i^*,k^*}$. To that end, we define $x = q'_{i^*,k^*} - q_{i^*,k^*}$. Note that $x \geq 0$ because $q'_{i^*,k^*} = \sum_{k \geq k^*} q_{i^*,k}$. By Eq. (7), we have

$$\begin{aligned}
&s'_{i^*,k^*}\\
&= \min\left\{\frac{q'_{i^*,k^*}\left(D(\pi'_{\text{eq}}) - \sum_{\{i,k:\ p'_{i,k} < \pi'_{\text{eq}}\}} q'_{i,k}\right)}{\sum_{\{i,k:\ p'_{i,k} = \pi'_{\text{eq}},\ i \neq i^*\}} q'_{i,k} + q'_{i^*,k^*}},\ q'_{i^*,k^*}\right\}\\
&= \min\left\{\frac{q'_{i^*,k^*}\left(D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k} < \pi_{\text{eq}}\}} q_{i,k}\right)}{\sum_{\{i,k:\ p_{i,k} = \pi_{\text{eq}},\ i \neq i^*\}} q_{i,k} + q'_{i^*,k^*}},\ q'_{i^*,k^*}\right\}\\
&\text{(since } \pi'_{\text{eq}} = \pi_{\text{eq}}, \text{ and } q_{i,k} = q'_{i,k} \text{ for all } i \neq i^*)\\
&= \min\left\{\frac{(q_{i^*,k^*} + x)\left(D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k} < \pi_{\text{eq}}\}} q_{i,k}\right)}{\sum_{\{i,k:\ p_{i,k} = \pi_{\text{eq}}\}} q_{i,k} + x},\right.\\
&\qquad \left. q_{i^*,k^*} + x\right\}\\
&\geq \min\left\{\frac{q_{i^*,k^*}\left(D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k} < \pi_{\text{eq}}\}} q_{i,k}\right)}{\sum_{\{i,k:\ p_{i,k} = \pi_{\text{eq}}\}} q_{i,k}},\ q_{i^*,k^*}\right\}\\
&\text{(applying Lemma C.9)}\\
&= s_{i^*,k^*} \text{ (By Eq. (7)).}
\end{aligned}$$

[7] Note that for every firm i, both $\{k :\ p_{i,k} = \pi_{\text{eq}}\}$ and $\{k :\ p'_{i,k} = \pi'_{\text{eq}}\}$ have at most one element, and (one or both) could be empty for some firms.

Then, it only remains to show $s'_{i^*,k^*} \le s_{i^*,k^*}$. Because $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium, the firm i^*'s payoff under $(\mathbf{p}', \mathbf{q}')$ should be less than or equal to her payoff under $(\mathbf{p}, \mathbf{q})$. In other words, we have

$$\begin{aligned}
&\pi'_{\text{eq}}\left(s'_{i^*,k^*} + \sum_{\{k:\, p'_{i^*,k}<\pi'_{\text{eq}}\}} s'_{i^*,k}\right) \\
&\le \pi_{\text{eq}}\left(s_{i^*,k^*} + \sum_{\{k:\, p_{i^*,k}<\pi_{\text{eq}}\}} s_{i^*,k}\right) \\
\Longrightarrow\ &\pi_{\text{eq}}\left(s'_{i^*,k^*} + \sum_{\{k:\, p_{i^*,k}<\pi_{\text{eq}}\}} q_{i^*,k}\right) \\
&\le \pi_{\text{eq}}\left(s_{i^*,k^*} + \sum_{\{k:\, p_{i^*,k}<\pi_{\text{eq}}\}} q_{i^*,k}\right) \\
\Longrightarrow\ &s'_{i^*,k^*} \le s_{i^*,k^*}.
\end{aligned}$$

Thus, we must have $s'_{i,k^*} = s_{i^*,k^*}$, i.e., $s'_{i,k^*} = s_{i^*,l}$.

Step 3: we prove that, if there exists l such that $p_{i^*,l} = \pi_{\text{eq}}$, then $s'_{i,k} = s_{i,k}$ for all i, k such that $p'_{i,k} = \pi'_{\text{eq}}$. By the definition of k^*, we have $l = k^*$. By Step 2, we have $s'_{i^*,k^*} = s_{i^*,k^*}$. Let $x = q'_{i^*,k^*} - q_{i^*,k^*} \ge 0$. Define $y = D(\pi_{\text{eq}}) - \sum_{\{i,k:\, p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}$. If $y = 0$, by Lemma C.5(c), we have $s'_{i,k} = s_{i,k} = 0$ for all i, k such that $p'_{i,k} = \pi'_{\text{eq}}$. Thus, it only remains to consider the situation of $y > 0$. From the conclusion of Step 2, we have

$$s_{i^*,k^*} = s'_{i^*,k^*},$$

which implies

$$\begin{aligned}
&\min\left\{\frac{q_{i^*,k^*}\left(D(\pi_{\text{eq}}) - \sum_{\{i,k:\, p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}\right)}{\sum_{\{i,k:\, p_{i,k}=\pi_{\text{eq}}\}} q_{i,k}},\ q_{i^*,k^*}\right\} \\
&= \min\left\{\frac{q'_{i^*,k^*}\left(D(\pi'_{\text{eq}}) - \sum_{\{i,k:\, p'_{i,k}<\pi'_{\text{eq}}\}} q'_{i,k}\right)}{\sum_{\{i,k:\, p'_{i,k}=\pi'_{\text{eq}}\}} q'_{i,k}},\ q'_{i^*,k^*}\right\},
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\min\left\{\frac{q_{i^*,k^*}\left(D(\pi_{\text{eq}}) - \sum_{\{i,k:\, p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}\right)}{\sum_{\{i,k:\, p_{i,k}=\pi_{\text{eq}}\}} q_{i,k}},\ q_{i^*,k^*}\right\} \\
&= \min\left\{\frac{(q_{i^*,k^*}+x)\left(D(\pi_{\text{eq}}) - \sum_{\{i,k:\, p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}\right)}{\sum_{\{i,k:\, p_{i,k}=\pi_{\text{eq}}\}} q_{i,k} + x},\right. \\
&\qquad \left. q_{i^*,k^*} + x\right\}.
\end{aligned}$$

Applying Lemma C.9, we have

$$x = 0 \text{ or } q_{i^*,k^*} = \sum_{\{i,k:\, p_{i,k}=\pi_{\text{eq}}\}} q_{i,k}.$$

If $x = 0$, then $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{p}', \mathbf{q}')$) are exactly the same. If $q_{i^*,k^*} = \sum_{\{i,k:\, p_{i,k}=\pi_{\text{eq}}\}} q_{i,k}$, then (i^*, k^*) is the only pair (i, k) that $p_{i,k} = \pi_{\text{eq}}$. In both cases, we always have $s'_{i,k} = s_{i,k}$ for all i, k that $p'_{i,k} = \pi'_{\text{eq}}$.

By Step 1 to 3, we have $\sum_{\{k:\, p_{i,k}=\pi_{\text{eq}}\}} s_{i,k} = \sum_{\{k:\, p'_{i,k}=\pi'_{\text{eq}}\}} s'_{i,k}$ for all i. Notice that $\sum_{\{k:\, p_{i,k}>\pi_{\text{eq}}\}} s_{i,k} = 0 = \sum_{\{k:\, p'_{i,k}>\pi'_{\text{eq}}\}} s'_{i,k}$ for all i. Thus, we have $\sum_{k=1}^{K'_i} s'_{i,k} = \sum_{k=1}^{K_i} s_{i,k}$ for all i. By Corollary C.8, we conclude that $(\mathbf{p}', \mathbf{q}')$ is an outcome-equivalent Nash equilibrium. □

Proposition C.11. *Assume that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium. Suppose that a firm i^* has at least one bid with the bidding price lower than π_{eq} and higher than 0. Let $k^* = \min\{k : 0 < p_{i^*,k} < \pi_{eq}\}$[8]. Consider the new bidding strategy $(\mathbf{p}', \mathbf{q}')$ when the firm i^* bids as follows*

$$\begin{cases}
K'_{i^*} = K_{i^*} - (k^* - 1), \\
p'_{i^*,1} = 0, \quad q'_{i^*,1} = \sum_{k\in\{1\}\cup\{k^*\}} q_{i^*,k}, \\
p'_{i^*,k} = p_{i^*,k+k^*-1},\ q'_{i^*,k} = q_{i^*,k+k^*-1},\ \textit{for all } k \ge 2.
\end{cases}$$

and other firms' bids do not change. Then, the new bidding strategy $(\mathbf{p}', \mathbf{q}')$ is an outcome-equivalent Nash equilibrium.

Proof. By assumptions, the firm i^* bids more aggressively and we have

$$\begin{aligned}
\sum_{\{i,k:\, p'_{i,k}<\pi_{\text{eq}}\}} q'_{i,k} &= \sum_{\{i,k:\, p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}, \\
\sum_{\{i,k:\, p'_{i,k}=\pi_{\text{eq}}\}} q'_{i,k} &= \sum_{\{i,k:\, p_{i,k}=\pi_{\text{eq}}\}} q_{i,k}.
\end{aligned}$$

Thus, by Lemma C.5(b)(d), we have $\pi'_{\text{eq}} = \pi_{\text{eq}}$. For any firm i, we have

$$\begin{aligned}
\sum_{\{k:\, p'_{i,k}<\pi_{\text{eq}}\}} q'_{i,k} &= \sum_{\{k:\, p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}, \\
q'_{i,k}\Big|_{k:\, p'_{i,k}=\pi_{\text{eq}}} &= q_{i,l}\Big|_{l:\, p_{i,l}=\pi_{\text{eq}}}.
\end{aligned}$$

Thus, by Algorithm 2, for any firm i, we have

$$\sum_{k=1}^{K'_i} s'_{i,k} = \sum_{k=1}^{K_i} s_{i,k}.$$

By Corollary C.8, the new bidding strategy $(\mathbf{p}', \mathbf{q}')$ is an outcome-equivalent Nash equilibrium. □

C.3 Proof of Theorem C.2

Now we are ready to prove Theorem C.2.

Proof. At any Nash equilibrium $(\mathbf{p}, \mathbf{q})$ in the multi-block-bid system, if a firm has more than 1 bid with the bidding price lower than π_{eq}, then we can repeatedly apply Proposition C.11 to combine all such bids with prices below π_{eq} to one bid with the price 0. If a firm has any bid with the price higher than or equal to π_{eq}, then we can apply Proposition C.10 to merge such bids into one bid at the price π_{eq}. Each step in those changes produces an outcome-equivalent Nash equilibrium. At the end, each firm only has at most two bids, one at the price π_{eq} and another at the price 0. The result of this theorem thus follows. □

[8] Note that, because there can be only one bid at price zero, k^* can only take values of either 1 or 2. Specifically, $k^* = 2$ when $p_{i^*,1} = 0$, and $k^* = 1$ when $p_{i^*,1} > 0$.

C.4 Preparation for the proof of Theorem 5.2

By Theorem C.2, when analyzing the Nash equilibrium $(\mathbf{p}, \mathbf{q})$ of the market, we can restrict our attention to $K_i \le 2,\ p_{i,k} \in \{0,\ \pi_{\text{eq}}\}$ for all i, k. We first obtain the following lemma on the relationship between the sold amount and the market price at any Nash equilibrium.

Lemma C.12. *$D^{-1}\left(\sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}\right) = \pi_{eq}$ at any Nash equilibrium $(\mathbf{p}, \mathbf{q})$.*

Proof. We prove by contradiction. Suppose on the contrary that $D^{-1}\left(\sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}\right) \neq \pi_{\text{eq}}$. Since $D^{-1}(D(\pi_{\text{eq}})) = \pi_{\text{eq}}$, it implies that we must have $D(\pi_{\text{eq}}) \neq \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}$. By Lemma C.5(c), we must have $D(\pi_{\text{eq}}) > \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}$. Recall from Theorem C.2 that we can restrict our attention to outcome-equivalent Nash equilibrium such that $p_{i,k} \in \{0, \pi_{\text{eq}}\}$ for all i and for all k. By Lemma C.5(c), we must have $s_{i,k} = q_{i,k}$ for all i, k. Thus, the payoff of the firm i equals $\pi_{\text{eq}} \sum_{k=1}^{K_i} q_{i,k}$. We also have

$$
\begin{aligned}
& D(\pi_{\text{eq}}) > \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k} \\
\Longrightarrow\ & D(\pi_{\text{eq}}) > \sum_{i=1}^{M}\sum_{k=1}^{K_i} q_{i,k} \\
\Longrightarrow\ & D(\pi_{\text{eq}}) > C \text{ (recall that } \sum_{k=1}^{K_i} q_{i,k} = C_i).
\end{aligned}
$$

We first show that there must exist π_0 such that $\pi_0 > \pi_{\text{eq}}$ and $D(\pi_{\text{eq}}) > D(\pi_0) > C$. (Recall that we have shown in Appendix B.3 that $D(\pi)$ is strictly monotone decreasing when $D(\pi) > 0$.) To see this, suppose on the contrary that $D(\pi) \le C$ for all $\pi > \pi_{\text{eq}}$. Then, because we have shown that $D(\pi_{\text{eq}}) > C$, $D(\cdot)$ must be discontinuous at π_{eq}. Thus, we have $\pi_{\text{eq}} = D^{-1}(C) = D^{-1}\left(\sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}\right)$, which contradicts our initial assumption that $D^{-1}\left(\sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}\right) \neq \pi_{\text{eq}}$. Thus, there must exist π_0 such that $\pi_0 > \pi_{\text{eq}}$ and $D(\pi_{\text{eq}}) > D(\pi_0) > C$.

Then, we let a firm i^* deviate to another bidding strategy $(\mathbf{p}', \mathbf{q}')$ that $K'_{i^*} = 1,\ p'_{i^*,1} = \pi_0,\ q'_{i^*,1} = C_i$ (i.e., bidding all her amount at the price π_0). Since $\pi_0 > \pi_{\text{eq}}$ and $p'_{i,k} = p_{i,k} \in \{0, \pi_{\text{eq}}\}$ for all $i \neq i^*$ and for all k, we have $p'_{i,k} \le \pi_0$ for all i and for all k. Thus, recalling that $\sum_{k=1}^{K'_i} q'_{i,k} = C_i$, we then have

$$
\sum_{\{i,k:\ p'_{i,k} \le \pi_0\}} q'_{i,k} = \sum_{i=1}^{M}\sum_{k=1}^{K'_i} q'_{i,k} = C.
$$

Because $D(\pi_0) > C$, we then have $D(\pi_0) > C = \sum_{\{i,k:\ p'_{i,k} \le \pi_0\}} q'_{i,k}$. By Lemma C.5(a), we have $\pi'_{\text{eq}} \ge \pi_0$. Since only the firm i^* bids at the price π_0 and other firms bids at π_{eq} or 0, by Algorithm 1, we have $\pi'_{\text{eq}} = \pi_0$. Because $D(\pi_0) > \sum_{\{i,k:\ p'_{i,k} \le \pi_0\}} q'_{i,k} \ge \sum_{i,k} s'_{i,k}$, by Lemma C.5(c), we have $s'_{i^*,1} = q'_{i^*,1} = C_i$. Thus, the profit of the firm i^* under $(\mathbf{p}', \mathbf{q}')$ equals to $\pi'_{\text{eq}} C_i = \pi_0 C_i > \pi_{\text{eq}} \sum_{k=1}^{K_i} q_{i^*,k}$, i.e., the new profit is greater than the profit of the original bidding strategy $(\mathbf{p}, \mathbf{q})$. This contradicts the assumption that the original bidding strategy is a Nash equilibrium. The conclusion of this lemma thus follows. □

Recall that in Theorem 5.2, we want to show that $\sum_{k=1}^{K_i} s_{i,k} = \sum_{k=1}^{K_i} q_{i,k} = C_i$ for all i and $\pi_{\text{eq}} = D^{-1}(C)$. Lemma C.12 implies that $\pi_{\text{eq}} = D^{-1}(C)$ would automatically hold if $\sum_{k=1}^{K_i} s_{i,k} = \sum_{k=1}^{K_i} q_{i,k} = C_i$ for all i. Therefore, in the rest of the proof, we only need to focus on proving $\sum_{k=1}^{K_i} s_{i,k} = \sum_{k=1}^{K_i} q_{i,k} = C_i$ for all i.

Proposition C.13. *If $C \le D(\max_n \overline{\pi}_n)$, then the outcome of the market could only be the outcome described in Theorem 5.2. Further, at any Nash equilibrium, we must have $\pi_{eq} = \max_n \overline{\pi}_n$.*

Proof. Step 1: We prove that the bidding strategy $(\mathbf{p}, \mathbf{q})$ defined as $K_i = 1,\ q_{i,1} = C_i,\ p_{i,1} = \max_n \overline{\pi}_n$ for all i is a Nash equilibrium. By Algorithm 1, we have $\pi_{\text{eq}} = \max_n \overline{\pi}_n$. By Algorithm 2, we have $s_{i,1} = q_{i,1} = C_i$ for all i. Thus, the payoff of the firm i is $C_i \max_n \overline{\pi}_n$. Because $D(\pi) = 0$ for all $\pi > \max_n \overline{\pi}_n$, we know the firm i has already gotten the maximum payoff compared to any bidding strategy. Thus, the bidding strategy $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium.

Step 2: We prove that, if a firm i^* bids as $K_{i^*} = 1,\ p_{i^*,1} = \max_n \overline{\pi}_n$, and $q_{i^*,1} = C_{i^*}$, then we must have $\pi_{\text{eq}} = \max_n \overline{\pi}_n$ and $s_{i^*,1} = C_{i^*}$. Because $D(\pi) = 0$ for all $\pi > \max_n \overline{\pi}_n$ and $D(\max_n \overline{\pi}_n) > C$, by Lemma C.5(a), we have $\pi_{\text{eq}} = \max_n \overline{\pi}_n$. Now, it remains to show that $s_{i^*,1} = C_{i^*}$. To that end, because $C \le D(\max_n \overline{\pi}_n)$ and $\pi_{\text{eq}} = \max_n \overline{\pi}_n$, we then have

$$
\begin{aligned}
& D(\pi_{\text{eq}}) \ge C \\
\Longrightarrow\ & D(\pi_{\text{eq}}) \ge \sum_{\{i,k:\ p_{i,k} < \pi_{\text{eq}}\}} q_{i,k} + \sum_{\{i,k:\ p_{i,k} = \pi_{\text{eq}}\}} q_{i,k} \\
\Longrightarrow\ & \frac{D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k} < \pi_{\text{eq}}\}} q_{i,k}}{\sum_{\{i,k:\ p_{i,k} = \pi_{\text{eq}}\}} q_{i,k}} \ge 1.
\end{aligned}
$$

By Eq. (7), we have $s_{i^*,1} = q_{i^*,1} = C_{i^*}$.

Step 3: We prove that at any Nash equilibrium, we must have $s_{i,k} = q_{i,k}$ for all i, k, and $\pi_{\text{eq}} = \max_n \overline{\pi}_n$. We prove by contradiction. Suppose on the contrary that, at a Nash equilibrium, there exist i^*, k^* such that $s_{i^*,k^*} < q_{i^*,k^*}$ or $\pi_{\text{eq}} \neq \max_n \overline{\pi}_n$. Then the firm i^*'s payoff is $s_{i^*,k^*}\pi_{\text{eq}} < C_i \max_n \overline{\pi}_n$. Now, let the firm i^* deviates to the bidding strategy described in Step 2. Then, the firm i^*'s payoff under the new bidding strategy is $C_i \max_n \overline{\pi}_n$, which is larger than her payoff under the original bidding strategy. This contradicts the assumption that the original bidding strategy is a Nash equilibrium. Thus, we must have $s_{i,k} = q_{i,k}$ for all i, k and $\pi_{\text{eq}} = \max_n \overline{\pi}_n$ at any Nash equilibrium. The result of the proposition thus follows. □

Proposition C.13 shows that if the total panel area is scarce, then every firm leases out all of her solar panels, and thus the market outcome must be the outcome described in Theorem 5.2.

By Proposition C.13, to finish the proof of Theorem 5.2, we only need to consider the case when $C > D(\max_n \overline{\pi}_n)$, i.e., the total panel area is plentiful.

C.5 Proof of Theorem 5.2

We first prove the following lemma.

Lemma C.14. *Let a, b, c, r, s be five positive real numbers. If $\frac{s}{r} \le \min\left\{\frac{a}{c},\ 1\right\}$ and $b > a$, then we must have*

$$r - \frac{b-a}{b}s \ge re^{-\frac{b-a}{c}}.$$

Proof. We consider two cases.

Case 1: $\frac{a}{c} \le 1$. Then, we have $\frac{s}{r} \le \min\left\{\frac{a}{c},\ 1\right\} = \frac{a}{c}$. Since

$$\frac{b}{a} - \frac{b-a}{c} \ge \frac{b}{a} - \frac{b-a}{a} = 1,$$

we have

$$\frac{\frac{b-a}{c}}{\frac{b}{a} - \frac{b-a}{c}} \le \frac{b-a}{c}.$$

Then, because $e^x \ge 1 + x$ for all $x \in \mathbb{R}$, we have

$$e^{\frac{b-a}{c}} \ge 1 + \frac{b-a}{c} \ge 1 + \frac{\frac{b-a}{c}}{\frac{b}{a} - \frac{b-a}{c}} = \frac{\frac{b}{a}}{\frac{b}{a} - \frac{b-a}{c}} = \frac{1}{1 - \frac{a}{c}\frac{b-a}{b}},$$

which implies

$$re^{-\frac{b-a}{c}} \le r - r\frac{a}{c}\frac{b-a}{b} \le r - s\frac{b-a}{b}$$
$$\left(\text{since } \frac{s}{r} \le \frac{a}{c}, \text{ then } r\frac{a}{c} \ge s\right).$$

Case 2: $\frac{a}{c} > 1$. Then, we have $\frac{s}{r} \le \min\left\{\frac{a}{c},\ 1\right\} = 1$. Let $c' = a > c$. We have

$$e^{-\frac{b-a}{c'}} \ge e^{-\frac{b-a}{c}}.$$

Thus, it is sufficient to show that

$$r - \frac{b-a}{b}s \ge re^{-\frac{b-a}{c'}} \text{ for all } \frac{s}{r} \le 1,$$

which is true by case 1 as $\frac{a}{c'} = 1$.

In conclusion, the result of this lemma thus follows. □

Now, we start to prove Theorem 5.2.

Proof of Theorem 5.2. Before entering the main part of our proof, we first show that $\pi_{\text{eq}} > 0$ for any Nash equilibrium. Because $\frac{\max_i C_i}{C} \le \frac{D^{-1}(C)}{\pi_g \gamma}$, we must have $D^{-1}(C) > 0$. By Lemma C.12, we then have $\pi_{\text{eq}} = D^{-1}(\sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}) \ge D^{-1}(C) > 0$.

We divide the proof into two parts according to the definition of the desired outcome. In Part 1, we prove the existence of Nash equilibrium. In Part 2, we prove the statements for any Nash equilibrium. (Notice that in Section 5.4, we omit Part 1 and provide the proof sketch of only the main steps of Part 2.)

Part 1: we show that the bidding strategy $(\mathbf{p}, \mathbf{q})$ defined as $K_i = 1,\ p_{i,1} = D^{-1}(C),\ q_{i,1} = C_i$ for all i is a Nash equilibrium. By Algorithm 1 and 2, we must have $\pi_{\text{eq}} = D^{-1}(C)$ and $s_{i,1} = C_i$ for all i. For any firm j, the payoff equals to $\pi_{\text{eq}} C_j$. We now prove by contradiction that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium. Suppose on the contrary that the current bidding strategy is not a Nash equilibrium. Then, there must exist a firm j that can deviate to another bidding strategy $(\mathbf{p}', \mathbf{q}')$ to increase her payoff. Thus, we must have $\pi'_{\text{eq}} > \pi_{\text{eq}}$ because the new sold amount of the firm j cannot exceed C_j. For any other firm $i \ne j$, since $p'_{i,1} = p_{i,1} = \pi_{\text{eq}} < \pi'_{\text{eq}}$, by Algorithm 2, we must have $s'_{i,1} = q_{i,1}$. Thus, by Lemma C.5(c), we have

$$\begin{aligned}\sum_{k=1}^{K'_j} s'_{j,k} &\le D(\pi'_{\text{eq}}) - \sum_{i\ne j} s'_{i,1} \\ &= D(\pi'_{\text{eq}}) - \sum_{i\ne j} q_{i,1} \\ &= D(\pi'_{\text{eq}}) - C + C_j.\end{aligned}$$

On the one hand, since the new payoff of the firm j is larger than her old payoff, we then have

$$\begin{aligned}&\pi'_{\text{eq}} \sum_{k=1}^{K'_j} s'_{j,k} > \pi_{\text{eq}} q_{j,1} \\ \Longrightarrow\ &\pi'_{\text{eq}} \left(D(\pi'_{\text{eq}}) - C + C_j\right) - \pi_{\text{eq}} C_j > 0 \\ \Longrightarrow\ &(\pi'_{\text{eq}} - \pi_{\text{eq}})C_j + \pi'_{\text{eq}} \left(D(\pi'_{\text{eq}}) - C\right) > 0 \\ \Longrightarrow\ &D(\pi'_{\text{eq}}) > C - \frac{\pi'_{\text{eq}} - \pi_{\text{eq}}}{\pi'_{\text{eq}}} C_j. \qquad (18)\end{aligned}$$

On the other hand, by Corollary B.9, we have

$$D(\pi'_{\text{eq}}) \le Ce^{-\frac{\pi'_{\text{eq}} - \pi_{\text{eq}}}{\pi_g \gamma}}.$$

Because $\frac{C_j}{C} \le 1$ and $\frac{C_j}{C} \le \frac{\max_i C_i}{C} \le \frac{\pi_{\text{eq}}}{\pi_g \gamma}$, we must have $\frac{C_j}{C} \le \min\left\{\frac{\pi_{\text{eq}}}{\pi_g \gamma}, 1\right\}$. Applying Lemma C.14, we have

$$D(\pi'_{\text{eq}}) \le C - \frac{\pi'_{\text{eq}} - \pi_{\text{eq}}}{\pi'_{\text{eq}}} C_j.$$

This contradicts Eq. (18). Thus, we have proven that the original bidding strategy $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium.

Part 2: we next prove that, at any Nash equilibrium, we must have $s_{i,k} = q_{i,k}$ for all i, k. We roughly follow the main steps sketched in Section 5.4. We prove by contradiction. Suppose on the contrary that at a Nash equilibrium $(\mathbf{p}, \mathbf{q})$, there exists at least one firm with an unsold/partly-sold bid, i.e.,

$$\left\{i^* :\ \sum_{k=1}^{K_{i^*}} s_{i^*,k} < C_{i^*}\right\} \ne \emptyset.$$

In the following, we will consider another bidding strategy $(\mathbf{p}', \mathbf{q}')$ that a carefully-chosen firm j deviates from the original bidding strategy $(\mathbf{p}, \mathbf{q})$ to another strategy with $K'_j = 1,\ p'_{j,1} = \pi^*$, and $q'_{j,1} = C_j$. We find j and π^* through the following steps 1 and 2. We then get some useful properties in steps 3, 4 and 5. In the end, we establish the contradiction to complete the proof in step 6.

Step 1: We prove that there exists a firm j such that $\sum_{k=1}^{K_j} s_{j,k} < C_j$ and $\frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})} \le \frac{C_j}{C}$. (Notice that this step is mentioned in conclusion ② of Section 5.4.) By Lemma C.5(c), we must have

$$D(\pi_{\text{eq}}) \ge \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k} = \left(\sum_{\left\{i^*:\ \sum_{k=1}^{K_{i^*}} s_{i^*,k} < C_{i^*}\right\}} \sum_{k=1}^{K_{i^*}} s_{i^*,k}\right) + \sum_{\left\{i:\ \sum_{k=1}^{K_i} s_{i,k} = C_i\right\}} C_i,$$

which implies that

$$\sum_{\left\{i^*:\ \sum_{k=1}^{K_{i^*}} s_{i^*,k} < C_{i^*}\right\}} \frac{\sum_{k=1}^{K_{i^*}} s_{i^*,k}}{D(\pi_{\text{eq}}) - \sum_{\left\{i:\ \sum_{k=1}^{K_i} s_{i,k}=C_i\right\}} C_i} \le 1.$$

On the other hand, we have

$$C = \sum_{\left\{i^*:\ \sum_{k=1}^{K_{i^*}} s_{i^*,k} < C_{i^*}\right\}} C_{i^*} + \sum_{\left\{i:\ \sum_{k=1}^{K_i} s_{i,k}=C_i\right\}} C_i$$

$$\Longrightarrow \sum_{\left\{i^*:\ \sum_{k=1}^{K_{i^*}} s_{i^*,k} < C_{i^*}\right\}} \frac{C_{i^*}}{C - \sum_{\left\{i:\ \sum_{k=1}^{K_i} s_{i,k}=C_i\right\}} C_i} = 1.$$

As a result, there must exist a firm j such that $\sum_{k=1}^{K_j} s_{j,k} < C_j$ and

$$\frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}}) - \sum_{\left\{i:\ \sum_{k=1}^{K_i} s_{i,k}=C_i\right\}} C_i} \le \frac{C_j}{C - \sum_{\left\{i:\ \sum_{k=1}^{K_i} s_{i,k}=C_i\right\}} C_i}$$

$$\Longrightarrow \left(\sum_{k=1}^{K_j} s_{j,k}\right)\left(C - \sum_{\left\{i:\ \sum_{k=1}^{K_i} s_{i,k}=C_i\right\}} C_i\right) \le C_j\left(D(\pi_{\text{eq}}) - \sum_{\left\{i:\ \sum_{k=1}^{K_i} s_{i,k}=C_i\right\}} C_i\right)$$

$$\Longrightarrow \left(\sum_{k=1}^{K_j} s_{j,k}\right) C \le C_j D(\pi_{\text{eq}}) \text{ (because } \sum_{k=1}^{K_j} s_{j,k} < C_j)$$

$$\Longrightarrow \frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})} \le \frac{C_j}{C}.$$

Step 2: We let the firm j found in step 1 change her bids in the way that we describe earlier (i.e., $K_j' = 1$, $p_{j,1}' = \pi^*$, and $q_{j,1}' = C_j$). In this step, we will prove that for all $\pi^* \in \left(\frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j},\ \pi_{\text{eq}}\right)$, the market outcome must satisfy $s_{j,1}' < C_j$. We prove by contradiction. Suppose on the contrary that there exists a price $\pi \in \left(\frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j},\ \pi_{\text{eq}}\right)$ such that $s_{j,1}' = C_j$ when the firm bids $p_{j,1}' = \pi$. The firm j's payoff with $(\mathbf{p}', \mathbf{q}')$ must then be equal to or greater than πC_j. Since $\pi > \frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j}$, we then have

$$\pi C_j > \pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k},$$

i.e., the firm j's payoff under $(\mathbf{p}', \mathbf{q}')$ is larger than that under $(\mathbf{p}, \mathbf{q})$. This contradicts the assumption that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium. Thus, we have proven that, for all $\pi^* \in \left(\frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j},\ \pi_{\text{eq}}\right)$, the market outcome under the new bid must satisfy $s_{j,1}' < C_j$.

Step 3: We now let firm j found in step 1 change her bid in the way that we describe earlier, with $\pi^* \in \left(\frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j},\ \pi_{\text{eq}}\right)$. In this step, we will prove that the market outcome must satisfy $\pi_{\text{eq}}' = \pi^*$. By Step 2, we have $s_{j,1}' < C_j = q_{j,1}'$. Thus, we have $\pi_{\text{eq}}' \le p_{j,1}' = \pi^*$. Because $p_{i,k}' \in \{0, \pi^*, \pi_{\text{eq}}\}$ for all (i,k) (notice that $\pi_{\text{eq}} > \pi^* > \frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j} \ge 0$ and we have restricted our attention to two-block bids by Theorem C.2), we have

$$\begin{aligned}
\sum_{\{i,k:\ p_{i,k}' < \pi^*\}} q_{i,k}' &= \sum_{\{i,k:\ p_{i,k}'=0\}} q_{i,k}' \\
&= \sum_{\{i,k:\ p_{i,k}=0,\ i\neq j\}} q_{i,k} \\
&\le \sum_{\{i,k:\ p_{i,k}=0\}} q_{i,k} \\
&\le \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k} \text{ (because } \pi_{\text{eq}} > 0) \\
&\le D(\pi_{\text{eq}}) \text{ (by Lemma C.5(a))} \\
&\le D(\pi^*).
\end{aligned}$$

Applying Lemma C.5(a), we have $\pi_{\text{eq}}' \ge \pi^*$. As a result, we conclude that $\pi_{\text{eq}}' = \pi^*$.

Step 4: Following step 3, we prove that $s_{j,1}' - \sum_{k=1}^{K_j} s_{j,k} \ge D(\pi^*) - \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}$. By step 2, we know that $s_{j,1}' < C_j$. By step 3, we know $\pi_{\text{eq}} = \pi^*$. (Notice that this step is mentioned in conclusion ③ of Section 5.4.) Thus, by Lemma C.5(c), we must have $D(\pi^*) = \sum_{i=1}^{M}\sum_{k=1}^{K_i'} s_{i,k}'$. As a result, we have

$$\begin{aligned}
D(\pi^*) - \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k} &= \sum_{i=1}^{M}\sum_{k=1}^{K_i'} s_{i,k}' - \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k} \\
&= s_{j,1}' - \sum_{k=1}^{K_j} s_{j,k} + \sum_{i\neq j}\left(\sum_{k=1}^{K_i'} s_{i,k}' - \sum_{k=1}^{K_i} s_{i,k}\right).
\end{aligned}$$

Because $\pi'_{\text{eq}} = \pi^* \in (0, \pi_{\text{eq}})$ and the bidding strategy of any other firm $i \neq j$ does not deviate, we have

$$\sum_{k=1}^{K'_i} s'_{i,k} = \sum_{\{k:\, p'_{i,k}=0\}} q'_{i,k} = \sum_{\{k:\, p_{i,k}=0\}} s_{i,k} \leq \sum_{k=1}^{K_i} s_{i,k} \text{ for all } i \neq j.$$

Thus, we have $s'_{j,1} - \sum_{k=1}^{K_j} s_{j,k} \geq D(\pi^*) - \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}$.

Step 5: We prove that $\pi_{\text{eq}} > D^{-1}(C)$ by contradiction. Suppose on the contrary that $\pi_{\text{eq}} \leq D^{-1}(C)$. Thus, we have $D(\pi_{\text{eq}}) \geq D\left(D^{-1}(C)\right) \geq C > \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}$. By Lemma C.5(c), we have $s_{i,k} = q_{i,k}$ for all i, k (because $p_{i,k} \in \{0, \pi_{\text{eq}}\}$). That contradicts the assumption that the firm j has an unsold/partly-sold bid. Thus, we have proven that $\pi_{\text{eq}} > D^{-1}(C)$. (Notice that Step 5 is mentioned in conclusion ① of Section 5.4.)

Step 6: We establish the contradiction for the initial assumption of the whole Part 2 that there exists at least one firm with an unsold/partly-sold bid at $(\mathbf{p}, \mathbf{q})$. Following the strategy $(\mathbf{p}', \mathbf{q}')$ stated in step 3, because $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium, we have

$$\pi^* s'_{j,1} \leq \pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}$$

$$\Longrightarrow \pi^* \left(s'_{j,1} - \sum_{k=1}^{K_j} s_{j,k} \right) \leq \left(\pi_{\text{eq}} - \pi^*\right) \sum_{k=1}^{K_j} s_{j,k}$$

$$\Longrightarrow \pi^* \left(D(\pi^*) - \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k} \right) \leq \left(\pi_{\text{eq}} - \pi^*\right) \sum_{k=1}^{K_j} s_{j,k}$$

(by step 4)

$$\Longrightarrow \pi^* \left(D(\pi^*) - D(\pi_{\text{eq}}) \right) \leq \left(\pi_{\text{eq}} - \pi^*\right) \sum_{k=1}^{K_j} s_{j,k}$$

(by Lemma C.5(c))

$$\Longrightarrow \frac{D(\pi^*) - D(\pi_{\text{eq}})}{D(\pi_{\text{eq}})} \leq \frac{\pi_{\text{eq}} - \pi^*}{\pi^*} \cdot \frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})}.$$

Note that the last inequality holds for any $\pi^* \in \left(\frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j},\ \pi_{\text{eq}} \right)$.

Letting π^* approach π_{eq}, we have

$$\left| \frac{\partial D(\pi_{\text{eq}})}{D(\pi_{\text{eq}})} \right| \leq \left| \frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}} \right| \frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})}$$

$$\leq \left| \frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}} \right| \frac{C_j}{C} \text{ (by step 1)}$$

$$\leq \left| \frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}} \right| \frac{D^{-1}(C)}{\pi_g \gamma}$$

$$< \left| \frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}} \right| \frac{\pi_{\text{eq}}}{\pi_g \gamma} \text{ (by step 5)},$$

which contradicts Proposition 5.1.

In conclusion, the result stated in this theorem thus holds. □

D SITUATION WITH HETEROGENEOUS $G_i(t)$

In this section, we discuss briefly the situation when the efficiency of solar generation $G(t)$ is not the same for all firms, in which case we use $G_i(t)$ to denote the generation per unit size of solar panel by firm i at time t. Such a situation occurs when the firms are not receiving the same level of irradiance, or they use solar panels with different conversion efficiency, or some prosumers use the solar generation for their own demand first (before selling the remaining generation to the market). We note that this situation has no impact to our results for real-time markets in Section 4. However, it does affect the rental markets in Section 5, as a consumer renting from a firm with low $G_i(t)$ will receive less solar energy in real-time, which complicates the calculation of the panel demand function in (4).

To resolve this issue, we propose that the rental market operator introduces a normalization procedure to complement the rental market mechanism presented in Section 5. This normalization procedure calculates an "effective panel size" C_i for each firm based on her historical generation efficiency $G_i(t)$ and her declared panel size C_i^r, and calculates a common efficiency $G(t)$ based on the generation efficiency of all firms in the market. Afterwards, the rental markets can operate based on the effective panel size C_i and common/normalized efficiency $G(t)$ only. The high-level goals of this normalization procedure are the following. First, firms with higher (correspondingly, lower) $G_i(t)$ will have larger (correspondingly, smaller) effective panel sizes C_i than their declared panel sizes C_i^r, so that the firms are compensated by their actual solar generation, not merely by what panel size they declare. Second, regardless of from which firm a consumer rents the solar panel from, the consumer will receive the same amount of solar energy in real-time per unit of effective panel size. Third, the normalized $G(t)$ and the effective panel sizes C_i^r are calculated in such a way that the total available solar energy to the market at each time equals to the actual solar generation (see (19) below), so that the normalized procedure does not lead to any surplus/deficit of solar energy to the market operator.

Specifically, we propose the following normalization procedure:

(1) The market operator estimates the overall efficiency of each firm's solar panels based on historical data over a past interval of length T. Recall that $G_i(t)$ denote the generation per unit area of firm i's solar panel at time t. The overall efficiency e_i of firm i is estimated by the average of $G_i(t)$ over the past interval, i.e.,

$$e_i = \frac{1}{T} \int_0^T G_i(t) dt.$$

(2) The market operator rates the effective panel size of each firm i by her overall efficiency e_i. Specifically, the market operator chooses a baseline efficiency e_0 in advance. (For example, e_0 could be the expected overall efficiency of a typical panel over the past interval.) Denote firm i's declared panel size as C_i^r. Its effective panel size can then be calculated as

$$C_i = \frac{e_i}{e_0} C_i^r.$$

Note that this formula ensures that firms with higher generation efficiency on average will be rated with larger effective panel sizes.

(3) The market operator calculates the normalized generation efficiency (per unit of effective panel size) for all firms in the market as

$$G(t) = \frac{\sum_{i=1}^{M} G_i(t) C_i^r}{\sum_{i=1}^{M} C_i}.$$

Note that this formula guarantees that the total generation calculated by the normalized $G(t)$ and the effective panel sizes equlas to the actual total generation at each time, i.e.,

$$G(t) \sum_{i=1}^{M} C_i = \sum_{i=1}^{M} G_i(t) C_i^r. \tag{19}$$

Once C_i and $G(t)$ are provided by the market operator, the rental market will operate in the same way as we presented in Section 5 for the next interval of length T. Specifically, the consumers will use the historical data of $G(t)$ to estimate its future probability distribution, assuming that the future time interval will have a similar distribution as the previous interval. Based on this distribution of $G(t)$, the consumers can calculate their panel demand functions in (4). Similarly, the firms submit their bids to the market using only their effective panel sizes. The rental market operator then determines, for the next interval of length T, the rental price (per unit of effective panel size), and settles the payment between firms and consumers based on the effective panel sizes that are cleared. In real-time, the market operator will distribute the solar generation according to the effective panel size. Specifically, at time t in the next interval of length T, we still denote $G_i(t)$ as the generation per unit area of firm i's solar panel at time t. The total generation is then $\sum_{i=1}^{M} G_i(t) C_i^r$. For each consumer n renting c_n units of effective panel, she will then "be credited" with the amount of solar generation equal to

$$\frac{c_n}{\sum_{i=1}^{M} C_i} \sum_{i=1}^{M} G_i(t) C_i^r.$$

In this way, the consumer will receive the same amount of solar energy per unit of effective panel size, regardless of from which firm she rents. At the end of this interval of length T, the above normalized procedure is repeated to update C_i and $G(t)$ for the next interval of rental market operation.

E SYSTEM SETUP OF FIG. 6

In this part, we describe the system setup of the forward energy market used in Fig. 6, and explain how to calculate the solar generation amount that is utilized in both the rental market and the forward energy market.

We first explain the market clearing mechanism of the forward energy market. For each firm i, the supply curve is determined by

$$s_i(\pi) = \arg\max_x \left(x\pi - \pi_g \mathbb{E}[x - C_i G(t)]^+ \right).$$

Due to our assumption that all solar panels have the same generation efficiency $G(t)$, we can easily prove that $s_i(\pi)$ is proportional to C_i. Thus, the aggregate supply curve is then

$$S(\pi) := \sum_{i=1}^{M} s_i(\pi) = \arg\max_x \left(x\pi - \pi_g \mathbb{E}[x - CG(t)]^+ \right).$$

Similarly, for the customers side, their aggregate demand curve is

$$D(\pi) := \sum_{i=1}^{N} d_i(\pi) = \arg\min_x \left(x\pi + \pi_g \mathbb{E}[L(t) - x]^+ \right).$$

Thus, when $G(t)$ and $L(t)$ are uniformly distributed in $[a, b]$, we have

$$S(\pi) = C \cdot \left((b-a) \cdot \frac{\pi}{\pi_g} + a \right), \quad D(\pi) = b - (b-a) \cdot \frac{\pi}{\pi_g}.$$

The market price π_{eq} is determined by the intersection of the aggregate supply curve and the aggregate demand curve, i.e., $S(\pi_{\text{eq}}) = D(\pi_{\text{eq}})$. We thus get $\frac{\pi_{\text{eq}}}{\pi_g} = \frac{1}{C+1} \cdot \frac{b-aC}{b-a}$.

We can then calculate the the expected amount of solar generation that is used. For the forward energy market (without any additional real-time market), the used generation amount can be calculated by

$$\mathbb{E}[\min\{CG(t), D(\pi_{\text{eq}}), L(t)\}].$$

In comparison, since the rental market experiences the outcome described in Theorem 5.2, we can calculate its expected used generation amount by

$$\mathbb{E}[\min\{CG(t), L(t)\}].$$

The curves in Fig. 6 correspond to these two expressions, assuming the two different settings of correlation between $L(t)$ and $G(t)$ described in Section 5.6.