

Efficient and stable data-sharing in a public transit oligopoly as a cooperative game

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Abstract

In this study, various forms of data sharing are axiomatized. A new way of studying coopetition, especially data-sharing coopetition, is proposed. The problem of the Bayesian game with signal dependence on actions is observed; and a method to handle such dependence is proposed. We focus on fixed-route transit service markets. A discrete model is first presented to analyze the data-sharing coopetition of an oligopolistic transit market when an externality effect exists. Given a fixed data sharing structure, a Bayesian game is used to capture the competition under uncertainty while a coalition formation model is used to determine the stable data-sharing decisions. A new method of composite coalition is proposed to study efficient markets. An alternative continuous model is proposed to handle large networks using simulation. We apply these models to various types of networks. Test results show that perfect information may lead to perfect selfishness. Sharing more data does not necessarily improve transit service for all groups, at least if transit operators remain noncooperative. Service complementarity does not necessarily guarantee a grand data-sharing coalition. These results can provide insights on policy-making, like whether city authorities should enforce compulsory data-sharing along with cooperation between operators or setup a voluntary data-sharing platform.

Keywords: oligopolistic transit market, data sharing, coopetition, coalition formation, Bayesian equilibrium, composite coalitional structure

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1 Introduction

Mobility-as-a-Service (MaaS) systems play an increasingly important role in providing urban transportation, especially in a smart cities context (Chow, 2018, Wong et al., 2020). Travelers may book one or more mobility services in getting from one location to another, particularly in congested areas where driving is not reliable or too costly. This is the case in many dense cities like New York, Tokyo, or London, which traditionally provide only fixed route public transit services. Due to the emergence of Internet of Things (IoT), many new privately operated mobility services have found their way to urban markets: peer-to-peer ridesharing, ride-hail, carshare, micromobility, microtransit, with variations that incorporate electric charging or automation.

Public agencies need to obtain data from private operators to measure and evaluate the market impact on travelers. This is nontrivial because operators risk losing competitive advantages if certain shared data is exploited by adversaries. As shown in He and Chow (2020), network data can be used by adversaries to reverse engineer an algorithm or operating policy. As such, operators require a certain level of privacy control when sharing data onto open data platforms to support public agency efforts. Due to this challenge, agencies in the industry have begun devising “Mobility Data Specifications” (MDS) to reach consensus on what types of data can and should be shared, and in what format, to best support mobility platform functions. An example is from the Los Angeles Department of Transportation (LADOT, 2020).

Addressing this concern from mobility operators can be important for cities with explicit oligopolistic competition, such as the MTR and private buses in Hong Kong (Today, 2018); regional competition between agencies like Port Authority (Path Train) and NJ Transit in the NJ/NYC metropolitan area; or considering private mobility companies like Uber Transit that provide service in coexistence with public transit agencies (Amtrak, 2017, TheVerge, 2021). This study investigates data sharing decisions in these contexts. Data sharing decisions cannot be separated from transit network operations; the value of data is in fact derived from transit operation.

We propose a new cooperative game methodology to study the joint decision of data sharing along with operational decisions like frequency setting between two or more coexisting operators (Chow and Sayarshad, 2014). An important feature of this type of game is that the payoff of an operator, or a coalition of operators who share data with each other, depends on the noncooperative actions of all other operators and the user route choices on a transit network. In other words, the novelty of the method lies in characterizing the data sharing as a type of coalition formation while simultaneously modeling the frequency setting as a noncooperative game with Bayesian equilibria for a given data-sharing coalition structure.

2 Literature review

2.1 Oligopolistic transit markets

Research studies on oligopolistic markets of transit operators have existed for decades. Harker (1988) studied deregulated transit systems with non-overlapping corridors to a central business district (CBD). He introduced the generalized Nash equilibrium concept and quasi-variational inequality (QVI) formulation. Fernandez and Marcotte (1992) first studied the transit market with a full network equilibrium model. They assumed a free competitive network – operators are free to offer service on any route. They combined the operator revenue maximization upper level model with user travel time-minimizing lower level model and expressed them as a system of variational inequalities (VI). Zubieta (1998) modified the Fernandez and Marcotte (1992) model to study the case where operators have exclusive right to run and adjust

frequencies. The transit route user equilibrium constraints expressed as VIs by Fernandez and Marcotte (1992) and Zubieta (1998) is an elegant way to capture the interactions between users and transit operators. This idea is used in this study.

Zhou et al. (2005) studied the oligopolistic transit market with elastic demand. They proposed a bi-level equilibrium model whose decision variables are prices. They transformed the problem to a QVI to prove the existence of Nash equilibria. Sun and Gao (2007) proposed a generalized Nash equilibrium model to study the joint behavior of operators and users. Clark et al. (2011) investigated the relationship between fares and trip lengths in a transit duopoly market equilibrium. They compared the results of collusion, Cournot, Stackelberg, Bertrand and sequential price competition models. Clark et al. (2014) studied the equilibrium of a duopoly providing complementary transport services differentiated by travel distances. Equilibria were derived for collusion and competition in price and quantity. Yao et al. (2019) focused on the problem of an upper-level transit authority assigning transit lines to lower-level operators making line frequency strategies. Rasulkhani and Chow (2019), Pantelidis et al. (2020b), and Ma et al. (2021) proposed different models under a stable matching framework to reveal the joint behavior of operator decisions (pricing and routes to serve) and user path choices. Pantelidis et al. (2020a) investigated the problem of how multiple operators can pool their risk together by setting up an insurance contract to share resources under disruption using cooperative game theory. The value of a coalition is computed by two-stage stochastic optimization.

Previous studies on transit markets mainly focus on using non-cooperative game theory to study the frequency or fare settings. There remains a research gap on modeling oligopolistic public transit design games involving data sharing as a strategy.

2.2 Data sharing game

Data, knowledge, or patent sharing games have much in common. Most of these studies use cooperative game theory. We remark that this implies some constraints on the data sharing form to be explained in Section 3. Muto et al. (1989) studied the *information market game* in which there is one veto player holding valuable information, such as patents. The payoff of a coalition $S \subseteq N$ is the profit which it can gain without the help of any outside firm. Aumann (1999) proposed five equivalent formalizations of the idea of knowledge: signal functions, information functions, information partitions, knowledge operators, and knowledge universal fields. Information partitions have been widely used to study information sharing since then. We adopt this interpretation of knowledge in this study.

Brânzei et al. (2001) studied *information collecting (IC) games*. This type of game has one decision-maker (the only veto player) and many information holders. The information partition structure from Aumann (1999) is adopted. Decision maker i chooses a single action for each element of the information partition that maximizes expected payoff. Slikker et al. (2003) extended IC games to *information sharing (IS) games*. Each player is a decision maker and receives a reward. The coalition in this game means a data-sharing coalition: members share all the data that they have with each other. The reward of a coalition is the sum of its members' expected rewards. This type of game has *monotonic allocation schemes*, which means as a coalition grows larger, every player involved can secure more rewards. Two facts lead to this property: 1) a player can identify the true state better with added information, and 2) the reward of one player is independent of the actions of the other players. The grand coalition is optimal for everyone; hence it is guaranteed to emerge.

Slikker et al.'s (2003) IS game cannot be directly applied to transit markets; the reward of a transit operator depends not just on its own operational decisions. This interaction is sometimes called the *externality effect* (Ray, 2007). Hence the transit game does not belong to

the class of games with monotonic allocation schemes generally and the grand coalition is not guaranteed. Ray (2007) and Diamantoudi and Xue (2007) studied the IS game with externality effect using *coalition formation* theory. The solution is a coalition structure that is stable from deviations. Deviation means a group of players within a coalition trying to break out. The group must also attempt to predict the coalition structure that arises elsewhere, otherwise it won't be able to compute its own payoffs since they depend on the actions of other players. Hence the set of coalitional equilibrium actions ("EBA set") are built recursively, starting from singletons. The final solution depends on the "initial" condition specified. Ray (2007) gave two types of initial conditions: starting from either a grand coalition or a singleton (every player on their own) structure. In this study, we will call them top-down and bottom-up solutions, respectively. Transportation applications of coalition formation models include shipper collaboration (e.g. Yilmaz and Savasaneril, 2012), connected vehicle infrastructure (Saad et al., 2010), and vehicle routing (Wang et al., 2018).

Bacchetta and Espinosa (1995) researched information sharing between governments where tax collection competition exists. Governments can choose information sharing and tax levels. A two-stage non-cooperative game is proposed. Tosh et al. (2015) analyzed the cyber security information sharing among firms with non-cooperative game theory. The study of Bacchetta and Espinosa (1995) and Tosh et al. (2015) represent a type of data-sharing game in which data have direct and obvious effects on player utility.

The idea of acting under uncertainty also appears in non-cooperative games under the name *Bayesian game* (Osborne and Rubinstein, 1994). Players' preferences are influenced by the unknown system state. Players can make partial observations and then choose actions depending on their posterior speculations. A Nash equilibrium arising from such a game consists of a set of actions corresponding to each pair (*player, observation*) such that no player has the incentive to choose other actions given the observations and other players' strategies. Bayesian games have been used in the literature to address cybersecurity in intelligent transportation systems under uncertainty (see Sedjelmaci et al., 2019), traffic route assignment under uncertainty (Gairing et al., 2008), and electric vehicle charging demand management (Liu et al., 2017). To date, information sharing action has not been studied using these games. The knowledge structure for each player is assumed and fixed. Bayesian games capture the competition among operators under uncertainty well, but they lack the data sharing cooperation strategy.

2.3 Coopetition

Coopetition is one kind of game in which both cooperation and competition exist. Coopetition is a subject on which the management literature has increasingly focused since Nalebuff and Brandenburger (1996). When data-sharing happens between competing companies, it is a common form of coopetition. "There is a paradox that the knowledge shared for cooperation may also be used for competition" (Loebecke et al., 1999). This situation is also called the *coopetition dilemma*. Loebecke et al. (1999) proposed a normal form game to study knowledge transfer actions between companies. The gains from the cooperation and the losses from competition are values assumed to be known. Smichowski (2018) analyzed the willingness of transit operators to participate in a MaaS platform which requires data-sharing. He described two effects: "losing rides through MaaS" effect due to competition from the operators on the platform and "winning rides through MaaS" effect since cooperation helps draw more users to the MaaS platform. They illustrated how the two effects change with the number of operators participating (Figure 1). They conclude that an operator is willing to join a platform once the size of the platform is over a threshold: a "minimum number of cooperating players that justify sharing data". The setup of the initial MaaS platform assumes simultaneous actions by

operators. Again, a normal form game is used to model this like in Loebecke et al. (1999). Smichowski's (2018) threshold assumes that more players involved in data sharing is always better for each player.

Carfi (2012, 2015) proposed another formulation of competition. The action profile in their games is (x, y, z) , where x and y are the independent actions of the first and second players respectively; z is an action to be chosen by players jointly. For a fixed z , the game is reduced to a normal form game, G_z , with solution set S_z . S_z is a set-valued function of z in the feasible solution space. A Nash bargaining model is then applied to find the setting of z .

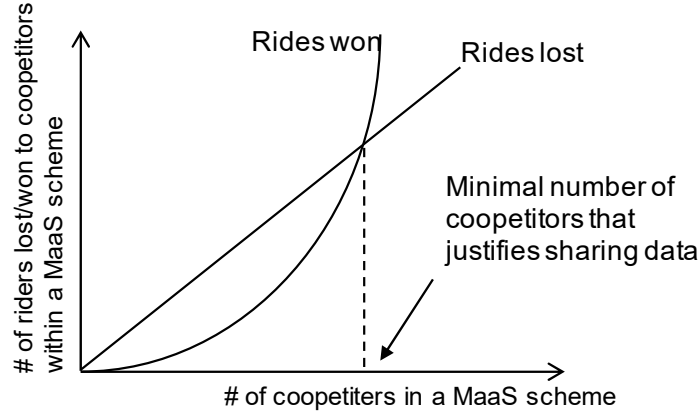


Figure 1. The choice of data sharing in the context of MaaS from the perspective of a single transportation operator, which assumes that larger data sharing coalitions are always better for all players. (Smichowski, 2018)

For a transit market, there is a popular belief that more data sharing means better services, as suggested by Smichowski (2018) and reflected by the efforts of Mobility Data Specification (LADOT, 2018), Open Mobility Foundation (Smart City Dive 2022), Open City Network (Pembleton, 2019). Should city authorities enforce compulsory data-sharing or setup voluntary data-sharing platforms? Our intuition is that competitive operators are not willing to share while complementary operators tend to cooperate. In practice, some parties are not willing to participate in data-sharing and it remains a question what kind of policy is needed. Many factors may influence data-sharing competition, like the network structure, user/operator cost settings, the distribution of lines between operators, the degree of complementarity between transport operators' lines, etc. Currently, there is no model to support policy-making decisions and factors like these.

2.4 Summary

To sum up, the main research gap is that there lacks an oligopolistic transit market competition model with cooperative data-sharing and noncooperative frequency setting under uncertainty that depends on the coalition. This model needs to handle externality effects. This study benefits from the ideas of Slikker et al. (2003), Bayesian games (Osborne and Rubinstein, 1994), Ray (2007), and Carfi (2012). In our competitive transit game, operators have two types of actions: information sharing decisions and transit operational decisions. As shown above, information sharing can be modeled using coalition formation to define partitions for players that choose to share information with one another. Meanwhile, the outcome equilibrium needs to consider the availability of incomplete information in a noncooperative game setting; this is typically modeled using Bayesian games. For each data sharing structure, there is an associated Bayesian game to capture the market competition. Nash equilibria are found for these Bayesian games. Then coalition formation theory is used to determine stable data-sharing decisions and

payoffs in which no operator has incentive to switch from. This type of coopetition is similar to that of Carfi (2012) but with the added complexities of the oligopolistic transit market. The Nash bargaining model is also replaced with a coalition formation model extended from Slikker et al. (2003) to determine stable coalitions for data sharing. The contributions of this paper are summarized below:

- Various forms of data sharing are axiomatized;
- A new way of studying coopetition, especially data-sharing coopetition, is proposed;
- A Bayesian game with signal dependence on actions is observed and a method to handle such dependence is proposed;
- A method to remedy coalition formation with market efficiency is proposed;
- Two alternative models—discrete and continuous—are proposed to analyze the data-sharing coopetition for a transit market;
- Test results on transportation networks provide insights on transportation market policy making.

While the intent of this research is to study the transit market data-sharing, many of the ideas presented extend beyond this scope. For example, the data-sharing axiomatization scheme, the way we study coopetition, the method to handle Bayesian games with signal dependence on actions, and the idea of coalition composition are all generally applicable.

3 Proposed methodology

Before getting started, we make, to the best of our knowledge, a first attempt in the literature to define properties of various forms of data sharing.

Definition 1: A *data sharing relation* is a set of ordered pairs where (i, j) belongs to this set if and only if i shares data to j . The data sharing relation is said to be:

- *reflexive*, if (i, i) belongs to this relation;
- *symmetric*, if (i, j) belongs to this relation, then (j, i) also belongs to this relation;
- *transitive*, if (i, j) and (j, k) belong to this relation, then (i, k) also belongs to this relation;
- *nonexclusive*, if (i, j) belongs to this relation for some $j \neq i$ then (i, k) belongs to this relation for any k .

The first three are commonly seen definitions in mathematics. Reflexivity is introduced for convenience. We define nonexclusive to mean that operator i can only choose to *publish* data or not. In this way, it's easier to see the connections between various forms of data sharing and to do the proofs. It opens a gateway to new forms of data sharing that have not been proposed in the literature or used in practice. New forms of data sharing can be identified by changing the set of axioms.

A freely constrained data-sharing form requires the data sharing relation to be reflexive, which we define as “free mode data sharing”. Any operator i can choose whether or not to share data to any other j . One common form of data sharing is to require the relation to be reflexive and symmetric. This relation is like “is-a-friend-of” relation (“friend mode data sharing”). For example, operator A and B are friends; B and C are friends; but A and C are not. So, A and B share their *own* data with each other; B and C also share with each other. But A and C don't. A very constrained form of data sharing is to require the relation to satisfy the nonexclusive property (“publishing mode data sharing”). Another commonly seen form is to require the relation to be reflexive, symmetric, and transitive. Namely, the data sharing relation is an *equivalence relation* (“equivalence relation mode data sharing”). It defines a partition.

Existing studies focus on either “free mode”, equivalence relation, or “publishing” types of data sharing. In this study, we focus on the equivalence relation.

Consider the bilevel formulation (Fernandez and Marcotte, 1992, Zubieta, 1998) for an oligopolistic transit market. Let N be the set of transit operators, indexed by $i = 1, \dots, n$. This market is defined by a graph J where different operators own subgraphs J_i , where $i = 1, \dots, n$ and $J = \bigcup_{i=1}^n J_i$. The market is assumed to have a deterministic, fixed origin-destination (OD) demand $(h_\theta)_{\theta \in \Theta}$, for OD set Θ , known by all the operators. In the upper level, given the user hyperpath flow $\mathbf{f} \in D$, where D is the set of feasible hyperpath flows, operators optimize line frequencies $\boldsymbol{\phi}$ independently of each other to maximize their own profits. In the lower level, given the frequency settings $\boldsymbol{\phi}$, user equilibrium hyperpath flows $\mathbf{f}$ are generated. The payoff r_i of operator i is a differentiable function of frequency settings $\phi_i \in \mathbb{R}_{\geq 0}^{L_i}$, where L_i is the number of lines per operator, and passenger flow vector $\mathbf{f}$. The upper-level maximization problem is shown in Eq. (1), where ϕ_{i-} refers to the frequencies set by operators other than i . We assume that operators are privately-owned and offer fixed-route transit services. This framework can be applied to other types of operators and objectives as well without loss of generality.

$$\max_{\phi_i \geq 0} r_i(\phi_i; \phi_{i-}, \mathbf{f}), \quad \forall i \in N \quad (1)$$

The KKT conditions of Eq. (1) are shown in Eqs. (2) to (4).

$$\phi_i^* \geq 0, \quad \forall i \in N \quad (2)$$

$$H_i(\phi_i^*; \phi_{i-}^*, \mathbf{f}^*) := - \frac{\partial r_i(\phi_i; \phi_{i-}, \mathbf{f})}{\partial \phi_i} \big|_{\phi_i^*, \mathbf{f}^*} \geq 0, \quad \forall i \in N \quad (3)$$

$$\phi_i^* \cdot H_i(\phi_i^*; \phi_{i-}^*, \mathbf{f}^*) = 0, \quad \forall i \in N \quad (4)$$

This is a nonlinear complementarity problem. We can transform it to a VI problem as shown in Eq. (5) (Karamardian, 1972). In Zubieta (1998) it is assumed that each operator has only one line. Here, this constraint is relaxed. The products in Eqs. (4) and (5) are inner products.

$$H_i(\phi_i^*; \phi_{i-}^*, \mathbf{f}^*) \cdot (\phi_i - \phi_i^*) \geq 0, \quad \forall \phi_i \geq 0 \quad (5)$$

Combining Eq. (5) for all operators leads to a VI in Eq. (6). The lower level user equilibrium model as formulated in Spiess and Florian (1989) is written as a VI in Eq. (7). Other types of transit user assignment models can be adopted so long as they can be formulated as VI. For example, stochastic user equilibrium models (Fisk, 1980) can be transformed from an optimization problem to a VI in the same way as described above. Eqs. (6) and (7) form a system of VIs that represent the equilibrium condition of an oligopolistic transit market under deterministic demand and perfect information.

$$\mathbf{H}(\boldsymbol{\phi}^*; \mathbf{f}^*) \cdot (\boldsymbol{\phi} - \boldsymbol{\phi}^*) \geq 0, \quad \forall \boldsymbol{\phi} \geq \mathbf{0} \quad (6)$$

$$\mathbf{C}(\mathbf{f}^*; \boldsymbol{\phi}^*) \cdot (\mathbf{f} - \mathbf{f}^*) \geq 0, \quad \forall \mathbf{f} \in D \quad (7)$$

where

$$H(\phi; f) = - \begin{bmatrix} \frac{\partial r_1}{\partial \phi_1} \\ \dots \\ \frac{\partial r_n}{\partial \phi_n} \end{bmatrix}$$

$$\phi = \begin{bmatrix} \phi_1 \\ \dots \\ \phi_n \end{bmatrix}$$

ϕ^* : equilibrium frequency setting of all operators;

f^* : the equilibrium hyperpath flow vector on the network;

C : the vector-valued hyperpath cost function;

D : the feasible set of hyperpath flows that satisfies demand h_θ given OD pair $\theta \in \Theta$; Θ is the set of all OD pairs.

On top of the oligopolistic market described by the VIs in Eqs. (6) – (7), we now assume there are some unknown information that may influence operator payoffs. Let the unknown state be the origin-destination (OD) demand $h_\theta(\omega), \theta \in \Theta$. Other types of unknown information such as user preferences can be handled in the same way in principle. Now we allow operators to form data sharing coalitions with each other. Note that players in the same coalition may share data but still act noncooperatively in frequency setting. The standard cooperative game theory starts with a characteristic function v . However, we encounter a problem with defining the worth, $v(S)$, of a data-sharing coalition S . This is because the profit of coalition S also depends on the data-sharing and frequency setting actions of players in $N \setminus S$ since travelers choose routes on the whole transit network - the *externality effect*.

We study this problem by using the idea of a *partition function form* (Kóczy, 2018). We make assumptions on how $N \setminus S$ would break into further sub-data-sharing-coalitions. Formally, a partition $\mathcal{P}$ of N defines a collection of subcoalitions formed by the players. Given a partition $\mathcal{P}$ of N , we use a *partition function* $\mathbf{u}_{S|\mathcal{P}}$ to represent the payoff vector of all operators in coalition S under $\mathcal{P}$ ($S \in \mathcal{P}$), where $u_{i,S|\mathcal{P}}$ refers to the element corresponding to operator i in the vector. The knowledge of each operator is determined by $\mathcal{P}$. Operators set frequencies independently under an uncertain state. This is modeled by a Bayesian game in Section 3.1. For each coalition structure, which we call a partition, we can find a Bayesian game equilibrium. Then we turn to coalition formation and coalition composition theory in Section 3.2 to evaluate the data-sharing decisions. The output is a composite coalition structure, which we characterize as $\mathcal{Q} = [\mathcal{P}^1, \mathcal{P}^2, \dots, \mathcal{P}^K]$. The main assumptions are summarized below and the steps are shown in *Algorithm 1*.

(Rationality assumption) Transit operators are selfish and rational;

(Data sharing partition assumption) Data sharing relation is an equivalence relation;

(Bayesian assumption) The system has an unknown state that affects transit operators' payoffs; operators share a common prior of the state; operators can gain partial information by observing link flows from their coalition (the *signal*);

(Data-sharing cooperation assumption) Transit operators can write binding agreements at no cost to form data-sharing coalitions;

(Competitive market assumption) All transit operators (even those sharing the same data sharing coalition) set transit frequencies independently.

Algorithm 1: Data-sharing coopetition

1: Initiate transit network J ;

- 2: For each partition $\mathcal{P}$:
 - 3: Apply *Algorithm 2* (discrete) or *Algorithm 3* (continuous) to find the Nash equilibrium expected payoffs $\mathbf{u}_{S|\mathcal{P}}$ of the Bayesian game G^* under $\mathcal{P}$;
 - 4: End for
 - 5: Apply *Algorithm 4* to find the composite coalition structure $\mathcal{Q} = [\mathcal{P}^1, \mathcal{P}^2, \dots, \mathcal{P}^K]$ and the corresponding action profile and payoffs $\mathbf{u}_{S|\mathcal{P}^K}$.
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3.1 Bayesian game model for frequency settings

3.1.1 Discrete setting

First, we briefly explain the Aumann (1999)'s representation of knowledge. Let Ω denote the set of possible states of the world. The true state $\omega \in \Omega$ is unknown. Assume that there is a common prior on Ω , described by probability space $(\Omega, \mathcal{F}, \mu)$, where $\mathcal{F}$ is the σ -algebra, i.e. the elements of $\mathcal{F}$ are events – subsets of Ω , and μ is the probability measure defined on $\mathcal{F}$. The information owned by a player $i \in N$ is represented by a partition $\mathbf{I}_i$ of Ω . A partition $\mathbf{I}$ of set Ω is a collection of non-overlapping subsets whose union is Ω . Each element in $\mathbf{I}_i$ should be a measurable set representing a possible observation by player $i \in N$. A new partition $\mathbf{I}_i \vee \mathbf{I}_j$ can be generated by the intersections of elements from $\mathbf{I}_i$ and $\mathbf{I}_j$. This idea can be extended to a coalition such that $\mathbf{I}_S = \bigvee_{i \in S} \mathbf{I}_i$.

Operators set frequencies depending on their collected data, which would be transit link flows in this study, which can be used to infer OD demand (e.g. Liu and Chow (2022)). T_i , the set of possible observations for operator i , is called the *signals* of operator i . Every operator i has a *signal function* $\tau_i: \Omega \rightarrow T_i$. $\mathbf{I}_i := \{\tau_i^{-1}(t_i) | t_i \in T_i\}$ is the information of operator i . Throughout Section 3.1.1, we assume that Ω and T_i are finite sets. For a data sharing coalition S , each member has the same signal function $\tau_S: \Omega \rightarrow \bigoplus_{i \in S} T_i$. For a Bayesian game, the set of *players* is the set of all (*operator, observation*) pairs $(i, \mathbf{t}_i)$, where $\mathbf{t}_i = \tau_i(\omega)$. The *action* $\phi_{i, \mathbf{t}_i}$ of each player $(i, \mathbf{t}_i)$ refers to the frequency set under observation $\mathbf{t}_i$. The action set of i is denoted by A_i . An *action profile* $\boldsymbol{\phi}$ is an element of $\times_{i \in N} (\times_{\mathbf{t}_i \in T_i} A_i)$. Given $\boldsymbol{\phi}$, we have a *lottery* $\mathcal{L}_i(\boldsymbol{\phi}, \mathbf{t}_i)$ over $A \times \Omega$ for each player $(i, \mathbf{t}_i)$, where $A = \prod_i A_i$. A lottery is a collection of (*action-profile, state*) pairs weighted by probabilities. We define player preference over lotteries instead of action profiles. The probability assigned by $\mathcal{L}_i(\boldsymbol{\phi}, \mathbf{t}_i)$ to $\left(\left(\phi_{j, \tau_j(\omega)} \right)_{j \in N}, \omega \right)$, denoted by $p_{i, \boldsymbol{\phi}}(\omega | \mathbf{t}_i)$, means player $(i, \mathbf{t}_i)$'s posterior belief of the state being ω and each operator j acting with $\phi_{j, \tau_j(\omega)}$.

Unlike Aumann's (1999) model, the signal functions also depend on the actions. The number (or proportion) of passengers that an operator observes depend on the frequency setting of all operators. In contrast, in a typical Bayesian game setting, the signal function is fixed and independent of the actions. We introduce the following definition of *consistency*.

Definition 2: An observation vector $\mathbf{t}$ is said to be *consistent* with frequency strategy $\boldsymbol{\phi}$ under demand $\mathbf{h}(\omega)$, $\omega \in \Omega$, if $\tau_i(\mathbf{h}(\omega), \boldsymbol{\phi}(\mathbf{t})) = \mathbf{t}_i$ for each operator $i \in N$. The *consistent set*, $E(\omega, \boldsymbol{\phi})$, includes all observations that are consistent with $\boldsymbol{\phi}$ under state $\omega \in \Omega$.

Consistency means that if operators observe $\mathbf{t}$ and then act with $\boldsymbol{\phi}(\mathbf{t})$, then they will observe $\mathbf{t}$ as a result of traffic assignment under the same state $\omega \in \Omega$. $E(\omega, \boldsymbol{\phi})$ can be found by making use of a transit assignment model (e.g. Eq. (7)) when the size of Ω and T are small. Let

$p_{i,\phi}(\omega, \mathbf{t} \mid \mathbf{t}_i)$ be the posterior probability of the state being $\omega \in \Omega$ and the observation vector of all operators being $\mathbf{t}$ (the i -th component of $\mathbf{t}$ is $\mathbf{t}_i$), for player $(i, \mathbf{t}_i)$ given the frequency vector ϕ . We compute $p_{i,\phi}(\omega, \mathbf{t} \mid \mathbf{t}_i)$ with Eq. (8).

$$\begin{aligned} p_{i,\phi}(\omega, \mathbf{t} \mid \mathbf{t}_i) &= \frac{p_{i,\phi}(\omega, \mathbf{t})}{p_{i,\phi}(\mathbf{t}_i)} \\ &= \frac{p_{i,\phi}(\mathbf{t} \mid \omega) p_{i,\phi}(\omega)}{\sum_{\omega'} p_{i,\phi}(\mathbf{t}_i \mid \omega') p_{i,\phi}(\omega')} \\ &= \begin{cases} \frac{\frac{1}{|E(\omega, \phi)|} p(\omega)}{\sum_{\omega'} \frac{|\{\mathbf{t}' \mid \mathbf{t}'_i = \mathbf{t}_i, \mathbf{t}' \in E(\omega', \phi)\}|}{|E(\omega', \phi)|} p(\omega')}, & \text{if } \mathbf{t} \in E(\omega, \phi) \\ 0, & \text{otherwise} \end{cases} \end{aligned} \quad (8)$$

where $p(\omega)$ is the common prior and $|E|$ means the cardinality of the set E . The set $\{\mathbf{t}' \mid \mathbf{t}'_i = \mathbf{t}_i, \mathbf{t}' \in E_{\omega', \phi}\}$ is the set of observations $\mathbf{t}'$ that is consistent with ϕ under $\omega' \in \Omega$ and the i -th component being $\mathbf{t}_i$. We assume that observations in $E(\omega, \phi)$ are equally possible in Eq. (8). Now we redefine the probability assigned by lottery $\mathcal{L}_i(\phi, \mathbf{t}_i)$ to $(\omega, \mathbf{t})$ to be $p_{i,\phi}(\omega, \mathbf{t} \mid \mathbf{t}_i)$. Assume that operators are risk-neutral. The *von Neumann–Morgenstern utility function* for a player $(i, \mathbf{t}_i)$ is represented by an expected profit in Eq. (9). We assume that the operational cost of a transit line is proportional to its frequency.

$$\begin{aligned} u_{i,S|\mathcal{P}}(\mathcal{L}_i(\phi, \mathbf{t}_i)) &= \mathbb{E}[r_i(\phi, \mathbf{f}) \mid \mathbf{t}_i] \\ &= \sum_{\omega \in \Omega, \mathbf{t} \in F(\omega)} \left(\sum_{l \in L_i} \alpha_l v_l(\omega, \phi(\mathbf{t})) - \beta_l \phi_l \right) p_{i,\phi}(\omega, \mathbf{t} \mid \mathbf{t}_i) \end{aligned} \quad (9)$$

where $\mathbf{t}_i$: the observation of operator $i \in N$;
 $\mathbf{t}$: observation vector of all operators;
 L_i : set of links (or lines) owned by $i \in N$;
 ϕ_l : frequency of link (or line) $l \in L_i$;
 α_l : the transit fare of link (or line) $l \in L_i$;
 v_l : flow on link (or line) $l \in L_i$;
 β_l : operational cost coefficient of link (or line) $l \in L_i$;
 $F(\omega)$: the set of feasible observation vector under ω .

The transit flow v_l is a function of OD demand $\mathbf{h}(\omega)$ and frequency settings $\phi(\mathbf{t})$. The summation in Eq. (9) is over all possible ω and $\mathbf{t}$ combinations weighted by the posterior probability $p_{i,\phi}(\omega, \mathbf{t} \mid \mathbf{t}_i)$. The preference ordering $\succsim_{(i, \mathbf{t}_i)}$ of each player $(i, \mathbf{t}_i)$ is defined as:

$$\phi \succsim_{(i, \mathbf{t}_i)} \phi' \quad \text{iff} \quad u_{i,S|\mathcal{P}}(L_i(\phi, \mathbf{t}_i)) \geq u_{i,S|\mathcal{P}}(L_i(\phi', \mathbf{t}_i))$$

This Bayesian game is denoted by the tuple $G^* = \langle N, \Omega, (A_i), (T_i), (\tau_i), \mathbf{p}, (\succsim_{(i, \mathbf{t}_i)}) \rangle$.

We use Eq. (10) and (11) to replace Eq. (6) and (7) to characterize the Nash equilibrium of Bayesian game G^* . Eq. (11) means that the user equilibrium holds under every $\mathbf{h}(\omega) - \mathbf{t}^\omega$ combination if $\mathbf{t}^\omega$ is feasible under ω . Notice that we need both ω and $\mathbf{t}^\omega$ to index the

equilibrium flow. Eq. (10) means that operator i sets frequency at observation $\mathbf{t}_i$ by maximizing its expected payoffs. This also holds under every feasible $\mathbf{t}$. The total number of possible states and observations are finite. Eq. (10) and (11) are used in the proof of existence of a Nash equilibrium (see the Appendix). The steps to find a Nash equilibrium are shown in Algorithm 2 below.

$$\sum_{i=1}^n -\frac{\partial}{\partial \phi_i} u_{i,S|\mathcal{P}}(\mathcal{L}_i(\phi_i; \phi_{i-}, \mathbf{t}_i))|_{\phi^{*,\mathbf{t}}} \cdot (\phi_i - \phi_i^{*,\mathbf{t}}) \geq 0, \quad \forall \phi_i \geq 0, \quad \forall \mathbf{t} \in F \quad (10)$$

$$C(\mathbf{f}^{*,\omega,\mathbf{t}^\omega}; \boldsymbol{\phi}^{*,\mathbf{t}^\omega}) \cdot (\mathbf{f} - \mathbf{f}^{*,\omega,\mathbf{t}^\omega}) \geq 0, \quad \forall \mathbf{f} \in D(\omega), \quad \forall \omega \in \Omega; \mathbf{t}^\omega \in F(\omega) \quad (11)$$

where $\boldsymbol{\phi}^{*,\mathbf{t}}$: the frequency vector under observation $\mathbf{t}$ at equilibrium;

$\mathbf{t}^\omega$: the observation vector of all operators under ω ;

$\mathbf{f}^{*,\omega,\mathbf{t}^\omega}$: the flow under state ω and $\mathbf{t}^\omega$ at equilibrium;

$D(\omega)$: set of feasible transit flow vectors under ω ;

$F(\omega)$: the set of feasible observation vectors under ω ;

F : the set of feasible observation vectors; $F = \bigcup_{\omega} F(\omega)$;

Algorithm 2: Computing the Nash equilibrium (discrete case)

- 1: Initialize the strategy $\phi_i^{(0)}$ for each operator i ;
 - 2: While the difference of strategies $|\phi^{(n)} - \phi^{(n-1)}| \geq \epsilon_d$:
 - 3: For operator i :
 - 4: For each possible observation $\mathbf{t}_i$;
 - 5: Fix the strategies of players other than $(i, \mathbf{t}_i)$, find $\arg \max_{\phi_i^{(n)}(\mathbf{t}_i)} u_{i,S|\mathcal{P}}(\mathcal{L}_i(\boldsymbol{\phi}, \mathbf{t}_i))$;
 - 6: Update the current strategy for operator i ;
 - 7: End for
 - 8: End for
 - 9: End while
-

The steps in Algorithm 2 correspond to players alternately updating their strategies. This is the usual way to find a pure-strategy Nash equilibrium if it exists. Unfortunately, the existence and uniqueness of such a solution cannot be guaranteed in general. In Algorithm 2, we stop iterating when the successive strategy change is smaller than ϵ_d . An alternative is to find a mixed strategy Nash equilibrium, whose existence is guaranteed. Implementation of that algorithm will be reserved for future research.

Suppose there are A levels of frequency settings and the observations are discretized into K levels. Also, suppose each operator i has M_i observations. There would be $\sum_{i=1}^n K^{M_i}$ players in the Bayesian game. This number increases exponentially as the network grows. We need to optimize frequencies for each such player based on its posterior perception. This is formidable work for a large network. The number of iterations is bounded by $A^{\sum_{i=1}^n K^{M_i}}$, the number of entries in the normal form of the Bayesian game.

3.1.2 Adaptations for large networks

While the previous model is proven to have Bayesian Nash equilibria (Appendix), the assumption of discrete distribution for random OD pairs and link flows is not convenient for large networks. As such, we adopt an alternative model with continuous random variables to

represent the OD demands and link flows. The strategy for operator i would be a vector-valued multivariate function $\phi_i(\mathbf{t}_i)$. This has a polynomial complexity of the number of observations (namely, the number of transit links) and the number of transit lines, which is more computationally tractable for large-scale instances. For example, suppose operator i has L_i lines. Operator i 's frequency setting is vector $\phi_i = [\phi_{i,1}, \dots, \phi_{i,l}, \dots, \phi_{i,L_i}]'$. i 's observation vector is $\mathbf{t}_i$ with dimension M_i , so $\mathbf{t}_i = [t_{i,1}, \dots, t_{i,m}, \dots, t_{i,M_i}]'$. Suppose we consider a polynomial of degree d . Then i 's line l frequency as a function of $\mathbf{t}_i$ has the following form (subscript “ i ” in all these notations neglected for simplicity) in Eq. (12). The dimension of this polynomial space is a constant $L_i \binom{d + M_i}{d}$.

$$\phi_l(\mathbf{t}) = c^0 + \sum_{m=1}^M c_m^1 t_m + \sum_{m^1=1}^M \sum_{m^2=m^1}^M c_{m^1, m^2}^2 t_{m^1} t_{m^2} + \dots + \sum_{m^1=1}^M \dots \sum_{m^d=m^{d-1}}^M c_{m^1, \dots, m^d}^d t_{m^1} \dots t_{m^d} \quad (12)$$

We propose *Algorithm 3* to determine the equilibrium.

Algorithm 3: Computing the Nash equilibrium (continuous case)

- 1: Initialize strategy (polynomial coefficients) $\phi^{(0)}$ for each operator i ;
- 2: While the difference of strategies $|\phi^{(n)} - \phi^{(n-1)}| \geq \epsilon_c$:
3. Draw N_{pool} samples from $(\omega, \mathbf{t})$ distribution (‘sample pool’) based on current strategies:
 - 3.a. Simulate truncated multivariate normal $\mathbf{h}(\omega)$ using combination of Choleski transformation and accept-reject method;
 - 3.b. Initialize observation $\mathbf{t}^{(0)}$, then obtain the initial assigned transit flow $\mathbf{f}^{(0)}$;
 - 3.c. While transit flow differences $|\mathbf{f}^{(k)} - \mathbf{f}^{(k-1)}| \geq \epsilon_s$:
 - 3.d. Update the frequencies by evaluating polynomial $\phi^{(k)}$ at observation $\mathbf{t}^{(k)}$ based on current strategies;
 - 3.e. Assign $\mathbf{h}(\omega)$ to network, update transit flow $\mathbf{f}^{(k+1)}$ and observation $\mathbf{t}^{(k+1)}$;
 - 3.f. End while;
4. For operator i :
5. For k in range(0, $param1_i$) where $param1_i \geq L_i \binom{d + M_i}{d}$:
6. Draw the k -th observation sample $\mathbf{t}_{i,k}$, by choosing one sample randomly from sample pool without replacement to simulate the marginal distribution $p(\mathbf{t}_i)$;
7. For player $(i, \mathbf{t}_{i,k})$:
8. Sample the conditional distribution $p(\omega, \mathbf{t} | \mathbf{t}_{i,k})$ by choosing observations from the sample pool whose i -th part $\mathbf{t}'_i$ being close enough to $\mathbf{t}_i$, namely $|\mathbf{t}'_i - \mathbf{t}_i| \leq \epsilon'_s$, or by choosing $param2_i$ closest samples;
9. Find $\phi_{i,k}^*(\mathbf{t}_{i,k})$ by maximizing $u_{i,S|P}(\mathcal{L}_i(\phi, \mathbf{t}_{i,k}))$ with sampled conditional distribution;
10. End for
11. End for
12. Solve the new polynomial fitting coefficients using pairs $\{\mathbf{t}_{i,k}, \phi_{i,k}^*(\mathbf{t}_{i,k})\} | k = 1, \dots, C_i\}$ by linear regression;

13. Use Method of Successive Averaging (MSA) step to update the strategy:

$$\phi_i^{(n+1)} = \phi_i^{(n)} + \frac{1}{n}(\phi_i^* - \phi_i^{(n)});$$
- 14: End for
- 15: End while
-

Simulations are used in step 3 and 6 instead of exact posterior probabilities. Step 3 is about sampling the marginal distribution $p(\mathbf{t}_i)$. We first simulate $p(\omega, \mathbf{t})$ to form a sample pool, then obtain the marginal distribution simulation. Step 6 is about sampling the conditional distribution $p(\omega, \mathbf{t}|\mathbf{t}_i)$. Gibb's sampling and Metropolis-Hastings are difficult to implement for $p(\mathbf{t}|\mathbf{t}_i, \omega)$ and $p(\omega|\mathbf{t})$ because we do not know the density formula of $p(\omega, \mathbf{t}|\mathbf{t}_i)$. It is also difficult to simulate $p(\omega, \mathbf{t}|\mathbf{t}_i)$ from samples of the joint distribution $p(\omega, \mathbf{t})$ since it is not possible to find enough sample points as shown in Figure 2(c). In this study, we use the illustrated in Figure 2(d) to construct these distributions. The payoff outputs of the continuous model are random variables since simulations are involved. In Algorithm 3, there are two parameters: the size of the samples drawn from marginal distribution from step 5 (*param1_i*) and the size of samples drawn from the conditional distribution from step 8 (*param2_i*). The variance of payoffs are functions of these two parameters, $var_{param1_i, param2_i}[payoff]$.

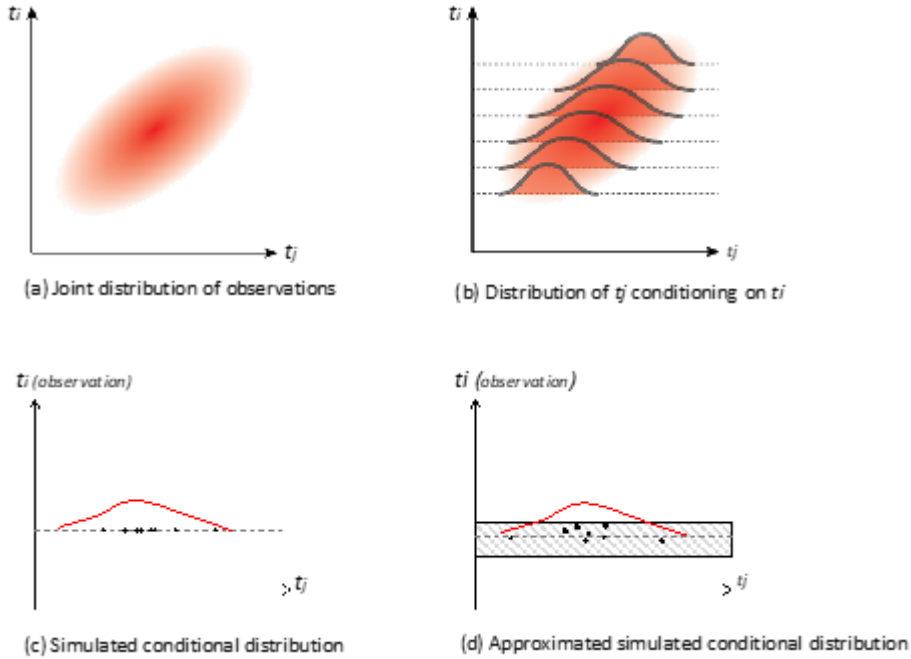


Figure 2. (a) the joint distributions of observations of operators; (b) the distribution of operator j conditioning on the observation of operator i; (c) the desired simulation of the conditional distribution; (d) the approximation approach to simulate the conditional distribution.

The coefficients of $\phi_i(\mathbf{t}_i)$ polynomial can be estimated as a linear regression using pairs $\{\mathbf{t}_{i,k}, \phi_{i,k}^*(\mathbf{t}_{i,k})\} | k = 1, \dots, C_i\}$. The bottleneck of Algorithm 3 is step 9: finding optimal $\phi_{t_{i,k}}^*$ given the simulated conditional distribution $p(\omega, \mathbf{t}|\mathbf{t}_{i,k})$. Eq. (13) is the formula meant to find ϕ , where the integration is conducted over the probability distribution of $\omega, \mathbf{t}|\mathbf{t}_i$. Meanwhile, Eq. (14) is the simulated one that we use.

$$\phi_{t_{i,k}}^* = \arg \max_{\phi} \int r_i(\phi|\omega, \mathbf{t}) d_{F(\omega, \mathbf{t}|\mathbf{t}_{i,k})} \quad (13)$$

$$\phi_{t_{i,k}}^* \approx \arg \max_{\phi} \frac{1}{M} \sum_{m=1}^M r_i(\phi | \omega^{(k,m)}, t^{(k,m)}) \quad (14)$$

where M is the number of samples simulated for $p(\omega, \mathbf{t} | \mathbf{t}_{i,k})$; $\omega^{(k,m)}, t^{(k,m)}$ are the m -th simulated demand and (complete) observation vector for the k -th simulation of $p(\mathbf{t}_i)$.

The profit function $r_i(\phi | \omega, t)$ function is generally not guaranteed to be monotone nor convex. Many optimization sub-problems like this take time. If the dimension of a frequency vector is not large, we can divide the frequency domain into $\left(\frac{\bar{\phi} - \phi}{\Delta\phi}\right)^{L_i}$ parts and try only one point in each part. The error is guaranteed to be less than or equal to $\Delta\phi$. This can speed up only if L_i is small.

We use MSA in step 13 of Algorithm 3, which is about updating the strategies. The ϕ_i^* found from step 9 is like the optimal moving direction in optimization. The best step size is difficult to obtain, since strategy changes impact the posterior probabilities. Hence a sequence of pre-determined step sizes is adopted. The number of sample points in step 5 is $O\left(L_i \binom{d + M_i}{d}\right)$ for operator i . This is the number of optimization programs (step 9) for i .

The running time of each optimization is proportional to the number of samples used to simulate the posterior probabilities. The number of samples is determined by the threshold ϵ'_s . ϵ'_s should not be too small, otherwise there will be not enough samples. Neither should it be too large or the simulated distribution in Figure 2(d) will be distorted. The number of samples is tested to be approximately inversely proportional to ϵ'_s . The total running time of Algorithm 3 has complexity $O\left(\sum_{i=1}^n L_i \binom{d + M_i}{d} / \epsilon'_s\right)$.

3.2 Data-sharing decision making

Now that we can compute the partition function $\mathbf{u}_{S|\mathcal{P}}$, our next step is to decide which partition(s) are stable. We turn to *coalition formation theory* (Ray, 2007). A *blocking approach* is adopted. The main idea is that larger coalitions may break into smaller pieces when some members seek their own benefits. However, when doing this they are aware of the avalanche of deviations that may occur. We want to find a partition, along with the strategy and payoff, that is stable from farsighted deviations. We try to find an *Equilibrium Binding Agreement* (EBA) set for every partition $\mathcal{P}$. For some $\mathcal{P}$, this set may be empty, which means that it is not stable. We can find EBA recursively, starting from the most basic structure.

Definition 3: The partition that is composed of only sets with exactly one element is called a *singleton structure*.

The singleton structure is written as $\mathcal{P}_0 = \{\{1\}, \{2\}, \dots, \{n\}\}$. We refer to Chapter 12 of Ray (2007) for the recursive method. After having found the EBA sets for all partitions, we start from the grand coalition, $\mathcal{P}_n$, to seek the solution. If EBA set of grand coalition is not empty, that means all operators are willing to share data. Otherwise, we look at the next lower level of partitions for non-empty EBA sets. The associated structure would be the stable partition that we expect to appear. We give this partition a name.

Definition 4: The partition $\mathcal{P}$ with non-empty EBA that is closest to the grand coalition is called the *top-down solution* partition $\mathcal{P}^*$ with payoffs $\mathbf{u}_{S|\mathcal{P}^*}$, where $(\mathcal{P}^*, \mathbf{u}_{S|\mathcal{P}^*}) = \pi(\mathcal{P}_0)$. $\pi(\cdot)$ represents the top-down method that depends on the singleton structure $\mathcal{P}_0$ and outputs a partition $\mathcal{P}^*$ and its corresponding partition function $\mathbf{u}_{S|\mathcal{P}^*}$.

The non-empty EBA aspect of a top-down solution implies that it is a stable outcome. If utility is assumed to be non-transferrable, the top-down solution is the final result and the data sharing decisions emerge from the partition as a binary matrix $\xi(\mathcal{P}^*)$ where $\xi_{ij} = 1$ means operator i shares data with j , determined as a coalition formed between them in partition $\mathcal{P}^*$.

The problem with the top-down solution method is that it does not necessarily ensure *market efficiency* (Ray, 2007) when utility is transferable. In fact, just as we expect of a Nash equilibrium, it is typically inefficient (i.e. an equilibrium might not generate the greatest value; players can cooperate to “beat” the market). However, the *Coase theorem* states that the market outcome should be efficient if a binding agreement can be written at no cost and all participants have complete and perfect information. As such, we propose an extension of the top-down solution method to achieve both stability and efficiency for a utility transferable (UT) case.

(Utility transferable assumption) The utility of transit operators is transferable.

The idea of the extension is that a coalition can be multi-leveled. For example, in the real world there exists multi-leveled coalitions in which some partners have prior binding agreements forming subcoalitions within a coalition: international trade agreements between a European bloc and the U.S., or material suppliers with transport providers, etc. We introduce a novel multi-level coalitional structure to capture this phenomenon. To the best of our knowledge this is a novel coalitional structure.

Definition 5: A *composite coalition structure* is a sequence of partitions $\mathcal{P}^1, \mathcal{P}^2, \dots, \mathcal{P}^K$ such that each $\mathcal{P}^{k-1}$ is a refinement of $\mathcal{P}^k$ for $k \leq K$, denoted as a tuple $\mathcal{Q} = [\mathcal{P}^1, \mathcal{P}^2, \dots, \mathcal{P}^K]$. K is called the *dimension* of $\mathcal{Q}$. Set $S^k \in \mathcal{Q}_k = \mathcal{P}^k$ is called a *level- k coalition* in the composite coalition structure $\mathcal{Q}$.

A level- $(k-1)$ coalition is “stronger” than a level- k coalition. Each coalition $S^{k-1} \in \mathcal{P}^{k-1}$ acts as a singleton in the formation of $\mathcal{P}^k$. Algorithm 4 summarizes this process. If UT is not assumed, then the output is just the result of step 1, i.e. the top-down solution method from Ray (2007) is a special case. Let $\mathcal{P}_0^{(k)}$ be the partition consisting of singleton “players” at the start of k -th iteration. Also, let $\mathcal{P}_n^{(k)}$ be the partition representing a single grand coalition of players at the k -th iteration. At the start of iteration k , we let the coalition structure from last iteration be the new singleton structure, $\mathcal{P}_0^{(k)} := \mathcal{P}^{*(k-1)}$ (step 3), and apply coalition formation $(\mathcal{P}^{*(k)}, \mathbf{u}_{S|\mathcal{P}^k}) = \pi(\mathcal{P}_0^{(k)})$ (step 4).

Definition 6: A composite coalition structure is said to be a *grand structure* if the last partition in $\mathcal{Q}$ is grand coalition, namely $\mathcal{P}^{*(K)} = \mathcal{P}_n^{(K)}$ where K is the dimension of $\mathcal{Q}$.

We construct the composite coalition recursively in this manner until we either reach the grand structure or the current iteration’s top-down solution remains the singleton structure.

Algorithm 4: Data-sharing coalition formation

-
- 1: Set $\mathcal{P}^{*(0)} = \{\{1\}, \{2\}, \dots, \{n\}\}$; set $k = 1$;
 - 2: For each k ,
 - 3: $\mathcal{P}_0^{(k)} := \mathcal{P}^{*(k-1)}$;
 - 4: $(\mathcal{P}^{*(k)}, \mathbf{u}_{S|\mathcal{P}^{*(k)}}) = \pi(\mathcal{P}_0^{(k)})$;
 - 4.a: Each element in $\mathcal{P}_0^{(k)}$ is regarded as a single player in this subroutine; let $\mathcal{P}_0 := \mathcal{P}_0^{(k)}$ in this subroutine;
 - 4.b: Find the equilibrium payoffs for each structure $\mathcal{P}$ that may appear in the coalition formation process;
 - 4.c: For each structure $\mathcal{P}$, find the EBA set – actions and payoffs that are not *sequentially blocked* by EBA set of refinement structures of $\mathcal{P}$; return the resulting $(\mathcal{P}^*, \mathbf{u}_{S|\mathcal{P}^*})$;
 - 5: Stop if $\mathcal{P}^{(k)} = \mathcal{P}_n^{(k)}$ or $\mathcal{P}^{(k)} = \mathcal{P}_0^{(k)}$ and go to Step 6. Else, $k = k + 1$; go to Step 2.
 - 6: Output the final composite coalition structure $\mathbf{Q} = [\mathcal{P}^{*(1)}, \mathcal{P}^{*(2)}, \dots, \mathcal{P}^{*(K)}]$, data sharing decision matrix $\xi(\mathcal{P}^{*(K)})$ and the payoff vector $\mathbf{u}_{S|\mathcal{P}^{*(K)}}$.
-

Note that step 4.c in Algorithm 4 is recursive. The tasks are partially ordered by the partition refinement relation. Figure 3 shows the coalition formation process tasks' precedence relation for a game with three players. Any task order that complies with the relation is fine, like ①②③④⑤ or ①③④②⑤, etc.

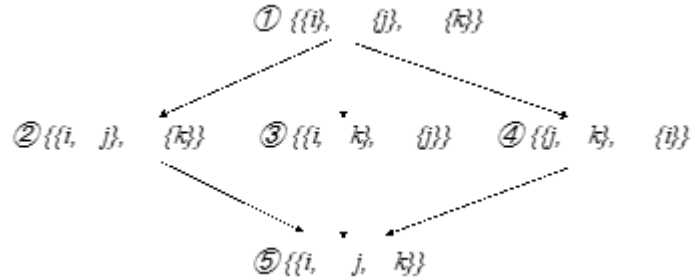


Figure 3. Algorithm 4 coalition formation process subroutine – task precedence relation illustration.

Algorithm 4 and the grand structure stopping condition is illustrated in Figure 4 for a market with three operators. They form two sub-coalitions $\{i, j\}$ and $\{k\}$ after the first iteration and then further connecting into a grand coalition in the second iteration. Each iteration in Figure 4 is resolved with a top-down method based on the updated singleton structure.

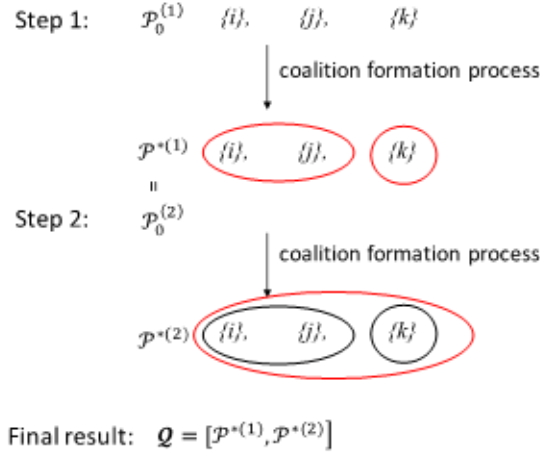


Figure 4. Illustration of singleton structures in each iteration of the coalition composition.

Proposition 1: Algorithm 4 leads to a stable and efficient result (action profile and payoffs).

Proof. The resulting $\mathcal{Q}$ with the actions and payoffs being not efficient means that some players in $\mathcal{P}^{*(K)}$ can cooperate to form coalition structure (say $\mathcal{P}$) such that everyone becomes better off. Then at the last iteration leading to $\mathcal{Q}$, the singleton structure should not be the equilibrium structure, since players are farsighted - they should not breach from $\mathcal{P}$ to end up with the singleton structure that is less advantages to all of them. As a result, the iteration should not stop. This leads to contradiction. ■

The composition process in Algorithm 4 can continue at most n times, where n is the number of operators. The extreme case happens only when two players form a coalition for each iteration. If T is the runtime of Algorithm 2 or 3, then the runtime of Algorithm 4 is bounded by nT .

Algorithm 4 does not necessarily reach a grand structure. That means the grand structure is not necessarily efficient. We may care under what circumstances Algorithm 4 would indeed arrive at a grand structure. We give sufficient condition in proposition 2. Before that, we assert Lemma 1 and introduce the concept of *grand-coalition super-additive*.

Lemma 1: If the payoff of a grand coalition $\mathcal{P}_n$, $\|\mathbf{u}_{N|\mathcal{P}_n}\|_1$, is larger than the sum of payoffs $\sum_{i \in N} u_{i|\mathcal{P}_0}$ of singleton structure $\mathcal{P}_0$, then the top-down solution is not the singleton structure, i.e. $(\mathcal{P}_0, \mathbf{u}_{S|\mathcal{P}_0}) \neq \pi(\mathcal{P}_0)$.

Proof. Under these conditions and UT assumption, the payoffs of any player under the grand coalition $\mathcal{P}_n$ could better off than in the singleton structure. If the singleton structure is the top-down solution, then players should not break from a grand coalition since they are far-sighted. Then the EBA set of grand coalition is non-empty. This is a contradiction to the fact that singleton structure is the top-down solution. ■

Definition 7: A partition function $\mathbf{u}_{S|\mathcal{P}}$ is said to be *grand-coalition super-additive* if it satisfies:

$$\|\mathbf{u}_{N|\mathcal{P}_n}\|_1 \geq \sum_{S \in \mathcal{P}} \|\mathbf{u}_{S|\mathcal{P}}\|_1, \quad \forall \mathcal{P}$$

Proposition 2: The coalition composition process (Algorithm 4) results in a grand composite

coalition structure for a grand-coalition super-additive transferable-utility game.

Proof: Grand-coalition super-additivity property ensures that the top-down solution is not a singleton structure at each step by Lemma 1. There are finitely many operators, so this process must stop. And it can stop only when $\mathcal{P}^{*(K)} = \mathcal{P}_n^{(K)}$ for some K . ■

Given Proposition 2, we present a class of games for which it can be applied to. The partition function $u_{S|\mathcal{P}}$ is said to be *derived from strategic form* if we start with a strategic game. Let $u_i(a)$ be the payoff of operator i given the action profile a . Then define $\mathbf{u}_{S|\mathcal{P}}$ by $\mathbf{u}_{S|\mathcal{P}} := (u_i(a))_{i \in S}$ such that $\mathbf{u}_{S|\mathcal{P}}$ satisfy the Nash equilibrium condition for every S in $\mathcal{P}$:

$$\mathbf{u}_{S|\mathcal{P}}(a'_S, a_{-S}) \geq \mathbf{u}_{S|\mathcal{P}}(a), \quad \forall a'_S \in A_S$$

Lemma 2. *Partition function $\mathbf{u}_{S|\mathcal{P}}$ derived from the strategic form satisfies grand-coalition super-additivity.*

Proof: The grand coalition can do everything a partition can do. Hence $u_{S|\mathcal{P}}$ is always smaller than or equal to $\sum_{i \in S} \mathbf{u}_{i, N|\mathcal{P}_n}$. Summing over S leads to $\sum_{S \in \mathcal{P}} \|\mathbf{u}_{S|\mathcal{P}}\|_1 \leq \|\mathbf{u}_{N|\mathcal{P}_n}\|$, which is the definition of grand-coalition super-additivity. ■

Proposition 3. *If partition function $\mathbf{u}_{S|\mathcal{P}}$ is derived from the strategic form and utility is transferrable, then the coalition composition process (Algorithm 4) results in a grand structure.*

Proof: Combine Proposition 2 with Lemma 2. ■

We illustrate Algorithm 4 and Proposition 3 with Example 1 in which the grand coalition's core in the traditional sense is empty but a stable outcome in a multi-level grand structure can be reached.

Example 1: Suppose there is a market with three companies. Each company has the same action set $A_i = \{l, m, h\}$ for “low”, “medium”, “high”. The strategic form is shown in Table 1. This game is symmetric. When there is no cooperation of any kind (namely the singleton structure), the Nash equilibrium action profile is (l, l, l) and the corresponding payoff vector is $(0.1, 0.1, 0.1)$. When a $\{i, j\}$ coalition is formed, the partition becomes $\{i, j\}, \{k\}$ (denoted by $\mathcal{P}'$). The corresponding strategic form is shown in Table 2. We can see that the Nash equilibrium becomes (m, m, l) with corresponding payoffs being $(0.35, 0.35, 0.22)$. If a grand coalition is formed, the highest total payoff they can earn is $0.99 (= 0.33 + 0.33 + 0.33)$ and the action is (h, h, h) .

Table 1. Strategic form of the game if there is no cooperation

Actions profile of i, j, k	Payoffs of i, j, k
l, l, l	0.1, 0.1, 0.1
m, l, l	0.05, 0.4, 0.4
m, m, l	0.35, 0.35, 0.22
m, m, m	0.2, 0.2, 0.2
h, l, l	0, 0.35, 0.35
h, m, l	0, 0.3, 0.5
h, m, m	0, 0.4, 0.4
h, h, l	0.1, 0.1, 0.3
h, h, m	0.15, 0.15, 0.55

h, h, h	0.33, 0.33, 0.33
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Table 2. Strategic form of the game if i and j cooperate

$k \setminus (i, j)$	(l, l)	(l, m)	(m, m)	(h, m)	(h, h)
l	0.1, (0.2)	0.4, (0.45)	0.22, (0.7)	0.5, (0.3)	0.3, (0.2)
h	0, (0.7)	0, (0.8)	0, (0.8)	0.15, (0.7)	0.33, (0.66)

Remark: the payoffs in parenthesis, like (0.7), is the summation of payoffs of i and j .

The partition function is shown as follows. Since we assume UT, we use a single value for coalition payoff instead of a vector or a set of vectors.

$$u_{\{i\}|\mathcal{P}_0} = 0.1$$

$$u_{\{i,j\}|\mathcal{P}'} = 0.7$$

$$u_{\{i,j,k\}|\mathcal{P}_N} = 0.99$$

The grand coalition is not stable because $\{i, j\}$ can form a coalition that can increase their payoffs from (0.33, 0.33) to (0.35, 0.35). Meanwhile, players $\{i\}$ or $\{j\}$ would not break from $\{i, j\}$ since they would only earn (0.1, 0.1) if they do so. Therefore, the stable structure of the first-round coalition formation would be $\{i, j\}, \{k\}$ with the action profile being (m, m, l) and payoff being (0.35, 0.35, 0.22). If utility transfer is not allowed, then we stop here and output this result; this is the top-down solution (corresponding to one iteration of Step 4 of Algorithm 4, and on its own is not efficient in this example).

We start the second-round coalition formation. $\{i, j\}$, as a single entity, could ask $\{k\}$ to choose action h altogether – (h, h, h) , and this results in a total payoff of 0.99 ($= 0.33 + 0.33 + 0.33$), larger than 0.92 ($= 0.35 + 0.35 + 0.22$) of $\{i, j\}, \{k\}$ under (m, m, l) . As long as $\{i, j\}$ can get more than 0.7 ($= 0.35 + 0.35$) and $\{k\}$ get more than 0.22, this cooperation is feasible. Therefore, the resulting action profile is (h, h, h) and the payoff is (x, x, y) where $x > 0.35, y > 0.22$ and $2x + y = 0.99$. Note that i or j cannot break out of the first-round coalition $\{i, j\}$. This is why the final result is not stable under the non-hierarchical coalitional structure but is stable under the composite coalitional structure. Here the partition function is a result of strategic form. And we indeed reach a grand structure as asserted by Proposition 3. The final payoff is not Nash equilibrium of the original strategic form game nor is it in the core. We cannot get this result from traditional game theory. But it is justified as the result of a dynamic coalition formation process.

The set of partition functions derived from a strategic form includes a large class of games. However, the proposed transit competition game is not derived from strategic form as defined above because of the existence of the always-competing frequency setting action. As such, a grand data-sharing structure may not always do better, i.e. a grand data-sharing coalition may not be efficient.

4 Numerical tests

In this section, we apply our previous models to some example networks. All the example data and code are shared in <https://github.com/BUILTNYU/transit-data-game>. We propose an index to evaluate the data-sharing level of a transit market.

Definition 8: Data sharing index $\gamma(\mathcal{P})$ is a function mapping the set of data-sharing partitions to $\mathbb{R}_{\geq 0}$, defined by Eq. (15).

$$\gamma(\mathcal{P}) = \ln \frac{|N|!}{\prod_{S \in \mathcal{P}} |S|!} \quad (15)$$

The grand data-sharing coalition, namely $\mathcal{P} = \{N\}$, has $\gamma(\{N\}) = \ln \frac{|N|!}{|N|!} = 0$. The singleton coalition has $\gamma(\{1\}, \{2\}, \dots, \{n\}) = \ln |N|!$, a larger number. The smaller the $\gamma(\mathcal{P})$, the higher the data-sharing level.

Example 2: For the two partitions in Figure 5, we have $|N| = 6$; $\gamma(\mathcal{P}_{(a)}) = \ln \frac{6!}{1!2!2!1!} \approx 5.1$; $\gamma(\mathcal{P}_{(b)}) = \ln \frac{6!}{4!2!} \approx 2.7 < \gamma(\mathcal{P}_a)$. The degree of data sharing in (b) is higher.

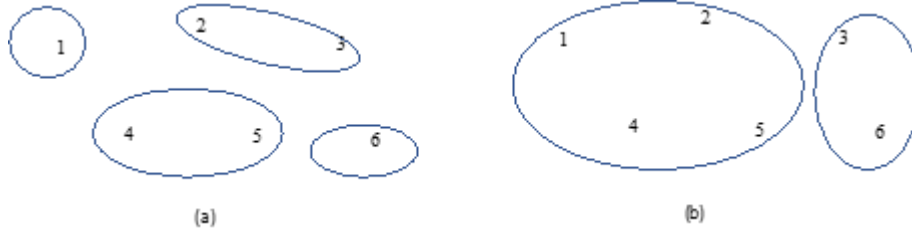


Figure 5. Partitions for example 2, where (b) has higher degree of data sharing than (a).

4.1 Discrete model example

4.1.1 Competitive services

In this section, we test our discrete game model from section 3.1.1 on simple networks. We consider a case shown in Figure 6. This example represents three identical operators providing perfectly competitive services. The parameters settings are shown in Table 3. The precision of measurement means the resolution of observations. Operators can only observe that the flow is within certain interval like $[1000, 2000]$. The interval $[1000(n-1), 1000n]$ is denoted by “ nK ” below.

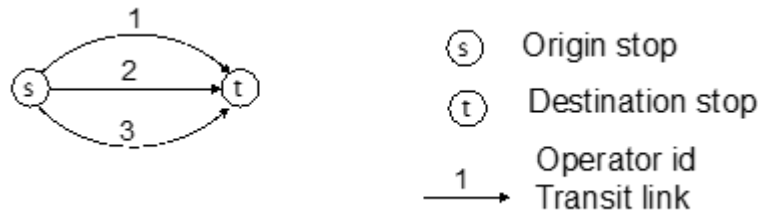


Figure 6. Example network with competitive services.

Table 3. The parameters settings

Parameter	Value
Demand	6000 (high) with probability $\frac{1}{2}$; 3000 (low) with probability $\frac{1}{2}$;
Transit price	\$2
Operational cost	\$24000 * frequency;
Frequency interval	$[0, 0.2]$ per min
User Logit choice model dispersion parameter	0.3

Transit travel time	20 min
Expected utility of no-entry	-30 (unit: min)
Precision of measurement	1000 vehicle units
Iteration difference threshold ϵ_d	0.005

There is only one OD pair (A-B) on this network. We have five possible partitions. We assume operators' roles being symmetric, hence we only need to consider three of them due to symmetry: $\{\{1\}, \{2\}, \{3\}\}$, $\{\{i, j\}, \{k\}\}$ and $\{\{1, 2, 3\}\}$.

Each user can choose to take transit services or not to use transit at all. Users' choices are modeled by a MNL model. The payoffs at Nash equilibrium for the three types of partitions are shown in Table 4. Operator i 's posterior probabilities $p(\omega, t_j, t_k | t_i)$ and the strategy (a vector) under singleton structure are shown in Table 5 and Table 6 respectively. The frequency "0.1000" in the tables is the initial frequency setting of the algorithm. Some observation combinations turn out to be inconsistent under the final strategy; hence their frequencies are not updated and left with the initial one. Operator j and k have the same results due to symmetry. For partition $\{\{i, j\}, \{k\}\}$, operator i 's posterior probabilities $p(\omega, t_k | t_i, t_j)$ and strategy (a matrix) are shown in Table 7 and Table 8 respectively. Operator j has the same results due to symmetry; Operator k has posterior probabilities similar to a singleton structure since its knowledge level remains the same. We can see that i and j 's posterior distributions are less dispersed when they share data. When i and j share data, their expected payoff would increase; the leftover operator k would become worse off. Operator i and j know the true demand level better and cater to the high demand more effectively. At high demand level, i and j set frequencies at 0.1255 (Table 8), so high that k will stop serving by setting frequency at 0. At low demand level, their frequencies 0.0819 are much lower than operator k 's 0.1294.

Under a grand data-sharing coalition, they have perfect information of the demand level and each other's observations. Operators have the same strategy as shown in

Table 10. Their payoffs are a bit less than the case they don't share data at all. This is because they compete too fiercely when the demand is high: they all set frequency at 0.1037 (

Table 10), and they care less about low demand, resulting in user loss. The solution is $\mathcal{P} = \{\{i, j\}, \{k\}\}$. Namely, only a proper subset of operators will share data with each other. The data sharing index is $\gamma(\mathcal{P}) = -\ln\left(\frac{1!2!}{3!}\right) = 0.4$. To realize a grand-coalition, a subsidy of $1590 - 1572 = 18$ is needed.

Table 4. Nash equilibrium for three partitions

Partition	$\{1\}, \{2\}, \{3\}$	$\{i, j\}, \{k\}$	$\{1, 2, 3\}$
Expected payoffs	571 571 571	593 593 404	524 524 524
Sum of payoffs	1713	1590	1572

Table 5. Operator i 's posterior probability of (ω, v_j, v_k) conditioning on observing t_i under singleton structure; the same is true for operator j and k .

Observation	Estimation			Posterior prob
t_j	ω	t_j	t_k	$p(\omega, v_j, v_k v_i)$
1K	3K	1K	1K	1/6
1K	3K	1K	2K	1/6
1K	3K	2K	1K	1/6
1K	3K	1K	3K	1/6
1K	3K	3K	1K	1/6
1K	6K	1K	1K	1/6
2K	3K	1K	1K	1/6
2K	3K	1K	1K	1/6
2K	3K	2K	1K	1/6

2K	6K	2K	2K	1/6
2K	6K	3K	2K	1/6
2K	6K	2K	3K	1/6
3K	6K	3K	1K	1/3
3K	6K	2K	2K	1/3
3K	6K	1K	3K	1/3

Table 6. The strategy (frequency setting) of each operator when observing t_i under singleton structure.

Observation t_i	1K	2K	3K
Frequency	0.1000	0.1022	0.1194

Table 7. Operator i 's posterior probability of (ω, v_k) conditioning on observing (t_i, t_j) under partition $\{\{i, j\}, \{k\}\}$; the same is true for operator j .

Observation		Estimation		Posterior prob
t_i	t_j	ω	t_k	$p(\omega, v_k v_i, v_j)$
1K	1K	3K	1K	1/3
1K	1K	3K	1K	1/3
1K	1K	3K	2K	1/3
2K	2K	3K	1K	1/2
2K	2K	6K	3K	1/2
3K	3K	6K	1K	1.0

Table 8. The strategy (frequency setting) of operator i when observing (t_i, t_j) under partition $\{\{i, j\}, \{k\}\}$; the same is true for operator j .

Observation $t_i \setminus t_j$	1K	2K	3K
1K	0.0000	0.1000	0.1000
2K	0.1000	0.0819	0.1000
3K	0.1000	0.1000	0.1255

Table 9. The strategy of operator k under partition $\{\{i, j\}, \{k\}\}$

Observation t_k	1K	2K	3K
Frequency	0.0000	0.1294	0.1162

Table 10. The strategy of each operator under grand data-sharing coalition

Demand level ω	3K	6K
Frequency	0.0602	0.1037

This is an example of the fact that *perfect knowledge may lead to perfect selfishness*. Since we do not assume frequency setting cooperation, the act of selfishness offsets the benefits gained by the ability of efficient frequency setting with more knowledge. Therefore, if operators plan to share data, we suggest them to strengthen their cooperation to another level by setting transit frequencies and fares under coordination. This result contrasts with the popular belief that more data sharing means better services for all operators and occurs because frequency setting remains noncooperative. *It implies that policymakers wanting market-wide data sharing should also encourage operators to cooperate in their operations as well.*

Sensitivity tests are conducted with respect to the ratio of operator cost coefficient β and transit fare coefficient α . The results are shown in Figure 7. The x-axes are the ratios: operator cost per minute frequency (\$) divided by $(1000 \times \text{transit fare per ride} (\$))$. The y-axis in (a), (b), (c) are Bayesian equilibrium payoffs (\$). As the ratio increases, the general trends of payoffs are decreasing for all structures and all operators as we would expect. But we can see that the curves are not monotone and fluctuations do exist. The fluctuations get larger as data sharing levels become lower ((c) $\rightarrow$ (b) $\rightarrow$ (a)). That means the Bayesian equilibrium is more difficult to estimate. This result makes sense. It is like when people interact in a very dark space,

everyone is less predictable of the others' actions as well as his or her own action (reaction). Figure 7(d) shows the results of a one-step coalition formation (green solid dots) and coalition composition process (black hollow dots). We don't see any easy trend due to the existence of payoff fluctuations. But we can make out three regions (surrounded by dashed lines) where the resulting data-sharing levels are relatively homogenous. The left and right regions have high data-sharing levels and the middle region has a low level. Our previous results in Table 4 to Table 10 belong to this middle area.

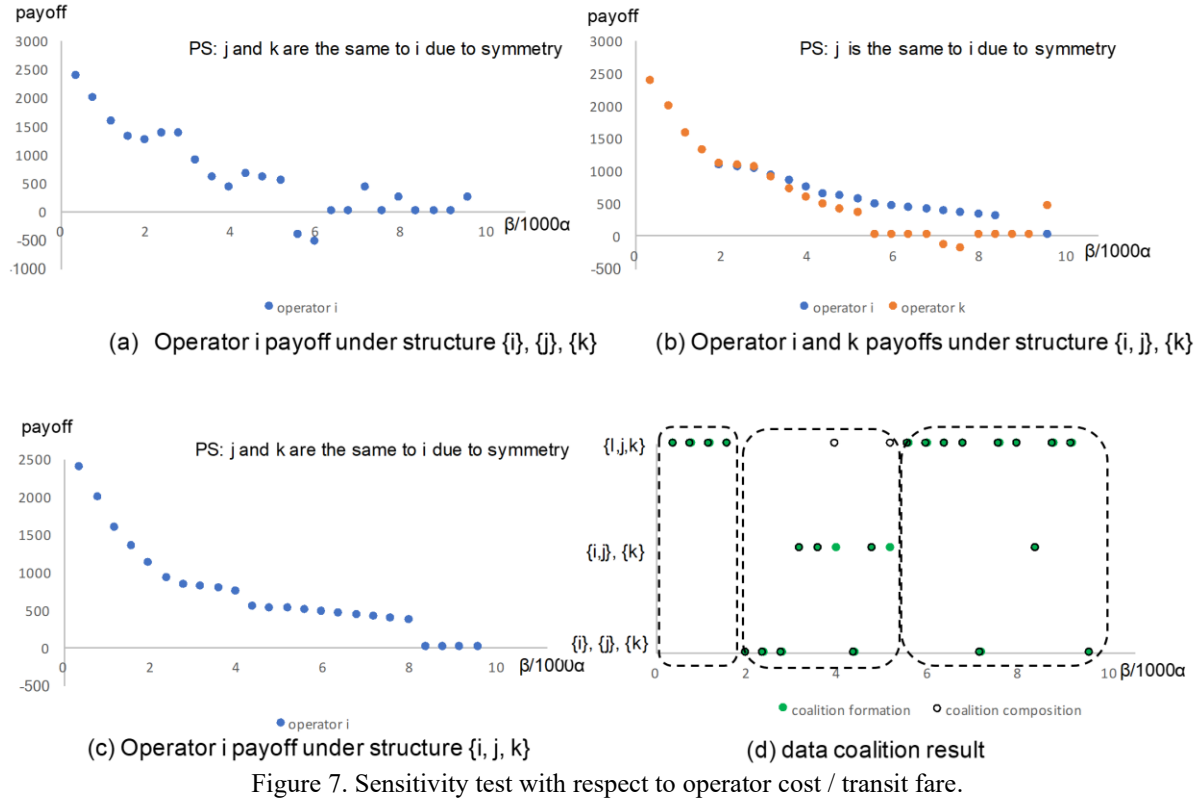


Figure 7. Sensitivity test with respect to operator cost / transit fare.

4.1.2 Complementary services

In this example, we consider a network with complementary services as shown in Figure 8. There are six OD pairs (A-B, A-C, A-D, B-C, B-D, C-D) on this small network. Compared to the parameter settings in Table 3, the demands change from “6000 (high)” and “3000 (low)” to “2000 (high)” and “1000 (low)”; the operator cost is changed from 24000 to 30000. Now the roles of operators are not symmetric. We need to consider all five possibilities of partitions. The payoffs are shown in

Table 11. Because of the fact that there are more OD pairs and the game is asymmetric, the detailed results in this case are much lengthier than before. We only list operator 1's strategy under singleton structure (Table 12) and structure $\{\{1,2\}, \{3\}\}$ (Table 13). “0.1” is still the default setting.

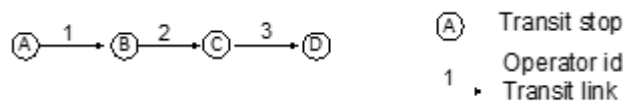


Figure 8. Example network with complementary services

Table 11. Nash equilibrium for each of the five coalition structures

Structure\Operator	Payoffs		
	Operator 1	Operator 2	Operator 3
$\{\{1\}, \{2\}, \{3\}\}$	1155	1792	1155
$\{\{1, 2\}, \{3\}\}$	2803	3878	804
$\{\{1, 3\}, \{2\}\}$	5322	1057	5324
$\{\{2, 3\}, \{1\}\}$	629	5937	3461
$\{\{1, 2, 3\}\}$	1136	1827	1136

Table 12. The strategy (frequency setting) of operator 1 when observing t_i under singleton structure

Observation t_1	1K	2K	3K	4K	5K	6K	7K	8K
Strategy	0.0000	0.0000	0.0000	0.1224	0.1282	0.1329	0.1000	0.1000

Table 13. The strategy (frequency setting) of operator 1 when observing (t_i, t_j) under structure $\{\{1, 2\}, \{3\}\}$

$t_1 \backslash t_2$	1K	2K	3K	4K	5K	6K	7K	8K
1K	0.0000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
2K	0.1000	0.0000	0.0000	0.0000	0.1000	0.1000	0.1000	0.1000
3K	0.1000	0.0000	0.0000	0.0000	0.0960	0.1000	0.1000	0.1000
4K	0.1000	0.1000	0.1000	0.0000	0.1255	0.1000	0.1430	0.1000
5K	0.1000	0.1000	0.1000	0.1000	0.1327	0.1313	0.1180	0.1000
6K	0.1000	0.1000	0.1000	0.1000	0.1000	0.1364	0.1000	0.1636
7K	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
8K	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000

When two operators i and j share data, their payoffs both increase significantly. The payoff of the remaining operator decreases slightly. This is explained by the two operators efficiently exploiting high demands and leaving out low demands although all three operators services are complementary. Each operator caters to demands they like most and know well, resulting in loss of potential transit users. This is verified by the actions of operator 1 under two structures. The demand that operator 1 cares about is from OD pairs A-B, A-C and A-D. Operator 1's frequency settings under singleton structure (Table 12) do not vary as drastically as under structure $\{\{1,2\}, \{3\}\}$ (Table 13) when it knows more. In the latter case, it sets frequency as high as 0.1636 when the demands it cares for are high and decays to zero more quickly when demands are low. Structure $\{\{1,2\}, \{3\}\}$ is the final output. A grand coalition does not form. This is quite counter-intuitive: a sharing-data grand coalition does not necessarily make operators' situations much better-off under perfectly complimentary services. Here we verified again the fact that perfect knowledge may lead to perfect selfishness. These results depend on parameter settings. There are other settings under which complementary operators do form grand data sharing coalitions.

There are still many other factors that can potentially influence data-sharing, like other types of network structures, the number of operators, the distribution of lines between operators, the degree of complementarity between transport operators' lines, etc. We leave the

testing of those factors for future work.

4.2 Continuous model example

The continuous model in Section 3.1.2 is tested on a new transit line version of the Nguyen Dupuis network. This network has two origin nodes, two destination nodes and nine intermediate nodes. There are four OD pairs (1-2, 1-3, 4-2, 4-3). The network is modified to include three operators with six lines, as shown in Figure 9(a). This physical network is converted to a transit network with boarding and alighting links as in Spiess and Florian (1989). The expanded network has 44 nodes and 75 links (25 traveling links, 25 boarding links and 25 alighting links). Line 1 in the expanded network is shown in Figure 9(b). Other lines follow the same pattern.

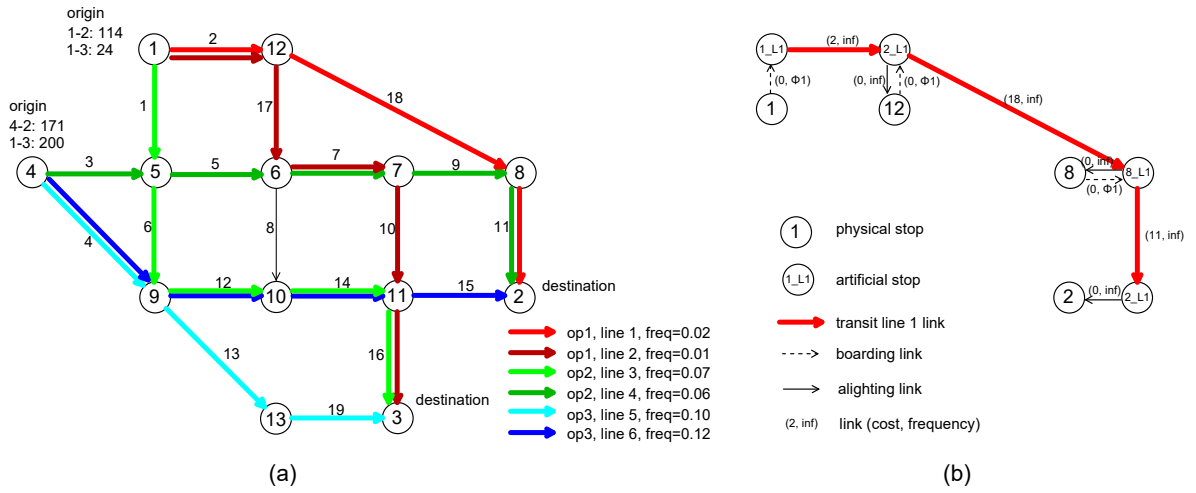


Figure 9. (a) Transit lines on Nguyen Dupuis network (b) line 1 in expanded transit network

The OD demands are *i.i.d.* normal random variables with $\mu = \sigma = 100$, truncated at zero to make sure that they are non-negative. Operators 1, 2 and 3 have 8, 10, and 7 directional links, respectively. The operators' services are more competitive than complementary and operator 2 has greater market power than the other two.

The degree of the polynomial is chosen to be 1 (linear approximation of $\phi_i(t_i)$). MSA style step-size is employed as indicated in Algorithm 3. We refer to the GitHub files for detailed settings of the network characteristics and operational parameters.

The payoffs over iterations are illustrated in Figure 10. The payoffs for five possible partitions are shown in Table 14. The results show that the top-down solution (after first-round of Step 4 of Algorithm 4) is: $\mathcal{P}^* = \{\{1, 3\}, \{2\}\}$. Namely, the two weaker operators would share data with each other; their payoffs both get better compared to the singleton structure. It's interesting to see that the payoff of operator 2 also gets slightly better (+2.5%). A grand coalition is not formed because operator 2 will break from this partition. Simulation results, including OD demands and frequency configurations, for $\mathcal{P}^*$ are shown in Table 15. The OD demands, link flows, and frequency settings corresponding to simulation #1 in Table 15 are shown in Figure 11.

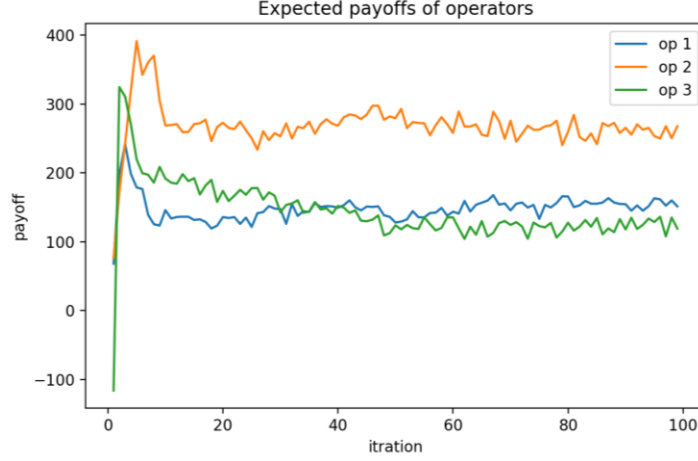


Figure 10. Simulated payoffs changing over iterations (partition: $\{\{1\}, \{23\}\}$)

Table 14. Payoffs under various data sharing coalitions.

Partition	$\{\{1\}, \{2\}, \{3\}\}$			$\{\{1, 2\}, \{3\}\}$		
Operator	1	2	3	1	2	3
# links	8	10	7	8	10	7
# measurements	8	10	7	18	18	7
Poly space dim	9	11	8	19	19	8
Expected payoff	149.47	277.03	89.87	109.25	259.67	157.32

Partition	$\{\{1, 3\}, \{2\}\}$			$\{\{1\}, \{2, 3\}\}$		
Operator	1	2	3	1	2	3
# links	8	10	7	8	17	17
# measurements	15	10	15	8	17	17
Poly space dim	16	10	16	9	18	18
Expected payoff	165.66	285.53	149.84	149.30	279.88	119.55

Partition	$\{\{1, 2, 3\}\}$		
Operator	1	2	3
# links	8	10	7
# measurements	25	25	25
Poly space dim	26	26	26
Expected payoff	156.46	225.57	123.82

If UT is assumed, then Algorithm 4 would compare $\|u_{\{1,2\}|\mathcal{P}^*}\|_1 + \|u_{\{3\}|\mathcal{P}^*}\|_1$ with $\|u_{N|\mathcal{P}_n}\|_1$. The latter is smaller; hence a grand structure is not formed in this case, i.e. the final solution $\mathcal{Q} = [\mathcal{P}^*]$ coincides with the top-down solution $\mathcal{P}^*$. In other words, in this case the level 1 top-down structure is already an efficient structure and cannot be improved further with higher compositions of coalitions. We output the 1-dimensional $[\mathcal{P}^*]$ along with the actions and payoffs.

Table 15. Eight simulated demands and frequency settings for partition $\{\{1,3\}, \{2\}\}$.

# simulation	1	2	3	4	5	6	7	8
OD Demands	1→2	114.4	200.0	57.0	66.5	188.1	66.7	0.0
	4→2	171.3	176.7	0.0	95.5	90.3	75.0	149.4
	1→3	23.9	123.3	145.4	141.8	108.0	121.7	121.5
	4→3	200.0	83.1	93.5	200.0	153.6	177.6	195.4
frequency	fl	0.02	0.17	0.02	0.01	0.08	0.20	0.01

settings	f2	0.01	0.18	0.06	0.08	0.07	0.15	0.03	0.07
	f3	0.07	0.12	0.09	0.05	0.09	0.09	0.11	0.14
	f4	0.06	0.12	0.05	0.04	0.07	0.06	0.08	0.12
	f5	0.10	0.01	0.08	0.11	0.08	0.20	0.10	0.10
	f6	0.12	0.01	0.05	0.08	0.08	0.01	0.11	0.06

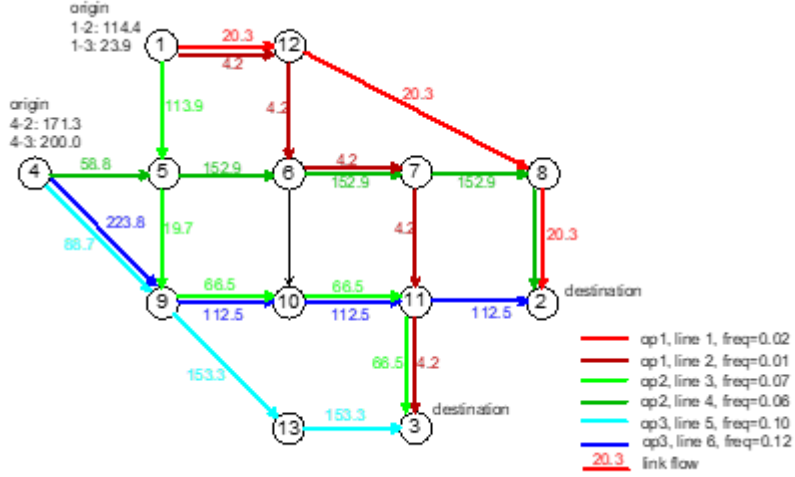


Figure 11. OD flows, link flows, and frequency settings of simulation #1 from Table 15.

The simulated relationships between $\text{var}[\text{payoff}]$ with param1 and param2 are shown in Figure 12 using the coalitional structure $\{\{1,3\}, \{2\}\}$. $\text{Var}[\text{payoff}]$ values are estimated using sample variance from 10 simulations. For Figure 12(a), the x-axis is the ratio of param1_i over $L_i \binom{d+M_i}{d}$. We can see that the $\text{var}[\text{payoff}]$ tends to decrease quickly as $\text{param1}_i / L_i \binom{d+M_i}{d}$ increases from 1 to 4 and stabilize as it increases to 8. For Figure 12(b), the x-axis is param2_i similarly shows $\text{Var}[\text{payoff}]$ decreasing and stabilizing even as the runtimes increase in Figures 12(c) and 12(d), respectively. The runtime increases linearly with param1_i and param2_i .

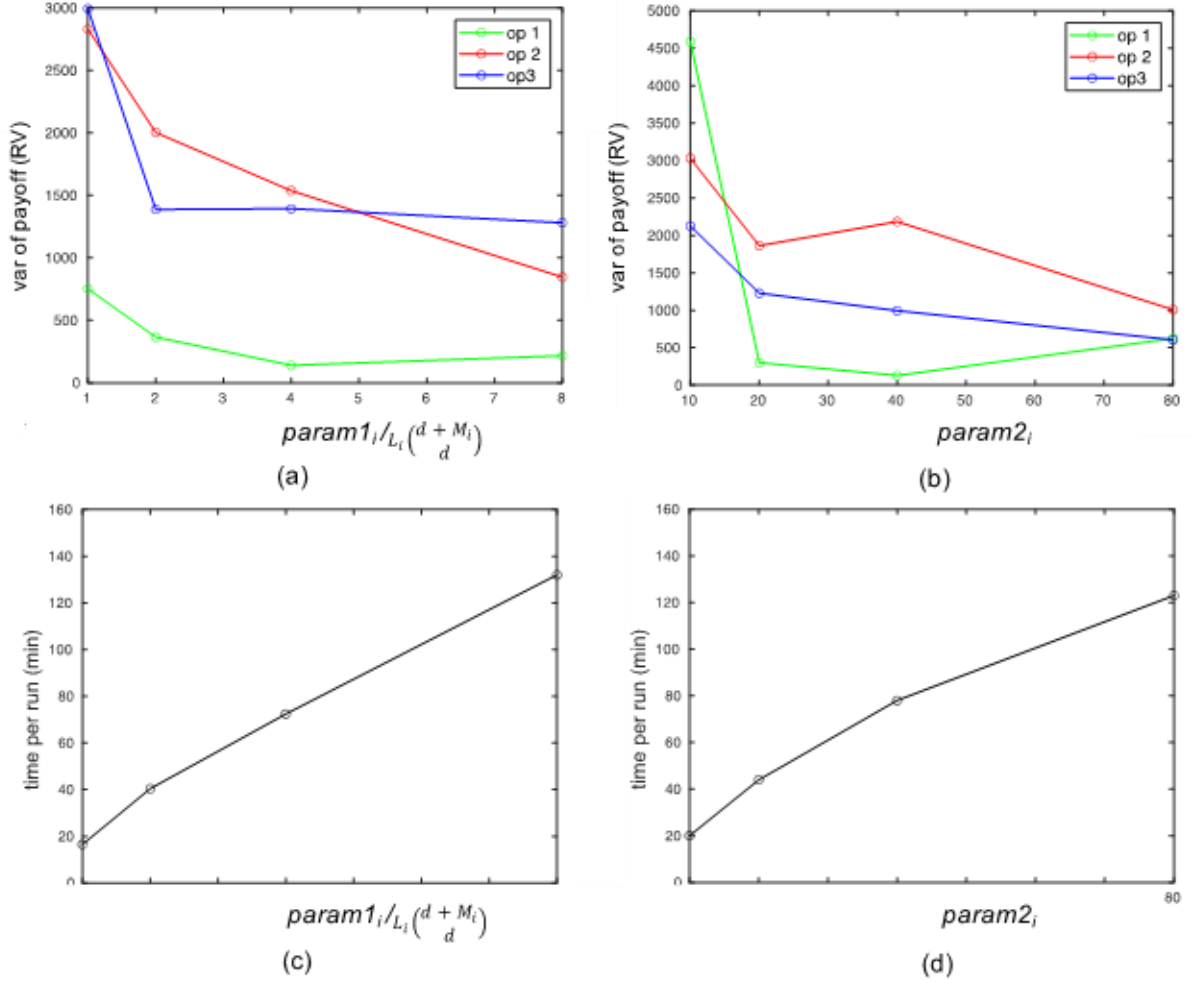


Figure 12. Payoff variances under different (a) param1 and (b) param2 settings; Runtimes under different (c) param1 and (d) param2 settings.

The solution has several implications for a policymaker in charge of this network. A stable grand coalition is not achievable if the policymaker does not also require cooperation in setting frequencies. A requirement for all three providers to share data with each other could alienate the larger provider, operator #2, as they can break from the coalition and do better. The payoff difference provides the policymaker with suggestions for subsidies that they might make to operator #2 to keep them in the data sharing coalition. If there is no data sharing at all, the policymaker can convince operators #1 and #3 to contribute some investment to setup the infrastructure for sharing data with each other, as long as the amount does not exceed their gains. In a MaaS environment with multiple operators, operators may not necessarily cooperate in providing bundled fares. An example is the pilot MaaS platform in Pittsburgh (Move PGH, 2022), which allows travelers to see all the mobility services (Port Authority transit, Spin e-scooters, Scoobi e-mopeds, Healthy Ride bikeshare, Zipcar carshare, and Waze carpool) but only the transit and bikeshare are integrated together. The different operators are not sharing data with each other. In those cases, the design of the platform architecture should not only allow for privacy for an operator but also make it possible for subgroups of operators to share data internally.

5 Conclusion

Data-sharing is a most common form of coopetition in which companies may choose to cooperate with data-sharing while simultaneously competing in other decisions. However, this data-sharing coopetition game has never been studied in the complex context of public transit oligopolies. These markets are unique in that coalitional values depend on a non-cooperative game between each of the operators regardless of whether they are in a coalition together.

This study proposes a new type of coopetition model to study the data-sharing of oligopolistic market with multiple transit operators when such an externality effect exists. It assumes multiple transit operators compete in frequency setting for an underlying population of travelers (assigned using a model like that of Spiess and Florian, 1989). The solution method obtains a partition initially based on a top-down rule that is extended to a multi-level composite coalition structure to ensure efficiency. Each coalition determines the data sharing which informs the signal functions of each player within a Bayesian game; each operator makes frequency setting decisions to achieve a Nash equilibrium in that game. We prove that the coalition composition process results in an efficient solution for a super-additive transferable utility market.

Computational tests are conducted on several instances. A simple 3-link network with a single OD pair is used to illustrate the mechanics of the model and solution algorithm. It shows that the best solution with three operators can result in only two of them sharing data with each other, even as all three compete for passengers. This underlines the need to study such games because the grand coalition may not be stable. A transit-customized version of the Nguyen-Dupuis network reinforces this point using a continuous model. Assuming three operators owning two lines each, the top-down partition results in having $\{1,3\}$ teaming up and leaving out operator 2. The example illustrates how operators can use the model to identify data-sharing strategies under competition which can also quantify the opportunity costs of not cooperating on operations like frequency setting.

Our results can provide insights for policy making. The results of our models can be used to evaluate the degree of data sharing. Perfect knowledge may lead to perfect selfishness. Sharing more data does not necessarily improve transit service for all groups if operators remain noncooperative. Service complementarity may not necessarily guarantee a grand coalition of data-sharing. For example, some city authorities may consider issuing laws to enforce data-sharing. In some cases, a voluntary data sharing platform is recommended instead of mandatory data-sharing with noncooperative operators, or to encourage cooperative operations if data sharing is enforced.

For future research, this model can be extended to include frequency setting coalitions or other forms of cooperation. Other forms of data sharing can be discussed. Sensitivity to other factors (e.g. network structure, the number of operators, the distribution of lines between operators) can be investigated. Data-sharing can be studied under the context of MaaS platforms, where operators and users form two-sided markets and stable matching are sought (Pantelidis et al., 2020b).

Acknowledgments

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Appendix: Existence of Bayesian game Nash equilibrium for finite state game

The first way to ensure the existence of a Bayesian Nash equilibrium is by allowing a mixed strategy. Suppose that the possible frequency settings for each link is within compact interval of $\mathbb{R}$, $[\underline{\phi}, \overline{\phi}]$. By discretizing $[\underline{\phi}, \overline{\phi}]$, this game G^* becomes a finite game. Then we can assert the existence of equilibrium *mixed* strategy.

Proposition A1: *If the set of feasible frequencies A is finite, then there exists a mixed strategy Nash equilibrium of the game G^* .*

Proof: every finite game has a mixed strategy Nash equilibrium (Nash, 1950). ■

The second approach is to impose some additional constraints to ensure the existence of a pure strategy. Given the formulation by Eqs. (10) and (11), it's natural to attempt to achieve this by requiring $\mathbb{E}[r_i(\boldsymbol{\phi}, \mathbf{f})|t_i]$ and $C(\mathbf{f}; \boldsymbol{\phi})$ to be continuously differentiable. $\mathbb{E}[r_i(\boldsymbol{\phi}, \mathbf{f})|t_i]$ is composed of products of posterior probability terms $p_{i,\boldsymbol{\phi}}(\omega, \mathbf{t} | t_i)$ and link flows $v_l(\omega, \boldsymbol{\phi}(\mathbf{t}))$. They have poor smoothness properties generally. First we look at $v_l(\omega, \boldsymbol{\phi}(\mathbf{t}))$, link flow as a function of $\boldsymbol{\phi}$ given fixed OD demand ω . For most transit assignment models, this function is not continuous.

Example A1. For the simple network Figure A1(a), we apply the assignment model from Spiess and Florian (1989) to assign unit flow from A to C. If $\phi_{BC} < \frac{1}{20}$, link AB is not included in the attractive set; hence $v_{BC} = 0$. If $\phi_{BC} \geq \frac{1}{20}$, all links are included in the attractive set. The relations between links flows and ϕ_{BC} are plotted in Figure (b).

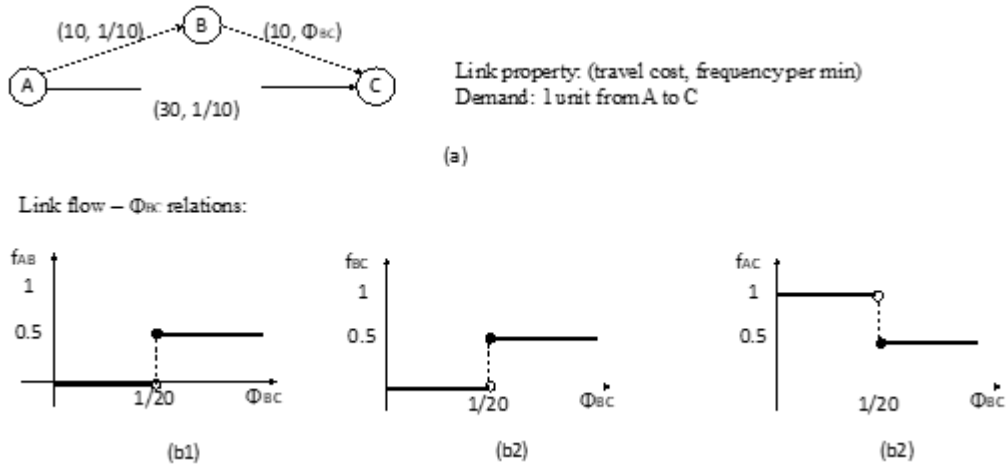


Figure A1. (Example 3) link flow – frequency relation under Spiess assignment model; (a) the network settings; (b1) AB flow – BC frequency relation; (b2) BC flow – BC frequency relation; (b3) AC flow – BC frequency relation.

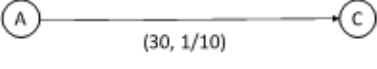
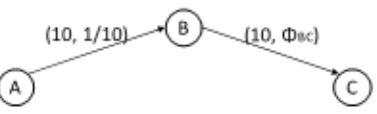
We can see that link flow is discontinuous when one link on the network drops out or joins in the attractive set as its frequency varies. Hence, link a flow as a function of link b frequency, $v_a(\phi_b)$, is generally not continuous. It is right- or left-continuous depending on whether or not a link is added to the attractive set when the expected costs draw. In our case, it's right continuous. Semi-continuity is a concept used frequently in proving Nash equilibrium existence. We note that when link a and b are complementary, like link AB and BC in our case,

$v_a(\phi_b)$ is increasing and upper semi-continuous; when a and b are competitive, like link AC and BC in our case, $v_a(\phi_b)$ is decreasing and lower semi-continuous. We introduce the following definition of smooth assignment.

Definition A1: A transit assignment model is said to be a *smooth assignment* if any link flow is a smooth (C^∞) function of any link frequency on the feasible interval.

A common way to realize smoothness is through stochastic assignment. For example, if we assume that users comply with a Logit model to select hyperpaths, then we can obtain the hyperpaths and probabilities as in Table A1 for Example A1. The resulting link flows are obviously smooth functions of ϕ_{BC} .

Table A1. (Example 3) Hyperpaths and probabilities of stochastic assignment

hyperpaths:	Expected cost:	Probabilities:
h1 	40	$\frac{e^{-40\theta}}{e^{-40\theta} + e^{-\left(30 + \frac{1}{2\phi_{BC}}\right)\theta} + e^{-\left(30 + \frac{1}{\phi_{BC}}\right)\theta}}$
h2 	$30 + 1/(2 \cdot \phi_{BC})$	$\frac{e^{-\left(30 + \frac{1}{2\phi_{BC}}\right)\theta}}{e^{-40\theta} + e^{-\left(30 + \frac{1}{2\phi_{BC}}\right)\theta} + e^{-\left(30 + \frac{1}{\phi_{BC}}\right)\theta}}$
h3 	$30 + 1/\phi_{BC}$	$\frac{e^{-\left(30 + \frac{1}{\phi_{BC}}\right)\theta}}{e^{-40\theta} + e^{-\left(30 + \frac{1}{2\phi_{BC}}\right)\theta} + e^{-\left(30 + \frac{1}{\phi_{BC}}\right)\theta}}$

There are some studies on applying Logit model to transit assignment, like Nguyen et al. (1998) and Florian and Constantin (2012). They apply the idea of Dial's algorithm to transit and hence their algorithms don't need hyperpath enumeration. However, we noticed that these assignments don't satisfy our previous definition. This is because a hyperpath may still appear or disappear from user choice set when link frequency varies, thus leading to discontinuous link flows. We give the following sufficient condition for smooth assignment.

Lemma A1: Suppose a stochastic (Logit or Probit) model is used for transit user choice, and if the hyperpath choice set H_θ for each OD pair θ is independent of frequencies ϕ , then the resulting assignment map is smooth.

Proof: Take Logit for example; Probit case can be proved similarly. The hyperpath h expected travel time $\mathbb{E}[T_h(\phi)]$ is continuous function of ϕ . When the hyperpath choice set H for each user is fixed, the hyperpath choice probability is:

$$p_h(\phi) = \frac{e^{-\mathbb{E}[T_h(\phi)]}}{\sum_{h' \in H} e^{-\mathbb{E}[T_{h'}(\phi)]}}$$

This is obviously a smooth function of ϕ . ■

The difficulty part of finding a smooth assignment algorithm is to about how to avoid hyperpath enumeration. We don't provide one in this study. Next, we look at the properties of $p_{i,\phi}(\omega, \mathbf{t} \mid t_i)$. It may be discontinuous as ϕ varies (Figure A2 (a)). Note that although we write

ϕ as a subscript in $p_{i,\phi}(\omega, \mathbf{t} \mid \mathbf{t}_i)$, it is a variable. We can smooth it by using *bump functions* (Eq. (A1)) or smooth kernels like a *Gaussian kernel*.

$$\Psi(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right), & \text{if } x \in (-1, 1) \\ 0, & \text{otherwise} \end{cases} \quad (\text{A1})$$

Take a one-dimensional case for example. Let there be two posterior functions $p_\phi(t)$ and $p_\phi(t')$. Suppose discontinuity happens at point ϕ_0 . We smooth them on interval $(\phi_0 - \epsilon, \phi_0 + \epsilon)$. We first construct g_1 and g_2 to be defined by Eq. (A2) and (A3). Then we define ψ_1 and ψ_2 as in Eq. (A3). The resulting ψ_1 and ψ_2 are the smooth partitions of unity on interval $(\phi_0 - \epsilon, \phi_0 + \epsilon)$ as we desired. The result is illustrated in Figure A2(b). Another way of using a smooth kernel is similar. The variables g_1 and g_2 are obtained by convolution of a smooth kernel with characteristic functions at the end of intervals, then ψ_1 and ψ_2 are obtained again by Eq. (A3).

$$g_1(\phi) = \begin{cases} 1, & \text{if } \phi \leq \phi_0 - \epsilon \\ \exp\left(1 - \frac{1}{1 - \frac{(\phi - \phi_0 + \epsilon)^2}{4\epsilon^2}}\right), & \text{if } \phi_0 - \epsilon < \phi < \phi_0 + \epsilon \\ 0, & \text{if } \phi \geq \phi_0 + \epsilon \end{cases} \quad (\text{A2})$$

$$g_2(\phi) = \begin{cases} 0, & \text{if } \phi \leq \phi_0 - \epsilon \\ e * \exp\left(1 - \frac{1}{1 - \frac{(\phi_0 + \epsilon - \phi)^2}{4\epsilon^2}}\right), & \text{if } \phi_0 - \epsilon < \phi < \phi_0 + \epsilon \\ 1, & \text{if } \phi \geq \phi_0 + \epsilon \end{cases} \quad (\text{A2})$$

$$\psi_1 = \frac{g_1}{g_1 + g_2}, \quad \psi_2 = \frac{g_2}{g_1 + g_2} \quad (\text{A3})$$

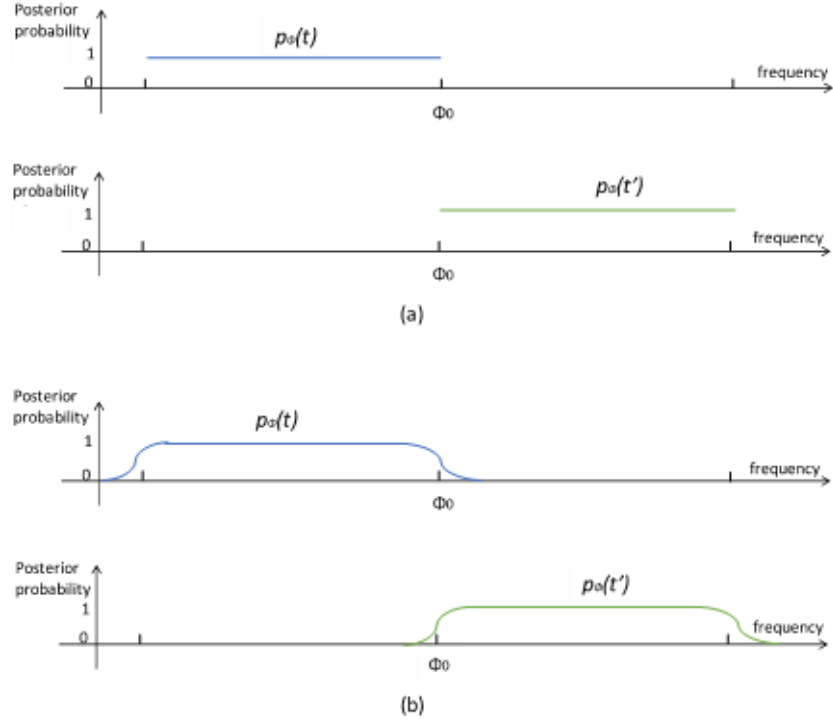


Figure A2. Posterior probability- frequency relation (1-dimensional case).

Proposition A2: *If the set of feasible transit frequency is compact, the transit assignment is smooth, and the posterior probabilities are smooth functions of frequencies, then there exists a pure strategy Nash equilibrium of game G^* .*

Proof: The problem can be formulated as VIs described by Eq. (10) and (11). They require that the state set to be finite; but there is no loss of generality in this assumption since traffic flows can be considered discrete. The gradient is continuous and defined on compact set under our assumption of finite OD demand. Hence there exist at least one solution by Theorem 3.1 from Kinderlehrer and Stampacchia (2000). ■

We do not really need such strong C^∞ condition to prove the existence of Nash equilibrium. A C^1 assignment map would suffice.

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