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REPRESENTATION STABILITY FOR PURE BRAID GROUP MILNOR FIBERS

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ABSTRACT. We prove a representation stability result for the Milnor fiber associated to the pure braid group. Our result connects previous work of Settepanella to representation stability in the sense of Church–Ellenberg–Farb, answering a question of Denham.

CONTENTS

1. INTRODUCTION

Let $\mathrm{Conf}_n(\mathbb{C})$ denote the configuration space of n points in the complex plane. This configuration space is a hyperplane complement and we will study the homology of the associated Milnor fiber

$$F_n = \left\{ (x_1, \ldots, x_n) \mid \prod_{i<j} (x_i - x_j) = 1 \right\} \subseteq \mathrm{Conf}_n(\mathbb{C}).$$

The Milnor fiber F_n admits two natural group actions: the alternating group A_n acts by permuting the coordinates, and the $\binom{n}{2}$th roots of unity act by multiplying the coordinates. In fact, we can extend these actions to the action of a single group

$$\widehat{\mathrm{S}}_n := \{(\sigma, d) \in \mathrm{S}_n \times \mathbb{Z} \mid d \text{ odd} \iff \mathrm{sgn}\ \sigma = -1\},\ n \geq 2.$$

The element (σ, d) acts by

$$(x_1, \ldots, x_n) \mapsto (\zeta^d_{n(n-1)}\ x_{\sigma(1)}, \ldots, \zeta^d_{n(n-1)}\ x_{\sigma(n)}),$$

where $\zeta_k := \exp(\frac{2\pi i}{k})$ is the distinguished primitive kth root of unity.

In [Set04, Theorem 1.1], Settepanella showed that, for $n \geq 3i - 2$, the action by roots of unity on $H_i(F_n, \mathbb{Q})$ is trivial. This result prompted Graham Denham to

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ask whether the homology of the Milnor fiber exhibits any form of representation stability [Den18, Problem 10]. In this paper, we establish a representation stability result for the homology of F_n which incorporates the action of $\widehat{\mathrm{S}}_n$.

To state this theorem, we use a category $\widehat{\mathrm{FI}}$ which is built out of quotients of $\widehat{\mathrm{S}}_n$ in the same way as the category FI, of finite sets and injections, is built from S_n. We define an action of $\widehat{\mathrm{FI}}$ on $H_*(F_n, \mathbb{Z})$, and prove a finite generation result.

Theorem A. *For all i, the sequence $\{H_i(F_n, \mathbb{Z})\}_n$ is a finitely generated $\widehat{\mathrm{FI}}$-module.*

See Definition 2.2 for a definition of finite generation and see Theorem 3.15 for a version of Theorem A with explicit stability bounds.

This theorem has several consequences. We show that if M_n is a finitely generated $\widehat{\mathrm{FI}}$-module, then for n sufficiently large, the subgroup $\mathbb{Z} \subset \widehat{\mathrm{S}}_n$ acts trivially and hence M_n agrees with a finitely generated FI-module in a stable range. In this way, Theorem A incorporates features of both the phenomenon Settepanella established and representation stability for symmetric group representations.

Theorem B. *For all $n \geq 5 + 11i + 3i^2$, the roots of unity $\mu_{\binom{n}{2}}$ act trivially on $H_i(F_n, \mathbb{Z})$. In this range, $H_i(F_n, \mathbb{Z})$ agrees with a finitely generated* FI*-module.*

In particular, the rational S_n representations $H_i(F_n, \mathbb{Q})$ exhibit representation stability in the sense of Church–Farb [CF13, Definition 1.1] (see Church–Ellenberg–Farb [CEF15, Theorem 1.13]). In [Set04, Theorem 1.2], Settepanella computed the groups $H_i(F_n, \mathbb{Q})$ in a stable range. Using our results, we are able to extend this to an integral calculation.

Theorem C. *For $n \geq 5 + 11i + 3i^2$, there is an S_n-equivariant injection:*

$$H_i(F_n, \mathbb{Z}) \to H_i(\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{Z}).$$

The cokernel agrees with a finitely generated FI*-module consisting of torsion abelian groups.*

In particular, the group $H_i(F_n, \mathbb{Z})$ is noncanonically isomorphic to $H_i(\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{Z})$ in a stable range. The homology of $\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*$ is canonically isomorphic to the homology of the moduli space of genus 0 curves with $n+1$ marked points, $M_{0,n+1}$. The S_n representation $H^i(M_{0,n+1})$ has been calculated by Getzler [Get95, Theorem 5.7]. In §5, we give a self-contained description of the homology.

Our method of proof of Theorem A involves considering highly connected semi-simplicial sets with actions of the groups $\widehat{\mathrm{S}}_n$ and $\pi_1(F_n)$. This is an adaptation of Quillen's approach to proving homological stability. The proof is in the spirit of Putman [Put15] and largely fits into the axiomatic framework of Patzt [Pat19].

Similar theorems are likely true for the Milnor fibers associated to the type B and type D braid groups. Additionally, we expect that the techniques of this paper apply to prove representation stability for homology of the subgroup of surface braid groups with total winding number zero. We will not consider these generalizations here.

1.1. **Description of Stabilization Maps.** The category $\widehat{\mathrm{FI}}$ does not act naturally on the Milnor fiber F_n; we only construct an action of $\widehat{\mathrm{FI}}$ up to homotopy. Our situation is analogous to the action of FI on $H_*(\mathrm{Conf}_n(\mathbb{C}))$ by adding points, where FI only acts on $\mathrm{Conf}_n(\mathbb{C})$ up to homotopy. For this FI action, a representative of the standard injection $[n] \hookrightarrow [n+1]$ is given by a map $\mathrm{Conf}_n(\mathbb{C}) \to \mathrm{Conf}_{n+1}(\mathbb{C})$ that adds an $n+1$st point to the right of the first n points.

Our stabilization map $F_n \to F_{n+1}$ is induced by the map $\mathrm{Conf}_n(\mathbb{C}) \to \mathrm{Conf}_{n+1}(\mathbb{C})$ in the following sense. The Milnor fiber F_n is a $K(\pi,1)$, and we have $\pi_1(F_n) \subset \pi_1(\mathrm{Conf}_n(\mathbb{C}))$. The FI stabilization map $\mathrm{Conf}_n(\mathbb{C}) \to \mathrm{Conf}_{n+1}(\mathbb{C})$ takes $\pi_1(F_n) \to \pi_1(F_{n+1})$, and we define the action of $\widehat{\mathrm{FI}}$ so that $[e] \in \widehat{\mathrm{S}}_{n+1}/\widehat{\mathrm{S}}_n = \widehat{\mathrm{FI}}(n, n+1)$ acts on π_1 by this inclusion. This suffices to determine the action $e : F_n \to F_{n+1}$ up to homotopy.

We have two other, more geometric, descriptions of this stabilization map. To describe the first, we replace the Milnor fiber F_n by the covering space of $\mathrm{Conf}_n(\mathbb{C})$ associated to the inclusion of fundamental groups $\pi_1(F_n) \hookrightarrow \pi_1(\mathrm{Conf}_n(\mathbb{C}))$, F'_n. This space can be described as

$$F'_n = \left\{ (x_i)_{i=1}^n \in \mathrm{Conf}_n\mathbb{C}, z \in \mathbb{C} \mid \prod_{i<j}(x_i - x_j) = \exp(z) \right\},$$

since taking log shows that this is a cover and there is a deformation retraction of the map $F'_n \to \mathrm{Conf}_n\mathbb{C}$ to $F_n \to \mathrm{Conf}_n\mathbb{C}$, given by taking $z \to \lambda z$ and $x_i \to x_i \exp(\lambda/\binom{n}{2})$ for $\lambda \in [0,1]$. There is a unique lift of any choice of stabilization map $\mathrm{Conf}_n(\mathbb{C}) \to \mathrm{Conf}_{n+1}(\mathbb{C})$ to a map of covers $F'_n \to F'_{n+1}$. On homology, this induces the action of $[e] \in \widehat{\mathrm{FI}}(n, n+1)$.

Second, Gadish has described a stabilization map on the Milnor fiber F_n itself, which induces the action of $[e] \in \widehat{\mathrm{FI}}(n, n+1)$ on homology. Gadish's observation is that given a configuration $(x_i)_{i=1}^n$ such that $\prod_{1 \le i < j \le n}(x_j - x_i) = 1$, if we add a point $x_{n+1} \in \mathbb{R} \subset \mathbb{C}$ such that x_{n+1} is $\gg 0$, then the complex number

$$a = \prod_{1 \le i < j \le n+1} (x_j - x_i) = \prod_{i=1}^n (x_{n+1} - x_i)$$

has argument > 0 and we can choose a branch of the function that takes $\binom{n+1}{2}$th roots, and divide each x_i by $a^{1/\binom{n+1}{2}}$ to continuously obtain a point in F_{n+1}.

To formally define Gadish's map $e : F_n \to F_{n+1}$, we fix a branch of the $\binom{n+1}{2}$th root function with branch cut along the negative real axis. Then we define $e(x_1, \ldots, x_n)$ to be $(y_1, \ldots, y_{n+1})$ where $y_i = x_i/a^{1/\binom{n+1}{2}}$, $a = \prod_{i=1}^n (x_{n+1} - x_i)$, and

$$x_{n+1} = 1 + \sum_{i=1}^n \mathrm{re}(x_i) + \sum_{i=1}^n \frac{\mathrm{im}(x_i)}{\tan(\pi/n)}.$$

2. Algebraic preliminaries

In this section, we define $\widehat{\mathrm{FI}}$. We recall some facts concerning the theory of FI-modules and describe their implications for $\widehat{\mathrm{FI}}$-modules.

2.1. $\widehat{\mathrm{FI}}$-modules.

We begin by constructing a monoidal structure on the groupoid

$$\widehat{\mathrm{S}} := \coprod_n \widehat{\mathrm{S}}_n.$$

Here we define $\widehat{\mathrm{S}}_0$ and $\widehat{\mathrm{S}}_1$ to be trivial groups. The monoidal structure is given by the maps

$$m_{n_1,n_2} : \widehat{\mathrm{S}}_{n_1} \times \widehat{\mathrm{S}}_{n_2} \to \widehat{\mathrm{S}}_{n_1+n_2},\ (\sigma_1, d_1) \times (\sigma_2, d_2) \mapsto (\sigma_1\sigma_2, d_1 + d_2).$$

When n_1, n_2 are clear from context, we write $m = m_{n_1,n_2}$. We will write $i_1(\widehat{\mathrm{S}}_{n_1}) \subset \widehat{\mathrm{S}}_{n_1+n_2}$ for $m(\widehat{\mathrm{S}}_{n_1} \times e)$, and $i_2(\widehat{\mathrm{S}}_{n_2})$ for $m(e \times \widehat{\mathrm{S}}_{n_2})$. Since the subgroup $i_r(\widehat{\mathrm{S}}_{n_r})$ is isomorphic to $\widehat{\mathrm{S}}_{n_r}$, when it is clear which embedding we are taking, we suppress i_r from our notation.

The category $\widehat{\mathrm{S}}$ has a braided monoidal structure induced by the surjection $p_n : \mathrm{Br}_n \to \widehat{\mathrm{S}}_n$. More precisely, the braid $\sigma_{a,b}$ which braids the first a strands over the last b strands conjugates $m_{a,b}$ to $m_{a,b}$. See §3.1.1 for the definition of p_n and our conventions on braid groups. Since the maps $i_r : \widehat{\mathrm{S}}_{n_r} \to \widehat{\mathrm{S}}_{n_1+n_2}$ are inclusions, the construction of Randall-Williams Wahl [RWW17, Theorem 1.10] applies to produce a monoidal category $\widehat{\mathrm{FI}} = \mathcal{U}\widehat{\mathrm{S}}$. We will make the definition of $\widehat{\mathrm{FI}}$ and its monoidal structure explicit.

The category $\widehat{\mathrm{FI}}$ has objects indexed by natural numbers, and morphisms given by the right cosets

$$\widehat{\mathrm{FI}}(n, m) = \widehat{\mathrm{S}}_m / i_2(\widehat{\mathrm{S}}_{m-n}).$$

The composition $\widehat{\mathrm{FI}}(n, m) \times \widehat{\mathrm{FI}}(m, l) \to \widehat{\mathrm{FI}}(n, l)$ is given by $[s] \times [t] \mapsto [ti_1(s)]$. It is well defined because elements of $i_2(\widehat{\mathrm{S}}_{l-m})$ commute with $i_1(a)$, and is associative because $ui_1(ti_1(s)) = ui_1(t)i_1(s)$.

The monoidal structure is given on objects by $n_1 \times n_2 \mapsto n_1 + n_2$, and on morphisms by

$$\widehat{\mathrm{FI}}(n_1, m_1) \times \widehat{\mathrm{FI}}(n_2, m_2) \mapsto \widehat{\mathrm{FI}}(n_1 + n_2, m_1 + m_2), [s] \times [t] \mapsto [i_1(s)i_2(t)] \circ [\tau_{m_1-n_1,n_2}],$$

where $\tau_{m_1-n_1,n_2}$ denotes the element of $\mathrm{Br}_{m_1+m_2}$ defined as follows. Writing

$$[m_1 + m_2] = [n_1] \sqcup [m_1 - n_1] \sqcup [n_2] \sqcup [m_2 - n_2],$$

we let $\tau_{m_1-n_1,n_2}$ be the element which braids the strands of $[m_2 - n_2]$ over the strands of $[n_1]$.

Remark 2.1. To obtain the monoidal category $\widehat{\mathrm{FI}}$ as we have defined it from [RWW17, Theorem 1.10], apply their construction to braided monoidal groupoid defined by:

$$\widehat{\mathrm{S}}_a \times \widehat{\mathrm{S}}_b \xrightarrow{\text{switch}} \widehat{\mathrm{S}}_b \times \widehat{\mathrm{S}}_a \xrightarrow{m_{b,a}} \widehat{\mathrm{S}}_{a+b}.$$

Given a category $\mathcal{C}$, the term $\mathcal{C}$-module will mean functor from the category $\mathcal{C}$ to the category of abelian groups. Let $\mathrm{Mod}_{\mathcal{C}}$ denote the category of $\mathcal{C}$-modules. For an $\widehat{\mathrm{FI}}$-module or $\widehat{\mathrm{S}}$-module M and n a natural number, let M_n denote the value of M on n. There is a functor from $\widehat{\mathrm{S}}$ to $\widehat{\mathrm{FI}}$ which identifies $\widehat{\mathrm{S}}$ with the largest subcategory of $\widehat{\mathrm{FI}}$ such that every morphism is invertible. This gives a forgetful functor $\mathrm{Mod}_{\widehat{\mathrm{FI}}} \to \mathrm{Mod}_{\widehat{\mathrm{S}}}$.

Definition 2.2. Let $\mathbf{I} : \mathrm{Mod}_{\widehat{\mathrm{S}}} \to \mathrm{Mod}_{\widehat{\mathrm{FI}}}$ be the left adjoint to the forgetful functor. An $\widehat{\mathrm{FI}}$-module M is called *induced* if $M \cong \mathbf{I}(W)$ for some $\widehat{\mathrm{S}}$-module W. We say M has generation degree $\leq d$ if there is a short exact sequence of $\widehat{\mathrm{FI}}$-modules:

$$\mathbf{I}(W) \to M \to 0$$

with $W_n \cong 0$ for $n > d$. We say M is finitely generated if there is a short exact sequence of $\widehat{\mathrm{FI}}$-modules:

$$\mathbf{I}(W) \to M \to 0$$

with $\bigoplus_n W_n$ a finitely generated abelian group. We say M has presentation degree $\leq r$ if there is a short exact sequence of $\widehat{\mathrm{FI}}$-modules:

$$\mathbf{I}(V) \to \mathbf{I}(W) \to M \to 0$$

with $W_n \cong V_n \cong 0$ for $n > r$.

Note that if each M_n is finitely generated as an abelian group, then M is finitely generated if and only if it has finite generation degree. Many definitions appearing in this paper, including the above definitions, are adaptations of definitions for FI-modules which have appeared in other papers. For the sake of brevity, we will often only state definitions for $\widehat{\mathrm{FI}}$-modules but will often also use the corresponding definition for FI-modules.

2.2. **Central stability homology and regularity.** Central stability homology is a construction which often appears on E^2-pages of spectral sequences used to establish representation stability. When certain semi-simplicial sets are highly connected, central stability homology controls degrees of higher syzygies [Pat19, Theorem 5.7].

Definition 2.3. Let M be an $\widehat{\mathrm{FI}}$-module and n a natural number. For $p \geq -1$, let

$$C_p^{cs,\widehat{\mathrm{FI}}}(M)_n = \mathrm{Ind}_{i_1(\widehat{\mathrm{S}}_{n-(p+1)})}^{\widehat{\mathrm{S}}_n} M_{n-(p+1)}.$$

These groups assemble to form an augmented semi-simplicial $\widehat{\mathrm{FI}}$-module, defined in terms of the following maps.

The $\widehat{\mathrm{FI}}$-module structure of M gives maps $x_n : \mathbb{Z}\widehat{\mathrm{FI}}(n, n+1) \otimes M_n \to M_{n+1}$. The automorphism group $\widehat{\mathrm{FI}}(n, n) = \widehat{\mathrm{S}}_n$ acts on M_n on the left and on $\widehat{\mathrm{FI}}(n, n+1)$ on the right. And the map x_n factors the quotient to yield

$$x_n : \mathrm{Ind}_{i_1(\widehat{\mathrm{S}}_n)}^{\widehat{\mathrm{S}}_{n+1}} M_n = \mathbb{Z}\widehat{\mathrm{FI}}(n, n+1) \otimes_{\mathbb{Z}\widehat{\mathrm{S}}_n} M_n \to M_{n+1}.$$

For a braid $b \in \mathrm{Br}_m$, right multiplication by b gives an automorphism $\mathbb{Z}\widehat{\mathrm{S}}_{n+m} \to \mathbb{Z}\widehat{\mathrm{S}}_{n+m}$ as an $\widehat{\mathrm{S}}_{n+m}, \widehat{\mathrm{S}}_n$ bi-module. There is an induced automorphism of $\mathrm{Ind}_{\widehat{\mathrm{S}}_n}^{\widehat{\mathrm{S}}_{n+m}} M_n$, which we will also denote b. Let $u_i \in \mathrm{Br}_{p+1}$ be the element that braids the ith strand over all the others to the left, $u_i := \sigma_{i-1,i}^{-1}\sigma_{i-2,i-1}^{-1} \cdots \sigma_{1,2}^{-1}$.

The ith face operator $f_i : C_p(M)_n \to C_{p-1}(M)_n$ is given by $f_i = (\mathrm{Ind}_{\widehat{\mathrm{S}}_{n-p}}^{\widehat{\mathrm{S}}_n} x_{n-(p+1)}) \circ u_i$:

$$\begin{aligned} \mathrm{Ind}_{\widehat{\mathrm{S}}_{n-(p+1)}}^{\widehat{\mathrm{S}}_n} M_{n-(p+1)} &\to \mathrm{Ind}_{\widehat{\mathrm{S}}_{n-(p+1)}}^{\widehat{\mathrm{S}}_n} M_{n-(p+1)} \\ &\cong \mathrm{Ind}_{\widehat{\mathrm{S}}_{n-p}}^{\widehat{\mathrm{S}}_n} \mathrm{Ind}_{\widehat{\mathrm{S}}_{n-(p+1)}}^{\widehat{\mathrm{S}}_{n-p}} M_{n-(p+1)} \to \mathrm{Ind}_{\widehat{\mathrm{S}}_{n-p}} \widehat{\mathrm{S}}_n M_{n-p} \end{aligned}$$

These f_i satisfy the semi-simplicial identites. This fact is a consequence of the general definition of central stability chains given in §5.1.

We call the associated chain complex $C_*^{cs,\widehat{\mathrm{FI}}}(M)$ *central stability chains.* We call the homology of this chain complex *central stability homology* and denote it by $H_*^{cs,\widehat{\mathrm{FI}}}(M)$. We note that because induction of representations from a subgroup is exact, the functor $M \mapsto C_*^{cs,\widehat{\mathrm{FI}}}(M)$ is exact. We will use this fact tacitly throughout the paper.

One can define central stability chains and homology for FI-modules by analogous formulas. We will use the notation $C_*^{cs,\mathrm{FI}}(M)$ and $H_*^{cs,\mathrm{FI}}(M)$ respectively for the central stability chains and homology of an FI-module M. The following is [MW19, Corollary 2.25].

Proposition 2.4. *Let M be an induced* FI*-module with generation degree $\leq d$. Then $\left(H_i^{cs,\mathrm{FI}}(M)\right)_n = 0$ for $i \leq n-2-d$.*

We will need a slight generalization of this result to a larger class of FI-modules.

Definition 2.5. An FI-module is called semi-induced if it has a filtration with filtration quotients induced FI-modules.

Corollary 2.6. *Let M be a semi-induced* FI*-module with generation degree $\leq d$. Then $\left(H_i^{cs,\mathrm{FI}}(M)\right)_n \leq n-2-d$.*

Proof. By definition, semi-induced FI-modules have filtrations with filtration quotients induced FI-modules. Central stability chains is an exact functor. The claim follows by induction on this filtration using Proposition 2.4 and the long exact sequence in homology induced by a short exact sequence of chain complexes. □

Remark 2.7. In §5,we give more conceptual definitions the central stability homology chain complex for $\widehat{\mathrm{FI}}$-modules, in terms of (braided) commutative monoids in a braided monoidal category.

Vanishing of central stability homology controls the generation and presentation degree.

Proposition 2.8. *Let M be an $\widehat{\mathrm{FI}}$-module, and let $r \geq d$. Then $H_{-1}^{cs,\widehat{\mathrm{FI}}}(M)_n \cong 0$ for all $n > d$ if and only if M has generation degree $\leq d$. Additionally, $H_{-1}^{cs,\widehat{\mathrm{FI}}}(M)_n \cong 0$ for all $n > d$ and $H_0^{cs,\widehat{\mathrm{FI}}}(M)_n \cong 0$ for all $n > r$ if and only if M has generation degree $\leq d$ and presentation degree $\leq r$.*

We will defer the proof until the next section. For FI-modules, a similar theorem is true.

Proposition 2.9. *Let M be an* FI*-module. Then $H_{-1}^{cs,\mathrm{FI}}(M)_n \cong 0$ for all $n > d$ if and only if M has generation degree $\leq d$. Additionally, $H_{-1}^{cs,\mathrm{FI}}(M)_n \cong H_0^{cs,\mathrm{FI}}(M)_n \cong 0$ for all $n > r$, if and only if M has presentation degree $\leq r$.*

Proof. Let H_i^{FI} denote the *ith* left derived functor of $H_{-1}^{cs,\mathrm{FI}}$. By definition, $H_{-1}^{cs,\mathrm{FI}}(M) \cong H_0^{\mathrm{FI}}(M)$. It follows from the proof of [CMNR18, Proposition 2.4] that $H_0^{cs,\mathrm{FI}}(M)$ vanishes if and only if $H_1^{\mathrm{FI}}(M)$ vanishes. The claim now follows from Church–Ellenberg [CE17, Proposition 4.2]. □

2.3. **Relationship between $\widehat{\mathrm{FI}}$ and FI.** There is a functor $\widehat{\mathrm{FI}} \to \mathrm{FI}$ given by $\widehat{\mathrm{S}}_n/\widehat{\mathrm{S}}_{n-m} \to \mathrm{S}_n \,/\, \mathrm{S}_{n-m}$, which intertwines the monoidal structures on them. This map is an isomorphism $\widehat{\mathrm{FI}}(n,m) \to \mathrm{FI}(n,m)$ whenever $m \geq n+2$. We say that an $\widehat{\mathrm{FI}}$-module M is an FI-module if the functor $M : \widehat{\mathrm{FI}} \to \mathrm{Mod}_{\mathbb{Z}}$ factors through $\widehat{\mathrm{FI}} \to \mathrm{FI}$.

Proposition 2.10. *An* $\widehat{\mathrm{FI}}$*-module* M *is an* FI*-module if and only if* $(e,2)_n \in \widehat{\mathrm{FI}}(n,n)$ *acts trivially on* M_n *for all* n*.*

Proof. Clearly, if M is an FI-module, then $(e,2)_n$ acts trivially on FI. Conversely, we have $\widehat{\mathrm{S}}_n/(e,2)_n \cong \mathrm{S}_n$ and $(e,2)_n\backslash\widehat{\mathrm{FI}}(n,n+1) \cong \mathrm{FI}(n,n+1)$, so that if $(e,2)_n$ acts trivially, the action factors through $\mathrm{FI}(n,n+1)$. □

Definition 2.11. The monoidal structure $-\oplus- : \widehat{\mathrm{FI}}\times\widehat{\mathrm{FI}} \to \widehat{\mathrm{FI}}$ gives rise to a functor $-\oplus 1 : \widehat{\mathrm{FI}} \to \widehat{\mathrm{FI}}$. We define the suspension ΣM to be the restriction of M along $-\oplus 1$. The unique map $0 \to 1 \in \widehat{\mathrm{FI}}(0,1)$, induces a natural transformation $M_n = M_{n\oplus 0} \to M_{n\oplus 1} = \Sigma M$.

Definition 2.12. For M an $\widehat{\mathrm{FI}}$-module, we define ${}_{\geq d}M \subset M$ to be the submodule

$$({}_{\geq d}M)_n = \begin{cases} M_n & \text{if } n \geq d \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2.13. *Let* M *be an* $\widehat{\mathrm{FI}}$*-module generated in degree* $\leq a$*. Then* ${}_{\geq a+2}M$ *and* $\Sigma^a M$ *are* FI*-modules. Let* W *be an* $\widehat{\mathrm{FI}}$*-module with* $\Sigma^n W$ *an* FI*-module. Then* ${}_{\geq n+2}W$ *is an* FI*-module.*

Proof. If $n \geq a+2$, then $(e,2)_n$ acts trivially on M_n, since $(e,2)_n$ acts trivially on $\widehat{\mathrm{FI}}(a,n)$ and the map $\mathbb{Z}\widehat{\mathrm{FI}}(a,n)\otimes M_a \to M_n$ is a surjective map of $\widehat{\mathrm{S}}_n$ modules. Therefore ${}_{\geq a+2}M$ is an FI-module. Also $(e,2)_n$ acts trivially on $\Sigma^a M$ for all $n \geq 2$. So $\Sigma^a M$ is an FI-module.

For the last statement, let $(e,2)_j \in \widehat{\mathrm{S}}_j$ for $j \geq n+2$. Then the element $(e,2)_{j-n}$ acts on $(\Sigma^n M)_{j-n} = M_j$ via its image under $i_1 : \widehat{\mathrm{S}}_{j-n} \to \widehat{\mathrm{S}}_j$, which is $(e,2)_j$. Since $\Sigma^n M$ is an FI module Proposition 2.10 implies the action of $(e,2)_{j-n}$ on $(\Sigma^n M)_{j-n}$ is trivial. Therefore $(e,2)_j$ acts on M_j, so applying Proposition 2.10 again we see that ${}_{\geq n+2}M$ is an FI module. □

We now compare $\widehat{\mathrm{FI}}$-module central stability homology with FI-module central stability homology.

Proposition 2.14. *Let* N *be an* FI*-module. There is a canonical map of central stability complexes* $C_p^{cs,\widehat{\mathrm{FI}}}(N)_n \to C_p^{cs,\mathrm{FI}}(N)_n$ *which induces an isomorphism for* $p \leq n-3$*.*

Proof. When $n-(p+1) \geq 2$, the projection map $\widehat{\mathrm{S}}_n \to \mathrm{S}_n$ gives a canonical bijection $\widehat{\mathrm{S}}_n/\widehat{\mathrm{S}}_{n-(p+1)} \to \mathrm{S}_n/\mathrm{S}_{n-(p+1)}$. The projection map induces $\mathbb{Z}\widehat{\mathrm{S}}_n \otimes_{\widehat{\mathrm{S}}_{n-(p+1)}} N_n \to \mathbb{Z}\,\mathrm{S}_n \otimes_{\mathrm{S}_{n-(p+1)}} N_n$, and writing this map in terms of coset representatives for $\widehat{\mathrm{S}}_n/\widehat{\mathrm{S}}_{n-(p+1)}$, we see that it is an isomorphism. □

Now we return to the proof of Proposition 2.8. The key input is the following Lemma.

Lemma 2.15. *We have that* $H_0^{cs,\widehat{\mathrm{FI}}}(\mathbf{I}(V)) = 0, H_{-1}^{cs,\widehat{\mathrm{FI}}}(\mathbf{I}(V)) = V$ *for all* $\widehat{\mathrm{S}}$ *representations* V*.*

Proof. To simplify notation, we will write H_i for $H_{i-1}^{cs,\widehat{\mathrm{FI}}}$ and C_i for $C_{i-1}^{cs,\widehat{\mathrm{FI}}}$ throughout the proof.

First we show the claim in the case $V = \mathbb{Z}_0$, where $\mathbb{Z}_0$ the $\widehat{\mathrm{S}}$ representation that is $\mathbb{Z}$ in degree 0 and 0 elsewhere. In this case $\mathbf{I}(V)_n = \mathbb{Z}\widehat{\mathrm{FI}}(0,n)$. We need only consider the first three terms of the complex. Except for $n \leq 3$, this complex agrees with the FI central stability complex of $\mathbb{Z}\mathrm{FI}(0,-)$, and hence is exact by a special case of Patzt [Pat19, Theorem 5.7]. Therefore we have $H_0(\mathbf{I}(\mathbb{Z}_0))_n = H_1(\mathbf{I}(\mathbb{Z}_0))_n = 0$ for all $n \geq 3$. It is easy to check directly that $H_0(\mathbf{I}(\mathbb{Z}_0))_0 = \mathbb{Z}$ and $H_1(\mathbf{I}(\mathbb{Z}_0))_0 = H_*(\mathbf{I}(\mathbb{Z}_0))_1 = 0$. In the case $n = 2$ the complex takes the form $\mathbb{Z}\widehat{\mathrm{S}}_2 \to \mathbb{Z}\widehat{\mathrm{S}}_2 \to \mathbb{Z} \to 0$ where the generator $[e] \mapsto [e] - [(\sigma,1)]$. Using the identification $\widehat{\mathrm{S}}_2 \cong \mathbb{Z}$, it is clear that this complex is exact. Finally, in the case $n = 3$, we have

$$\mathbb{Z}\widehat{\mathrm{FI}}(2,3) \to \mathbb{Z}\widehat{\mathrm{FI}}(1,3) = \mathbb{Z}\mathrm{FI}(1,3) \to \mathbb{Z}\widehat{\mathrm{FI}}(0,3) = \mathbb{Z}\mathrm{FI}(0,3) \to 0.$$

Since leftmost map factors through the surjection $\mathbb{Z}\widehat{\mathrm{FI}}(2,3) \to \mathbb{Z}\mathrm{FI}(2,3)$, we may again use the exactness of the FI central stability complex of $\mathbb{Z}\mathrm{FI}(0,-)$ to show exactness. This finishes the case $V = \mathbb{Z}_0$.

To extend to the general case, notice that the complex of $\widehat{\mathrm{S}}$ representations $C_*(\mathbf{I}(V))$ takes the form $V * C_*(\mathbf{I}(\mathbb{Z}_0))$, where $V * M$ denotes the induction product or Day convolution

$$(V * M)_n = \bigoplus_{a+b=n} \mathrm{Ind}_{\widehat{\mathrm{S}}_a \times \widehat{\mathrm{S}}_b}^{\widehat{\mathrm{S}}_n} V_a \otimes M_b.$$

Write C for the complex $C_*(\mathbf{I}(\mathbb{Z}_0))$. Using the first homology spectral sequence for $V *^{\mathbb{L}} C$, we see that $H_0(V *^{\mathbb{L}} C) = H_0(V * C)$ and $H_1(V *^{\mathbb{L}} C)$ has a two-step filtration with graded pieces $H_1(V * C)$ and $H_0(\mathrm{Tor}_1^*(V, C))$. Here Tor_i^* denotes the i^{th} derived functor for $- * -$, which can be computed by resolving either factor. To prove the lemma, it suffices to show that $H_0(V *^{\mathbb{L}} C) = V$ and $H_1(V *^{\mathbb{L}} C) = 0$.

Using the second hyperhomology spectral sequence, we see that $H_1(V *^{\mathbb{L}} C)$ has homology bounded above by $\mathrm{Tor}_1^*(V, H_0(C))$ and $\mathrm{Tor}_0^*(V, H_1(C))$. Both of these groups vanish: in the first case $H_0(C) = \mathbb{Z}_0$ and the functor $V \mapsto V * \mathbb{Z}_0 = V$ is exact, and in the second case because $H_1(C) = 0$. Thus $H_1(V *^{\mathbb{L}} C)$ vanishes. Similarly, the hyperhomology spectral sequence gives us $H_0(V *^{\mathbb{L}} C) = V * H_0(C) = V * \mathbb{Z}_0 = V$, completing the proof. ∎

Proof of Proposition 2.8. Again, to simplify notation, we will write H_i for $H_{i-1}^{cs,\widehat{\mathrm{FI}}}$ and C_i for $C_{i-1}^{cs,\widehat{\mathrm{FI}}}$. Let

$$\mathbf{I}(T) \to \mathbf{I}(W) \to \mathbf{I}(V) \to M \to 0$$

be an exact sequence, which is the beginning of a resolution of M by induced modules. Then by Lemma 2.15, and a hyperhomology spectral sequence we have $H_0(M)$ and $H_1(M)$ are computed by H_1 and H_0 of the complex

$$D := (H_0(\mathbf{I}(T)) \to H_0(\mathbf{I}(W)) \to H_0(\mathbf{I}(V))) = (T \to W \to V).$$

Suppose that $V_n = 0$ for all $n > d$ and $W_n \geq 0$ for all $n > r$. Then $H_0(M)_n = 0$ for all $n > d$ and $H_1(M)_n = 0$ for all $m > r$. Thus, the generation degree of M is at most d and the presentation degree is at most r. This implies the vanishing of homology.

To prove the converse, we use the notion of a minimal surjection. For any $\widehat{\mathrm{FI}}$-module M we say that a surjection $\mathbf{I}(V) \twoheadrightarrow M$ is *minimal* if $V \subset \mathbf{I}(V) \to$

M is an inclusion, and for all n we have that $V_n = H_0(\mathbf{I}(V))_n \to H_0(M)_n = M_n/\widehat{\mathrm{FI}}(n-1,n)M_{n-1}$ is a surjection, and if $H_0(M)_n = 0$ then $V_n = 0$.

For any $\widehat{\mathrm{FI}}$-module M, we construct a minimal surjection $\mathbf{I}(V) \to M$ as follows. Let $V_n \subset M_n$ be an $\widehat{\mathrm{S}}_n$ subrepresentation that surjects onto $H_0(M)_n = M_n/\widehat{\mathrm{FI}}(n, n+1)M_{n-1}$, such that $V_n = 0$ for all n such that $H_0(M)_n = 0$. Then the $\widehat{\mathrm{S}}$ representation V gives a map $\mathbf{I}(V) \to M$. This map is surjective by the graded Nakayama lemma: clearly $\mathbf{I}(V)_0 = V_0 \to M_0 = H_0(M)_0$ is surjective and inductively $\mathbf{I}(V)_{n-1} \twoheadrightarrow M_{n-1}$ implies that $\widehat{\mathrm{FI}}(n-1,n)\mathbf{I}(V)_{n-1} \twoheadrightarrow \widehat{\mathrm{FI}}(n-1,n)M_{n-1}$ and so $V_n \twoheadrightarrow M_n/\widehat{\mathrm{FI}}(n-1,n)M_{n-1}$ implies that $\mathbf{I}(V)_n \twoheadrightarrow M_n$. By construction it is minimal.

Let M be a module such that $H_0(M)_n = 0$ for $n > d$ and $H_1(M)_n = 0$ for $n > r \geq d$. Choose a minimal surjection $p : \mathbf{I}(V) \twoheadrightarrow M$ and a minimal surjection $\mathbf{I}(W) \twoheadrightarrow K := \ker p$, to obtain a presentation $\mathbf{I}(W) \to \mathbf{I}(V) \to M$. By minimality of $\mathbf{I}(V) \twoheadrightarrow M$, V is nonzero only in degrees $\leq d$. Further, the long exact sequence in homology induced by $0 \to K \to \mathbf{I}(V) \to M \to 0$ gives an exact sequence

$$0 \to H_1(M) \to H_0(K) \to V \to H_0(M) \to 0$$

Thus in degrees $> r \geq d$ we have that $H_0(K)$ vanishes. By minimality of $\mathbf{I}(W) \twoheadrightarrow K$, we have that $W_n = 0$ for $n > r$. Thus the generation degree of M is $\leq d$ and the presentation degree is $\leq r$. ☐

2.4. **Stable degree and local degree.** In this subsection, we describe how the theory of stable and local degree of FI-modules can be adapted to $\widehat{\mathrm{FI}}$-modules.

Definition 2.16. Let M be an $\widehat{\mathrm{FI}}$-module. The local degree of M is the smallest number $N \geq -1$ such that $\Sigma^{N+1}M$ is a semi-induced FI-module.

Following [CMNR18], we denote the local degree of M by $h^{max}(M)$.

Definition 2.17. Let M be an $\widehat{\mathrm{FI}}$-module. Let ΔM be the cokernel of the natural map $M \to \Sigma M$. We say that M is torsion if for all n and all $x \in M_n$, there is an element $f \in \mathrm{Hom}_{\widehat{\mathrm{FI}}}(n, m)$ with $f_*(x) = 0$. The stable degree of M is the smallest number $N \geq -1$ such that $\Delta^{N+1}M$ is torsion.

Following [CMNR18], we denote the stable degree of M by $\delta(M)$. In [CMNR18], the stable degree of an FI-module was defined using an analogous formula. Note that the functors Σ and Δ in the category of FI-modules defined in [CMNR18] agrees with their $\widehat{\mathrm{FI}}$ analogs on the subcategory of the category of $\widehat{\mathrm{FI}}$-modules with trivial $\mathbb{Z}$-action. Similarly, an FI-module is torsion if and only if it is torsion when viewed as an $\widehat{\mathrm{FI}}$-module. Thus, for M an $\widehat{\mathrm{FI}}$-module which is also an FI-module, $\delta(M)$ as defined in [CMNR18] agrees with $\delta(M)$ as defined here. Thus, we will not distinguish between the FI-module and the $\widehat{\mathrm{FI}}$-module versions of these notions.

Proposition 2.18. *Let M be an $\widehat{\mathrm{FI}}$-module. If $\Sigma^N M$ is an* FI*-module with generation degree $\leq d$, then $\delta(M) \leq d$.*

Proof. By [CMNR18, Page 2, Equation $(\star\star)$], $\delta(\Sigma^N M) \leq d$. To prove the claim, we just need to show that $\delta(\Sigma^N M) = \delta(M)$. This is the case because Δ commutes with Σ^N and the fact that an $\widehat{\mathrm{FI}}$-module T is torsion if and only if $\Sigma^N T$ is torsion. Also see [CMNR18, Proposition 2.9]. ☐

The following proposition is immediate from the definition.

Proposition 2.19. *Let M be an $\widehat{\mathrm{FI}}$-module. Then $h^{max}(M) \leq N + h^{max}(\Sigma^N M)$.*

Lemma 2.20. *Let M be an $\widehat{\mathrm{FI}}$-module with $\Sigma^N M$ generated in degrees $\leq d$. Then M has generation degree $\leq d+N$.*

Proof. Since $\Sigma^N M$ is generated in degrees $\leq d$,

$$\mathrm{Ind}_{\widehat{\mathrm{S}}_d}^{\widehat{\mathrm{S}}_n}(\Sigma^N M_d) \to \Sigma^N M_n$$

is surjective for all $n \geq d$. This is equivalent to the statement that

$$\mathrm{Ind}_{\widehat{\mathrm{S}}_d}^{\widehat{\mathrm{S}}_n} M_{d+N} \to (\Sigma^N M)_n$$

is surjective for all $n \geq d$. This implies that

$$\mathrm{Ind}_{\widehat{\mathrm{S}}_{d+N}}^{\widehat{\mathrm{S}}_{n+N}} M_{d+N} \to M_{n+N}$$

is surjective for all $n \geq d$ and so M is generated in degree $\leq d+N$. ☐

The following is an adaptation of [CMNR18, Proposition 3.1] to the case of $\widehat{\mathrm{FI}}$-modules.

Proposition 2.21. *Let M be an $\widehat{\mathrm{FI}}$-module with local degree N and stable degree d. Then the generation degree of M is $\leq d+N+3$ and the presentation degree is $\leq 2N+d+6$.*

Proof. As noted in the proof of Proposition 2.18, stable degree is independent of shifts. Thus, $\delta(\Sigma^{N+1}M) = d$. Since $\Sigma^{N+1}M$ is a semi-induced FI-module, [CMNR18, Proposition 2.9 (1)] implies that $\Sigma^{N+1}M$ has generation degree equal to d. By Lemma 2.20, M has generation degree $\leq d+N+1$. This implies $H_{-1}^{cs,\widehat{\mathrm{FI}}}(M)_n \cong 0$ for $n > d+N+1$ by Proposition 2.8.

By Proposition 2.13, ${}_{\geq N+3}M$ is an FI-module. The stable degrees of ${}_{\geq N+3}M$ and M agree since they agree after sufficiently many shifts. Since $\Sigma^{N+3}\,{}_{\geq N+3}M = \Sigma^{N+3}M$ is a semi-induced FI-module, ${}_{\geq N+3}M$ has local degree $\leq N+2$. By [CMNR18, Page 2, Equation ($\star$)], ${}_{\geq N+3}M$ has generation degree $\leq N+d+3$ and presentation degree $\leq 2N+d+6$ as an FI-module. By Proposition 2.9, we have that

$$H_{-1}^{cs,\mathrm{FI}}({}_{\geq N+3}M)_n \cong 0 \text{ for } n > N+d+3$$

and

$$H_{0}^{cs,\mathrm{FI}}({}_{\geq N+3}M)_n \cong 0 \text{ for } n > 2N+d+6.$$

By Proposition 2.14, we have that

$$H_{-1}^{cs,\widehat{\mathrm{FI}}}({}_{\geq N+3}M)_n \cong 0 \text{ for } n > N+d+3$$

and

$$H_{0}^{cs,\widehat{\mathrm{FI}}}({}_{\geq N+3}M)_n \cong 0 \text{ for } n > 2N+d+6.$$

The natural map of $\widehat{\mathrm{FI}}$-modules ${}_{\geq N+3}M \to M$ induces an isomorphism on $(C_p^{cs,\widehat{\mathrm{FI}}})_n$ for $n \geq p+N+4$. Thus,

$$H_{0}^{cs,\widehat{\mathrm{FI}}}(M)_n \cong 0 \text{ for } n > 2N+d+6.$$

The claim now follows by Proposition 2.8. ☐

3. Representation stability

In this section, we will prove that $\{H_i(F_n)\}_n$ assemble to form an $\widehat{\mathrm{FI}}$-module which is generated in finite degree. We adapt the algebraic techniques of [CMNR18] to the case of $\widehat{\mathrm{FI}}$-modules. We use connectivity results of Hatcher–Wahl [HW10].

3.1. **Fundamental group of the Milnor fiber and $\widehat{\mathrm{S}}_n$.** Our first goal is describe an action up to homotopy of $\widehat{\mathrm{FI}}$ on the spaces F_n. We begin with a discussion of braid groups and fundamental groups of Milnor fibers.

3.1.1. *Braid Conventions.* Let Br_n be the braid group on n strands. We will write $\{\sigma_{i,i+1}\}_{i=1}^{n-1}$ for the Artin generators of Br_n. Diagramatically, our convention is that strands are numbered $1,\dots,n$ from left to right, and we read the braids from top to bottom. The element $\sigma_{i,i+1}$ braids the ith strand over the $i+1$st strand. The element b_1b_2 denotes the braid b_1 followed by the braid b_2. Similarly, given a decomposition of $[n]$ into disjoint subsets $[n] = A \sqcup B$, such that $b \geq a$ for all $a \in A, b \in B$, we let $\sigma_{A,B} \in B_n$ denote the element that braids the strands of A over the strands of B.

We will write PBr_n for the pure braid group. The pure braid group is generated by elements $a_{i,j} \subset \mathrm{Br}_n$ which braids the ith strand over and around the jth strand. That is, we have $a_{i,j} = \sigma_{i,i+1}\dots\sigma_{j,j+1}\sigma_{j,j+1}\sigma_{j-1,j}^{-1}\dots\sigma_{i,i+1}^{-1}$.

The groups Br_n and PBr_n are the fundamental groups of the unordered and ordered configuration spaces of $\mathbb{C}$ respectively. When we take the fundamental group of a configuration space, we implicitly choose a base point where all of the points are on the x-axis, and are in order if the configuration space is ordered.

Throughout, for $n \in \mathbb{N}$, we define $[n] = \{1,\dots,n\}$ to be the distinguished set with n elements. If we speak of an order on $[n]$ it will be the standard order.

Let $q_n : \mathrm{Conf}_n(\mathbb{C}) \to \mathbb{C}^*$ be the map $q_n(x_i) = \prod_{i<j}(x_i - x_j)$. Then the nth type A Milnor fiber is $F_n = q_n^{-1}(1)$. We write $\widehat{\mathrm{PBr}}_n := \pi_1(F_n)$ for its fundamental group. To compute the map $\pi_1 q_n : \mathrm{PBr}_n \to \mathbb{Z}$, notice that it factors through the abelianization, so we may compute the map on H_1, or its dual on H^1. The form $dz/z = d\log(z)$ that generates $H^1_{DR}(\mathbb{C}^*)$ pulls back along q_n to $d\log(\prod_{i<j}(x_i - x_j)) = \sum_{i<j} d\log(x_i - x_j)$. So the map on H^1 is $1 \mapsto \sum_{i,j} w_{ij}$, where $w_{ij} \in H^1(\mathrm{PBr}_n, \mathbb{Z})$ is the cohomology class that gives the winding number between two points. Thus $\pi_1 q_n$ is given by $a_{i,j} \mapsto 1$ for the generators of the pure braid group $a_{i,j}$ dual to w_{ij}.

Since the map q_n is a fibration and $\mathrm{PBr}_n \to \mathbb{Z}$ is surjective, we have a short exact sequence

$$1 \to \widehat{\mathrm{PBr}}_n \to \mathrm{PBr}_n \to \mathbb{Z} \to 1,$$

where $\mathbb{Z}$ is the fundamental group of $\mathbb{C}^*$. From the long exact sequence in homotopy groups associated to the fiber sequence

$$F_n \to \mathrm{Conf}(\mathbb{C}) \to \mathbb{C}^*,$$

we see that the Milnor fiber is homotopy equivalent to the classifying space of $\widehat{\mathrm{PBr}}_n$.

We have the following chain of inclusions $\widehat{\mathrm{PBr}}_n \subset \mathrm{PBr}_n \subset \mathrm{Br}_n$. The next proposition explains how $\widehat{\mathrm{S}}_n$ relates to $\widehat{\mathrm{PBr}}_n$.

Proposition 3.1. *Let $p_n : \mathrm{Br}_n \to \widehat{\mathrm{S}}_n$ be the map defined on generators by $\sigma_{i,i+1} \mapsto ((i,i+1),1)$, which takes a braid to its associated permutation and winding number.*

The map p_n gives a short exact sequence:

$$1 \to \widehat{\mathrm{PBr}}_n \to \mathrm{Br}_n \xrightarrow{p_n} \widehat{\mathrm{S}}_n \to 1.$$

Proof. The kernel of p_n equals the kernel of its restriction to the pure braid group $\mathrm{PBr}_n \to e \times 2\mathbb{Z}$. This map takes the generator of the pure braid group $a_{i,j}$ to $(e, 2)$. Thus the map agrees with $\pi_1 p_n$, and the kernel is $\widehat{\mathrm{PBr}}_n$. □

The category $\widehat{\mathrm{FI}}$ acts on F_n up to homotopy. We may see this by producing an action of $\widehat{\mathrm{FI}}$ on $\widehat{\mathrm{PBr}}_n$ up to inner automorphisms. The most direct way to define this action is as follows.

For $[s] \in \widehat{\mathrm{S}}_m / i_2(\widehat{\mathrm{S}}_{n-m}) = \widehat{\mathrm{FI}}(n, m)$, choose a lift $\tilde{s} \in \mathrm{Br}_m$ such that $p_m(\tilde{s}) = s$. Then $[s] : F_n \to F_m$ is given by $f \mapsto \tilde{s} i_1(f) \tilde{s}^{-1}$. This map is well defined up to conjugation by elements of $\widehat{\mathrm{PBr}}_m$ since

(1) any other lift of s differs by an element of $\widehat{\mathrm{PBr}}_m$,
(2) if $[t] = [s]$, then $t = su$ for $u \in i_2(\widehat{\mathrm{S}}_{n-m})$ and every lift of u to an element $\tilde{u}$ in $i_2(\mathrm{Br}_{n-m})$ commutes with elements of $\widehat{\mathrm{PBr}}_n$.

These maps compose properly to give a functor from $\widehat{\mathrm{FI}}$ to the category of groups modulo inner automorphisms. Composing with the ith homology functor gives a functor from $\widehat{\mathrm{FI}}$ to abelian groups. We denote this $\widehat{\mathrm{FI}}$-module by $H_i(F)$. We use the convention that if we do not specify coefficients for homology, then the statement we make is true with any choice of untwisted coefficients.

3.2. **Construction of a spectral sequence.** Our first goal is to construct spectral sequences, $E^*_{*,*}(N)_n$ such that the E^2 page is

$$E^2_{p,q}(N)_n \cong H_p^{cs,\widehat{\mathrm{FI}}}(\Sigma^N H_q(F))_n$$

and which converges to zero for $p + q \leq n - 3$. We do this by constructing an augmented semi-simplicial space which is highly connected. The main input that we use is a connectivity result of Hatcher–Wahl [HW10, Proposition 7.2].

Definition 3.2. Let $\mathrm{Br}_{k,N}$ be the preimage of $i_1(\mathrm{S}_k)$ under the map $\mathrm{Br}_{k+N} \to \mathrm{S}_{k+N}$.

Definition 3.3. Let $Z_\bullet(N)_n$ be the following semi-simplicial set. We let $Z_p(N)_n = \mathrm{Br}_{n,N} / i_1(\mathrm{Br}_{n-(p+1),N})$ for $n \leq p - 1$, and $Z_p(N)_n = \emptyset$ for $p \geq n$. The kth face map is induced by

$$- \cdot u_k : \mathrm{Br}_{n,N} \to \mathrm{Br}_{n,N},$$

where $u_k \in \mathrm{Br}_{p+1}$ is as in Definition 2.3, and Br_{p+1} is included into $\mathrm{Br}_{n,N} \subset \mathrm{Br}_{n+N}$ by

$$\mathrm{Br}_{p+1} \overset{i_2}{\hookrightarrow} \mathrm{Br}_n \overset{i_1}{\hookrightarrow} \mathrm{Br}_{n+N}.$$

We will show that the complex $Z_\bullet(N)_n$ agrees with a simplicial complex that Hatcher–Wahl [HW10] proved is highly connected.

Proposition 3.4. *$Z_\bullet(N)_n$ is $(n-2)$-connected.*

Proof. We will write Z_p for $Z_p(N)_n$ throughout the proof.

Elements of $\mathrm{Br}_{k,N}$ are isotopy classes of braids on $k + N$ strands which return the last N strands to themselves. Therefore, $\mathrm{Br}_{k,N}$ is the fundamental group of $\mathrm{Conf}_{k+N}(\mathbb{C}) / \mathrm{S}_k$.

Let

$$\mathrm{Arc}_{p+1}(\mathrm{Conf}_{n+N}(D^2)/\,\mathrm{S}_n)$$

be the space of configurations of $n+N$ points, where the first n points are unlabelled, and $p+1$ arcs connecting a subset of the n points to an interval on the boundary of D^2, see Kupers–Miller [KM14, Appendix] and Miller–Wilson [MW19, Section 3.2]. This space is homotopy-equivalent to $\mathrm{Conf}_{n-(p+1)+N}(\mathbb{C})/\,\mathrm{S}_{n-(p+1)}$, and so we have

$$\mathrm{Br}_{n-(p+1),N} = \pi_1\left(\mathrm{Arc}_{p+1}(\mathrm{Conf}_{n+N}(D^2)/\,\mathrm{S}_n)\right).$$

Further, the map $\mathrm{Arc}_{p+1}(\mathrm{Conf}_{n+N}(D^2)/\,\mathrm{S}_n) \to \mathrm{Conf}_{n+N}(D^2)/\,\mathrm{S}_n)$, given by forgetting the $p+1$ arcs is a fibration, and on fundamental groups it is given by the inclusion $\mathrm{Br}_{n-p+1,N} \to \mathrm{Br}_{n,N}$ used to defined $Z_\bullet$. By the long exact sequence in homotopy,

$$\mathrm{fib}(\mathrm{Arc}_{p+1}(\mathrm{Conf}_{n+N}(D^2)/\,\mathrm{S}_n) \to \mathrm{Conf}_{n+N}(D^2)/\,\mathrm{S}_n),$$

is homotopy discrete and its connected components are identified with

$$\mathrm{Br}_{n,N}\,/\,\mathrm{Br}_{n-(p+1),N} = Z_p.$$

Simultaneously, the connected components of the fiber are of isotopy classes of $p+1$ arcs from the boundary, connecting to the first n of $N+n$ points, so that the arcs are not allowed to cross or pass through the points. Under this identification, the face maps of Z_p correspond to forgetting arcs, and so $Z_\bullet$ is isomorphic to the complex $A(D^2-[N];[n],[n])_\bullet$ appearing in Hatcher–Wahl [HW10, Section 7]. Hatcher–Wahl [HW10, Proposition 7.2] implies states $A(D^2-[N];[n],[n])$ $(n-2)$-connected. □

It is convenient to use the variant of $\mathrm{Br}_{k,N}$ for $\widehat{\mathrm{S}}_n$, and of the semi simplicial set $Z_\bullet(N)_n$.

Definition 3.5. Let $\widehat{\mathrm{Br}}_{k,N}$ be the preimage of $i_1(\widehat{\mathrm{S}}_k)$ under the map $\mathrm{Br}_{k+N} \to \widehat{\mathrm{S}}_{k+N}$.

The sets $\widehat{Z}_p(N)_n := \widehat{\mathrm{Br}}_{n,N}/\widehat{\mathrm{Br}}_{n-(p+1),N}$ form a semi-simplicial set defined by the same formulas as in Definition 3.3.

Notice that $\widehat{\mathrm{Br}}_{k,N} \subset \mathrm{Br}_{k,n}$. In fact, the two groups are often the same.

Lemma 3.6. *We have $\widehat{\mathrm{Br}}_{k,N} \cong \mathrm{Br}_{k,N}$ for $k \geq 2$.*

Proof. We need that the preimage of S_k under the projection $\widehat{\mathrm{S}}_{k+N} \to \mathrm{S}_{k+N}$ is $\widehat{\mathrm{S}}_k$. This follows from the fact that $\widehat{\mathrm{S}}_{k+N}/\widehat{\mathrm{S}}_k \to \mathrm{S}_{k+N}\,/\,\mathrm{S}_k$ is an isomorphism for $k \geq 2$. □

Definition 3.7. Let $\widehat{\mathrm{PBr}}_{n,N} \subset \widehat{\mathrm{Br}}_{n,N}$ be the kernel of the surjection $\widehat{\mathrm{Br}}_{n,N} \twoheadrightarrow \widehat{\mathrm{S}}_n$. Then $\widehat{\mathrm{PBr}}_{n,N}$ acts on the semi-simplicial set $Z_p(N)_n = \widehat{\mathrm{Br}}_{n,N}/\widehat{\mathrm{Br}}_{n-(p+1),N}$ by left multiplication. Let $X_p(N)_n := \widehat{\mathrm{PBr}}_{n,N}\backslash\backslash Z_p(N)_n$, where $\widehat{\mathrm{PBr}}_{n,N}\backslash\backslash -$ denotes a functorial homotopy quotient. Then $X_\bullet(N)_n$ is a semisimplicial space, which is augmented by the map $X_p(N)_n \to X_{-1}(N)_n := \widehat{\mathrm{PBr}}_{n,N}\backslash\backslash *$.

We will suppress n, N from the notation for $X_p(N)_n$ when the context is clear.

Proposition 3.8. *For every j, the associated chain complex $H_j(X_\bullet(N)_n)$ is canonically isomorphic to the central stability chains $C_\bullet^{cs,\widehat{\mathrm{FI}}}(\Sigma^N H_j(F_-))$.*

Proof. We have that

$$\widehat{\mathrm{PBr}}_{n,N}\backslash\backslash\widehat{\mathrm{Br}}_{n,N}/\widehat{\mathrm{Br}}_{n-(p+1),N} \simeq \widehat{\mathrm{PBr}}_{n,N}\backslash\backslash\widehat{\mathrm{Br}}_{n,N}//\widehat{\mathrm{Br}}_{n-(p+1),N}$$
$$\simeq \widehat{\mathrm{PBr}}_{n,N}\backslash\widehat{\mathrm{Br}}_{n,N}//\widehat{\mathrm{Br}}_{n-(p+1),N} \simeq \widehat{\mathrm{S}}_n//\widehat{\mathrm{Br}}_{n-(p+1),N}.$$

Thus $H_j(X_p) \cong H_j(\widehat{\mathrm{Br}}_{n-(p+1),N}, \mathbb{Z}\widehat{\mathrm{S}}_n)$. Under this identification, the ith face operator acts by restricting $\mathbb{Z}\widehat{\mathrm{S}}_n$ to Br_{n-p} and multiplying by u_i, where u_i is as in §2.2.

Now the map $\widehat{\mathrm{Br}}_{n-(p+1),N} \to \widehat{\mathrm{S}}_n$ factors as $\widehat{\mathrm{Br}}_{n-(p+1),N} \twoheadrightarrow \widehat{\mathrm{S}}_{n-(p+1)} \hookrightarrow \widehat{\mathrm{S}}_n$. By definition of $\widehat{\mathrm{Br}}_{n-(p+1),N} \subseteq \mathrm{Br}_{n-(p+1)+N}$, the kernel of this map equals the kernel of $\mathrm{Br}_{n-(p+1)+N} \to \widehat{\mathrm{S}}_{n-(p+1)+N}$ which is $\widehat{\mathrm{PBr}}_{n-(p+1)+N}$.

We have

$$H_a(\widehat{\mathrm{S}}_{n-(p+1)}, H_b(\widehat{\mathrm{PBr}}_{n-(p+1)+N}, \mathbb{Z}\widehat{\mathrm{S}}_n)) \cong H_a(\widehat{\mathrm{S}}_{n-(p+1)}, \mathbb{Z}\widehat{\mathrm{S}}_n \otimes H_b(\widehat{\mathrm{PBr}}_{n-(p+1)+N}))$$

since the coefficients are free, this last term vanishes for $a \neq 0$, and for $a = 0$ is equal to $\mathrm{Ind}_{\widehat{\mathrm{S}}_{n-(p+1)}}^{\widehat{\mathrm{S}}_n}(H_b(\widehat{\mathrm{PBr}}_{n-(p+1)+N}))$. By the Serre spectral sequence, this shows that $H_j(X_p, \mathbb{Z}) \cong \mathrm{Ind}_{\widehat{\mathrm{S}}_{n-(p+1)}}^{\widehat{\mathrm{S}}_n}(H_j(\widehat{\mathrm{PBr}}_{n-(p+1)+N}))$, as desired. □

We now show that the augmented semi-simplicial space is connected in a range growing in n.

Proposition 3.9. *Let $n \geq 2$. Then the augmentation map $|X_\bullet| \to X_{-1}$ induces an isomorphism on H_i for $i \leq n-3$.*

Proof. It suffices to prove that the semi-simplicial set $\widehat{Z_\bullet}$ is $n-3$ connected since it has the homotopy type of the homotopy fiber of $|X_\bullet| \to X_{-1}$. The inclusion $\widehat{\mathrm{Br}}_{k,N} \subset \mathrm{Br}_{k,N}$ induces a map of simplical sets $\widehat{Z}_\bullet \to Z_\bullet$. By Lemma 3.6, this map is an isomorphism on p simplices for $p \leq n-3$. Since we have assumed $n \geq 2$, it is a surjection on $n-2$ simplices. Thus $\widehat{Z}_\bullet$ is $n-3$ connected if $Z_\bullet$ is. Finally by Proposition 3.4, we have that $Z_\bullet$ is $n-2$ connected (and hence $n-3$ connected), completing the proof. □

From the above two propositions, we obtain a spectral sequence with the desired properties.

Proposition 3.10. *For all n and N, there is a homologically graded spectral sequence $E^r_{p,q}(N)_n$ with*

$$E^2_{p,q}(N)_n \cong \left(H_p^{cs,\widehat{\mathrm{FI}}}(\Sigma^N H_q(F))\right)_n \text{ and}$$
$$E^\infty_{p,q}(N)_n \cong 0 \text{ for } p+q \leq n-3.$$

Here we take $p \geq -1$ and $q \geq 0$.

3.3. **Proof of stability.** The following three lemmas will be used in an induction argument to prove representation stability for Milnor fibers.

Lemma 3.11. *We have that $\delta(H_0(F)) \leq 0$ and $h^{max}(H_0(F)) = -1$.*

Proof. Since $H_0(F_n) \cong \mathbb{Z}$ and all of the stabilization maps are isomorphisms, $\delta(H_0(F))$ is generated in degree 0 and so $\delta(H_0(F)) \leq 0$. Since $\Sigma^0 H_0(F)$ is an induced FI-module, $h^{max}(H_0(F)) = -1$. □

The following lemma is an adaptation of the arguments in [CMNR18, Theorem 5.1, Part 1)].

Lemma 3.12. *Let $i \geq 1$ and suppose $\delta(H_q(F)) \leq 2q+1$ for all $q < i$ and $h^{max}(H_q(F))$ is finite for all $q < i$. Then $\delta(H_i(F)) \leq 2i+1$.*

Proof. Let N be a number larger than $h^{max}(H_q(F))$ for all $q < i$ and take $n > 2i+1$. By Proposition 3.10, we have that

$$E^{\infty}_{-1,i}(N)_n \cong 0$$

and

$$E^2_{p,q}(N)_n \cong H_p^{cs,\widehat{\mathrm{FI}}}(\Sigma^N H_q(F))_n.$$

By Proposition 2.18 and Proposition 2.9, it suffices to show $H_{-1}^{cs,\widehat{\mathrm{FI}}}(\Sigma^N H_i(F))_n \cong 0$. Since $E^2_{-1,i}(N)_n \cong H_{-1}^{cs,\widehat{\mathrm{FI}}}(\Sigma^N H_i(F))_n$ and $E^{\infty}_{-1,i}(N)_n \cong 0$, it suffices to show that $E^2_{-1,i}(N)_n \cong E^{\infty}_{-1,i}(N)_n$. To do this, we will show that $E^2_{t,i-t}(N)_n \cong 0$ for all $0 \leq t \leq i+1$.

We have $E^2_{p,q}(N)_n \cong H_p^{cs,\widehat{\mathrm{FI}}}(\Sigma^N H_q(F))$. Consider $q < i$. Since $N > h^{max}(H_q(F))$, $\Sigma^N H_q(F)$ is a semi-induced FI-module. Since $\Sigma^N H_q(F)$ has generation degree $\leq 2q+1$ and is semi-induced, Corollary 2.6 implies that

$$\left(H_p^{cs,\mathrm{FI}}(\Sigma^N H_q(F))\right)_n \cong 0 \text{ for } p \leq n-2-2q-1.$$

By Proposition 2.14,

$$\left(H_p^{cs,\widehat{\mathrm{FI}}}(\Sigma^N H_q(F))\right)_n \cong \left(H_p^{cs,\mathrm{FI}}(\Sigma^N H_q(F))\right)_n \text{ for } p \leq n-3.$$

Consider $t \leq i+1, i > 0$. Since $n > 2i+1$, and $i \geq 1$, we have that $t \leq n-3$. Thus $E^2_{t,i-t}(N)_n \cong 0$ for all $0 \leq t \leq i+1$ and so the claim follows. □

The following lemma is an adaptation of the arguments in [CMNR18, Theorem 5.1, Part 2)].

Lemma 3.13. *Let $i > 0$ and assume $\delta(H_q(F)) \leq 2q+1$ for $q \leq i$ and $h^{max}(H_q(F)) \leq f(q)$ for $q < i$ for some increasing function f. Then $h^{max}(H_i(F)) \leq f(i-1) + 6i + 6$.*

Proof. Let $N = f(i-1)+1$ and let $N' \geq N$. As in the proof of Lemma 3.12, we have that

$$E^2_{p,q}(N')_n \cong 0 \text{ for } p \leq n-2-2q-1 \text{ and } q < i.$$

This means there are no nontrivial differentials into or out of $E^r_{-1,i}(N')_n$ for $r \geq 2$ and $n > 2i+1$ since $E^2_{i-q,q}(N')_n \cong 0$ for $q < i$ and $n > 2i+1$. Thus

$$H_{-1}^{cs,\widehat{\mathrm{FI}}}(\Sigma^{N'} H_i(F))_n \cong E^2_{-1,i}(N')_n \cong E^{\infty}_{p,q}(N')_n \cong 0 \text{ for } n > 2i+1.$$

By considering $E^2_{0,i}(N')_n$ instead of $E^2_{-1,i}(N')_n$, we get the inequality

$$H_0^{cs,\widehat{\mathrm{FI}}}(\Sigma^{N'} H_i(F))_n \cong 0 \text{ for } n > 2i+2.$$

Let $M = \Sigma^{N+2i+3} H_i(F)$. By Proposition 2.13, M is an FI-module. By Proposition 2.14, $H_{-1}^{cs,\mathrm{FI}}(M) \cong 0$ for $n > \max(2i+1, 2) = 2i+1$ and $H_0^{cs,\mathrm{FI}}(M) \cong 0$ for $n > \max(2i+2, 3) = 2i+2$. By Proposition 2.9, we have that M has generation degree $\leq 2i+1$ and presentation degree $\leq 2i+2$. By [CMNR18, Page 2, Equation ($\star\star$)], the local degree of M is $\leq 4i+2$. Note that M is an $N+2i+3$-fold shift of

$H_i(F)$ and $N + 2i + 3 = f(i-1) + 2i + 4$. Proposition 2.19 implies that the local degree of $H_i(F)$ is $\leq f(i-1) + 6i + 6$. □

Solving the recurrence and combining Lemma 3.11, Lemma 3.12, Lemma 3.13 gives the following.

Proposition 3.14. *The stable degree of $H_i(F)$ is $\leq 2i+1$ and the local degree is $\leq -1 + 9i + 3i^2$.*

Combining this with Proposition 2.21 gives the following.

Theorem 3.15. *The generation degree of $H_i(F)$ is $\leq 3 + 11i + 3i^2$ and the presentation degree is $\leq 5 + 20i + 6i^2$.*

Since F_n is an algebraic variety, $H_i(F_n)$ is finitely generated as an abelian group for all i and n, Theorem A follows from Theorem 3.15. Theorem B follows from Theorem 3.15 and Proposition 2.13.

Remark 3.16. It seems very plausible that a linear stable range is in fact optimal. Gan–Li [GL17] were able to prove linear stable ranges for congruence subgroups of general linear groups. Can one adapt their techniques to the case of Milnor fibers? One major obstacle to doing this is the fact that the chains on the Milnor fibers do not seem to be homotopy equivalent to an $\widehat{\mathrm{FI}}$-chain complex.

4. Stable calculations

In this section, we will study $H_i(F_n, \mathbb{Z})$ in the range where the action of $\mu_{\binom{n}{2}}$ is trivial. In particular, we will compare its homology to $H_i(\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{Z})$ using the fact that $\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^* = F_n/\mu_{\binom{n}{2}}$.

Theorem 4.1. *Suppose that $\mu_{\binom{n}{2}}$ acts trivially on $H_i(F_n, \mathbb{Z})$ for $i \leq k$. Then $H_i(F_n, \mathbb{Z})$ is torsion free for $i \leq k$, and the map $F_n \to \mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*$ induces an S_n equivariant isomorphism on rational homology in degrees $\leq k$.*

The main content of the above theorem is that the homology of F_n is torsion free. The rank of the group was already determined by Settepanella [Set04, Theorem 1.2].

4.1. Comparing $\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*$ and F_n.

First we note that with $\mathbb{Q}$ coefficients, the homology of F_n is canonically isomorphic to the homology of $\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*$ in the range where $\mu_{\binom{n}{2}}$ acts trivially.

Proposition 4.2. *If $\mu_{\binom{n}{2}}$ acts trivially on $H_i(F_n, \mathbb{Q})$ for $i \leq k$, then $H_i(F_n, \mathbb{Q}) \cong H_i(\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{Q})$ for $i \leq k$.*

Proof. The group $\mu_{\binom{n}{2}}$ acts freely on F_n and its quotient is $\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*$. Thus the rational homology of $\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*$ is canonically identified with the coinvariants $H_i(F_n, \mathbb{Q})_{\mu_{\binom{n}{2}}}$ under the pushforward map. Since $\mu_{\binom{n}{2}}$ acts trivially for $i \leq k$, we obtained the desired isomorphism. □

Proposition 4.3. *If $\mu_{\binom{n}{2}}$ acts trivially on $H_i(F_n, \mathbb{Z})$ for all $i \leq k$, then $H_i(F_n, \mathbb{Z})$ is torsion free for all $i \leq k-1$.*

To prove this proposition, we will need the following lemma.

Lemma 4.4. *Let $\mathbb{Z}/m$ be an abelian group, and let C be a chain complex of $\mathbb{F}_p[\mathbb{Z}/m] = \mathbb{F}_p[x]/(x^m - 1)$ modules concentrated in homological degree ≥ 0. Assume that $H_i(C)$ is finite dimensional for all $i \leq k$, and $x-1$ acts nilpotently on $H_i(C)$. Then there exists a chain complex G_* of projective $\mathbb{F}_p[\mathbb{Z}/m]$ modules and a quasi-isomorphism $f : G_* \xrightarrow{\simeq} C_*$ such that:*

(1) *for all $i \leq k$, G_i is isomorphic $P^{\oplus r}$ for some $r \in \mathbb{N}$, where P is the module $P := \mathbb{F}_p[x]/(x-1)^{p^d}$, and p^d is the largest power of p dividing m,*
(2) *for all $i \leq k+1$, the differential $d_i : G_i \to G_{i-1}$ is zero mod $x-1$.*

Proof. First, note that $(x^{p^d} - 1) = (x-1)^{p^d}$ is the largest power of $x-1$ dividing $x^m - 1$. Therefore P is a summand of $\mathbb{F}_p[x]/(x^m-1)$ by the Chinese remainder theorem, and so is projective. Since $x-1$ acts invertibly on all of the other factors, if a power of $(x-1)$ annihilates an element of a $\mathbb{F}_p[\mathbb{Z}/m]$-module, then $(x-1)^{p^d}$ annihilates it.

We construct the resolution G_* inductively in the usual way. To determine G_0, choose $m_1, \ldots, m_{r_0}$ a collection generators of $H_0(C_*)$ which is minimal in the sense that the associated map $P^{\oplus r_0} \to H_0(C_*)$ is an isomorphism mod $x-1$. We let $G_0 = P^{\oplus r_0}$, and choose a lift of $G_0 \to H_0(C_*)$ to $f_0 : G_0 \to Z_0(C_*) \subset C_0$.

To determine G_1, we consider $H_1(\text{cone}(G_0 \to C_*)) = \ker(G_0 \oplus C_1 \to C_0)/d(C_2)$, and again choose a collection of minimal generators which give a map $P^{r_1} \to H_1(\text{cone}(G_0 \to C_*))$, which lifts to a map $d_1 \oplus f_0 : P^{r_1} \to G_0 \oplus C_1$. The map from the two term complex induces an isomorphism of homology groups in degree 0, and a surjection on homology in degree 1. The map $d_1 : G_1 \to G_0$ is minimal because its image is $\{g \in G_0 \mid \exists c \in C_1,\ f_0(g) = d(c)\}$ and we have that for every such g the homology class of $f_0(g)$ vanishes and so g must be divisible by $(x-1)$ by the minimality of f_0.

To determine G_3, we choose minimal generators of the second homology of the cone, $\ker(G_1 \oplus C_2 \to G_0 \oplus C_1)/d(C_3)$, and so on. We continue in this way until determining G_{k+1}, where we replace the role of the module P by the free module $\mathbb{F}_p[x]/(x^m-1)$, and no longer require minimality of generators. □

Proof of Proposition 4.3. Fix $i \leq k-1$ and let $\mu = \mu_{\binom{n}{2}}$. We have that $H_j(F_n, \mathbb{Z})$ is a finitely generated abelian group, with rank equal to the rank of the group $H_j(\text{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{Z})$ for all $j \leq k$ by Proposition 4.2. The group $H_j(\text{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{Z})$ is torsion free, see §5.3. Thus by the universal coefficient theorem, to show that $H_i(F_n, \mathbb{Z})$ is torsion free, it suffices to show that the dimension of $H_{i+1}(F_n, \mathbb{F}_p)$ equals the dimension of $H_{i+1}(\text{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{F}_p)$ for all primes p and all $i \leq k-1$.

Since $\text{Conf}_n(\mathbb{C})/\mathbb{C}^*$ is a quotient of F_n by a free μ action, we have that $C_*(\text{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{F}_p)$ is quasi-isomorphic to $G_* \otimes_{\mathbb{F}_p[\mu]} \mathbb{F}_p$ with G_* any chain complex of projective $\mathbb{F}_p[\mu]$-modules quasi-isomorphic to $C_*(F_n; \mathbb{F}_p)$. We will choose G_* so it satisfies the conditions of Lemma 4.4.

For $j \leq k$, we have $G_j = P^{\oplus r_j}$. By Condition 2 of Lemma 4.4, we have $G_* \otimes_{\mathbb{F}_p[\mu]} \mathbb{F}_p$ has zero differential in degrees $\leq k$. Thus r_j is the dimension of $H_j(\text{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{F}_p)$.

The dimension of $H_j(F_n, \mathbb{F}_p)$ is the dimension of $H_j(G_*)$. Call this number c_j. We want to show that $r_j = c_j$. We have that $r_j \leq c_j$ because $H_j(\text{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{Z})$ is torsion free and agrees with $H_j(F_n, \mathbb{Z})$ rationally and because the dimension of $H_j(F_n, \mathbb{F}_p)$ is at least as large as the dimension of $H_j(F_n, \mathbb{Q})$.

To show that $c_j \leq r_j$, we show that any subquotient of $P^{\oplus r_j}$ (in particular $H_j(G_*)$) can be generated by less than or equal to r_j elements. It suffices to show this for submodules, and the preimage every submodule of $M \subseteq P^{\oplus r_j}$ under the projection $\mathbb{F}_p[x]^{\oplus r_j} \to P^{\oplus r_j}$ is an $\mathbb{F}_p[x]$ submodule $\tilde{M} \subseteq \mathbb{F}_p[x]^{\oplus r_j}$. Since $\mathbb{F}_p[x]$ is a PID, the submodule $\tilde{M}$ is free and thus generated by at most r_j elements. The images of these elements in $P^{\oplus r_j}$ generate M and so the claim is proved, $r_j = c_j$, and we are done. □

From Proposition 4.3 and Proposition 4.2, we immediately obtain Theorem 4.1. Theorem B and Theorem 4.1 imply Theorem C.

5. Appendix

5.1. **Central Stability Homology of Braided Monoidal Groupoids.** Let $K_n \subset \mathrm{Br}_n$ be a sequence of normal subgroups such that the image of $K_a \times K_b$ under the map $m_{a,b} : \mathrm{Br}_a \times \mathrm{Br}_b \to \mathrm{Br}_{a+b}$ is contained in K_{a+b}. Denote the quotient by G_n. Then $\{G_n\}_{n\in\mathbb{N}}$ forms a braided monoidal groupoid. We have maps $m_{a,b} : G_a \times G_b \to G_{a+b}$. The braiding is the natural transformation $m_{a,b} \to m_{b,a}$ induced by multiplication by $\sigma_{a,b} \in \mathrm{Br}_{a+b}$.

Write $\mathcal{A} = \mathrm{Rep} \sqcup_n G_n$ for the category of sequences of abelian groups A_n with a G_n action. The induction product makes $\mathcal{A}$ into a braided monoidal category as follows (see e.g. Joyal–Street [JS93]).

(1) We define
$$M_m * N_n = \mathrm{Ind}_{G_m \times G_n}^{G_{m+n}} M_m \otimes N_n = \mathbb{Z}G_{m+n} \otimes_{\mathbb{Z}G_n \times G_m} M_m \otimes N_n.$$

(2) We define the map $t_{m,n} : M_m * N_n \to N_n * M_m$ from the action of $\sigma_{n,m}$ on $\mathbb{Z}G_{m+n}$ by right multiplication.

As usual, in a monoidal category associative algebras and modules can be defined diagramatically. From the braided monoidal structure on $\mathcal{A} = \mathrm{Rep} \sqcup_n G_n$, we can define a *commutative algebra* to be a unital associative algebra A, with a multiplication $\mu : A * A \to A$ such that $\mu \circ t = m$. [1]

Let V be an object of $\mathcal{A}$, and let $\mathrm{Sym}_q(V) = \bigoplus_n V^{*n}/\mathrm{Br}_n$ be the free commutative algebra. A right module over $\mathrm{Sym}_q(V)$ consists of $M \in \mathcal{A}$ and a map $a : M * V \to M$, such that $a \circ (a * \mathrm{id}_V) : M * V * V \to M$ equals $a \circ (a * \mathrm{id}_V) \circ (\mathrm{id}_M * t)$.

Then for any $\mathrm{Sym}_q(V)$-module M, there is a chain complex of $\mathrm{Sym}_q(V)$-modules $C_*^{cs}(M)$:
$$M \xleftarrow{d_0} M * V \xleftarrow{d_1} M * V * V \xleftarrow{d_2} M * V * V * V \leftarrow \dots,$$
defined as follows. We have $C_p^{cs}(M) = M * V^{*p+1}$ for $p \geq -1$. The differential $d_p = \sum_{i=1}^{p+1} (-1)^i f_i$ is defined from an augmented semisimplicial set where the face operator $f_i : M * V^{*p+1} \to M * V^{*p}$ acts by using the braiding to move the ith factor of V over the other factors to M and then applying the multiplication $a : M * V \to M$.

More formally, write $u_i \in \mathrm{Br}_{p+1}$ for the element $\sigma_{i-1,i}^{-1} \sigma_{i-2,i-1}^{-1} \dots \sigma_{1,2}^{-1}$ that braids the ith strand over all the others to the left. Then f_i acts by $(a * \mathrm{id}_V^{*p}) \circ (\mathrm{id}_M * u_i)$. The semisimplicial identities hold because the multiplication map $M * V * V \to M$ factors through $M * (V * V)/\mathrm{Br}_2 \to M$. For similar reasons $C_*^{cs}(M)$ has the

[1] More properly, we could call A a braided commutative algebra, or an E_2-algebra.

structure of a chain complex of right $\mathrm{Sym}_q(V)$ modules, where $\mathrm{Sym}_q(V)$ acts on $M * V^{*i}$ by using the braiding to move over the factors of V.

Remark 5.1. The construction generalizes to produce a semisimplicial object for any object with an action of a free commutative monoid in a braided monoidal category.

In the cases we consider, V is $\mathbb{Z}$, the trivial representation of G_1 concentrated in degree 1. That is, we have

$$V = \{V_n\}_{n\in\mathbb{N}} = \begin{cases} V_1 = \mathbb{Z} \\ V_i = 0 \quad i \neq 1 \end{cases}.$$

Further, we will only be concerned with cases corresponding to $\widehat{\mathrm{FI}}$ and FI.

Example 5.2. Let $G_n = \widehat{\mathrm{S}}_n$, and $V = \mathbb{Z}$ as above. Then right $\mathrm{Sym}_q(V)$-module are canonically equivalent to $\widehat{\mathrm{FI}}$-modules: the data of a right $\mathrm{Sym}_q(V)$-module is given by maps $M_n * \mathrm{Sym}^i_q(V) \to M_{n+i}$, which correspond to

$$\mathrm{Ind}^{\widehat{\mathrm{S}}_{n+i}}_{\widehat{\mathrm{S}}_n \times \widehat{\mathrm{S}}_i} M_n \otimes \mathbb{Z} \cong \mathbb{Z}\widehat{\mathrm{S}}_{n+i}/i_2(\widehat{\mathrm{S}}_i) \otimes_{\widehat{\mathrm{S}}_n} M_n \to M_{n+i}.$$

Further, we have that $(M * V^{*p})_n = \mathrm{Ind}^{\widehat{\mathrm{S}}_n}_{i_1(\widehat{\mathrm{S}}_{n-p})} M_{n-p}$, and $C^{cs}_*(M)$ agrees with $C^{cs,\widehat{\mathrm{FI}}}_*(M)$ as defined in Definition 2.3.

Example 5.3. For $G_n = \mathrm{S}_n$, $\mathrm{Sym}_q(V)$-modules are the same as FI-modules, and we obtain the FI central stability complex in the same way.

For any inclusion of subgroups $J_n \subset K_n$ with quotient $p : H_n \twoheadrightarrow G_n$, there is a pullback $p^* : \mathrm{Rep} \sqcup_n G_n \to \mathrm{Rep} \sqcup_n H_n$. The pullback is braided lax monoidal in the sense that there is a canonical map $p^*M * p^*N \to p^*(M * N)$, and this map is compatible with the braiding.

Using this structure, $\mathrm{Sym}_q(V)$-modules pull back to $\mathrm{Sym}_q(V)$-modules. Because central stability complexes are defined in terms of tensor powers of V, the braiding, and the action of V on M, there is an induced map of semisimplicial complexes of $\mathrm{Sym}_q(V)$-modules $C^{cs}_*(p^*M) \to p^*C^{cs}_*(M)$.

Example 5.4. In the case of $p : \widehat{\mathrm{S}}_n \to \mathrm{S}_n$, the map of central stability complexes agrees with the map of Proposition 2.14.

5.2. **Comparison with the central stability complex of Patzt.** Let $\mathcal{C}, \oplus$, be a monoidal category such that the unit object $0 \in \mathcal{C}$ is initial. Let $\mathbf{I}_x : \mathrm{Mod}\ \mathcal{C} \to \mathrm{Mod}\ \mathcal{C}$ denote the left adjoint to $\mathbf{S}_x$, defined to be restriction along the functor $- \oplus x : \mathcal{C} \to \mathcal{C}$. Let M be a $\mathcal{C}$-module. Let Δ_{inj} denote the category governing augmented semi-simplicial objects. That is, Δ_{inj} is the category with objects finite ordered sets and morphisms given by order preserving injections. Patzt defines the *central stability chains* of M with respect to x, to be the chain complex associated to the augmented semi-simplicial abelian group

$$\Delta^{\mathrm{op}}_{inj} \to \mathrm{Mod}\ \mathcal{C},\ [n] \mapsto \mathbf{I}_{x^{\oplus n}} M.$$

For an ordered injection $f : [n] \to [m] \in \Delta_{inj}([n],[m])$, the associated map $\mathbf{I}_{x^{\oplus n}} \leftarrow \mathbf{I}_{x^{\oplus m}}$ is adjoint to the natural transformation $\mathbf{S}_{x^{\oplus n}} \to \mathbf{S}_{x^{\oplus m}}$ induced by the morphism $f : x^{\oplus n} \to x^{\oplus m}$.

In our setting $\mathcal{C}$ is the category $\widehat{\mathrm{FI}}$, and x is the object 1. To compute the functor $\mathbf{I}_1^{\widehat{\mathrm{S}}}$ in this case, we note that there are restriction and induction functors $\mathbf{S}_1^{\widehat{\mathrm{S}}}$ and $\mathbf{I}_1^{\widehat{\mathrm{S}}}$ defined on the category Mod$\widehat{\mathrm{S}}$.

In fact, when M is an $\widehat{\mathrm{FI}}$-module $\mathbf{I}_1^{\widehat{\mathrm{S}}}M$ carries a canonical $\widehat{\mathrm{FI}}$-module structure. To see this, we identify $\widehat{\mathrm{FI}}$-modules with $\mathrm{Sym}_q(V)$-modules, where V is the $\widehat{\mathrm{S}}$ representation consisting of V concentrated in degree 1. Observe that $\mathbf{I}_1^{\widehat{\mathrm{S}}}M = M * V$, where $*$ denotes the induction tensor product of $\widehat{\mathrm{S}}$-modules. Then $M * V$ becomes a right $\mathrm{Sym}_q(V)$-module through the map

$$M * V * \mathrm{Sym}_q(V) \to^{\mathrm{id}_M * t} M * \mathrm{Sym}_q(V) * V \to^{a * \mathrm{id}_V} M * V,$$

where t denotes the braiding, and a denotes the action map for the $\widehat{\mathrm{FI}}$-modules structure on M.

This lifts $\mathbf{I}_1^{\widehat{\mathrm{S}}}$ to a functor Mod $\widehat{\mathrm{FI}} \to$ Mod $\widehat{\mathrm{FI}}$, and we have that $\mathbf{I}_1^{\widehat{\mathrm{S}}} \simeq \mathbf{I}_1^{\widehat{\mathrm{FI}}}$ is adjoint to $\mathbf{S}_1^{\widehat{\mathrm{FI}}}$. In other words, let M, N be $\widehat{\mathrm{FI}}$-modules. Then a map of $\widehat{\mathrm{S}}$-modules $\mathbf{I}_1^{\widehat{\mathrm{S}}}M \to N$ is a map of FI-modules if and only if $M \to \mathbf{S}_1^{\widehat{\mathrm{S}}}N = \mathbf{S}_1^{\widehat{\mathrm{FI}}}M$ is a map of $\widehat{\mathrm{FI}}$-modules.

Under this identification, the central stability complex of Patzt corresponds to the central stability complex of §2.2. The presence of a braiding in the differentials of our central stability complex corresponds to the braiding used to define the $\widehat{\mathrm{FI}}$-module structure of $\mathbf{I}_1^{\widehat{\mathrm{S}}}M$, and thus the corresponding adjoint maps $\mathbf{I}_{1^{\oplus r}}^{\widehat{\mathrm{S}}}M \to \mathbf{I}_{1^{\oplus s}}^{\widehat{\mathrm{S}}}M$.

5.3. **Combinatorial description of the homology of** $\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*$**.** There is a well known homeomorphism $\mathrm{Conf}_n(\mathbb{C})/(\mathbb{C}^* \ltimes \mathbb{C}) \cong \mathrm{Conf}_{n+1}(\mathbb{P}^1)/\mathrm{PGL}_2 = M_{0,n+1}$ for all $n \geq 2$. Because $\mathbb{C}$ is contractible, this gives an isomorphism $H_i(\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{Z}) = H_i(M_{0,n+1}, \mathbb{Z})$. These homology groups were first computed by Getzler [Get95]. These groups have also appeared in the representation stability literature in the work of Hyde–Lagarias [HL17].

In this section, we describe $H_d(\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{Z})$ combinatorially, following Getzler [Get96, Sec 1.17]. We write C_k for the k^{th} graded piece of the Arnold ring, also known as the Orlik–Solomon algebra associated to the braid arrangement. C_* is the free graded commutative algebra generated by classes $\{\omega_{ij}\}_{i \neq j \in \{1,\dots,n\}}$ in degree 1, subject to the relations $\omega_{ij} = \omega_{ji}$ and $\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0$ for all $i,j,k \in \{1,\dots,n\}$. The group C_k was computed to be the homology of $H^k(\mathrm{Conf}_n(\mathbb{C}))$ by Arnold [Arn69], it is a free abelian group of rank $(-1)^k s(n, n-k)$ where $s(n,k)$ denotes the signed Stirling number of the first kind.

Define a differential $d : C_\bullet \to C_{\bullet - 1}$ by setting $d(\omega_{ij}) = 1$ for all i, j and extending to all of $C_\bullet$ by linearity and the Leibniz rule. We write $d_k : C_k \to C_{k-1}$ for the degree k component of the differential.

Theorem 5.5 (Getzler)**.** *The map $d_k : C_k \to C_{k-1}$ makes $C_\bullet$ into an exact chain complex of S_n representations. Further,* $\mathrm{coker}(d_k^\vee) \cong H_k(\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{Z})$ *as S_n representations.*

Proof. Getzler [Get96, Sec 1.17] shows that $(C_\bullet, d_\bullet)$ is exact, and identifies d_k with the action of the fundamental class, ϵ, of $\mathbb{C}^* \simeq S^1$ on $H^\bullet(\mathrm{Conf}_n(\mathbb{C}))$. This suffices to determine the integral cohomology of the quotient S_n equivariantly. One method of computation is as follows. The E^2 page of the Moore spectral sequence

$\mathrm{Tor}_\bullet^{H_\bullet(\mathbb{C}^*)}(\mathbb{Z}, H_\bullet(\mathrm{Conf}_n(\mathbb{C})))$ (see e.g. McCleary [McC01, Theorem 7.28]) can be computed using $H_\bullet(\mathbb{C}^*) \cong \mathbb{Z}[\epsilon]/\epsilon^2$ and the minimal resolution of $\mathbb{Z}$ over this ring, to be a direct sum of shifts of truncations of the complex $C^\vee$. This, together with Getzler's exactness, shows that the spectral sequence degenerates at E^2 and H_k is isomorphic to $\operatorname{coker} d_k$. □

As a consequence of the exactness of C_k, we can compute the rank of $\operatorname{coker}(d_k^\vee)$, as well as the S_n character as an alternating sum.

Corollary 5.6. *The group* $H_d(\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{Z})$ *is free abelian of rank*

$$r_d = (-1)^d \sum_{i \le d} s(n, n-i).$$

As S_n *representations we have*

$$\mathrm{ch}(H_d(\mathrm{Conf}_n(\mathbb{C})/\mathbb{C}^*, \mathbb{Q})) = (-1)^d \sum_{k \le d} (-1)^k \mathrm{ch}(C_k),$$

where ch *denotes the Frobenius character.*

The character $\mathrm{ch}(C_k)$ may also be described in terms of Lie characters as follows. For any set S, we define $\mathrm{Lie}(S)$ to be the free abelian group on all bracketings of the elements of S modulo insertions of the anticommutativity and the Jacobi relations.

Example 5.7. As an $\mathbf{S}_3$ representation, may write a presentation of $\mathrm{Lie}(\{1,2,3\})$ as

$$\mathrm{Lie}(\{1,2,3\}) = \frac{\mathbb{Z}\mathbf{S}_3\{[[12]3],[1[23]]\}}{[[12]3] = -[[21]3],\ [[12]3] = -[3[12]],\ [1[23]] + [3[12]] + [2[31]] = 0}$$

Define $\mathrm{lie}_n^\vee := \mathrm{ch}(\mathrm{sgn} \otimes \mathrm{Lie}(\{1,\dots,n\}))$. Then Sundaram–Welker proved the following theorem [SW97], stated in this form in [HR17, Sec 2.3].

Proposition 5.8. *The Frobenius character of* C_k *is given by the symmetric function*

$$\mathrm{ch}(C_k) = \sum_{(m_1, m_2, \dots)\ |\ \sum_i i m_i = n,\ \sum_i (i-1) m_i = k} \ \prod_{i\ even} h_{m_i}[\mathrm{lie}_i^\vee] \prod_{i\ odd} e_{m_i}[\mathrm{lie}_i^\vee]$$

where h_m *is the homogeneous symmetric function, and* $f[g]$ *denotes plethysm of symmetric functions.*

There are formulas that express the symmetric functions $\mathrm{lie}^\vee$ in terms of power sums and mobius numbers (of $\mathbb{N}$), which can be used to make the above formula more explicit.

Theorem 5.9 (Stanley, [Sta82]). *We have* $lie_n^\vee = \frac{(-1)^n}{n} \sum_{d|n} \mu(d)(-1)^{n/d} p_d^{n/d}$.

Remark 5.10. Because $s(n, n-i)$ is a degree $2i$ polynomial in n, it follows from the arguments of this section that the ranks of $H_i(F_n, Z)$ eventually agree with a degree $2i$ polynomial in n. This can be used to show that the stable degree of $H_i(F, \mathbb{Z})$ is exactly $2i$. One can use this improved bound on stable degree to slightly improve the bounds for local degree, generation degree, and presentation degree of $H_i(F, \mathbb{Z})$. Plausibly, these improved ranges are also suboptimal. In fact, we conjecture that these quantities grow linearly with i.

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