Waving arms around to teach quantum mechanics

Kelby T. Hahn[a)]
College of Education, Oregon State University, Corvallis, Oregon 97331-6507

Elizabeth Gire[b)]
Department of Physics, Oregon State University, Corvallis, Oregon 97331-6507

(Received 5 October 2021; accepted 22 June 2022)

Kinesthetic (or embodied) representations help students build intuition and deep understanding of concepts. This paper presents a series of kinesthetic activities for a spins-first undergraduate quantum mechanics course that supports students in reasoning and developing intuition about the complex-valued vectors of spin states. The arms representation, used in these activities, was developed as a tangible representation of complex numbers: Students act as an Argand diagram, using their left arm to represent numbers in the complex plane. The arms representation is versatile and can be expanded to depict complex-valued vectors with groups of students. This expansion enables groups of students to represent quantum mechanical state vectors with their arms. We have developed activities using the arms representation that parallel the progression of a spins-first approach by starting with complex numbers, then representing two- and three-state systems, considering time-dependence, and, eventually, extending to approximate wavefunctions. Each activity illustrates the complex nature of quantum states and provides a tangible manipulative from which students can build intuition about quantum phenomena.

https://doi.org/10.1119/5.0073946

I. INTRODUCTION

A particularly challenging aspect of quantum mechanics is the fundamental role complex numbers play in concepts, mathematics, and geometry.[1,2] In a spins-first approach,[3–5] also known as the Stern–Gerlach-first approach,[6] students are introduced to a finite Hilbert space for spin systems and explore quantum mechanics through Stern–Gerlach experiments. This spins-first approach is increasing in popularity and has a distinct advantage, compared to the wavefunction- or position-first approach, of providing a conceptually rich introduction to the quantum postulates without requiring the advanced mathematics needed to solve differential equations.

One noteworthy mathematical trade-off of the spins-first approach is that students work with the complex nature of quantum states right at the start of the course when they represent quantum spin states as complex-valued vectors. Matrix notation and bra-ket notation are used from the beginning. The traditional mathematics courses required for physics majors (i.e., calculus, infinite series and sequences, differential equations, and linear algebra) may not provide students with the in-depth understanding of complex numbers necessary to parse the significance and geometry of these complex vector components.[7] To help students build intuition concerning these mathematical objects, many visualization options have been proposed.[8–16] Although these visualizations support aspects of students' reasoning, they may impose additional burdens on (1) students (to coordinate multiple representations for a single quantum state[8,9] or to parse complex nested diagrams[14]), (2) educators (to write, maintain, and acclimate to new textbooks[15,16]), and (3) classroom technology (to generate higher dimensional graphs, which is resource-intensive[10–13]). Despite strengthening student understanding, these visualizations still require students to navigate coordinating between representations and, in turn, further convolute the already abstract concept of quantum states. In an effort to lessen the abstraction and highlight key geometric relationships, the arms representation was designed as a tangible metaphor for quantum states.[17]

In this paper, we present the arms representation, a kinesthetic, Argand-based representation for complex-valued vectors, and the accompanying arms activities, a cluster of classroom exercises to support a spins-first approach to teaching quantum mechanics. These are instructor-guided, whole-class, active-engagement activities where students stand up and use their left arms to represent complex numbers. Arms was designed to develop student understanding while acting as a real-time formative assessment. Arms has the distinct advantage of not requiring specialized equipment or technical training.

The arms representation is a kinesthetic representation where each student embodies a complex number, and a group of students collaboratively represents quantum states. Researchers have demonstrated that student learning in physics can be enhanced by integrating classroom activities that provide physical experience with physics content, activate sensorimotor brain systems, or support tactile knowledge building.[18–22] Many kinesthetic physics activities focus on kinematics broadly,[20–28] but some have also been developed for other or more focused physics concepts: linear and angular momentum,[19,27] projectile motion,[28] electric field,[29] centripetal force,[30] astronomy,[31–33] Newton's second law,[34] torque,[35] solar cells,[36] energy,[37–39] mechanics,[40] electricity and magnetism,[41] wave dynamics,[42] and more.[43,44]

In addition to being kinesthetic, the arms activities are social active-engagement activities that are often at odds with students' expectations of a quantum mechanics class. A meta-analysis of 225 studies on STEM classrooms found that active-engagement activities, like arms, improve student performance.[45] Additionally, activities that students do not expect to happen in class often lead to a ripple of giggles, and a review of 40 years of studies found that infusing classes with positive humor bolsters engagement and affect.[46] Finally, unlike many of the aforementioned, primarily 2D, quantum mechanics representations, the arms representation takes advantage of all three spatial dimensions, involves multiple students, and uses time effectively to convey the high dimensionality of the quantum systems.

In this paper, we present how we have used arms in a junior-level, spins-first quantum mechanics course to support an array of topics, starting with complex numbers, then introducing two- and three-state systems, considering time-dependence, and, eventually, extending to approximate wavefunctions.[47] We begin with an introduction to the arms representation (Sec. II). Then, we provide in-depth descriptions of five activities (Sec. III). In Sec. IV, we discuss how these activities are integrated into the course and report informal observations of student engagement and learning with this method. We conclude the paper with a discussion of the advantages of the arms activities and provide implementation suggestions and additional resources (Sec. V).

II. ARMS REPRESENTATION OVERVIEW

In its simplest form, the arms representation consists of one person's left arm embodying a complex number, as can be seen in Fig. 1(a). The arms representation is essentially an embodied Argand diagram, that is, a complex plane. The origin is at the shoulder; the real axis is parallel to the ground, with positive numbers in front of the person; the imaginary axis is perpendicular to the ground with positive values above the shoulder, and the left fist then represents a complex number in the complex plane. The left arm is preferred

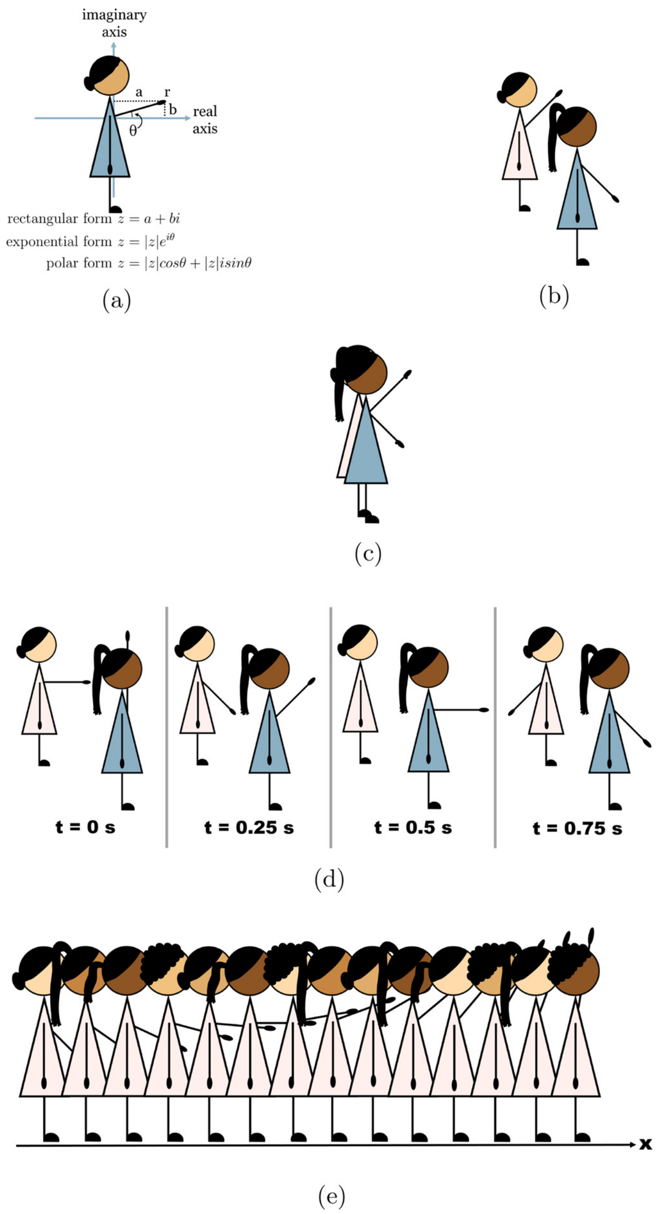


Fig. 1. Stick figure examples of how students can use the arms representation to represent various quantum mechanics phenomena. For (b)–(d), the student in pink (at the back) represents the $|+\rangle_z$ coefficient, and the student in blue represents the $|-\rangle_z$ coefficient. (a) A student representing $z = a + bi$ with the arms representation superimposed to an Argand diagram. (b) Two students working together to represent the spin-1/2 quantum state $|\Psi\rangle = \frac{1}{\sqrt{2}}e^{i\pi/4}|+\rangle_z + \frac{1}{\sqrt{2}}e^{-i\pi/4}|-\rangle_z$. (c) Two students working together to represent a state with a relative phase of $\pi/2$. (d) Representative snapshots in time, of two students working together to represent the spin 1/2 quantum state $|\Psi(t)\rangle = e^{-i\pi t}(|+\rangle_z + e^{i\pi/2}|-\rangle_z)$ evolving with time. (e) Infinitely, many students infinitesimally close together using the arms representation to approximate a wavefunction in the position basis.

to the right, so that the person embodying the number sees a correctly oriented Argand diagram.

Figure 1(a) outlines three of the symbolic notations we use to represent complex numbers. The rectangular form represents the complex number (z) as the sum of a real component (a) and an imaginary component (b). In the exponential form, z is expressed as its norm ($|z|$) multiplied by a complex phase factor, $e^{i\theta}$, where θ is the angle that the complex number makes with the positive real axis. The polar form corresponds to $z = |z|\cos\theta + i|z|\sin\theta$. Each of these notations can be related to the arms representation. The magnitude ($|z|$) is represented by the length of the arm, θ is the angle between the arm and horizontal, and a (i.e., $|z|\cos\theta$) and b (i.e., $|z|\sin\theta$) are the projections onto the real and imaginary axes, respectively.

The arms representation is versatile and scalable. It can be expanded to represent n-dimensional complex-valued vectors by having n partners use their left arm to each represent one of the components of the vector. Moreover, participants' arms can be rotated in time to represent time-dependent complex-valued vectors.

III. ACTIVITIES USING ARMS

In this section, we discuss some of the active-engagement classroom activities we have designed using the arms representation. We present five instructor-guided activities: complex number (Sec. III A), spin-1/2 quantum state (Sec. III B), relative and overall phase (Sec. III C), time evolution (Sec. III D), and going from spin to wavefunction (Sec. III E). For each activity, we outline the goals and prerequisite knowledge, and we discuss how we use the activity in our course.

We note that our descriptions of the activities make them seem highly scripted. In fact, we advocate that the instructor be flexible and responsive to what students are doing in the classroom. We consider the versions of the activities presented here akin to platonic ideals rather than a record of what occurred in any actual class. Our goals are to show the general scope and sequence of each activity as well as suggest some prompts that instructors may use. We also provide notes about how activities can be modified and share anecdotes from our classrooms. This paper is intended to give potential adapters a flavor of these activities, but we strongly encourage educators to read the instructor's guides on our website for more detailed implementation guidance and updates as we modify and refine the activities.[48] Tables I–V outline the approximate duration and central goal for each activity. These durations reflect the time students spend engaged in the arms representation. In Sec. IV, we discuss how the activities can be integrated into a spins-first quantum mechanics course and provide insights into how students interact and learn during these activities. This course utilizes McIntyre's quantum mechanics: A paradigms approach textbook.[4] Also, an outline of the learning goals associated with each activity can be found in the supplementary material.[51]

A. Complex number activity

During the complex number activity, each student is asked to represent a series of complex numbers with their arm. We introduce the arms representation to students for the first time at the beginning of the activity. Students should already be familiar with complex numbers and Argand diagrams. In our course, this activity is done during the first few days of class. Figure 1(a) and Table I provide an overview of this activity.

Table I. Overview of the complex number activity.

Central prompt	Represent the complex number $1+i$ with your arms
Duration	10 min
When used	Week 1, after complex numbers in rectangular and exponential forms have been introduced
Topics to review	Complex numbers Argand diagrams
Key concepts	Forms of complex numbers Arms representation
Related homework	Circle trig complex[a] Phase[b]

[a]Find this homework problem in the supplementary material.[51]
[b]Find this homework problem in the supplementary material.[51]

The instructor begins by reminding students of the rectangular and exponential forms used to represent a complex number, and by drawing an Argand diagram on the board. Students are then asked to stand and face sideways, so that their left shoulder points towards the board. By orienting them in this way, when a student sweeps out the complex plane with their arm, their complex plane aligns with the Argand diagram on the board. The arms representation is then introduced. The instructor begins by outlining the basics of arms by acting out and describing the representation:

> "Swing your left arm in a circle. You just swept out the complex plane."
>
> "Put your left arm straight forward, parallel to the ground. This is the positive, real axis."
>
> "Put your left arm straight upward, perpendicular to the ground. This is the positive imaginary axis."

The instructor then asks the students to practice sweeping out the complex plane with their arm. We have found it necessary to remind students to rotate their shoulder as their arm goes behind them in order to avoid injury.

After introducing students to the basics of the arms representation, the instructor asks them to represent a specific complex number given in a form the students are familiar with,

Table II. Overview of spin-1/2 quantum state activity.

Central prompt	Working in pairs, represent $|\Psi\rangle = \frac{1}{\sqrt{2}}e^{i\pi/4}|+\rangle_z + \frac{1}{\sqrt{2}}e^{-i\pi/4}|-\rangle_z$ with your arms
Duration	5 min
When used	Week 2, after students have been introduced to the S_z basis, knowing that the expansion coefficients are complex numbers, and know how to represent the eigenstates of S_x and S_y in the S_z basis
Topics to review	Arms representation basics Complex-valued vectors
Key concepts	Components of quantum state are complex numbers How to represent common spin-1/2 quantum states
Textbook section (s)	McIntyre 1.2 and 1.3
Related homework	Unknown spin-1/2 brief[a]

[a]Find this homework problem in the supplementary material.[51]

Table III. Overview of relative and overall phase activity.

Central prompt	Geometrically show the relative phase of $\|+\rangle_x$ in the S_z basis. Now multiply it by an overall phase factor $e^{\pi/4}$.
Duration	15 min
When used	Week 2, after students have had some practice in representing spin-1/2 states in the S_z basis and have been introduced to the exponential form of complex numbers, and can multiply complex phase factors.
Topics to review	Arms representation
	How to multiply complex phase factors
Key concepts	Relative phase defines a quantum state
Textbook section (s)	McIntyre 1.2
Related homework	McIntyre 1.3
	Phase 2[a]

[a]Find this homework problem in the supplementary material.[51]

often rectangular form, e.g., $z = 1 + i$ as seen in Fig. 1(a). Occasionally, the instructor will ask students to close their eyes before representing the complex number, so that they will not be influenced by what other students are doing.

As students represent complex numbers with arms, the instructor leads a whole-class discussion highlighting some geometric elements of complex numbers, such as the real and imaginary components as projections onto the respective axes. When asked to represent a complex number with a magnitude other than 1, students will often attempt to stretch their arm (for $|z| > 1$) or bend their elbow (for $|z| < 1$). This quickly leads to a discussion of how arms poorly represents the norm of a complex number, since an arm has a fixed length. We encourage students to think of their arm as an object of whatever length it needs to be. Although the arms representation is not suited to illustrate the magnitude of complex numbers, it allows students to depict approximate phase angles. Note that bending one's arm to account for the magnitude $|z|$ makes these angles more difficult to visualize.

Once students are comfortable representing a complex number in arms from an instruction given in a familiar symbolic form, the instructor may use prompts given in other symbolic forms such as rectangular, polar, and exponential (see Fig. 1(a)). Each new symbolic form should include a

Table IV. Overview of time evolution activity.

Central prompt	Show how the spin-1/2 state: $\|\Psi(t)\rangle = e^{-i(\pi/4)t}\left(\frac{1}{\sqrt{2}}\|+\rangle_z + \frac{1}{\sqrt{2}}e^{i(\pi/2)}\|-\rangle_z\right)$ evolves with time.
Duration	10 min.
When used	Week 4, after students have seen the general solution to the Schrödinger equation for a time-independent Hamiltonian.
Topics to review	Arms representation
	How quantum states are defined
Key concepts	Time-dependent overall phases preserve the quantum state
Textbook section (s)	McIntyre Chap. 3
Related homework	McIntyre 3.2
	McIntyre 3.5
	Frequency (McIntyre 3.12)[a]

[a]Find this homework problem in the supplementary material.[51]

Table V. Overview of going from spin to wavefunction activity.

Central prompt	Using as many peers as you need, approximate a quantum state represented in the position basis using the arms representation.
Duration	10 min
When used	Week 5, after students have been introduced to two-state and n-state quantum systems, and to operators that correspond to observable quantities.
Topics to review	Arms representation
	Position basis
Key concepts	Position is continuous and uncountably infinite
Textbook section (s)	McIntyre 5.2 and 5.3
Related homework	Wavefunctions[a]

[a]Find this homework problem in the supplementary material.[51]

whole-class discussion of the geometric elements. In particular, relating the phase angle to the exponent of the exponential form gives a tangible meaning to θ. This can lead to an additional discussion about how multiplying by a complex phase factor causes a complex number to rotate in the complex plane. For this, the instructor should encourage students to rotate their arm as θ increases and to practice multiplying a given complex number by a complex phase factor.

The complex number activity is also particularly well suited for discussing complex conjugation. The instructor incites the students to represent a complex number and its complex conjugate. The instructor then facilitates a discussion of how, symbolically, complex conjugation corresponds to a change of sign in of the imaginary number, while, geometrically, it is a reflection over the real axis.

1. Noteworthy anecdotes about this activity

This activity typically involves a lot of giggling and a bit of disruption to class. We view this as a good thing. Students are taken off guard by the kinesthetic nature of the activity that often falls outside the realm of what they consider "doing physics." We have found that students adjust quickly, and that this activity helps set the tone for other active engagement activities they encounter in our courses. Even the most stubborn students typically realize that our "silly" activities are conceptually rigorous and buy into them. One tip for building this buy-in is for the instructor to fully commit; let it be silly and fun while also focusing on the concepts.

B. Spin-$1/2$ quantum state activity

The spin-$1/2$ quantum state activity builds off of the complex number activity by having students work in pairs to represent the complex-valued vectors of spin-1/2 quantum states. Before this activity, students should be familiar with quantum mechanical state vectors. They should also know that the eigenstates for the z-component of the spin of a spin-1/2 particle operator S_z are $|+\rangle_z$ and $|-\rangle_z$ (and $|+\rangle_x$ and $|-\rangle_x$ are those of the S_x operator, and $|+\rangle_y$ and $|-\rangle_y$ are those of the S_y operator), and that $(|+\rangle_z, |-\rangle_z)$, $(|+\rangle_y, |-\rangle_y)$, or $(|+\rangle_x, |-\rangle_x)$ can serve as a basis to express any spin-$1/2$ state. This can be extended to spin-n particles with a basis of $2n+1$ eigenvectors. In particular, students should know how to write the $|+\rangle_x$ and $|-\rangle_x$ and $|+\rangle_y$ and $|-\rangle_y$ eigenstates in the $(|+\rangle_z, |-\rangle_z)$ basis,

$$|+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle_z + \frac{1}{\sqrt{2}}|-\rangle_z, \quad |+\rangle_y = \frac{1}{\sqrt{2}}|+\rangle_z + \frac{i}{\sqrt{2}}|-\rangle_z,$$
$$|-\rangle_x = \frac{1}{\sqrt{2}}|+\rangle_z - \frac{1}{\sqrt{2}}|-\rangle_z, \quad |-\rangle_y = \frac{1}{\sqrt{2}}|+\rangle_z - \frac{i}{\sqrt{2}}|-\rangle_z.$$

We typically do this activity in week 2 of our five-week intensive course. Figure 1(b) and Table II provide an overview of this activity.

For the spin-$1/2$ quantum state activity, pairs of students represent a spin quantum state, so that together they can represent the complex coefficient (i.e., probability amplitude) of both $|+\rangle_z$ and $|-\rangle_z$. Arms is used to emphasize that quantum states are complex-valued vectors, and that multiple complex numbers are needed to represent a single quantum state.

We find that, at the beginning of each of the subsequent activities, it is useful to remind students of the basics of the representation. The instructor should then pair the students and designate the student standing on the left as representing the $|+\rangle_z$ component and the student on the right as representing the $|-\rangle_z$ component. At the beginning of the course, our students are most comfortable representing spin-1/2 states in the S_z-basis. A visualization of the spin-$1/2$ quantum state activity can be seen in Fig. 1(b).

To help reinforce what each student in the pair represents, we first ask the pairs to represent the two eigenstates of S_z, so that one of the students in each pair should display 0 with their arm. Next, the instructor prompts the pairs to represent $|+\rangle_x$, $|-\rangle_x$, $|+\rangle_y$, $|-\rangle_y$, $|+\rangle_z$, and $|-\rangle_z$ in the S_z-basis. This practice with different quantum states supplements the understanding developed with Stern–Gerlach experiment simulations.[49] As with the complex number activity, the instructor asks students to translate states given in various notations (bra-ket, matrix, and exponential) into the arms representation and highlights the geometric relationships between the various components. We typically reserve the discussion of the relative phase or angle between the students in each pair for subsequent activities, but it could be integrated here as well.

At the end of the spin-$1/2$ quantum state activity, students should understand that quantum states are complex-valued vectors and should be able to represent a spin-1/2 state with arms and with multiple symbolic notations.

1. *Noteworthy anecdotes about this activity*

Students can be a bit perplexed when they have to represent zero for the $|+\rangle_z$ or $|-\rangle_z$ states because their arm cannot have zero size and hanging straight down means pure imaginary. This can lead to some giggles and/or creativity about how to handle this. We encourage this giggling and consideration by students and let them represent zero either with their arm relaxed to their side or scrunched in small at their shoulder.

C. Relative and overall phase activity

In our course, the relative and overall phase activity follows quickly after the spin-$1/2$ quantum state activity, and they could be done consecutively, as one larger activity. In this activity, the instructor facilitates a conversation about the geometric interpretation of a quantum state and highlights the convention of choosing the first component to be real and positive by factoring out an overall phase. Before this activity, students should be able to multiply complex phase factors. Figure 1(c) and Table III provide an overview of this activity.

We recommend reminding students that they will work in pairs to represent a spin-1/2 state, and that the student on the left represents the component of $|+\rangle_z$ and the student on the right represents the component of $|-\rangle_z$.

The instructor begins the activity by asking students to express a state and identify the relative phase angle, both verbally and geometrically, as seen in Fig. 1(c). They should then ask the students to multiply this state by a (constant) overall phase, like $e^{i\pi/4}$. Students must think through what multiplication means for the quantum state, and in what direction to rotate their arms. Reminding students of any rotations done in the complex number activity can help.

After rotating to accommodate the overall phase, the instructor again invites the students to determine the relative phase angle. Students are then prompted to discuss whether they are in a new quantum state and how they can tell if they are. Through this line of inquiry and the geometric representations, students learn that a quantum spin-$1/2$ state is defined by the magnitudes and relative phase angle of the two coefficients, and that an overall phase does not change the quantum state. From here, a discussion about conventional choices for phase factors ensues.

1. *Noteworthy anecdotes about this activity*

We have found that this activity benefits from the geometric nature of arms as well as from the malleability of the representation. Letting students rotate and move with their partner to best represent the relative phase can be useful. When asked to rotate or multiply the state, students will occasionally rotate at the waist instead of, or in addition to, rotating at the shoulder. This phenomenon typically signals a misunderstanding of the representation, but it may also indicate confusion about the physical system.

D. Time evolution activity

The time evolution activity takes advantage of the kinesthetic nature of the arms representation and especially of the ability to use real-world time as a representational dimension. In this activity, students again work in pairs to represent quantum spin-$1/2$ states, but they are now requested to rotate their arms at the rate indicated by a time-dependent complex coefficient they have to represent. Before the time evolution activity, students should: (1) know that the solution to the Schrödinger equation for a time-independent Hamiltonian is a linear combination of energy eigenstates, $|E_n\rangle$, with time-dependent coefficients that depend on the energy, E_n, $|\psi(t)\rangle = \sum_n c_n(0)e^{-iE_nt/\hbar}|E_n\rangle$, where $c_n(0)$ is the expansion coefficients of the state in the energy basis at time $t=0$ and $\hbar$ is the reduced Planck's constant, (2) be comfortable multiplying complex phase factors, and (3) be familiar with both overall and relative phases of a quantum state. We typically do the time evolution activity about two-thirds of the way through our course, before discussing wavefunction notation. Figure 1(d) and Table IV provide an overview of this activity.

When using arms to teach about time evolution, we find it is crucial to begin the activity with a review of key states such as $|+\rangle_x$, $|-\rangle_x$, $|+\rangle_y$, and $|-\rangle_y$ in the S_z-basis. After students have been re-familiarized with these states, the instructor asks them to represent a state, such as $|+\rangle_x$ in the S_z basis with an arbitrary overall phase. The instructor guides the students through a quick review of what the relative phase is and reminds them that the relative phase determines the state (along with the norms of the expansion coefficients).

After this initial exercise, the instructor initiates a discussion of stationary states. The instructor asks students to represent a state, such as $|+\rangle_x$, by writing the state on the board. Then, the instructor multiplies the state by a time-dependent overall phase, $e^{i\omega t}|+\rangle_x$, and asks the students to represent this state with their arms.

Students typically spend a few minutes discussing how to represent the time component and which way to rotate their arms (counterclockwise for positive ω). After most students have figured out how the time evolution part of arms works, the instructor asks the students to start over representing their state, so that the whole class begins representing, and rotating, together. Once they have been rotating for a while, the instructor asks them to pause their rotation and share what state they are in, or what the relative phase is. Snapshots of this process can be seen in Fig. 1(d). A whole class discussion ensues about how the vector itself changes with time (i.e., the overall phase changes), but the state and any associated measurement or probability does not (i.e., the relative phase does not change). The term "stationary states" is introduced to describe this phenomenon.

The instructor discusses that if the initial state of the quantum system is an eigenstate of the Hamiltonian, the time-dependent complex phase factor acts as an overall phase, and the state is a stationary state. For example, in a case where the Hamiltonian is proportional to the $\hat{S}_x$ spin component operator (e.g., a spin-1/2 particle in a uniform magnetic field pointing in the x direction), the $|+\rangle_x$ is an energy eigenstate and, therefore, a stationary state.

The instructor then initiates a discussion of more general time evolution. The instructor asks the students, what kind of motion would result in a state that changes with time. The whole-class discussion should be guided to come to the agreement that each student in the vector pair would need to be rotating at a different rate or in a different direction. Mathematically, that would correspond to a quantum vector with different time dependent coefficients for each basis state. Students should then be asked to represent a specific time-dependent, non-stationary quantum state in the form, $|\Psi(t)\rangle = e^{i\omega_1 t}c_+|+\rangle_z + e^{i\omega_2 t}c_-|-\rangle_z$. The instructor then invites the students to summarize why a state changes with time and encourages them to articulate that if the relative phase changes, then the state changes.

In the final sequence of this activity, the instructor supposes that each student pair, initially representing $|+\rangle_x$ in the $(|+\rangle_z, |-\rangle_z)$ basis, is subjected to a Hamiltonian that is proportional to the spin component operator $\hat{S}_z$, as is the case of a spin-$1/2$ system in a z-oriented magnetic field. The instructor leads a discussion about how, in this case, the two students in the pair represent the two energy eigenstates, i.e., the eigenstates of $\hat{S}_z$, $|+\rangle_z$ and $|-\rangle_z$. Therefore, to represent the time-evolved state, each person needs to enact the unique time-dependent phase factor of the eigenstate they symbolize. The time dependence can be expressed as (if starting in the state $|+\rangle_x$ at $t=0$)

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\hbar t/2}|+\rangle_z + \frac{1}{\sqrt{2}}e^{+i\hbar t/2}|-\rangle_z.$$

This is an opportunity to emphasize that the energy-dependent complex phase factor (here $e^{\mp i\hbar t/2}$) is applied to the corresponding eigenstate of the Hamiltonian, here $|\pm\rangle_z$. One must be sure that the initial state is expressed in the basis of energy eigenstates before multiplying each component by the relevant time-dependent phase factors to obtain the time-dependent state.

The students then act out this time dependence. To emphasize how the state changes, the instructor can ask students to pause at key configurations that represent $|+\rangle_x$, $|-\rangle_x$, $|+\rangle_y$, $|-\rangle_y$, the states reviewed at the beginning of the activity. The students then identify that the state rotates through $|+\rangle_x$, $|+\rangle_y$, $|-\rangle_x$, $|-\rangle_y$, which can be conceptualized as a spin vector precessing around the z-axis in the xy-plane.

1. Noteworthy anecdotes about this activity

We have tried structuring this activity to prompt students to represent systems where the state evolved with time at the beginning of the activity (rather than starting with stationary states) and observed a student who stood still, looking perplexed. When questioned, the student asked "How can it stay the same state, if the arms move in opposite directions?" The instructor reinforced this idea that the arms, each embodying given eigenstates, indeed, rotates independently, resulting in a change in the relative phase, and, thus, the state was not conserved. We have found that inverting the prompts is advantageous for building to the idea of a state changing with time, rather than confronting it head-on as we used to. However, either prompt order guides the students to the ah-ha moment, confronting the cognitive dissonance, that time evolution of systems can change the quantum state.

E. Going from spin to wavefunction activity

The going from spin to wavefunction arms activity seeks to help students transition their thinking from discrete spin systems to continuous systems that can be described with wavefunctions. Before the going from spin to wavefunction activity, students should be familiar with spin-1 and spin-n systems, understanding that observables are represented by operators, and that the eigenstates of operators form a basis. We typically do this activity in the last week of the course, but it could also be done before the time evolution activity, if desired. Figure 1(e) and Table V provide an overview of this activity.

We encourage beginning the new activity with a review of bra-ket notation, probability histograms of measurement outcomes, and matrix notation. One way to do this is to ask students to represent a specific spin-1/2 state, say in the $(|+\rangle_z, |-\rangle_z)$ basis, given in the bra-ket notation, with their arms, then have them draw the probability histogram of the possible outcomes of the measurement of S_z for that state, and write the state in matrix notation.

Once students have successfully expressed a spin-1/2 state with arms, bra-ket notation, probability histograms, and matrix notation, the conversation shifts to spin-1 states. The instructor asks the class "What would we have to do in order to represent a spin-1 state with the arms representation?" The students will eventually suggest adding a third person (or arm) to the initial pair of students. Students should then be regrouped into triads, with each student representing one of the three basis kets ($|1\rangle_z$, $|0\rangle_z$, $|-1\rangle_z$). Now in triads, students are asked to translate from bra-ket notation to the arms representation, then draw probability histograms of the measurement results of S_z, and write the state in matrix notation.[50]

The process of considering a higher-state system, translating bra-ket notation to arms, probability histograms, and matrix notation, is repeated for at least one higher-state discrete system. For example, groups of seven students would represent a spin-3 system. After progressing through discrete states with an increasing number of basis kets, the instructor asks the students to imagine a particle where the position is the observable of interest. Students are asked how they could use the arms representation to enact the basis that would allow us to measure position, the position basis. The instructor encourages all of the students to work together to represent a state in the position basis. They then solicit ideas about how many students we need and where they should stand. The discussion continues until a consensus of needing an infinite number of people standing infinitely close together is reached. A visualization of this can be seen in Fig. 1(e).

The instructor then leads a discussion about the similarities and differences between probability amplitudes in the spin case and a wavefunction (i.e., a probability density amplitude). After representing a state in the position basis with the arms representation, the instructor works with the students to create a probability histogram of this state, as they did with spin measurements. Students determine that this histogram is a function of x and depicts the probability density. The instructor then introduces the parallel between spin states (where probabilities are the norm squared of probability amplitudes) and wavefunctions (where the probability density is the norm squared of the probability density amplitude, i.e., the wavefunction or the complex function the students had just been representing with their arms).

Next, the class considers how to represent this state in matrix notation, as they did with spin systems, concluding that, for a wavefunction, an infinitely long column matrix is needed, with each entry corresponding to a position eigenstate. The discussion should include how even writing down values for this matrix is an approximation of the state itself. Just like with the arms representation for this system, between any two entries you write down, there are infinitely more entries. The instructor then frames this as a motivation for wavefunction form, which can represent all of the entries at once.

The instructor encourages a discussion of the parameters of a position basis; how the basis, even when the region is bounded, has an uncountably infinite number of basis kets. Additionally, the students are encouraged to think about what an individual position basis ket would look like; we have found drawing parallels to spiky delta functions to be useful. At this stage, the instructor should emphasize some differences between wavefunctions and probability amplitudes, including that squaring the norm of these quantities leads to probability densities and probabilities, respectively, and that, therefore, a wavefunction and a probability amplitude have different physical dimensions. We find it helpful to draw a parallel between this situation and other discrete vs. continuous situations students might be familiar with, like the relationship between discrete charges and charge density. The activity wraps up with a review of the main learning goals outlined in the supplementary material.[51]

1. *Noteworthy anecdotes about this activity*

This activity is most powerful if all students engage in representing one state together. The probability density histogram is also particularly potent in connecting the arms representation to wavefunction notation. We strongly urge taking the time to re-represent the student's 3D arms wavefunction as a 2D probability density histogram.

IV. EMBEDDING IN INSTRUCTION

We typically weave these five activities throughout our five-week intensive quantum fundamentals course. An example schedule of the quantum fundamentals course can be found in the supplementary material.[51] Throughout the course, we leverage the advantages of the arms representation to support key conceptual understandings. The features of arms highlight key physics concepts: for example, the spatial separation of the basis kets reinforces the complex nature of each coefficient. Geometric relations between arms highlight the significance of relative phase in defining a state. The ability to enact the temporal changes of time-evolution helps students to understand stationary states. Enacting the transition from discrete to continuous reinforces the similarities between discrete spin states and continuous position wavefunctions and provides conceptual anchors for learning about wavefunctions. The arms activities are not the core of the curriculum but rather one tool in our instructor's tool box to support student reasoning.

Due to their kinesthetic nature, students are unable to take notes during the activities. At first, this can cause some discomfort for some students, but we have found that students quickly develop skills to reflect on their learning in their notes after an activity, supporting both the physics content and students' metacognitive skills. This reflection is reinforced through whole-class wrap-up discussions after each activity. These discussions/mini-lectures summarize the main points of the activity and provide a space for students to take notes if they choose. Additionally, these activities tend to be easy for students to remember, even without extensive notes. This is a particularly interesting aspect of activities-paired curricula: the activities continually build upon the previously learnt physics concepts, so that students rarely need detailed notes to solidify their understanding of physics concepts. Indeed, the concepts presented with the arms activities are continually reinforced throughout the course through later arms activities, small-group activities, homework, and exams.

A final note on embedding the arms activities in instruction is that it is crucial that we do not force students to participate in activities. Often, a student or two will sit out of any given activity, including the kinesthetic ones. For the arms activities, we often begin with, "If it is physically safe for you to do so, please move your arm in a circle." We aim to normalize the idea that not everyone does everything, to make it easier for people to decide for themselves when to participate.

V. CONCLUSIONS

The novelty of the arms representation comes from its simplicity. The lack of specialized equipment and training means arms has a low barrier of entry for instructors and students alike. Arms is easily manipulable and provides geometric insights that are difficult to convey in two-dimensional representations. We have shown how the various activities can bolster students' understanding of quantum mechanics and help them gain a physical intuition of the various phenomena. The kinesthetic nature of the activities brings fun and engagement to quantum mechanics courses. This paper outlines a few classroom tested activities that use the arms representation. We envision many more possibilities for integrating the arms representation into our quantum mechanics courses, which will be the subject of future publications.

The arms representation does have some inherent limitations. For example, arms are a fixed length, and, consequently, the arms representation does not convey the norm of a complex number. Additionally, while the arms representation does not require specialized equipment, it does require significant space, multiple people, and high participant mobility. We have explored options for representing length, such as using tinker toys rather than arms to represent the complex numbers, but found that the added benefit is not worth the cost, logistical constraints, and removed embodiment that comes with adding equipment. We have also explored alternative modes of engagement for students with mobility restrictions, such as using equipment rather than their arm: tinker toys, pens, and meter sticks. These alternatives may work for many students, but, if not, we recommend grouping those students with students who can act out the representation and encourage their participation in the group discussions about how to move. Despite these limitations, we have found the arms representation and activities to support students' conceptual understanding, and we hope to see them taken up by other quantum mechanics instructors. Beyond the professional reflections offered here, a qualitative analysis of student reasoning and experiences with these activities is underway, though the results from this analysis are beyond the scope of this manuscript. Support for adopting these and other arms activities may be found on our website.[48]

ACKNOWLEDGMENTS

The Paradigms @ OSU group, particularly Dr. Corinne Manogue and Dr. Elizabeth Gire, developed and refined the arms representation and activities. Additional refinement of ideas for this article came from the OSUPER group, particularly David Roundy, Christian Solorio, Dustin Treece, and Maggie Greenwood. This material is based upon work supported by the National Science Foundation under Grant No. DUE-1836604.

[a]Electronic mail: kelby.hahn@oregonstate.edu, ORCID: 0000-0002-3067-5573.
[b]Electronic mail: giree@oregonstate.edu, ORCID: 0000-0002-6126-7863.

[1]Tong Wan, Paul Emigh, and Peter Shaffer, "Student understanding of the measurable effects of relative phases in superposition states," in *Physics Education Research Conference 2017, PER Conference* (PERC Proceedings, Cincinnati, OH, 2017), pp. 420–423.

[2]Ricardo Karam, "Why are complex numbers needed in quantum mechanics? Some answers for the introductory level," Am. J. Phys. **88**(1), 39–45 (2020).

[3]John S. Townsend, *A Modern Approach to Quantum Mechanics* (University Science Books, Sausalito, CA, 2000).

[4]David H. McIntyre, Corinne A. Manogue, and Janet Tate, *Quantum Mechanics: A Paradigms Approach* (Pearson, New York, 2012).

[5]Mark Beck, *Quantum Mechanics: Theory and Experiment* (Oxford U. P., Oxford, 2012).

[6]William F. Courtney, Lucas B. Vieira, Paul S. Julienne, and James K. Freericks, "Incorporating the Stern–Gerlach delayed-choice quantum eraser into the undergraduate quantum mechanics curriculum," Am. J. Phys. **88**(4), 298–307 (2020).

[7]Emily M. Smith, Justyna Zwolak, and Corinne Manogue, "Student difficulties with complex numbers," in *Physics Education Research Conference, PER Part of the PER Conference Series* (PERC Proceedings, College Park, MD, 2015), pp. 311–314.

[8]M. Belloni and W. Christian, "Physlets for quantum mechanics," Comput. Sci. Eng. **5**(1), 90–97 (2003).

[9]Gina Passante and Antje Kohnle, "Enhancing student visual understanding of the time evolution of quantum systems," Phys. Rev. Phys. Educ. Res. **15**(1), 010110 (2019).

[10]John R. Hiller, Ian D. Johnston, and Daniel F. Styer, *Quantum Mechanics Simulations: The Consortium for Upper-Level Physics Software* (Wiley, New York, 1995).

[11]Antje Kohnle, Donatella Cassettari, Tom J. Edwards, Callum Ferguson, Alastair D. Gillies, Christopher A. Hooley, Natalia Korolkova, Joseph Llama, and Bruce D. Sinclair, "A new multimedia resource for teaching quantum mechanics concepts," Am. J. Phys. **80**(2), 148–153 (2012).

[12]S. B. McKagan, K. K. Perkins, M. Dubson, C. Malley, S. Reid, R. LeMaster, and C. E. Wieman, "Developing and researching PhET simulations for teaching quantum mechanics," Am. J. Phys. **76**(4), 406–417 (2008).

[13]Abraham Goldberg, Harry M. Schey, and Judah L. Schwartz, "Computer-generated motion pictures of one-dimensional quantum-mechanical transmission and reflection phenomena," Am. J. Phys. **35**(3), 177–186 (1967).

[14]Hunter Close, Catherine Schiber, Eleanor Close, and Dave Donnelly, "Students' dynamic geometric reasoning about quantum spin-1/2 states," in *Physics Education Research Conference* (PERC Proceedings, Portland, OR, 2013), pp. 93–96.

[15]Siegmund Brandt and Hans Dieter Dahmen, *The Picture Book of Quantum Mechanics* (Springer, New York, 2012).

[16]Bernd Thaller, *Visual Quantum Mechanics: Selected Topics with Computer-Generated Animations of Quantum-Mechanical Phenomena* (Springer, New York, 2002).

[17]Corinne Manogue, Elizabeth Gire, and David Roundy, "Tangible metaphors," in *Physics Education Research Conference* (PERC Proceedings, Portland, OR, 2013), pp. 27–30.

[18]J. Solomon, R. Bevan, A. Frost, H. Reynolds, M. Summers, and C. Zimmerman, "Can pupils learn through their own movements? A study of the use of a motion sensor interface," Phys. Educ. **26**(6), 345–349 (1991).

[19]Carly Kontra, Daniel J. Lyons, Susan M. Fischer, and Sian L. Beilock, "Physical experience enhances science learning," Psychol. Sci. **26**(6), 737–749 (2015).

[20]Carolien Duijzer, Marja Van den Heuvel-Panhuizen, Michiel Veldhuis, Michiel Doorman, and Paul Leseman, "Embodied learning environments for graphing motion: A systematic literature review," Educ. Psychol. Rev. **31**(3), 597–629 (2019).

[21]William Struck and Randy Yerrick, "The effect of data acquisition-probeware and digital video analysis on accurate graphical representation of kinetics in a high school physics class," J. Sci. Educ. Technol. **19**(2), 199–211 (2010).

[22]Robert J. Beichner, "The effect of simultaneous motion presentation and graph generation in a kinematics lab" J. Res. Sci. Teach. **27**(8), 803–815 (1990).

[23]Zachary Dale, Paul R. DeStefano, Lori Shaaban, Cora Siebert, and Ralf Widenhorn, "A step forward in kinesthetic activities for teaching kinematics in introductory physics," Am. J. Phys. **88**(10), 825–830 (2020).

[24]Janice L. Anderson and Steven D. Wall, "Kinecting physics: Conceptualization of motion through visualization and embodiment," J. Sci. Educ. Technol. **25**(2), 161–173 (2016).

[25]Heather Brasell, "The effect of real-time laboratory graphing on learning graphic representations of distance and velocity," J. Res. Sci. Teach. **24**(4), 385–395 (1987).

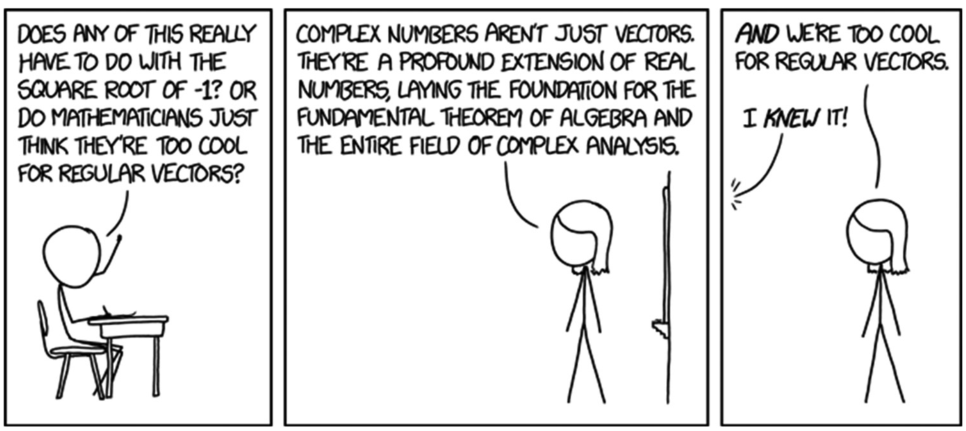
[26]Insook Han and John B. Black, "Incorporating haptic feedback in simulation for learning physics," Comput. Educ. **57**(4), 2281–2290 (2011).

[27]Elliot Mylott, Justin Dunlap, Lester Lampert, and Ralf Widenhorn, "Kinesthetic activities for the classroom," Phys. Teach. **52**(9), 525–528 (2014).

[28]Brooke A. Whitworth, Jennifer L. Chiu, and Randy L. Bell, "Kinesthetic investigations in the physics classroom," Phys. Teach. **52**(2), 91–93 (2014).

[29]Mina C. Johnson-Glenberg and Colleen Megowan-Romanowicz, "Embodied science and mixed reality: How gesture and motion capture affect physics education," Cognit. Res. **2**(1), 24 (2017).

[30]Mina C. Johnson-Glenberg, Colleen Megowan-Romanowicz, David A. Birchfield, and Caroline Savio-Ramos, "Effects of embodied learning and digital platform on the retention of physics content: Centripetal force," Front. Psychol. **7**, 1819 (2016).
[31]Robb Lindgren, Michael Tscholl, and J. Moshell, *MEteor: Developing Physics Concepts Through Body-based Interaction With a Mixed Reality Simulation* (PERC Proceeedings, 2014), pp. 217–220.
[32]Bor Gregorcic and Jesper Haglund, "Conceptual blending as an interpretive lens for student engagement with technology: Exploring celestial motion on an interactive whiteboard," Res. Sci. Educ. **51**(2), 235–275 (2021).
[33]Cherilynn A. Morrow, "Kinesthetic astronomy: The sky time lesson," Phys. Teach. **38**(4), 252–253 (2000).
[34]Vincent P. Coletta, Josh Bernardin, Daniel Pascoe, and Anatol Hoemke, "Feeling Newton's second law," Phys. Teach. **57**(2), 88–90 (2019).
[35]Padma Ganesh, "Connecting torque and friction using tactile learning activities," Phys. Teach. **58**(9), 659–662 (2020).
[36]A. J. Richards and Eugenia Etkina, "Kinaesthetic learning activities and learning about solar cells," Phys. Educ. **48**(5), 578–585 (2013).
[37]Hunter G. Close and Rachel E. Scherr, "Enacting conceptual metaphor through blending: Learning activities embodying the substance metaphor for energy," Int. J. Sci. Educ. **37**(5–6), 839–866 (2015).
[38]Abigail Daane, Lindsay Wells, and Rachel Scherr, "Energy theater," Phys. Teach. **52**, 020105 (2014).
[39]Cecily Criminale, Neda Esfan, and Mariam Mathew, "Energy bowlerama," Sci. Scope **30**(2), 30–33 (2006), http://www.jstor.org/stable/43181012.
[40]A. J. Richards, "Teaching mechanics using kinesthetic learning activities," Phys. Teach. **57**(1), 35–38 (2019).
[41]A. J. Richards, "Teaching electricity and magnetism using kinesthetic learning activities," Phys. Teach. **58**(8), 572–576 (2020).
[42]Panagiotis Pantidos and Stamatis Patapis, "Kinesthetic transverse wave demonstration," Phys. Teach. **43**(6), 344–345 (2005).
[43]Paradigms @ osu kinesthetic activities.
[44]Carole Escobar, "Amusement park physics," Phys. Teach. **28**(7), 446–453 (1990).
[45]Scott Freeman, Sarah L. Eddy, Miles McDonough, Michelle K. Smith, Nnadozie Okoroafor, Hannah Jordt, and Mary Pat Wenderoth, "Active learning increases student performance in science, engineering, and mathematics," Proc. Natl. Acad. Sci. U. S. A. **111**(23), 8410–8415 (2014).
[46]John A. Banas, Norah Dunbar, Dariela Rodriguez, and Shr-Jie Liu, "A review of humor in educational settings: Four decades of research," Commun. Educ. **60**(1), 115–144 (2011).
[47]Paul J. Emigh, Elizabeth Gire, Corinne A. Manogue, Gina Passante, and Peter S. Shaffer, "Research-based quantum instruction: Paradigms and tutorials," Phys. Rev. Phys. Educ. Res. **16**(2), 020156 (2020).
[48]OSUPER, <Paradigms@osu>, 2019.
[49]David McIntyre, Spins java homepage, http://sites.science.oregonstate.edu/~mcintyre/ph425/spins/index.html.
[50]These probability histograms are similar to those in McIntyre's textbook, with each of the possible measurement result presented on the horizontal axis, and then bars representing the probability of that measurement result. See Fig. 1.9 in McIntyre for an example (Ref. 4, p. 16).
[51]See supplementary material at https://www.scitation.org/doi/suppl/10.1119/5.0073946 for a sample schedule for Quantum Fundamentals, Example homework problems, and a list of Arms activity learning goals.



Im trying to prove that mathematics forms a meta-abelian group, which would finally confirm my suspicions that algebraic geometry and geometric algebra are the same thing. (Source: https://xkcd.com/2028/)