Functional central limit theorem via nonstationary projective conditions

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Abstract

In this paper we survey some recent progress on the Gaussian approximation for nonstationary dependent structures via martingale methods. First we present general theorems involving projective conditions for triangular arrays of random variables and then present various applications for rho-mixing and alpha-dependent triangular arrays, stationary sequences in a random time scenery, application to the quenched FCLT, application to linear statistics with alpha-dependent innovations, application to functions of a triangular stationary Markov chain.

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1 Introduction and notations

A time dependent series, in a discretized form, consists of a triangular array of random variables. Examples of this kind are numerous and we can cite, for instance, the time varying regression model. On another hand, a Markov chain with stationary transition operator is not stationary when it does not start from its equilibrium and it rather starts at a point. Nonstationary type of behavior also appears when we study evolutions in random media. It is also well-known that the blocking procedure, used to weaken the dependence for studying a stationary process or a random field, introduces triangular arrays of variables. Furthermore, many of the results for functions of stationary random fields, often incorporate in their proofs complicated inductions, which lead to triangular arrays of random variables.

Historically, the most celebrated limit theorems in nonstationary setting are, among others, the limit theorems involving nonstationary sequences of martingale differences. For more general dependent sequences one of the basic techniques is to approximate them with martingales. A remarkable early result obtained by using this technique is due to Dobrushin [8], who studied the central limit theorem for nonstationary Markov Chains. In order to treat

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more general dependent structures, McLeish [23, 24] introduced the notion of mixingales, which are martingale-like structures, and imposed conditions to the moments of projections of an individual variable on past sigma fields to derive the functional form of the central limit theorem. This method is very fruitful, but still involves a large degree of stationarity. In general, the theory of nonstationary martingale approximation has remained much behind the theory of martingale methods for stationary processes. In the stationary setting, the theory of martingale approximations was steadily developed. We mention the well-known results, such as the celebrated results by Gordin [13], Heyde [19], Maxwell and Woodroofe [22] and the more recent results by Peligrad and Utev [31], Zhao and Woodroofe [44], Gordin and Peligrad [15], among many others. In the context of random fields, the theory of martingale approximation has been developed in the last decade, with several results by Gordin [14], Volný and Wang [42], Cuny et al. [3], El Machkouri and Giraudo [11], Peligrad and Zhang [33, 34, 35], Giraudo [12] and Volný [40, 41]. Due to these results we know now necessary and sufficient conditions for various types of martingale approximations which lead to a variety of maximal inequalities and limit theorems.

The goal of this paper is to survey some results obtained in the recent book [26] and the recent papers [25, 27] concerning the functional form of the central limit theorem for non necessarily stationary dependent structures. These results are obtained by using nonstationary martingale techniques and, as we shall see, the results are in the spirit of those obtained by McLeish [23, 24]. More precisely the conditions can be compared to the mixingales conditions imposed in his paper.

Still concerning Gaussian approximation for non necessarily stationary dependent structures, we would like to mention the paper by Wu and Zhou [43] who show that, under mild conditions, the partial sums of a non homogeneous function of an i.i.d. sequence can be approximated, on a richer probability space, by sums of independent Gaussian random variables with nearly optimal errors in probability. As a byproduct, a CLT can be derived provided the underlying random variables have moments of order $2+\delta$, $\delta > 0$. Their proof combines martingale approximation with m-dependent approximation. The fact that the random variables are functions of an i.i.d. sequence is a crucial assumption in their paper.

We shall point out classes of nonstationary time series, satisfying certain projective criteria (i.e. conditions imposed to conditional expectations), which benefit from a martingale approximation. We shall stress the nonstationary version of the Maxwell-Woodroofe condition, which will be essential for obtaining maximal inequalities and asymptotic results for the following examples: functions of linear processes with nonstationary innovations; quenched version of the functional central limit theorem for a stationary sequence; evolutions in random media such as a process sampled by a shifted Markov chain; nonstationary $\rho-$mixing and $\alpha-$mixing processes.

The basic setting will be mostly of a sequence of real-valued random variables $(X_k)_{k\geq 1}$ defined on the probability space $(\Omega, \mathcal{K}, P)$, adapted to an increasing filtration $\mathcal{F}_k \subset \mathcal{K}$. Set $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$ and $S_0 = 0$.

We shall also consider real-valued triangular arrays $(X_{k,n})_{1\leq k\leq n}$ adapted to $\mathcal{F}_{k,n} \subset \mathcal{K}$. This means that $X_{k,n}$ is $\mathcal{F}_{k,n}$ measurable and $\mathcal{F}_{k-1,n} \subset \mathcal{F}_{k,n}$ for all $n \geq 1$ and all $1 \leq k \leq n$.

In this case we set $S_k = S_{k,n} = \sum_{i=1}^{k} X_{i,n}$ $n \geq 1$, and $S_0 = 0$.

We shall be interested in both CLT, i.e.

$$\frac{S_n - a_n}{b_n} \Rightarrow N(0, \sigma^2),$$

where $\Rightarrow$ denotes the convergence in distribution and N is a normal distributed variable, and also in its functional (FCLT) form, i.e.

$$\{W_n(t), t \in [0,1]\} \Rightarrow |\sigma| W \text{ in } (D([0,1]), \|\cdot\|_\infty),$$

where $W_n(t) = b_n^{-1}(S_{[nt]} - a_{[nt]})$ and W is a standard Brownian motion (here and everywhere in the paper $[x]$ denotes the integer part of x).

We shall consider centered real-valued random variables which are square integrable. The normalizations will be taken $a_n = 0$ and $b_n^2 = n$ or $b_n^2 = \sigma_n^2 = \mathrm{Var}(S_n)$.

In the sequel, we shall often use the notation $\mathbb{E}_i(X) = \mathbb{E}(X|\mathcal{F}_i)$, to replace the conditional expectation. In addition all along the paper we shall use the notation $a_n \ll b_n$ to mean that there exists a universal constant C such that, for all n, $a_n \leq C b_n$.

2 Projective criteria for nonstationary time series

One of the first projection condition, in the nonstationary setting, goes back to McLeish [23]. To simplify the exposition let us state it in the adapted case, i.e. when $(\mathcal{F}_i)_{i\geq 0}$ is a non-decreasing sequence of σ-algebras such that X_i is $\mathcal{F}_i$-measurable for any $i \geq 1$.

Theorem 1 *Let $(X_k)_{k\in\mathbb{Z}}$ be a sequence of random variables, centered, with finite second moment and adapted to a non-decreasing sequence $(\mathcal{F}_k)_{k\in\mathbb{Z}}$ of σ-algebras. Assume that $(X_k^2)_{k\in\mathbb{Z}}$ is uniformly integrable and that, for any k and i,*

$$\|\mathbb{E}(X_{i+k}|\mathcal{F}_i)\|_2 \leq C k^{-1/2} (\log k)^{-(1+\varepsilon)} \ , \tag{1}$$

and there exists a nonnegative constant c^2 such that

$$\frac{\mathbb{E}(S_{[nt]}^2)}{n} \to c^2 t \text{ for any } t \in [0,1] \text{ and } \frac{\mathbb{E}_{k-m}(S_{k+n} - S_k)^2}{n} \to c^2 \text{ in } \quad \mathbb{L}_1 \, ,$$

as $\min(k, m, n) \to \infty$. Then $\{n^{-1/2} S_{[nt]}, t \in [0,1]\} \Rightarrow cW$ in $(D([0,1]), \|\cdot\|_\infty)$, where W is a standard Brownian motion.

However, in the stationary case, a more general projection condition than (1) is known to be sufficient for both CLT and its functional form. Let us describe it briefly.

Let $(X_k)_{k\in\mathbb{Z}}$ be a strictly stationary and ergodic sequence of centered real-valued random variables in $\mathbb{L}^2$, adapted to a strictly stationary filtration $(\mathcal{F}_k)_{k\in\mathbb{Z}}$ and such that

$$\sum_{k\geq 1} \frac{\|\mathbb{E}_0(S_k)\|_2}{k^{3/2}} < \infty. \tag{2}$$

Under condition (2), Maxwell-Woodroofe [22] proved the CLT under the normalization $\sqrt{n}$

and Peligrad-Utev [31] proved its functional form, namely:

$$\{n^{-1/2}S_{[nt]}, t \in [0,1]\} \Rightarrow cW \text{ in } (D([0,1]), \|\cdot\|_\infty),$$

where $c^2 = \lim_n n^{-1}\mathbb{E}(S_n^2)$.

It is known that (2) is equivalent to $\sum_{k\geq 0} 2^{-k/2}\|\mathbb{E}_0(S_{2^k})\|_2 < \infty$ and it is implied by

$$\sum_{k>0} k^{-1/2}\|\mathbb{E}_0(X_k)\|_2 < \infty. \tag{3}$$

It should be noted that condition (2) is a sharp condition in the sense that if it is barely violated, then the sequence $(n^{-1/2}S_n)$ fails to be stochastically bounded (see [31]).

The Maxwell-Woodroofe condition is very important for treating the class of $\rho-$mixing sequences whose definition is based on maximum coefficient of correlation. In the stationary case this is

$$\rho(k) = \sup \operatorname{corr}\big(f(X_{i,}, \ i \leq 0), g(X_j, j \geq k)\big) \to 0,$$

where sup is taken over all functions f, g which are square integrable.

It can be shown that condition (2) is implied by $\sum_{k\geq 0} \rho(2^k) < \infty$ (which is equivalent to $\sum_{k\geq 1} k^{-1}\rho(k) < \infty$). It is therefore well adapted to measurable functions of stationary Gaussian processes. To give another example of a sequence satisfying (2) let

$$X_k = f\Big(\sum_{i\geq 0} a_i\varepsilon_{k-i}\Big) - \mathbb{E}f\Big(\sum_{i\geq 0} a_i\varepsilon_{k-i}\Big),$$

where (ε_k) are i.i.d. with variance σ^2 and let f be a function such that

$$|f(x) - f(y)| \leq c(|x-y|) \text{ for any } (x,y) \in \mathbb{R}^2,$$

where c is a concave non-decreasing function such that

$$\sum_{k\geq 1} k^{-1/2} c\Big(2\sigma \sum_{i\geq k} |a_i|\Big) < \infty\,.$$

Then (3) holds (and then (2) also).

The question is, could we have similar results, which extend condition (2) to the nonstationary case and improve on Theorem 1?

2.1 Functional CLT under the standard normalization $\sqrt{n}$

We shall discuss first FCLT in the non-stationary setting under the normalization $\sqrt{n}$. With this aim, we impose the Lindeberg-type condition in the form:

$$\sup_{n\geq 1} \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}(X_j^2) \leq C < \infty \text{ and, for any } \varepsilon > 0,\ \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\{X_k^2 I(|X_k| > \varepsilon\sqrt{n})\} = 0\,. \tag{4}$$

For any $k \geq 0$, let

$$\delta(k) = \max_{i\geq 0} \|\mathbb{E}(S_{k+i} - S_i|\mathcal{F}_i)\|_2$$

and for any $k, m \geq 0$, let

$$\theta_k^m = m^{-1} \sum_{i=1}^{m-1} \mathbb{E}_k(S_{k+i} - S_k).$$

The following FCLT in the non-stationary setting under the normalization $\sqrt{n}$ was proven by Merlevède et al. [25, 26].

Theorem 2 *Assume that the Lindeberg-type condition (4) holds. Suppose also that*

$$\sum_{k \geq 0} 2^{-k/2} \delta(2^k) < \infty \tag{5}$$

and there exists a constant c^2 such that, for any $t \in [0,1]$ and any $\varepsilon > 0$,

$$\lim_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}\Big(\Big| \frac{1}{n} \sum_{k=1}^{[nt]} \big(X_k^2 + 2X_k \theta_k^m \big) - tc^2 \Big| > \varepsilon \Big) = 0 \,. \tag{6}$$

Then $\{n^{-1/2} S_{[nt]}, t \in [0,1]\} \Rightarrow cW$ in $(D([0,1]), \|\cdot\|_\infty)$.

We mention that (5) is equivalent to $\sum_{k>0} k^{-3/2} \delta(k) < \infty$ and it is implied by

$$\sum_{k>0} k^{-1/2} \sup_{i \geq 0} \|\mathbb{E}_i(X_{k+i})\|_2 < \infty. \tag{7}$$

About condition (6) we would like to mention that in the stationary and ergodic case, it is verified under condition (2). Indeed, by the ergodic theorem, for any $k \geq 0$,

$$\lim_{n \to \infty} \mathbb{E} \Big| \frac{1}{n} \sum_{k=1}^{[nt]} (X_k^2 + 2X_k \theta_k^m) - c^2 t \Big| = t \big| \mathbb{E} X_0^2 + 2\mathbb{E}(X_0 \theta_0^m) - c^2 \big| \,.$$

and note that, under condition (2), it has been proved in [31] that

$$\frac{1}{m} \mathbb{E}(S_m^2) = \mathbb{E}(X_0^2) + 2\mathbb{E}(X_0 \theta_0^m) \to c^2 \text{ as } m \to \infty.$$

Therefore Theorem 2 is indeed a generalization of the results in Peligrad and Utev [31].

A first application of Theorem 2 is the following:

Example 3 ***Application to stationary sequences in a random time scenery.***

We are interested to investigate the limiting behavior of the partial sums associated with the process defined by

$$X_k = \zeta_{k + \phi_k} \,,$$

where $\{\zeta_j\}_{j \in \mathbb{Z}}$ is a stationary sequence (observables/random scenery), and $\{\phi_k\}_{k \geq 0}$ is a Markov chain (random time).

The sequence $\{\phi_n\}_{n \geq 0}$ is a "renewal"-type Markov chain defined as follows: $\{\phi_k; k \geq 0\}$ is a discrete Markov chain with the state space $\mathbb{Z}^+$ and transition matrix $P = (p_{i,j})$ given by $p_{k,k-1} = 1$ for $k \geq 1$ and $p_{0,j-1} := p_j = \mathbb{P}(\tau = j)$, $j = 1, 2, \ldots$.

We assume that $\mathbb{E}[\tau] < \infty$ which ensures that $\{\phi_n\}_{n\geq 0}$ has a stationary distribution $\pi = (\pi_i, i \geq 0)$ given by

$$\pi_j = \pi_0 \sum_{i=j+1}^{\infty} p_i \ , \ j = 1, 2 \dots \text{ where } \pi_0 = 1/\mathbb{E}(\tau)\,.$$

We also assume that $p_j > 0$ for all $j \geq 0$. Hence the Markov chain is irreducible.

We are interesting by the asymptotic behavior of

$$\big\{n^{-1/2} \sum_{k=1}^{[nt]} X_k, t \in [0,1]\big\}$$

when the Markov chain starts at 0 (so under $\mathbb{P}_{\phi_0=0}$).

Under $\mathbb{P}_{\phi_0=0}$, one can prove that $\mathbb{E}(X_1X_2) \neq \mathbb{E}(X_2X_3)$ and hence stationarity is ruled out immediately. Let assume the following assumption on the random time scenery:

Condition (A_1) $\{\zeta_j\}_{j\geq 0}$ is a strictly stationary sequence of centered random variables in $\mathbb{L}^2$, independent of $(\phi_k)_{k\geq 0}$ and such that

$$\sum_{k\geq 1} \frac{\|\mathbb{E}(\zeta_k|\mathcal{G}_0)\|_2}{\sqrt{k}} < \infty \text{ and } \lim_{n\to\infty} \sup_{j\geq i\geq n} \|\mathbb{E}(\zeta_i\zeta_j|\mathcal{G}_0) - \mathbb{E}(\zeta_i\zeta_j)\|_1 = 0,$$

where $\mathcal{G}_i = \sigma(\zeta_k, k \leq i)$.

Corollary 4 *Assume that* $\mathbb{E}(\tau^2) < \infty$ *and that* $\{\zeta_j\}_{j\geq 0}$ *satisfies condition* (A_1)*. Then, under* $\mathbb{P}_{\phi_0=0}$*,* $\{n^{-1/2}S_{[nt]}, t \in [0,1]\}$ *converges in distribution in* $D[0,1]$ *to a Brownian motion with parameter* c^2 *defined by*

$$c^2 = \mathbb{E}(\zeta_0^2)\Big(1 + 2\sum_{i\geq 1} i\pi_i\Big) + 2\sum_{m\geq 1} \mathbb{E}(\zeta_0\zeta_m) \sum_{j=1}^{m} (P^j)_{0,m-j}\,,$$

where $(P^j)_{0,b} = \mathbb{P}_{\phi_0=0}(\phi_j = b)$*.*

The idea of proof is the following. We take $\mathcal{A} = \sigma(\phi_k, k \geq 0)$ and $\mathcal{F}_k = \sigma(\mathcal{A}, X_j, 1 \leq j \leq k)$. One can show that

$$\sup_{k\geq 0} \|\mathbb{E}(X_{k+m}|\mathcal{F}_k)\|_2^2 \leq b^2([m/2]) + b^2(0)\mathbb{P}(\tau > [m/2]),$$

where $b(k) = \|\mathbb{E}(\zeta_k|\mathcal{G}_0)\|_2$. To prove that condition (6) holds, we use in particular the ergodic theorem for recurrent Markov chains (together with many tedious computations).

An additional comment. In the stationary case, other projective criteria can be considered to get the FCLT such as the so-called Hannan's condition [18]:

$$\mathbb{E}(X_0|\mathcal{F}_{-\infty}) = 0 \text{ a.s. and } \sum_{i\geq 0} \|\mathbf{P}_0(X_i)\|_2 < \infty\,,$$

where $\mathbf{P}_0(\cdot) = \mathbb{E}_0(\cdot) - \mathbb{E}_{-1}(\cdot)$.

The Hannan's condition and condition (2) have different areas of applications and are not comparable (see [10]).

If the scenery is a sequence of martingale difference sequence and the process is sampled by the renewal Markov Chain, then under $\mathbb{P}_{\phi_0=0}$, one can prove that

$$\sup_{k\geq 0} \|\mathbf{P}_{k-m}(X_k)\|_2 \sim C\sqrt{\mathbb{P}(\tau > m)}.$$

Hence, in this case, $\sup_{k\geq 0} \|\mathbf{P}_{k-m}(X_k)\|_2$ and $\sup_{k\geq 0} \|\mathbb{E}(X_{k+m}|\mathcal{F}_k)\|_2$ are of the same order of magnitude and

$$\sum_{m\geq 0} \sup_{k\geq 0} \|\mathbf{P}_{k-m}(X_k)\|_2 < \infty \iff \sum_{m\geq 0} \sqrt{\mathbb{P}(\tau > m)} < \infty\,.$$

On the other hand (7) holds provided $\sum_{k\geq 1} \sqrt{\mathbb{P}(\tau > k)}/\sqrt{k} < \infty$.

2.2 A more general FCLT for triangular arrays

Let $\{X_{i,n}, 1 \leq i \leq n\}$ be a triangular array of square integrable ($\mathbb{E}(X_{i,n}^2) < \infty$), centered ($\mathbb{E}(X_{i,n}) = 0$), real-valued random variables adapted to a filtration $(\mathcal{F}_{i,n})_{i\geq 0}$.

We write as before $\mathbb{E}_{j,n}(X) = \mathbb{E}(X|\mathcal{F}_{j,n})$ and set

$$S_{k,n} = \sum_{i=1}^{k} X_{i,n} \text{ and } \theta_{k,n}^m = m^{-1} \sum_{i=1}^{m-1} \mathbb{E}_{k,n}(S_{k+i,n} - S_{k,n})\,. \tag{8}$$

We assume that the triangular array satisfies the following triangular Lindeberg-type condition:

$$\sup_{n\geq 1} \sum_{j=1}^{n} \mathbb{E}(X_{j,n}^2) \leq C < \infty, \text{ and } \lim_{n\to\infty} \sum_{k=1}^{n} \mathbb{E}\{X_{k,n}^2 I(|X_{k,n}| > \varepsilon)\} = 0\,, \text{ for any } \varepsilon > 0. \tag{9}$$

For a non-negative integer u and positive integers ℓ, m, define the following martingale-type dependence characteristics:

$$A^2(u) = \sup_{n\geq 1} \sum_{k=0}^{n-1} \|\mathbb{E}_{k,n}(S_{k+u,n} - S_{k,n})\|_2^2$$

and

$$B^2(\ell, m) = \sup_{n\geq 1} \sum_{k=0}^{[n/\ell]} \|\bar{S}_{k,n}(\ell, m)\|_2^2\,,$$

where

$$\bar{S}_{k,n}(\ell, m) = \frac{1}{m} \sum_{u=0}^{m-1} \left(\mathbb{E}_{(k-1)\ell+1,n}(S_{(k+1)\ell+u,n} - S_{k\ell+u,n})\right).$$

We mention that if $X_{k,n} = X_k/\sqrt{n}$,

$$A^2(u) \le \delta^2(u) \text{ and } B^2(\ell, m) \le \delta^2(\ell-1)/\ell.$$

The next theorem was proven by Merlevède et al. [25].

Theorem 5 *Assume that the Lindeberg condition (9) holds and that*

$$\lim_{j\to\infty} 2^{-j/2} A(2^j) = 0 \text{ and } \liminf_{j\to\infty} \sum_{\ell\ge j} B(2^\ell, 2^j) = 0\,. \tag{10}$$

Moreover, assume that there exists a sequence of non-decreasing and right-continuous functions $v_n(\cdot) : [0,1] \to \{0,1,2,\dots,n\}$ and a non-negative real c^2 such that, for any $t \in (0,1]$,

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\Big| \sum_{k=1}^{v_n(t)} \big(X_{k,n}^2 + 2X_{k,n}\theta_{k,n}^m \big) - tc^2 \Big| > \varepsilon \Big) = 0\,. \tag{11}$$

Then $\big\{ \sum_{k=1}^{v_n(t)} X_{k,n}, t \in [0,1] \big\}$ converges in distribution in $D([0,1])$ to cW where W is a standard Brownian motion.

The proof is based on a suitable triangular (non-stationary) martingale approximation. More precisely, for any fixed integer m, we write

$$X_{\ell,n} = D_{\ell,n}^m + \theta_{\ell-1,n}^m - \theta_{\ell,n}^m + Y_{\ell-1,n}^m\,, \tag{12}$$

where $\theta_{\ell,n}^m$ is defined in (8), $Y_{\ell,n}^m = \frac{1}{m}\mathbb{E}_{\ell,n}(S_{\ell+m,n} - S_{\ell,n})$ and, with the notation $\mathbf{P}_{\ell,n}(\cdot) = \mathbb{E}_{\ell,n}(\cdot) - \mathbb{E}_{\ell-1,n}(\cdot)$,

$$D_{\ell,n}^m = \frac{1}{m}\sum_{i=0}^{m-1} \mathbf{P}_{\ell,n}(S_{\ell+i}) = \frac{1}{m}\sum_{i=0}^{m-1} \mathbf{P}_{\ell}(S_{\ell+i} - S_{\ell-1})\,. \tag{13}$$

Then we show that the FCLT for $\big\{ \sum_{k=1}^{v_n(t)} X_{k,n}, t \in [0,1] \big\}$ is reduced to prove the FCLT for sums associated to a triangular array of martingale differences, namely for $\big\{ \sum_{k=1}^{v_n(t)} D_{k,n}^{m_n}, t \in [0,1] \big\}$, where (m_n) is a suitable subsequence.

Comment 6 Let us make some comments on the Lindeberg-type condition (9) which is commonly used to prove the CLT when we deal with dependent structures. We refer for instance to the papers by Neumann [28] or Rio [37] where this condition is also imposed and examples satisfying such a condition are provided. In addition, in many cases of interest, the considered triangular array takes the following form: $X_{k,n}/\sigma_n$ where $\sigma_n^2 = \mathrm{Var}(S_n)$ and then the first part of (9) reads as : there exists a positive constant C such that for any $n \ge 1$,

$$\sum_{k=1}^{n} \mathbb{E}(X_{k,n}^2) \le C\mathrm{Var}(S_n)\,, \tag{14}$$

which then imposed a certain growth of the variance of the partial sums. Let us give another example where this condition is satisfied. Assume that $X_i = f_i(Y_i)$ where Y_i is a Markov chain satisfying $\rho_Y(1) < 1$, then according to [29, Proposition 13], $C \leq (1+\rho_Y(1))(1-\rho_Y(1))^{-1}$. Here $(\rho_Y(k))_{k\geq 0}$ is the sequence of ρ-mixing coefficients of the Markov chain $(Y_i)_i$. On another hand, avoiding a condition as (14) is a big challenge and is one of the aims of the Hafouta's recent paper [17]. His main new idea is a linearization of the variance of the partial sums, which, to some extent, allows us to reduce the limit theorems to the case when $\mathrm{Var}(S_n)$ grows linearly fast in n. To give more insights, the partial sums are partitioned into blocks, so we write $S_n = \sum_{i=1}^{k_n} Y_{i,n}$, where k_n is of order $\mathrm{Var}(S_n)$ and the summands $Y_{i,n}$ are uniformly bounded in some $\mathbb{L}^p$ (see [17, section 1.4] for more details). Then the FCLT has to be obtained for the new triangular array $(Y_{i,n}, 1 \leq i \leq k_n)$.

To verify condition (11), one can use the following proposition proved in [25].

Proposition 7 *Assume that the Lindeberg-type condition* (9) *holds. Assume also that for any non-negative integer* ℓ*,*

$$\lim_{b\to\infty} \limsup_{n\to\infty} \sum_{k=b+1}^{n} \|\mathbb{E}_{k-b,n}(X_{k,n}X_{k+\ell,n}) - \mathbb{E}_{0,n}(X_{k,n}X_{k+\ell,n})\|_1 = 0$$

and, for any $t \in [0,1]$*,*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\Big| \sum_{k=1}^{v_n(t)} \big(\mathbb{E}_{0,n}(X_{k,n}^2) + 2\mathbb{E}_{0,n}(X_{k,n}\theta_{k,n}^m)\big) - tc^2 \Big| > \varepsilon \Big) = 0 \,. \qquad (15)$$

Then condition (11) is satisfied.

Starting from (12) and summing over ℓ, we get

$$\sum_{\ell=1}^{v_n(t)} (X_{\ell,n}^2 + 2X_{\ell,n}\theta_{\ell,n}^m) = \sum_{\ell=1}^{v_n(t)} (D_{\ell,n}^m)^2 + (\theta_{0,n}^m)^2 - (\theta_{v_n(t),n}^m)^2 + \sum_{\ell=1}^{v_n(t)} 2D_{\ell,n}^m(\theta_{\ell-1}^m + Y_{\ell-1,n}^m) + R_n \,,$$

where

$$R_n = \sum_{\ell=0}^{v_n(t)-1} (Y_{\ell,n}^m)^2 + 2 \sum_{k=0}^{v_n(t)-1} \theta_k^m Y_{k,n}^m \,.$$

Clearly

$$\sum_{\ell=1}^{v_n(t)} \mathbb{E}(X_{\ell,n}^2 + 2X_{\ell,n}\theta_{\ell,n}^m) = \sum_{\ell=1}^{v_n(t)} \mathbb{E}(D_{\ell,n}^m)^2 + \mathbb{E}(\theta_{0,n}^m)^2 - \mathbb{E}(\theta_{v_n(t),n}^m)^2 + \mathbb{E}(R_n) \,.$$

The Lindeberg's condition implies that $\mathbb{E}(\theta_{0,n}^m)^2 + \mathbb{E}(\theta_{v_n(t),n}^m)^2$ is tending to zero as $n \to \infty$, whereas

$$\mathbb{E}(R_n) \ll m^{-2}\big(A^2(m) + A(m)\sum_{i=1}^{m} A(i)\big) \,.$$

Hence, if we assume that $m^{-1}A^2(m) \to 0$ as $m \to \infty$, we derive

$$\lim_{m\to\infty} \limsup_{n\to\infty} \Big| \sum_{\ell=1}^{v_n(t)} \mathbb{E}(X_{\ell,n}^2 + 2X_{\ell,n}\theta_{\ell,n}^m) - \sum_{\ell=1}^{v_n(t)} \mathbb{E}(D_{\ell,n}^m)^2 \Big| = 0\,.$$

Note also that under the Lindeberg's condition and the following reinforced version of condition (10)

$$\lim_{m\to\infty} m^{-1/2}A(m) = 0 \text{ and } \lim_{m\to\infty} \sum_{\ell \geq [\log_2(m)]} B(2^\ell, m) = 0\,, \tag{16}$$

we have

$$\lim_{m\to\infty} \limsup_{n\to\infty} \Big\| \sum_{\ell=1}^{v_n(t)} X_{\ell,n} - \sum_{\ell=1}^{v_n(t)} D_{\ell,n}^m \Big\|_2 = 0\,.$$

Lemma 5.4 in [25] can be used to see this (note that in this lemma, there is a misprint in the statement since in the last term of the RHS of its inequality, the term $2^{-j/2}$ has to be deleted, as it can be clearly derived from their inequality (5.22)). Therefore, as soon as we consider $\mathcal{F}_{0,n} = \{\emptyset, \Omega\}$ (so $\mathbb{E}_{0,n}(\cdot) = \mathbb{E}(\cdot)$), condition (15) can be verified with the help of the following proposition.

Proposition 8 *Assume that the Lindeberg-type condition* (9) *holds and that* (16) *is satisfied. Assume also that there exists a constant* c^2 *such that, for any* $t \in [0,1]$,

$$\mathbb{E}(S_{v_n(t),n}^2) \to c^2 t\,. \tag{17}$$

Then

$$\lim_{m\to\infty} \limsup_{n\to\infty} \Big| \sum_{k=1}^{v_n(t)} \mathbb{E}\big(X_{k,n}^2 + 2X_{k,n}\theta_{k,n}^m\big) - tc^2 \Big| = 0\,.$$

3 Applications

3.1 Application to ρ-mixing triangular arrays

Theorem 5 gives the following result for ρ-mixing triangular arrays.

Let $\{X_{i,n}, 1 \leq i \leq n\}$ be a triangular array of square integrable centered real-valued random variables. Denote by $\sigma_{k,n}^2 = \mathrm{Var}\big(\sum_{\ell=1}^k X_{\ell,n}\big)$ and $\sigma_n^2 = \sigma_{n,n}^2$. For $0 \leq t \leq 1$, let

$$v_n(t) = \inf\Big\{k;\, 1 \leq k \leq n \colon \frac{\sigma_{k,n}^2}{\sigma_n^2} \geq t\Big\} \text{ and } W_n(t) = \sigma_n^{-1} \sum_{i=1}^{v_n(t)} X_{i,n}\,. \tag{18}$$

Assume that the triangular array is ρ-mixing in the sense that

$$\rho(k) = \sup_{n\geq 1} \max_{1\leq j\leq n-k} \rho\big(\sigma(X_{i,n}, 1 \leq i \leq j), \sigma(X_{i,n}, j+k \leq i \leq n)\big) \to 0$$

where $\rho(U,V) = \sup\{|corr(X,Y)| : X \in L^2(U), Y \in L^2(V)\}$.

The following is a FCLT for ρ-mixing triangular arrays:

Theorem 9 *Assume that*

$$\sup_{n\geq 1}\sigma_n^{-2}\sum_{j=1}^{n}\mathbb{E}(X_{j,n}^2)\leq C<\infty\,,$$

$$\lim_{n\to\infty}\sigma_n^{-2}\sum_{k=1}^{n}\mathbb{E}\{X_{k,n}^2 I(|X_{k,n}|>\varepsilon\sigma_n)\}=0\,,\ \ \textit{for any } \varepsilon>0$$

and

$$\sum_{k\geq 0}\rho(2^k)<\infty\,.$$

Then $\{W_n(t), t\in[0,1]\}$ *converges in distribution in* $D([0,1])$ *(equipped with the uniform topology) to* W.

This is the functional version of the CLT obtained by Utev [39]. It answers an open question raised by Ibragimov in 1991.

Theorem 9 follows from an application of Theorem 5 to the triangular array $\{\sigma_n^{-1}X_{k,n}, 1\leq k\leq n\}_{n\geq 1}$ and the σ-algebras $\mathcal{F}_{k,n}=\sigma(X_{i,n}, 1\leq i\leq k)$ for $k\geq 1$ and $\mathcal{F}_{k,n}=\{\emptyset,\Omega\}$ for $k\leq 0$.

In what follows, to soothe the notations, we omit the index n involved in the variables and in the σ-algebras.

To check condition (5), we used the fact that, by the definition of the ρ-mixing coefficient, for any $b>a\geq 0$,

$$\|\mathbb{E}_k(S_{k+b}-S_{k+a})\|_2\leq\rho(a)\|S_{k+b}-S_{k+a}\|_2\,,$$

and that, under $\sum_{k\geq 0}\rho(2^k)<\infty$, by the variance inequality of Utev [39], there exists κ such that for any integers a and b,

$$\|S_b-S_a\|_2^2\leq\kappa\sum_{i=a+1}^{b}\|X_i\|_2^2\,.$$

We then obtain

$$m^{-1}A^2(m)\ll\{\rho^2([\sqrt{m}])+m^{-1/2}\}\ \text{and}\ B(2^r,m)\ll\rho(2^r-1)\,.$$

Since $\rho(n)\to 0$, in order to prove condition (11), we use both Proposition 7 (by recalling that $\mathcal{F}_{0,n}$ is the trivial field $\{\emptyset,\Omega\}$) and Proposition 8. Therefore the proof of (11) is reduced to show that

$$\sigma_n^{-2}\mathbb{E}(S_{v_n(t)}^2)\to t\,,\ \ \text{as } n\to\infty,$$

which holds by the definition of $v_n(t)$ and the Lindeberg's condition (9).

For the $\rho-$mixing sequences we also obtain the following corollary:

Corollary 10 *Let* $(X_n)_{n\geq 1}$ *be a sequence of centered random variables in* $\mathbb{L}^2(\mathbb{P})$. *Let* $S_n=\sum_{k=1}^{n}X_k$ *and* $\sigma_n^2=\mathrm{Var}(S_n)$. *Suppose that the Lindeberg condition is satisfied and that* $\sum_{k\geq 0}\rho(2^k)<\infty$. *In addition assume that* $\sigma_n^2=nh(n)$ *where* h *is a slowly varying function at infinity. Then* $W_n=\{\sigma_n^{-1}\sum_{k=1}^{[nt]}X_k, t\in(0,1]\}$ *converges in distribution in* $D([0,1])$ *to* W *where* W *is a standard Brownian motion.*

If W_n converges weakly to a standard Brownian motion, then necessarily $\sigma_n^2 = nh(n)$ where $h(n)$ is a slowly varying function. If in Corollary 10 we assume that $\sigma_n^2 = n^\alpha h(n)$ where $\alpha > 0$, then one can prove that $W_n \Rightarrow \{G(t), t \in [0,1]\}$ in $D([0,1])$ where $G(t) = \sqrt{\alpha}\int_0^t u^{(\alpha-1)/2} dW(u)$.

In the strictly stationary case, condition $\sum_{k\geq 0} \rho(2^k) < \infty$ implies that $\sigma_n^2/n \to \sigma^2$ and if $\sigma_n^2 \to \infty$ then $\sigma > 0$. Therefore the functional limit theorem holds under the normalization $\sqrt{n}\sigma$. We then recover the FCLT obtained by Shao [38] (the CLT was first proved by Ibragimov [20]). In this context, condition $\sum_{k\geq 0} \rho(2^k) < \infty$ is minimal as provided by several examples by Bradley, which are discussed in [1, Chap. 34] .

Comment 11 In a recent paper, denoting by P_X the law of a random variable X and by G_a the normal distribution $N(0,a)$, Dedecker et al. [7] have proved quantitative estimates for the convergence of P_{S_n/σ_n} to G_1, where S_n is the partial sum associated with either martingale differences sequences or more general dependent sequences, and $\sigma_n^2 = \mathrm{Var}(S_n)$. In particular they considered the case of ρ-mixing sequences and, under reinforced conditions compared to those imposed in Theorem 9 or in Corollary 10, they obtained rates in the CLT. Let us describe their result. Let $(X_i)_{i\geq 1}$ be a sequence of centered ($\mathbb{E}(X_i) = 0$ for all i), real-valued bounded random variables, which are ρ-mixing in the sense that

$$\rho(k) = \sup_{j\geq 1} \sup_{v>u\geq j+k} \rho\big(\sigma(X_i, 1 \leq i \leq j), \sigma(X_u, X_v)\big) \to 0 \,, \text{ as } k \to \infty \,,$$

where $\sigma(X_t, t \in A)$ is the σ-field generated by the r.v.'s X_t with indices in A. Let us assume the following set of assumptions

$$(H) := \begin{cases} 1) \ \Theta = \sum_{k\geq 1} k\rho(k) < \infty \,. \\ 2) \text{ For any } n \geq 1, \ C_n := \max\limits_{1\leq \ell\leq n} \dfrac{\sum_{i=\ell}^n \mathbb{E}(X_i^2)}{\mathbb{E}(S_n - S_{\ell-1})^2} < \infty \,. \end{cases}$$

Denoting by $K_n = \max_{1\leq i\leq n} \|X_i\|_\infty$, they proved in their Section 4.2 that if K_n is uniformly bounded then, for any positive integer n,

$$\int_{\mathbb{R}} |F_n(t) - \Phi(t)| dt \ll C_n \sigma_n^{-1} \log(2 + C_n \sigma_n^2) \text{ and } \|F_n - \Phi\|_\infty \ll \sigma_n^{-1/2} \sqrt{C_n \log(2 + C_n \sigma_n^2)} \,,$$

where F_n is the c.d.f. of S_n/σ_n and Φ is the c.d.f. of a standard Gaussian r.v. We also refer to [17, Section 2.2] for related results concerning rates in the FCLT in terms of Prokhorov distance.

3.2 Application to functions of linear processes

Assume that

$$X_k = f_k\big(\sum_{i\geq 0} a_i \varepsilon_{k-i}\big) - \mathbb{E} f_k\big(\sum_{i\geq 0} a_i \varepsilon_{k-i}\big),$$

where $(\varepsilon_i)_{i\in\mathbb{Z}}$ are independent random variables such that $(\varepsilon_i^2)_{i\in\mathbb{Z}}$ is a uniformly integrable family and $\sup_{i\in\mathbb{Z}} \|\varepsilon_i\|_2 := \sigma$. The functions f_k are such that, for any k,

$$|f_k(x) - f_k(y)| \leq c(|x - y|) \text{ for any } (x, y) \in \mathbb{R}^2 \,,$$

where c is concave, non-decreasing and such that $\lim_{x\to 0} c(x) = 0$ (we shall say that $f_k \in \mathcal{L}(c)$).

Applying Theorem 5 with $X_{k,n} = X_k/\sigma_n$, we derive the following FCLT.

Corollary 12 *Assume that $\sigma_n^2 = nh(n)$ where $h(n)$ is a slowly varying function at infinity such that $\liminf_{n\to\infty} h(n) > 0$ and*

$$\sum_{k\geq 1} k^{-1/2} c\Big(2\sigma \sum_{i\geq k} |a_i|\Big) < \infty\,. \tag{19}$$

Then $\big\{\sigma_n^{-1}\sum_{k=1}^{[nt]} X_k, t\in[0,1]\big\}$ converges in distribution in $D([0,1])$ to a standard Brownian motion.

The detailed proof can be found in Section 5.6 of [25] but let us briefly describe the arguments allowing to verify conditions (7) and (11) with $v_n(t) = [nt]$ and $X_{k,n} = X_k/\sigma_n$ (recall that (7) implies (5), which in turn implies (10) since $\sigma_n^2 = nh(n)$ with $\liminf_{n\to\infty} h(n) > 0$).

We first consider the following choice of $(\mathcal{F}_i)_{i\geq 0}$: $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_i = \sigma(X_1,\dots,X_i)$, for $i \geq 1$. Denote by $\mathbb{E}_\varepsilon$ the expectation with respect to $\varepsilon := (\varepsilon_i)_{i\in\mathbb{Z}}$ and note that since $\mathcal{F}_i \subset \mathcal{F}_{\varepsilon,i}$ where $\mathcal{F}_{\varepsilon,i} = \sigma(\varepsilon_k, k\leq i)$, for any $i\geq 0$, $\|\mathbb{E}(X_{k+i}|\mathcal{F}_i)\|_2 \leq \|\mathbb{E}(X_{k+i}|\mathcal{F}_{\varepsilon,i})\|_2$. Next, for any $i \geq 0$, note that

$$\big|\mathbb{E}(X_{k+i}|\mathcal{F}_{\varepsilon,i})\big| = \Big|\mathbb{E}_\varepsilon\Big(f\Big(\sum_{\ell=0}^{k-1} a_\ell\varepsilon'_{k+i-\ell} + \sum_{\ell\geq k} a_\ell\varepsilon_{k+i-\ell}\Big)\Big) - \mathbb{E}_\varepsilon\Big(f\Big(\sum_{\ell=0}^{k-1} a_\ell\varepsilon'_{k+i-\ell} + \sum_{\ell\geq k} a_\ell\varepsilon'_{k+i-\ell}\Big)\Big)\Big|\,,$$

where $(\varepsilon'_i)_{i\in\mathbb{Z}}$ is an independent copy of $(\varepsilon_i)_{i\in\mathbb{Z}}$. Therefore, using [6, Lemma 5.1],

$$\|\mathbb{E}(X_{k+i}|\mathcal{F}_i)\|_2 \leq \Big\|c\Big(\sum_{\ell\geq k}|a_\ell||\varepsilon_{k+i-\ell} - \varepsilon'_{k+i-\ell}|\Big)\Big\|_2 \leq c\Big(2\sigma_\varepsilon \sum_{\ell\geq k}|a_\ell|\Big)\,,$$

proving that (7) holds under (19).

On another hand, to verify condition (11) with $v_n(t) = [nt]$ and $X_{k,n} = X_k/\sigma_n$, Proposition 7 can be used. Hence, because of the Lindeberg's condition and the choice of the filtration $(\mathcal{F}_i)_{i\geq 0}$ it is sufficient to prove

$$\lim_{b\to\infty}\limsup_{n\to\infty} \sigma_n^{-2} \sum_{k=b+1}^{n} \|\mathbb{E}_{k-b}(X_kX_{k+\ell}) - \mathbb{E}(X_kX_{k+\ell})\|_1 = 0 \tag{20}$$

and that, for any $t\in[0,1]$,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{\sigma_n^2}\Big|\sum_{k=1}^{[nt]}\Big\{\mathbb{E}(X_k^2) + 2\mathbb{E}(X_k\theta_k^m)\Big\} - t\Big| = 0\,. \tag{21}$$

Condition (20) can be proved by using similar arguments as those leading to (7). On another hand, (21) follows from an application of Proposition 8 with $v_n(t) = [nt]$ and $X_{k,n} = X_k/\sigma_n$. Indeed, the Lindeberg's condition can be verified, (7) is satisfied and it is also assumed that $\sigma_n^2 = nh(n)$ where $h(n)$ is a slowly varying function at infinity with $\liminf_{n\to\infty} h(n) > 0$.

3.3 Application to the quenched FCLT

We should also note that the general FCLT in Theorem 2 also leads as an application to the quenched FCLT under Maxwell-Woodroofe condition (previously proved by Cuny-Merlevède [2], with a completely different proof).

More precisely the result is the following:

Corollary 13 *Let $(X_k)_{k\in\mathbb{Z}}$ be an ergodic stationary sequence of $\mathbb{L}^2$ centered random variables, adapted to $(\mathcal{F}_k)$ and satisfying*

$$\sum_{k>0} k^{-3/2}\|\mathbb{E}_0(S_n)\|_2 < \infty.$$

then $\lim_{n\to\infty} n^{-1/2}\mathbb{E}(S_n^2) = c^2$ and, on a set of probability one, for any continuous and bounded function f from $(D([0,1),\|\cdot\|_\infty)$ to $\mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}_0(f(W_n)) = \int f(zc)W(dz)\,,$$

where $W_n = \{n^{-1}\sum_{k=1}^{[nt]} X_k, t\in[0,1]\}$ and W is the distribution of a standard Wiener process.

The idea of proof is to work under $\mathbb{P}_0$ (the conditional probability given $\mathcal{F}_0$) and verify that the conditions of our general FCLT hold with probability one. For instance, we need to verify (6), that is: with probability one, there exists a constant c^2 such that, for any $t\in[0,1]$

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}_0\Big(\Big|\frac{1}{n}\sum_{k=1}^{[nt]}\big(X_k^2+\frac{2}{m}X_k\sum_{i=1}^{m-1}\mathbb{E}_k(S_{k+i}-S_k)\big)-tc^2\Big|>\varepsilon\Big)=0\,.$$

But, by the ergodic theorem,

$$\lim_{m\to\infty}\lim_{n\to\infty}|\frac{1}{n}\sum_{k=1}^{[nt]}(X_k^2+\frac{2}{m}X_k\sum_{i=1}^{m-1}\mathbb{E}_k(S_{k+i}-S_k))-tc^2|=0 \ \ a.s.$$

Hence, by the properties of the conditional expectation, the desired convergence follows.

3.4 Application to locally stationary processes

Let consider $\{n^{-1/2}\sum_{k=1}^{[nt]}X_{k,n}, t\in[0,1]\}$ when $(X_{k,n}, 1\le k\le n)$ is a locally stationary process in the sense that $X_{k,n}$ can be locally approximated by a stationary process $\tilde{X}_k(u)$ in some neighborhood of u, i.e. for those k where $|(k/n)-u|$ is small.

Assume that $\mathbb{E}(X_{k,n})=0$. For each $u\in[0,1]$, let $\tilde{X}_k(u)$ be a stationary and ergodic process such that

(S_0) $\max_{1\le j\le n} n^{-1/2}\Big|\sum_{k=1}^{j}X_{k,n}-\sum_{k=1}^{j}\tilde{X}_k(k/n)\Big|\to^{\mathbb{P}} 0.$

(S_1) $\sup_{u\in[0,1]}\|\tilde{X}_k(u)\|_2<\infty$ and

$$\lim_{\varepsilon\to 0}\sup_{|u-v|\le\varepsilon}\|\tilde{X}_k(u)-\tilde{X}_k(v)\|_2=0.$$

(D) There exists a stationary non-decreasing filtration $(\mathcal{F}_k)_{k\geq 0}$ such that, for each $u \in [0,1]$, $\tilde{X}_k(u)$ is adapted to $\mathcal{F}_k$ and the following condition holds: $\sum_{k\geq 0} 2^{-k/2}\ \tilde{\delta}(2^k) < \infty$, where $\tilde{\delta}(k) = \sup_{u\in[0,1]} \|\mathbb{E}(\tilde{S}_k(u)|\mathcal{F}_0)\|_2$ and $\tilde{S}_k(u) = \sum_{i=1}^k \tilde{X}_i(u)$.

Let us give an example. For any $u \in [0,1]$, let

$$Y_k(u) = \sum_{i\geq 0} (\alpha(u))^i \varepsilon_{k-i} \text{ and } \tilde{X}_k(u) = f(Y_k(u)) - \mathbb{E}f(Y_k(u))$$

with $f \in \mathcal{L}(c)$ (this space of functions has been defined in subsection 3.2) and $\alpha(\cdot)$ a Lipschitz continuous function such that $\sup_{u\in[0,1]} |\alpha(u)| = \alpha < 1$.

Define

$$X_{k,n} = \tilde{X}_k(k/n) + n^{-3/2} u_n(\varepsilon_k + \cdots + \varepsilon_{k-n})$$

where $u_n \to 0$.

Condition (S_0) is satisfied and conditions (S_1) and (D) also, provided

$$\int_0^1 \frac{c(t)}{t\sqrt{|\log t|}} dt < \infty\,.$$

Theorem 14 *Assume the above conditions. Then there exists a Lebesgue integrable function* $\sigma^2(\cdot)$ *on* $[0,1]$ *such that, for any* $u \in [0,1]$, *where*

$$\lim_{m\to\infty} \mathbb{E}(\tilde{S}_m(u))^2 = \sigma^2(u)$$

and the sequence of processes $\{n^{-1/2} W_n(t),\ t \in [0,1]\}$ *converges in distribution in* $D([0,1])$ *to*

$$\Big\{ \int_0^t \sigma(u) dW(u),\ t \in [0,1] \Big\},$$

where W *is a standard Brownian motion.*

Compared to the results in Dahlhaus, Richter and Wu [4], this result has a different range of applications. In addition, we do not need to assume that $\| \sup_{u\in[0,1]} |\tilde{X}_k(u)| \|_2 < \infty$ nor that $\tilde{X}_k(u)$ takes the form $H(u, \eta_k)$ with H a measurable function and $\eta_k = (\varepsilon_j, j \leq k)$ where $(\varepsilon_j)_{j\in\mathbb{Z}}$ a sequence of iid real-valued random variables.

4 The case of α-dependent triangular arrays

We start this section by defining weak forms of strong-mixing-type coefficients for a triangular array of random variables $(X_{i,n})$. For any integer $i \geq 1$, let $f_{i,n}(t) = \mathbf{1}_{\{X_{i,n} \leq t\}} - \mathbb{P}(X_{i,n} \leq t)$. For any non-negative integer k, set

$$\alpha_{1,n}(k) = \sup_{i\geq 0} \max_{i+k\leq u} \sup_{t\in\mathbb{R}} \big\| \mathbb{E}\big(f_{u,n}(t)|\mathcal{F}_{i,n}\big) \big\|_1\,,$$

and

$$\alpha_{2,n}(k) = \sup_{i\geq 0} \max_{i+k\leq u\leq v} \sup_{s,t\in\mathbb{R}} \big\| \mathbb{E}\big(f_{u,n}(t) f_{v,n}(s)|\mathcal{F}_{i,n}\big) - \mathbb{E}\big(f_{u,n}(t) f_{v,n}(s)\big) \big\|_1\,,$$

where, for $i \geq 1$, $\mathcal{F}_{i,n} = \sigma(X_{j,n} 1 \leq j \leq i\}$ and $\mathcal{F}_{0,n} = \{\emptyset, \Omega\}$. In the definitions above we extend the triangular arrays by setting $X_{i,n} = 0$ if $i > n$. Assume that

$$\sigma_{n,n}^2 = \mathrm{Var}\Big(\sum_{\ell=1}^{n} X_{\ell,n}\Big) = 1\,, \tag{22}$$

and, for $0 \leq t \leq 1$, define $v_n(t)$ and $W_n(t)$ as in (18).

We shall now introduce two conditions that combine the tail distributions of the variables with their associated α-dependent coefficients:

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k=1}^{n} \sum_{i=m}^{n} \int_0^{\alpha_{1,n}(i)} Q_{k,n}^2(u) du = 0 \tag{23}$$

and

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{k=1}^{n} \int_0^{\alpha_{2,n}(m)} Q_{k,n}^2(u) du = 0\,, \tag{24}$$

where $Q_{k,n}$ is the quantile function of $X_{k,n}$ i.e., the inverse function of $t \mapsto \mathbb{P}(|X_{k,n}| > t)$.

Under the conditions above and using a similar martingale approximation approach as in the proof of Theorem 2, the following result holds (see [27]):

Theorem 15 *Suppose that (9),* (22)*,* (23) *and* (24) *hold. Then* $\big\{W_n(t), t \in [0,1]\big\}$ *converges in distribution in* $D([0,1])$ *(equipped with the uniform topology) to* W*, where* W *is a standard Brownian motion.*

Under the assumptions of Theorem 15, we then get that $\sum_{k=1}^{n} X_{k,n} \Rightarrow N(0,1)$. To see this, it suffices to notice that by (22), proving that $\big\|W_n(1) - \sum_{k=1}^{n} X_{k,n}\big\|_2 \to 0$ is reduced to prove that $\mathrm{Cov}(\sum_{k=1}^{v_n(1)} X_{k,n}, \sum_{k=1+v_n(1)}^{n} X_{k,n}) \to 0$ which follows from (23) by using Rio's covariance inequality [36] and taking into account the Lindeberg's condition.

Very often, for the sake of applications, it is convenient to express the conditions in terms of mixing rates and moments:

Corollary 16 *Assume that conditions (9) and (22) hold. Suppose in addition that, for some* $\delta \in (0, \infty]$,

$$\sup_n \sum_{k=1}^{n} \|X_{k,n}\|_{2+\delta}^2 < \infty \text{ and } \sum_{i\geq 1} i^{2/\delta} \alpha_1(i) < \infty$$

and that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \alpha_{2,n}(m) = 0\,.$$

Then the conclusion of Theorem 15 holds.

There are numerous counterexamples to the CLT, involving stationary strong mixing sequences, in papers by Davydov [5], Bradley [1], Doukhan et al. [9], Häggström [16] among others. We know that in the stationary case our conditions reduce to the minimal ones. These examples show that we cannot just assume that only the moments of order 2 are finite. Furthermore the mixing rate is minimal in some sense (see [9]).

We also would like to mention that a central limit theorem was obtained by Rio [37] which also implies the CLT in Corollary 16.

4.1 Application to functions of α-dependent Markov chains.

Let $Y_{i,n} = f_{i,n}(X_i)$ where $\mathbf{X} = (X_i)_{i\in\mathbb{Z}}$ is a stationary Markov process with Kernel operator K and invariant measure ν and, for each i and n, $f_{i,n}$ is such that $\nu(f_{i,n}) = 0$ and $\nu(f_{i,n}^2) < \infty$. Let $\sigma_n^2 = \mathrm{Var}(\sum_{i=1}^n Y_{i,n})$ and $X_{i,n} = \sigma_n^{-1} Y_{i,n}$. Note that the weak dependent coefficients $\alpha_1(i)$ of $\mathbf{X}$ can be rewritten as follows: Let BV_1 be the class of bounded variation functions h such that $|h|_v \leq 1$ (where $|h|_v$ is the total variation norm of the measure dh). Then

$$\alpha_1(i) = \frac{1}{2} \sup_{f\in BV_1} \nu\big(|K^i(f) - \nu(f)|\big)\,.$$

We mention that $\alpha_2(i)$ will have the same order of magnitude as $\alpha_1(i)$ if the space BV_1 is invariant under the iterates K^n of K, uniformly in n, i.e., there exists a positive constant C such that, for any function f in BV_1 and any $n \geq 1$,

$$|K^n(f)|_v \leq C|f|_v\,.$$

The Markov chains such that $\alpha_2(n) \to 0$, as $n \to \infty$, are not necessarily mixing in the sense of Rosenblatt.

Let us give an example. In what follows, for $\gamma \in]0,1[$, we consider the Markov chain $(X_k)_{k\geq 1}$ associated with the transformation T_γ defined from $[0,1]$ to $[0,1]$ by

$$T_\gamma(x) = \begin{cases} x(1+2^\gamma x^\gamma) & \text{if } x \in [0,1/2[\\ 2x-1 & \text{if } x \in [1/2,1]\,. \end{cases}$$

This is the so-called LSV [21] map with parameter γ. There exists a unique T_γ-invariant measure ν_γ on $[0,1]$, which is absolutely continuous with respect to the Lebesgue measure with positive density denoted by h_γ. We denote by K_γ the Perron-Frobenius operator of T_γ with respect to ν_γ (recall that for any bounded measurable functions f and g, $\nu_\gamma(f\cdot g\circ T_\gamma) = \nu_\gamma(K_\gamma(f)g)$). Then $(X_i)_{i\geq 0}$ will be the stationary Markov chain with transition Kernel K_γ and invariant measure ν_γ. In addition, we assume that, for any i and n fixed, $f_{i,n}$ is monotonic on some open interval and 0 elsewhere. It follows that the weak dependence coefficients associated with $(X_{i,n})$ are such that $\alpha_{2,n}(k) \leq Ck^{1-1/\gamma}$, where C is a positive constant not depending on n. By applying Corollary 16, we derive that if the triangular array $(X_{i,n})$ satisfies the Lindeberg condition (9) and if

$$\gamma \in (0,1/2) \text{ and } \sup_{n\geq 1} \frac{1}{\sigma_n^2} \sum_{i=1}^n \Big(\int_0^1 f_{i,n}^{2+\delta}(x) x^{-\gamma} dx\Big)^{2/(2+\delta)} < \infty \text{ for some } \delta > \frac{2\gamma}{1-2\gamma}\,,$$

then the conclusion of Theorem 15 is satisfied for the triangular array $(X_{i,n})$ defined above.

4.2 Application to linear statistics with $\alpha-$dependent innovations

We consider statistics of the type

$$S_n = \sum_{j=1}^{n} d_{n,j} X_j, \tag{25}$$

where $d_{n,j}$ are real valued weights and (X_j) is a strictly stationary sequence of centered real-valued random variables in $\mathbb{L}^2$. This model is also useful to analyze linear processes with dependent innovations and regression models. It was studied in Peligrad and Utev [30], Rio [37] and also in Peligrad and Utev [32], where a central limit theorem was obtained by using a stronger form of the mixing coefficients.

We assume that the sequence of constants satisfy the following two conditions:

$$\sum_{i=1}^{n} d_{n,i}^2 \to c^2 \text{ and } \sum_{i=1}^{n} (d_{n,j} - d_{n,j-1})^2 \to 0 \text{ as } n \to \infty\,, \tag{26}$$

where $c^2 > 0$. Also, we impose the conditions

$$\sum_{i \geq 0} \int_0^{\alpha_1(i)} Q^2(u) du < \infty \tag{27}$$

and

$$\alpha_2(m) \to 0, \tag{28}$$

where Q is the quantile function assciated with X_0.

Condition (27) implies that $\sum_{k \geq 0} |\mathrm{Cov}(X_0, X_k)| < \infty$ and therefore that the sequence (X_j) has a continuous spectral density $f(x)$. Note also that if the spectral density f is continuous and (26) is satisfied then

$$\sigma_n^2 = \mathrm{Var}(S_n) \to 2\pi c^2 f(0), \text{ as } n \to \infty\,.$$

We refer for instance to [26, Lemma 1.5] for a proof of this fact. Note also that (26) implies the Lindeberg condition (4). Indeed, condition (26) entails that $\max_{1 \leq \ell \leq n} |d_{n,\ell}| \to 0$, as $n \to \infty$ (see [26, Lemma 12.12]).

By applying Theorem 15 we obtain the following result (see Merlevède-Peligrad [27]).

Theorem 17 *Let $S_n = \sum_{j=1}^{n} d_{n,j} X_j$, where $d_{n,j}$ are real valued weights and (X_j) is a strictly stationary sequence. Assume that (26), (27) and (28) are satisfied. Then S_n converges in distribution to $\sqrt{2\pi f(0)}|c|N$ where N is a standard Gaussian random variable. Let $v_{k,n}^2 = \sum_{i=1}^{k} d_{n,i}^2$. Define*

$$v_n(t) = \inf\left\{k;\, 1 \leq k \leq n:\; v_{k,n}^2 \geq c^2 t\right\} \text{ and } W_n(t) = \sum_{i=1}^{v_n(t)} d_{n,i} X_i\,.$$

Then $W_n(\cdot)$ converges weakly to $\sqrt{2\pi f(0)}|c|W$ where W is the standard Brownian motion.

Comment 18 To apply Theorem 15, we do not need to impose condition (26) in its full generality. Indeed, this condition can be replaced by the following ones

$$\sum_{i=1}^{n} d_{n,i}^2 \to c^2 \text{ and } \max_{1\le \ell \le n} |d_{n,\ell}| \to 0 \text{ as } n \to \infty , \tag{29}$$

and, for any positive k, there exists a constant c_k such that

$$\lim_{n\to\infty} \frac{\sum_{\ell=1}^{n-k} d_{n,\ell} d_{n,\ell+k}}{\sum_{\ell=1}^{n} d_{n,\ell}^2} \to c_k . \tag{30}$$

Indeed condition (29) implies the Lindeberg condition (4) whereas condition (30) together with $\sum_{k\ge 0} |\mathrm{Cov}(X_0, X_k)| < \infty$ (which is, in particular, implied by (27)) entail that

$$\frac{\sigma_n^2}{\sum_{\ell=1}^{n} d_{n,\ell}^2} \to \sigma^2 = \mathrm{Var}(X_0) + 2 \sum_{k\ge 1} c_k \mathrm{Cov}(X_0, X_k) , \text{ as } n \to \infty . \tag{31}$$

Note that if condition (26) holds then (30) is satisfied with $c_k = 1$ for all positive integer k and therefore $\sigma^2 = 2\pi f(0)$. Hence, if in the statement of Theorem 17, condition (26) is replaced by conditions (29) and (30) then, its conclusions hold with σ^2 replacing $2\pi f(0)$, where σ^2 is defined in (31). To end this comment, let us give an example where conditions (29) and (30) are satisfied but the second part of (26) fails. With this aim, let x be a real such that $x \notin \pi\mathbb{Z}$ and let $d_{n,k} = \sin(xk)/\sqrt{n}$. For this choice of triangular array we have $\sum_{i=1}^{n} d_{n,i}^2 \to 1/2$ and, for any positive k, $\sum_{\ell=1}^{n-k} d_{n,\ell} d_{n,\ell+k} \to 2^{-1}\cos(xk)$. Therefore (30) is satisfied with $c_k = \cos(xk)$ and (26) does not hold.

Remark 19 *We refer to Dedecker et al. [7, Section 4] for various results concerning rates of convergence in the central limit theorem for linear statistics of the above type with dependent innovations. In particular, they proved the following result (see their corollary 4.1 and their remark 4.2). Let* $p \in (2, 3]$. *Assume that*

$$\mathbb{P}(|X_0| \ge t) \le C t^{-s} \text{ for some } s > p \text{ and } \sum_{k\ge 1} k(\alpha_2(k))^{2/p-2/s} < \infty ,$$

and that the spectral density of (X_i) *satisfies* $\inf_{t\in[-\pi,\pi]} |f(t)| = m > 0$. *Then, setting* $m_n = \max_{1\le \ell\le n} |d_{n,\ell}|$, *the following upper bounds holds: for any positive integer* n,

$$\int_{\mathbb{R}} |F_n(t) - \Phi(t)| dt \ll C(n,p) := \begin{cases} \dfrac{m_n^{p-2}}{\sigma_n} \Big(\displaystyle\sum_{\ell=1}^{n} d_{n,\ell}^2 \Big)^{(3-p)/2} & \text{if } p \in (2,3) \\ \dfrac{m_n}{\sigma_n} \log \Big(m_n^{-1} \displaystyle\sum_{\ell=1}^{n} d_{n,\ell}^2 \Big) & \text{if } p = 3, \end{cases} \tag{32}$$

where we recall that F_n *is the c.d.f. of* S_n/σ_n *and* Φ *is the c.d.f. of a standard Gaussian r.v. Note that if we replace the condition that the spectral density has to be bounded away from 0 by the weaker one:* $f(0) > 0$, *and if, as a counterpart, we assume the additional condition*

$\sum_{k>0} k^2 |\mathrm{Cov}(X_0, X_k)| < \infty$, *then an additional term appears in* (32)*; namely, we get*

$$\int_{\mathbb{R}} |F_n(t) - \Phi(t)| dt \ll C(n,p) + \frac{\left(\sum_{k=1}^{n+1} (d_{n,k} - d_{n,k-1})^2 \right)^{1/2}}{\sigma_n}.$$

See [7, Corollary 4.2].

In what follows, we apply Theorem 17 to the model of the nonlinear regression with fixed design. Our goal is to estimate the function $\ell(x)$ such that

$$y(x) = \ell(x) + \xi(x),$$

where ℓ is an unknown function and $\xi(x)$ is the noise. If we fix the design points $x_{n,i}$ we get

$$Y_{n,i} = y(x_{n,i}) = \ell(x_{n,i}) + \xi_i(x_{n,i}).$$

According to [32], the nonparametric estimator of $\ell(x)$ is defined to be

$$\hat{\ell}_n(x) = \sum_{i=1}^{n} w_{n,i}(x) Y_{n,i}, \tag{33}$$

where

$$w_{n,i}(x) = K\Big(\frac{x_{n,i} - x}{h_n}\Big) / \sum_{i=1}^{n} K\Big(\frac{x_{n,i} - x}{h_n}\Big).$$

We apply Theorem 17 to find sufficient conditions for the convergence of the estimator $\hat{\ell}_n(x)$. To fix the ideas we shall consider the following setting: The kernel K is a density function, continuous with compact support $[0,1]$. The design points will be $x_{n,i} = i/n$ and $(\xi_i(x_{n,1}), \ldots, \xi_i(x_{n,i}))$ is distributed as $(X_1, \ldots, X_n)$, where $(X_k)_{k\in\mathbb{Z}}$ is a stationary sequence of centered sequence of random variables satisfying (27) and (28). We then derive the normal asymptotic limit for

$$V_n(x) = \left(\sum_{i=1}^{n} w_{n,i}^2(x) \right)^{-1/2} \left(\hat{\ell}_n(x) - \mathbb{E}(\hat{\ell}_n(x)) \right).$$

The following theorem was established in Merlevède-Peligrad [27].

Theorem 20 *Assume for x fixed that $\hat{\ell}_n(x)$ in defined by (33) and the sequence (X_j) is a stationary sequence satisfying (27) and (28). Assume that the kernel K is a density, it is square integrable, has compact support and is continuous. Assume $nh_n \to \infty$ and $h_n \to 0$. Then $\sqrt{nh_n}(\hat{\ell}_n(x) - \mathbb{E}(\hat{\ell}_n(x)))$ converges in distribution to $\sqrt{2\pi f(0)}|c|N$ where N is a standard Gaussian random variable and c^2 is the second moment of K.*

4.3 Application to functions of a triangular stationary Markov chain

Let us consider a triangular version of the Markov chain defined in Example 3.

For any positive integer n, $(\xi_{i,n})_{i\geq 0}$ is an homogeneous Markov chain with state space $\mathbb{N}$ and transition probabilities given by

$$\mathbb{P}(\xi_{1,n} = i | \xi_{0,n} = i+1) = 1 \text{ and } \mathbb{P}(\xi_{1,n} = i | \xi_{0,n} = 0) = p_{i+1,n} \text{ for } i \geq 1,$$

where, for $i \geq 2$, $p_{i,n} = c_a/(v_n i^{a+2})$ with $a > 0$, $c_a \sum_{i\geq 2} 1/i^{a+2} = 1/2$, $(v_n)_{n\geq 1}$ a sequence of positive reals and $p_{1,n} = 1 - 1/(2v_n)$. $(\xi_{i,n})_{i\geq 0}$ has a stationary distribution $\pi_n = (\pi_{j,n})_{j\geq 0}$ satisfying

$$\pi_{0,n} = \Big(\sum_{i\geq 1} i p_{i,n}\Big)^{-1} \text{ and } \pi_{j,n} = \pi_{0,n} \sum_{i\geq j+1} p_{i,n} \text{ for } j \geq 1.$$

Let $Y_{i,n} = I_{\xi_{i,n}=0} - \pi_{0,n}$. Let $b_n^2 = \mathrm{Var}\Big(\sum_{k=1}^n Y_{k,n}\Big)$ and set $X_{i,n} = Y_{i,n}/b_n$. Provided that $a > 1$ and $v_n/n \to 0$, $(X_{k,n})_{k>0}$ satisfies the functional central limit theorem given in Theorem 15.

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