Categorifications of rational Hilbert series
and characters of FS^{op} modules

Philip Tosteson

Categorifications of rational Hilbert series and characters of $\textbf{\textit{FS}}^{\mathrm{op}}$ modules

Philip Tosteson

We introduce a method for associating a chain complex to a module over a combinatorial category such that if the complex is exact then the module has a rational Hilbert series. We prove homology-vanishing theorems for these complexes for several combinatorial categories including the category of finite sets and injections, the opposite of the category of finite sets and surjections, and the category of finite-dimensional vector spaces over a finite field and injections.

Our main applications are to modules over the opposite of the category of finite sets and surjections, known as $\textbf{\textit{FS}}^{\mathrm{op}}$ modules. We obtain many constraints on the sequence of symmetric group representations underlying a finitely generated $\textbf{\textit{FS}}^{\mathrm{op}}$ module. In particular, we describe its character in terms of functions that we call character exponentials. Our results have new consequences for the character of the homology of the moduli space of stable marked curves, and for the equivariant Kazhdan–Lusztig polynomial of the braid matroid.

1. Introduction

Prompted by the work of Snowden [2013] and Church, Ellenberg, and Farb [2015], there have been many recent applications of the representation theory of categories to sequences of group representations arising in topology, algebra, and combinatorics.

Let $(V_n)_{n \in \mathbb{N}}$ be a sequence of representations of groups G_n over a field k. In these applications, $(V_n)_{n \in \mathbb{N}}$ can be lifted to a representation of a category $\mathcal{C}$ whose objects (or isomorphism classes) are indexed by natural numbers such that the automorphism group of the n-th object of $\mathcal{C}$ is G_n. A $\mathcal{C}$ *representation*, or $\mathcal{C}$ *module*, is a functor $\mathcal{C} \to \operatorname{Mod} k$. One attempts to use the representation theory of $\mathcal{C}$ to show that the whole sequence of representations is determined by a finite amount of data, and to discover universal patterns which the sequence must satisfy. Often, these patterns are expressed in terms of generating functions.

One coarse invariant of a sequence of representations is its sequence of dimensions: $(\dim V_n)_{n \in \mathbb{N}}$. The data can be recorded as a generating function, the *Hilbert series* of $(V_n)_{n \in \mathbb{N}}$,

$$h_V(t) := \sum_n (\dim_k V_n) t^n \in \mathbb{Z}[[t]].$$

Often, theorems from the representation theory of $\mathcal{C}$ imply that $h_V(t)$ is rational with denominator of a specific form. We list three important instances.

- Let **FI** be the category of finite sets and injections. If V is a finitely generated **FI** module then $h_V(t)$ is a rational function with denominator $(1-t)^d$ for some $d \in \mathbb{N}$.

  Equivalently, the sequence $(\dim V_n)_{n \in \mathbb{N}}$ eventually agrees with a polynomial in n. This was proved in characteristic 0 in [Church et al. 2015; Sam and Snowden 2016] and for all fields by Church, Ellenberg, Farb, and Nagpal [2014].

- Let **FS**$^{\mathrm{op}}$ be the opposite of the category of finite sets and surjections. If V is a finitely generated **FS**$^{\mathrm{op}}$ module, then $h_V(t)$ is a rational function with denominator a power of $\prod_{j=0}^{d-1}(1-jt)$ for some $d \in \mathbb{N}$. This was proved by Sam and Snowden [2017].

- Let **VI**$_q$ be the category of finite-dimensional vector spaces over $\mathbb{F}_q$ and linear injections between them. If V is a finitely generated **VI**$_q$ module and char $k \neq$ char $\mathbb{F}_q$, then $h_V(t)$ is rational with denominator $\prod_{j=0}^{d-1}(1-q^j t)$ for some $d \in \mathbb{N}$.

  Equivalently, $(\dim V_n)_{n \in \mathbb{N}}$ eventually agrees with a polynomial in q^n, This was proved by Nagpal [2019].

In this paper we *categorify* these three results, in a uniform way.

Let $\mathcal{C} = \textbf{FI}$, $\textbf{FS}^{\mathrm{op}}$, or $\textbf{VI}_q$, and let $w_d(t) = (1-t)^d$, $\prod_{j=0}^{d-1}(1-jt)$, or $\prod_{j=0}^{d-1}(1-q^j t)$, respectively. Let V be a $\mathcal{C}$ module. Using methods from poset topology, we construct a chain complex of $\mathcal{C}$ modules $\mathrm{K}_d(V)$ for each $d \in \mathbb{N}$ such that exactness of $\mathrm{K}_d(V)$ categorifies the equation $w_d(t)h_V(t) = p(t)$ for p a polynomial. In each case, we prove a theorem showing if V is finitely generated, either K_d or a power of K_d applied to V is exact (modulo torsion).

In particular, our categorification explains why the denominators $(1 - t)^d$, $\prod_{j=0}^{d-1}(1 - jt)$, and $\prod_{j=0}^{d-1}(1 - q^j t)$ appear. They are the *Whitney polynomials* of the poset of subsets of a d-element set, of set partitions of a d-element set, and of subspaces of a d-dimensional vector space, respectively (see Definition 2.6).

1A. ***FS^{op} modules.*** Our main applications are in the case where $\mathcal{C}$ is the opposite of the category of finite sets and surjections, FS^{op}.

In practice, FS^{op} modules have been used to study sequences of symmetric group representations $(V_n)_{n \in \mathbb{N}}$ such that $\dim(V_n)$ grows at an exponential rate — too quickly for representation stability in the sense of Church and Farb [2013] to hold. Proudfoot and Young [2017] used FS^{op} to study the Kazhdan–Lusztig polynomial of the braid matroid, by constructing an action of FS^{op} on the intersection homology of the reciprocal plane of the braid arrangement. In [Tosteson 2021], we showed that the homology of the moduli space of stable curves is a finitely generated FS^{op} module. In forthcoming work, we extend this result to the Kontsevich space of stable maps. Proudfoot and Ramos [2021] constructed finitely generated FS^{op} modules in the cohomology of the resonance arrangement.

In other directions, FS^{op} was used by Sam and Snowden [2017] to prove the Lannes–Schwartz Artinian conjecture: for a finite ring R any finitely generated VI_R module (Definition 3.10) restricts to a finitely generated FS^{op} module. The category of FS^{op} modules is the category of right modules over the commutative operad, and thus is closely tied to homology theories for commutative algebras, and FS^{op} may be used to compute the homology groups of pointed mapping spaces [Pirashvili 2000]. In forthcoming work with Sam and Snowden, we relate FS^{op} modules to representations of the Witt Lie algebra and the monoid of endomorphisms of $\mathbb{A}^n$.

Despite these applications, the representation theory of FS^{op} is not well understood. For finitely generated FI modules and VI_q modules in nondescribing characteristic, Nagpal [2015; 2019] has proved structure theorems, known as shift theorems, which we use to establish our results. But FS^{op} modules do not satisfy a shift theorem, and are qualitatively different: the category of finitely generated FS^{op} modules has infinite Krull–Gabriel dimension, as opposed to dimension 1. Little has been known about the sequence of symmetric group representations underlying a finitely generated FS^{op} module.

1B. ***Categorification for FS^{op} modules.*** For an FS^{op} module M, the complex $K_d(M)$ we construct takes the form

$$\Sigma^d M \leftarrow \Sigma^{d-1} M^{\oplus\binom{d}{2}} \leftarrow \cdots \leftarrow \Sigma^1 M^{\oplus(d-1)!}.$$

Here $\Sigma^n M$ denotes the FS^{op} module $x \mapsto M_{x \sqcup [n]}$, where $[n] := \{1, \ldots, n\}$ is the distinguished finite set with n elements. In general, the degree i term of $K_d(M)$ is $\Sigma^{d-i} M^{\oplus s(d, d-i)}$, where $s(d, d-i)$ is the unsigned *Stirling number of the first kind*: by definition, $(-1)^i s(d, d-i)$ is the coefficient of t^i in the product $\prod_{j=0}^{d-1}(1 - jt)$. We may extend the definition of K_d to chain complexes of FS^{op} modules, by applying K_d degreewise and forming the total complex.

To state our categorification for FS^{op} modules, we recall the following definition.

Definition 1.1. For $d \in \mathbb{N}$, we say that M is *finitely generated in degree* $\leq d$ if M_x is a finitely generated k module for all $x \in \mathbf{FS}^{\mathrm{op}}$ with $|x| \leq d$, and for any $y \in \mathbf{FS}^{\mathrm{op}}$, M_y is spanned by elements of the form $M_f(v)$, where $v \in M_x$, $f : y \twoheadrightarrow x$ and $|x| \leq d$.

Theorem 1.2. *Let M be an $\mathbf{FS}^{\mathrm{op}}$ module over a field which is a subquotient of an $\mathbf{FS}^{\mathrm{op}}$ module that is finitely generated in degree $\leq d$. Then there exists $s \in \mathbb{N}$ such that for all $r \geq s$ and all $(\ell_t) \in \mathbb{N}^r$ satisfying $\ell_t \geq d$ for all t and $\ell_r \geq d + 1$, the complex*

$$\mathrm{K}_{\ell_1} \circ \mathrm{K}_{\ell_2} \circ \cdots \circ \mathrm{K}_{\ell_r}(M)$$

is exact.

In particular, there exists an $s \geq 1$ such that $\mathrm{K}_d^{\circ s - 1}(\mathrm{K}_{d+1}(M))$ is exact. This categorifies the theorem of Sam and Snowden that $h_M(t)$ is rational with denominator a power of $\mathrm{W}_{d+1}(t) := \prod_{j=0}^{d}(1 - jt)$, by categorifying the equation $\mathrm{W}_d(t)^{s-1}\mathrm{W}_{d+1}(t)h_M(t) = p(t)$ for p a polynomial. We prove Theorem 1.2 by using Sam–Snowden Gröbner theory to reduce it to a combinatorial statement about regular languages associated to M. The value of s is related to the minimal number of states in the DFAs accepting these languages.

We list two cases in which Theorem 1.2 applies.

Example 1.3. For $i, g \in \mathbb{N}$, the sequence of S_n representations $n \mapsto H_i(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$, where $\overline{\mathcal{M}}_{g,n}$ is the moduli space of marked stable curves, can be extended to an $\mathbf{FS}^{\mathrm{op}}$ module. In [Tosteson 2021], we showed that this $\mathbf{FS}^{\mathrm{op}}$ module is a subquotient of one that is finitely generated in degree less than or equal to $8g^2i^2 + 29g^2i + 16gi^2 + 21g^2 + 10gi$.

Example 1.4. The i-th coefficient of the equivariant Kazhdan–Lusztig polynomial of the n-th braid arrangement is the character of an S_n representation. For fixed i, Proudfoot and Young [2017] extended this sequence of S_n representations to an $\mathbf{FS}^{\mathrm{op}}$ module and proved it is a subquotient of one that is finitely generated in degree $\leq 2i$.

1C. *Characters of $\mathbf{FS}^{\mathrm{op}}$ modules.* For each $n, d \in \mathbb{N}$ the group $\mathrm{S}_d \times \mathrm{S}_n$ acts on $\mathrm{K}_d(M)_n$. The action of S_d is closely related to its action on the Whitney homology of the partition lattice. Via this action, Theorem 1.2 decategorifies to a system of differential equations for the Frobenius character of M. By solving these equations, we obtain new results about the character of a finitely generated $\mathbf{FS}^{\mathrm{op}}$ module. To state them, we take $k = \mathbb{Q}$.

Associated to every $\mathbf{FS}^{\mathrm{op}}$ module M is its *character* $\bigsqcup_n \mathrm{S}_n \to \mathbb{Q}$, given by $\sigma \in \mathrm{S}_n \mapsto \mathrm{Tr}(\sigma, M_{[n]})$. Our first result describes the character of an $\mathbf{FS}^{\mathrm{op}}$ module in terms of character polynomials and new functions that we call character exponentials.

Definition 1.5. Let $d \in \mathbb{N}$, $d \geq 1$. Then $\mathbb{X}_d : \bigsqcup_n \mathrm{S}_n \to \mathbb{Q}$ is the class function

$$\mathbb{X}_d(\sigma) := \#\{d\text{-cycles of } \sigma\}.$$

A *character polynomial* is any class function that is a polynomial in the functions $\{\mathbb{X}_d\}_{d \in \mathbb{N}}$.

Definition 1.6. Let $A = 1^{a_1} 2^{a_2} \ldots$ be an integer partition. We define the *character exponential* of A to be class function

$$A^{\mathbb{X}} := \prod_{n \geq 1} \left(\sum_{d \mid n} d a_d \right)^{\mathbb{X}_n},$$

where the sum is over all natural numbers $d \geq 1$ which divide n. The product is well defined because any $\sigma \in S_m$ has finitely many cycles, so $A^{\mathbb{X}}(\sigma)$ is a natural number.

In this definition, it is important that $0^0 = 1$ and $0^n = 0$ for all $n > 0$. We give examples to clarify:

Example 1.7. If $a_i = 0$ for $i > 1$, then $A^{\mathbb{X}} = (a_1)^{\sum_{n \geq 1} \mathbb{X}_n}$.

Example 1.8. If $a_1 = 0$ and $a_i = 0$ for $i > 2$, then

$$A^{\mathbb{X}}(\sigma) = \begin{cases} (2a_2)^{\sum_{n \geq 1} \mathbb{X}_{2n}(\sigma)} & \text{if all cycles of } \sigma \text{ have even length,} \\ 0 & \text{otherwise.} \end{cases}$$

In fact, to state our results we need a more general function, which is similar to (but not equal to) a product of a character polynomial and a character exponential.

Definition 1.9. Let $A = 1^{a_1} 2^{a_2} \ldots$ and $\nu = 1^{m_1} 2^{m_2} \ldots$ be integer partitions. We define

$$\binom{\mathbb{X}}{\nu} A^{\mathbb{X} - \nu} := \prod_{n \geq 1} \binom{\mathbb{X}_n}{m_n} \left(\sum_{d \mid n} d a_d \right)^{\mathbb{X}_n - m_n},$$

where $\binom{\mathbb{X}_i}{m_i}(\sigma) = \binom{\mathbb{X}_i(\sigma)}{m_i}$. We define the *rank of ν* to be $\text{rank}(\nu) := \sum_{i \geq 1} m_i$, so that $\binom{\mathbb{X}}{\nu}$ is a character polynomial of degree $\text{rank}(\nu)$.

The function $\binom{\mathbb{X}}{\nu} A^{\mathbb{X} - \nu}$ specializes to the character polynomial $\binom{\mathbb{X}}{\nu}$ when $A = 1$ and to the character exponential $A^{\mathbb{X}}$ when $\nu = 0$.

Example 1.10. If $a_i = 0$ for all i and $\nu = 1^{m_1} 2^{m_2} \ldots$, then

$$\binom{\mathbb{X}}{\nu} A^{\mathbb{X} - \nu}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is in the conjugacy class corresponding to } \nu, \\ 0 & \text{otherwise.} \end{cases}$$

Let $d, s \in \mathbb{N}$. We say that an *FS*^{op} module has *class* (d, s) if it satisfies the hypothesis and the conclusion of Theorem 1.2 for d and s. Then our first result on characters is:

Theorem 1.11. *Let M be an FS*^{op} *module of class (d, s). The character function of M_n takes the form*

$$\sum_{\nu, A \text{ partitions}} c(\nu, A) \binom{\mathbb{X}}{\nu} A^{\mathbb{X} - \nu},$$

where $c(\nu, A) \in \mathbb{Q}$ and $c(\nu, A) = 0$ if $|A| > d$; or $\text{rank}(\nu) \geq s$; or $|A| = d$ and $\text{rank}(\nu) > 1$.

In the case $s = 1$ this sum is finite and Theorem 1.11 sharp, in the sense that the characters of degree d projective *FS*^{op} modules span the space of character functions. We emphasize that the sum in Theorem 1.11 is not finite when $s > 1$. So by itself Theorem 1.11 does not reduce the computation of the character of M to a finite problem. However if we restrict the character of M to elements in S_n

whose cycles all have length $\leq C$ (which corresponds to setting $\mathbb{X}_i = 0$ for $i > C$), then there are only finitely many terms in the sum. In this case Theorem 1.11 can be rephrased as a rationality theorem for a multivariate generating function (left to the reader). When $C = 1$, we precisely recover the rationality theorem of Sam and Snowden.

Example 1.12. Let M be an FS^{op} module of class $(2, 2)$. Theorem 1.11 states that the character of M takes the form

$$c_1 2^{\sum_{i \geq 1} \mathbb{X}_i} + c_2 \, \delta_{\mathrm{even}} \, 2^{\sum_{i \geq 1} \mathbb{X}_{2i}} + c_3 + \sum_{n \geq 1} b_n \mathbb{X}_n + \sum_{n \geq 0} f_n \delta_{n\text{-cycle}},$$

where $c_1, c_2, b_n, f_n \in \mathbb{Q}$. Here $\delta_{\mathrm{even}}(\sigma)$ (resp. $\delta_{n\text{-cycle}}(\sigma)$) is 1 if all of the cycles of σ have even length (resp. if σ is an n-cycle) and zero otherwise.

There are further constraints on FS^{op} modules of class (d, s): if M is a subquotient of an FS^{op} module generated in degree d we show the character of M is *uniquely determined* by its restriction to elements of S_n whose cycles have length $\leq d$. Using this, we can prove the following.

Theorem 1.13. *Let* $\hat{\Lambda}$ *be the completed ring of symmetric functions. There is a finite-dimensional subspace* $\mathcal{U}_{d,s} \subseteq \hat{\Lambda}$ *such that if M is an FS^{op} module of class (d, s), then its Frobenius character* ch(M) *is an element of* $\mathcal{U}_{d,s}$. *The dimension of* $\mathcal{U}_{d,s}$ *is* $p(d) + \binom{d+s-1}{s-1} \sum_{i=0}^{d-1} p(i)$, *where $p(i)$ is the number of integer partitions of i.*

This reduces the computation of the character of an FS^{op} module of class (d, s) to a finite problem. We describe (refined versions of) the space $\mathcal{U}_{d,s}$ in more detail in Sections 10 and 11. In particular, we construct bases for $\mathcal{U}_{d,s}$. We also show that if F is a (restricted) class function, defined on elements of S_n whose cycles have length $\leq d$, which takes the form of Theorem 1.11, then there is a unique lift of F to a (full) class function taking the same form. This lift corresponds to an element of $\mathcal{U}_{d,s}$, and we describe how to compute it.

Using these methods, we can refine Example 1.12.

Example 1.14. Let M be an FS^{op} module of class $(2, 2)$. Then its character takes the form

$$c_1 2^{\sum_{i \geq 1} \mathbb{X}_i} + c_2 2^{\sum_{i \geq 1} \mathbb{X}_{2i}} \delta_{\mathrm{even}} + c_3 + c_4 \mathbb{X}_1 + c_5 \sum_{n \geq 2} n \mathbb{X}_n + \sum_{n=0}^{2} f_n \delta_{n-\text{cycle}}, \tag{1.15}$$

where $c_n, f_n \in \mathbb{Q}$. This behavior is representative: in general the coefficients $c(\nu, A)$ of Theorem 1.11 are governed by recurrence relations. There are class $(2, 2)$ FS^{op} modules whose characters span the space of characters of the form (1.15) with $c_5 = 0$. We do not know an example of a class $(2, 2)$ FS^{op} module with $c_5 \neq 0$, but heuristic comparisons with FI_d modules suggest that they may exist.

Our final result on FS^{op} module characters is an analog of multiplicity stability for FI modules.

Definition 1.16. For λ an integer partition $\lambda_1 \geq \lambda_2 \geq \cdots$ and $n \geq \lambda_1$, we let (n, λ) be the integer partition $n \geq \lambda_1 \geq \lambda_2 \geq \cdots$. We write $|\lambda| = \sum_i \lambda_i$. Given an $S_{|\lambda|}$ representation V over a field of characteristic zero, we write mult$_\lambda(V)$ to denote the multiplicity of the irreducible representation corresponding to λ in V.

Theorem 1.17. *Let λ be an integer partition. Let M be a finitely generated $\textbf{\textit{FS}}^{\text{op}}$ module over a field of characteristic zero. Then the generating function*

$$\sum_{n \geq |\lambda|} \text{mult}_{(n,\lambda)}(M_{|\lambda|+n}) \, t^{n+|\lambda|}$$

is rational. If M is of class (d, s), then the roots of the denominator are roots of unity of order $\leq d$ and $\text{mult}_{(n,\lambda)}(M_{|\lambda|+n})$ is eventually a quasipolynomial of degree $\leq ds$.

Whereas for $\textbf{\textit{FI}}$ modules the multiplicities in Theorem 1.17 are eventually constant, for $\textbf{\textit{FS}}^{\text{op}}$ modules they are eventually quasipolynomials.

1D. *Construction of chain complexes.* Let $(\mathcal{C}, \oplus)$ be a small monoidal category. Let $d \in \mathcal{C}$ be an object, and $\mathcal{C}/d$ be the overcategory. To construct functors $\text{Mod}\,\mathcal{C} \to \text{Ch}(\text{Mod}\,\mathcal{C})$ from $\mathcal{C}$ modules to chain complexes of $\mathcal{C}$ modules, we proceed in the following simple way. First we restrict along $\oplus : \mathcal{C} \times \mathcal{C}/d \to \mathcal{C}$ to obtain a functor $\text{Mod}\,\mathcal{C} \to \text{Mod}(\mathcal{C} \times \mathcal{C}/d)$. Then we apply a functor $F : \text{Mod}(\mathcal{C}/d) \to \text{Ch}(\text{Mod}\,k)$ in the second factor to obtain $\text{Mod}(\mathcal{C} \times \mathcal{C}/d) \to \text{Ch}(\text{Mod}\,\mathcal{C})$. The functors we use are the composite, $(\text{id} \times F) \circ \text{Res}^{\oplus} : \text{Mod}\,\mathcal{C} \to \text{Ch}(\text{Mod}\,\mathcal{C})$.

Of course, there are many possible choices for F: up to quasi-isomorphism they are parametrized by chain complexes of $(\mathcal{C}/d)^{\text{op}}$ modules. When $\mathcal{C}/d$ is equivalent to a poset P there is a particularly natural choice: a bar construction $\text{B}_P : \text{Mod}\,P \to \text{Mod}\,k$, which we recall in Section 2. When P satisfies a Cohen–Macaulay property, the functor B_P can be replaced by a smaller functor K_P, which is quasi-isomorphic to B_P. Corresponding to these two constructions, there are functors $\text{K}_d, \text{B}_d : \text{Mod}\,\mathcal{C} \to \text{Ch}(\text{Mod}\,\mathcal{C})$. The conditions for B_d and K_d to exist are satisfied for any object $d \in \mathcal{C}$ when $\mathcal{C}$ is one of the categories $\textbf{\textit{FI}}$, $\textbf{\textit{FS}}^{\text{op}}$, $\textbf{\textit{VI}}_L$, or $\textbf{\textit{VIC}}_L$. Here L is a field and $\textbf{\textit{VI}}_L$ (resp. $\textbf{\textit{VIC}}_L$) denotes the category of finite-dimensional vector spaces over L and with injections (resp. injections together with a splitting; see [Putman and Sam 2017]).

When $\mathcal{C}$ is an EI category, there is a natural notion of $\mathcal{C}$ module homology, which in the case of $\textbf{\textit{FI}}$ modules reduces to $\textbf{\textit{FI}}$ module homology [Church and Ellenberg 2017]. The complexes we use admit a more general definition in terms of $\mathcal{C}$ module homology, which we discuss in Section 4, although we do not use it elsewhere in the paper.

One feature of our arguments is that, for the most part, we do not deal with the complexes K_d or B_d directly. Instead, we work with poset representations before applying the functors K_d or B_d, and use a couple of general facts to prove exactness. So although, when it exists, the complex K_d is more elegant than B_d, for the first half of the paper the two are interchangeable. The applications to $\textbf{\textit{FS}}^{\text{op}}$ modules in the second part use K_d and the homology of the partition lattice.

1E. *Relation to other work.* Beyond the theorem of Sam and Snowden on the rationality of Hilbert series, we are only aware of two previously known statements about the characters of finitely generated $\textbf{\textit{FS}}^{\text{op}}$ modules. They are both elementary: they follow from decomposition of the free $\textbf{\textit{FS}}^{\text{op}}$ module generated in degree d into irreducibles. The first states that if M is a subquotient of an $\textbf{\textit{FS}}^{\text{op}}$ module

generated in degree d, then the irreducible representations appearing in its decomposition have at most d rows. The second is that the multiplicity of (n, λ) in $M_{n+|\lambda|}$ is bounded above by a polynomial.

There is a long history of using poset topology for what we now call categorification, going at least back to Rota [1964]. Our only novelty is in the specific application of these methods. The complexes associated to poset representations that we use were first introduced by Baclawski [1976; 1980], in terms of the cohomology of constructible sheaves on the Alexandroff space of the poset. Baclawski also gave explicit Bar constructions, and the smaller complex K_P in the case of Cohen–Macaulay posets. They may also be thought of as computing derived functors for the tensor product with a representation of the opposite poset, as discussed in Section 4. To minimize technical overhead, we use explicit complexes.

In the case $\mathcal{C} = \textbf{\textit{FI}}$ the complex $K_d(M)$ is simply the complex obtained by iterating the functor

$$M \mapsto \operatorname{cone}(M \to \Sigma M)$$

d times. This construction and its categorification of rationality is well known. The complex $K_d(M)$ appears implicitly in many of arguments about the structure theory of $\textbf{\textit{FI}}$ modules, which proceed inductively using M, ΣM, and the cokernel of $M \to \Sigma M$.

The way we construct chain complexes is, heuristically, adjoint to the central stability complexes which appear in the representation stability literature, introduced in [Putman and Sam 2017] and axiomatized in [Patzt 2020]. Whereas central stability complexes are constructed by induction along the monoidal functor $- \oplus - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, ours are constructed via restriction. We do not know of a direct relationship between these two types of complexes, although there is one with $\textbf{\textit{FI}}$ module homology [Church and Ellenberg 2017; Putman 2015] (which goes under the name central stability homology in Putman's original paper on central stability).

Sam and Snowden [2018, §5] prove that the Hilbert series of a finitely generated $\textbf{\textit{FI}}_r$ module (see [Sam and Snowden 2017, §7.1] for the definition) is a sum of functions of the form $\binom{X}{\nu} A^{\mathbb{X}-\nu}$ for $A = 1^d$ and $d \leq r$. They also prove a statement that categories the rationality of Hilbert series of finitely generated $\textbf{\textit{FI}}_r$ modules. Our categorifications are in the same spirit as those of Sam and Snowden, but their complexes do not fit into our framework. Given a finitely generated $\textbf{\textit{FI}}_r$ module M, the complex Sam and Snowden use to categorify the rationality of $h_M(t)$ depends on the choice of subspaces of $k^{\oplus r}$ specifically adapted to M. In contrast, the complexes we use depend on a discrete set of choices. By generalizing our definitions slightly, it is possible to obtain the complexes of Sam and Snowden corresponding to the subspaces of $k^{\oplus r}$ that are linearizations of subsets $s \subseteq [r]$.

The Gröbner theory developed by Sam and Snowden [2017] is the source of many results about Hilbert series of representations of combinatorial categories. Our proof of Theorem 1.2 relies heavily on their ideas: we reduce to the case of monomial $\textbf{\textit{OS}}^{\mathrm{op}}$ modules, and eventually to proving a combinatorial statement about poset ideals associated to certain ordered DFAs (concepts they introduced). Sam and Snowden [2017, §11.2] ask whether it is possible to prove rationality results for enhanced Hilbert series for the combinatorial categories they consider. We view our results as an affirmative answer in the case of $\textbf{\textit{FS}}^{\mathrm{op}}$ modules, and the framework of this paper as a new strategy for approaching other cases.

1F. *Guide to the paper.* For a reader who would like to extract detailed results about the characters of FS^{op} modules as quickly as possible, we suggest the following. Read Definitions 9.1, 9.7, and 9.9 for the symmetric functions y_n, the type of an FS^{op} module, and the space $\mathcal{V}_{A,r} \subseteq \hat{\Lambda}$. Then read Definitions 10.1 and 10.2 for the rank of an FS^{op} module and the space $\mathcal{F}_{\leq k} \subseteq \hat{\Lambda}$. Then read Section 11, referring back as necessary. Our conventions for symmetric functions are in Section 8A, and we explain how to translate from symmetric functions to character functions in Section 9C.

In Section 2, following Baclawski [1980] we discuss how to construct chain complexes associated to representations of a poset. In Section 3 we apply the constructions of Section 2 to build chain complexes associated to modules over combinatorial categories. Then we prove a general result, Theorem 3.18, which gives a condition for our complexes to be exact when applied to projective modules. Combining Theorem 3.18 with the work of Nagpal, we obtain our categorifications for VI_q and FI modules. In Section 4 we discuss a general definition of chain complexes in the setting of small monoidal EI categories.

The next few sections of the paper are devoted to proving Theorem 1.2. In Section 5, we prove a proposition that we will use to reduce to the case of monomial OS^{op} modules. The heart of the proof is in Section 6, which shows that the poset homology of certain ideals associated to ordered regular languages vanishes. In Section 7 we introduce OS^{op} and prove Theorem 1.2, using Sam–Snowden Gröbner theory and Section 5 to reduce to the combinatorial result of Section 6.

The second half of the paper is about the characters of FS^{op} modules. In Section 8 we compute the Frobenius character of the complex $\mathrm{K}_d(M)$ in terms of $\mathrm{ch}(M)$ and certain differential operators. Using this computation, Theorem 1.2 translates into a system of differential equations for $\mathrm{ch}(M)$. In Section 9 we solve these differential equations, introduce the notion of the *type* of an FS^{op} module, and prove Theorem 1.11. In Section 10, we study the intersection between the solution space of Section 9 and the space spanned by characters of S_n representations whose Young diagrams have $\leq k$ rows, and prove Theorem 1.13. In Section 11 we summarize the results of previous sections, and use generating function methods to compute a basis of $\mathcal{U}_{d,s}$ and prove Theorem 1.17.

1G. *Conventions and notation.*

Categories. Given a category $\mathcal{C}$ and an object $c \in \mathcal{C}$, we write $\mathcal{C}/c$ for the *overcategory* of C. This is the category whose objects are morphisms $f : d \to c$ and whose morphisms are morphisms $g : d \to d'$ such that $f' \circ g = f$. For $c, d \in \mathcal{C}$ we write $\mathcal{C}(c, d)$ for the set of morphisms from c to d.

We write $\mathrm{Mod}\,\mathcal{C} = \mathrm{Mod}\,k^{\mathcal{C}}$ to denote the category of functors from $\mathcal{C}$ to k modules and $\mathrm{Ch}(\mathrm{Mod}\,\mathcal{C})$ for chain complexes of $\mathcal{C}$ modules. We refer to elements of $\mathrm{Mod}\,\mathcal{C}$ as $\mathcal{C}$ representations or $\mathcal{C}$ modules. For any functor $M : \mathcal{C} \to \mathrm{Mod}\,k$ we write M_c for the value of M at $c \in \mathcal{C}$ and $M_f : M_c \to M_d$ for any $f \in \mathcal{C}(c, d)$. If M is a $\mathcal{C}$ module and $d \in \mathcal{C}$, we write $M|_{\mathcal{C}/d}$ for the functor $\mathcal{C}/d \to \mathrm{Mod}\,k$ defined by $f : c \to d \mapsto M_c$.

When a group G acts (strictly) on the category $\mathcal{C}$, a G-equivariant $\mathcal{C}$ representation is a representation M of $\mathcal{C}$ together with morphisms $M_c \to M_{g(c)}$ for all $g \in G$ which are compatible with the maps $M_f : M_c \to M_d$ for $f \in \mathcal{C}(c, d)$, and which satisfy the identity and associativity conditions. Concisely, M is a representation of the Grothendieck construction $G \ltimes \mathcal{C}$.

On size issues: as usually defined $\textbf{FS}^{\text{op}}$, $\textbf{VI}_q$ and $\textbf{FI}$ are essentially small, but not small. However, we may replace these categories by equivalent small full subcategories containing all of the objects that we consider. We do this implicitly, and treat these categories as small categories.

If $\mathcal{C}$ and $\mathcal{D}$ are categories and F is a functor $F : \text{Mod}\,\mathcal{C} \to \text{Ch}(\text{Mod}\,\mathcal{D})$, there is a canonical extension of F to a functor $\text{Ch}(\text{Mod}\,\mathcal{C}) \to \text{Ch}(\text{Mod}\,\mathcal{D})$ given by applying F levelwise and forming the total complex of the resulting bicomplex (using sums). We will also denote this functor by F. We do not specify our sign conventions for the total complex, since our results are independent of this choice.

Posets. For p, q elements of a poset P, we write $p \prec q$ if $p < q$ and there does not exist $r \in P$ such that $p < r < q$. If P has a top (resp. bottom) element, we denote it by $\hat{1}$ (resp. $\hat{0}$). An *upward ideal* or simply *ideal* of a poset P is a subset $S \subseteq P$ which is *upward closed*: if $s \in S$ and $p \geq s$ then $s \in S$. The ideal S may be empty. All of the poset ideals appearing in this paper are upward closed: we do not use downward closed ideals. We write $P_{\geq x}$ for the ideal $\{p \mid p \geq x\}$.

To any poset we may associate a category. The objects of the category are the elements of the poset, and there is a unique morphism $p \to q$ if $p \leq q$, and no morphisms otherwise. We treat the poset and its associated category interchangeably and use the same symbol to denote both of them.

Integer partitions. An integer partition λ is a descending sequence of nonnegative numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. We may also specify λ by its vector of multiplicities in $\mathbb{N}^{\oplus\infty}$: we write $\lambda = 1^{m_1} 2^{m_2} \ldots$, where m_i is the number of times i appears in the sequence $\lambda_1 \geq \cdots$. We say that λ is a partition of n and write $\lambda \vdash n$ or $|\lambda| = n$ when $n = \sum_{i \geq 1} \lambda_{i \geq 1} = \sum_i i m_i$. We will also refer to the Young diagram associated to λ; see [Stanley 1999, §7.2].

Miscellaneous.

- For an abelian category $\mathcal{A}$, we write $\text{Ch}(\mathcal{A})$ (resp. $\text{Ch}^-(A)$) for the category of chain complexes in $\mathcal{A}$ (resp. nonnegatively graded complexes).

- If X is a set and k is a ring, we write kX for the free k module with basis X. Given a functor $F : C \to \text{Set}$, we say that the *linearization of F* is the functor $kF : C \to \text{Mod}\,k$ given by $(kF)_c := k(F_c)$.

- We write $\mathbb{N}$ for the set of nonnegative natural numbers, and $\mathbb{N}^{\oplus\infty}$ for the set of eventually zero sequences of natural numbers.

Part I. Chain complexes categorifying rationality

2. Poset homology

In this section, following Baclawski [1980], we associate chain complexes to poset representations. Then we discuss several examples, and recall elementary properties of these complexes. Throughout, we let k be a commutative ring.

Definition 2.1. For any poset Q, we write $\mathrm{N}Q$ for the *order complex of Q*, the simplicial complex whose r simplices are the set of chains

$$q_0 < q_1 < \cdots < q_r \in Q.$$

Equivalently $\mathrm{N}Q$ is the geometric realization of the simplicial set $[r] \mapsto \{q_0 \leq \cdots \leq q_r \in Q\}$.

Definition 2.2. When Q has top and bottom elements $\hat{1}$ and $\hat{0}$, we define $Z_Q \subseteq \mathrm{N}Q$ to be the subcomplex $\mathrm{N}(Q - \hat{0}) \cup \mathrm{N}(Q - \hat{1})$. In other words, Z_Q is the union of all simplices corresponding to chains $q_0 < \cdots < q_r$ where either $q_0 \neq \hat{0}$ or $q_1 \neq \hat{1}$. When $\hat{0} = \hat{1}$, $Z_Q = \varnothing$.

Definition 2.3. We say that a poset P is *graded* if for any $p \leq q \in P$ all maximal chains $p = p_0 \prec p_1 \prec \cdots \prec p_\ell = q$ have the same length. When P is graded, we define $\mathrm{r}(p, q)$ to be ℓ. When P has a top element, we put $\mathrm{r}(p) := \mathrm{r}(p, \hat{1})$.

Definition 2.4. Let P be a finite poset, with top element $\hat{1}$. For $p \in P$ we define the *Möbius number* of p, denoted by $\mu(p)$, to be the Euler characteristic of the pair $(\mathrm{N}[p, \hat{1}], Z_{[p,\hat{1}]})$. When P is graded, we define the *unsigned Möbius number* to be $\widetilde{\mu}(p) = (-1)^{\mathrm{r}(p)} \mu(p)$.

Remark 2.5. Our definition of $\mu(p)$ agrees with $\mu(p, \hat{1})$ as usually defined (as for example in [Wachs 2007, §1.2]). Instead of using the pair $(\mathrm{N}[p, \hat{1}], Z_{[p,\hat{1}]})$, it is more common to interpret the Möbius function topologically in terms of the reduced homology of $\mathrm{N}(p, \hat{1})$. Whenever $p \neq \hat{1}$, the homology groups of these complexes are related by

$$\widetilde{H}_{s-2}(\mathrm{N}(p, \hat{1})) = H_s(\mathrm{N}[p, \hat{1}], Z_{[p,\hat{1}]}) \quad \text{for all } s \geq 0,$$

as can be seen by writing the simplicial chains for both. In this paper, we prefer the latter homology groups because they lead to cleaner formulas and behave more simply for products of posets.

Definition 2.6. Let P be a finite graded poset with top element $\hat{1}$. The *Whitney Polynomial* of P is

$$W_P(t) := \sum_{p \in P} \mu(p) t^{\mathrm{r}(p)}.$$

Now we define the Bar complex associated to a P representation.

Definition 2.7. A *representation of P* is a functor $P \to \mathrm{Mod}\, k$. For a representation M, we write M_p for the value of M at p, and if $p \leq q \in P$ we write M_{pq} for the associated morphism $M_p \to M_q$.

Definition 2.8. Let P be a poset with top element $\hat{1}$, and let M be a P representation. We define $\mathrm{B}_P(M)$ be the following chain complex. In degree $s \in \mathbb{N}$,

$$\mathrm{B}_P(M)_s := \bigoplus_{\hat{1} > p_1 > p_2 > \cdots > p_s} M_{p_s}.$$

The differential of $B_P(M)$ is given by an alternating sum $d_s = \sum_{i=0}^{s}(-1)^i \partial_i$, where if we represent an element of $B_P(M)_s$ as a tuple $(\hat{1} > \cdots > p_s, m)$ for $m \in M_{p_s}$, the boundary ∂_i is defined by

$$\partial_i(\hat{1} > \cdots > p_s, m) = \begin{cases} 0 & \text{if } i = 0, \\ (\hat{1} > \cdots > p_{i-1} > p_{i+1} > \cdots > p_s, m) & \text{if } s > i > 0, \\ (\hat{1} > \cdots > p_{s-1}, M_{p_s p_{s-1}}(m)) & \text{if } i = s. \end{cases}$$

Definition 2.9. Let P be a poset, and G a group acting on P. We say that M is a *G-equivariant P representation* if M is a P representation and there are morphisms $g : M_p \to M_{g(p)}$ for every $g \in G$ which are compatible with the maps $M_{pq} : M_p \to M_q$ for $p \leq q \in P$ and satisfy the obvious associativity and identity properties.

When P is a poset with a top element and M is a G-equivariant P representation for some G-action on P, the chain complex $B_P(M)$ is naturally a complex of G representations. The element $g \in G$ acts by taking $(p_0 < \cdots < p_r, m)$ to $(g(p_0) < \cdots < g(p_r), g(m))$.

2A. *Koszul complexes.* Following Baclawski [1980], when the poset P satisfies a connectivity condition it is possible to replace $B_P(M)$ by a smaller complex $K_P(M)$, which we define in this subsection.

Definition 2.10. We say that a poset P is *upper CM over k* if it is graded, has a top element $\hat{1}$, and satisfies the following conditions for every $p \in P$:

$$H_j(N[p, \hat{1}], Z_{[p,\hat{1}]}; k) = 0 \quad \text{for all } 0 \leq j < r(p),$$

and $H_{r(p)}(N[p, \hat{1}], Z_{[p,\hat{1}]}; k)$ is a free k module.

Example 2.11. If P is Cohen–Macaulay over k then by definition $\widetilde{H}_{s-2}(N(x, y)) = 0$ for $s < \text{rank}(x, y)$ [Björner et al. 1982, Definition 3.1]. Therefore, P is upper CM over k. Examples include arbitrary geometric lattices [Folkman 1966].

Since $N[p, \hat{1}]$ is $r(p)$-dimensional, when P is upper CM the unsigned Möbius number computes the rank of the top nonvanishing homology group of $(N[p, \hat{1}], Z_{[p,1]})$. In other words we have

$$H_{r(p)}(N[p, \hat{1}], Z_{[p,\hat{1}]}; k) \cong k^{\oplus \widetilde{\mu}(p)}.$$

Definition 2.12. When P is upper CM over k, there is a functor from P representations to chain complexes of k modules, K_P, defined as follows.

Let M be a P representation. We filter $B_P(M)$ by defining

$$F_i(B_P(M)_s) := \bigoplus_{\{\hat{1} > \cdots > p_s \,|\, r(p_s) \leq i\}} M_{p_s}.$$

The complex $K_P(M)$ is defined to be the E_1 page of the spectral sequence associated to this filtration.

Let us describe $K_P(M)$ more concretely, by computing the E_1 page. The associated graded complex of the filtration F_i has the same chain groups as $B_P(M)$ but ∂_s acts by zero: in other words the differential

of the E_0 page is $d_0^s = \sum_{i=0}^{s-1}(-1)^i \partial_i$. Since d_0^s preserves the grading by $p \in P$, the E_0 page splits as a direct sum of complexes $\bigoplus_p C(p) \otimes M_p$ for $p \in P$, where

$$C(p)_s := \bigoplus_{\hat{1} > p_1 > \cdots > p_{s-1} > p} k.$$

By definition, we see $C(p)$ is the chain complex computing the simplicial homology of $(\mathrm{N}[p, \hat{1}], Z_{[p,\hat{1}]})$. Because P is upper CM, the homology of $C(p)$ is concentrated in degree $\mathrm{r}(p)$, and it has rank $\widetilde{\mu}(p)$. So the spectral sequence associated to the filtration degenerates after the E_1 page, and on the E_1 page it takes the form

$$M_{\hat{1}} \leftarrow \bigoplus_{\substack{p \in P \\ \mathrm{r}(p,\hat{1})=1}} M_p \leftarrow \cdots \leftarrow \bigoplus_{\substack{p \in P \\ \mathrm{r}(p,\hat{1})=s}} H_s(\mathrm{N}[p, \hat{1}], Z_{[p,\hat{1}]}; k) \otimes M_p \leftarrow \cdots .$$

This is the complex $\mathrm{K}_P(M)$. Noncanonically it takes the form

$$M_{\hat{1}} \leftarrow \bigoplus_{\substack{p \in P \\ r(p)=1}} M_p \leftarrow \bigoplus_{\substack{p \in P \\ r(p)=2}} M_p^{\oplus \widetilde{\mu}(p)} \leftarrow \cdots .$$

The differential is induced by the action of $(-1)^s \partial_s$.

Both $\mathrm{K}_P(M)$ and $\mathrm{B}_P(M)$ compute the same homology groups, but $\mathrm{K}_P(M)$ depends more closely on the combinatorics of P and is much smaller than $\mathrm{B}_P(M)$.

Proposition 2.13. *There is an edge map $\mathrm{K}_P(M) \hookrightarrow \mathrm{B}_P(M)$ which induces a canonical isomorphism $H_\bullet(\mathrm{K}_P(M)) \cong H_\bullet(\mathrm{B}_P(M))$.*

Proof. The complex $\mathrm{K}_P(M)$ can be identified with the subcomplex of $\mathrm{B}_P(M)$ which in degree s is

$$\ker\left(\sum_{i=0}^{s-1}(-1)^i \partial_i\right) \subseteq \bigoplus_{\hat{1} \succ p_1 \succ \cdots \succ p_s} M_{p_s}.$$

Since the spectral sequence degenerates, this inclusion is a quasi-isomorphism. $\square$

When P is an upper CM poset, G is a group acting on P, and M is a G-equivariant P representation, the G action on $\mathrm{B}_P(M)$ induces one on $\mathrm{K}_P(M)$. The degree s term,

$$\bigoplus_{\substack{p \in P \\ \mathrm{r}(p,\hat{1})=s}} H_s(\mathrm{N}[p, \hat{1}], Z_{[p,\hat{1}]}; k) \otimes M_p,$$

is a G representation in the obvious way: $g \in G$ acts by tensoring and summing the maps $g : M_p \to M_{g(p)}$ and $g : H_s(\mathrm{N}[p, \hat{1}], Z_{[p,\hat{1}]}; k) \to H_s(\mathrm{N}[g(p), \hat{1}], Z_{[g(p),\hat{1}]}; k)$ induced by the functoriality of homology.

Remark 2.14. From an abstract perspective both $\mathrm{B}_P(M)$ and $\mathrm{K}_P(M)$ compute the derived functors of tensoring with a P^{op} representation $S(\hat{1})$ (see Definition 4.6). The complex $\mathrm{B}_P(M)$ arises from the Bar resolution of $S(\hat{1})$, and the complex $\mathrm{K}_P(M)$ arises from a minimal resolution (assuming that k is a field).

Essentially, the complex $K_P(M)$ is a Koszul complex: there is a generalization of the theory of Koszul duality from graded algebras to graded linear categories such that the category kP is Koszul if P is Cohen–Macaulay [Woodcock 1998; Li 2014]. In this case, $K_P(M)$ is a Koszul complex.

2B. *Examples.* We discuss the posets that arise in the main examples of this paper.

Definition 2.15. Let x be a finite set. A (*set*) *partition* p of x is a set of nonempty subsets of x such that each $i \in x$ is contained in exactly one element of p. We call the elements of p the *blocks of p*. Let $P(x)$ be the set of all partitions of x. Given $p, q \in P(x)$, we define $p \le q$ if every block of q is contained in a block of p. Under this relation, $P(x)$ is a finite lattice, called the *partition lattice of x*. The top element is the discrete partition and the bottom element is the indiscrete partition. When $n \in \mathbb{N}$, we define $P(n) := P([n])$.

Example 2.16. The partition lattice $P(n)$ is Cohen–Macaulay over any ring, and for $p \in P(n)$,

$$\widetilde{\mu}(p) = \prod_{b \text{ block of } p} (|b| - 1)!,$$

and its Whitney polynomial is

$$W_{P(n)}(t) = \prod_{j=1}^{n-1} (1 - jt);$$

see [Stanley 1982, §7].

Definition 2.17. For a finite set x the *Boolean lattice* $\boldsymbol{B}(x)$ is the poset of subsets of x. We write $\boldsymbol{B}(n) := \boldsymbol{B}([n])$

Example 2.18. The Boolean lattice $\boldsymbol{B}(n)$ is Cohen–Macaulay and $\widetilde{\mu}(s) = 1$ for every $s \in \boldsymbol{B}(n)$. Its Whitney polynomial is $W_{\boldsymbol{B}(n)}(t) = (1 - t)^n$. See [Stanley 1982, §4].

Definition 2.19. Let V be finite-dimensional vector space over the finite field $\mathbb{F}_q$. We write $\boldsymbol{B}_q(V)$ for the poset of subspaces of V and $\boldsymbol{B}_q(n) := \boldsymbol{B}_q(\mathbb{F}_q^{\oplus n})$.

Example 2.20. The poset $\boldsymbol{B}_q(n)$ is Cohen–Macaulay and $\widetilde{\mu}(w) = q^{\binom{c}{2}}$, where $c = r(w)$ is the codimension of $w \subseteq \mathbb{F}_q^n$. It follows from the q-binomial theorem that its Whitney polynomial is

$$W_{\boldsymbol{B}_q(n)}(t) = \prod_{i=0}^{n-1} (1 - q^i t).$$

See [Stanley 1982, §5]. It is natural to think of $\boldsymbol{B}(n)$ as a specialization of $\boldsymbol{B}_q(n)$ to $q = 1$.

In fact, if V is a finitely generated module over L, where L is a field or a finite ring, then the poset of submodules of V is Cohen–Macaulay by the Solomon–Tits theorem and the Cohen–Macaulayness of supersolvable lattices, respectively. There is another example, related to the category $\boldsymbol{VIC}_L$.

Example 2.21. Let L be a field, and V be a vector space over L. There is a poset whose elements are pairs of subspaces (W, W') satisfying $V = W \oplus W'$, with relation defined by $(W_1, W_2') \le (W_2, W_2')$ if $W_1 \subseteq W_2$ and $W_1' \supseteq W_2'$. Lehrer and Rylands [1993] proved that this poset is Cohen–Macaulay.

2C. *Complexes associated to product posets.* Let $P \times Q$ be a product of two posets with top elements $\hat{1}_P$ and $\hat{1}_Q$. Suppose we are given a $P \times Q$ module M. We may consider M as a functor $P \to \mathrm{Mod}\, Q$, and postcomposing with B_Q we obtain a chain complex of P modules, $p \mapsto \mathrm{B}_Q(M_{(p,-)})$. We may apply B_P to this complex, as described in Section 1G, to obtain a chain complex of k modules.

Definition 2.22. We define $\mathrm{B}_{(P,Q)}(M)$ to be the chain complex $\mathrm{B}_P(p \mapsto \mathrm{B}_Q(M_{(p,-)}))$ and $\mathrm{B}_{(Q,P)}(M)$ to be the chain complex $\mathrm{B}_Q(q \mapsto \mathrm{B}_P(M_{(-,q)}))$.

There is an isomorphism of complexes $\mathrm{B}_{(P,Q)}(M) \cong \mathrm{B}_{(Q,P)}(M)$. More concretely, their degree n term is

$$\bigoplus_{i+j=n} \; \bigoplus_{p_i < \cdots < \hat{1}_P} \; \bigoplus_{q_j < \cdots < \hat{1}_Q} M_{(p_i, q_j)},$$

and their differentials agree up to a sign introduced by the definition of the total complex.

Proposition 2.23. *Let P and Q be finite posets with top elements. Let M and N be P and Q representations, respectively. Let $M \boxtimes N$ be the $P \times Q$ representation $(p, q) \mapsto M_p \otimes N_q$. Then $\mathrm{B}_{(P,Q)}(M \boxtimes N) \cong \mathrm{B}_P(M) \otimes \mathrm{B}_Q(N)$. In particular, if $\mathrm{B}_Q(N)$ is exact, then $\mathrm{B}_{(P,Q)}(M \boxtimes N)$ is exact.*

Proof. This is true by the definition of the tensor product of chain complexes. $\square$

We may extend the definition of $\mathrm{B}_{(P,Q)}$ to a finite product of posets with top elements $P_1 \times P_2 \times \cdots \times P_r$ in the obvious way to obtain a functor $\mathrm{B}_{(P_1,\ldots,P_r)}$. Proposition 2.23 carries over to this setting, and we also have the following.

Proposition 2.24. *Let $P_1, \ldots, P_r$ be finite posets with top elements and let L be a $P_1 \times \cdots \times P_r$ representation. We put $Q := P_2 \times \cdots \times P_r$ and for $p \in P_1$ write $L|_{p \times Q}$ for the Q representation $q \mapsto L_{(p,q)}$. Suppose that for every $p \in P_1$ we have that $\mathrm{B}_{(P_2,\ldots,P_r)}(L|_{p \times Q})$ is exact. Then $\mathrm{B}_{(P_1,\ldots,P_r)}(L)$ is exact.*

Proof. Applying $\mathrm{B}_{(P_2,\ldots,P_r)}$ to L, by hypothesis we obtain an exact complex of P_1 representations $p \mapsto \mathrm{B}_{(P_2,\ldots,P_r)}(L|_{p \times Q})$. Denote this complex by C. Since B_{P_1} is exact, $\mathrm{B}_{P_1}(C) \cong \mathrm{B}_{(P_1,\ldots,P_r)}(L)$ is exact. $\square$

Remark 2.25. If the definition of $\mathrm{B}_{(P,Q)}$ seems ad hoc, we remark that for any $P \times Q$ representation L there is a canonical quasi-isomorphism $\mathrm{EZ} : \mathrm{B}_{(P,Q)}(L) \xrightarrow{\sim} \mathrm{B}_{P \times Q}(L)$. To construct it, note that $\mathrm{B}_{(P,Q)}(L)$ is the totalization of a normalized bicomplex of a bisimplicial abelian group, and $\mathrm{B}_{P \times Q}(L)$ is the normalized chain complex of that bisimplicial group's diagonal. Then apply the Eilenberg–Zilber theorem stated in [Weibel 1994, §8.5]. In fact, if P and Q are upper CM, then so is $P \times Q$ and EZ induces an *isomorphism* between the subcomplexes $\mathrm{K}_{(P,Q)}(L)$ and $\mathrm{K}_{P \times Q}(L)$. We use $\mathrm{B}_{(P,Q)}$ rather than $\mathrm{B}_{P \times Q}$, because all the properties we need are immediate for it. $\square$

2D. *Poset homology of ideals.*

Definition 2.26. Associated to any upward ideal I in a poset P, there is a functor $\mathrm{S}I : P \to \mathrm{Set}$ given by

$$p \mapsto \begin{cases} * & \text{if } p \in I, \\ \varnothing & \text{otherwise,} \end{cases}$$

where $*$ denotes the terminal set $\{\{\varnothing\}\}$. (The action of SI on morphisms is uniquely determined by its value on objects.) Postcomposing with the free functor from sets to k modules we obtain a P representation, denoted by kI, such that $kI_p = k$ if $p \in I$ and $kI_p = 0$ otherwise.

The order complex $N(I - \hat{1})$ is a subcomplex of $N(I)$, and by definition $B(kI)$ is the chain complex computing the relative simplicial homology $(NI, N(I - \hat{1}))$ with coefficients in k. We need to know that the homology of the bar complex applied to a principal ideal is zero.

Proposition 2.27. *Let P be a finite poset with top element $\hat{1}$. Let $x \in P$ such that $x \neq \hat{1}$, and let $P_{\geq x} = \{p \in P \mid p \geq x\}$ be the principal ideal generated by x. Then the homology of $B(kP_{\geq x})$ vanishes.*

Proof. Since both I and $I - \hat{1}$ contain x as their smallest element, NI and $N(I - \hat{1})$ are both contractible, and so the relative homology vanishes. $\square$

We will also need a simple fact about P-sets.

Proposition 2.28. *Let P be a poset with top element $\hat{1}$. Let $F : P \to$ Set be a functor such that F_{pq} is an injection for all $p \leq q \in P$. Given $x \in F_{\hat{1}}$, let $F\langle x \rangle_p := \{y \in F_p \mid F_{p\hat{1}}(y) = x\}$. Then $F\langle x \rangle$ is a subfunctor, and there are isomorphisms $\bigsqcup_{x \in F_{\hat{1}}} F\langle x \rangle \cong F$ and $F\langle x \rangle \cong SI\langle x \rangle$, where $I\langle x \rangle$ is the ideal*

$$I\langle x \rangle := \{p \in P \mid \text{ there exists } y \in F_p \text{ such that } F_{p\hat{1}}(y) = x\}.$$

Proof. This is straightforward. The map $\bigsqcup_{x \in F_{\hat{1}}} F\langle x \rangle \to F$ is induced by the inclusions $F\langle x \rangle \subseteq F$, and it is an injection because if x and y are distinct elements of $F_{\hat{1}}$ the fact that the transition maps are inclusions implies that $F\langle x \rangle \cap F\langle y \rangle = \varnothing$. The isomorphism $SI\langle x \rangle \to F\langle x \rangle$ is given by defining the image of $*$ in $F\langle x \rangle_p$ to be the unique element y such that $F_{p\hat{1}}(y) = x$, if it exists. $\square$

3. Chain complexes associated to categories

In this section, we apply the definitions of B_P and K_P to construct chain complexes associated to modules over combinatorial categories.

3A. *Poset versus posetal.* In our main examples, FI, VI_q, and FS^{op}, the overcategories of objects are equivalent to posets, but *not* isomorphic to posets. Although this may seem to be a fussy distinction, we want our complexes to be canonical, so that they carry group actions. Accordingly, we recall the following notion.

Definition 3.1. A small category $\mathcal{J}$ is *posetal* if there is at most one morphism between any two objects of $\mathcal{J}$.

When $\mathcal{J}$ is posetal, its set of isomorphism classes $|\mathcal{J}|$ forms a poset, by defining $[j] \leq [j']$ if there is a morphism $j \to j'$. There is a canonical functor

$$\mathcal{J} \to (|\mathcal{J}|, \leq), \quad j \mapsto [j],$$

which is an equivalence of categories. Inducing and restricting along this functor, we obtain an equivalence of categories between Mod $\mathcal{J}$ and Mod $|\mathcal{J}|$. Given a $\mathcal{J}$ module M, the associated $|\mathcal{J}|$ module takes an equivalence class $x \in |\mathcal{J}|$ to

$$|M|_x := \frac{\bigsqcup_{\{j \in \mathcal{J} : [j] = x\}} M_j}{(j, m) \sim (j', m') \text{ if } M_f(m) = m' \text{ for the unique map } f : j \to j'}.$$

For any $j \in \mathcal{J}$ such that $[j] = x$ there is a canonical isomorphism $M_j \to |M|_x$. And if $x \leq y \in |\mathcal{J}|$ there is a unique $|M|_x \to |M|_y$ such for any $j, k \in \mathcal{J}$ with $[j] = x$ and $[k] = y$, the obvious diagram commutes.

Definition 3.2. Let $\mathcal{J}$ be a posetal category. For any functor $F_{|\mathcal{J}|} : \mathrm{Mod}\,|\mathcal{J}| \to \mathcal{D}$, we define $F_\mathcal{J}$ to be the functor $F_\mathcal{J}(M) := F_{|\mathcal{J}|}(|M|)$.

Thus, when $\mathcal{J}$ is posetal and has a terminal object (equivalently $|\mathcal{J}|$ has a top element), we have a definition of $\mathrm{B}_\mathcal{J}(M)$. And when $|\mathcal{J}|$ is upper CM we have a definition of $\mathrm{K}_\mathcal{J}(M)$. Similarly, given a product of posetal categories with terminal objects $\mathcal{J}_1 \times \cdots \times \mathcal{J}_r$, we may define $\mathrm{B}_{(\mathcal{J}_1, \ldots, \mathcal{J}_r)}$.

Finally, suppose that G is a group which acts on $\mathcal{J}$ by isomorphisms. Then G acts on $|\mathcal{J}|$, and if M is a G-equivariant $\mathcal{J}$ representation, $|M|$ is a G-equivariant $|\mathcal{J}|$ representation. If $x \in |\mathcal{J}|$, $j \in \mathcal{J}$ and $[j] = x$, then for all $g \in G$ we have $g(x) = g([j])$ and $g : M_x \to M_{g(x)}$ is the unique morphism compatible with $g : M_j \to M_{g(j)}$.

3B. *Construction of complexes.* Let $(\mathcal{C}, \oplus)$ be a monoidal category and $d \in \mathcal{C}$. Restricting along the functor $\oplus : \mathcal{C} \times \mathcal{C}/d \to \mathcal{C}$ given by $(c, c' \to d) \mapsto c \oplus c'$, we obtain a functor

$$\mathrm{Mod}\,\mathcal{C} \to (\mathrm{Mod}\,\mathcal{C}/d)^\mathcal{C},$$

taking a $\mathcal{C}$ module M to the functor $c \mapsto (M_{c \oplus -})|_{\mathcal{C}/d}$. Note that $\mathrm{id} : d \to d$ is always a terminal object of $\mathcal{C}/d$, so if $\mathcal{C}/d$ is posetal, it contains a top element.

Definition 3.3. If $\mathcal{C}/d$ is posetal, we define $\mathrm{B}_d : \mathrm{Mod}\,\mathcal{C} \to \mathrm{Ch}(\mathrm{Mod}\,\mathcal{C})$ to be the composite

$$\mathrm{Mod}\,\mathcal{C} \to (\mathrm{Mod}\,\mathcal{C}/d)^\mathcal{C} \xrightarrow{\mathrm{B}_{\mathcal{C}/d}} \mathrm{Ch}(\mathrm{Mod}\,k)^\mathcal{C} = \mathrm{Ch}(\mathrm{Mod}\,\mathcal{C}).$$

Thus for every $c \in \mathcal{C}$, the complex $\mathrm{B}_d(M)_c \in \mathrm{Ch}(\mathrm{Mod}\,k)$ is $\mathrm{B}_{\mathcal{C}/d}(M_{c \oplus -}|_{\mathcal{C}/d})$.

We define the complex K_d similarly.

Definition 3.4. If $\mathcal{C}/d$ is equivalent to an upper CM poset, then we define $\mathrm{K}_d : \mathrm{Mod}\,\mathcal{C} \to \mathrm{Mod}\,\mathcal{C}$ to be the composite functor $\mathrm{Mod}\,\mathcal{C} \to (\mathrm{Mod}\,\mathcal{C}/d)^\mathcal{C} \xrightarrow{\mathrm{K}_{\mathcal{C}/d}} \mathrm{Ch}(\mathrm{Mod}\,\mathcal{C})$.

The following elementary proposition shows that B_d is always defined for categories in which all morphisms are monomorphisms.

Proposition 3.5. *Let $\mathcal{C}$ be a category. Then $\mathcal{C}/d$ is posetal for all $d \in \mathcal{C}$ if and only if every morphism of $\mathcal{C}$ is a monomorphism.*

For $d \in \mathcal{C}$, the group $\mathrm{Aut}(d)$ acts on the category $\mathcal{C}/d$ by postcomposition. For N a $\mathcal{C}$ module, the $\mathcal{C}/d$ module $N|_{\mathcal{C}/d}$ is $\mathrm{Aut}(d)$ equivariant: an element $g \in \mathrm{Aut}(d)$ acts by $(f, n) \mapsto (g \circ f, n)$ for $f : c \to d$ and $n \in N_f$. Therefore, when $\mathcal{C}/d$ is posetal (resp. $\mathcal{C}/d$ is equivalent to an upper CM poset) the group $\mathrm{Aut}(d)$ acts on $\mathrm{B}_d(M)$ (resp. $\mathrm{K}_d(M)$) for all $\mathcal{C}$ modules M.

Assume that $\mathcal{C}/d$ is posetal. Instead of passing through $|\mathcal{C}/d|$, we may obtain a complex that is isomorphic to $\mathrm{B}_d(M)$ by restricting $(M_{c \oplus -})|_{\mathcal{C}/d}$ to a skeleton $\mathcal{S} \subseteq \mathcal{C}/d$, and applying $\mathrm{B}_{\mathcal{S}}$ (since $\mathcal{S}$ is necessarily a poset). Recall that a full subcategory $\mathcal{S}$ is a *skeleton of $\mathcal{C}/d$* if any two isomorphic objects of $\mathcal{S}$ are equal, and every object of $\mathcal{C}/d$ is isomorphic to one in $\mathcal{S}$.

3C. *Examples.* Now we discuss the complexes arising in our main three examples.

Definition 3.6. When $(\mathcal{C}, \oplus)$ is a monoidal category, for any $c \in \mathcal{C}$ we define $\Sigma^c M$ to be the $\mathcal{C}$ module $(\Sigma^c M)_{c'} := M_{c \oplus c'}$.

Example 3.7. Let $d \in \boldsymbol{FS}^{\mathrm{op}}$. Then $|\boldsymbol{FS}^{\mathrm{op}}/d|$ is $\mathrm{P}(d)$, the lattice of set partitions of d. A surjection $f : d \twoheadrightarrow y$ corresponds to the following partition of d: $i, j \in d$ are in the same block of p if $f(i) = f(j)$. Two surjections are isomorphic if and only if their associated partitions agree.

Let M be an $\boldsymbol{FS}^{\mathrm{op}}$ module. Then $\mathrm{K}_d(M)$ takes the form

$$\Sigma^{[d]} M \leftarrow \bigoplus_{\substack{p \in \mathrm{P}(d) \\ r(p)=1}} \Sigma^p M \leftarrow \cdots \leftarrow \bigoplus_{\substack{p \in \mathrm{P}(d) \\ r(p)=s}} \Sigma^p M \otimes H_s(\mathrm{N}[p, \hat{1}], Z_{[p,\hat{1}]}) \leftarrow \cdots.$$

(Recall that for us the partition p equals its set of blocks.) The first differential is given by summing the maps induced by the surjection $d \twoheadrightarrow p$ for each partition p.

Definition 3.8. Let $\boldsymbol{FI}$ be the category of finite sets and injections.

Example 3.9. Let $d \in \boldsymbol{FI}$. Then $|\boldsymbol{FI}/d|$ is the poset of subsets of d, and for an $\boldsymbol{FI}$ module M, the complex $\mathrm{K}_d(M)$ takes the form

$$\Sigma^d M \leftarrow \bigoplus_{i \in d} \Sigma^{d-i} M \bigoplus_{\{i,j\} \subseteq d} \leftarrow \Sigma^{d-\{i,j\}} M \leftarrow \cdots \leftarrow M,$$

where, up to signs, the differentials are induced by the inclusions of subsets. Notice that the action by S_d is nontrivial: for instance, S_d acts on the highest term of the complex by the $\mathrm{id}_M \otimes \mathrm{sgn}$.

Definition 3.10. Let R be a ring. Then $\boldsymbol{VI}_R$ is the category of R modules that are isomorphic to $R^{\oplus n}$ for some $n \in \mathbb{N}$, and injections between them. Let q be a prime power. We define $\boldsymbol{VI}_q := \boldsymbol{VI}_R$.

Example 3.11. For $w \in \boldsymbol{VI}_q$ the poset $|\boldsymbol{VI}_q/w|$ is the poset of subspaces of w. Then $\mathrm{K}_w(M)$ takes the form

$$\Sigma^w M \leftarrow \cdots \leftarrow M \otimes \mathrm{Stein}(w),$$

where $\mathrm{Stein}(w)$ is the Steinberg representation of $\mathrm{GL}(w)$. The degree s term is

$$\bigoplus_{\substack{v \subseteq w \\ \mathrm{codim}(v)=s}} \Sigma^v M \otimes \mathrm{Stein}(w/v).$$

More generally, the complex K_d is defined for any object $d \in \textbf{VI}_L$ where L is a field or a finite ring, since, as discussed in Section 2B, the poset of submodules of a finitely generated module is Cohen–Macaulay.

Example 3.12. Let L be a field and $\textbf{VIC}_L$ be the category of vector spaces and complemented injections of [Putman and Sam 2017]. For any $V \in \textbf{VIC}_L$, the overcategory $\textbf{VIC}_L / V$ is equivalent to the poset defined in Example 2.21. Therefore, the complex K_V is defined.

3D. *Rational Hilbert series*. We discuss how exactness of the complex $\mathrm{B}_d(M)$ categorifies rationality, in an axiomatic setting. (In our three examples, it is easy to see the categorification directly using $\mathrm{K}_d(M)$.) The key is the following general identity.

Proposition 3.13. *Let $\mathcal{C}$ be a small monoidal category, and let $d \in \mathcal{C}$ be an object with $\mathcal{C}/d$ equivalent to a finite poset. Let M be a $\mathcal{C}$ module over a commutative Noetherian ring k such that M_c is finitely generated for all $c \in \mathcal{C}$. Then for every $c \in \mathcal{C}$ we have*

$$[\mathrm{B}_d(M)_c] = \sum_{[f:c' \to d] \in |\mathcal{C}/d|} \mu([f])[M_{c \oplus c'}]$$

in the Grothendieck group of finitely generated k modules.

Proof. We may neglect the differentials of $\mathrm{B}_d(M)$. Then, as in Section 2A, the complex $\mathrm{B}_d(M)$ splits as a sum of $C([f]) \otimes M_{c \oplus c'}$ over $[f : c' \to d] \in |\mathcal{C}/d|$. By the definition of μ as an Euler characteristic, $[C([f])] = \mu([f])[k]$, so the result follows. $\qquad\square$

We specialize to the following situation, which holds in our main examples.

Let k be a field, and let $\mathcal{C}$ be a monoidal category all of whose morphisms are monomorphisms. Its set of isomorphism classes $|\mathcal{C}|$ is a monoid. Suppose that we have an isomorphism $s : (|\mathcal{C}|, \oplus) \to (\mathbb{N}, +)$, which we use to identify isomorphism classes of objects with natural numbers. Suppose that if there is a morphism from the n-th object to the m-th one, then $n \leq m$. For any $f : c \to d \in \mathcal{C}/d$ define $r(f) = s(d) - s(c)$.

Definition 3.14. The *Hilbert series* of a pointwise finite-dimensional $\mathcal{C}$ module (relative to s) is

$$H_M(t) := \sum_{n \in \mathbb{N}} (\dim M_n) t^n \in \mathbb{Z}[\![t]\!].$$

Definition 3.15. We define the *Whitney polynomial* of $\mathcal{C}/d$ to be

$$W_{\mathcal{C}/d}(t) := \sum_{p \in |\mathcal{C}/d|} \mu(p) t^{r(p)} \in \mathbb{Z}[t].$$

In the cases $\mathcal{C} = \textbf{FI}$, $\textbf{VI}_q$, and $\textbf{FS}^{\text{op}}$, this definition of the Whitney polynomial agrees with the previous one. The following statement shows that when $\mathrm{B}_d(M)$ is exact the Hilbert series of M is rational.

Proposition 3.16. *Let M be a $\mathcal{C}$ module, and suppose that $\mathrm{B}_{d_1} \circ \cdots \circ \mathrm{B}_{d_r}(M)_c$ is exact for all $c \in \mathcal{C}$ with $s(c) \gg 0$. Then $H_M(t)$ is rational, with denominator $\prod_{i=1}^r W_{\mathcal{C}/d_i}(t)$.*

Proof. We consider the special case $r = 1$, and put $d := d_1$. We write $h_1(t) \approx h_2(t) \in \mathbb{Z}[\![t]\!]$ if their difference is a polynomial. Then, by hypothesis,

$$0 \approx \sum_{n \geq 0} \chi(\mathrm{B}_d(M)_n) t^{n+s(d)} = \sum_{n \geq 0} \sum_{[f:c \to d] \in |\mathcal{C}/d|} \mu(f) \chi(M_{n+s(c)}) t^{n+s(d)}$$

$$= \sum_{[f:c \to d] \in |\mathcal{C}/d|} \sum_{m \geq s(c)} \chi(M_m) \mu(f) t^{m+r(f)} \approx H_M(t) W_{\mathcal{C}/d}(t),$$

where the first equality follows from Proposition 3.13. (Here $\chi(\mathrm{B}_d(M)_n)$ denotes $\chi(\mathrm{B}_d(M)_c)$ for some $c \in \mathcal{C}$ with $s(c) = n$.) Therefore, $H_M(t) W_{\mathcal{C}/d}(t)$ is a polynomial, and so $H_M(t)$ is rational with denominator $W_{\mathcal{C}/d}(t)$. The general case is similar, iterating applications of Proposition 3.13. $\qquad\square$

3E. *Composites and products.* Suppose that $\mathcal{C}$ is a small monoidal category and every morphism is a monomorphism. Given $d_1, \ldots, d_r \in \mathcal{C}$ we may iterate B_{d_i} to obtain a complex

$$\mathrm{B}_{d_1} \circ \mathrm{B}_{d_2} \circ \cdots \circ \mathrm{B}_{d_r}(M)$$

for every $M \in \mathrm{Mod}\,\mathcal{C}$. When the overcategories of $\mathcal{C}$ are equivalent to upper CM posets, we may do the same with K_{d_i}. Alternately, we may construct this complex in one step, by using the functor

$$\oplus : \mathcal{C} \times \mathcal{C}/d_1 \times \cdots \times \mathcal{C}/d_r \to \mathcal{C}$$

to restrict a $\mathcal{C}$ module to an object of $\mathrm{Mod}(\mathcal{C}/d_1 \times \cdots \times \mathcal{C}/d_r)^{\mathcal{C}}$ and applying the construction $\mathrm{B}_{(\mathcal{C}/d_1, \ldots, \mathcal{C}/d_r)}$. The complex we obtain is isomorphic to $\mathrm{B}_{d_1} \circ \cdots \circ \mathrm{B}_{d_r}(M)$.

Similarly, we note the following elementary proposition.

Proposition 3.17. *Suppose that $(\mathcal{C}, \oplus)$ is a symmetric monoidal small category, and every morphism is a monomorphism. Then there are natural isomorphisms $\mathrm{B}_d(\Sigma^c M) \cong \Sigma^c \mathrm{B}_d(M)$ and $\mathrm{B}_{d_1} \circ \mathrm{B}_{d_2}(M) \cong \mathrm{B}_{d_2} \circ \mathrm{B}_{d_1}(M)$ for every $c, d, d_1, d_2 \in \mathcal{C}$.*

3F. *Categorifications for FI and VI_q.* Let k be a commutative ring. Let $\mathcal{C}$ be a category such that every morphism is a monomorphism, and every endomorphism is an isomorphism. Because every endomorphism is an isomorphism, the isomorphism classes of $\mathcal{C}$ inherit a natural partial order defined by $[c] \leq [d]$ if there is a map from c to d. We write $\boldsymbol{P}(c)$ for the projective $\mathcal{C}$ module $d \mapsto k\mathcal{C}(c, d)$. Then we have the following general result.

Theorem 3.18. *Let $c \in \mathcal{C}$. Suppose that $\mathrm{Res}^{\oplus} \boldsymbol{P}(c)$ splits as a sum of $\mathcal{C} \times \mathcal{C}$ modules of the form $\boldsymbol{P}(c_1) \otimes_k \boldsymbol{P}(c_2)$, where $c_i \in \mathcal{C}$ and $[c_i] \leq [c]$. Then $\mathrm{B}_d(\boldsymbol{P}(c))$ is exact for all $d \in \mathcal{C}$ such that $[d] > [c]$.*

Proof. Fix $c' \in \mathcal{C}$. To show that $\mathrm{B}_d(\boldsymbol{P}(c))_{c'}$ is exact, we consider the representation $\mathcal{C}/d \to \mathrm{Mod}\,k$,

$$(b \to d) \mapsto \boldsymbol{P}(c)_{c' \oplus b},$$

from which $\mathrm{B}_d(\boldsymbol{P}(c))_{c'}$ is constructed by applying $\mathrm{B}_{\mathcal{C}/d}$. By our hypothesis, this representation splits as a sum of representations of the form $(\boldsymbol{P}(c_1)_{c'}) \otimes_k \boldsymbol{P}(c_2)|_{\mathcal{C}/d}$ for some choice of $c_i \in \mathcal{C}$ with $[c_i] \leq [c] < [d]$.

Since $(\boldsymbol{P}(c_1)_{c'})$ is a free k module and $\mathrm{B}_{\mathcal{C}/d}(-)$ preserves sums and tensor products with k modules, it suffices to show that the homology of $\mathrm{B}_{\mathcal{C}/d}(\boldsymbol{P}(c_2)|_{\mathcal{C}/d})$ vanishes for all $c_2 \in \mathcal{C}$ with $[c_2] < [d]$. Further, note that

$$\boldsymbol{P}(c_2)|_{\mathcal{C}/d} = \bigoplus_{f:c_2 \to d} k(\mathcal{C}/d)(f, -).$$

Under the equivalence between $\mathcal{C}/d$ modules and $|\mathcal{C}/d|$ modules, $k(\mathcal{C}/d)(f, -)$ corresponds to the $|\mathcal{C}/d|$ module associated to the principal ideal $\{x \in |\mathcal{C}/d| : x \geq [f]\}$. Since $[c_2] < [d]$, f is never an isomorphism, and so $[f] < \hat{1}_{|\mathcal{C}/d|}$. So, by Proposition 2.27, it follows that $\mathrm{B}_{\mathcal{C}/d}(k\mathcal{C}/d(f, -))$ is exact. Thus $\mathrm{B}_d(\mathrm{P}(c))$ is exact. $\qquad\square$

Theorem 3.18 is formal, and can be extended to a general setting where we no longer assume every morphism of $\mathcal{C}$ is a monomorphism, as discussed in Section 4. Using it, we may prove our categorification theorems for **FI** and **VI**$_q$ modules.

Theorem 3.19. *Let k be a commutative noetherian ring and let q be a prime power that is invertible in k. Let M be a finitely generated **VI**$_q$ module over k. Then there exists $D \geq 0$ and $n \geq 0$ such for all $d \geq D$ and $W \in$ **VI**$_q$ with $\dim W \geq n$ the homology of*

$$(\mathrm{K}_{\mathbb{F}_q^d}(M))_W$$

vanishes.

Proof. By Proposition 2.13, we may prove the statement for $\mathrm{B}_{\mathbb{F}_q^d}(M)$. Since the direct sum functor is symmetric monoidal, by Proposition 3.17 we have that $\mathrm{B}_{\mathbb{F}_q^d}(\Sigma^V M) \cong \Sigma^V \mathrm{B}_{\mathbb{F}_q^d}(M)$. Therefore, to prove the statement for M, it suffices to prove that $\mathrm{B}_{\mathbb{F}_q^d}(\Sigma^V M)$ is exact for $\dim V \gg 0$ and $d \gg 0$. Theorem 1.2 of Nagpal [2019] states that if q is invertible in k there is $V \in$ **VI**$_q$ such that $\Sigma^V M$ admits a finite filtration whose associated graded modules are induced. The spectral sequence for this filtration reduces us to the case where M is an induced module. Recall that M is *induced* if there is a sequence of $\mathrm{GL}_n(\mathbb{F}_q)$ representations, $(W_n)_{n \in \mathbb{N}}$ with $W_n = 0$ for $n \gg 0$, such that

$$M \cong \bigoplus_{n \in \mathbb{N}} W_n \otimes_{\mathrm{GL}_n(\mathbb{F}_q)} \boldsymbol{P}(\mathbb{F}_q^n).$$

We have that

$$\mathrm{B}_{\mathbb{F}_q^d}(M) \cong \bigoplus_n W_n \otimes_{\mathrm{GL}_n(\mathbb{F}_q)} \mathrm{B}_{\mathbb{F}_q^d}(\boldsymbol{P}(\mathbb{F}_q^n)).$$

For every $V \in$ **VI**$_q$ the $k[\mathrm{GL}_n(\mathbb{F}_q)]$ module $\boldsymbol{P}(\mathbb{F}_q^n)_V$ is flat, because $\mathrm{GL}_n(\mathbb{F}_q)$ acts freely on the set of linear injections **VI**$(\mathbb{F}_q^n, V)$. Thus $\mathrm{B}_{\mathbb{F}_q^d}(\boldsymbol{P}(\mathbb{F}_q^n))$ is a complex of flat $\mathrm{GL}_n(\mathbb{F}_q)$ modules, and so it suffices to prove that the homology of $\mathrm{B}_{\mathbb{F}_q^d}(\boldsymbol{P}(\mathbb{F}_q^n))$ vanishes for $d > n$.

To complete the proof, we verify the hypothesis of Theorem 3.18 for $($**VI**$_q, \oplus)$. Indeed we have that for $U, V_1, V_2 \in$ **VI**$_q$,

$$\boldsymbol{VI}_q(U, V_1 \oplus V_2) = \bigsqcup_{\substack{A_1, A_2 \subseteq U \\ A_1 \cap A_2 = 0}} \boldsymbol{VI}_q(U/A_1, V_1) \times \boldsymbol{VI}_q(U/A_2, V_2).$$

Thus

$$\mathrm{Res}^{\oplus}(\boldsymbol{P}(U)) \cong \bigoplus_{\substack{A_1, A_2 \subseteq U \\ A_1 \cap A_2 = 0}} \boldsymbol{P}(U/A_1) \otimes \boldsymbol{P}(U/A_2),$$

and $U/A_i \leq U$ in the order on the set of objects of $\boldsymbol{VI}_q$. $\square$

The analog for $\boldsymbol{FI}$ modules has an identical proof, using the identity

$$\boldsymbol{FI}(x, a \sqcup b) = \bigsqcup_{x = r \sqcup s} \boldsymbol{FI}(r, a) \times \boldsymbol{FI}(s, b)$$

to apply Theorem 3.18 and Nagpal's shift theorem for $\boldsymbol{FI}$ modules [2015, Theorem A].

Theorem 3.20. *Let k be a commutative noetherian ring. Let M be a finitely generated $\boldsymbol{FI}$ module over K. Then there exist $D \geq 0$ and $n \geq 0$ such that for every $d \geq D$ the complex $(\mathrm{K}_{[d]}(M))_x$ is exact for all $x \in \boldsymbol{FI}$ of size $\geq n$.*

Remark 3.21. Theorem 3.19 is a categorification of the result, due to Nagpal, that if M is a finitely generated $\boldsymbol{VI}_q$ module in nondescribing characteristic then $h_M(t)$ is rational with denominator $\prod_{i=0}^{d-1}(1 - q^d t)$ for some $d > 0$. However, when k has the same characteristic as $\mathbb{F}_q$, this is false. For instance, consider the case $k = \mathbb{F}_q$. Then if M is the $\boldsymbol{VI}_q$ module defined by the identity functor $V \mapsto V$, we have

$$h_M(t) = \sum_i i t^i = \frac{t}{(1-t)^2}.$$

Therefore, Theorem 3.19 cannot extend to representations of $\boldsymbol{VI}_q$ in equal characteristic. However, in analogy with Theorem 1.2, we make the following conjecture.

Conjecture 3.22. *Let k be an arbitrary field. Let M be a $\boldsymbol{VI}_q$ module which is a subquotient of a module generated in degree d. Then there exists an $r \in \mathbb{N}$ such that the homology of $(\mathrm{K}_{\mathbb{F}_q^{d+1}})^{\circ r}(M)$ vanishes in sufficiently large degrees.*

4. Extension to EI categories

In this section, we explain how the complexes B_d can be interpreted using $\mathcal{C}$ module homology. This allows us to define the complexes B_d more generally when $\mathcal{C}$ is a monoidal EI category. The material in this section is not used elsewhere in the paper, and it is also more technical. We include it since it may be of interest to some readers, but others may prefer to skip it.

4A. *Homology of EI categories.* First we recall the definition of the homology of EI categories, paralleling the discussion for $\boldsymbol{FI}$ module homology in [Church and Ellenberg 2017, §5].

Definition 4.1. An *EI category* is a category in which every endomorphism is an isomorphism. If $\mathcal{D}$ is an EI category, then the set of isomorphism classes of $\mathcal{D}$ carries a natural partial order: define $[d_1] \leq [d_2]$ if there is a morphism $d_1 \to d_2$ in $\mathcal{D}$.

For the remainder of this section, $\mathcal{D}$ denotes a small EI category.

Definition 4.2. Let $M : \mathcal{D} \to \mathrm{Mod}\, k$ be a $\mathcal{D}$ module. Define

$$H_0^{\mathcal{D}}(M)_d := \frac{M_d}{\{M_f(m) \mid [c] < [d],\ m \in M_c\}}.$$

The left derived functors of $H_0^{\mathcal{D}}(-)_d$ are denoted by $H_i^{\mathcal{D}}(-)_d$ and called the $\mathcal{D}$ module homology in degree d.

The functor $H_0^{\mathcal{D}}(-)_d$ is naturally isomorphic to $S(d) \otimes_{\mathcal{D}} -$ for the following $\mathcal{D}^{\mathrm{op}}$ module.

Definition 4.3. Let $d \in \mathcal{D}$. We let $S(d) : \mathcal{D}^{\mathrm{op}} \to \mathrm{Mod}\, k$ be the $\mathcal{D}^{\mathrm{op}}$ representation

$$c \mapsto \frac{k\mathcal{D}(c, d)}{k\{f \mid f : c \to d,\ [d] < [c]\}}.$$

Concretely, it takes c to $k\mathcal{D}(c, d)$ if c is isomorphic to d, and zero otherwise.

For any category $\mathcal{C}$, using $H_0^{\mathcal{D}}(-)_d : \mathrm{Mod}\,\mathcal{D} \to \mathrm{Mod}\, k$ we may construct a functor $\mathrm{id} \times H_0^{\mathcal{D}}(-)_d$:

$$\mathrm{Mod}(\mathcal{C} \times \mathcal{D}) = \mathrm{Mod}(\mathcal{D})^{\mathcal{C}} \xrightarrow{H_0^{\mathcal{D}}(-)} (\mathrm{Mod}\, k)^{\mathcal{C}} = \mathrm{Mod}\,\mathcal{C}.$$

The following proposition interprets B_d in terms of $\mathcal{D}$ module homology.

Proposition 4.4. *Let $\mathcal{D}$ be a monoidal EI category, and $d \in D$. If $\mathcal{D}/d$ is equivalent to a poset, then $M \mapsto \mathrm{B}_d(M)$ is a model for the derived functor*

$$\left(\mathrm{L}(\mathrm{id} \times H_0^{\mathcal{D}}(-)_d)\right) \circ \mathrm{Res}^{\oplus} : \boldsymbol{D}^-(\mathrm{Mod}\,\mathcal{D}) \to \boldsymbol{D}^-(\mathrm{Mod}\,\mathcal{D}).$$

Concretely, if $\mathrm{C}_d^{\mathcal{D}}(-)$ is a functorial complex computing $\mathcal{D}$ module homology in degree d, coming from a projective $\mathcal{D}^{\mathrm{op}}$ resolution of $S(d)$, then there is a quasi-isomorphism

$$(\mathrm{id} \times \mathrm{C}_d^{\mathcal{D}}(-)) \circ \mathrm{Res}^{\oplus}(M) \simeq \mathrm{B}_d(M)$$

for any $\mathcal{D}$ module M.

Remark 4.5. We must derive the product functor $\mathrm{id} \times H_0^{\mathcal{D}}(-)_d$ in the statement of Proposition 4.4 instead of using the functor

$$\mathrm{id} \times \mathrm{L}H_0^{\mathcal{D}}(-)_d : \boldsymbol{D}(\mathrm{Mod}\,\mathcal{D})^{\mathcal{D}} \to \boldsymbol{D}(\mathrm{Mod}(k))^{\mathcal{D}},$$

because for ordinary (unenhanced) derived categories there is no equivalence $\boldsymbol{D}(\mathrm{Mod}\,\mathcal{D}) \simeq \boldsymbol{D}(\mathrm{Mod}(k))^{\mathcal{D}}$. However, if N is a $\mathcal{D} \times \mathcal{D}$ module such that the $\mathcal{D}$ module $N_{(c,-)}$ is $\mathrm{L}H_0^{\mathcal{D}}(-)_d$ acyclic for all $c \in \mathcal{D}$, then N is $\mathrm{L}(\mathrm{id} \times H_0^{\mathcal{D}}(-)_d)$ acyclic. So we may compute $\mathrm{L}(\mathrm{id} \times H_0^{\mathcal{D}}(-)_d)$ exactly how we would compute $\mathrm{id} \times \mathrm{L}H_0^{\mathcal{D}}(-)_d$.

We also note that from a more sophisticated point of view, Proposition 4.4 follows immediately by interpreting $\mathcal{D}$ module homology as the homology of

$$\mathrm{cone}((\mathrm{L}j_! j^* M)_d \to M_d),$$

where $j : \mathcal{D}_{<d} \to \mathcal{D}$ is the inclusion of the subcategory spanned by objects $< d$, and j^* and $j_!$ are the

restriction and left Kan extension functors. Then $(Lj_! j^* M)_d$ can be computed as a homotopy colimit over $\mathcal{D}_{<d}/d$, which is modeled by a bar construction. $\square$

Before proving Proposition 4.4 we introduce some definitions.

4B. *General bar complexes.* Let $\mathcal{J}$ be a small category containing a terminal object and such that there are no morphisms in $\mathcal{J}$ from a terminal object to a nonterminal object.

Definition 4.6. Define $S(\hat{1}_{\mathcal{J}})$ to be the $\mathcal{J}^{\mathrm{op}}$ module given by

$$S(\hat{1})_x = \begin{cases} k & \text{if } x \text{ is terminal}, \\ 0 & \text{otherwise}, \end{cases}$$

where for $f \in \mathcal{J}$, $S(\hat{1})_f$ is id_k if f is a morphism between two terminal objects, and the 0 map otherwise.

Write $\mathrm{Ob}(\mathcal{J})$ for the subcategory of $\mathcal{J}$ consisting of the set of objects of $\mathcal{J}$ and identity maps. Note $\mathrm{Ob}(\mathcal{J}) = \mathrm{Ob}(\mathcal{J}^{\mathrm{op}})$. Via the adjunction

$$(\mathrm{Mod}\,k)^{\mathrm{Ob}(\mathcal{J})} \underset{\mathrm{Res}}{\overset{\mathrm{Free}}{\rightleftarrows}} (\mathrm{Mod}\,k)^{\mathcal{J}^{\mathrm{op}}},$$

we may resolve $S(\hat{1}_{\mathcal{J}})$ using the associated monad $\perp = \mathrm{Free} \circ \mathrm{Res}$ as in [Weibel 1994, §8.6]. We obtain a simplicial abelian $\mathcal{J}^{\mathrm{op}}$ module F whose r simplices are

$$x \in \mathcal{J}^{\mathrm{op}} \mapsto k\{x \to x_r \to x_{r-1} \to \cdots \xrightarrow{f_1} x_0 \mid x_0 \text{ is terminal}\}.$$

Let $\widetilde{F}$ be the normalized chain complex associated to F. Then $\widetilde{F} \to S(\hat{1})$ is a projective resolution, and so the derived functor of $S(\hat{1}_{\mathcal{J}}) \otimes_{\mathcal{J}} -$ is modeled by $\widetilde{F} \otimes_{\mathcal{J}} -$. For any $\mathcal{J}$ module M, we have that $\widetilde{F} \otimes_{\mathcal{D}} M$ is a chain complex, which in degree r is

$$(\widetilde{F} \otimes_{\mathcal{D}} M)_r = \bigoplus_{\{x_r \xrightarrow{f_r} x_{r-1} \to \cdots \to x_0 \mid x_0 \text{ terminal}, \, f_i \neq \mathrm{id}_{x_i}\}} M_{x_r}.$$

Definition 4.7. $$\mathrm{Bar}_{\mathcal{J}}(M) := (\widetilde{F} \otimes_{\mathcal{D}} M).$$

It is clear that when $\mathcal{J}$ is a poset with top element $\hat{1}$, we have $\mathrm{Bar}_{\mathcal{J}}(M) \cong \mathrm{B}_{\mathcal{J}}(M)$. As a warning, note that if there is an equivalence of categories $f : \mathcal{J}' \xrightarrow{\sim} \mathcal{J}$, we do not have $\mathrm{Bar}_{\mathcal{J}}(M) \cong \mathrm{Bar}_{\mathcal{J}'}(f^* M)$. However, because there is an isomorphism of $(\mathcal{J}')^{\mathrm{op}}$ modules $f^* S(\hat{1}_{\mathcal{J}}) \cong S(\hat{1}_{\mathcal{J}'})$, both complexes compute the same derived functor, and thus there is a quasi-isomorphism

$$\mathrm{Bar}_{\mathcal{J}}(M) \simeq \mathrm{Bar}_{\mathcal{J}'}(f^* M).$$

Definition 4.8. Using $\mathrm{Bar}_{\mathcal{J}}$ we may extend our construction of chain complexes to any small monoidal category $\mathcal{C}$ satisfying the retraction–isomorphism property

($\bullet$) if $c \in \mathcal{C}$ and $c \xrightarrow{f} c' \xrightarrow{g} c$ is a retraction (i.e., $g \circ f = \mathrm{id}$), then f is an isomorphism,

by defining

$$\mathrm{Bar}_d(N) := (\mathrm{id} \times \mathrm{Bar}_{\mathcal{C}/d}(-)) \circ \mathrm{Res}^{\oplus}(N)$$

for any C module N. Indeed, the overcategory C/c always contains $\mathrm{id}_c : c \to c$ as a terminal object, and the condition ($\bullet$) corresponds to the statement that any morphism from id_c to $c' \to^g c$ in C/c is an isomorphism.

When the overcategories of C are posetal, the discussion above implies that there is a quasi-isomorphism of complexes of C modules $\mathrm{Bar}_d(N) \simeq \mathrm{B}_d(N)$ for every $d \in C$.

4C. _Proof of Proposition 4.4._ With the general definitions of Bar complexes in hand, we may prove Proposition 4.4. In fact, we may weaken our assumption that the overcategories of $\mathcal{D}$ are posets, to the assumption that $\mathcal{D}$ satisfies property ($\bullet$).

Proof of Proposition 4.4. Let $F : \mathrm{Ch}^-(\mathrm{Mod}(\mathcal{D} \times \mathcal{D})) \to \mathrm{Ch}^-(\mathrm{Mod}\,\mathcal{D})$ be an exact sum-preserving functor from bounded below chain complexes $\mathcal{D} \times \mathcal{D}$ modules to bounded below chain complexes of $\mathcal{D}$ modules. To show that F models $\mathrm{L}(\mathrm{id} \times H_0^{\mathcal{D}}(-)_d)$ it suffices to prove that there is a natural isomorphism $H_0(F(N)) \cong \mathrm{id} \times H_0^{\mathcal{D}}(-)_d$ for all $\mathcal{D} \times \mathcal{D}$ modules N, and that for every $c_1, c_2 \in \mathcal{D}$ we have that $H_i(F(k(\mathcal{D}(c_1, -) \times \mathcal{D}(c_2, -)))) = 0$ for $i > 0$. This follows from the fact that, by the Yoneda lemma, $\{k(\mathcal{D}(c_1, -) \times \mathcal{D}(c_2, -))\}_{c_1, c_2 \in \mathcal{D}}$ form a collection of projective generators of $\mathrm{Mod}(\mathcal{D} \times \mathcal{D})$.

The equivalence between the abstract statement of the proposition and the concrete one follows from the fact that if $P_\bullet \to S(d)$ is a projective resolution of $S(d)$ and we define $C(-) := P_\bullet \otimes_{\mathcal{D}} -$, then $F := \mathrm{id} \times C(-) : \mathrm{Mod}(\mathcal{D} \times \mathcal{D}) \to \mathrm{Mod}(\mathcal{D})$ satisfies the above criteria.

Let $r : \mathcal{D}/d \to \mathcal{D}$ be the functor that forgets the morphism to d. For any $\mathcal{D}$ module M there is a natural isomorphism

$$S(\hat{1}) \otimes_{\mathcal{D}/d} r^* M \cong \frac{M_d}{\{M_g(m) \mid g : c \to d, \, m \in M_c, \, g \text{ not an isomorphism}\}} \cong H_0^{\mathcal{D}}(M)_d.$$

Further we have that

$$r^*(k\mathcal{D}(c, -)) = \bigoplus_{f : c \to d} k(\mathcal{D}/d)(f, -).$$

Because $\mathrm{Bar}_{\mathcal{D}/d}(-)$ models $S(\hat{1}) \otimes_{\mathcal{D}}^{\mathrm{L}} -$ and $k(\mathcal{D}/d)(f, -)$ is a projective $\mathcal{D}/d$ module by Yoneda, it follows that $H_i(\mathrm{Bar}_{\mathcal{D}/d}(r^*(k\mathcal{D}(c, -))) = 0$ for all $i > 0$. Therefore, $\mathrm{id} \times \mathrm{Bar}_{\mathcal{D}/d}(-)$ satisfies the criteria to model $\mathrm{L}(\mathrm{id} \times H_0^{\mathcal{D}}(-)_d)$. When $\mathcal{D}/d$ is equivalent to a poset Q, the quasi-isomorphism between $\mathrm{Bar}_{\mathcal{D}/d}$ and B_Q finishes the proof of the proposition. $\square$

Remark 4.9. Using the material in this section, we may extend the definition of the complexes $\mathrm{B}_d(-)$ to an arbitrary small monoidal EI category $\mathcal{D}$, by taking Proposition 4.4 as a definition. It is natural to ask: which results from Section 3 continue to hold in this level of generality?

The answer is that Theorem 3.18 holds for formal reasons. However, at this level of generality, exactness of complexes does not imply rationality of Hilbert series. Of course, exactness always encodes _some_ relation between the virtual $\mathrm{Aut}(c)$ representations $[M_c]$. But this relation only implies a relation between the dimensions $\dim M_c$ if for every $d \in \mathcal{D}$ the overcategory $\mathcal{D}/d$ is _directed_: every endomorphism of an object in $\mathcal{D}/d$ is the identity. (Directedness of $\mathcal{D}/d$ is equivalent to $\mathrm{Aut}(c)$ acting freely on $\mathcal{D}(c, d)$ for all c.)

In fact, all of Sections 2 and 3 generalizes easily if we replace "posetal" by "directed" and "poset" by "skeletal directed category". We chose not to work at that level of generality because posets are more familiar, and all of the examples we consider have posetal overcategories. □

5. Gröbner theory

In this section, we use the Sam–Snowden Gröbner theory of categories to give a criterion for chain complexes such as $K_d(M)$ and $B_d(M)$ to be exact. For background, see [Sam and Snowden 2017, §4].

Let $(\mathcal{C}, \sqcup)$ be a small monoidal category. In this paper, we will only consider the case $\mathcal{C} = \boldsymbol{OS}^{\mathrm{op}}$ (see Definition 7.1), but in the interest of future applications work in a more general setting. Fix $d \in \mathcal{C}$, and let $\boldsymbol{S}(d) : \mathcal{C} \to \mathrm{Set}$ be the functor $c \mapsto \mathcal{C}(d, c)$. We assume that $\mathcal{C}(d, c)$ is finite.

Definition 5.1. An *ordering* on $\boldsymbol{S}(d)$ consists of a total ordering $\leq$ on the set of functions $\mathcal{C}(d, c)$ for every in c such that postcomposition is strictly order-preserving: if $f_1 < f_2$ then $g \circ f_1 < g \circ f_2$ for all $g \in \mathcal{C}(c, c')$.

We assume there is an ordering on $\boldsymbol{S}(d)$, and let $\boldsymbol{P}(d) := k\boldsymbol{S}(d)$. For any $c \in \mathcal{C}$, the k module $\boldsymbol{P}(d)_c$ is freely spanned by monomials e_f for $f \in \mathcal{C}(d, c)$.

Definition 5.2. The *initial term* of an element $\sum_f \lambda_f e_f \in \boldsymbol{P}(d)_c$ is $\lambda_{f_0} e_{f_0}$, where $f_0 \in \boldsymbol{P}(d)_c$ is the largest element such that $\lambda_{f_0} \neq 0$.

Definition 5.3. Let $J \subseteq \boldsymbol{P}(d)$ be a $\mathcal{C}$ submodule. Let $\mathrm{init}(J)_c \subseteq \boldsymbol{P}(d)_c$ be the subspace $\{\mathrm{init}(j) \mid j \in J_c\}$. By [Sam and Snowden 2017, Theorem 4.2.1], $c \mapsto \mathrm{init}(J)_c$ is a *monomial submodule* of $\boldsymbol{P}(d)$. This means that $\mathrm{init}(J)_c$ is spanned by the elements $e_f \in \boldsymbol{P}(d)_c$ that it contains. We say that $\mathrm{init}(J)$ is the *monomial ideal* associated to J.

Let c and x be objects of $\mathcal{C}$. Let $\mathcal{Q}$ be a subcategory of $\mathcal{C}/c$, and let $\mathcal{B}$ be an exact functor from $\mathcal{Q}$ representations to chain complexes. Consider the $\mathcal{Q}$ representations $J_{x \sqcup -}$ and $\mathrm{init}(J)_{x \sqcup -}$ defined by $(g : y \to c) \mapsto J_{x \sqcup y}$ and $(g : y \to c) \mapsto \mathrm{init}(J)_{x \sqcup y}$ respectively. In this context, we prove the following result.

Proposition 5.4. *If $\mathcal{B}(\mathrm{init}(J)_{x \sqcup -})$ is exact, then $\mathcal{B}(J_{x \sqcup -})$ is exact.*

Proof. Using the order on $\boldsymbol{S}(d)$ we define a filtration of the $\mathcal{Q}$ representation $\boldsymbol{P}(d)_{x \sqcup -}$ as follows. Given $f : d \to x \sqcup c$, let $F_{\leq f} \boldsymbol{P}(d)_{x \sqcup -}$ be the subrepresentation

$$(g : y \to c) \mapsto \mathrm{span}\{e_h \mid h : d \to x \sqcup y \text{ such that } (\mathrm{id}_x \sqcup g) \circ h \leq f\}.$$

This is a subrepresentation because a morphism $g \to g'$ consists of a morphism $s : y \to y'$ such that $g' \circ s = g$, and thus $(\mathrm{id}_x \sqcup g') \circ (\mathrm{id}_x \sqcup s) \circ h = (\mathrm{id}_x \sqcup g) \circ h$. If $f \leq f'$ then $F_{\leq f} \boldsymbol{P}(d)_{x \sqcup -} \subseteq F_{\leq f'} \boldsymbol{P}(d)_{x \sqcup -}$, so we obtain a filtration using the total order on $\mathcal{C}(d, x \sqcup c)$.

We define $F_{\leq f} J_{x \sqcup -}$ to be the intersection of $F_{\leq f} \boldsymbol{P}(d)_{x \sqcup -}$ and $J_{x \sqcup -}$. Then we have:

Lemma 5.5. *The associated graded $\mathcal{Q}$ representation is isomorphic to* $\mathrm{init}(J)_{x \sqcup -}$.

Proof. Define a map $(F_{\leq f} J_{x \sqcup -})_g \to (\text{init}(J)_{x \sqcup -})_g$ by

$$\sum_h \lambda_h e_h \mapsto \begin{cases} \lambda_{h_f} e_{h_f} & \text{if there exists } h_f : d \to x \sqcup y \text{ such that } (\text{id}_x \sqcup g) \circ h_f = f, \\ 0 & \text{otherwise.} \end{cases}$$

Because $\text{id}_x \sqcup g$ strictly preserves the total order on functions, this yields a well-defined morphism of representations

$$\bigoplus_{f \in \textbf{\textit{P}}(d)_{x \sqcup c}} \frac{F_{\leq f} J_{x \sqcup -}}{F_{< f} J_{x \sqcup -}} \to \text{init}(J)_{x \sqcup -}.$$

It is injective because for each g, the monomials e_{h_f} are linearly independent for every f. And it is surjective, because each $\lambda_h e_h \in \text{init}(J)_{x \sqcup y}$ is hit by the map corresponding to $f = (\text{id}_x \sqcup g) \circ h$. $\qquad \square$

Thus we have a filtration of $J_{x \sqcup -}$ whose associated graded is isomorphic to $\text{init}(J)_{x \sqcup -}$. Since $\mathcal{B}$ is an exact functor, this filtration induces a filtration of the complex $\mathcal{B}(J_{x \sqcup -})$ whose associated graded is $\mathcal{B}(\text{init}(J)_{x \sqcup -})$. This filtration is finite, because we have assumed $\textbf{\textit{P}}(d)_{x \sqcup c}$ has finitely many elements. Therefore, the spectral sequence associated to a filtered complex shows that if $\mathcal{B}(\text{init}(J)_{x \sqcup -})$ is exact, then so is $\mathcal{B}(J_{x \sqcup -})$. $\qquad \square$

6. Poset ideals associated to ordered languages

Let $\Sigma = \{a_1, \ldots, a_d\}$ be a finite alphabet with d letters.

Definition 6.1. Let $w \in \Sigma^*$ be a length n word. We consider w as a function $w : [n] \to \Sigma$. Given a set partition $p \in P(n)$ with m blocks, order the blocks according to their smallest element to obtain an identification between p and $[m]$. We write $f_p : [n] \to [m]$ for the surjection that takes $i \in [n]$ to the block that contains it. We say that w *factors through* p if there exists a word $w_p : [m] \to \Sigma$ such that $w_p \circ f_p = w$. Because f_p is a surjection, w_p is necessarily unique if it exists. If w factors through p, then we call w_p the *quotient word of w by p*.

Let $L \subseteq \Sigma^*$ be a regular language. Throughout we assume that L satisfies the following property.

$(*)$ Let $w_1, w_2, w_3 \in \Sigma^*$ and $a \in \Sigma$. If $w_1 a w_2 w_3 \in L$, then $w_1 a w_2 a w_3 \in L$.

Definition 6.2. We define an upward ideal, $\textbf{\textit{I}}(w, L) \subseteq P(n)$, associated to $w \in \Sigma^*$ and L a language satisfying property $(*)$. A partition p is contained in $\textbf{\textit{I}}(w, L)$ if and only if w factors through p and $w_p \in L$.

The subset $\textbf{\textit{I}}(w, L)$ is upward-closed because of our assumption $(*)$ on L. If $w \notin L$, then $\textbf{\textit{I}}(w, L) = \varnothing$.

Example 6.3. Let $\Sigma = \{a, b\}$ and $w = abba$. Then w factors through the partitions $1|2|3|4$, $14|2|3$, $1|23|4$ and $14|23$. The associated quotient words are $abba$, abb, aba, and ab respectively. If L is the language defined by the regular expression $ab^*a(a^*b^*)^*$, then L satisfies property $(*)$ and we have that $I(w, L) = \{1|2|3|4, 1|23|4\}$.

Recall from [Sam and Snowden 2017, §5.2] that an *ordered DFA* is a deterministic finite automaton A, together with a partial order on the set of states of A, such that $x_1 \leq x_2$ if and only if there is a word $u \in \Sigma^*$

such that running A with start state x_1 on input u results in the final state x_2; symbolically we will write $x_1 \to^u x_2$. A regular language L is *ordered* if and only if there is an ordered DFA that accepts it. All the DFAs we consider will be *connected*: if α is the start state of A and β is some other state of A, then there is some word $u \in \Sigma^*$ such that $\alpha \to^u \beta$. Any (ordered) DFA may be replaced by a connected (ordered) DFA accepting the same language, by discarding the states which are not connected to the start state. It is convenient to introduce the following operation, in order to make inductive arguments.

Definition 6.4. Let A be an ordered DFA and let α be a state of A. Then we define the *truncation of A at α*, denoted by $A_{\geq \alpha}$, to be the following ordered DFA. The (accept) states of $A_{\geq \alpha}$ are the (accept) states of A that are $\geq \alpha$ in the partial order. The transition function of $A_{\geq \alpha}$ is the restriction of the transition function of A to $A_{\geq \alpha}$: this is well defined because A is ordered. The start state of $A_{\geq \alpha}$ is α.

Let A be an ordered DFA accepting a language $L \subseteq \Sigma^*$ which satisfies property (∗). Let $w_1, \ldots, w_r \in \Sigma^*$ be words of length $\ell_1, \ldots, \ell_r \in \mathbb{N}$, and put $w = w_1 \ldots w_r$. Let $\ell = \sum_t \ell_t$ be the length of w.

There is an embedding of posets $\prod_{t=1}^{r} P(\ell_t) \to P(\ell)$, constructed using the bijection $[\ell_1] \sqcup \cdots \sqcup [\ell_r] \to [\ell]$ which maps $[\ell_t]$ to the interval $\left(\sum_{u=1}^{t-1} \ell_u, \sum_{u=1}^{t} \ell_u \right]$. Given partitions $q_t \in P(\ell_t)$ for $t = 1, \ldots, r$ the associated element of $P(\ell)$ is $q_1 \sqcup \cdots \sqcup q_r$, considered as a partition of $[\ell]$ via the bijection. Then $\prod_{t=1}^{r} P(\ell_t)$ is isomorphic to its image in $P(\ell)$ (with subposet structure induced by $P(\ell)$), so we identify both posets and write $\prod_{t=1}^{r} P(\ell_t) \subseteq P(\ell)$.

Definition 6.5. Let $\boldsymbol{J}(w, L)$ be the ideal $\boldsymbol{I}(w, L) \cap \prod_{t=1}^{r} P(\ell_t)$.

The key combinatorial result underlying Theorem 1.2 is the following.

Theorem 6.6. *Let $d \geq 1$ and let Σ be an alphabet of size d. Let A be a connected, finite, ordered DFA, accepting a language $L \subseteq \Sigma^*$ which satisfies property (∗). Let $w_1, \ldots, w_r \in \Sigma^*$ be words of length $\ell_1, \ldots, \ell_r$. Suppose that r is greater than or equal to the length of A (considered as a poset), $\ell_t \geq d$ for $t = 1, \ldots, r - 1$ and $\ell_r \geq d + 1$. Then $\mathrm{B}_{(P(\ell_1), \ldots, P(\ell_r))}(k\boldsymbol{J}(w, L))$ is exact.*

Proof. We write $\mathcal{B}_i$ for the functor $\mathrm{B}_{(P(\ell_i), \ldots, P(\ell_r))}$ from $\operatorname{Rep} \prod_{t=i}^{r} P(\ell_t)$ to chain complexes.

We induct on the length of A. In the base case, the length of A is 1 and A consists of a single state. If the state is a reject state, then $k\boldsymbol{J}(w, L) = 0$ so the statement is trivially true, so assume that the state is an accept state. Then A accepts every word and $\boldsymbol{J}(w, L)$ is the ideal consisting of all $q \in \prod_{i=t}^{\ell} P(\ell_t)$ such that w factors through q. Let p_t be the partition of $[\ell_t]$ associated to the function $w_t : [\ell_t] \to \Sigma$ (in other words, i and j are in the same block of p_t if and only if the i-th letter of w_t equals the j-th letter of w_t). Then $\boldsymbol{J}(w, l) = \prod_{t=1}^{r} P(\ell_t)_{\geq p_t}$. So $k\boldsymbol{J}(w, l) = k\left(\prod_{t=1}^{r-1} P(\ell_t)_{\geq p_t} \right) \boxtimes k P(\ell_r)_{\geq p_r}$. Because $\ell_r > d$ and $|\Sigma| = d$, the pigeon-hole principle implies at least one letter of Σ must be repeated in w_r. Therefore, $p_r \neq \hat{1}_{P(\ell_r)}$, and so by Proposition 2.27 the complex $\mathrm{B}(k P(\ell_r)_{\geq p_r})$ is exact. Then by Proposition 2.23, $\mathcal{B}_1(k\boldsymbol{J}(w, l))$ is exact.

For the inductive step assume that the length of A is ≥ 2, and let α be the starting state of A. Consider the word w_1. There are two cases: either (1) $\alpha \to^{w_1} \alpha$ or (2) $\alpha \to^{w_1} \beta$ for some state $\beta > \alpha$.

The argument in case (1) is similar to the base case. Let $p_1 \in P(n)$ be the partition associated to the function $w_1 : [\ell_1] \to \Sigma$. We claim that

$$\boldsymbol{J}(w, L) = P(\ell_1)_{\geq p_1} \times \boldsymbol{J}(w_2 \ldots w_r, L).$$

Indeed, we may write an element of $q \in P(\ell_1) \times \prod_{t=2}^{r} P(\ell_t)$ uniquely as $q_1 \sqcup q'$, where $q_1 \in P(\ell_1)$ and $q' \in \prod_{t=2}^{r} P(\ell_t)$. Then w factors through q if and only if w_1 factors through q_1 and $w' := w_2 \ldots w_r$ factors through q'. If w factors through q, then $w_q = (w_1)_{q_1} w'_{q'}$. Because $(w_1)_{q_1}$ contains only letters which appear in w_1, and $\alpha \to^t \alpha$ for every letter of w, we have that $\alpha \to^{(w_1)_{q_1}} \alpha$. Therefore, A accepts w_q if and only if A accepts $w'_{q'}$. Because w_1 factors through q_1 if and only if $q_1 \geq p_1$, we have that $q \in \boldsymbol{J}(w, L)$ if and only if $q_1 \in P(n)_{\geq p_1}$ and $q' \in \boldsymbol{J}(w', L)$. This establishes the claim.

So we have

$$k\boldsymbol{J}(w, L) = kP(\ell_1)_{\geq p_1} \boxtimes k\boldsymbol{J}(w_2 \ldots w_r, L).$$

The word w_1 only contains letters $a \in \Sigma$ which satisfy $\alpha \to^a \alpha$. Since α is not the only state of A, there are at most $d - 1$ such letters. Thus by the pigeon-hole principle it follows that w_1 must contain a repeat letter and therefore $p_1 \neq \hat{1}_{P(\ell_1)}$. So Proposition 2.27 implies that $B(kP(\ell_1)_{\geq p_1})$ is exact. Therefore, by Proposition 2.23, $\mathcal{B}_1(k\boldsymbol{J}(w, L))$ is exact in case (1).

In case (2), let $q_1 \in P(\ell_1)$, and consider the ideal of $\prod_{t=2}^{r} P(\ell_t)$

$$I_{q_1} := \boldsymbol{J}(w, L) \cap \left(q_1 \times \prod_{t=2}^{r} P(\ell_t) \right).$$

We note that

$$kI_{q_1} = (k\boldsymbol{J}(w, L))|_{q_1 \times \prod_{t=2}^{r} P(\ell_t)},$$

so by Proposition 2.24, it suffices to show that the homology $\mathcal{B}_2(kI_{q_1})$ vanishes for all q_1. If w_1 does not factor through q_1, then $I_{q_1} = \varnothing$, and $kI_{q_1} = 0$, so $\mathcal{B}_2(kI_{q_1}) = 0$. So suppose that w_1 factors through q_1, and let $(w_1)_{q_1}$ be the quotient word. Notice that $\alpha \to^{(w_1)_{q_1}} \beta'$ for some state $\beta' > \alpha$. Indeed, if a is the first letter of w_1 which causes A to transition to a state distinct from α, then $(w_1)_{q_1}$ also contains the letter a, and so also transitions to a state distinct from α.

Now let $A' := A_{\geq \beta'}$ be the truncation at β'. Let L' be the language recognized by A'. We claim that

$$I_{q_1} = \boldsymbol{J}(w', L'),$$

where $w' = w_2 \ldots w_r$. Indeed since w_1 factors through q_1, we have that for all $q' \in \prod_{t=2}^{r} P(\ell_t)$, w factors through $q_1 \sqcup q'$ if and only if w' factors through q'. If w' factors through q', then $w_{q_1 \sqcup q'} = (w_1)_{q_1} w'_{q'}$. Therefore, $w_{q_1 \sqcup q'}$ is accepted by A if and only if $\alpha \to^{w_1 q_1} \beta' \to^{w'_{q'}} \gamma$, where γ is an accept state of A. This occurs if and only if $w'_{q'} \in L'$. This shows the claim.

The length of A' is less than the length of A, hence is $\leq r - 1$. The language L' satisfies property $(*)$ because L does and $u \in L'$ if and only if $w_1 u \in L$. So the induction hypothesis gives that the homology of $\mathcal{B}_2(k\boldsymbol{J}(w', L'))$ vanishes. Thus the homology of $\mathcal{B}_2(kI_{q_1})$ vanishes for all $q_1 \in P(\ell_1)$, so we are done. $\square$

7. OS^{op} modules and Theorem 1.2

In this section, we combine the results of the previous sections in order to prove Theorem 1.2.

Definition 7.1. Let OS be the category of *ordered surjections* of [Sam and Snowden 2017, §8]. The objects of OS are finite sets equipped with a total order. A morphism from X to Y in OS is an *ordered surjection*: a surjection $f : X \twoheadrightarrow Y$ such that if $y_1 < y_2$ then $\min f^{-1}(y_1) < \min f^{-1}(y_2)$.

FS^{op} is equivalent to the subcategory spanned by the sets $[n] := \{1, \ldots, n\}$ for $n \in \mathbb{N}$. Similarly OS^{op} is equivalent to the subcategory spanned by the sets $[n]$ equipped with their canonical order $1 \le 2 \le \cdots \le n$. A functor with domain FS^{op} or OS^{op} is determined by its restriction to the corresponding subcategory, so we will freely pass between the ambient categories and their respective subcategories. When we use a natural number $n \in \mathbb{N}$ to denote an object of OS^{op} or FS^{op}, we are referring to the object $[n]$.

Disjoint union endows OS^{op} with a monoidal structure

$$- \sqcup - : OS^{\mathrm{op}} \times OS^{\mathrm{op}} \to OS^{\mathrm{op}},$$

where we define the total order on $S \sqcup T$ by extending the order on S and T and declaring every element of S to be less than every element on T. There is an essentially surjective, monoidal functor $\pi : OS^{\mathrm{op}} \to FS^{\mathrm{op}}$, given by forgetting the total ordering. Thus we may restrict any FS^{op} module M to obtain an OS^{op} module $\pi^* M$. We now prove two propositions relating OS^{op} and FS^{op} modules.

Proposition 7.2. *For $X \in OS^{\mathrm{op}}$, the functor $OS^{\mathrm{op}}/X \to FS^{\mathrm{op}}/X$ is an isomorphism of categories. Consequently,* K_X *and* B_X *are defined for OS^{op} modules, and for any FS^{op} module M there are natural isomorphisms* $\mathrm{B}_X(\pi^* M) \cong \pi^* \mathrm{B}_X(M)$ *and* $\mathrm{K}_X(\pi^* M) \cong \pi^* \mathrm{K}_X(M)$.

Proof. Let $X \in OS^{\mathrm{op}}$ be a set with a total order. Given any surjection $f : X \twoheadrightarrow Y$ there is a unique total order on Y such that f is an ordered surjection: define $y \le y'$ if $\min f^{-1}(y) \le \min f^{-1}(y')$. And given any $X \twoheadrightarrow Y_1 \twoheadrightarrow Y_2$, the orders induced on Y_1 and Y_2 are compatible, so that $Y_1 \twoheadrightarrow Y_2$ is an ordered surjection. Thus $OS^{\mathrm{op}}/X \to FS^{\mathrm{op}}/X$ is an isomorphism and the other statements follow immediately. □

Proposition 7.3. *Let N be an FS^{op} module generated in degree $\le d$. Then $\pi^* N$ is generated in degree $\le d$.*

Proof. Let $l \in \mathbb{N}$ and let $x \in N_l$ be an element. Then for any $m \in \mathbb{N}$, any surjection $f : [m] \twoheadrightarrow [l]$ factors as an ordered surjection, followed by a bijection $[l] \to [l]$. Thus the FS^{op} submodule generated by x equals the OS^{op} module generated by $\{\sigma x\}_{\sigma \in S_l}$. □

For $d \in \mathbb{N}$, let $P(d)$ be the free OS^{op} module generated in degree d: $P(d)_m := k OS^{\mathrm{op}}(d, m)$. The next proposition applies Theorem 6.6 and Proposition 5.4 to prove a variant of Theorem 1.2 for submodules of $P(d)$.

Proposition 7.4. *Let k be a field. Let J be an OS^{op} submodule of $P(d)$. Then there exists an $s \in \mathbb{N}$ such that* $\mathrm{K}_{\ell_1} \circ \cdots \circ \mathrm{K}_{\ell_r}(J)$ *is exact for all $r \ge s$ and all $(\ell_t)_{t=1}^r \in \mathbb{N}^r$ satisfying $\ell_t \ge d$ for $t = 1, \ldots, r-1$ and $\ell_r \ge d + 1$.*

Proof. By [Sam and Snowden 2017, Theorem 8.1.1], the category $\boldsymbol{OS}^{\mathrm{op}}$ is *Gröbner*: in particular the functor $S(d)$, defined by $n \mapsto \boldsymbol{OS}^{\mathrm{op}}(d, n)$, carries an ordering satisfying the conditions of Section 5. (We will not use the particular definition of the ordering, so we do not recall it.) We let $\mathrm{init}(J)$ be the monomial ideal associated to J with respect to this ordering.

For $n \in \mathbb{N}$, let $T_n \subseteq \mathrm{init}(J)_n$ be the set of $f \in S(d)_n$ such that $e_f \in \mathrm{init}(J)$. Then $n \mapsto T_n$ is functor $\boldsymbol{OS}^{\mathrm{op}} \to \mathrm{Set}$, and there is a canonical morphism $kT \to \mathrm{init}(J)_n$ given by $e_f \mapsto e_f$. This map is injective for any ring, and because k is a field it is surjective: if $\lambda e_f \in \mathrm{init}(J)$ is a nonzero element, then $e_f = \lambda^{-1} \lambda e_f \in \mathrm{init}(J)$.

Let $\Sigma = [d]$ and $\iota : \bigsqcup_{n \in \mathbb{N}} \boldsymbol{OS}^{\mathrm{op}}(n, d) \to \Sigma^*$ be the tautological embedding taking $f : [n] \to [d]$ to its word. In the terminology of Sam and Snowden [2017],

$$\bigsqcup_{n \in \mathbb{N}} \boldsymbol{OS}^{\mathrm{op}}(n, d) = |S(d)| = |\mathcal{C}_{[d]}|,$$

where $\mathcal{C} = \boldsymbol{OS}^{\mathrm{op}}$. Sam and Snowden [2017, §8.2] show that the embedding $\iota : |S(d)| \to \Sigma^*$ endows $S(d)$ with an *O-lingual structure* [loc. cit., Definition 6.2.1]. In particular, this implies there is an ordered DFA A which accepts the language $L := \iota\left(\bigsqcup_n T_n\right) \subseteq \Sigma^*$ (we take A to be connected). We will prove the statement for s equal to the length of A, considered as a poset. So fix $r \geq s$.

There is a quasi-isomorphism $\mathrm{K}_{\ell_1} \circ \cdots \circ \mathrm{K}_{\ell_r} \simeq \mathrm{B}_{\ell_1} \circ \cdots \circ \mathrm{B}_{\ell_r}$ by Proposition 2.13. Fix $x \in \boldsymbol{OS}^{\mathrm{op}}$. As discussed in Section 3E, it suffices show that $\mathrm{B}_{(\boldsymbol{OS}^{\mathrm{op}}/\ell_1, \ldots, \boldsymbol{OS}^{\mathrm{op}}/\ell_r)}(J_{x \sqcup -})$ is exact. To do this, we identify $\mathrm{P}(\ell_t)$ with the full subcategory of $\boldsymbol{OS}^{\mathrm{op}}/\ell_t$ spanned by the canonical ordered surjections $[\ell_t] \twoheadrightarrow p$ for every $p \in \mathrm{P}(\ell_t)$. Then

$$\mathrm{B}_{(\boldsymbol{OS}^{\mathrm{op}}/\ell_1, \ldots, \boldsymbol{OS}^{\mathrm{op}}/\ell_r)}(J_{x \sqcup -}) \cong \mathrm{B}_{(\mathrm{P}(\ell_1), \ldots, \mathrm{P}(\ell_r))}(J_{x \sqcup -}),$$

and by Proposition 5.4 it suffices to show that

$$\mathrm{B}_{(\mathrm{P}(\ell_1), \ldots, \mathrm{P}(\ell_r))}((\mathrm{init}\, J)_{x \sqcup -}) \cong \mathrm{B}_{(\mathrm{P}(\ell_1), \ldots, \mathrm{P}(\ell_r))}(kT_{x \sqcup -})$$

is exact. So we consider the functor $T_{x \sqcup -} : \mathrm{P}(\ell_1) \times \cdots \times \mathrm{P}(\ell_r) \to \mathrm{Set}$,

$$(q_1, \ldots, q_r) \mapsto T_{x \sqcup q_1 \sqcup \cdots \sqcup q_r}.$$

Since $S(d)_f$ is injective for all $\boldsymbol{OS}^{\mathrm{op}}$ morphisms f, by Proposition 2.28 we see that $T_{x \sqcup -} = \bigsqcup_u SI\langle u \rangle$, where the sum is over words $u : x \sqcup [\ell_1] \sqcup \cdots \sqcup [\ell_r] \to [d]$.

Fix $u : x \sqcup [\ell_1] \sqcup \cdots \sqcup [\ell_r] \to [d]$ and write $u = zw_1 \ldots w_r$, where $z : x \to [d]$ and w_t is a word on ℓ_t letters for $t = 1, \ldots, r$. Let α be the state obtained by running A with input z. Let L' be the language accepted by $A_{\geq \alpha}$.

Because T is an $\boldsymbol{OS}^{\mathrm{op}}$ set, the languages L and L' both satisfy property $(*)$ of Section 6: under the identification between words and functions, the substitution $w_1 a w_2 w_3 \mapsto w_1 a w_2 a w_3$ corresponds to precomposition with the obvious ordered surjection $[n] \twoheadrightarrow [n-1]$. Therefore, the ideal $\boldsymbol{J}(w_1 \ldots w_r, L') \subset \mathrm{P}(\ell_1) \times \cdots \times \mathrm{P}(\ell_r)$ is defined.

Lemma 7.5. *The ideal $I\langle u \rangle$ equals the ideal $\boldsymbol{J}(w_1 \ldots w_r, L')$.*

Proof. Let $\ell = \sum \ell_t$. By definition, a partition $p \in P(\ell_1) \times \cdots \times P(\ell_r)$ lies in $I\langle u \rangle$ if and only if there is an element $u' \in T_{x \sqcup p}$ such that for the ordered surjection $f_p : [\ell] \twoheadrightarrow p$ we have $T_{\mathrm{id}_x \sqcup f_p}(u') = u$. This occurs if and only if (1) there is an element of $u' \in S(d)_{x \sqcup p}$ satisfying $S(d)_{\mathrm{id}_x \sqcup f}(u')$ and (2) $u' \in T_{x \sqcup p}$. These two conditions correspond to (1) the word $w_1 \ldots w_r$ factoring through p and (2) the word $z(w_1 \ldots w_r)_p$ lying in L. (Here $(w_1 \ldots w_r)_p$ is the quotient word.) Since L' is the language accepted by $A_{\geq \alpha}$, (2) holds if and only if $w_1 \ldots w_r \in L'$. □

Therefore, by Theorem 6.6, we have that $B_{(P(\ell_1),\ldots,P(\ell_r))}(kI_u)$ is exact. Summing over u, we see that $B_{(P(\ell_1),\ldots,P(\ell_r))}(kT_{x \sqcup -})$ is exact, completing the proof. □

The remainder of the proof of Theorem 1.2 is a dévissage.

Proof of Theorem 1.2. Let M be an $\boldsymbol{FS}^{\mathrm{op}}$ module that is a subquotient of one generated in degree $\leq d$. By Proposition 7.2 it suffices to show that there is an $s \in \mathbb{N}$ such that the complex

$$\mathrm{K}_{\ell_1} \circ \cdots \circ \mathrm{K}_{\ell_r}(\pi^* M)$$

is exact for all $r \geq s$ and choices of $\ell_t \geq d$ for $t = 1, \ldots, r-1$ and $\ell_r \geq d+1$.

Embed M into an $\boldsymbol{FS}^{\mathrm{op}}$ module N which is finitely generated in degree $\leq d$. By Proposition 7.3, $\pi^* N$ is also generated in degree $\leq d$. Choose a surjection $p : G \twoheadrightarrow \pi^* N$, where $G = \bigoplus_{i=1}^R \boldsymbol{P}(n_i)$ for $\{n_i\}_{i=1}^R$ a finite list of natural numbers $n_i \leq d$. We obtain the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & \pi^* N & \xleftarrow{\ p\ } & G & \longleftarrow & \ker p & \longleftarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & \pi^* M & \longleftarrow & J & \longleftarrow & J' & \longleftarrow & 0
\end{array}
$$

Here $J = p^{-1}(\pi^* M)$ and $J' = \ker p \cap p^{-1}(\pi^* M)$. If $\mathcal{B}$ is any exact functor from $\boldsymbol{OS}^{\mathrm{op}}$ modules to chain complexes, we have that $\mathcal{B}(\pi^* M)$ is exact if both $\mathcal{B}(J)$ and $\mathcal{B}(J')$ are. Now filter G by defining $F_I G = \bigoplus_{i=1}^I \boldsymbol{P}(n_i)$. Intersecting with J and J' we obtain filtrations $F_I J$ and $F_I J'$. To show that $\mathcal{B}(\pi^* M)$ is exact, it suffices to show that the complexes obtained by applying $\mathcal{B}$ to the associated graded modules $F_I J / F_{I-1} J$ and $F_I J'/F_{I-1} J'$ are exact. These associated graded modules are submodules of $F_I G / F_{I-1} G = \boldsymbol{P}(n_I)$. By Proposition 7.4 there exists an $s_I \in \mathbb{N}$ such that for all $r \geq s_I$ and $(\ell_t)_{t=1}^r \in \mathbb{N}^s$ with $\ell_t \geq n_I$ and $\ell_r \geq n_I + 1$, the complex $\mathrm{K}_{(\ell_1,\ldots,\ell_r)}(F_I J / F_{I-1} J)$ is exact. Identically, there exists an $s_I' \in \mathbb{N}$ such that for all $r' \geq s_I'$ and $(\ell_t')_{t=1}^{r'} \in \mathbb{N}^{s_I'}$ with $\ell_t' \geq n_I$ and $\ell_{r'}' \geq n_I + 1$, the complex $\mathrm{K}_{(\ell_1,\ldots,\ell_{r'})}(F_I J'/F_{I-1} J')$ is exact. Take s to be $\max(\bigcup_I \{s_I, s_I'\})$. Then we conclude that $\mathrm{K}_{(\ell_1,\ldots,\ell_r)}(M)$ is exact for all choices of $r \geq s$, $\ell_t \geq d$ and $\ell_r \geq d+1$. □

Part II. Characters of $\boldsymbol{FS}^{\mathrm{op}}$ modules

8. The character of $\mathrm{K}_d(M)$

Let M be a chain complex of $\boldsymbol{FS}^{\mathrm{op}}$ modules over a commutative ring R. The purpose of this section

is, when R is a field of characteristic zero, to compute the virtual $S_d \times S_n$ representation associated to $K_d(M)_n$, in terms of certain differential operators on the ring of symmetric functions.

Definition 8.1. We write $S_\bullet$ to denote the groupoid $\bigsqcup_{n \in \mathbb{N}} S_n$. Any *FS*ᵒᵖ module M determines a representation of $S_\bullet$ by $n \mapsto M_{[n]}$.

The category $\text{Ch}(\text{Mod } R)$ of chain complexes of R modules is symmetric monoidal, using the Koszul braiding. Thus there is a symmetric monoidal induction product

$$\circledast : \text{Ch}(\text{Mod } R)^{S_\bullet} \times \text{Ch}(\text{Mod } R)^{S_\bullet} \to \text{Ch}(\text{Mod } R)^{S_\bullet},$$

constructed by tensoring and applying the adjoint to the restriction functor

$$\text{Res}^{\sqcup} : \text{Ch}(\text{Mod } R)^{S_\bullet \times S_\bullet} \leftarrow \text{Ch}(\text{Mod } R)^{S_\bullet}.$$

Concretely, if M and N are $S_\bullet$ representations, then

$$(M \circledast N)_n = \bigoplus_{i+j=n} \text{Ind}_{S_i \times S_j}^{S_n} M_i \otimes N_j.$$

Finally, there is a composition product

$$\odot : \text{Ch}(\text{Mod } R)^{S_\bullet} \times \text{Ch}(\text{Mod } R)^{S_\bullet} \to \text{Ch}(\text{Mod } R)^{S_\bullet},$$

defined by

$$M \odot N := \bigoplus_r M_r \otimes_{S_r} N^{\circledast r},$$

where S_r acts on $N^{\circledast r}$ in the way induced by the Koszul sign rule.   □

Our first step will be to describe the underlying representation of $K_d(M)$ in terms of these operations and the top Whitney homology of the partition lattice.

Definition 8.2. For b a set of size ≥ 1, let $\mathcal{W}_b$ be the degree $n-1$ homology group of the pair $\big(N(P(b)), Z_{P(b)}\big)$ with coefficients in R. We consider $\mathcal{W}_n := \mathcal{W}_{[n]}$ to be a chain complex concentrated in homological degree $n-1$, so that $\mathcal{W}_n \in \text{Ch}(\text{Mod } R)^{S_n}$. Define $\mathcal{W}_0 = 0$ and $\mathcal{W} := \bigoplus_{n \in \mathbb{N}} \mathcal{W}_n \in \text{Ch}(\text{Mod } R)^{S_\bullet}$.

Definition 8.3. Let M be a $S_\bullet \times S_\bullet$ representation. Then for an $S_\bullet$ representation N we write $M \odot_2 N$ to denote the $S_\bullet \times S_\bullet$ representation

$$\bigoplus_{\{(n,r):r \geq 1\}} M_{(n,r)} \otimes_{S_r} N^{\circledast r},$$

where the subscript 2 indicates that we are performing the composition product with respect to the second factor.

Proposition 8.4. *Let M be a chain complex of *FS*ᵒᵖ modules. The $S_\bullet \times S_\bullet$ complexes $\bigoplus_{d \geq 1} K_d(M)$ and*

$$(\text{Res}^{\sqcup} M) \odot_2 \mathcal{W}$$

have isomorphic underlying graded $S_\bullet \times S_\bullet$ representations.

Proof. First we describe $(\mathrm{Res}^{\sqcup} M) \odot_2 \mathcal{W}$ more concretely. We have an isomorphism of S_d representations

$$(\mathcal{W}^{\circledast r})_d = \bigoplus_{\substack{e_1 + \cdots + e_r = d \\ e_i \geq 1}} \mathrm{Ind}_{S_{e_1} \times \cdots \times S_{e_r}}^{S_d} (\mathcal{W}_{e_1} \otimes \cdots \otimes \mathcal{W}_{e_r}) \cong \bigoplus_{[d] = b_1 \sqcup \cdots \sqcup b_r} \mathcal{W}_{b_1} \otimes \cdots \otimes \mathcal{W}_{b_r},$$

where the second sum is over partitions of $[d]$ into r labeled blocks. To construct this isomorphism, note S_d acts on the set of such ordered partitions, with orbits in bijection with $\{(e_1, \ldots, e_r) \mid e_i \geq 1, \sum_i e_i = e\}$, and each orbit contains a distinguished element with stabilizer $S_{e_1} \times \cdots \times S_{e_r}$. Now, S_r acts on the right by relabeling the blocks and permuting the tensor factors by the Koszul sign rule. And we have that

$$((\mathrm{Res}^{\sqcup} M) \odot_2 \mathcal{W})_{n,d} \cong \bigoplus_r M_{[n] \sqcup [r]} \otimes_{S_r} \left(\bigoplus_{[d] = b_1 \sqcup \cdots \sqcup b_r} \mathcal{W}_{b_1} \otimes \cdots \otimes \mathcal{W}_{b_r} \right).$$

Next, we have that

$$K_d(M)_n = \bigoplus_{p \in P(d)} M_{[n] \sqcup p} \otimes H_{r(p)}(N[p, \hat{1}], Z_{[p, \hat{1}]}; R).$$

Let $p \in P(d)$ be a partition with r blocks. There is an isomorphism of posets

$$\prod_{b \text{ block of } p} P(b) \to [p, \hat{1}],$$

given by disjoint union. Choose an arbitrary labeling of the blocks of p as $b_1, \ldots, b_r$, giving a bijection $\sigma_p : p \to [r]$. Since for any pair of posets P and Q with top and bottom elements, we have that $Z_{P \times Q} = NP \times Z_Q \cup Z_P \times NQ$, and the Künneth formula for pairs gives

$$\mathrm{Kun}_{\sigma_p} : H_{r(p)}(N([p, \hat{1}]), Z_{[p, \hat{1}]}) \cong H_{|b_1|-1}(NP(b_1), Z_{P(b_1)}) \otimes \cdots \otimes H_{|b_r|-1}(NP(b_r), Z_{P(b_r)})$$

$$= \mathcal{W}_{b_1} \otimes \cdots \otimes \mathcal{W}_{b_r}.$$

Here the coefficients are in R, and we have used the fact that by the universal coefficient theorem, $\mathcal{W}_b$ is a free R module. For a different ordering of the blocks, the Künneth isomorphism we obtain differs by a permutation of the tensor factors modified by a sign according to the Koszul sign rule. Then summing the morphism $M_{\mathrm{id}_{[n] \sqcup \sigma_p}} \otimes \mathrm{Kun}_{\sigma_p}$ over all partitions p of $[d]$ with r blocks, we obtain an isomorphism

$$\bigoplus_p M_{[n] \sqcup p} \otimes H_{r(p)}(N([p, \hat{1}]), Z_{[p, \hat{1}]}) \cong M_{[n] \sqcup [r]} \otimes_{S_r} \left(\bigoplus_{[d] = b_1 \sqcup \cdots \sqcup b_r} \mathcal{W}_{b_1} \otimes \cdots \otimes \mathcal{W}_{b_r} \right).$$

This isomorphism does not depend on the choices of σ_p, because any two choices differ by an element of S_r, and in the target we have quotiented by the action of S_r. It is compatible with the action of S_n and S_d, and hence an isomorphism of representations.

To check the compatibility with $\tau \in S_d$, let $p \in P(d)$ and label the blocks of p and $\tau(p)$ such that τ maps the i-th block of p to the i-th block of $\tau(p)$. Then both pairs of maps $M_{\mathrm{id}_{[n] \sqcup \sigma_p}}, M_{\mathrm{id}_{[n] \sqcup \sigma_{\tau(p)}}}$ and $\mathrm{Kun}_{\sigma_p}, \mathrm{Kun}_{\sigma_{\tau(p)}}$ are compatible with the action of τ. $\square$

8A. *Symmetric functions and Frobenius characters.* We now specialize to $R = \mathbb{Q}$. For the remainder of this paper, we will work with **FS**^{op} modules over $\mathbb{Q}$. Since all of the irreducible representations of S_n are defined over $\mathbb{Q}$, all of our arguments and results carry over to **FS**^{op} modules over any field of characteristic zero.

Definition 8.5. Let be Λ be the ring of symmetric functions with $\mathbb{Q}$ coefficients, Λ_n the subspace of symmetric functions of degree n, and $\hat{\Lambda} := \prod_{n \in \mathbb{N}} \Lambda_n$ its completion with respect to the filtration by degree.

We use standard notation for symmetric functions. Let $\lambda = 1^{m_1} 2^{m_2} \ldots$ be an integer partition. We write p_λ for the power-sum monomial $\prod_i p_i^{m_i}$. Similarly e_λ, h_λ, and s_λ, denote respectively the elementary, homogeneous, and Schur symmetric functions corresponding to λ. An element of $f \in \hat{\Lambda}$ admits a unique expansion as a power series with respect to any of these collections of monomials. For instance, we may write

$$f = \sum_\lambda a_\lambda p_\lambda$$

for a unique vector $(a_\lambda) \in \prod_{\lambda \text{ integer partition}} \mathbb{Q}$.

We set $\lambda! := \prod_i m_i!$ and $z_\lambda := \lambda! \prod_i (i)^{m_i}$ and $\text{sgn}(\lambda) = \prod_{i \text{ even}} (-1)^{m_i}$. We define $|\lambda| := \sum_i i m_i$ and $\text{rank}(\lambda) := \sum_i m_i$ and write σ_λ to denote an element of the conjugacy class of $S_{|\lambda|}$ corresponding to λ. Thus $\text{sgn}(\lambda)$ is the trace of σ_λ on the sign representations of $S_{|\lambda|}$.

The Hall inner product on Λ is defined by

$$\langle p_\lambda / z_\lambda, \, p_\nu \rangle = \delta(\lambda, \nu) = \langle s_\lambda, s_\nu \rangle,$$

for $\delta(\lambda, \nu)$ the Kronecker delta. The Hall inner product extends uniquely to a pairing $\langle -, - \rangle : \Lambda \times \hat{\Lambda} \to \mathbb{Q}$, which induces an isomorphism $\hat{\Lambda} \to \Lambda^*$, given by $s \in \hat{\Lambda} \mapsto \langle -, s \rangle$. □

Definition 8.6. Let N be a finite-dimensional S_n representation over $\mathbb{Q}$. Then its *Frobenius character* $\text{ch}(N) \in \Lambda_n$ is defined to be

$$\text{ch}(N) := \sum_{\lambda \vdash n} \text{Tr}(\sigma_\lambda, \text{ch}(N)) \frac{p_\lambda}{z_\lambda} = \sum_{\lambda \vdash n} \text{mult}_\lambda(N) s_\lambda.$$

Let M be a finite-dimensional representation of $S_{n_1} \times \cdots \times S_{n_r}$. Then

$$\text{ch}(M) \in \Lambda_{n_1} \otimes \cdots \otimes \Lambda_{n_r}$$

is uniquely defined by setting $\text{ch}\big(\bigotimes_{i=1}^r (N_i)\big) = \bigotimes_{i=1}^r \text{ch}(N_i)$ and extending bilinearly. For D a bounded finite-dimensional chain complex of $S_{n_1} \times \cdots \times S_{n_r}$ representations, we define

$$\text{ch}(D) := \sum_i (-1)^i \, \text{ch}(H_i(D)) = \sum_i (-1)^i \, \text{ch}(D_i).$$

Finally, if C is a chain complex of finite-dimensional $\big(\bigsqcup_n S_n\big)^r$ representations which is bounded for each $(n_1, \ldots, n_r)$, we define $\text{ch}(C) \in \widehat{\Lambda^{\otimes r}} = \prod_{(n_1, \ldots, n_r)} \Lambda_{n_1} \otimes \cdots \otimes \Lambda_{n_r}$ to be $\sum_{(n_1, \ldots, n_r)} \text{ch}(C_{n_1, \ldots, n_r})$.

The Frobenius character intertwines operations on chain complexes of $S_\bullet$ representations and operations on symmetric functions. We have that $\text{ch}(M \odot N) = \text{ch}(M)[\text{ch}(N)]$, where the square brackets denote plethysm [Stanley 1999, A2.6]. And $\text{ch}(\text{Res}^{\sqcup} M) = \Delta(\text{ch}(M))$, where $\Delta : \hat{\Lambda} \to \widehat{\Lambda^{\otimes 2}}$ is the coproduct, characterized by $\Delta(p_i) = 1 \otimes p_i + p_i \otimes 1$ for $i \in \mathbb{N}$. These compatibilities are standard for ordinary S_n representations: for a discussion of their extension to chain complexes, see for instance [Getzler 1995, §5].

8B. *Character computation.* To translate Proposition 8.4 into a statement about Frobenius characters, we need the following.

Proposition 8.7 [Stanley 1982]. *Let $\mu : \mathbb{N}_{\geq 1} \to \{1, 0, -1\}$ be the arithmetic Möbius function. Then*

$$\text{ch}(\mathcal{W}) = w := \sum_{d \geq 1} \frac{\mu(d)}{d} \log(1 + p_d).$$

The element w is the plethystic inverse of $\sum_{n \geq 1} h_n$.

Definition 8.8. We write ∂_n for the differential operator on $\partial/\partial p_n : \hat{\Lambda} \to \hat{\Lambda}$, given by expressing $s \in \hat{\Lambda}$ as a power series in the p_i and differentiating with respect to p_n. We define

$$D_n := \sum_{d \mid n} \frac{\mu(d)}{d} \partial_{n/d},$$

where $\mu : \mathbb{N} \to \{+1, 0, -1\}$ is the arithmetic Möbius function. To an integer partition $\lambda = 1^{m_1} 2^{m_2} \ldots$, we associate a differential operator, which is a polynomial in the D_i defined by

$$\binom{D}{\lambda} := \prod_i \binom{D_i}{m_i}.$$

The main theorem of this section describes $\text{ch}(K_d(M))$ in terms of $\text{ch}(M)$ and the differential operators $\binom{D}{\lambda}$.

Theorem 8.9. *Let M be a chain complex of $\textbf{FS}^{\text{op}}$ modules which is bounded and finite-dimensional in each degree. Let $d \in \mathbb{N}$ such that $d \geq 1$. Then there is an identity in $\widehat{\Lambda \otimes \Lambda}$,*

$$\text{ch}(K_d(M)) = \sum_{\lambda \vdash d} \binom{D}{\lambda} \text{ch}(M) \otimes p_\lambda.$$

Proof. In terms of symmetric functions, Proposition 8.4 implies that

$$\sum_d \text{ch}(K_d(M)) = \Delta(\text{ch}(M))[w]_2.$$

The brackets $[-]_2$ denote plethysm in the second factor.

To calculate this expression, note that since $\Delta(p_i) = p_i \otimes 1 + 1 \otimes p_i$ for any $s \in \hat{\Lambda}$, expanding s as a power series in the p_λ we have

$$\Delta(s) = \sum_{\lambda \text{ integer partition}} \frac{\partial_\lambda(s)}{\lambda!} \otimes p_\lambda,$$

where if $\lambda = 1^{m_1} 2^{m_2} \ldots$, then $\partial_\lambda := \prod_i \partial_i^{m_i}$. Therefore

$$\Delta(\mathrm{ch}(M)) = \sum_{\lambda \text{ integer partition}} \frac{\partial_\lambda (\mathrm{ch}(M))}{\lambda!} \otimes p_\lambda.$$

Since $p_\lambda[w] = \prod_i (w[p_i])^{m_i}$ and $w[p_i] = \sum_{d \geq 1} (\mu(d)/d) \log(1 + p_{id})$ we have

$$\sum_{d \geq 1} \mathrm{ch}(\mathrm{K}_d(M)) = \sum_{\lambda = 1^{m_1} 2^{m_2} \ldots} \prod_{i \geq 1} \frac{\partial_i^{m_i}(\mathrm{ch}(M))}{m_i!} \otimes \prod_{i \geq 1} \left(\sum_{d \geq 1} \frac{\mu(d)}{d} \log(1 + p_{id}) \right)^{m_i}.$$

To rearrange this sum, we introduce formal variables t_i which commute with p_i and consider the formal expression

$$\sum_{\lambda = 1^{m_1} 2^{m_2} \ldots} \prod_{i \geq 1} \frac{t_i^{m_i}}{m_i!} \left(\sum_{d \geq 1} \frac{\mu(d)}{d} \log(1 + p_{id}) \right)^{m_i} = \exp \left(\sum_i t_i \left(\sum_d \frac{\mu(d)}{d} \log(1 + p_{id}) \right) \right)$$

$$= \exp \left(\sum_{n \geq 1} \log(1 + p_n) \sum_{d \mid n} \frac{\mu(d)}{d} t_{n/d} \right) = \prod_{n \geq 1} (1 + p_n)^{T_n},$$

where $T_n := \sum_{d \mid n} (\mu(d)/d) t_{n/d}$ and the final expression is interpreted as a power series via the binomial theorem $(1 + p_n)^{T_n} = \sum_k \binom{T_n}{k} p_n^k$. Expanding the product, we see that this expression equals

$$\sum_{\lambda = 1^{m_1} 2^{m_2} \ldots} p_\lambda \prod_i \binom{T_i}{m_i}.$$

Now, we may interpret the formal variables t_i (resp. p_i) as the operators $\widehat{\Lambda \otimes \Lambda} \to \widehat{\Lambda \otimes \Lambda}$ defined by $a \otimes b \mapsto \partial_i(a) \otimes b$ (resp. $a \otimes b \mapsto a \otimes p_i b$), because these operators commute with each other. Applying the operator defined by our formal expression in p_i, t_i to $\mathrm{ch}(M) \otimes 1$, we obtain an equality

$$\sum_d \mathrm{ch}(\mathrm{K}_d(M)) = \sum_\lambda \binom{D}{\lambda} \mathrm{ch}(M) \otimes p_\lambda.$$

Equating degree d pieces in the second factor, we obtain the desired result. $\qquad \square$

9. Differential equations and characters of bounded type

In this section, motivated by Theorem 8.9, we first describe the solution space to a system of differential equations involving $\binom{D}{\lambda}$. Then we introduce a notion of *type* of **FS**^{op} module and rephrase the results so far in terms of the type. We also prove Theorem 1.11.

9A. *Solution space to a system differential equations.* It is convenient to introduce generators of $\hat{\Lambda}$ adapted to D_i.

Definition 9.1. For $n \in \mathbb{N}$ define

$$y_n := \sum_{k \geq 1} \frac{p_{nk}}{k} = \sum_i -\log(1 - x_i^n).$$

The elements y_n satisfy several elementary identities: for example, $y_1 = \log\left(\sum_{n \geq 0} h_n\right)$ and $y_n = y_1[p_n]$. The most important one follows from Möbius inversion:

Proposition 9.2. *For all $n, m \geq 1$ we have $D_n(y_m) = \delta(n, m)$, where $\delta(n, m)$ is the Kronecker delta.*

Proof. We have

$$y_m = \sum_{e, \, m \mid e} (m/e) p_e.$$

So

$$D_n(y_m) = \sum_{d \mid n} \frac{\mu(n/d)}{n/d} \partial_d \sum_{e, \, m \mid e} (m/e) p_e = \sum_{e, \, m \mid e \mid n} \frac{\mu(n/e)}{(n/e)} (m/e).$$

Factoring out m/n and applying Möbius inversion for the poset $(\mathbb{N}_{\geq 1}, \mid)$, we see that this sum is 1 if $n = m$ and zero otherwise. $\square$

A similar Möbius inversion to Proposition 9.2 yields

$$p_n = \sum_{d \geq 1} \frac{\mu(d)}{d} y_{nd}.$$

Definition 9.3. Let $(\hat{\Lambda})_i^{\text{pow}}$ denote the closure of the span of the degree i monomials in the power-sum symmetric functions. Thus an arbitrary element of $(\hat{\Lambda})_i^{\text{pow}}$ takes the form

$$\sum_{\substack{(m_1, m_2, \ldots) \\ \sum m_j = i}} a_{(m_1, \ldots)} \prod_j p_j^{m_j},$$

for some choice of $a_{(m_1, \ldots)} \in \mathbb{Q}$. We also write $(\hat{\Lambda})_{\leq i}^{\text{pow}} := \bigoplus_{j \leq i} (\hat{\Lambda})_j^{\text{pow}}$.

Our main theorem of this section describes the space of solutions to a system of differential equations.

Theorem 9.4. *Let $j_1 \geq j_2 \geq \cdots \geq j_r \in \mathbb{N}$. Then the space of solutions in $\hat{\Lambda}$ to the system of linear differential equations*

$$\binom{D}{\lambda_1} \cdots \binom{D}{\lambda_r} s = 0 \quad \text{for all } (\lambda_1, \ldots, \lambda_r) \text{ integer partitions such that } |\lambda_t| \geq j_t \qquad (9.5)$$

is the subspace

$$\sum_{t=1}^r (\hat{\Lambda})_{t-1}^{\text{pow}} \left\{ \exp\left(\sum_{i \geq 1} a_i y_i \right) \, \middle| \, A = 1^{a_1} 2^{a_2} \ldots \text{ and } |A| < j_t \right\}.$$

Example 9.6. Let $r = 2$, $j_1 = 3$ and $j_2 = 2$. Then Theorem 9.4 states that the space of solutions to (9.5) is the closure of the span of

$$\{1, \exp(y_1), \exp(y_2), \exp(2y_1)\} \cup \{p_k, \, p_k \exp(y_1)\}_{k \in \mathbb{N}}.$$

The proof of Theorem 9.4 is straightforward, but occupies some space. We defer the proof until the end of this section, and first explain the consequences of Theorem 9.4.

9B. *Spaces of characters of type $< J$.* To summarize the results so far, we introduce a notion of type for **FS**$^{\mathrm{op}}$ modules.

Definition 9.7. Let M be an **FS**$^{\mathrm{op}}$ module let J be the integer partition $j_1 \geq j_2 \geq \cdots \geq j_r$. We say that M has *type $< J$* if

$$\mathrm{K}_{\ell_r} \circ \cdots \circ \mathrm{K}_{\ell_1}(M)$$

is exact for all choices of $\ell_t \geq j_t$.

Remark 9.8. By Proposition 3.17 and the quasi-isomorphism $\mathrm{K}_d \xrightarrow{\sim} \mathrm{B}_d$, we may equivalently define M to have type $< J$ if $\mathrm{K}_{\ell_1} \circ \cdots \circ \mathrm{K}_{\ell_r}(M)$ is exact for all choices of $\ell_t \geq j_t$. $\qquad\square$

In terms of the type, Theorem 1.2 states that if M is a subquotient of an **FS**$^{\mathrm{op}}$ module generated in degree d, then there exists an $r \geq 0$ such that M has type $< (d+1)d^{r-1}$. (Here $(d+1)d^{r-1}$ is the partition $d+1 \geq d \geq \cdots \geq d$, with d repeated $r-1$ times.) Theorem 8.9 implies that if M has type $< J$, then $s = \mathrm{ch}(M)$ satisfies the system of differential equations (9.5). (Note that the collection of operators $\left\{ \binom{D}{\lambda} \right\}_{\lambda \text{ partition}}$ all commute.) Theorem 9.4 describes the space of solutions to this system of equations, and thus constrains characters of **FS**$^{\mathrm{op}}$ modules of type $< J$. We introduce some terminology in order to restate these results in a convenient way.

Definition 9.9. Let $A = 1^{a_1} 2^{a_2} \ldots$ be an integer partition and $r \in \mathbb{N}$ such that $r > 1$. We define $\mathcal{V}_{A,r} \subseteq \hat{\Lambda}$ to be the subspace

$$\mathcal{V}_{A,r} := \exp\left(\sum_i a_i y_i \right) (\hat{\Lambda})^{\mathrm{pow}}_{<r}.$$

By convention $\mathcal{V}_{A,0} = 0$. Note that $\mathcal{V}_{A,r}$ is 1-dimensional for $r = 1$ and infinite-dimensional for $r > 1$.

Let $J = j_1 \geq \cdots \geq j_r$ be an integer partition. Theorems 8.9 and 9.4 combine to show:

Theorem 9.10. *If M is an **FS**$^{\mathrm{op}}$ module of type $< J$ for $J = j_1 \geq \cdots \geq j_r$, then*

$$\mathrm{ch}(M) \in \bigoplus_{A \text{ integer partition}} \mathcal{V}_{A,t(A,J)} \subseteq \hat{\Lambda},$$

where $t(A, J)$ is the number of $i \in \{1, \ldots, r\}$ such that $j_i > |A|$.

Proof. By Theorem 8.9, if M has type $< J$ then

$$0 = \mathrm{ch}(\mathrm{K}_{\ell_1} \circ \mathrm{K}_{\ell_2} \cdots \circ \mathrm{K}_{\ell_r}(M)) = \sum_{\lambda_1 \vdash \ell_1, \ldots, \lambda_r \vdash \ell_r} \left(\prod_{i=1}^r \binom{D}{\lambda_i} \mathrm{ch}(M) \right) \otimes p_{\lambda_1} \otimes \cdots \otimes p_{\lambda_r},$$

for all $\ell_t \geq j_t$. Since the terms $p_{\lambda_1} \otimes \cdots \otimes p_{\lambda_r}$ are linearly independent, $\mathrm{ch}(M)$ satisfies the equations (9.5). Since $\bigoplus_{A \text{ integer partition}} \mathcal{V}_{A,t(A,J)}$ is simply an alternate description of the solution space of Theorem 9.4, the result follows. $\qquad\square$

In particular, we have the following immediate corollary, in the case $J = d$.

Corollary 9.11. *If M is an $\mathbf{FS}^{\mathrm{op}}$ module and $\mathrm{K}_d(M)$ is exact, then $\mathrm{ch}(M)$ is contained in the finite-dimensional subspace of $\hat{\Lambda}$ spanned by $\exp(\sum_i a_i y_i)$ for all integer partitions $A = 1^{a_1} 2^{a_2} \ldots$ such that $|A| < d$.*

In fact, the statement of Corollary 9.11 is sharp, in the sense that the characters of projective $\mathbf{FS}^{\mathrm{op}}$ modules generated in degree d span the subspace.

In contrast, when $J = j_1 \geq \cdots \geq j_r$ for $r > 1$ the space of solutions to (9.5) is no longer finite-dimensional. For instance, the space $\mathcal{V}_{0,1}$ consists of arbitrary linear combinations of the power-sum symmetric functions.

However, for any $d > 0$ the image of $\mathcal{V}_{A,r}$ under the ring homomorphism

$$\hat{\Lambda} \to \widehat{\mathbb{Q}[p_1, p_2, \ldots, p_d]}$$

setting $p_i = 0$ for $i > d$ is finite-dimensional. In the next section, we combine this fact together with the constraint that the irreducible representations appearing in a finitely generated $\mathbf{FS}^{\mathrm{op}}$ module have bounded rank to produce finite-dimensional spaces of $\mathbf{FS}^{\mathrm{op}}$ characters.

Finally, in the special case $J = d^{s-1}(d+1)^1$, Theorem 9.10 implies Theorem 1.11, as follows.

9C. *From symmetric functions to class functions.* We show how to recover the results of the introduction, stated in terms of class functions, from the results of the body of the paper, stated in terms of symmetric functions.

There is a one-to-one correspondence between elements of $\hat{\Lambda}$ and class functions on $\bigsqcup_n S_n$, under which $\mathrm{ch}(M)$ corresponds to the character of M, $\sigma \mapsto \mathrm{Tr}(\sigma, M)$. If $s \in \hat{\Lambda}$, then s corresponds to the class function $\sigma_\lambda \mapsto \langle p_\lambda, s \rangle$, where $\sigma_{|\lambda|} \in S_{|\lambda|}$ is an element in the conjugacy class of the integer partition λ.

The following fact allows us to translate into the language of the introduction.

Proposition 9.12. *Let $A = 1^{a_1} 2^{a_2} \ldots$ and $\nu = 1^{\ell_1} 2^{\ell_2} \ldots$ integer partitions. Then the symmetric function $(p_\nu / z_\nu) \exp(\sum_i a_i y_i)$ corresponds to the class function*

$$\binom{\mathbb{X}}{\nu} A^{\mathbb{X}-\nu} = \prod_{n \geq 1} \binom{\mathbb{X}_n}{\ell_n} \left(\sum_{d \mid n} d a_d \right)^{\mathbb{X}_n - \ell_n}.$$

Proof. Let λ be an integer partition. We wish to determine $\langle p_\lambda, (p_\nu / z_\nu) \exp(\sum_i a_i y_i) \rangle$, in other words, the coefficient of p_λ / z_λ in the power series expansion of $(p_\nu / z_\nu) \exp(\sum_i a_i y_i)$. To do this, it is convenient to introduce the variables $t_n := p_n / n$, so that if $\lambda = 1^{m_1} 2^{m_2} \ldots$, we have $(p_\lambda / z_\lambda) = \prod_i (t_i^{m_i} / m_i!)$, and

$$\exp\left(\sum_{i \geq 1} a_i y_i \right) = \prod_{n \geq 1} \exp\left(t_n \sum_{d \mid n} d a_d \right) = \left(\prod_{n \geq 1} \sum_{m=0}^{\infty} \frac{(t_n)^m}{m!} \left(\sum_{d \mid n} d a_d \right)^m \right).$$

Now, if $\nu = 1^{\ell_1} 2^{\ell_2} \ldots$, we have

$$\frac{p_\nu}{z_\nu} \exp\left(\sum_i a_i y_i \right) = \prod_{n \geq 1} \left(\sum_{m=0}^{\infty} \frac{t_n^{\ell_n}}{\ell_n!} \frac{(t_n)^m}{m!} \left(\sum_{d \mid n} d a_d \right)^m \right).$$

So the coefficient of $\prod_n (t_n^{m_n}/m_n!)$ is

$$\prod_n \binom{m_n}{\ell_n} \left(\sum_{d \mid n} d a_d \right)^{m_n - \ell_n}. \qquad \square$$

Proof of Theorem 1.11. Let M be an $\boldsymbol{FS}^{\mathrm{op}}$ module of class (d, s). Then, by definition, M has type $< d^{s-1}(d+1)^1$. Thus, by Theorem 9.10,

$$\mathrm{ch}(M) \in \bigoplus_{A \vdash d} \mathcal{V}_{A,1} \oplus \bigoplus_{\substack{A \text{ integer partition} \\ |A| < d}} \mathcal{V}_{A,s}.$$

Concretely, this means that $\mathrm{ch}(M)$ is an infinite sum of terms proportional to $p_\nu \exp(\sum_i a_i y_i)$, with restrictions on the ν and a_i appearing. Using Proposition 9.12 to translate, we obtain Theorem 1.11. $\square$

9D. *Proof of Theorem 9.4.*

Proof of Theorem 9.4. Using the transition matrix between $\{p_i\}$ and $\{y_i\}$, we may write any $s \in \hat{\Lambda}$ as a power series in the variables $\{y_i\}_{i \geq 1}$:

$$s = \sum_{(d_1, d_2, \ldots) \in \mathbb{N}^{\oplus \infty}} c_{d_1, d_2, \ldots} \prod_i y_i^{d_i}.$$

We cut down the space of possible solutions to (9.5) in several steps.

We first note that if s is a solution to (9.5) and $(d_i) \in \mathbb{N}^{\oplus \infty}$ satisfies $\sum_{i \geq j_1} d_i \geq r$ then $c_{d_1, d_2, \ldots} = 0$. To see this apply the operator $D' := \prod_i (D_i^{d_i}/d_i!)$ to s. On the one hand, the constant term of $D'(s)$ is $c_{d_1, d_2, \ldots}$. On the other hand, by our hypothesis on (d_i), we may choose a monomial $y_{i_1} y_{i_2} \cdots y_{i_r}$ dividing $\prod_i y_i^{d_i}$ with $i_t \geq j_1$. Then $D_{i_1} \cdots D_{i_r}$ divides D' and taking $\lambda_t = i_t$ we see that (9.5) implies that $D_{i_1} \cdots D_{i_r}(s) = 0$ and thus $D'(s) = 0$.

Therefore, assuming that s satisfies (9.5), we may write

$$s = \sum_w w f_w(y_1, \ldots, y_{j_1 - 1}), \tag{9.13}$$

where the sum is over monomials w of degree $< d$ in the variables $y_{j_1}, y_{j_1+1}, \ldots$ (Throughout this proof, if $\prod_i y_i^{d_i}$ is a monomial we define its degree to be $\deg(m) := \sum_i d_i$). Now consider the operators $\binom{D_i}{j_1}^r$ for $i = 1, \ldots, j_1 - 1$. Equation (9.5) implies that $\binom{D_i}{j_1}^r(s)$ equals zero. Because D_i only affects the y_i variable,

$$\binom{D_i}{j_1}^r f_w = 0 \quad \text{for all } i = 1, \ldots, j_1 - 1.$$

These $j_1 - 1$ equations have the following consequence.

Lemma 9.14. *The function f_w is a linear combination of functions of the form*

$$\exp\left(\sum_{i=1}^{j_1 - 1} a_i y_i \right) \prod_{i=1}^{j_1 - 1} y_i^{d_i},$$

where $(a_i), (d_i) \in \mathbb{N}^{j_1 - 1}$, and $a_i < j_1$ and $d_i < r$ for all i.

For each $A = 1^{a_1} 2^{a_2} \ldots$, let $\mathcal{V}_{A,r} \subseteq \hat{\Lambda}$ be the subspace of functions of the form

$$\exp\left(\sum_i a_i y_i\right)\left(\sum_m c_m m\right), \tag{9.15}$$

where the sum is over monomials m of degree $< r$ and $c_m \in \mathbb{Q}$. Then (9.13) and Lemma 9.14 imply the crude statement that if s is a solution to (9.5) then

$$s \in \sum_{A=(a_i)\in\{0,\ldots,j_1-1\}^r} \mathcal{V}_{A,r}.$$

Note that the linear independence of finitely many exponentials of distinct orders implies that the right-hand sum is in fact a direct sum. And for any $A = 1^{a_1} 2^{a_2} \ldots$, the action of $\{D_i\}_{i\in\mathbb{N}}$ preserves $\mathcal{V}_{A,r}$. This means that the solution space to the system (9.5) is a sum over $(a_i) \in \{0, \ldots, j-1\}^r$ to the space of solutions in $\mathcal{V}_{A,r}$. We compute these spaces of solutions in the following lemma.

Lemma 9.16. *Let $(a_i) \in \mathbb{N}^{\oplus\infty}$. Let $\sum_m c_m m$, where $c_m \in \mathbb{Q}$, be a sum over monomials in the y_i of bounded degree. By convention we set $j_t = 0$ if $t > r$.*

The symmetric function $s := \exp\left(\sum a_i y_i\right)\left(\sum_m c_m m\right)$ is a solution to (9.5) if and only if for every monomial m appearing in the sum with $c_m \neq 0$ we have $\sum_i i a_i < j_{\deg(m)+1}$.

Assuming the lemma, we have that the space of solutions is spanned by functions of the form $\exp\left(\sum_i a_i y_i\right)\left(\sum_m c_m m\right)$, where the sum is over monomials m in the y_i of degree $t-1$ and $(a_i) \in \mathbb{N}^{\oplus\infty}$ satisfies $\sum_i i a_i < j_t$. Since the change of coordinates between p_i and y_i is linear, this completes the proof. $\square$

Proof of Lemma 9.14. Set $j := j_1$ and $f := f_w$. Note that the space of solutions to the one-variable differential equation $\left(\frac{\partial_u}{j}\right)^r = 0$ in the ring $\mathbb{Q}[[u]]$ is spanned by the functions $u^d \exp(au)$ for $a, d \in \mathbb{N}$, $d < r$ and $a < j$, since $\left(\frac{\partial_u}{j}\right)^r$ is proportional to $\partial_u^r(\partial_u - 1)^r \cdots (\partial_u - j + 1)^r$.

To obtain the multivariable statement from the single-variable one, first expand $f(y_1, \ldots, y_j)$ as $\sum_m m f_m(y_1)$, where m is a monomial in $y_2, \ldots, y_j$. Then each $f_m(y_1)$ satisfies the single-variable equation, and hence is a linear combination of $y_1^{d_1} \exp(a_1 u)$ for $a_1 < j$ and $d_1 < r$. So we can write

$$f = \sum_m m \sum_{a_1,d_1} c_m(a_1, d_1) y_1^{d_1} \exp(a_1 y_1) = \sum_{a_1,d_1} y^{d_1} \exp(a_1 y_1) f_{a_1,d_1}(y_2, \ldots, y_{j-1}),$$

where $c_m(a_1, d_1) \in \mathbb{Q}$ is some choice of coefficients; the sum is over a_1 (resp. d_1) ranging from 0 to $j-1$ (resp. $r-1$); and $f_{a_1,d_1}(y_2, \ldots, y_j) = \sum_m c_m(a_1, d_1) m$. Applying this same process to the variable y_2 in the function f_{a_1,d_1} and continuing on like this, expanding out the resulting sums, we obtain the result. $\square$

Proof of Lemma 9.16. First suppose that there is a monomial m in the sum such that $\sum_i i a_i \geq j_{\deg(m)+1}$. We may assume that m has maximal degree, since any monomial of degree $\geq \deg(m)$ will also satisfy the inequality. Write $m = \prod_i y_i^{d_i}$ and $d = \deg(m)$.

We will choose a sequence of partitions $\lambda_1, \ldots, \lambda_{\max(r,d)}$ such that $|\lambda_t| \geq j_t$ and $\prod_{t=1}^{\max(r,d)} \binom{D}{\lambda_t}$ does not annihilate s. This suffices for the forward direction, because it shows that $\prod_{t=1}^r \binom{D}{\lambda_t}$ does not annihilate s. Let $\nu := 1^{a_1} 2^{a_2} \ldots$, and write $\prod_i y_i^{d_i}$ as $y_{i_1} \cdots y_{i_d}$. For $1 \leq t \leq d$ we choose $\lambda_t = i_t^{\kappa_t}$, where $\kappa_t \in \mathbb{N}$ is

sufficiently large that $\kappa_t > a_{i_t}$ and $\kappa_t i_t \geq j_t$. If $d < t \leq r$, we set $\lambda_t := \nu$. In both cases, by construction we have $|\lambda_t| \geq j_t$, since $|\lambda_t| = \kappa_t i_t \geq j_t$ and $|\lambda_t| = \sum_i i a_i \geq j_{d+1} \geq j_t$ respectively.

That the operator corresponding to this choice of $(\lambda_t)_{t=1}^{\max(r,d)}$ does not annihilate s follows from two claims:

(1) $\binom{D}{\lambda_1} \cdots \binom{D}{\lambda_d}(s)$ is a nonzero scalar times $\exp(\sum_i a_i y_i)$.

(2) $\exp(\sum_i a_i y_i)$ is an eigenvector for $\binom{D}{\nu}$ with nonzero eigenvalue.

To see claim (1), we factor

$$\binom{D}{\lambda_1} \cdots \binom{D}{\lambda_d} = \binom{D_{i_1}}{\kappa_1} \cdots \binom{D_{i_d}}{\kappa_d}$$

as $D' \prod_i (D_i - a_i)^{d_i}$, where D' is a nonzero scalar times a product of terms of the form $(D_i - b_i)$ for $b_i \neq a_i$. Let $n = \prod_i y_i^{e_i}$ be a monomial distinct from m. Then by maximality of the degree of m there exists an $i_0 \geq 1$ such that $e_{i_0} < d_{i_0}$, so the factor $(D_{i_0} - a_{i_0})^{d_{i_0}}$ annihilates $n \exp(\sum_i a_i y_i)$. Therefore we have that $\binom{D}{\lambda_1} \cdots \binom{D}{\lambda_d}(s) = \binom{D}{\lambda_1} \cdots \binom{D}{\lambda_d}(c_m m \exp(\sum_i a_i y_i))$. Further, we have that $(\prod_i (D_i - a_i)^{d_i} m \exp(\sum_i a_i y_i))$ is a nonzero scalar times $\exp(\sum_i a_i y_i)$. And because $(D - b_k) \exp(\sum_i a_i y_i) = (b_k - a_k) \exp(\sum_i a_i y_i)$, we see that $\exp(\sum_i a_i y_i)$ is an eigenvector of D' with nonzero eigenvalue, establishing the first claim.

To see the second claim, notice that $\binom{D}{\nu}$ is a nonzero scalar times a product of operators of the form $(D_i - b_i)$ for $b_i < a_i$.

For the converse, let $\prod_i y_i^{d_i}$ be a monomial of degree d and suppose that $\sum_i i a_i < j_{d+1}$. (Note that this implies $d < r$.) Let $(\lambda_1, \ldots, \lambda_{d+1})$ be a sequence of integer partitions with $|\lambda_t| \geq j_t$. We show that

$$\left(\prod_{t=1}^{d+1} \binom{D}{\lambda_t} \right) \exp\left(\sum_i a_i y_i \right) \left(\prod_i y_i^{d_i} \right) = 0. \tag{9.17}$$

The converse follows from this statement via linearity and the fact that if the product of the first d operators annihilates s then so does the full product.

To see that (9.17) holds, note that for each $\lambda_t = 1^{m_{1,t}} 2^{m_{2,t}} \ldots$, there must exist an i such that $m_{i,t} > a_i$, since otherwise we would have

$$|\lambda_t| = \sum_i i m_{i,t} \leq \sum_i i a_i < j_{d+1} \leq j_t,$$

contradicting the hypothesis on λ_t. For each t choose such an i, and denote it by i_t. Then $\prod_{t=1}^{d+1}(D_{i_t} - a_t)$ is a factor of $(\prod_{t=1}^{d+1} \binom{D}{\lambda_t})$. Because $(D_k - a_k)(m \exp(\sum_i a_i y_i)) = \exp(\sum_i a_i y_i) D_k(m)$, we see that each factor in the product $\prod_{t=1}^{d+1}(D_{i_t} - a_t)$ reduces the degree of the coefficient of $\exp(\sum_i a_i y_i)$ by one. Since there are $d+1$ factors and $\prod_i y_i^{d_i}$ has degree d we have that

$$\left(\prod_{t=1}^{d+1}(D_{i_t} - a_t) \right) \exp\left(\sum_{a_i} y_i \right) \left(\prod_i y_i^{d_i} \right) = 0,$$

finishing the proof of the converse. $\qquad \square$

10. Characters of bounded rank

In Sections 8 and 9, we examined the constraints on the characters of $\mathbf{FS}^{\mathrm{op}}$ modules of type $< J$ for J an integer partition. The purpose of this section is to study an additional constraint on the characters of $\mathbf{FS}^{\mathrm{op}}$ modules.

Definition 10.1. Let M be an $S_\bullet$ module over $\mathbb{Q}$. Let $k \in \mathbb{N}$. We say that $\mathrm{rank}(M) \leq k$ if, in the decomposition of M_n into irreducibles, every irreducible representation appearing corresponds to a Young diagram with $\leq k$ rows. We say that an $\mathbf{FS}^{\mathrm{op}}$ module has rank $\leq k$ when its underlying $S_\bullet$ representation does.

Definition 10.2. We write $\mathcal{F}_{\leq k} \subseteq \hat{\Lambda}$ for the closure of the span of Schur functions s_λ such that the Young diagram of λ has $\leq k$ rows.

Notice that an $S_\bullet$ representation M has rank $\leq k$ if and only if $\mathrm{ch}(M) \in \mathcal{F}_{\leq k}$. The reason that the rank is useful is the following folklore proposition, which appears in [Proudfoot and Young 2017, Theorem 4.1].

Proposition 10.3. *Let M be an $\mathbf{FS}^{\mathrm{op}}$ module over $\mathbb{Q}$. If M is a subquotient of a module generated in degree $\leq d$, then $\mathrm{rank}(M) \leq d$.*

Therefore, by Theorems 1.2 and 9.10, if M is a subquotient of an $\mathbf{FS}^{\mathrm{op}}$ module generated in degree d, there exists an $r \in \mathbb{N}$ such that $\mathrm{ch}(M) \in \hat{\Lambda}$ lies in the subspace

$$\mathcal{F}_{\leq d} \cap \left(\bigoplus_{A \vdash d} \mathcal{V}_{A,1} \oplus \bigoplus_{\substack{A \text{ integer partition} \\ |A| < d}} \mathcal{V}_{A,r} \right).$$

In this section, we prove that this intersection is finite-dimensional, and characterize it using the Hall inner product.

Definition 10.4. Write $\epsilon_k : \hat{\Lambda} \to \widehat{\mathbb{Q}[p_1, \ldots, p_k]}$ for the specialization homomorphism $p_i \mapsto p_i$ for $i \leq k$ and $p_i \mapsto 0$ for $i > k$.

We prove the following theorem over the course of this section.

Theorem 10.5. *Let A_i for $i \in I$ be a finite collection of distinct integer partitions, and let $r_i \in \mathbb{N}$ for $i \in I$ be a finite collection of natural numbers (not necessarily distinct). Let $k \in \mathbb{N}$ and $k \geq \max_i(|A_i|)$. Then*

$$\mathcal{F}_{\leq k} \cap \bigoplus_{i \in I} \mathcal{V}_{A_i, r_i} = \bigoplus_{i \in I} \mathcal{F}_{\leq k} \cap \mathcal{V}_{A_i, r_i}.$$

Further, given any element $s \in \hat{\Lambda}$, there is a unique element $\pi_k(s) \in \mathcal{F}_{\leq k}$ such that

$$\epsilon_k(\pi_k(s)) = \epsilon_k(s).$$

And if $s \in \mathcal{V}_{A_i, r_i}$, then $\pi_k(s) \in \mathcal{V}_{A_i, r_i}$.

Because the space $\epsilon_k(\mathcal{V}_{A,r})$ is finite-dimensional, Theorem 10.5 implies that $\mathcal{F}_{\leq k} \cap \mathcal{V}_{A,r}$ is finite-dimensional, since the elements of this intersection are uniquely determined by their image in $\epsilon_k(\mathcal{V}_{A,r})$. In

particular, the elements $\pi_k\big(p_\nu \exp(\sum_i a_i y_k)\big)$, where p_ν ranges over degree $< r$ monomials in $p_1, \ldots, p_k$, form a basis of $\mathcal{F}_{\leq k} \cap \mathcal{V}_{A,r}$, because $\epsilon_k\big(p_\nu \exp(\sum_i a_i y_k)\big)$ is a basis for $\epsilon_k(\mathcal{V}_{A,r})$.

Under the correspondence between class functions and symmetric functions, the specialization ϵ_k corresponds to restricting the class function to elements $\sigma \in S_n$ whose cycles all have length $\leq k$. So Theorem 10.5 provides a unique lift from restricted class functions to class functions.

Assuming Theorem 10.5, we can prove Theorem 1.13.

Proof of Theorem 1.13. Let $(d, s) \in \mathbb{N}^2$. If M has type (d, s) then by Proposition 10.3 we have that the rank of M is $\leq d$ and M has type $< J$ for $J = d^{s-1}(d + 1)$. Define

$$\mathcal{U}_{d,s} := \bigoplus_{A \vdash d} \mathcal{F}_{\leq d} \cap \mathcal{V}_{A,1} \oplus \bigoplus_{\substack{A \text{ integer partition} \\ |A| < d}} \mathcal{F}_{\leq d} \cap \mathcal{V}_{A,s}.$$

By Theorems 10.5 and 9.10, we have that $\mathrm{ch}(M) \in \mathcal{U}_{d,s}$. Since the elements $\pi_k\big(p_\nu \exp(\sum_i a_i y_k)\big)$, where p_ν ranges over degree $< r$ monomials in $p_1, \ldots, p_k$, form a basis of $\mathcal{F}_{\leq k} \cap \mathcal{V}_{A,r}$, we see that $\mathcal{U}_{d,s}$ has dimension $p(d) + \binom{d+s-1}{s-1} \sum_{i<d} p(i)$, where $p(i)$ is the number of integer partitions of i. $\square$

To prove Theorem 10.5 and describe the projection operator $\pi_k : \hat{\Lambda} \to \mathcal{F}_{\leq k}$, we characterize the spaces $\mathcal{F}_{\leq k}$ and $\mathcal{V}_{A,r}$ using the Hall inner product.

Definition 10.6. If W is a subspace of Λ, we write $W^\perp \subseteq \hat{\Lambda}$ for the *perpendicular space* with respect to the Hall inner product, characterized by the property $t \in W^\perp$ if and only if $\langle w, t \rangle = 0$ for all $w \in W$.

We record the following general fact from linear algebra.

Proposition 10.7. *Let $W \subseteq \Lambda$. The Hall inner product induces an isomorphism $W^\perp \to (\Lambda/W)^*$, by $t \mapsto \langle -, t \rangle$. In particular, there is a unique element of $W^\perp$ corresponding to every linear function $L : \Lambda/W \to \mathbb{Q}$, namely,*

$$\sum_{\lambda \text{ integer partition}} L([p_\lambda]) \, p_\lambda,$$

where $[p_\lambda]$ denotes the equivalence class of p_λ mod W.

The next two propositions characterize $\mathcal{F}_{\leq k}$ and $\mathcal{V}_{A,r}$ as the perpendicular spaces to two different ideals in Λ. The first is certainly known, but we do not have a reference so we include the proof.

Proposition 10.8. *Let $(e_{k+1}, e_{k+2}, \ldots)\Lambda$ be the ideal of Λ generated by e_i for $i > k$. Then*

$$\mathcal{F}_{\leq k} = ((e_{k+1}, e_{k+2}, \ldots)\Lambda)^\perp,$$

with respect to the Hall inner product.

Proof. For λ an integer partition, we write $\mathrm{rank}(\lambda)$ for the number of rows in the Young diagram of λ. Write $t = \sum_\lambda b_\lambda s_\lambda$ for $b_\lambda \in \mathbb{Q}$. Let $w \in \Lambda$ and $i > k$. By [Stanley 1999, 7.15.4],

$$\langle we_i, t \rangle = \sum_\lambda b_\lambda \langle we_i, s_\lambda \rangle = \sum_\lambda b_\lambda \langle w, s_{\lambda/1^i} \rangle.$$

If $t \in \mathcal{F}_{\leq k}$ then $b_\lambda = 0$ for all λ of rank $> k$. By the (transpose of) the Pieri rule [Stanley 1999, 7.15.9], we have that $s_{\lambda/1^i}$ is zero if λ does not have rank $\geq i$. Hence $\langle we_i, t \rangle = 0$, and so $\mathcal{F}_{\leq k} \subseteq (e_{k+1}, \dots)\Lambda^\perp$.

For the converse note that the degree n component of the ideal $(e_{k+1}, \dots)\Lambda$ is spanned by the set of e_λ where $\lambda \vdash n$ is an integer partition with $> k$ columns in its Young diagram. Thus its perpendicular space in $\hat{\Lambda}_n$ has the same dimension as the number of integer partitions of n with $\leq k$ columns. Taking a transpose, we see that this is the same as the dimension of the degree n component of $\mathcal{F}_{\leq k}$. Thus the containment $\mathcal{F}_{\leq k} \subseteq (e_{k+1}, \dots)\Lambda^\perp$ is an equality. $\qquad\square$

Proposition 10.9. *Let $A = 1^{a_1} 2^{a_2} \dots$, and let $r \in \mathbb{N}$ such that $r > 1$. Define $u_n := p_n - \sum_{i|n} i a_i$ for all $n \geq 1$. Then Let $(u_1, u_2, \dots)^r \Lambda$ be the r-th power of the ideal generated by the u_n. Then*

$$\mathcal{V}_{A,r} = ((u_1, u_2, \dots)^r \Lambda)^\perp,$$

with respect to the Hall inner product.

Proof. Any element $t \in \hat{\Lambda}$ can be expanded uniquely as

$$t = \sum_{\lambda \text{ integer partition}} c_\lambda p_\lambda \exp\left(\sum_i a_i y_i\right)$$

for $c_\lambda \in \mathbb{Q}$, because the lowest degree term of $p_\lambda \exp(\sum_i a_i y_i)$ is p_λ. Then $t \in \hat{\Lambda}$ if and only if $c_\lambda = 0$ for all λ whose Young diagrams have $\geq r$ rows.

For $\lambda = 1^{m_1} 2^{m_2} \dots$, put $u_\lambda := \prod_i u_i^{m_i}$.

Lemma 10.10. *We have that*

$$\left\langle \frac{u_\lambda}{z_\lambda}, p_\nu \exp\left(\sum_i a_i y_i\right)\right\rangle = \delta(\lambda, \nu),$$

where $\delta(\lambda, \nu)$ is the Kronecker delta.

Proof. First notice that for any λ and ν we have

$$\langle \partial_i (p_\lambda/z_\lambda), p_\nu \rangle = \langle p_\lambda/z_\lambda, (p_i/i)p_\nu \rangle,$$

using the fact that $\langle p_\lambda/z_\lambda, p_\nu \rangle = \delta(\lambda, \nu)$. By linearity, it follows that the operator ∂_i and multiplication by p_i/i are adjoint. Thus for any pair of symmetric functions $f \in \Lambda$, $g \in \hat{\Lambda}$ we have that

$$\left\langle f, g \exp\left(\sum_{d \geq 1} a_d y_d\right)\right\rangle = \left\langle f, g \exp\left(\sum_{d \geq 1} a_d d \sum_{d|k} p_k/k\right)\right\rangle = \left\langle \exp\left(\sum_d a_d d \sum_{d|k} \partial_k\right) f, g \right\rangle.$$

Taylor expanding with respect to p_k, we see that for any $k \in \mathbb{N}$, the operator $\exp(\partial_k)$ acts on Λ by the ring homomorphism $p_k \mapsto p_k + 1$ and $p_n \mapsto p_n$ for $d \neq k$. Thus we have that $\exp(\sum_d d a_d \sum_k \partial_k)$ acts on Λ by the substitution $p_n \mapsto p_n + \sum_{d|n} d a_d$. In particular, we have that $\exp(\sum_d d a_d \sum_k \partial_k)u_\lambda = p_\lambda$. So

$$\left\langle u_\lambda/z_\lambda, p_\nu \exp\left(\sum_d a_d y_d\right)\right\rangle = \langle p_\lambda/z_\lambda, p_\nu \rangle = \delta(\lambda, \nu). \qquad\square$$

Applying Lemma 10.10 to the expansion of t, we see that $t \in \mathcal{V}_{A,r}$ if and only if $\langle u_\lambda/z_\lambda, t \rangle = 0$ for all $\lambda = 1^{m_1} 2^{m_2} \ldots$ such that $\sum_i m_i \geq r$. Rescaling by z_λ, we see that $t \in \mathcal{V}_{A,r}$ if and only if every degree $\geq r$ monomial in the u_i pairs to zero with it. Taking linear combinations of monomials finishes the proof. $\square$

Since sums of subspaces correspond to intersections of perpendicular spaces, Propositions 10.8 and 10.9 imply:

Proposition 10.11. *Let $A = 1^{a_1} 2^{a_2} \ldots$, let $u_n = p_n - \sum_{i \mid n} i a_i$, and let $r \in \mathbb{N}$. Then*

$$\mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k} = ((u_1, u_2, \ldots)^r + (e_{k+1}, e_{k+2}, \ldots))^\perp.$$

Next we need:

Proposition 10.12. *The ring homomorphism*

$$\iota_k : \mathbb{Q}[p_1, \ldots, p_k] \to \Lambda/(e_{k+1}, \ldots),$$

given by $\iota_k(p_i) := p_i$, is an isomorphism. In particular, the images of $\{p_\lambda/z_\lambda\}_{\lambda = 1^{m_1} \ldots k^{m_k}}$ in $\Lambda/(e_{k+1}, \ldots)$ form a basis. Here λ ranges over partitions whose associated Young diagrams have $\leq k$ columns.

Proof. We have that $\Lambda/(e_{k+1}, \ldots) = \mathbb{Q}[e_1, \ldots, e_k] \cong \mathbb{Q}[x_1, \ldots, x_k]^{S_k}$ is isomorphic to the ring of symmetric functions in k variables. It is well known that the power sums $p_1, \ldots, p_k$ form an independent set of generators of this ring. $\square$

Now Propositions 10.7, 10.8, and 10.12 allow us to define $\pi_k(s)$.

Definition 10.13. Let $s \in \hat{\Lambda}$. We define $\pi_k(s) \in \mathcal{F}_{\leq k}$ to be the element corresponding (as in Proposition 10.7) to the linear function $\Lambda/(e_{k+1}, \ldots) \to \mathbb{Q}$ defined by $p_\lambda/z_\lambda \mapsto \langle p_\lambda/z_\lambda, s \rangle$ for all integer partitions $\lambda = 1^{m_1} \ldots k^{m_k}$ whose Young diagrams have $\leq k$ columns.

Because $\langle p_\lambda/z_\lambda, t \rangle$ is precisely the coefficient of p_λ in the power-sum expansion of t, it is clear that s and $\pi_k(s)$ have the same image under ϵ_k. And by Propositions 10.7, 10.8, and 10.12, $\pi_k(s)$ is the only element of $\mathcal{F}_{\leq k}$ with this property.

To actually compute $\pi_k(s)$ as a sum of power functions, as in Proposition 10.7, we need to know how to write p_ν as a sum of monomials in $p_1, \ldots, p_k$ under the specialization $\Lambda \to \Lambda/(e_{k+1}, \ldots) \cong \mathbb{Q}[x_1, \ldots, x_k]^{S_k}$.

Proposition 10.14. *Let $A = 1^{a_1} 2^{a_2} \ldots$ be an integer partition and $r, k \in \mathbb{N}$, and let u_n denote the difference $p_n - \sum_{d \mid n} d a_d$. Suppose that $k \geq n$. Let $\overline{(u_1, u_2, \ldots)^r}$ be the image of $(u_1, u_2, \ldots)^r$ in $\Lambda/(e_{k+1}, \ldots)$. Then $\iota_k^{-1}(\overline{(u_1, u_2, \ldots)^r}) = (u_1, \ldots, u_k)^r$.*

Proof. It suffices to show that $\iota_k^{-1}(\overline{(u_1, u_2, \ldots)}) = (u_1, \ldots, u_k)$, where $\overline{(u_1, u_2, \ldots)}$ is the image of $(u_1, u_2, \ldots)$ in $\Lambda/(e_{k+1}, \ldots)$.

Now $\iota_k^{-1}(\overline{(u_1, u_2, \ldots)})$ clearly contains $(u_1, \ldots, u_k)$, which is a maximal ideal. So it suffices to prove that $\overline{(u_1, u_2, \ldots)}$ is a proper ideal of $\Lambda/(e_{k+1}, \ldots)$. To do this, we construct a nontrivial homomorphism $\Lambda/(e_{k+1}, \ldots) \to \mathbb{Q}$ under which u_i vanishes.

Consider the $\mathbb{Z}$-set $S := \bigsqcup_{d \geq 1}(\mathbb{Z}/d \times [a_d])$. Linearizing it we obtain an $|A|$-dimensional vector space $\mathbb{Q}S$ and an operator $T : \mathbb{Q}S \to \mathbb{Q}S$ defined to be the action of $+1 \in \mathbb{Z}$. Then there is a specialization homomorphism $\kappa : \Lambda \to \mathbb{Q}$ defined by

$$p_n \mapsto \mathrm{Tr}(T^n, \mathbb{Q}S).$$

Since $\mathbb{Q}S$ has dimension $|A| \leq k$ we have that $\bigwedge^i \mathbb{Q}S = 0$ for all $i > k$. Thus $\kappa(e_i) = \mathrm{Tr}(T, \bigwedge^i \mathbb{Q}S) = 0$ for all $i > k$. So κ factors through $\Lambda/(e_{k+1}, \dots)$. Finally, $\kappa(p_n)$ is the number of fixed points of the action of $+n \in \mathbb{Z}$ on S. If $d \mid n$ then $+n$ fixes all of the points of $\mathbb{Z}/d$, otherwise it fixes none of them. So we see that the number of fixed points of $+n$ is $\sum_{d \mid n} d a_d$. Therefore $\kappa(u_n) = \kappa\big(p_n - \sum_{d \mid n} d a_d\big) = 0$.

Thus κ is the desired homomorphism with kernel $(u_1, u_2, \dots) + (e_{k+1}, \dots)$ and so we are done. $\square$

Remark 10.15. The homomorphism κ constructed in the proof of Proposition 10.14 can also be directly given by $p_n \mapsto p_n(\zeta_1, \zeta_2, \dots, 0, 0, \dots)$, where the ζ_i are the roots of unity obtained by diagonalizing T. $\square$

Proposition 10.14 has the following immediate consequence.

Proposition 10.16. *Let $A = 1^{a_1} 2^{a_2} \dots$, and let $u_n = p_n - \sum_{i \mid n} i a_i$. There is an isomorphism*

$$\frac{\mathbb{Q}[u_1, \dots, u_k]}{(u_1, \dots, u_k)^r} \to \frac{\Lambda}{(u_1, u_2, \dots)^r + (e_{k+1}, e_{k+2}, \dots)}$$

induced by the maps $u_i \mapsto u_i$ for $i = 1, \dots, k$.

Proof. The morphism is surjective because its image equals the image of $\mathbb{Q}[p_1, \dots, p_k]$, and it is injective by Proposition 10.14. $\square$

Proposition 10.17. *Let A be an integer partition such that $|A| \leq k$. If $s \in \mathcal{V}_{A,r}$ then $\pi_k(s) \in \mathcal{V}_{A,r}$.*

Proof. Let $A = 1^{a_1} 2^{a_2} \dots$, and let $u_i = p_i - \sum_i i a_i$. By Proposition 10.9, it suffices to show that $\langle b, \pi_k(s) \rangle = 0$ for every $b \in (u_1, u_2, \dots)^r$. Consider $\iota_k^{-1}(b) \in \mathbb{Q}[p_1, \dots, p_k]$. By Proposition 10.14 it lies in the ideal $(u_1, \dots, u_k)^r$. By Propositions 10.9 and 10.7, it suffices to show that $\langle m, \pi_k(s) \rangle = 0$, where m is any degree $> r$ monomial in $u_1, \dots, u_k$. Because m is a linear combination of monomials in $p_1, \dots, p_k$, by the definition of $\pi_k(m)$ we have that $\langle m, \pi_k(s) \rangle = \langle m, s \rangle = 0$. $\square$

Proof of Theorem 10.5. We have already constructed π_k and shown that it is unique. Proposition 10.17 states that π_k preserves the subspace $\mathcal{V}_{A,r}$ when $k \geq |A|$. It remains to show that $\mathcal{F}_{\leq k} \cap \bigoplus_{i \in I} \mathcal{V}_{A_i,r_i} = \bigoplus_{i \in I} \mathcal{F}_{\leq k} \cap \mathcal{V}_{A_i,r_i}$. Suppose that we have $s_i \in \mathcal{V}_{A_i,r_i}$ such that $\sum_{i \in I} s_i \in \mathcal{F}_{\leq k}$. The operator π_k is projection onto the subspace $\mathcal{F}_{\leq k}$, so we have $\sum_{i \in I} s_i = \pi_k\big(\sum_{i \in I} s_i\big) = \sum_i \pi_k(s_i)$. By linear independence of $\mathcal{V}_{A_i,r_i}$ and the fact that $\pi_k(s_i) \in \mathcal{V}_{A_i,r_i}$, it follows that $\pi_k(s_i) = s_i$ for all i. Thus all of the s_i are in $\mathcal{F}_{\leq k}$. This completes the proof. $\square$

Finally, we include an alternate description of $\mathcal{F}_{\leq k}$, which can be used to compute π_k quickly some cases.

Proposition 10.18. *Let $(\hat{\Lambda})_{\leq k}^{\mathrm{homog}}$ denote the closure of the span of monomials of degree $\leq k$ in the homogenous symmetric functions. Then $\mathcal{F}_{\leq k} = (\hat{\Lambda})_{\leq k}^{\mathrm{homog}}$.*

Proof. We need to show that the closure of the span s_λ, where λ are integer partitions with $\leq d$ rows, equals $(\hat{\Lambda})^{\text{homog}}_{\leq d}$. By the Pieri rule [Stanley 1999, 7.15.9], we have that every monomial h_i of length $\leq d$ is a sum of such s_λ, showing one direction of the containment. To prove equality, observe that in degree n the two subspaces have the same dimension: the number of integer partitions of n with $\leq d$ rows. □

We conclude this section with an example.

Example 10.19. Consider the space $\mathcal{V}_{1,2}$, which is the closure of the span of $\exp(y_1)$ and $\{p_n \exp(y_1)\}_{n \geq 1}$. Then, because

$$\exp(y_1) = \sum_{n \geq 0} h_n,$$

we have $\pi_2(\exp(y_1)) = \exp(y_1)$ and $\pi_2(p_1 \exp(y_1)) = p_1 \exp(y_1)$. We also have that $\pi_2(p_n \exp(y_1)) = 0$ for $n \geq 3$, because $\epsilon_2(p_n \exp(y_1)) = 0$. So the interesting case is $\pi_2(p_2 \exp(y_1))$. Observe that we have

$$\sum_{n \geq 2} p_n \exp(y_1) = \sum_{n \geq 2} n h_n$$

by Newton's identity. Hence this element lies in $\mathcal{F}_{\leq 2}$ and its image under ϵ_2 agrees with $p_2 \exp(y_1)$. So it must be $\pi_2(p_2 \exp(y_1))$.

It is also possible to compute $\pi_2(p_2 \exp(y_1))$ directly from Definition 10.13. We describe a systematic method for computing π_k in Section 11D.

11. Generating functions valued in the dual character space

We begin with a summary of what we know so far. First, Theorems 10.5 and 9.10 immediately imply the following.

Theorem 11.1. *Let $J = j_1 \geq \cdots \geq j_r$ be an integer partition, and let $k \in \mathbb{N}$ satisfy $k \geq j_1$. Then if M is an $\textbf{\textit{FS}}^{\text{op}}$ module of type $< J$ and rank $\leq k$, we have*

$$\text{ch}(M) \in \bigoplus_{A \text{ integer partition}} \mathcal{F}_{\leq k} \cap \mathcal{V}_{A,t(A,J)}, \tag{11.2}$$

where $t(A, J) = \#\{i \in \{1, \ldots, r\} \mid j_i > |A|\}$.

In this section, we study each summand of (11.2) individually. Thus, for the remainder of this section, we fix an integer partition $A = 1^{a_1} 2^{a_2} \ldots$ and $r, k \in \mathbb{N}$ such that $|A| \leq k$. We set $u_n := p_n - \sum_{d \mid n} d a_d$ and $u_\lambda := \prod_i u_i^{m_i}$ for $\lambda = 1^{m_1} \ldots$ an integer partition. We will consider $\mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k}$ and the projection operator $\pi_k : \mathcal{V}_{A,r} \to \mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k}$.

Now, Propositions 10.11 and 10.16 show that there are isomorphisms

$$\mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k} \cong \left(\frac{\Lambda}{(u_1, u_2, \ldots)^r + (e_{k+1}, e_{k+2}, \ldots)} \right)^* \cong \left(\frac{\mathbb{Q}[u_1, \ldots, u_k]}{(u_1, \ldots, u_k)^r} \right)^*, \tag{11.3}$$

which take an element $s \in \mathcal{F}_{\leq k} \cap \mathcal{V}_{a,r}$ to the function $\langle -, s \rangle : \Lambda \to \mathbb{Q}$, and to the restriction of this function to degree $< r$ monomials in $u_1, \ldots, u_k$.

11A. *An interpretation.* We can interpret the isomorphisms of (11.3) in the following way, which we find clarifying. Let $\mathrm{ch}(V) \in \hat{\Lambda}$ be the character of some sequence of S_n representations V. Then elements $m \in \Lambda$ correspond to different *measurements* that we can perform on V, by taking the inner product $\langle m, \mathrm{ch}(V) \rangle$. For example:

- The element $m = p_\lambda$ measures the trace of an element of $S_{|\lambda|}$ with cycle type λ:

$$\langle p_\lambda, \mathrm{ch}(V) \rangle = \mathrm{Tr}(\sigma_\lambda, V_{|\lambda|}).$$

- The element $m = h_n$ measures the dimension of the space of S_n invariants:

$$\langle h_n, \mathrm{ch}(V) \rangle = \dim(V_n)^{S_n}.$$

- The element $m = u_\lambda$ measures the coefficient c_λ in the unique expansion

$$\mathrm{ch}(V) = \sum_{\lambda \text{ integer partition}} c_\lambda \frac{p_\lambda}{z_\lambda} \exp\left(\sum_{i \geq 1} a_i y_i\right),$$

by Lemma 10.10.

In these terms, the first isomorphism states that if $\mathrm{ch}(V) \in \mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k}$ then the value of the measurement $m \in \Lambda$ only depends on its image in the quotient $\Lambda/((u_1, u_2, \dots)^r + (e_{k+1}, e_{k+2}, \dots))$. (Measurements that were independent for arbitrary characters are correlated for elements of $\mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k}$.) And the second isomorphism shows that to determine the value of an arbitrary measurement $m \in \Lambda$ we can proceed in two steps:

(1) Compute the measurements, which are monomials in $u_1, \dots, u_k$.

(2) Express the image of m in the quotient ring as a linear combination of monomials in $u_1, \dots, u_k$.

Step (2) is nontrivial, since we have to actually compute the isomorphism which Proposition 10.16 provides abstractly. One can use Gröbner bases to algorithmically reduce m into a normal form, but this does not yield explicit formulas. The purpose of this section is to introduce a strategy for accomplishing step (2) in the cases $m = u_\lambda$ and $m = h_n$, using generating functions. We use this strategy to construct an explicit basis for $\mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k}$, to describe how to compute π_k, and to prove Theorem 1.17.

11B. *A generating function identity.* We let $R := \Lambda/((u_1, u_2, \dots)^r + (e_{k+1}, e_{k+2}, \dots))$ be the quotient ring, and write $f : \Lambda \to R$ for the quotient homomorphism.

Let λ be an integer partition. We wish to express $f(u_\lambda)$ as a linear combination of $f(u_\nu)$, where $\nu = 1^{\ell_1} \dots k^{\ell_k}$ has $\leq k$ columns (in other words u_ν is a monomial in $u_1, \dots, u_k$). Our strategy is to first compute this expression for $f(u_n)$ and then use the homomorphism property to compute it for $f(u_\lambda)$. Accordingly, we compute an expression for the R-valued generating function

$$\sum_{n \geq 0} \frac{f(u_n)}{n} t^n \in R[\![t]\!].$$

We will also consider generating functions valued in Λ (i.e., elements of $\Lambda[\![t]\!]$). There is a specialization homomorphism $f : \Lambda[\![t]\!] \to R[\![t]\!]$, given by applying $f : \Lambda \to R$ coefficientwise. Now, consider

$$\exp\left(-\sum_n u_n t^n/n\right) \in \Lambda[\![t]\!].$$

We have

$$\exp\left(-\sum_{n\geq 1} u_n t^n/n\right) = \exp\left(-\sum_{n\geq 1} p_n t^n/n\right) \exp\left(\sum_{n\geq 1}\sum_{i\mid n} i a_i t^n/n\right)$$

$$= \exp\left(-\sum_{n\geq 1} p_n t^n/n\right) \exp\left(\sum_{e,i\geq 1} a_i (t^i)^e/e\right)$$

$$= \left(1 + \sum_{n\geq 1} (-1)^n e_n t^n\right) \exp\left(\sum_{i\geq 1} -a_i \log(1 - t^i)\right)$$

$$= \frac{1 + \sum_{n\geq 1}(-1)^n e_n t^n}{\prod_{i\geq 1}(1 - t^i)^{a_i}}, \tag{11.4}$$

where we have used the well-known identity

$$\left(1 + \sum_{n\geq 1}(-1)^n e_n t^n\right) = \exp\left(-\sum_{n\geq 1} p_n t^n/n\right), \tag{11.5}$$

which can be proved by writing $p_n = \sum_j x_j^n$ and using the power series expansion of log.

Applying the specialization homomorphism $f : \Lambda[\![t]\!] \to R[\![t]\!]$, we obtain the identity

$$\exp\left(-\sum_n \frac{f(u_n)}{n} t^n\right) = \frac{1 - f(e_1)t + \cdots + (-1)^k f(e_k)t^k}{\prod_{i\geq 1}(1 - t^i)^{a_i}}.$$

Now we wish to express the numerator in terms of the variables $u_1, \ldots, u_k \in R$. Before doing this, we introduce some notation.

Definition 11.6. For $m \in \mathbb{Z}$, let c_m be the coefficient of t^m in $\prod_{i\geq 1}(1 - t^i)^{a_i}$. Concretely, c_m is an alternating sum of products of binomial coefficients. (We may also interpret it as the trace of an operator constructed as in the proof of Proposition 10.14.)

Definition 11.7. For $m \in \mathbb{N}$ we define $E_m \in \Lambda$ to be $E_m := -(-1)^m e_m + c_m$. Notice that $E_0 = 0$ and $E_m = -(-1)^m e_m$ for $m > |A|$. We extend this definition to integer partitions, by defining $E_\lambda := \prod_i E_i^{m_i}$ if $\lambda = 1^{m_1} 2^{m_2} \ldots$.

By definition, we have that

$$1 + \sum_{n\geq 1}(-1)^n e_n t^n = -\sum_{m\geq 0} E_m t^m + \prod_{i\geq 1}(1 - t^i)^{a_i}.$$

Therefore

$$\exp\left(-\sum_{n\geq 1} u_n t^n/n\right) = 1 - \frac{\sum_{m\geq 1} E_m t^m}{\prod_{i\geq 1}(1 - t^i)^{a_i}}.$$

Proposition 11.8. *We have that*

$$E_m = -\sum_{n=1}^{m}(-1)^n c_{m-n}\sum_{\lambda\vdash n}\mathrm{sgn}(\lambda)\frac{u_\lambda}{z_\lambda}.$$

In particular, $E_m \in (u_1, u_2, \dots)\Lambda$.

Proof. By (11.4), we have

$$\exp\left(-\sum_{n\geq 1}u_n t^n/n\right)\prod_{i\geq 1}(1-t^i)^{a_i} = 1 - \sum_{n\geq 1}(-1)^n e_n t^n.$$

We have the well-known identity

$$\exp\left(-\sum_{n\geq 1}v_n t^n/n\right) = \sum_{n}(-1)^n\sum_{\lambda\vdash n}\mathrm{sgn}(\lambda)\frac{v_\lambda}{z_\lambda},$$

for any family of independent variables v_n. (In particular, for $v_n = p_n$ this identity follows from (11.5) and the character of the sign representation.)

Thus the coefficient of t^m in $\exp\left(-\sum u_n t^n/n\right)\prod_i(1-t^i)^{a_i}$ is

$$c_m + \sum_{n\geq 1}c_{m-n}(-1)^n\sum_{\lambda\vdash n}\mathrm{sgn}(\lambda)\frac{u_\lambda}{z_\lambda}.$$

Subtracting c_m and multiplying by -1, we obtain the result. $\qquad\square$

Since $|A|\leq k$ we have that $E_m = \pm e_m$ for all $m > k$. Hence $f(E_m) = 0$ for all $m \geq k$, and specializing, we obtain

$$\exp\left(-\sum_n f(u_n)t^n/n\right) = 1 - \frac{\sum_{m=1}^{k}f(E_m)t^m}{\prod_{i\geq 1}(1-t^i)^{a_i}}.$$

Therefore

$$\sum_n \frac{f(u_n)}{n}t^n = -\log\left(1 - \frac{\sum_{m=1}^{k}E_m t^m}{\prod_{i\geq 1}(1-t^i)^{a_i}}\right).$$

Because the numerator $\sum_{m=1}^{k}f(E_m)t^m$ is nilpotent of order r, we obtain a finite Taylor expansion

$$\sum_n \frac{f(u_n)}{n}t^n = \frac{\sum_{m=1}^{k}f(E_m)t^m}{\prod_{i\geq 1}(1-t^i)^{a_i}} + \frac{\left(\sum_{m=1}^{k}f(E_m)t^m\right)^2}{2\prod_{i\geq 1}(1-t^i)^{2a_i}} + \cdots + \frac{\left(\sum_{m=1}^{k}f(E_m)t^m\right)^{r-1}}{(r-1)\prod_{i\geq 1}(1-t^i)^{(r-1)a_i}}$$

$$= \sum_{\lambda\in\mathrm{Part}(r,k)}f(E_\lambda)t^{|\lambda|}\frac{(\mathrm{rank}(\lambda)-1)!}{\lambda!\left(\prod_i(1-t^i)^{a_i}\right)^{\mathrm{rank}(\lambda)}}, \qquad (11.9)$$

where the sum in the second line is over all integer partitions whose Young diagrams have $\leq k$ columns and $< r$ rows. In the passage to the second line we have used the multinomial identity

$$\left(\sum_{i=1}^{k}x_i\right)^n = \sum_{\substack{\lambda=1^{m_1}\dots k^{m_k}\\ \sum_i m_i = n}}\binom{n}{m_1,\dots,m_k}x_\lambda.$$

Equation (11.9) is our desired expression for $\sum_n (f(u_n)/n)t^n$. The remaining subsections are independent of each other. In the next two subsections we apply (11.9) to construct a basis for $\mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k}$, then describe how to use (11.9) to compute the operator π_k. After that, we prove Theorem 1.17 using a related identity. Finally, we discuss the relationship between characters of $\textbf{\textit{FS}}^{\text{op}}$ and $\textbf{\textit{FI}}_r$ modules.

11C. *A convenient basis for* $\mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k}$.

Definition 11.10. Let $\text{Part}(r, k)$ be the set of integer partitions whose Young diagrams have $\leq k$ columns and $< r$ rows.

From the point of view of character functions, the elements $\pi_k \big(p_\nu \exp(\sum_i a_i y_k) \big)$ for $\nu \in \text{Part}(r, k)$ are a natural basis for $\mathcal{V}_{A,r}$. However, we do not know a simple expression for these elements. In this subsection we construct a different basis, better adapted to (11.9).

Proposition 11.11. *There is an isomorphism*

$$R \cong \frac{\mathbb{Q}[E_1, \ldots, E_k]}{(E_1, \ldots, E_k)^r},$$

defined by $E_i \mapsto f(E_i)$ *for* $i = 1, \ldots, k$. *Consequently, the set* $\{f(E_\nu)\}_{\nu \in \text{Part}(r,k)}$ *is a basis for* R.

Proof. We already know, by Proposition 10.16, that $u_i \mapsto f(u_i)$ for $i = 1, \ldots, k$ gives an isomorphism $\mathbb{Q}[u_1, \ldots, u_k]/(u_1, \ldots, u_k)^r \to R$. Since $E_m \in (u_1, \ldots)$ we have that $f(s) = 0$ for all $s \in (E_1, \ldots, E_k)^r$, and thus the homomorphism is well defined.

To see that it is surjective, note that, by Proposition 11.8,

$$E_m \equiv u_m/m + \sum_{1 \leq n < m} c_{m-n} u_n/n \quad \mod (u_1, u_2, \ldots)^2.$$

Using this equation, by upper-triangularity we may write any $f(u_m)$ as a linear combination of $f(E_m)$ modulo $(u_1, \ldots, u_k)^2$. Since any collection of elements $x_1, \ldots, x_k \in \mathbb{Q}[u_1, \ldots, u_k]/(u_1, \ldots, u_k)^r$ satisfying $x_i \equiv u_i \mod (u_1, \ldots, u_k)^2$ generate the ring, it follows that these linear combinations of $f(E_m)$ generate it. Thus the homomorphism is surjective, and comparing dimensions, it is also injective. $\qquad\square$

Since by Proposition 11.11 the elements $\{f(E_\nu)\}_{\nu \in \text{Part}(r,k)}$ form a basis of R, we may make the following definition.

Definition 11.12. We define $\{L_\nu\}_{\nu \in \text{Part}(r,k)}$ to be the basis of $\mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k}$ which is dual to $\{f(E_\nu)\}_{\lambda \in \text{Part}(r,k)}$.

Thus L_ν is the unique element of $\mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k}$ satisfying $\langle E_\lambda, L_\nu \rangle = \delta(\lambda, \nu)$ for all integer partitions $\lambda, \nu \in \text{Part}(r, k)$. We will now an explicit expression for L_ν.

Definition 11.13. Let $B = 1^{b_1} 2^{b_2} \ldots$ be an integer partition. For $n \in \mathbb{Z}$ we define $g_B(n)$ be the coefficient of t^n in $\prod_i (1 - t^i)^{-b_i}$. In particular, $g_B(n) = 0$ for $n < 0$. For $n \geq 0$ we have

$$g_B(n) = \sum_{\substack{\lambda \vdash n \\ \lambda = 1^{m_2} 2^{m_2} \ldots}} \prod_{i \geq 1} \left(\binom{b_i}{m_i} \right).$$

The function $g_B(n)$ equals a quasipolynomial of degree $\text{rank}(B)$ for all $n \geq 0$.

Definition 11.14. If $v = 1^{\ell_1} 2^{\ell_2} \ldots$ and $v' = 1^{\ell'_1} 2^{\ell'_2} \ldots$, we write $v' + v$ for the partition $1^{\ell_1 + \ell'_1} 2^{\ell_2 + \ell'_2} \ldots$, making the set of partitions into a commutative monoid (corresponding to multiplication of monomials). For $n \in \mathbb{N}$ write $n \cdot v$ for the n-fold addition of v in this monoid.

Definition 11.15. $\qquad\qquad H_v(n) := n \, g_{\mathrm{rank}(v) \cdot A}(n - |v|)(\mathrm{rank}(v) - 1)!/v!.$

By (11.9), we have that $f(u_n) = \sum_{v \in \mathrm{Part}(r,k), |v| > 0} H_v(n) f(E_v)$. So, for $\lambda = n_1 \geq n_2 \geq \cdots$,

$$f(u_\lambda) = \prod_i \left(\sum_{v \in \mathrm{Part}(r,k)} H_v(n_i) f(E_v) \right) = \sum_{v \in \mathrm{Part}(r,k)} f(E_v) \sum_{\substack{v = \sum_i v_i \\ |v_i| > 0}} \prod_{i \geq 1} H_{v_i}(n_i).$$

Pairing with L_v we get a formula for $\langle u_\lambda, L_v \rangle$. By Lemma 10.10 we obtain:

Proposition 11.16. *There is an explicit formula for L_v,*

$$L_v = \exp\left(\sum_i a_i y_i \right) \sum_{\lambda = n_1 \geq n_2 \geq \cdots} \frac{p_\lambda}{z_\lambda} \sum_{\substack{v = \sum_i v_i \\ |v_i| > 0}} \prod_i H_{v_i}(n_i),$$

where the last sum is over all finite tuples of integer partitions $(v_1, v_2, \ldots, v_l)$ with $|v_i| > 0$ and $v = \sum_i v_i$.

Example 11.17. Let v be the partition $3 \geq 2$. Then L_v is $\exp\left(\sum_i a_i y_i \right)$ times

$$\sum_{n \geq 1} \frac{p_n}{n} H_{3 \geq 2}(n) + \sum_{n > m \geq 1} \frac{p_n p_m}{nm} \left(H_3(n) H_2(m) + H_2(n) H_3(m) \right) + \sum_{n \geq 1} \left(\frac{p_n}{n} \right)^2 H_3(n) H_2(n).$$

11D. *Computing the operator π_k.* Let λ be an integer partition. In this section, we discuss how to compute $\pi_k \left(p_v \exp\left(\sum_i a_i y_k \right) \right)$.

We assume that $p_v \exp\left(\sum_i a_i y_k \right) \in \mathcal{V}_{A,r}$ or, equivalently, that $\mathrm{rank}(v) < r$. If the Young diagram of λ has $> k$ columns, then $\epsilon_k(p_v) = 0$ and thus $\pi_k \left(p_v \exp\left(\sum_i a_i y_k \right) \right) = 0$.

Therefore, suppose that $v \in \mathrm{Part}(r, k)$. By Lemma 10.10 the coefficients $b_{\lambda, v}$ in the expansion

$$\pi_k \left(p_v \exp\left(\sum_i a_i y_i \right) \right) = \sum_\lambda b_{\lambda, v} \frac{p_\lambda}{z_\lambda} \exp\left(\sum_i a_i y_i \right)$$

are given by $\langle u_\lambda, \pi_k \left(p_v \exp\left(\sum_i a_i y_k \right) \right) \rangle$. Let

$$f(u_\lambda) = \sum_{v \in \mathrm{Part}(r,k)} \bar{b}_{\lambda, v} f(u_v) \qquad\qquad\qquad (11.18)$$

be the unique expansion $f(u_\lambda)$ in R. By Proposition 10.11, since $\pi_k \left(p_v \exp\left(\sum_i a_i y_i \right) \right) \in \mathcal{F}_{\leq k} \cap \mathcal{V}_{A,r}$, we have that $\langle u_\lambda, \pi_k \left(p_v \exp\left(\sum_i a_i y_k \right) \right) \rangle$ only depends on $f(u_\lambda)$. And by the definition of π_k, we have that, for every $\psi \in \mathrm{Part}(n, k)$,

$$\left\langle u_\psi, \pi_k \left(p_v \exp\left(\sum_i a_i y_i \right) \right) \right\rangle = \left\langle u_\psi, p_v \exp\left(\sum_i a_i y_i \right) \right\rangle = \delta(\psi, v).$$

Applying this to (11.18) we obtain

$$b_{\lambda,\nu} = \bar{b}_{\lambda,\nu}.$$

The coefficients $\bar{b}_{\lambda,\nu}$ can be computed from $\{\bar{b}_{n,\nu}\}_{n\in\mathbb{N}}$ by multiplication. And (11.9) can be used to compute $\bar{b}_{n,\nu}$. However, the equation for $\bar{b}_{\lambda,\nu}$ one obtains in this way is unwieldy. So we will just carry out this computation in an example.

Example 11.19. Suppose that $k = 2$, $r = 3$, $a_1 = 1$, and $a_i = 0$ for $i > 1$. Then $c_0 = 1$, $c_1 = -1$, and $c_i = 0$ for $i > 1$. So $f(E_1) = -f(u_1)$, $f(E_2) = \frac{1}{2}f(u_2) - \frac{1}{2}f(u_1)^2 + f(u_1)$, and $f(E_i) = 0$ for $i > 3$. Then

$$\frac{\sum_{m=1}^{k} f(E_m)t^m}{\prod_{i\geq 1}(1-t^i)^{a_i}} = -f(u_1)t + \frac{(f(u_2) + f(u_1)^2)t^2}{2(1-t)}.$$

So by (11.9) we compute that $\sum_{n\geq 1}(f(u_n)/n)t^n$ is

$$f(u_1)(-t) + f(u_2)\frac{t^2}{2(1-t)} + f(u_1^2)\left(t^2 + \frac{t^2}{2(1-t)}\right) + f(u_1u_2)\frac{-t^2}{2(1-t)} + f(u_2^2)\frac{t^4}{4(1-t)^2}.$$

Thus for $n \geq 4$ we have that

$$f(u_n) = \frac{1}{2}nf(u_2) + \frac{1}{2}nf(u_1^2) - \frac{1}{2}nf(u_1u_2) + \frac{1}{4}n(n+1)f(u_2^2).$$

This computes $\bar{b}_{n,\nu}$ for all ν and $n \geq 4$. To compute $\bar{b}_{\lambda,\nu}$ for the remaining cases, $\lambda = n \geq m$, simply perform the multiplication $f(u_n)f(u_m)$.

11E. *Proof of Theorem 1.17.* We use a generating function argument to prove Theorem 1.17.

Proof of Theorem 1.17. Fix an integer partition λ and consider the generating function

$$F_\lambda(t) = \sum_{k\geq 0} s_{(k,\lambda)} t^{k+|\lambda|} \in \Lambda[\![t]\!].$$

The following identity appears in the theory of $\textbf{\textit{FI}}$ modules:

$$F_\lambda(t) = q_\lambda(t) + \sum_{\{\nu\,|\,\lambda-\nu\in\text{VS}\}} (-1)^{|\lambda|-|\nu|} s_\nu t^{|\nu|} \sum_{m\geq 0} h_m t^m, \tag{11.20}$$

where the sum is over all ν such that the Young diagram of λ minus the Young diagram of ν is a vertical strip, and $q_\lambda(t) \in \Lambda[t]$ has degree $\leq |\lambda|$ (in both Λ and t). In [Sam and Snowden 2016, §5], these polynomials also go under the name q_λ, although they use the coordinates $t_i := p_i/i$ for the ring of symmetric functions. (They show that this identity corresponds to a resolution in the quotient of Mod $\textbf{\textit{FI}}$ / Mod $\textbf{\textit{FI}}^{\text{tors}}$ and interpret q_λ in terms of local cohomology [loc. cit., §7.4].)

Now fix an integer partition A and $r, k \in \mathbb{N}$ with $k \geq |A|$. We use the notation R and E_λ from above, relative to these choices of parameters. From here on we consider the image of our generating functions under the specialization $\Lambda[\![t]\!] \to R[\![t]\!]$. We have that

$$s_\nu t^{|\nu|} \sum_{m\geq 0} h_m t^m = \frac{s_\nu t^{|\nu|}}{1 + \sum_{m\geq 1}(-1)^m e_m t^m} = \frac{s_\nu t^{|\nu|}}{\prod_i(1-t^i)^{a_i} - \sum_{m=1}^{k} E_m t^m}.$$

Factoring out $\prod_i (1 - t^i)^{a_i}$ and applying a geometric series expansion we have that this equals

$$\frac{s_v t^{|v|}}{\prod_i (1 - t^i)^{a_i}} \sum_{n=0}^{r-1} \left(\frac{\sum_{m=1}^{k} E_m t^m}{\prod_i (1 - t^i)^{a_i}} \right)^n.$$

For any $u \in \mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k}$, applying $\langle -, u \rangle$ to this expression produces a rational function in $\mathbb{Z}[\![t]\!]$. Thus $\sum_{m \geq 0} \langle s_v h_m, u \rangle t^{m+|v|}$ is rational and therefore so is $\sum_{m \geq |v|} \langle s_{(m,v)}, u \rangle t^{m+|v|}$ by (11.20). The denominator of this function is of degree $\leq r|A|$ and its roots are roots of unity of order $\leq |A|$.

Finally, if M is an $\boldsymbol{FS}^{\mathrm{op}}$ module of type (d, s), then $\mathrm{ch}(M)$ is a finite sum of $u_{A,r,k} \in \mathcal{V}_{A,r} \cap \mathcal{F}_{\leq k}$ for $|A| \leq d$ and $r \leq s$. So the generating function $\sum_{m \geq |v|} \langle s_{(m,v)}, \mathrm{ch}(M) \rangle t^{m+|v|}$ is rational and its denominator has degree $\leq sd$, and the roots of the denominator are roots of unity of order $\leq d$. $\qquad \square$

Remark 11.21. In practice, the strategy of the proof of Theorem 1.17 can be used to make computations. For example, fix A an integer partition and $d, k \in \mathbb{N}$ with $d > |A|$. Then the proof of Theorem 1.17 yields a formula for $\sum_n \langle h_n, L_\lambda \rangle t^n$. Namely, it is

$$\sum_n \langle h_n, L_\lambda \rangle t^n = \frac{t^{|\lambda|} \, \mathrm{rank}(\lambda)!}{\lambda! \left(\prod_i (1 - t^i)^{a_i} \right)^{\mathrm{rank}(\lambda)+1}}. \qquad \square$$

11F. _Relationship with FI_d module characters._ Sam and Snowden [2018] characterized the Hilbert series of modules over the category $\boldsymbol{FI}_d$. They express these characters as polynomials in the elements $\sigma_n := \sum_{k \geq 0} \binom{n}{k} h_n$, and observe that σ_n takes the form $q_n(T_1, T_2, \dots) \exp(y_1)$, where q_n is a polynomial in $T_k := \sum_{n \geq 0} \binom{n}{k}(p_n/n)$. In fact, q_n is the n-th complete ordinary Bell polynomial. For $\lambda = 1^{m_1} 2^{m_2} \dots$, define $\sigma^\lambda := \prod_i \sigma_i^{m_i}$. Then if $k \in \mathbb{N}$ and λ satisfies $\mathrm{rank}(\lambda) \leq k$ we have that

$$\sigma^\lambda \sigma_0^{k-\mathrm{rank}(\lambda)} \in \mathcal{F}_{\leq k} \cap \mathcal{V}_{1^k, |\lambda|+1}.$$

Sam and Snowden show that these elements are linearly independent. Thus as λ ranges over partitions with $\leq k$ rows and $|\lambda| < r$, we obtain a collection of linearly independent elements in $\mathcal{F}_{\leq k} \cap \mathcal{V}_{1^k, r}$. Since the dimension of this space is the number of partitions with $\leq k$ rows and $< r$ columns, which is strictly larger than the number of $\sigma^\lambda \sigma_0^{k-\mathrm{rank}(\lambda)}$, this collection of elements is not spanning.

Sam and Snowden compute the character of an $\boldsymbol{FI}_d$ module to be a sum of elements $s_v \sigma^\lambda \sigma_0^{k-\mathrm{rank}(\lambda)}$. We do not know if it is possible to use such elements to construct a basis of $\mathcal{F}_{\leq k'} \cap \mathcal{V}_{1^k, r'}$ for some r' and k'.

12. Further questions

We list some questions related to our work. Many of these questions stem from the fact that we still do not understand the representation theory of $\boldsymbol{FS}^{\mathrm{op}}$ modules.

12A. _Behavior of the type under kernels and cokernels._ For applications, it would be useful to have more sophisticated methods of proving that a specific $\boldsymbol{FS}^{\mathrm{op}}$ module is of class (d, s).

There are two numerical invariants associated to an $\boldsymbol{FI}$ module M, the stable degree $\delta(M)$ and the local degree $h^{\max}(M)$, introduced and used to great effect in [Church et al. 2018]. The type of an $\boldsymbol{FS}^{\mathrm{op}}$

module is roughly analogous to these invariants. Given a morphism of $\textbf{\textit{FS}}^{\text{op}}$ modules $f : N_1 \to N_2$, is it possible to bound the type of ker f and coker f in terms of the type of N_1 and N_2 (or other invariants)?

12B. *Grothendieck group of FS$^{\textbf{op}}$ modules.* When do two finitely generated $\textbf{\textit{FS}}^{\text{op}}$ modules have the same character? In other words, what is the kernel of the map of Grothendieck groups $\text{K}_0(\textbf{\textit{FS}}^{\text{op}}) \to \prod_n \text{K}_0(\text{S}_n)$?

12C. *Are FS$^{\textbf{op}}$ module characters cycle-bounded?* We say that a symmetric function is *cycle-bounded* if it is a linear combination of finitely many functions of the form

$$\exp\left(\sum_{i \geq 1} a_i y_i \right) \sum_{\lambda \text{ integer partition}} c_\lambda p_\lambda,$$

where only finitely many c_λ are nonzero and $(a_1, a_2, \dots) \in \mathbb{N}^{\oplus \infty}$. Is the character of every finitely generated $\textbf{\textit{FS}}^{\text{op}}$ module cycle-bounded? If not, what is an example of a finitely generated $\textbf{\textit{FS}}^{\text{op}}$ module that is not cycle-bounded?

   We note that projective $\textbf{\textit{FS}}^{\text{op}}$ modules are cycle-bounded, and cycle-bounded characters are closed under sums, induction product and Kronecker product. And if t is cycle-bounded, then $s_\lambda[t]$ is also cycle-bounded. On the other hand, Sam and Snowden [2018] show that when $d > 1$ there are $\textbf{\textit{FI}}_d$ modules that are not cycle-bounded.

12D. *Characters of FS$^{\textbf{op}}$ modules in characteristic $> \textbf{0}$.* What are the consequences of Theorem 1.2 for fields of positive characteristic? Is it possible to carry out a similar combinatorial analysis for the characters of $\textbf{\textit{FS}}^{\text{op}}$ modules in this case?

12E. *Other combinatorial categories.* The arguments we use to prove Theorem 1.2 are mostly general. The specific combinatorics of $\textbf{\textit{FS}}^{\text{op}}$ only enters in Section 6. We are interested in extensions of the method of proof of Theorem 1.2 to other categories studied by Sam and Snowden [2017]. See for instance Conjecture 3.22. For which combinatorial categories does the method of proof of Theorem 1.2 work?

Acknowledgements

References

[Baclawski 1976] K. Baclawski, "Whitney numbers of geometric lattices", pp. 103–116 in *Surveys in applied mathematics* (*Proc. First Los Alamos Sympos. Math. in Natural Sci.*) (Los Alamos, N.M., 1974), edited by N. Metropolis, Academic Press, 1976. MR Zbl

[Baclawski 1980] K. Baclawski, "Cohen–Macaulay ordered sets", *J. Algebra* **63**:1 (1980), 226–258. MR Zbl

[Björner et al. 1982] A. Björner, A. M. Garsia, and R. P. Stanley, "An introduction to Cohen–Macaulay partially ordered sets", pp. 583–615 in *Ordered sets* (Banff, Alta., 1981), edited by I. Rival, NATO Adv. Study Inst. Ser. C: Math. Phys. Sci. **83**, Reidel, Dordrecht, 1982. MR Zbl

[Church and Ellenberg 2017] T. Church and J. S. Ellenberg, "Homology of FI-modules", *Geom. Topol.* **21**:4 (2017), 2373–2418. MR Zbl

[Church and Farb 2013] T. Church and B. Farb, "Representation theory and homological stability", *Adv. Math.* **245** (2013), 250–314. MR Zbl

[Church et al. 2014] T. Church, J. S. Ellenberg, B. Farb, and R. Nagpal, "FI-modules over Noetherian rings", *Geom. Topol.* **18**:5 (2014), 2951–2984. MR Zbl

[Church et al. 2015] T. Church, J. S. Ellenberg, and B. Farb, "FI-modules and stability for representations of symmetric groups", *Duke Math. J.* **164**:9 (2015), 1833–1910. MR Zbl

[Church et al. 2018] T. Church, J. Miller, R. Nagpal, and J. Reinhold, "Linear and quadratic ranges in representation stability", *Adv. Math.* **333** (2018), 1–40. MR Zbl

[Folkman 1966] J. Folkman, "The homology groups of a lattice", *J. Math. Mech.* **15** (1966), 631–636. MR Zbl

[Getzler 1995] E. Getzler, "Operads and moduli spaces of genus 0 Riemann surfaces", pp. 199–230 in *The moduli space of curves* (Texel Island, 1994), edited by R. Dijkgraaf et al., Progr. Math. **129**, Birkhäuser, Boston, 1995. MR Zbl

[Lehrer and Rylands 1993] G. I. Lehrer and L. J. Rylands, "The split building of a reductive group", *Math. Ann.* **296**:4 (1993), 607–624. MR Zbl

[Li 2014] L. Li, "A generalized Koszul theory and its application", *Trans. Amer. Math. Soc.* **366**:2 (2014), 931–977. MR Zbl

[Nagpal 2015] R. Nagpal, *FI-modules and the cohomology of modular representations of symmetric groups*, Ph.D. thesis, University of Wisconsin-Madison, 2015, available at https://www.proquest.com/docview/1681518656. MR

[Nagpal 2019] R. Nagpal, "VI-modules in nondescribing characteristic, I", *Algebra Number Theory* **13**:9 (2019), 2151–2189. MR Zbl

[Patzt 2020] P. Patzt, "Central stability homology", *Math. Z.* **295**:3-4 (2020), 877–916. MR Zbl

[Pirashvili 2000] T. Pirashvili, "Hodge decomposition for higher order Hochschild homology", *Ann. Sci. École Norm. Sup.* (4) **33**:2 (2000), 151–179. MR Zbl

[Proudfoot and Ramos 2021] N. Proudfoot and E. Ramos, "Stability phenomena for resonance arrangements", *Proc. Amer. Math. Soc. Ser. B* **8** (2021), 219–223. MR Zbl

[Proudfoot and Young 2017] N. Proudfoot and B. Young, "Configuration spaces, FS$^{\mathrm{op}}$-modules, and Kazhdan–Lusztig polynomials of braid matroids", *New York J. Math.* **23** (2017), 813–832. MR Zbl

[Putman 2015] A. Putman, "Stability in the homology of congruence subgroups", *Invent. Math.* **202**:3 (2015), 987–1027. MR Zbl

[Putman and Sam 2017] A. Putman and S. V. Sam, "Representation stability and finite linear groups", *Duke Math. J.* **166**:13 (2017), 2521–2598. MR Zbl

[Rota 1964] G.-C. Rota, "On the foundations of combinatorial theory, I: Theory of Möbius functions", *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **2** (1964), 340–368. MR Zbl

[Sam and Snowden 2016] S. V. Sam and A. Snowden, "GL-equivariant modules over polynomial rings in infinitely many variables", *Trans. Amer. Math. Soc.* **368**:2 (2016), 1097–1158. MR Zbl

[Sam and Snowden 2017] S. V. Sam and A. Snowden, "Gröbner methods for representations of combinatorial categories", *J. Amer. Math. Soc.* **30**:1 (2017), 159–203. MR Zbl

[Sam and Snowden 2018] S. V. Sam and A. Snowden, "Hilbert series for twisted commutative algebras", *Algebr. Comb.* **1**:1 (2018), 147–172. MR Zbl

[Snowden 2013] A. Snowden, "Syzygies of Segre embeddings and Δ-modules", *Duke Math. J.* **162**:2 (2013), 225–277. MR Zbl

[Stanley 1982] R. P. Stanley, "Some aspects of groups acting on finite posets", *J. Combin. Theory Ser. A* **32**:2 (1982), 132–161. MR Zbl

[Stanley 1999] R. P. Stanley, *Enumerative combinatorics, II*, Cambridge Studies in Advanced Mathematics **62**, Cambridge Univ. Press, 1999. MR Zbl

[Tosteson 2021] P. Tosteson, "Stability in the homology of Deligne–Mumford compactifications", *Compos. Math.* **157**:12 (2021), 2635–2656. MR Zbl

[Wachs 2007] M. L. Wachs, "Poset topology: tools and applications", pp. 497–615 in *Geometric combinatorics*, edited by E. Miller et al., IAS/Park City Math. Ser. **13**, Amer. Math. Soc., Providence, RI, 2007. MR Zbl

[Weibel 1994] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics **38**, Cambridge Univ. Press, 1994. MR Zbl

[Woodcock 1998] D. Woodcock, "Cohen–Macaulay complexes and Koszul rings", *J. London Math. Soc.* (2) **57**:2 (1998), 398–410. MR Zbl

ptoste@math.uchicago.edu   *Department of Mathematics, University of Chicago, Chicago, IL, United States*

Algebra & Number Theory

Algebra & Number Theory