

FROBENIUS-PERRON THEORY FOR PROJECTIVE SCHEMES

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ABSTRACT. The Frobenius-Perron theory of an endofunctor of a $\mathbb{k}$ -linear category (recently introduced in Chen et al. [Algebra Number Theory 13 (2019), pp. 2005–2055]) provides new invariants for abelian and triangulated categories. Here we study Frobenius-Perron type invariants for derived categories of commutative and noncommutative projective schemes. In particular, we calculate the Frobenius-Perron dimension for domestic and tubular weighted projective lines, define Frobenius-Perron generalizations of Calabi-Yau and Kodaira dimensions, and provide examples. We apply this theory to the derived categories associated to certain Artin-Schelter regular and finite-dimensional algebras.

INTRODUCTION

The Frobenius-Perron dimension of an endofunctor of a category was introduced by the authors in [CG]. It can be viewed as a generalization of the Frobenius-Perron dimension of an object in a fusion category introduced by Etingof-Nikshych-Ostrik [ENO05] in early 2000 (also see [EGNO15, EGO04, Nik04]). It is shown in [CG] that the Frobenius-Perron dimension of either Ext^1 or the suspension of a triangulated category is a useful invariant in several different topics such as embedding problem, Tame and wild dichotomy, complexity of categories. In particular, the Frobenius-Perron invariants have strong connections with the representation type of a category [CG, ZZ].

The definition of the Frobenius-Perron dimension of a category will be recalled in Section 2. In the present paper we continue to develop Frobenius-Perron theory, but we restrict our attention to the bounded derived category of coherent sheaves over a projective scheme. A projective scheme could be a classical (or commutative) one, or a noncommutative one in the sense of [AZ], or a weighted projective line in the sense of [GL]. We refer to Section 3 for some basics concerning weighted projective lines.

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Our first goal is to understand the Frobenius-Perron dimension, denoted by fpd , of the bounded derived category of coherent sheaves over a weighted projective line, which is also helpful for understanding the Frobenius-Perron dimension of the path algebra of an acyclic quiver of $\tilde{A}\tilde{D}\tilde{E}$ type via the derived equivalence given in Lemma 2.1(2). Let $\mathbb{X}$ be a weighted projective line (respectively, a commutative or noncommutative projective scheme). We use $\text{coh}(\mathbb{X})$ to denote the abelian category of coherent sheaves over $\mathbb{X}$ and $D^b(\text{coh}(\mathbb{X}))$ to denote the bounded derived category of $\text{coh}(\mathbb{X})$. Here is the main result in this topic.

Theorem 0.1 (Theorem 2.13). *Let $\mathbb{X}$ be a weighted projective line that is either domestic or tubular. Then $\text{fpd}(D^b(\text{coh}(\mathbb{X}))) = 1$.*

Note that Theorem 0.1 is a “weighted” version of [CG, Proposition 6.5(1,2)]. By Lemma 2.1(2), we obtain the Frobenius-Perron dimension of the bounded derived category of finite dimensional representations of acyclic quivers of $\tilde{A}\tilde{D}\tilde{E}$ type.

Our second goal is to introduce Frobenius-Perron (“fp” for short) version of some classical invariants. We will focus on fp-analogues of two important invariants in projective algebraic geometry, namely,

Calabi-Yau dimension, and
Kodaira dimension.

Let $\text{fp}\kappa$ (respectively, fpcy) denote the fp version of the Kodaira dimension [Definition 3.5(1)] (respectively, the Calabi-Yau dimension [Definition 3.1(3)]). Both are defined for bounded derived categories of smooth projective schemes or more generally triangulated categories with Serre functor. In algebraic geometry, a Calabi-Yau variety has the trivial canonical bundle. In noncommutative algebraic geometry, a “skew Calabi-Yau” scheme may not have a trivial canonical bundle. Our fp version of the Calabi-Yau dimension covers the case even when the canonical bundle is not trivial. Below is one of the main results in this direction. Note that the definition of $\text{fp}\kappa$ is dependent on a chosen structure sheaf.

Theorem 0.2 (Propositions 3.3 and 3.6). *Let $\mathbb{X}$ be a smooth projective scheme and $\mathcal{T}$ be the triangulated category $D^b(\text{coh}(\mathbb{X}))$ with structure sheaf $\mathcal{O}_{\mathbb{X}}$. Then the following hold.*

- (1) $\text{fp}\kappa(\mathcal{T}, \mathcal{O}_{\mathbb{X}}) = \kappa(\mathbb{X})$.
- (2) $\text{fpcy}(\mathcal{T}) = \dim \mathbb{X}$. As a consequence, if $\mathbb{X}$ is Calabi-Yau, then $\text{fpcy} \mathcal{T}$ equals the Calabi-Yau dimension of $\mathbb{X}$.

In the noncommutative case we have

Theorem 0.3 (Theorem 4.5). *Let A be a noetherian connected graded Artin-Schelter Gorenstein algebra of injective dimension $d \geq 2$ and AS index ℓ that is generated in degree 1. Suppose that $\mathbb{X} := \text{Proj } A$ has finite homological dimension and that the Hilbert series of A is rational. Let $\mathcal{T}$ be the bounded derived category of $\text{coh}(\mathbb{X})$ and $\mathcal{A}$ be the image of A in $\text{Proj } A$.*

- (1) $\text{fpcy}(\mathcal{T}) = d - 1$.
- (2) If $\ell > 0$, then $\text{fp}\kappa(\mathcal{T}, \mathcal{A}) = -\infty$ and $\text{fp}\kappa^{-1}(\mathcal{T}, \mathcal{A}) = \text{GKdim } A - 1$.
- (3) If $\ell < 0$, then $\text{fp}\kappa(\mathcal{T}, \mathcal{A}) = \text{GKdim } A - 1$ and $\text{fp}\kappa^{-1}(\mathcal{T}, \mathcal{A}) = -\infty$.
- (4) If $\ell = 0$, then $\text{fp}\kappa(\mathcal{T}, \mathcal{A}) = \text{fp}\kappa^{-1}(\mathcal{T}, \mathcal{A}) = 0$.

Similar results are proved for Piontkovski projective lines, see Section 4.

Note that the definitions of fp Calabi-Yau dimension and fp Kodaira dimension make sense for bounded derived category of left modules over a finite dimension algebra of finite global dimension. However, we don't have a complete result for that case. Some examples are given in Section 5.

When we are working with finite dimensional algebras, it is well-known that the global dimension is not a derived invariant. By definition, fp Calabi-Yau dimension is a derived invariant. This suggests that the fp Calabi-Yau dimension is nicer than the global dimension in some aspects. So it is important to study this new invariant. To start we ask the following questions.

Question 0.4. Let A be a finite dimensional algebra of finite global dimension and let $\mathcal{T}$ be the derived category $D^b(\text{Mod}_{f,d} - A)$.

- (1) Is $\text{fpcy}(\mathcal{T})$ always finite?
- (2) What is the set of possible values of $\text{fpcy}(\mathcal{T})$ when A varies?
- (3) What is the set of possible values of $\text{fp}\kappa(\mathcal{T}, A)$ when A varies?

More questions and some partial answers are given in Section 5. Some other examples are given in [Wi, ZZ].

We have outlined several important applications of Frobenius-Perron invariants in [CG]. Next we would like to mention one surprising application of the Frobenius-Perron curvature defined in [CG, Definition 2.3(4)].

Theorem 0.5 ([ZZ, Corollary 0.6]). *Suppose that the bounded derived categories of representations of two finite acyclic quivers are equivalent as tensor triangulated categories. Then the quivers are isomorphic.*

This result is striking because it is well-known that, for two Dynkin quivers with the same underlying Dynkin diagram, their derived categories are triangulated equivalent [BGP, Ha], even if the quivers are non-isomorphic. We hope that different Frobenius-Perron invariants will become effective tools in the study of triangulated categories and monoidal (or tensor) triangulated categories.

This paper is organized as follows. Some background material are provided in Section 1. In Section 2, we review some facts about weighted projective lines and prove Theorem 0.1. In Section 3, we introduce fp-version of Calabi-Yau dimension and Kodaira dimension for Ext-finite triangulated categories with Serre functor and prove Theorem 0.2. In Section 4, fp Calabi-Yau dimension and fp Kodaira dimension are studied for noncommutative projective schemes and Theorem 0.3 is proved there. In Section 5, some partial results, comments and examples are given concerning finite dimensional algebras. Sections 6 and 7 are appendices. The proof of Theorem 0.1 is dependent on some linear algebra computation given in Section 6. This paper can be viewed as a sequel of [CG].

1. PRELIMINARIES AND DEFINITIONS

Throughout let $\mathbb{k}$ be a base field that is algebraically closed. Let everything be over $\mathbb{k}$.

We are mainly interested in the derived category $D^b(\text{coh}(\mathbb{X}))$ where $\mathbb{X}$ is a smooth commutative or noncommutative projective scheme, but most definitions work for more general pre-triangulated (or abelian) categories.

Part of this section is copied from [CG].

1.1. Spectral radius of a square matrix. Let A be an $n \times n$ -matrix over complex numbers $\mathbb{C}$. The *spectral radius* of A is defined to be

$$\rho(A) = \max\{|r_1|, |r_2|, \dots, |r_n|\},$$

where $\{r_1, r_2, \dots, r_n\}$ is the complete set of eigenvalues of A . When each entry of A is a positive real number, $\rho(A)$ is also called the *Perron root* or the *Perron-Frobenius eigenvalue* of A .

In order to include the “infinite-dimensional” setting, we extend the definition of the spectral radius in the following way.

Let $A = (a_{ij})_{n \times n}$ be an $n \times n$ -matrix with entries a_{ij} in $\mathbb{R}^+ := \mathbb{R} \cup \{\pm\infty\}$. Define $A' = (a'_{ij})_{n \times n}$ where

$$a'_{ij} = \begin{cases} a_{ij} & a_{ij} \neq \pm\infty, \\ x_{ij} & a_{ij} = \infty, \\ -x_{ij} & a_{ij} = -\infty. \end{cases}$$

In other words, we are replacing ∞ in the (i, j) -entry by a finite real number, called x_{ij} , in the (i, j) -entry. Or x_{ij} are considered as function or a variable mapping $\mathbb{R} \rightarrow \mathbb{R}$.

Definition 1.1. Let A be an $n \times n$ -matrix with entries in $\mathbb{R}^+$. The *spectral radius* of A is defined to be

$$\rho(A) := \liminf_{\text{all } x_{ij} \rightarrow \infty} \rho(A').$$

See [CG, Remark 1.3 and Example 1.4].

1.2. Frobenius-Perron dimension of a quiver.

Definition 1.2 ([CG, Definition 1.6]). Let Q be a quiver.

- (1) If Q has finitely many vertices, then the *Frobenius-Perron dimension* of Q is defined to be

$$\text{fpd } Q := \rho(A(Q))$$

where $A(Q)$ is the adjacency matrix of Q .

- (2) Let Q be any quiver. The *Frobenius-Perron dimension* of Q is defined to be

$$\text{fpd } Q := \sup\{\text{fpd } Q'\}$$

where Q' runs over all finite subquivers of Q .

1.3. Frobenius-Perron dimension of an endofunctor. Let $\mathcal{C}$ denote a $\mathbb{k}$ -linear category. For simplicity, we use $\dim(A, B)$ for $\dim \text{Hom}_{\mathcal{C}}(A, B)$ for any two objects A and B in $\mathcal{C}$. Here the second $\dim$ is $\dim_{\mathbb{k}}$.

The set of finite subsets of nonzero objects in $\mathcal{C}$ is denoted by Φ and the set of subsets of n nonzero objects in $\mathcal{C}$ is denoted by Φ_n for each $n \geq 1$. It is clear that $\Phi = \bigcup_{n \geq 1} \Phi_n$. We do not consider the empty set as an element of Φ .

Definition 1.3 ([CG, Definition 2.1]). Let $\phi := \{X_1, X_2, \dots, X_n\}$ be a finite subset of nonzero objects in $\mathcal{C}$, namely, $\phi \in \Phi_n$. Let σ be an endofunctor of $\mathcal{C}$.

- (1) The *adjacency matrix* of (ϕ, σ) is defined to be

$$A(\phi, \sigma) := (a_{ij})_{n \times n}, \quad \text{where } a_{ij} := \dim(X_i, \sigma(X_j)) \quad \forall i, j.$$

- (2) An object M in $\mathcal{C}$ is called a *brick* [ASS, Definition 2.4, Ch. VII] if

$$\mathrm{Hom}_{\mathcal{C}}(M, M) = \mathbb{k}.$$

If $\mathcal{C}$ is a pre-triangulated category [Ne, Definition 1.1.2] with suspension Σ , an object M in $\mathcal{C}$ is called an *atomic* object if it is a brick and satisfies

$$\mathrm{Hom}_{\mathcal{C}}(M, \Sigma^{-i}(M)) = 0, \quad \forall i > 0.$$

- (3) $\phi \in \Phi$ is called a *brick set* (respectively, an *atomic set*) if each X_i is a brick (respectively, atomic) and

$$\dim(X_i, X_j) = \delta_{ij}$$

for all $1 \leq i, j \leq n$. The set of brick (respectively, atomic) n -object subsets is denoted by $\Phi_{n,b}$ (respectively, $\Phi_{n,a}$). We write $\Phi_b = \bigcup_{n \geq 1} \Phi_{n,b}$ (respectively, $\Phi_a = \bigcup_{n \geq 1} \Phi_{n,a}$).

Definition 1.4 ([CG, Definition 2.3]). Retain the notation as in Definition 1.3, and we use Φ_b as the testing objects. When $\mathcal{C}$ is a pre-triangulated category, Φ_b is automatically replaced by Φ_a unless otherwise stated.

- (1) The *n th Frobenius-Perron dimension* of σ is defined to be

$$\mathrm{fpd}^n(\sigma) := \sup_{\phi \in \Phi_{n,b}} \{\rho(A(\phi, \sigma))\}.$$

If $\Phi_{n,b}$ is empty, then, by convention, $\mathrm{fpd}^n(\sigma) = 0$.

- (2) The *Frobenius-Perron dimension* of σ is defined to be

$$\mathrm{fpd}(\sigma) := \sup_n \{\mathrm{fpd}^n(\sigma)\} = \sup_{\phi \in \Phi_b} \{\rho(A(\phi, \sigma))\}.$$

- (3) The *Frobenius-Perron growth* of σ is defined to be

$$\mathrm{fpg}(\sigma) := \sup_{\phi \in \Phi_b} \{\limsup_{n \rightarrow \infty} \log_n(\rho(A(\phi, \sigma^n)))\}.$$

By convention, $\log_n 0 = -\infty$.

- (4) The *Frobenius-Perron curvature* of σ is defined to be

$$\mathrm{fpv}(\sigma) := \sup_{\phi \in \Phi_b} \{\limsup_{n \rightarrow \infty} (\rho(A(\phi, \sigma^n)))^{1/n}\}.$$

In this paper, we only use Φ_b and Φ_a as the testing objects. But in principal one can use other testing objects, see Section 7. We continue to review definitions from [CG].

Definition 1.5 ([CG, Definition 2.7]).

- (1) Let $\mathfrak{A}$ be an abelian category. The *Frobenius-Perron dimension* of $\mathfrak{A}$ is defined to be

$$\mathrm{fpd} \mathfrak{A} := \mathrm{fpd}(E^1)$$

where $E^1 := \mathrm{Ext}_{\mathfrak{A}}^1(-, -)$ is defined as in [CG, Example 2.6(1)]. The *Frobenius-Perron theory* of $\mathfrak{A}$ is the collection

$$\{\mathrm{fpd}^m(E^n)\}_{m \geq 1, n \geq 0}$$

where $E^n := \mathrm{Ext}_{\mathfrak{A}}^n(-, -)$ is defined as in [CG, Example 2.6(1)].

- (2) Let $\mathcal{T}$ be a pre-triangulated category with suspension Σ . The *Frobenius-Perron dimension* of $\mathcal{T}$ is defined to be

$$\mathrm{fpd} \mathcal{T} := \mathrm{fpd}(\Sigma).$$

(3) The *fp-global dimension* of $\mathcal{T}$ is defined to be

$$\mathrm{fp}\mathrm{gldim}\, \mathcal{T} := \sup\{n \mid \mathrm{fpd}(\Sigma^n) \neq 0\}.$$

Fix an endofunctor σ of a category $\mathcal{C}$. For a set of bricks B in $\mathcal{C}$ (or a set of atomic objects when $\mathcal{C}$ is triangulated), we define

$$\mathrm{fpd}^n|_B(\sigma) = \sup\{\rho(A(\phi, \sigma)) \mid \phi := \{X_1, \dots, X_n\} \in \Phi_{n,b}, \text{ and } X_i \in B \ \forall i\}.$$

Let $\Lambda := \{\lambda\}$ be a totally ordered set. We say a set of bricks B in $\mathcal{C}$ has a σ -*decomposition* $\{B^\lambda\}_{\lambda \in \Lambda}$ (based on Λ) if the following holds.

- (1) B is a disjoint union $\bigcup_{\lambda \in \Lambda} B^\lambda$.
- (2) If $X \in B^\lambda$ and $Y \in B^\delta$ with $\lambda < \delta$, $\mathrm{Hom}_{\mathcal{C}}(X, \sigma(Y)) = 0$.

The following is [CG, Lemma 6.1].

Lemma 1.6 ([CG, Lemma 6.1]). *Let n be a positive integer. Suppose that B has a σ -decomposition $\{B^\lambda\}_{\lambda \in \Lambda}$. Then*

$$\mathrm{fpd}^n|_B(\sigma) \leq \sup_{\lambda \in \Lambda, m \leq n} \{\mathrm{fpd}^m|_{B^\lambda}(\sigma)\}.$$

2. FROBENIUS-PERRON THEORY OF WEIGHTED PROJECTIVE LINES

The main goal of this section is to recall some facts about weighted projective lines and then to prove Theorem 0.1.

2.1. Weighted projective lines. First we recall the definition and some basics about weighted projective lines. Details can be found in [GL, Section 1].

For $t \geq 1$, let $\mathbf{p} := (p_0, p_1, \dots, p_t)$ be a $(t+1)$ -tuple of positive integers, called the *weight sequence*. Let $\mathbf{D} := (\lambda_0, \lambda_1, \dots, \lambda_t)$ be a sequence of distinct points of the projective line $\mathbb{P}^1$ over $\mathbb{k}$. We normalize $\mathbf{D}$ so that $\lambda_0 = \infty$, $\lambda_1 = 0$ and $\lambda_2 = 1$ (if $t \geq 2$). Let

$$S := \mathbb{k}[X_0, X_1, \dots, X_t] / (X_i^{p_i} - X_1^{p_1} + \lambda_i X_0^{p_0}, i = 2, \dots, t).$$

The image of X_i in S is denoted by x_i for all i . Let $\mathbb{L}$ be the abelian group of rank 1 generated by $\vec{x}_i$ for $i = 0, 1, \dots, t$ and subject to the relations

$$p_0 \vec{x}_0 = \dots = p_i \vec{x}_i = \dots = p_t \vec{x}_t =: \vec{\tau}.$$

The algebra S is $\mathbb{L}$ -graded by setting $\deg x_i = \vec{x}_i$. The corresponding *weighted projective line*, denoted by $\mathbb{X}(\mathbf{p}, \mathbf{D})$ or simply $\mathbb{X}$, is a noncommutative space whose category of coherent sheaves is given by the quotient category

$$\mathrm{coh}(\mathbb{X}) := \frac{\mathrm{gr}^{\mathbb{L}} S}{\mathrm{gr}_{f.d.}^{\mathbb{L}} S}.$$

The weighted projective lines are classified into the following three classes:

$$\mathbb{X} \text{ is } \begin{cases} \textit{domestic} & \text{if } \mathbf{p} \text{ is } (p, q), (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5); \\ \textit{tubular} & \text{if } \mathbf{p} \text{ is } (2, 3, 6), (3, 3, 3), (2, 4, 4), (2, 2, 2, 2); \\ \textit{wild} & \text{otherwise.} \end{cases}$$

In [Sc, Section 4.4], domestic (respectively, tubular, wild) weighted projective lines are called *parabolic* (respectively, *elliptic*, *hyperbolic*). Let $\mathbb{X}$ be a weighted projective line. A sheaf $F \in \mathrm{coh}(\mathbb{X})$ is called *torsion* if it is of finite length in $\mathrm{coh}(\mathbb{X})$. Let $\mathrm{Tor}(\mathbb{X})$ denote the full subcategory of $\mathrm{coh}(\mathbb{X})$ consisting of all torsion objects. By

[Sc, Lemma 4.16], the category $Tor(\mathbb{X})$ decomposes as a direct product of orthogonal blocks

$$(E2.0.1) \quad Tor(\mathbb{X}) = \prod_{x \in \mathbb{P}^1 \setminus \{\lambda_0, \lambda_1, \dots, \lambda_t\}} Tor_x \times \prod_{i=0}^t Tor_{\lambda_i}$$

where Tor_x is equivalent to the category of nilpotent representations of the Jordan quiver (with one vertex and one arrow) over the residue field $\mathbb{k}_x$ and where Tor_{λ_i} is equivalent to the category of nilpotent representations over $\mathbb{k}$ of the cyclic quiver of length p_i , see Example 2.3. A simple object in $coh(\mathbb{X})$ is called *ordinary simple* (see [GL]) if it is the skyscraper sheaf $\mathcal{O}_x$ of a closed point $x \in \mathbb{P}^1 \setminus \{\lambda_0, \lambda_1, \dots, \lambda_t\}$.

Let $Vect(\mathbb{X})$ be the full subcategory of $coh(\mathbb{X})$ consisting of all vector bundles. Similar to the elliptic curve case [BB, Section 4], one can define the concepts of *degree*, *rank* and *slope* of a vector bundle on a weighted projective line $\mathbb{X}$; details are given in [Sc, Section 4.7] and [LM, Section 2]. For each $\mu \in \mathbb{Q}$, let $Vect_\mu(\mathbb{X})$ be the full subcategory of $Vect(\mathbb{X})$ consisting of all semistable vector bundles of slope μ . By convention, $Vect_\infty(\mathbb{X})$ denotes $Tor(\mathbb{X})$. By [Sc, Comments after Corollary 4.34], when $\mathbb{X}$ is a domestic or tubular weighted projective line, every indecomposable object in $coh(\mathbb{X})$ is in

$$\bigcup_{\mu \in \mathbb{Q} \cup \{\infty\}} Vect_\mu(\mathbb{X}).$$

Below we collect some nice properties of weighted projective lines. The definition of a stable tube (or simply tube) was introduced in [Ri].

Lemma 2.1 ([CG, Lemma 7.9]). *Let $\mathbb{X} = \mathbb{X}(\mathbf{p}, \mathbf{D})$ be a weighted projective line.*

- (1) *$coh(\mathbb{X})$ is noetherian and hereditary.*
- (2)

$$D^b(coh(\mathbb{X})) \cong \begin{cases} D^b(\text{Mod}_{f.d.} -\mathbb{k}\tilde{A}_{p,q}) & \text{if } \mathbf{p} = (p, q), \\ D^b(\text{Mod}_{f.d.} -\mathbb{k}\tilde{D}_n) & \text{if } \mathbf{p} = (2, 2, n), \\ D^b(\text{Mod}_{f.d.} -\mathbb{k}\tilde{E}_6) & \text{if } \mathbf{p} = (2, 3, 3), \\ D^b(\text{Mod}_{f.d.} -\mathbb{k}\tilde{E}_7) & \text{if } \mathbf{p} = (2, 3, 4), \\ D^b(\text{Mod}_{f.d.} -\mathbb{k}\tilde{E}_8) & \text{if } \mathbf{p} = (2, 3, 5). \end{cases}$$

- (3) *Let $\mathcal{S}$ be an ordinary simple object in $coh(\mathbb{X})$. Then $\text{Ext}_{\mathbb{X}}^1(\mathcal{S}, \mathcal{S}) = \mathbb{k}$.*
- (4) *$\text{fpd}^1(coh(\mathbb{X})) \geq 1$.*
- (5) *If $\mathbb{X}$ is tubular or domestic, then $\text{Ext}_{\mathbb{X}}^1(X, Y) = 0$ for all $X \in Vect_{\mu'}(\mathbb{X})$ and $Y \in Vect_\mu(\mathbb{X})$ with $\mu' < \mu$.*
- (6) *If $\mathbb{X}$ is domestic, then $\text{Ext}_{\mathbb{X}}^1(X, Y) = 0$ for all $X \in Vect_{\mu'}(\mathbb{X})$ and $Y \in Vect_\mu(\mathbb{X})$ with $\mu' \leq \mu < \infty$. As a consequence, $\text{fpd}(\Sigma|_{Vect_\mu(\mathbb{X})}) = 0$ for all $\mu < \infty$.*
- (7) *Suppose $\mathbb{X}$ is tubular or domestic. Then every indecomposable vector bundle $\mathbb{X}$ is semi-stable.*
- (8) *Suppose $\mathbb{X}$ is tubular and let $\mu \in \mathbb{Q}$. Then each $Vect_\mu(\mathbb{X})$ is a uniserial category. Accordingly indecomposables in $Vect_\mu(\mathbb{X})$ decomposes into Auslander-Reiten components, which all are stable tubes of finite rank. In fact, for every $\mu \in \mathbb{Q}$,*

$$Vect_\mu(\mathbb{X}) \cong Vect_\infty(\mathbb{X}) = Tor(\mathbb{X}).$$

Proof. (1) This is well-known, see [Le, Theorem 2.2].

(2) See [KLM, Proposition 5.1(i)] and [GL, 5.4.1].

(3) Let $\mathcal{S}$ be an ordinary simple object which is of the form $\mathcal{O}_x$ for some $x \in \mathbb{P}^1 \setminus \{\lambda_0, \dots, \lambda_t\}$. Then $\mathcal{S}$ is a brick and $\text{Ext}_{\mathbb{X}}^1(\mathcal{S}, \mathcal{S}) = \text{Ext}_{\mathbb{X}}^1(\mathcal{O}_x, \mathcal{O}_x) = \mathbb{k}$.

(4) Follows from (3) by taking $\phi := \{\mathcal{S}\}$.

(5) This is [Sc, Corollary 4.34(i)].

(6) This is [Sc, Comments after Corollary 4.34]. The consequence is clear.

(7) [GL, Theorem 5.6(i)].

(8) See [Sc, Theorem 4.42] and [GL, Theorem 5.6(iii)]. $\square$

Our main goal in this section is to prove Theorem 0.1. Eventually one should ask Question 2.2.

Question 2.2 ([CG, Question 7.11]). Let $\mathbb{X}$ be a weighted projective line. What is the exact value of $\text{fpd}^n D^b(\text{coh}(\mathbb{X}))$, for $n \geq 1$, in terms of other invariants of $\mathbb{X}$?

2.2. Standard stable tubes [LS, Ri]. In this subsection we would like to understand the (standard) stable tubes in Tor_{λ_i} in the decomposition (E2.0.1), which is the Auslander-Reiten quiver of the x -nilpotent (or x -torsion) representations of the algebra in Example 2.3.

Example 2.3. Let ξ be a primitive n th root of unity. Let T_n be the algebra

$$T_n := \frac{\mathbb{k}\langle g, x \rangle}{(g^n - 1, gx - \xi xg)}.$$

This algebra can be expressed by using a group action. Let G be the group

$$\{g \mid g^n = 1\} \cong \mathbb{Z}/(n)$$

acting on the polynomial ring $\mathbb{k}[x]$ by $g \cdot x = \xi x$. Then T_n is naturally isomorphic to the skew group ring $\mathbb{k}[x] * G$. Let $\overrightarrow{A_{n-1}}$ denote the cycle quiver with n vertices, namely, the quiver with one oriented cycle connecting n vertices. It is also known that T_n is isomorphic to the path algebra of the quiver $\overrightarrow{A_{n-1}}$.

Let $\mathfrak{A}$ be the category of finite dimensional left T_n -modules that are x -torsion. In this subsection we will show that $\text{fpd}(\mathfrak{A}) = 1$ [Corollary 2.12]. We start with somewhat more general setting.

Let A be any algebra and let $\text{Mod}_{f.d.} - A$ be the category of finite dimensional left A -modules. Let $\Gamma(\text{Mod}_{f.d.} - A)$ denote the Auslander-Reiten quiver with Auslander-Reiten translation τ .

Let $\mathfrak{C}$ be a component of $\Gamma(\text{Mod}_{f.d.} - A)$. We say $\mathfrak{C}$ is a *self-hereditary component* of $\Gamma(\text{Mod}_{f.d.} - A)$ if for each pair of indecomposable A -modules X and Y in $\mathfrak{C}$, we have $\text{Ext}_A^2(X, Y) = 0$.

We now recall some results from the book [SS]. The definitions can be found in [SS]. Let $\phi := \{E_1, \dots, E_r\}$ be a brick set in $\mathfrak{C}$. (In [SS], ϕ is called a finite family of pairwise orthogonal bricks.) The *extension category* [SS, p.13] of ϕ , denoted by $\mathcal{E}$, $\mathcal{E}_A$, or $\mathcal{EXT}_A(E_1, \dots, E_r)$, is defined to be the full subcategory of $\text{Mod}_{f.d.} - A$ whose nonzero objects are all the objects M such that there exists a chain of submodules

$$M = M_0 \supsetneq M_1 \supsetneq \dots \supsetneq M_l = 0,$$

for some $l \geq 1$, with M_i/M_{i+1} isomorphic to one of the bricks $E_1, \dots, E_r$ for all $0 \leq i < l$. We say $\{E_1, \dots, E_r\}$ is a τ -cycle if $\tau(E_i) = E_{i-1}$ for all $i \in \mathbb{Z}/(r)$.

We will be using the notation introduced in [SS]. For example, $E_i[j]$ represents some uniserial object, which is nothing to do with the j th suspension of E_i .

Theorem 2.4 ([SS, Lemma 2.1 and Theorem 2.2 in Ch. X]). *Let $\phi := \{E_1, \dots, E_r\}$, with $r \geq 1$, be a brick set on $\text{Mod}_{f.d.} - A$. Suppose that ϕ is a τ -cycle and a self-hereditary family [SS, p.14]. Then the extension category $\mathcal{E}$ is an abelian category with the following properties.*

- (1) *For each pair (i, j) , with $1 \leq i \leq r$ and $j \geq 1$, there exist a uniserial object $E_i[j]$ of $\mathcal{E}$ -length $l_{\mathcal{E}}(E_i[j]) = j$ in the category $\mathcal{E}$, and homomorphisms*

$$u_{ij} : E_i[j-1] \longrightarrow E_i[j], \quad \text{and} \quad p_{ij} : E_i[j] \longrightarrow E_{i+1}[j-1],$$

for $j \geq 2$, such that we have two short exact sequences in $\text{Mod}_{f.d.} - A$

$$0 \longrightarrow E_i[j-1] \xrightarrow{u_{ij}} E_i[j] \xrightarrow{p'_{ij}} E_{i+j-1}[1] \longrightarrow 0,$$

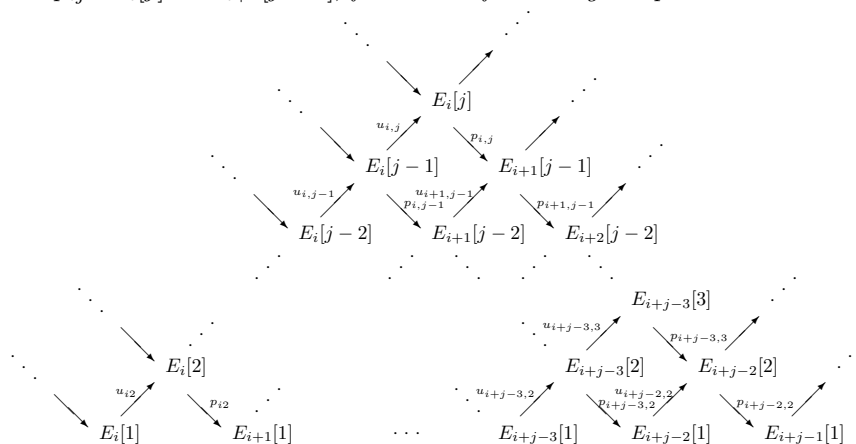
$$0 \longrightarrow E_i[1] \xrightarrow{u'_{ij}} E_i[j] \xrightarrow{p_{ij}} E_{i+1}[j-1] \longrightarrow 0,$$

where $p'_{ij} = p_{i+j-2,2} \circ \dots \circ p_{ij}$ and $u'_{ij} = u_{ij} \circ \dots \circ u_{i2}$. Moreover, for each $j \geq 2$, there exists an almost split sequence

$$0 \rightarrow E_i[j-1] \xrightarrow{\begin{bmatrix} p_{i,j-1} \\ u_{ij} \end{bmatrix}} E_{i+1}[j-2] \oplus E_i[j] \xrightarrow{(u_{i+1,j-1} \quad p_{ij})} E_{i+1}[j-1] \rightarrow 0,$$

in $\text{Mod}_{f.d.} - A$, where we set $E_i[0] = 0$ and $E_{i+kr}[m] = E_i[m]$, for $m \geq 1$ and all $k \in \mathbb{Z}$.

- (2) *The indecomposable uniserial objects $E_i[j]$, with $i \in \{1, \dots, r\}$ and $j \geq 1$, of the category $\mathcal{E}$, connected by the homomorphisms $u_{ij} : E_i[j-1] \rightarrow E_i[j]$ and $p_{ij} : E_i[j] \rightarrow E_{i+1}[j-1]$, form the infinite diagram presented below.*



- (3) $\text{Ext}_A^2(X, Y) = 0$, for each pair of objects X and Y of $\mathcal{E}$.

Theorem 2.5 ([SS, Theorem 2.6 in Ch. X]). *Retain the hypothesis as in Theorem 2.4. Then the abelian category $\mathcal{E}$ has the following properties.*

- (1) *Every indecomposable object M of the category $\mathcal{E}$ is uniserial and is of the form $M \cong E_i[j]$, where $i \in \{1, \dots, r\}$ and $j \geq 1$.*

- (2) The collection of indecomposable objects forms a self-hereditary component, denoted by $\mathfrak{T}_{\mathcal{E}}$, of $\Gamma(\text{Mod}_{f.d.} -A)$.
- (3) The component $\mathfrak{T}_{\mathcal{E}}$ is a standard stable tube of rank r [SS, Definition 1.1 in Ch. X].
- (4) The objects $E_1, \dots, E_r$ form the complete set of objects lying on the mouth [SS, Definition 1.2 in Ch. X] of the tube $\mathfrak{T}_{\mathcal{E}}$.

Let $\mathfrak{T}$ be the $\mathfrak{T}_{\mathcal{E}}$ defined as in Theorem 2.5. Let $D := \text{Hom}_{\mathbb{k}}(-, \mathbb{k})$ be the usual $\mathbb{k}$ -linear dual.

Corollary 2.6 ([SS, Corollary 2.7 in Ch. X]). *Retain the hypothesis as in Theorem 2.4.*

- (1) The only homomorphisms between two indecomposable modules in $\mathfrak{T}$ are $\mathbb{k}$ -linear combinations of compositions of the homomorphisms u_{ij}, p_{ij} , and the identity homomorphisms, and they are only subject to the relations arising from the almost split sequences in Theorem 2.4(1).
- (2) Given $i \in \{1, \dots, r\}$ and $j \geq 1$, we have
 - (2a) $\text{End}_A(E_i[j]) \cong \mathbb{k}[t]/(t^m)$, for some $m \geq 1$,
 - (2b) $\text{End}_A(E_i[j]) \cong \mathbb{k}$ if and only if $j \leq r$, and
 - (2c) $\text{Ext}_A^1(E_i[j], E_i[j]) \cong D \text{Hom}_A(E_i[j], \tau E_i[j]) = 0$ if and only if $j \leq r-1$.
- (3) If the tube $\mathfrak{T}$ is homogeneous (namely, $r = 1$), then $\text{Ext}_A^1(M, M) \neq 0$, for any indecomposable M in $\mathfrak{C}$.

Brick objects in $\mathfrak{T}$ are determined by Corollary 2.6(2b). To work out all brick sets in $\mathfrak{T}$, we need to understand the Hom between brick objects. Part (2) of Theorem 2.7 describes these Homs.

Theorem 2.7. *Let $\mathfrak{T}$ be a standard stable tube of rank r as used in Theorem 2.5 and Corollary 2.6. Keep the notation as above and assume $1 \leq i, j, i', j' \leq r$. Then the following hold.*

- (1) $\text{End}_{\mathfrak{T}}(E_i[j]) \cong \mathbb{k}$.
 - (2) $\text{Hom}_{\mathfrak{T}}(E_i[j], E_{i'}[j']) \neq 0$ if and only if (i', j') satisfies one of the following conditions:
 - (2a) $i \leq i' \leq i + j - 1$ and $i + j \leq i' + j'$,
 - (2b) $i' \leq i + j - 1 - r$ and $i + j \leq i' + j' + r$. Here, if $i + j - 1 - r < 1$, then $\{i' \mid i' \leq i + j - 1 - r\}$.
- Moreover, if $\text{Hom}_{\mathfrak{T}}(E_i[j], E_{i'}[j']) \neq 0$, then $\text{Hom}_{\mathfrak{T}}(E_i[j], E_{i'}[j']) \cong \mathbb{k}$.

Proof. (1) See Corollary 2.6(2).

(2) Since $\mathfrak{T}$ is a standard stable tube, it is a mesh category [SS, Definition 2.4 in Ch. X]. Moreover, by Corollary 2.6(a), the only homomorphisms between two indecomposable modules in $\mathfrak{T}$ are $\mathbb{k}$ -linear combinations of compositions of the homomorphisms u_{ij}, p_{ij} , and the identity homomorphisms, which subject to the relations arising from the almost split sequences in Theorem 2.4(a). By mesh relationship [SS, Definition 2.4 in Ch. X], we obtain the description (2a) and (2b) for all objects $E_{i'}[j']$ that satisfy $\text{Hom}_{\mathfrak{T}}(E_i[j], E_{i'}[j']) \neq 0$.

Moreover, if $j' < j$, $\text{Hom}_{\mathfrak{T}}(E_i[j], E_{i'}[j'])$ is generated by composition morphisms

$$u_{i',j'} \cdots u_{i',i+j-i'+1+l} \cdots u_{i',i+j-i'+2} u_{i',i+j-i'+1} p_{i'-1,i+j-i'+1} \cdots p_{i+k,j-k} \cdots p_{i+1,j-1} p_{ij},$$

where $0 \leq k \leq i' - i, 0 \leq l \leq (i' + j') - (i + j) - 1$. Here, if $i + k > r$, then the index $i + k$ means $i + k - r$.

If $j' \geq j$, $\text{Hom}_{\mathfrak{T}}(E_i[j], E_{i'}[j'])$ is generated by

$$p_{i'-1,j'+1} \cdots p_{i+l,i'+j'-i-l} \cdots p_{i+1,i'+j'-i-1} p_{i,i'+j'-i} u_{i,i'+j'-i} \cdots u_{i,j+k} \cdots u_{i,j+2} u_{i,j+1},$$

where $0 \leq k \leq (i' + j') - (i + j), 0 \leq l \leq i' - i - 1$. Here, if $i + l > r$, then the index $i + l$ means $i + l - r$. Therefore $\text{Hom}_{\mathfrak{T}}(E_i[j], E_{i'}[j']) \cong \mathbb{k}$. $\square$

Part (1) of Corollary 2.8 next is just a re-interpretation of Theorem 2.7(2).

Corollary 2.8. *Let $\mathfrak{T}$ be a standard stable tube of rank r . Keep the notation as above and put $E_i[j], 1 \leq i, j \leq r$ in order*

$$E_1[1], E_2[1], \dots, E_r[1]; E_1[2], E_2[2], \dots, E_r[2]; \dots; E_1[r], E_2[r], \dots, E_r[r],$$

and denote them by $X_1, \dots, X_n$ where $n = r^2$.

(1) *The $n \times n$ matrix*

$$(\dim \text{Hom}_{\mathfrak{T}}(X_j, X_i))_{n \times n}$$

has the following form

$$(E2.8.1) \quad \begin{pmatrix} P^0 & P^1 & P^2 & P^3 & \cdots & P^{r-1} \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=1}^2 P^i & \sum_{i=2}^3 P^i & \cdots & \sum_{i=r-2}^{r-1} P^i \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=0}^2 P^i & \sum_{i=1}^3 P^i & \cdots & \sum_{i=r-3}^{r-1} P^i \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=0}^2 P^i & \sum_{i=0}^3 P^i & \cdots & \sum_{i=r-4}^{r-1} P^i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=0}^2 P^i & \sum_{i=0}^3 P^i & \cdots & \sum_{i=0}^{r-1} P^i \end{pmatrix}$$

where

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{r \times r}$$

and where P^0 is the identity matrix $I_{r \times r}$ of order r .

(2) The $n \times n$ -matrix

$$(\dim \operatorname{Ext}_{\mathfrak{T}}^1(X_j, X_i))_{n \times n}$$

has the following form

$$(E2.8.2) \quad \begin{pmatrix} P^{r-1} & P^{r-1} & P^{r-1} & P^{r-1} & \cdots & P^{r-1} \\ P^{r-2} & \sum_{i=r-2}^{r-1} P^i & \sum_{i=r-2}^{r-1} P^i & \sum_{i=r-2}^{r-1} P^i & \cdots & \sum_{i=r-2}^{r-1} P^i \\ P^{r-3} & \sum_{i=r-3}^{r-2} P^i & \sum_{i=r-3}^{r-1} P^i & \sum_{i=r-3}^{r-1} P^i & \cdots & \sum_{i=r-3}^{r-1} P^i \\ P^{r-4} & \sum_{i=r-4}^{r-3} P^i & \sum_{i=r-4}^{r-2} P^i & \sum_{i=r-4}^{r-1} P^i & \cdots & \sum_{i=r-4}^{r-1} P^i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=0}^2 P^i & \sum_{i=0}^3 P^i & \cdots & \sum_{i=0}^{r-1} P^i \end{pmatrix}.$$

Proof. (1) This follows from Theorem 2.7(2).

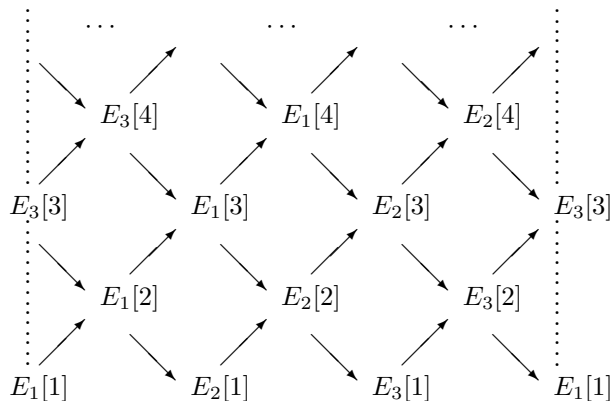
(2) The assertion follows from part (1) and the Serre duality

$$\operatorname{Ext}_{\mathfrak{T}}^1(E_i[j], E_{i'}[j']) \cong D \operatorname{Hom}_{\mathfrak{T}}(E_{i'}[j'], \tau E_i[j]) = D \operatorname{Hom}_{\mathfrak{T}}(E_{i'}[j'], E_{i-1}[j]).$$

Some detailed matching of entries is omitted. □

We use Example 2.9 to illustrate the results in Corollary 2.8.

Example 2.9. Let $\mathfrak{T}$ be a standard stable tube of rank 3:



Put $E_i[j]$, $1 \leq i, j \leq 3$ in order

$$E_1[1], E_2[1], E_3[1]; E_1[2], E_2[2], E_3[2]; E_1[3], E_2[3], E_3[3].$$

Denote this list as $X_1, \dots, X_9$. Then we have

TABLE 1. $\text{Hom}_{\mathfrak{T}}(X_j, X_i)$

$\text{Hom}_{\mathfrak{T}}(X_j, X_i)$	$X_j = E_1[1]$	$E_2[1]$	$E_3[1]$	$E_1[2]$	$E_2[2]$	$E_3[2]$	$E_1[3]$	$E_2[3]$	$E_3[3]$
$X_i = E_1[1]$	$\mathbb{k}$	0	0	0	0	$\mathbb{k}$	0	$\mathbb{k}$	0
$E_2[1]$	0	$\mathbb{k}$	0	$\mathbb{k}$	0	0	0	0	$\mathbb{k}$
$E_3[1]$	0	0	$\mathbb{k}$	0	$\mathbb{k}$	0	$\mathbb{k}$	0	0
$E_1[2]$	$\mathbb{k}$	0	0	$\mathbb{k}$	0	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$
$E_2[2]$	0	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	0	$\mathbb{k}$	0	$\mathbb{k}$
$E_3[2]$	0	0	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$	0
$E_1[3]$	$\mathbb{k}$	0	0	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$
$E_2[3]$	0	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$
$E_3[3]$	0	0	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$

TABLE 2. $\text{Ext}_{\mathfrak{T}}^1(X_j, X_i)$

$\text{Ext}_{\mathfrak{T}}^1(X_j, X_i)$	$X_j = E_1[1]$	$E_2[1]$	$E_3[1]$	$E_1[2]$	$E_2[2]$	$E_3[2]$	$E_1[3]$	$E_2[3]$	$E_3[3]$
$X_i = E_1[1]$	0	$\mathbb{k}$	0	0	$\mathbb{k}$	0	0	$\mathbb{k}$	0
$E_2[1]$	0	0	$\mathbb{k}$	0	0	$\mathbb{k}$	0	0	$\mathbb{k}$
$E_3[1]$	$\mathbb{k}$	0	0	$\mathbb{k}$	0	0	$\mathbb{k}$	0	0
$E_1[2]$	0	0	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$
$E_2[2]$	$\mathbb{k}$	0	0	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	0	$\mathbb{k}$
$E_3[2]$	0	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	0
$E_1[3]$	$\mathbb{k}$	0	0	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$
$E_2[3]$	0	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$
$E_3[3]$	0	0	$\mathbb{k}$	0	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$	$\mathbb{k}$

The corresponding Hom-dimension and Ext^1 -dimension matrices are

$$(\dim \text{Hom}_{\mathfrak{T}}(X_j, X_i))_{9 \times 9} = \begin{pmatrix} P^0 & P^1 & P^2 \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=1}^2 P^i \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=0}^2 P^i \end{pmatrix}$$

and

$$(\dim \text{Ext}_{\mathfrak{T}}^1(X_j, X_i))_{9 \times 9} = \begin{pmatrix} P^2 & P^2 & P^2 \\ P^1 & \sum_{i=1}^2 P^i & \sum_{i=1}^2 P^i \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=0}^2 P^i \end{pmatrix}$$

where

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Lemma 2.10 shows that if $H(i_1, \dots, i_s)$ is a principal submatrix of (E2.8.1) such that $H(i_1, \dots, i_s) = I_{s \times s}$, then the corresponding principal submatrix of (E2.8.2) (with the same rows and columns) has spectral radius at most 1.

Lemma 2.10 (Lemma 6.4). *Retain the above notation. Suppose the Hom-matrix is given as in (E2.8.1) and Ext^1 -matrix is given as in (E2.8.2), then $\rho(A(\phi)) \leq 1$ for every brick set ϕ .*

The proof of Lemma 2.10 is given in Appendix 6. Note that (E2.8.1)–(E2.8.2) are in fact the transpose of the usual Hom and Ext-matrices. By [CG, Lemma 3.7] (by considering the opposite category), Lemma 2.10 holds for the transpose matrices of (E2.8.1)–(E2.8.2) too.

Theorem 2.11. *Let $\mathcal{E}$ and $\mathfrak{T}$ be as in Theorem 2.5. Then $\text{fpd } \mathcal{E} = 1$.*

Proof. By Corollary 2.6(2b), all brick objects in $\mathcal{E}$ are $E_i[j]$ for all $i \in \mathbb{Z}/(r)$ and $1 \leq j \leq r$. We can determine all brick sets by using the matrix in Corollary 2.8(1). For each brick set, its Ext^1 -matrix M was determined by using (E2.8.2). By Lemma 2.10, $\rho(M) \leq 1$. On the other hand, let $\phi = \{E_1[1], \dots, E_r[1]\}$, then $M = P^{r-1}$ and hence $\text{fpd } \mathcal{E} = \rho(A(\phi, \text{Ext}^1)) = 1$. Therefore the assertion follows. $\square$

We have an immediate consequence. Let T_r be the algebra in Example 2.3.

Corollary 2.12. *Let $\mathfrak{A}$ be the category of finite dimensional left x -nilpotent T_r -modules. Then $\text{fpd } \mathfrak{A} = 1$.*

Proof. It is well-known that $\mathfrak{A}$ is equivalent to the category $\mathcal{E}$ of rank r . To see this we set $\deg x = 1$ and $\deg g = 0$. Then the degree zero component of T_r is isomorphic to $\mathbb{k}^{\oplus r}$ with primitive idempotents $\{e_1, \dots, e_r\}$. Under this setting, $E_i[j]$ is identify with $(T_r/(x^i))e_j$ for all i, j . Now the result follows from Theorem 2.11. $\square$

2.3. Proof of Theorem 0.1. Now we are ready to show Theorem 0.1.

Theorem 2.13. *Let $\mathbb{X}$ be a domestic or tubular weighted projective line. Then $\text{fpd } D^b(\text{coh}(\mathbb{X})) = 1$.*

Proof. By Lemma 2.1(4), it suffices to show that $\text{fpd}(D^b(\text{coh}(\mathbb{X}))) \leq 1$. By [CG, Theorem 3.5(4)], it is enough to show that

$$\text{fpd}(\sigma) = \text{fpd}(\text{coh}(\mathbb{X})) \leq 1$$

where σ is $\text{Ext}_{\mathbb{X}}^1(-, -)$.

By [Sc, Corollary 4.34(iii)], every brick (or indecomposable) object is semistable. By Lemma 2.1(5), the class $\{Vect_{\mu}(\mathbb{X})\}_{\mu \in \mathbb{Q} \cup \{\infty\}}$ is a σ -decomposition (see the definition before Lemma 1.6). By Lemma 1.6, it is enough to show the claim that $\text{fpd } |_{Vect_{\mu}(\mathbb{X})}(\sigma) \leq 1$ for every μ .

Case 1. $\mathbb{X}$ is domestic. If μ is finite, then $\text{fpd } |_{Vect_{\mu}(\mathbb{X})}(\sigma) = 0$ by Lemma 2.1(6). If $\mu = \infty$, then, by (E2.0.1), $Vect_{\infty}(\mathbb{X}) := \text{Tor}(\mathbb{X})$ has a decomposition into Auslander-Reiten components, which all are tubes of finite rank. By Theorem 2.11, $\text{fpd } |_{\text{Tor}(\mathbb{X})}(\sigma) = 1$. The claim follows.

Case 2. $\mathbb{X}$ is tubular. By Lemma 2.1(8),

$$Vect_{\mu}(\mathbb{X}) \cong Vect_{\infty}(\mathbb{X}) = \text{Tor}(\mathbb{X})$$

for all μ . Then the proof of Case 1 applies. Therefore the claim follows. $\square$

As usual we use $\mathcal{O}_{\mathbb{X}}$ for the structure sheaf of $\mathbb{X}$ [GL, Sect. 1.5].

Proposition 2.14. *Let $\mathbb{X}$ be a weighted projective line of wild type. Then*

$$\mathrm{fpd} \, D^b(\mathrm{coh}(\mathbb{X})) \geq \dim \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}(\vec{\omega}))$$

where $\vec{\omega}$ is the dualizing element [GL, Sec. 1.2].

Proof. Let $\phi = \{\mathcal{O}_{\mathbb{X}}\}$ which is a brick and atomic set. Then, by definition and Serre duality [GL, Theorem 2.2],

$$\begin{aligned} \mathrm{fpd} \, D^b(\mathrm{coh}(\mathbb{X})) &\geq \rho(A(\phi, \mathrm{Ext}^1)) = \dim \mathrm{Ext}_{\mathbb{X}}^1(\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}) \\ &= \dim \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}(\vec{\omega})). \end{aligned}$$

□

3. DIMENSION THEORY FOR CLASSICAL PROJECTIVE SCHEMES

The aim of this section is to introduce Frobenius-Perron versions of two important and related invariants – Calabi-Yau dimension and Kodaira dimension of $D^b(\mathrm{coh}(\mathbb{X}))$ where $\mathbb{X}$ is a smooth projective scheme.

3.1. A result from [CG]. Let $\mathbb{X}$ be a smooth (irreducible) projective scheme over $\mathbb{C}$ of positive dimension. By [CG, Proposition 6.5 and 6.7],

$$\mathrm{fpd}(D^b(\mathrm{coh}(\mathbb{X}))) = \begin{cases} 1 & \text{if } \mathbb{X} \text{ is } \mathbb{P}^1 \text{ or an elliptic curve,} \\ \infty & \text{otherwise.} \end{cases}$$

3.2. Calabi-Yau dimension. Recall from [Ke1, Section 8.1] that if a Hom-finite category $\mathcal{C}$ has a Serre functor S , then there is a natural isomorphism

$$D \mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(Y, S(X))$$

for all $X, Y \in \mathcal{C}$. A (pre-)triangulated Hom-finite category $\mathcal{C}$ with Serre functor S is called *n-Calabi-Yau* if there is a natural isomorphism

$$S \cong \Sigma^n =: [n]$$

where Σ is the suspension of $\mathcal{C}$. In this case n is called the Calabi-Yau dimension of $\mathcal{C}$. (In [Ke2, Section 2.6] it is called *weakly n-Calabi-Yau*.) More generally, $\mathcal{C}$ is called a *fractional Calabi-Yau category* if there is an $m > 0$ and there is a natural isomorphism

$$S^m \cong \Sigma^n = [n]$$

for some n , see [vR, p.2708] and [Ku, Definition 1.2]. In this case we say $\mathcal{C}$ has Calabi-Yau dimension $\frac{n}{m}$. Abelian hereditary fractionally Calabi-Yau categories are classified in [vR]. One key property of Calabi-Yau varieties is that the canonical bundle of these varieties are trivial. However, our definition of fp Calabi-Yau dimension (see Definition 3.1 below) applies to projective schemes that do not have the trivial canonical bundle.

If a Serre functor exists, then it is unique up to isomorphism. In this case we usually use S for the Serre functor. Throughout the rest of this section, let $\mathcal{T}$ be a Hom-finite (pre-)triangulated category with Serre functor S . We will define a version of (fractional) Calabi-Yau dimension for $\mathcal{T}$ which is not necessarily a (fractional) Calabi-Yau category.

Recall from Definition 1.4(3) that the Frobenius-Perron growth of a functor σ is defined to be

$$\mathrm{fpg}(\sigma) := \sup_{\phi \in \Phi_a} \{ \limsup_{n \rightarrow \infty} \log_n(\rho(A(\phi, \sigma^n))) \}.$$

By convention, $\log_n 0 = -\infty$. Similarly, we define a slightly modified version of fpg as follows, which is used to define the fp Calabi-Yau dimension.

Definition 3.1. Let σ be an endofunctor of $\mathcal{T}$ with Serre functor S .

- (1) The *lower Frobenius-Perron growth* of σ is defined to be

$$\underline{\text{fpg}}(\sigma) := \sup_{\phi \in \Phi_a} \left\{ \liminf_{n \rightarrow \infty} \log_n(\rho(A(\phi, \sigma^n))) \right\}.$$

By convention, $\log_n 0 = -\infty$.

- (2) The *spectrum* of $\mathcal{T}$ is defined to be

$$Sp(\mathcal{T}) := \{(m, n) \in \mathbb{Z}^{\times 2} \mid \underline{\text{fpg}}(S^m \circ \Sigma^{-n}) > -\infty\}.$$

- (3) The *fp Calabi-Yau dimension* of $\mathcal{T}$ is defined to be

$$\text{fpcy}(\mathcal{T}) := \lim_{M \rightarrow \infty} \left\{ \sup_{|m| \geq M} \left\{ \frac{n}{m} \mid (m, n) \in Sp(\mathcal{T}) \right\} \right\}.$$

Next we show that the fp Calabi-Yau dimension exists for various cases.

Lemma 3.2. Suppose $\mathcal{T}$ is a pre-triangulated category satisfying the following conditions:

- (a) $\mathcal{T}$ is Ext-finite [BV, Definition 2.1], namely, for all objects $X, Y \in \mathcal{T}$,

$$\sum_{s \in \mathbb{Z}} \dim \text{Hom}_{\mathcal{T}}(X, \Sigma^s(Y)) < \infty.$$

- (b) $\mathcal{T}$ has a Serre functor S .

- (c) $\mathcal{T}$ is fractional Calabi-Yau of dimension $d = a/b \in \mathbb{Q}$.

Then the following holds.

- (1) $Sp(\mathcal{T}) \subseteq (b, a)\mathbb{Q}$.
- (2) If $\mathcal{T}$ contains at least one atomic object, there exists $w \in \mathbb{N}$ such that $(bw, awt) \in Sp(\mathcal{T})$ for all $t \in \mathbb{Z}$.
- (3) Under the hypothesis of part (2), we have $\text{fpcy}(\mathcal{T}) = d$.

Proof. (1) Let (m, n) be a pair of integers that is not in $(b, a)\mathbb{Q}$, and let $\sigma = S^m \circ \Sigma^{-n}$. Since $\mathcal{T}$ is fractional Calabi-Yau of dimension a/b , we can assume that $S^b \circ \Sigma^{-a}$ is the identity functor. Then, for each t , $\sigma^{tb} = \Sigma^{t(ma-nb)}$ where $ma-nb \neq 0$. By hypothesis (a), for any two objects X and Y in $\mathcal{T}$, $\text{Hom}_{\mathcal{T}}(X, \Sigma^{t(ma-nb)}Y) = 0$ for $|t| \gg 0$. Then, for every atomic set ϕ , it implies that $A(\phi, \sigma^{tb}) = 0$ for $|t| \gg 0$. Thus $\log_{tb}(\rho(A(\phi, \sigma^{tb}))) = -\infty$ for all $|t| \gg 0$. This implies, by definition, that $\underline{\text{fpg}}(\sigma) = -\infty$, or $(m, n) \notin Sp(\mathcal{T})$. The assertion follows.

(2) Let ϕ be the set of a single atomic object X in $\mathcal{T}$. Replacing (b, a) by (bw, aw) for some positive integer w if necessary, we can assume that $\sigma := S^b \circ \Sigma^{-a}$ is the identity functor. Then $A(\phi, \sigma^n)$ is the 1×1 -identity matrix I_1 for all n . Then $\log_n(\rho(A(\phi, \sigma^n))) = 0$ for all n . This implies that $\underline{\text{fpg}}(\sigma) = 0 > -\infty$ or $(b, a) \in S(\mathcal{T})$. Similarly, one sees that $(bt, at) \in S(\mathcal{T})$ for all integer t . The assertion follows.

- (3) This follows from the definition and parts (1,2). □

The next proposition is part (2) of Theorem 0.2, which shows that fpcy is indeed a generalization of Calabi-Yau dimension.

Proposition 3.3. *Let $\mathbb{X}$ be a smooth irreducible projective variety of dimension $d \in \mathbb{N}$ and let $\mathcal{T}$ be $D^b(\text{coh}(\mathbb{X}))$. Then $\text{fpcy}(\mathcal{T}) = d$.*

Proof. When $d = 0$, then $\mathcal{T} = D_{f,d}^b(\text{Vect}_k)$. It is easy to see that the Serre functor S is the identity. So the assertion is easily shown. Now we assume that $d > 0$.

It suffices to show that $Sp(\mathcal{T}) = \{(t, td) \mid t \in \mathbb{Z}\}$.

By [BO, (7)], the Serre functor S is equal to $-\otimes_{\mathbb{X}} \omega_{\mathbb{X}}[d]$ where d is the dimension of $\mathbb{X}$ and $\omega_{\mathbb{X}}$ is the canonical bundle of $\mathbb{X}$. Let $\sigma = S \circ \Sigma^{-d}$. Then σ is the functor $-\otimes_{\mathbb{X}} \omega_{\mathbb{X}}$. Let $\mathcal{O}_a$ be the skyscraper sheaf of a closed point $a \in \mathbb{X}$. Then it is an atomic object and $\sigma(\mathcal{O}_a) \cong \mathcal{O}_a$. Let $\phi = \{\mathcal{O}_a\}$. Then $A(\phi, \sigma^n)$ is the 1×1 -identity matrix I_1 . This implies that $\log_n(\rho(A(\phi, \sigma^n))) = 0$ and that $\underline{\text{fpg}}(\sigma) = 0 > -\infty$. Therefore $(1, d) \in S(\mathcal{T})$. Similarly, one sees that $(t, dt) \in S(\mathcal{T})$ for all $t \geq 0$.

For the other implication, let $(m, n) \in \mathbb{Z}^{\times 2} \setminus \{(t, td) \mid t \in \mathbb{Z}\}$. Let $\sigma = S^m \circ \Sigma^{-n}$. We need to show that $\underline{\text{fpg}}(\sigma) = -\infty$. Note that $\sigma = -\otimes_{\mathbb{X}} \omega_{\mathbb{X}}^{\otimes m} \circ \Sigma^{-n+dm}$ where $-n + dm \neq 0$. Since $\text{coh}(\mathbb{X})$ has global dimension d , for all objects A and B in $\mathcal{T}$,

$$\text{Hom}_{\mathcal{T}}(A, \sigma^t(B)) = \text{Hom}_{\mathcal{T}}(A, (B \otimes_{\mathbb{X}} \omega_{\mathbb{X}}^{\otimes tm})[t(-n + dm)]) = 0$$

for all $t \gg 0$. This implies that $\underline{\text{fpg}}(\sigma) = -\infty$ as required. $\square$

3.3. Kodaira dimension. First we review the classical definition of the Kodaira dimension.

Definition 3.4 ([La, Definition 2.1.3 and Example 2.1.5]). Let $\mathbb{X}$ be a smooth projective variety and let $\omega_{\mathbb{X}}$ be the canonical bundle of $\mathbb{X}$.

- (1) The *Kodaira dimension* of $\mathbb{X}$ is defined to be

$$\kappa(\mathbb{X}) := \lim_{n \rightarrow \infty} \log_n (\dim H^0(\mathbb{X}, \omega_{\mathbb{X}}^{\otimes n})).$$

- (2) More generally, for a line bundle $\mathcal{M}$, the *Kodaira-Iitaka dimension* of $\mathcal{M}$ is defined to be

$$\kappa(\mathbb{X}, \mathcal{M}) := \lim_{n \rightarrow \infty} \log_n (\dim H^0(\mathbb{X}, \mathcal{M}^{\otimes n})).$$

- (3) The *anti-Kodaira dimension* of $\mathbb{X}$ is defined to be

$$\kappa^{-1}(\mathbb{X}) := \kappa(\mathbb{X}, \omega_{\mathbb{X}}^{-1}).$$

The anti-Kodaira dimension of a scheme was defined in [Sa]. It is classical and well-known that $\kappa(\mathbb{X}), \kappa(\mathbb{X}, \mathcal{M}) \in \{-\infty, 0, 1, \dots, \dim \mathbb{X}\}$ and that there are $0 < c_1 < c_2$ such that,

$$(E3.4.1) \quad c_1 n^{\kappa(\mathbb{X}, \mathcal{M})} \leq \dim \text{Hom}_{\mathbb{X}}(\mathcal{O}_{\mathbb{X}}, \mathcal{M}^{\otimes n}) \leq c_2 n^{\kappa(\mathbb{X}, \mathcal{M})} \quad \forall n \gg 0.$$

See [La, Corollary 2.1.37]. By Proposition 3.3, $\dim \mathbb{X} = \text{fpcy}(\mathcal{T})$, which suggests Definition 3.5.

In the following we use the order $t_1 < t_2$ if t_1 divides t_2 . Since we are mainly interested in commutative and noncommutative projective schemes, $\mathcal{T}$ is equipped with the structure sheaf, denoted by $\mathcal{O}$.

Definition 3.5. Let $\mathcal{T}$ be a pre-triangulated category and $\mathcal{T}_*$ denote the pair $(\mathcal{T}, \mathcal{O})$ where $\mathcal{O}$ is a given special object in $\mathcal{T}$. Suppose that $\mathcal{T}$ is Hom-finite with Serre functor S and that $\mathrm{fpcy}(\mathcal{T}) = d = a/b$ for some integers a, b .

- (1) The *fp Kodaira dimension* of $\mathcal{T}_*$ is defined to be

$$\mathrm{fp}\kappa(\mathcal{T}_*) := \lim_t \left\{ \limsup_{n \rightarrow \infty} \dim \mathrm{Hom}_{\mathcal{T}}(\mathcal{O}, (S^{bnt} \circ \Sigma^{-ant})(\mathcal{O})) \right\}$$

where the first limit ranges over all positive integers t with order $<$ as defined before the definition.

- (2) The *fp anti-Kodaira dimension* of $\mathcal{T}_*$ is defined to be

$$\mathrm{fp}\kappa^{-1}(\mathcal{T}_*) := \lim_t \left\{ \limsup_{n \rightarrow \infty} \dim \mathrm{Hom}_{\mathcal{T}}(\mathcal{O}, (S^{-bnt} \circ \Sigma^{ant})(\mathcal{O})) \right\}$$

where the first limit ranges over all positive integers t with order $<$ as defined before the definition.

Proposition 3.6 justifies the above definition.

Proposition 3.6. Let $\mathbb{X}$ be a smooth irreducible projective variety of dimension $d \in \mathbb{N}$ and let $\mathcal{T}$ be $D^b(\mathrm{coh}(\mathbb{X}))$ with structure sheaf $\mathcal{O} := \mathcal{O}_{\mathbb{X}}$. Then

$$\mathrm{fp}\kappa(\mathcal{T}_*) = \kappa(\mathbb{X}) \quad \text{and} \quad \mathrm{fp}\kappa^{-1}(\mathcal{T}_*) = \kappa^{-1}(\mathbb{X}).$$

Proof. By Proposition 3.3, $\mathrm{fpcy}(\mathcal{T}) = d = \dim \mathbb{X}$. So we take $b = 1$ and $a = d$ in Definition 3.5. By (E3.4.1), for each $t \geq 1$,

$$\limsup_{n \rightarrow \infty} \dim \mathrm{Hom}_{\mathcal{T}}(\mathcal{O}, (S^{bnt} \circ \Sigma^{-ant})(\mathcal{O})) = \kappa(\mathbb{X}).$$

The first assertion follows by the definition.

The proof for anti-Kodaira dimension is similar. □

Theorem 0.2 follows from Propositions 3.3 and 3.6.

Remark 3.7. Assume the hypothesis of Lemma 3.2. Then one can check easily that

$$(E3.7.1) \quad \mathrm{fp}\kappa(\mathcal{T}_*) = \mathrm{fp}\kappa^{-1}(\mathcal{T}_*) = 0.$$

So abstractly (E3.7.1) should be part of the definition of a fractional Calabi-Yau variety (even in the noncommutative setting).

Proposition 3.8. If $\mathcal{T}$ is a triangulated category such that either $\mathrm{fp}\kappa(\mathcal{T}_*) = \infty$ or $\mathrm{fp}\kappa^{-1}(\mathcal{T}_*) = \infty$ or $\mathrm{fpcy}(\mathcal{T}) = \infty$ or $-\infty$, then $\mathcal{T}_*$ is not triangulated equivalent to the bounded derived category of a smooth projective scheme.

Proof. By definition, fpcy is an invariant of a triangulated category, and $\mathrm{fp}\kappa$ is an invariant of a triangulated category with $\mathcal{O}$. The assertion follows from Propositions 3.3 and 3.6. □

4. INVARIANTS OF NONCOMMUTATIVE PROJECTIVE SCHEMES

In this section we study fp Calabi-Yau dimension and fp Kodaira dimension of noncommutative projective schemes in the sense of [AZ]. An algebra A is said to be *connected graded* over $\mathbb{k}$ if $A = \mathbb{k} \oplus A_1 \oplus A_2 \oplus \cdots$ with $A_i A_j \subseteq A_{i+j}$ for all $i, j \in \mathbb{N}$. Let A be a noetherian connected graded algebra. The *noncommutative projective scheme* associated to A is denoted by $\mathbb{X} := \mathrm{Proj} A$, see [AZ] for the detailed definition of a noncommutative projective scheme. Let $\mathrm{coh}(\mathbb{X})$ be the

category of noetherian objects in $\text{Proj } A$ and let $\mathcal{T}$ be the triangulated category $D^b(\text{coh}(\mathbb{X}))$. Here is a restatement of a nice result of Bondal-Van den Bergh [BV, Theorem 4.2.13].

Theorem 4.1 ([BV, Theorem 4.2.13]). *Suppose A is noetherian and has a balanced dualizing complex and that $\text{Proj } A$ has finite homological dimension. Then $\mathcal{T}$ has a Serre functor.*

One special class of connected graded algebras are the Artin-Schelter regular algebras (see the next definition).

Definition 4.2 ([AS]). Let A be a connected graded algebra over the base field $\mathbb{k}$. We say A is *Artin-Schelter Gorenstein* (or *AS Gorenstein*) if the following conditions hold:

- (a) A has finite injective dimension d on both sides,
- (b) $\text{Ext}_A^i(\mathbb{k}, A) = \text{Ext}_{A^{op}}^i(\mathbb{k}, A) = 0$ for all $i \neq d$ where $\mathbb{k} = A/A_{\geq 1}$, and
- (c) $\text{Ext}_A^d(\mathbb{k}, A) \cong \mathbb{k}(\ell)$ and $\text{Ext}_{A^{op}}^d(\mathbb{k}, A) \cong \mathbb{k}(\ell)$ for some integer ℓ . This integer ℓ is called the *AS index* of A .

If moreover

- (d) A has finite global dimension,

then A is called *Artin-Schelter regular* (or *AS regular*).

We collect some well-known facts below. Let $\pi : \text{Gr } A \rightarrow \text{Proj } A$ be the canonical quotient functor. By abuse of notation, we also apply π to some graded A -bimodules such as ${}^\mu A^1$ in Lemma 4.3.

Lemma 4.3. *Let A be a noetherian connected graded algebra, let $\mathbb{X}$ be $\text{Proj } A$, and let $\mathcal{T}$ be $D^b(\text{coh}(\mathbb{X}))$.*

- (1) [Ye, Corollary 4.14] *If A is Artin-Schelter Gorenstein, then A has a balanced dualizing complex.*
- (2) [YZ, Corollary 4.3] *Suppose A is Artin-Schelter Gorenstein such that $\mathbb{X}$ has finite homological dimension. Let $d = \text{injdim } A$, ℓ be the AS index of A and μ be the Nakayama automorphism of A . Then the Serre functor of $\mathcal{T}$ is $-\otimes_{\mathcal{O}} \pi({}^\mu A^1)(-\ell)[d-1]$.*

Let M be a locally finite $\mathbb{Z}$ -graded module or vector space. The Hilbert series of M is defined to be

$$H_M(t) := \sum_{n \in \mathbb{Z}} \dim M_n t^n.$$

Let A be a graded algebra and s be a positive integer. The *sth Veronese subalgebra* of A is defined to be

$$A^{(s)} := \bigoplus_{n \in \mathbb{Z}} A_{sn}.$$

Lemma 4.4 is well-known.

Lemma 4.4. *Let A be a noetherian connected graded algebra generated in degree 1. Let s be a positive integer.*

- (1) *A is a finitely generated module over $A^{(s)}$ on both sides.*
- (2) *$\text{GKdim } A = \text{GKdim } A^{(s)}$.*
- (3) *If the Hilbert series of A is a rational function, then so is the Hilbert series of $A^{(s)}$.*

(4) If the Hilbert series of A is a rational function, then

$$\limsup_{n \rightarrow \infty} \log_n(\dim A_n) = \mathrm{GKdim} A - 1.$$

(5) If A is an Artin-Schelter regular algebra, then the Hilbert series of A is a rational function.

Proof. (1) Connected graded noetherian algebras are finitely generated. So A is generated by $\bigoplus_{i=0}^{s-1} A_i$ over $A^{(s)}$ on both sides.

(2) It follows from part (1) and [MR, Proposition 8.2.9(i)].

(3) This follows from the fact that

$$H_A(t) = \frac{1}{s} \sum_{i=0}^{s-1} H_A(\xi^i t)$$

where ξ is an s th primitive root of unity.

(4) When $H_A(t)$ is a rational function, $\dim A_n$ is a multi-polynomial function of n in the sense of [Zh1, p.399]. Say d is the degree of the multi-polynomial function $\dim A_n$ of n . Then $\mathrm{GKdim} A = d + 1$ by using [Zh1, (E7)]. This implies that

$$\limsup_{n \rightarrow \infty} \log_n(\dim A_n) = d = \mathrm{GKdim} A - 1.$$

(5) This is [StZ, Proposition 3.1]. □

Theorem 4.5. *Let A be a noetherian connected graded Artin-Schelter Gorenstein algebra of injective dimension $d \geq 2$ that is generated in degree 1. Suppose that $\mathbb{X} := \mathrm{Proj} A$ has finite homological dimension. Let $\mathcal{T}$ be the bounded derived category of $\mathrm{coh}(\mathbb{X})$. In parts (2,3,4,5,6), we further assume that the Hilbert series of A is rational. Let ℓ be the AS index of A .*

(1) $\mathrm{fpcy}(\mathcal{T}) = d - 1$.

(2) If $\ell > 0$, then $\mathrm{fp}\kappa(\mathcal{T}_*) = -\infty$ and $\mathrm{fp}\kappa^{-1}(\mathcal{T}_*) = \mathrm{GKdim} A - 1$.

(3) If $\ell < 0$, then $\mathrm{fp}\kappa(\mathcal{T}_*) = \mathrm{GKdim} A - 1$ and $\mathrm{fp}\kappa^{-1}(\mathcal{T}_*) = -\infty$.

(4) If $\ell = 0$, then $\mathrm{fp}\kappa(\mathcal{T}_*) = \mathrm{fp}\kappa^{-1}(\mathcal{T}_*) = 0$.

(5) For all objects $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{T}$,

$$\limsup_{n \rightarrow \infty} \log_n(\dim \mathrm{Hom}_{\mathcal{T}}(\mathcal{C}, (S \circ \Sigma^{-d})^n(\mathcal{D}))) \leq \mathrm{GKdim} A - 1.$$

(6) For all objects $\mathcal{C}$ and $\mathcal{D}$ in $\mathcal{T}$,

$$\limsup_{n \rightarrow \infty} \log_n(\dim \mathrm{Hom}_{\mathcal{T}}(\mathcal{C}, (S \circ \Sigma^{-d})^{-n}(\mathcal{D}))) \leq \mathrm{GKdim} A - 1.$$

Proof. (1) Let ℓ be the AS index of A . There are two different cases: $\ell \leq 0$ and $\ell \geq 0$. The proofs are similar, so we only prove the assertion for the first case.

First we claim that $Sp(\mathcal{T}) \subseteq (1, d - 1)\mathbb{Z}$. Suppose $(m, n) \notin (1, d - 1)\mathbb{Z}$. Let $\sigma = S^m \circ \Sigma^{-n} = - \otimes \mathcal{M}^{\otimes m}[(d - 1)m - n]$ where $\mathcal{M} = \pi(\mu A^1)(-\ell)$ and where $(d - 1)m - n \neq 0$. Since $\mathrm{Proj} A$ has finite global dimension, we have that, for all objects A and B in $\mathcal{T}$,

$$\mathrm{Hom}_{\mathcal{T}}(A, \sigma^t(B)) = \mathrm{Hom}_{\mathcal{T}}(A, (B \otimes \mathcal{M}^{\otimes tm})[t((d - 1)m - n)]) = 0$$

for all $t \gg 0$. This implies that $\underline{\mathrm{fpg}}(\sigma) = -\infty$. Hence $(m, n) \notin Sp(\mathcal{T})$ and hence we have proven the claim.

Second we claim that $(1, d-1)\mathbb{N} \subseteq Sp(\mathcal{T})$. Let $\mathcal{O} = \pi(A)$. It is a brick object as $\text{Hom}_{\mathcal{T}}(\mathcal{O}, \mathcal{O}) = A_0 = \mathbb{k}$. For every $(m, n) = (s, s(d-1)) \in (1, d-1)\mathbb{N}$, let $\sigma = S^m \circ \Sigma^{-n} = - \otimes \mathcal{M}^{\otimes s}$. Then

$$\sigma(\mathcal{O}) \cong \mathcal{O}(-\ell s)$$

and

$$\text{Hom}_{\mathcal{T}}(\mathcal{O}, \sigma(\mathcal{O})) \cong A_{-\ell s} \neq 0$$

when $\ell \leq 0$. This implies that $\rho(A(\phi, \sigma)) \geq 1$ where $\phi = \{\mathcal{O}\}$. Similarly, $\rho(A(\phi, \sigma^n)) \geq 1$ for all n . Consequently, $\underline{\text{fpg}}(\sigma) \geq 0 > -\infty$. Therefore $(m, n) \in Sp(\mathcal{T})$ as desired. Now we have

$$(1, d-1)\mathbb{N} \subseteq Sp(\mathcal{T}) \subseteq (1, d-1)\mathbb{Z}$$

which implies that $\text{fpcy}(\mathcal{T}) = d-1$.

(2) Let $w = \text{GKdim } A$ which is at least 2. By Lemma 4.4(2,3,4), for every integer $s \geq 1$,

$$(E4.5.1) \quad \limsup_{n \rightarrow \infty} \log_n(\dim A_{sn}) = \text{GKdim } A - 1 = w - 1.$$

Now assume that ℓ is positive. By Lemma 4.3(2), $\sigma := S \circ \Sigma^{-(d-1)}$ is equivalent to $- \otimes \mathcal{O}(-\ell)$ when applied to $\mathcal{O}$. Thus

$$(E4.5.2) \quad \text{Hom}_{\mathcal{T}}(\mathcal{O}, \sigma^n(\mathcal{O})) = A_{-\ell n} = 0$$

when $n \geq 0$ and

$$(E4.5.3) \quad \text{Hom}_{\mathcal{T}}(\mathcal{O}, \sigma^{-n}(\mathcal{O})) = A_{\ell n}$$

for $n \geq 0$. Now (E4.5.2) implies that $\kappa(\mathcal{T}, \mathcal{O}) = -\infty$, and (E4.5.3) together with (E4.5.1) implies that $\kappa^{-1}(\mathcal{T}, \mathcal{O}) = w - 1 = \text{GKdim } A - 1$.

(3) Similar to the proof of (2).

(4) Assume that $\ell = 0$. Then $\sigma := S \circ \Sigma^{-(d-1)}$ is equivalent to $- \otimes \mathcal{O}(-\ell)$ when applied to $\mathcal{O}$. Thus

$$\text{Hom}_{\mathcal{T}}(\mathcal{O}, \sigma^n(\mathcal{O})) = A_{-\ell n} = A_0 = \mathbb{k}$$

for all $n \in \mathbb{Z}$. This implies that $\text{fp}\kappa(\mathcal{T}_*) = \text{fp}\kappa^{-1}(\mathcal{T}_*) = 0$.

(5,6) The proofs are similar. We only consider (5). Note that $S \circ \Sigma^{-(d-1)} = - \otimes \pi({}^\mu A^1)(-\ell)$. By [BV, Lemmas 4.2.3 and 4.3.2], $\mathcal{T}$ is generated by $\{\mathcal{O}(n)\}_{n \in \mathbb{Z}}$. Hence we can assume that $\mathcal{C} = \mathcal{O}$ and $\mathcal{D} = \mathcal{O}(a)[b]$ for some a and b . Note that [AZ, Theorem 8.1(3)] holds for Artin-Schelter Gorenstein algebras. Then [AZ, Theorem 8.1(3)] implies that

$$\dim \text{Ext}_{\mathbb{X}}^b(\mathcal{O}, (S \circ \Sigma^{-(d-1)})^{\otimes n}(\mathcal{O}(a))) \leq cn^{w-1}$$

for some constant c only dependent on a, b . Therefore the assertion follows. $\square$

The noncommutative projective scheme in the sense of [AZ] can be defined for connected graded coherent algebras that are not necessarily noetherian. Here we consider a family of noncommutative projective schemes of non-noetherian Artin-Schelter regular algebras of global dimension two.

Let W_n be the Artin-Schelter regular algebra $\mathbb{k}\langle x_1, \dots, x_n \rangle / (\sum_{i=1}^n x_i^2)$ of global dimension 2. When $n \geq 3$, this algebra is non-noetherian [Zh, Theorem 0.2(1)], but coherent [Pi, Theorem 1.2]. Let $\mathbb{P}_n^1$ denote the noncommutative projective scheme associated to W_n defined in [Pi], which is also denoted by $\text{Proj } W_n$. We call $\mathbb{P}_n^1$ a *Piontkovski projective line* of rank n . See [Pi, SSm] for basic properties of $\mathbb{P}_n^1$. The

main result concerning $\mathbb{P}_n^1$ is the following. See [CG] for the definition of fpcx and fpv .

Theorem 4.6. *Let $\mathbb{P}_n^1$ be a Piontkovski projective line of rank $n \geq 2$. Let $\mathcal{T}_n$ be the derived category $D^b(\text{coh}(\mathbb{P}_n^1))$. Then*

- (1) $\text{fpd}(\mathcal{T}_n) = 1$ for all $n \geq 2$.
- (2) $\text{fpgldim}(\mathcal{T}_n) = 1$ for all $n \geq 2$.
- (3) $\text{fpcx}(\mathcal{T}_n) = 0$ for all $n \geq 2$.
- (4) $\text{fpv}(\mathcal{T}_n) = 0$ for all $n \geq 2$.
- (5) $\text{fpcy}(\mathcal{T}_n) = 1$ for all $n \geq 2$.
- (6) $\text{fp}\kappa^{-1}((\mathcal{T}_n)_*) = \begin{cases} 1 & n = 2, \\ \infty & n \geq 3. \end{cases}$
- (7) $\text{fp}\kappa((\mathcal{T}_n)_*) = -\infty$ for all $n \geq 2$.

Proof. Let $\mathcal{O}$ denote the object $\pi(W_n)$ in $\mathbb{X} := \text{Proj } W_n$. Then $\text{coh}(\mathbb{X})$ is hereditary, and $\mathcal{T}_n$ is Ext-finite with Serre functor $S := - \otimes \mathcal{O}(-2)[1]$ [Pi, Proposition 1.5].

(1) By [CG, Theorem 3.5(4)], $\text{fpd}(\mathcal{T}_n) = \text{fpd}(\text{coh}(\mathbb{X}))$. Since $\text{coh}(\mathbb{X})$ is hereditary, $\text{fpd}(\text{coh}(\mathbb{X})) \leq 1$. It remains to show that $\text{fpd}(\text{coh}(\mathbb{X})) \geq 1$. Let A be the simple object in $\text{coh}(\mathbb{X})$ of the form $\pi(W/I)$ where I is the right ideal of W generated by $W(x_3, \dots, x_n) + x_2$. Then it is routine to verify that $\text{Ext}_{\text{coh}(\mathbb{X})}^1(A, A) = \mathbb{k}$. This implies that $\text{fpd}(\text{coh}(\mathbb{X})) \geq 1$ as required.

(2) By [CG, Theorem 3.5(1)], $\text{fpgldim}(\mathcal{T}_n) \leq \text{gldim } \text{coh}(\mathbb{X}) = 1$. By part (1), $\text{fpgldim}(\mathcal{T}_n) \geq 1$. The assertion follows.

(3,4) These follow from the fact that $\text{coh}(\mathbb{X})$ is hereditary.

(5) Since the Serre functor S is of the form $- \otimes \mathcal{O}(-2)[1]$, we can almost copy the proof of Proposition 3.3.

(6,7) Using the special form of the Serre functor S , we can adapt the proofs of Proposition 3.6 and Theorem 4.5. $\square$

5. COMMENTS ON FP-INVARIANTS OF FINITE DIMENSIONAL ALGEBRAS

In this section we give some remarks, comments and examples concerning finite dimensional algebras. Let $\mathcal{T}(A)$ be the derived category $D^b(\text{Mod}_{f,d} A)$. The fp Calabi-Yau dimension $\text{fpcy}(\mathcal{T})$ is defined as in Definition 3.1.

Example 5.1. Let Q be a finite acyclic quiver and A be the path algebra $\mathbb{k}Q$. Let $\mathcal{T}$ be the bounded derived category $D^b(\text{Mod}_{f,d} A)$.

- (1) If Q is of ADE type, then $\mathcal{T}$ is fractional Calabi-Yau and by Lemma 3.2(3) and [Ke1, Example 8.3(2)],

$$\text{fpcy}(\mathcal{T}) = \frac{h-2}{h}$$

where h is the Coxeter number of Q . In this case, $\text{fpcy}(\mathcal{T})$ is strictly between 0 and 1.

- (2) If Q is of $\widetilde{AD}\widetilde{E}$ type, then, by using Lemma 2.1(2), $\mathcal{T}$ is equivalent to $D^b(\text{coh}(\mathbb{X}))$ for a weighted projective line $\mathbb{X}$. Similar to the proof of Theorem 4.5(1), one can show that

$$\text{fpcy}(D^b(\text{coh}(\mathbb{X}))) = 1$$

for every weighted projective line $\mathbb{X}$. (The proof is slightly more complicated and details are omitted). Thus we obtain that

$$\mathrm{fpcy}(\mathcal{T}) = 1.$$

If Q is of wild representation type, in some cases, one can show that

$$(E5.1.1) \quad \mathrm{fpcy}(\mathcal{T}) = 1.$$

Such an example is the Kronecker quiver with two vertices and n arrows from the first vertex to the second, see Example 5.7. It is not clear to us if (E5.1.1) holds for all acyclic quivers of wild representation type.

Lemma 5.2. *Let A and B be finite dimensional algebras of finite global dimension. Suppose that both $\mathcal{T}(A)$ and $\mathcal{T}(B)$ are fractional Calabi-Yau of dimension d_1 and d_2 respectively. Then $\mathcal{T}(A \otimes B)$ is fractional Calabi-Yau of dimension $d_1 + d_2$.*

The proof of Lemma 5.2 is straightforward and hence omitted. One immediate question is

Question 5.3. Let A and B be finite dimensional algebras of finite global dimension. Is then

$$\mathrm{fpcy}(\mathcal{T}(A \otimes B)) = \mathrm{fpcy}(\mathcal{T}(A)) + \mathrm{fpcy}(\mathcal{T}(B))?$$

To define fp (anti-)Kodaira dimension of $\mathcal{T}$ as in Definition 3.5, we need to specify an object $\mathcal{O}$ which plays the role of the structure sheaf in algebraic geometry. One choice for $\mathcal{O}$ is the left A -module A . So we let $\mathcal{T}(A)_*$ be $(\mathcal{T}(A), A)$.

Definition 5.4. Let A be a finite dimensional algebra of finite global dimension. Suppose that $\mathcal{T}(A)$ has a fractional fp Calabi-Yau dimension $\frac{a}{b} \in \mathbb{Q}$.

(1) The *fp Kodaira dimension* of A is defined to be

$$\mathrm{fp}\kappa(A) := \mathrm{fp}\kappa(\mathcal{T}(A)_*) = \lim_t \left\{ \limsup_{n \rightarrow \infty} \dim \mathrm{Hom}_{\mathcal{T}}(A, (S^{bnt} \circ \Sigma^{-ant})(A)) \right\}$$

where the first limit ranges over all positive integers t with order $<$ as defined before Definition 3.5.

(2) The *fp anti-Kodaira dimension* of A is defined to be

$$\mathrm{fp}\kappa^{-1}(A) := \mathrm{fp}\kappa^{-1}(\mathcal{T}(A)_*) = \lim_t \left\{ \limsup_{n \rightarrow \infty} \dim \mathrm{Hom}_{\mathcal{T}}(A, (S^{-bnt} \circ \Sigma^{ant})(A)) \right\}$$

where the first limit ranges over all positive integers t with order $<$ as defined before Definition 3.5.

Remark 5.5. Suppose $\mathcal{T}(A)$ is fractional Calabi-Yau. Then one can check easily that

$$(E5.5.1) \quad \mathrm{fp}\kappa(A) = \mathrm{fp}\kappa^{-1}(A) = 0.$$

So if there is a notion of a fractional Calabi-Yau algebra, (E5.5.1) should be a part of the definition.

Let A be a finite dimensional algebra of finite global dimension. Then the Serre functor is given by $-\otimes_A^L DA$. It is unknown if $\mathrm{fpcy}(\mathcal{T}(A))$ always exists. If $\mathrm{fpcy}(\mathcal{T}(A))$ exists and is a rational number, then we can define and calculate fp

Kodaira dimension (respectively, anti-Kodaira dimension) of A . Here is a list of questions that are related to the fp Kodaira dimension.

Question 5.6. Let A and B be two finite dimensional algebra of finite global dimension. Suppose that $\mathrm{fpcy}(\mathcal{T}(A))$ and $\mathrm{fpcy}(\mathcal{T}(B))$ are rational numbers.

- (1) Are $\mathrm{fp}\kappa(A)$ and $\mathrm{fp}\kappa^{-1}(A)$ less than ∞ ?
- (2) If $\mathcal{T}(A)$ is triangulated equivalent to $\mathcal{T}(B)$, is then $\mathrm{fp}\kappa(A) = \mathrm{fp}\kappa(B)$ and $\mathrm{fp}\kappa^{-1}(A) = \mathrm{fp}\kappa^{-1}(B)$?
- (3) Suppose both $\mathrm{fp}\kappa(A)$ and $\mathrm{fp}\kappa(B)$ are finite, is then

$$\mathrm{fp}\kappa(A \otimes B) = \mathrm{fp}\kappa(A) + \mathrm{fp}\kappa(B)?$$

The first question has a negative answer.

Example 5.7. Let Q_n be the Kronecker quiver with two vertices and n arrows from the first vertex to the second. Let U_n be the path algebra of Q_n . By [Min, Theorem 0.1], if $n \geq 2$,

$$(E5.7.1) \quad D^b(\mathrm{Mod}_{f.d.} - U_n) \cong D^b(\mathrm{coh}(\mathbb{P}_n^1)) =: \mathcal{T}_n$$

where $\mathbb{P}_n^1$ is given in Theorem 4.6. By Theorem 4.6(3), $\mathrm{fpcy}(D^b(\mathrm{Mod}_{f.d.} - U_n)) = 1$ for all $n \geq 2$. On the other hand, $\mathrm{fpcy}(D^b(\mathrm{Mod}_{f.d.} - U_1)) = \frac{h-2}{h} = \frac{1}{3}$ where $h = 3$ is the Coxeter number of the quiver A_2 (which is Q_1), see [Ke1, Example 8.3(2)].

We claim that $\mathrm{fp}\kappa(U_n) = -\infty$ and that $\mathrm{fp}\kappa^{-1}(U_n) = \infty$ when $n \geq 3$. We only prove the second assertion. By the noncommutative Beilinson's theorem given in [Min, Theorem 0.1] (also see (E5.9.1)), the equivalent (E5.7.1) sends U_n to $\mathcal{O} \oplus \mathcal{O}(1)$ where $\mathcal{O}$ is the structure sheaf of $\mathbb{P}_n^1$.

Note that the Serre functor is $S := - \otimes \mathcal{O}(-2)[1]$. Let $A = U_n$. Then

$$\begin{aligned} \mathrm{fp}\kappa^{-1}(A) &= \lim_t \left\{ \limsup_{m \rightarrow \infty} \dim \mathrm{Hom}_{\mathcal{T}(A)}(A, (S^{-tm} \circ \Sigma^{tm})(A)) \right\} \\ &= \lim_t \left\{ \limsup_{m \rightarrow \infty} \dim \mathrm{Hom}_{\mathcal{T}_n}(\mathcal{O} \oplus \mathcal{O}(1), (S^{-tm} \circ \Sigma^{tm})(\mathcal{O} \oplus \mathcal{O}(1))) \right\} \\ &\geq \lim_t \left\{ \limsup_{m \rightarrow \infty} \dim \mathrm{Hom}_{\mathcal{T}_n}(\mathcal{O}, (S^{-tm} \circ \Sigma^{tm})(\mathcal{O})) \right\} \\ &= \mathrm{fp}\kappa^{-1}((\mathcal{T}_n)_*) \\ &= \infty \end{aligned}$$

where the last equation is Theorem 4.6(6).

We add a few more questions to Question 0.4.

Question 5.8. Let A be a finite dimensional algebra of finite global dimension and let $\mathcal{T}$ be the derived category $D^b(\mathrm{Mod}_{f.d.} - A)$.

- (1) By Example 5.7, if A is the path algebra of the quiver Q_1 , then $\mathrm{fpcy}(\mathcal{T}) = \frac{1}{3}$. Is the minimum value of $\mathrm{fpcy}(\mathcal{T})$ equal to $\frac{1}{3}$ for an arbitrary A ?

- (2) Is there a value of $\text{fpcy}(\mathcal{T})$ outside of the set

$$R := \left(\sum_{h \geq 3} \frac{h-2}{h} \mathbb{N} \right) \cap \mathbb{Q}_{>0}?$$

By Lemma 5.2 and [Ke1, Example 8.3(2)], every number in R can be realized as $\text{fpcy}(\mathcal{T})$ for some finite dimensional algebra. But we don't have examples of fpcy that are outside this range.

Recall that

Definition 5.9 ([HP, p. 1230]). Let $\mathbb{X}$ be a smooth projective scheme.

- (1) A coherent sheaf $\mathcal{E}$ on $\mathbb{X}$ is called *exceptional* if $\text{Hom}_{\mathbb{X}}(\mathcal{E}, \mathcal{E}) \cong \mathbb{k}$ and $\text{Ext}_{\mathbb{X}}^i(\mathcal{E}, \mathcal{E}) = 0$ for every $i \geq 0$.
- (2) A sequence $\mathcal{E}_1, \dots, \mathcal{E}_n$ of exceptional sheaves is called an *exceptional sequence* if $\text{Ext}_{\mathbb{X}}^k(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all k and for all $i > j$.
- (3) If an exceptional sequence generates $D^b(\text{coh}(\mathbb{X}))$, then it is called *full*.
- (4) If an exceptional sequence satisfies

$$\text{Ext}_{\mathbb{X}}^k(\mathcal{E}_i, \mathcal{E}_j) = 0$$

for all $k > 0$ and all i, j , then it is called a *strongly exceptional sequence*.

The existence of a full exceptional sequence has been proved for many projective schemes. However, on Calabi-Yau varieties there are no exceptional sheaves. When $\mathbb{X}$ has a full exceptional sequence $\mathcal{E}_1, \dots, \mathcal{E}_n$, then there is a triangulated equivalence

$$(E5.9.1) \quad D^b(\text{coh}(\mathbb{X})) \cong D^b(\text{Mod}_{f.d} - A)$$

where A is the finite dimensional algebra $\text{End}_{\mathbb{X}}(\oplus_{i=1}^n \mathcal{E}_i)$. In this setting, the fp Calabi-Yau dimension of $D^b(\text{Mod}_{f.d} - A)$ is equal to $\dim \mathbb{X}$, which exists and is finite. In many examples in algebraic geometry, a full exceptional sequence consists of line bundles. Assume this is true. Via (E5.9.1), one sees easily that $\text{fp}\kappa^{\pm 1}(A) \geq \text{fp}\kappa^{\pm 1}(\mathbb{X})$. In fact, in many examples, we have $\text{fp}\kappa^{\pm 1}(A) = \text{fp}\kappa^{\pm 1}(\mathbb{X})$.

6. APPENDIX A: PROOF OF LEMMA 2.10

As in Section 2, let r be a positive integer. Suppose that a hereditary abelian category $\mathfrak{T}$ has r^2 brick objects as given in Corollary 2.8, now labeled as

$$1, 2, 3, \dots, r^2 - 1, r^2,$$

where the matrices

$$H = (H_{ij})_{r^2 \times r^2} = (\dim \text{Hom}(i, j))_{r^2 \times r^2},$$

see (E2.8.1), and

$$E = (E_{ij})_{r^2 \times r^2} = (\dim \operatorname{Hom}(i, \Sigma j))_{r^2 \times r^2},$$

see (E2.8.2), are given by the following block matrices:

$$(E6.0.1) \quad H = \begin{matrix} & \begin{matrix} (1) & (2) & (3) & (4) & \cdots & (r) \end{matrix} \\ \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ \vdots \\ (r) \end{matrix} & \begin{pmatrix} P^0 & P^1 & P^2 & P^3 & \cdots & P^{r-1} \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=1}^2 P^i & \sum_{i=2}^3 P^i & \cdots & \sum_{i=r-2}^{r-1} P^i \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=0}^2 P^i & \sum_{i=1}^3 P^i & \cdots & \sum_{i=r-3}^{r-1} P^i \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=0}^2 P^i & \sum_{i=0}^3 P^i & \cdots & \sum_{i=r-4}^{r-1} P^i \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=0}^2 P^i & \sum_{i=0}^3 P^i & \cdots & \sum_{i=0}^{r-1} P^i \end{pmatrix} \end{matrix},$$

$$E = \begin{matrix} & \begin{matrix} (1) & (2) & (3) & (4) & \cdots & (r) \end{matrix} \\ \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ \vdots \\ (r) \end{matrix} & \begin{pmatrix} P^{r-1} & P^{r-1} & P^{r-1} & P^{r-1} & \cdots & P^{r-1} \\ P^{r-2} & \sum_{i=r-2}^{r-1} P^i & \sum_{i=r-2}^{r-1} P^i & \sum_{i=r-2}^{r-1} P^i & \cdots & \sum_{i=r-2}^{r-1} P^i \\ P^{r-3} & \sum_{i=r-3}^{r-2} P^i & \sum_{i=r-3}^{r-1} P^i & \sum_{i=r-3}^{r-1} P^i & \cdots & \sum_{i=r-3}^{r-1} P^i \\ P^{r-4} & \sum_{i=r-4}^{r-3} P^i & \sum_{i=r-4}^{r-2} P^i & \sum_{i=r-4}^{r-1} P^i & \cdots & \sum_{i=r-4}^{r-1} P^i \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ P^0 & \sum_{i=0}^1 P^i & \sum_{i=0}^2 P^i & \sum_{i=0}^3 P^i & \cdots & \sum_{i=0}^{r-1} P^i \end{pmatrix} \end{matrix}$$

$$= \begin{matrix} & \begin{matrix} (1) & (2) & (3) & (4) & \cdots & (r) \end{matrix} \\ \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ \vdots \\ (r) \end{matrix} & \begin{pmatrix} P^{r-1} & P^{r-1} & P^{r-1} & P^{r-1} & \cdots & P^{r-1} \\ P^{r-2} & \sum_{i=1}^2 P^{r-i} & \sum_{i=1}^2 P^{r-i} & \sum_{i=1}^2 P^{r-i} & \cdots & \sum_{i=1}^2 P^{r-i} \\ P^{r-3} & \sum_{i=2}^3 P^{r-i} & \sum_{i=1}^3 P^{r-i} & \sum_{i=1}^3 P^{r-i} & \cdots & \sum_{i=1}^3 P^{r-i} \\ P^{r-4} & \sum_{i=3}^4 P^{r-i} & \sum_{i=2}^4 P^{r-i} & \sum_{i=1}^4 P^{r-i} & \cdots & \sum_{i=1}^4 P^{r-i} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ P^0 & \sum_{i=r-1}^r P^{r-i} & \sum_{i=r-2}^r P^{r-i} & \sum_{i=r-3}^r P^{r-i} & \cdots & \sum_{i=1}^r P^{r-i} \end{pmatrix},$$

where P is the $r \times r$ permutation matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and P^0 denotes the $r \times r$ identity matrix. Note that $P^r = P^0$. We will show that $\rho(A(\phi)) \leq 1$ for all brick sets ϕ .

The Hom and Ext^1 matrices given in (E2.8.1)–(E2.8.2) are actually the transpose of the usual Hom and Ext^1 matrices since they follow the convention of Corollary 2.8. See the remark before Theorem 2.11.

First, notice that

$$H^T = \begin{matrix} & \begin{matrix} (1) & (2) & (3) & (4) & \cdots & (r) \end{matrix} \\ \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ \vdots \\ (r) \end{matrix} & \begin{pmatrix} P^r & P^r & P^r & P^r & \cdots & P^r \\ P^{r-1} & \sum_{i=0}^1 P^{r-i} & \sum_{i=0}^1 P^{r-i} & \sum_{i=0}^1 P^{r-i} & \cdots & \sum_{i=0}^1 P^{r-i} \\ P^{r-2} & \sum_{i=1}^2 P^{r-i} & \sum_{i=0}^2 P^{r-i} & \sum_{i=0}^2 P^{r-i} & \cdots & \sum_{i=0}^2 P^{r-i} \\ P^{r-3} & \sum_{i=2}^3 P^{r-i} & \sum_{i=1}^3 P^{r-i} & \sum_{i=0}^3 P^{r-i} & \cdots & \sum_{i=0}^3 P^{r-i} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ P^1 & \sum_{i=r-2}^{r-1} P^{r-i} & \sum_{i=r-3}^{r-1} P^{r-i} & \sum_{i=r-4}^{r-1} P^{r-i} & \cdots & \sum_{i=0}^{r-1} P^{r-i} \end{pmatrix} \end{matrix}.$$

Define the following non-negative $r^2 \times r^2$ matrices:

$$F := \begin{matrix} & \begin{matrix} (1) & (2) & (3) & (4) & \cdots & (r) \end{matrix} \\ \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ \vdots \\ (r) \end{matrix} & \begin{pmatrix} P^{r-1} & P^{r-1} & P^{r-1} & P^{r-1} & \cdots & P^{r-1} \\ P^{r-2} & P^{r-2} & P^{r-2} & P^{r-2} & \cdots & P^{r-2} \\ P^{r-3} & P^{r-3} & P^{r-3} & P^{r-3} & \cdots & P^{r-3} \\ P^{r-4} & P^{r-4} & P^{r-4} & P^{r-4} & \cdots & P^{r-4} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ P^0 & P^0 & P^0 & P^0 & \cdots & P^0 \end{pmatrix} \end{matrix},$$

$$G := \begin{matrix} & \begin{matrix} (1) & (2) & (3) & (4) & \cdots & (r) \end{matrix} \\ \begin{matrix} (1) \\ (2) \\ (3) \\ (4) \\ \vdots \\ (r) \end{matrix} & \begin{pmatrix} P^r & P^r & P^r & P^r & \cdots & P^r \\ P^{r-1} & P^r & P^r & P^r & \cdots & P^r \\ P^{r-2} & P^{r-1} & P^r & P^r & \cdots & P^r \\ P^{r-3} & P^{r-2} & P^{r-1} & P^r & \cdots & P^r \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ P^1 & P^2 & P^3 & P^4 & \cdots & P^r \end{pmatrix} \end{matrix}.$$

We can see that

$$\begin{aligned}
 & \text{(E6.0.2)} \quad H^T + F \\
 &= \begin{pmatrix} (1) & (2) & (3) & (4) & \cdots & (r) \\ (1) & \sum_{i=0}^1 P^{r-i} & \sum_{i=0}^1 P^{r-i} & \sum_{i=0}^1 P^{r-i} & \cdots & \sum_{i=0}^1 P^{r-i} \\ (2) & \sum_{i=1}^2 P^{r-i} & \sum_{i=0}^2 P^{r-i} & \sum_{i=0}^2 P^{r-i} & \cdots & \sum_{i=0}^2 P^{r-i} \\ (3) & \sum_{i=2}^3 P^{r-i} & \sum_{i=1}^3 P^{r-i} & \sum_{i=0}^3 P^{r-i} & \cdots & \sum_{i=0}^3 P^{r-i} \\ (4) & \sum_{i=3}^4 P^{r-i} & \sum_{i=2}^4 P^{r-i} & \sum_{i=1}^4 P^{r-i} & \cdots & \sum_{i=0}^4 P^{r-i} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ (r) & \sum_{i=r-1}^r P^{r-i} & \sum_{i=r-2}^r P^{r-i} & \sum_{i=r-3}^r P^{r-i} & \cdots & \sum_{i=0}^r P^{r-i} \end{pmatrix} \\
 &= E + G.
 \end{aligned}$$

To finish the proof we need a few lemmas.

Lemma 6.1. *Let ϕ be a brick set which is a subset of $\{1, 2, \dots, r^2\}$. Suppose that $I, J \in \phi$ and that the (I, J) -entry $E_{IJ} \neq 0$. Then $F_{IJ} = E_{IJ} = 1$.*

Proof. Since E is a matrix whose entries consist of zeros and ones, $E_{IJ} = 1$. We consider two different cases.

Case 1 ($I = J$). By (E6.0.2),

$$1 + F_{IJ} = (H^T)_{IJ} + F_{IJ} = E_{IJ} + G_{IJ} = E_{IJ} + 1$$

which implies that $F_{IJ} = E_{IJ} = 1$.

Case 2 ($I \neq J$). Since I, J are distinct elements in the same brick set, we have $(H^T)_{IJ} = 0$. By (E6.0.2),

$$F_{IJ} = (H^T)_{IJ} + F_{IJ} = E_{IJ} + G_{IJ} \geq 1.$$

Since F is a matrix whose entries are contained in $\{0, 1\}$, $F_{IJ} = 1$. □

Lemma 6.2. *If ϕ is a brick set, for each row I there exists at most one column J such that $A(\phi)_{IJ} \neq 0$. If there exists a row J such that $A(\phi)_{IJ} \neq 0$, then $A(\phi)_{IJ} = 1$.*

Proof. Assume for contradiction that we have a brick set ϕ containing elements I, J, J' such that $J \neq J'$ and $E_{IJ} \neq 0 \neq E_{IJ'}$. Notice that we have the following general formula for $n, m \in \{0, \dots, r-1\}, i, j \in \{1, \dots, r\}$:

$$F_{nr+i, mr+j} = (P^{r-1-n})_{ij}.$$

We can always write

$$I = nr + i, \quad J = mr + j, \quad J' = m'r + j',$$

for some $i, j, j' \in \{1, \dots, r\}, n, m, m' \in \{0, \dots, r-1\}$.

Assume without loss of generality that $m \geq m'$. We will show that

$$H_{JJ'} = H_{mr+j, m'r+j'} \neq 0.$$

Therefore, J, J' cannot be in the same brick set, a contradiction.

By Lemma 6.1,

$$\begin{aligned} 1 &= F_{IJ} = F_{nr+i, mr+j} = (P^{r-1-n})_{ij} = (P^{n+1-r})_{ji}, \\ 1 &= F_{IJ'} = F_{nr+i, m'r+j'} = (P^{r-1-n})_{ij'}. \end{aligned}$$

Then

$$\begin{aligned} 1 &= (P^{n+1-r})_{ji} (P^{r-1-n})_{ij'} = \sum_{k=1}^r (P^{n+1-r})_{jk} (P^{r-1-n})_{kj'} \\ &= (P^{n+1-r} P^{r-1-n})_{jj'} = \delta_{jj'} \end{aligned}$$

which implies that $j = j'$.

By examination of the H matrix (E6.0.1), we can see that for $m \geq m'$,

$$H_{JJ'} = H_{mr+j, m'r+j'} = (P^0)_{jj'} + A_{jj'}$$

for some non-negative $r \times r$ matrix A . Therefore,

$$H_{JJ'} = H_{mr+j, m'r+j'} = H_{mr+j, m'r+j} = (P^0)_{jj} + A_{jj} = \delta_{jj} + A_{jj} \geq 1$$

as required. $\square$

Lemma 6.3. *If ϕ is a brick set, for each column J there exists at most one row I such that $A(\phi)_{IJ} \neq 0$. If there exists a row I such that $A(\phi)_{IJ} \neq 0$, then $A(\phi)_{IJ} = 1$.*

Proof. The proof is similar to the proof of Lemma 6.2 and omitted. $\square$

Lemma 6.4. *If ϕ is a brick set, $\rho(A(\phi)) \leq 1$.*

Proof. By Lemmas 6.2 and 6.3, we have shown that if ϕ is a brick set, $A(\phi)$ is a matrix with at most one non-zero entry in each row and at most one non-zero entry in each column, such that if any entry is non-zero, it is 1. Then $A(\phi)$ is almost a permutation matrix. In this case the quiver corresponding to $A(\phi)$ has cycle number at most 1. By [CG, Theorem 1.8], $\rho(A(\phi)) \leq 1$. $\square$

7. APPENDIX B: SOME VARIANTS

Recall that the set of subsets of n non-zero objects in $\mathcal{C}$ is denoted by Φ_n for each $n \geq 1$. And let $\Phi = \bigcup_{n \geq 1} \Phi_n$. In this paper, we use either Φ_b or Φ_a as testing objects in the definition of fp-invariants [Definition 1.4]. Depending on the situation, we might want to choose a testing set different from Φ_b or Φ_a . Here is a list of possible alternative testing sets.

Example 7.1.

- (1) $\Phi = \bigcup_{n \geq 1} \Phi_n$.
- (2) If the category $\mathcal{C}$ is abelian, we can consider “simple sets” as follows. Let $\Phi_{n,s}$ be the set of n -object subsets of $\mathcal{C}$, say $\phi := \{X_1, X_2, \dots, X_n\}$, where the X_i are non-isomorphic simple objects in $\mathcal{C}$. Let $\Phi_s = \bigcup_{n \geq 1} \Phi_{n,s}$.
- (3) A subset $\phi = \{X_1, X_2, \dots, X_n\}$ is called a *triangular brick set* if each X_i is a brick object and, up to a permutation, $\text{Hom}_{\mathcal{C}}(X_i, X_j) = 0$ for all $i < j$. Let $\Phi_{n,tb}$ be the set of all triangular brick n -sets, and let $\Phi_{tb} = \bigcup_{n \geq 1} \Phi_{n,tb}$.

- (4) Now assume that $\mathcal{C}$ is a triangulated category with suspension functor Σ . A subset $\phi = \{X_1, X_2, \dots, X_n\}$ is called a *triangular atomic set* if each X_i is an atomic object and, up to a permutation, $\mathrm{Hom}_{\mathcal{C}}(X_i, X_j) = 0$ for all $i < j$. Let $\Phi_{n,ta}$ be the set of all triangular atomic n -sets, and let $\Phi_{ta} = \bigcup_{n \geq 1} \Phi_{n,ta}$.

Basically, for any property P , we can define $\Phi_{n,Pb}$ (respectively, $\Phi_{n,Pa}$) and let Φ_{Pb} (respectively, Φ_{Pa}) be $\bigcup_{n \geq 1} \Phi_{n,Pb}$ (respectively, $\bigcup_{n \geq 1} \Phi_{n,Pa}$). All of the definitions in this paper and in [CG] can be modified after we redefine $\rho(A(\phi, \sigma))$ as follows.

Definition 7.2. Let $\mathcal{C}$ be a $\mathbb{k}$ -linear category and let ϕ be a set of n non-zero objects, say $\{X_1, \dots, X_n\}$, in $\mathcal{C}$. Let σ be a $\mathbb{k}$ -linear endofunctor of $\mathcal{C}$. We define

$$\rho(A(\phi, \sigma)) := \frac{\rho((\dim \mathrm{Hom}_{\mathcal{C}}(X_i, \sigma(X_j)))_{n \times n})}{\rho((\dim \mathrm{Hom}_{\mathcal{C}}(X_i, X_j))_{n \times n})}.$$

Note that $\rho(A(\phi, \sigma))$ agrees with the original definition when ϕ is a brick set. One reason to introduce P -versions of fp-invariants is to extend these invariants even if the category contains no brick objects.

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