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THE JORDAN–CHEVALLEY DECOMPOSITION FOR G-BUNDLES ON ELLIPTIC CURVES

DRAGOŞ FRĂŢILĂ, SAM GUNNINGHAM, AND PENGHUI LI

ABSTRACT. We study the moduli stack of degree 0 semistable G-bundles on an irreducible curve E of arithmetic genus 1, where G is a connected reductive group in arbitrary characteristic. Our main result describes a partition of this stack indexed by a certain family of connected reductive subgroups H of G (the E-pseudo-Levi subgroups), where each stratum is computed in terms of H-bundles together with the action of the relative Weyl group. We show that this result is equivalent to a Jordan–Chevalley theorem for such bundles equipped with a framing at a fixed basepoint. In the case where E has a single cusp (respectively, node), this gives a new proof of the Jordan–Chevalley theorem for the Lie algebra $\mathfrak{g}$ (respectively, algebraic group G).

We also provide a Tannakian description of these moduli stacks and use it to show that if E is not a supersingular elliptic curve, the moduli of framed unipotent bundles on E are equivariantly isomorphic to the unipotent cone in G. Finally, we classify the E-pseudo-Levi subgroups using the Borel–de Siebenthal algorithm, and compute some explicit examples.

CONTENTS

1. INTRODUCTION

1.1. **Overview and main results.** Fix an irreducible projective curve E of arithmetic genus 1 over an algebraically closed field k. There are three possibilities:

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(1) E has a single ordinary cusp;
(2) E has a single node;
(3) E is smooth.

We also fix a basepoint x_0 in the smooth locus of E. We will refer to these cases as the *cuspidal*, *nodal*, or *elliptic* cases respectively.[1]

Fix also a connected reductive group G over k. We consider the moduli stack

$$\underline{G}_E := \mathrm{Bun}_G^{0,\mathsf{ss}}(E)$$

of degree 0, semistable G-bundles over E. We denote by G_E the moduli stack of such bundles $\mathcal{P}$ together with a framing - that is, a trivialization of the fiber $\mathcal{P}_{x_0}$.

We will see that G_E is a smooth algebraic variety with an action of G (changing the trivialization), and that $\underline{G}_E = G_E/G$. In fact, by results of Friedman–Morgan [FM01], we have:

- $G_E \cong \mathfrak{g} = \mathrm{Lie}(G)$ if E is cuspidal, and
- $G_E \cong G$ if E is nodal.

In these cases, the action of G on G_E corresponds to the adjoint/conjugation action. In the elliptic case, we will see that the stack $\underline{G}_E$ shares many of the properties of the adjoint quotient stacks $\mathfrak{g}/G$ and G/G.

1.1.1. *The Jordan–Chevalley decomposition.* One of our main results is a form of the Jordan–Chevalley decomposition for G_E, which recovers the usual Jordan–Chevalley decomposition for $\mathfrak{g}$ and G in the cuspidal and nodal cases respectively.

An element $p \in G_E$ is called *semisimple* if its G-orbit is closed. We say that an element $p \in G_E$ is *unipotent* if its orbit closure contains the trivial bundle p_0. We denote by G_E^{uni} the subvariety of unipotent elements of G_E and by $\underline{G}_E^{\mathsf{uni}}$ the corresponding substack in $\underline{G}_E$. We note that these definitions recover the usual notion of semisimple and unipotent/nilpotent elements in the nodal/cuspidal cases.

The Jordan–Chevalley decomposition essentially states that every framed bundle $p \in G_E$ can be uniquely decomposed as $p = p_s \cdot p_u$ where p_s is semisimple, p_u is unipotent, and p_s commutes with p_u. As it stands this statement is not well formed as it is not clear what it means to "multiply" elements in G_E (the usual statement of Jordan decomposition in the nodal and cuspidal cases uses multiplication in G and addition in $\mathfrak{g}$). However, it does make sense to multiply a $Z(G)$-bundle $\mathcal{P}'$ and a G-bundle $\mathcal{P}$: we define $\mathcal{P}' \cdot \mathcal{P}$ to be the bundle induced from the external product $\mathcal{P}' \times_E \mathcal{P}$ via the multiplication map $Z(G) \times G \to G$ (which is a group homomorphism). Moreover, this construction is naturally compatible with framings, giving rise to an abelian algebraic group structure on $Z(G)_E$ and an action of $Z(G)_E$ on G_E.

We will also show that for a reductive subgroup H of G, the induction map on framed semistable bundles $H_E \to G_E$ is a closed embedding (see Proposition 2.19). Therefore we can identify H_E with the corresponding closed subvariety of G_E. We are now ready to state the first result, a form of Jordan–Chevalley decomposition:

Theorem 1.1 (See Theorem 4.8). *Given $p \in G_E$, there is a unique triple (H, p_s, p_u), where H is a connected reductive subgroup of G, $p_s \in Z(H)_E$ with* $\mathrm{Stab}_G(p_s)^\circ = H$*, $p_u \in H_E^{\mathsf{uni}}$, and $p = p_s \cdot p_u$.*

The subgroups of G which occur as connected stabilizers of semisimple elements of G_E will be called *E-pseudo-Levi subgroups.* In the cuspidal case, these are

[1]Note that in the first two cases, unlike the third, the pair (E, x_0) has no moduli.

precisely the Levi subgroups (centralizers of semisimple elements of $\mathfrak{g}$) and in the nodal case, these are pseudo-Levi subgroups (connected centralizers of semisimple elements of G). We give a classification of E-pseudo-Levi subgroups in Appendix A: in the elliptic case we get precisely the intersections of two pseudo-Levi subgroups.

Remark 1.2. For simply connected groups over $\mathbb{C}$, a similar Jordan–Chevalley decomposition was proved in [BEG03, Theorem 5.6] using an algebraic uniformization of $\underline{G}_E$ through loop groups (see [BG96]).

1.1.2. *The partition according to E-pseudo-Levis.* Typically, in the statements of the Jordan–Chevalley decomposition for G and for $\mathfrak{g}$, the subgroup H is not explicitly mentioned, though it may be easily recovered as the connected centralizer of the semisimple element.

However, in our proof of Theorem 1.1, the subgroup H will be the key player and the decomposition into semisimple and unipotent elements will play a subsidiary role. In fact, the semisimple and unipotent elements may be recovered from the subgroup H in the following sense.

Given a subgroup H of G we define the G-regular loci

$$(Z(H)_E)^{\mathsf{reg}} = \{p \in Z(H)_E \mid \mathrm{Stab}_G(p)^\circ = H\} \subseteq Z(H)_E$$

and

$$(H_E)^{\mathsf{reg}}_{\heartsuit} = (Z(H)_E)^{\mathsf{reg}} \cdot H_E^{\mathsf{uni}} \subseteq H_E.$$

The subgroup H is an E-pseudo-Levi precisely when $(Z(H)_E)^{\mathsf{reg}}$ is non-empty. Note that the conditions in Theorem 1.1 are precisely stipulating that $p \in (H_E)^{\mathsf{reg}}_{\heartsuit}$. On the other hand, any element $p \in (H_E)^{\mathsf{reg}}_{\heartsuit}$ may be uniquely written as a product $p_s \cdot p_u$ where $p_s \in (Z(H)_E)^{\mathsf{reg}}$ and $p_u \in H_E^{\mathsf{uni}}$ (see Proposition 4.10). We denote by $(G_E)_H$ the image of $(H_E)^{\mathsf{reg}}_{\heartsuit}$ in G_E.

In other words, Theorem 1.1 states that every element $p \in G_E$ lies in the image of $(H_E)^{\mathsf{reg}}_{\heartsuit}$ for a unique E-pseudo-Levi subgroup H. Thus we may rephrase Theorem 1.1 as follows:

Theorem 1.3. *There is a locally closed partition:*

$$G_E = \bigsqcup_H (G_E)_H$$

indexed by E-pseudo-Levi subgroups $H \subseteq G$. Moreover, the natural embeddings $H_E \to G_E$ restrict to isomorphisms

$$(H_E)^{\mathsf{reg}}_{\heartsuit} \xrightarrow{\sim} (G_E)_H.$$

We will reformulate this result in one final way (in the form that will actually be proved in Section 4). Recall that $\underline{G}_E$ denotes the quotient stack G_E/G, and write $(\underline{H}_E)^{\mathsf{reg}}_{\heartsuit}$ for $(H_E)^{\mathsf{reg}}_{\heartsuit}/H$. We write $W_{G,H}$ for the relative Weyl group $N_G(H)/H$ (a finite group if H is an E-pseudo- Levi subgroup of G).

Theorem 1.4. *The stack $\underline{G}_E$ carries a locally closed partition*

$$\underline{G}_E = \bigsqcup_{[H]} (\underline{G}_E)_{[H]}$$

indexed by conjugacy classes $[H]$ of E-pseudo-Levi subgroups $H \subseteq G$. Moreover, the natural induction maps $\underline{H}_E \to \underline{G}_E$ restrict to equivalences

$$(\underline{H}_E)^{\mathsf{reg}}_{\heartsuit}/W_{G,H} \xrightarrow{\sim} (\underline{G}_E)_{[H]}.$$

Our proof of Theorem 1.4 (and hence of Theorem 1.1 and Theorem 1.3) involves a geometric analysis of the induction map $\underline{H}_E \to \underline{G}_E$ (see Section 4.2).

Remark 1.5. A key difference between the statements of Theorem 1.4 and Theorem 1.3 is that, unlike the subvarieties $(G_E)_H$ in G_E, the substacks $(\underline{G}_E)_{[H]}$ have an a priori definition that does not make reference to the induction map $\underline{H}_E \to \underline{G}_E$. More precisely, given a G-bundle $\mathcal{P} \in \underline{G}_E$, pick a framed lift $p \in G_E$ and then choose any semisimple bundle in the closure of its G-orbit. The underlying G-bundle defines a conjugacy class $[H]$ that is independent of the chosen framing and is hence canonically associated to $\mathcal{P}$. The content of Theorem 1.3 is that, in the presence of a framing, there is a canonical choice of subgroup H within its conjugacy class, and (equivalently) a canonical choice of semisimple element p_s in the orbit closure of p.

1.1.3. *Unipotent bundles.* Theorem 1.4 and Theorem 1.1 allow us to reduce the study of degree 0, semistable G-bundles on E to semisimple and unipotent bundles.

Our next result shows that, under certain hypotheses, the collection of unipotent bundles in G_E is insensitive to the isomorphism type of E. We let $J(E)$ denote the Jacobian of E, which is either isomorphic to $\mathbb{G}_a$, $\mathbb{G}_m$, or E itself in the cuspidal, nodal, or elliptic cases respectively. We denote by G^{uni} the unipotent cone of G (which is the same as G_E^{uni} for E a nodal curve).

Theorem 1.6. *An isomorphism of formal group $\hat{J}(E) \simeq \hat{\mathbb{G}}_m$ induces an isomorphism of G-varieties*

$$G_E^{\mathsf{uni}} \cong G^{\mathsf{uni}}.$$

Moreover this isomorphism extends over a formal neighbourhood of G_E^{uni} in G_E:

$$(G_E)^\wedge_{\mathsf{uni}} \cong G^\wedge_{\mathsf{uni}}.$$

Remark 1.7. In characteristic zero, there is always an isomorphism $\hat{J}(E) \simeq \hat{\mathbb{G}}_m$. In fact, there is also an isomorphism $\hat{J}(E) \simeq \hat{\mathbb{G}}_a$ which gives the same result but with the unipotent cone in G replaced by the nilpotent cone $\mathcal{N}$ in $\mathfrak{g}$.

In characteristic $p > 0$, if E is an elliptic curve, an isomorphism $\hat{J}(E) \simeq \hat{\mathbb{G}}_m$ exists precisely when E is ordinary (i.e. not supersingular).

Remark 1.8. As a special case of Theorem 1.6 we recover a G-equivariant isomorphism $\mathcal{N} \cong \mathcal{U}$ between the unipotent and nilpotent cones for each isomorphism $\hat{\mathbb{G}}_a \cong \hat{\mathbb{G}}_m$. The latter isomorphisms exist only in characteristic zero, and are given by exponential maps. On the other hand, there exist G-equivariant isomorphisms (the so-called Springer isomorphisms) $\mathcal{N} \cong \mathcal{U}$ under very mild conditions on the characteristic, even though $\hat{\mathbb{G}}_a \not\cong \hat{\mathbb{G}}_m$ in positive characteristic. It seems reasonable to expect that G_E^{uni} is isomorphic to $\mathcal{U}$ (and $\mathcal{N}$) under much more general conditions than in Theorem 1.6. From [GSB19, Theorem 3.11] one can deduce that the varieties G_E^{uni} and G^{uni} are smoothly equivalent for uniformizable elliptic curves under some restrictions on G and the characteristic. See also [GSB19, Corollary 8.8] where they show it fails for $G = E_8$, E supersingular in characteristic 2, 3 or 5.

1.1.4. *Semisimple bundles and the classification of E-pseudo-Levi subgroups.* Fix a maximal torus T of G and let $\Phi \subset \mathbb{X}^*$ denote the corresponding space of roots sitting inside the character lattice $\mathbb{X}^* = \mathbb{X}^*(T)$. Then

$$T_E \cong \mathrm{Hom}_{gp}(\mathbb{X}^*, J(E)) \cong J(E)^r,$$

where r is the rank of T.

Note that a G-bundle $\mathcal{P} \in \underline{G}_E$ is semisimple if and only if it admits a reduction to T. We may understand the partition into E-pseudo-Levi subgroups root theoretically as follows.

First note that any character $\alpha \in \mathbb{X}^*$ defines a homomorphism $\alpha_* : T_E \to J(E)$, taking a T-bundle on E to its induced line bundle via α. Given $p \in T_E$, we let $\Sigma_p = \{\alpha \in \Phi \mid \alpha_*(p) = \mathbf{1}_{J(E)}\}$. The subsets Σ of Φ which occur in this way will be called E-root subsystems of Φ.

It turns out that any such E-root subsystem $\Sigma \subset \Phi$ is a closed root subsystem and so it corresponds to a connected reductive subgroup H of G (see also Section 3.2 for Borel–de Siebenthal theory). In fact, we have $H = \mathrm{Stab}_G(p)^\circ$ and

Proposition 1.9. *There is a one-to-one correspondence between:*

- *E-pseudo-Levi subgroups H of G containing T, and*
- *E-root subsystems $\Sigma \subseteq \Phi$.*

Moreover, in each of the cases cuspidal, nodal and elliptic one can characterize precisely the E-root subsystems of Φ (see Appendix A *and Proposition* A.8 *for the elliptic case).*

Remark 1.10. The theory of semisimple bundles becomes increasingly complicated as one passes from the cuspidal to the nodal and then to the elliptic cases. For example, the centralizer of a semisimple element of $\mathfrak{g}$ is a Levi subgroup, and in particular connected. The centralizer of a semisimple element in G is connected (but not necessarily simply-connected) whenever G is simply connected. On the other hand, the automorphism group of a semisimple G-bundle $\mathcal{P} \in \underline{G}_E$ where E is smooth may be disconnected, even if G is simply connected! An example is given for G of type D_4, see Example 6.6 (this example also appears in [BEG03, p. 18]).

Fortunately, our result provides control over the component groups of automorphisms of semisimple bundles in terms of Weyl group combinatorics (just as Lusztig's stratification does in the group case). More precisely, let $p \in T_E$ with $\Sigma_p = \Sigma$ for some E-root subsystem Σ of Φ, and let $\mathcal{P}_G$ be the induced G-bundle. Then Theorem 1.4 implies that

$$\pi_0 \operatorname{Aut}(\mathcal{P}_G) \cong \mathrm{Stab}_{N_W(\Sigma)}(p)/W_\Sigma,$$

where $N_W(\Sigma) = \{w \in W \mid w(\Sigma) = \Sigma\}$.

1.1.5. *The Lusztig stratification.* Putting these results together, we can refine the partition of G_E in Theorem 1.4 as follows.

Corollary 1.11. *Suppose either that char$(k) = 0$ or that E is ordinary. There is a stratification*

$$\underline{G}_E = \bigsqcup_{[H,\mathcal{O}]} (\underline{G}_E)_{[H,\mathcal{O}]}$$

indexed by G-conjugacy classes of pairs $(H, \mathcal{O})$ where H is an E-pseudo-Levi subgroup of G, and $\mathcal{O} \subseteq H_E^{\mathsf{uni}}$ is a unipotent H-orbit. For each such pair $[H, \mathcal{O}]$ we have an isomorphism:

$$(\underline{G}_E)_{[H,\mathcal{O}]} \cong (Z(H)_E^{\mathsf{reg}} \times \mathcal{O})/N_G(H) \cong (Z(H)_E^{\mathsf{reg}} \times \underline{\mathcal{O}})/W_{G,H}.$$

Remark 1.12. To make the comparison with Lusztig's work more evident, note that to each E-pseudo-Levi subgroup H, one can associate a Levi $L = L_H = C_G(Z(H)^\circ)$.

This is the smallest Levi subgroup which contains H. A bundle $\mathcal{P} \in \underline{G}_E$ is called *isolated* if it is contained in $(\underline{G}_E)_{[H]}$ and H is not contained in any proper Levi. (It follows from Proposition 1.9 that there are finitely many isomorphism classes of isolated bundles.) Instead of parameterizing the strata using subgroups of elliptic type and unipotent bundles, one may use Levi subgroups and isolated bundles. This is how Lusztig describes the stratification of G in [Lus84].

1.2. **Motivation: The geometric Langlands program and elliptic Springer theory.**

1.2.1. *Global geometric Langlands.* Recall that in the global geometric Langlands program, one aims to describe the derived category $\mathcal{D}(\mathrm{Bun}_G(C))$ of D-modules or constructible sheaves on the moduli stack of G-bundles on a smooth projective curve C in terms of the derived category of quasi-coherent (or more general Ind-coherent) sheaves on the moduli stack of ${}^LG$-local systems, where ${}^LG$ is the Langlands dual group to G.

Recall that for every parabolic subgroup $P \subseteq G$ with Levi factor L, we have *Eisenstein* and *constant term* functors:

$$\mathrm{Eis}^G_{L,P} : \mathcal{D}(\mathrm{Bun}_L(C)) \leftrightarrows \mathcal{D}(\mathrm{Bun}_G(C)) : \mathrm{CT}^G_{L,P}\,.$$

Generally speaking, these functors are defined via a pull-push construction involving the diagram[2]:

(1.1)
$$\begin{tikzcd} & \mathrm{Bun}_P(C) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Bun}_L(C) & & \mathrm{Bun}_G(C) \end{tikzcd}$$

An object of $\mathcal{D}(\mathrm{Bun}_G(C))$ is called *cuspidal* if it is in the kernel of the constant term functor for every proper parabolic subgroup P of G. As per Harish-Chandra's philosophy of cusp forms, one may think of the category $\mathcal{D}(\mathrm{Bun}_G(C))$ as built up from Eisenstein series of cuspidal objects in $\mathcal{D}(\mathrm{Bun}_L(C))$ as L ranges over Levi subgroups of G. In this way one can hope to understand the category $\mathcal{D}(\mathrm{Bun}_G(C))$ inductively in terms of smaller reductive groups.

Typically, the Eisenstein series from different Levi subgroups will interact in a complicated way, and this does not lead to a straightforward description of $\mathcal{D}(\mathrm{Bun}_G(C))$. We will now describe a closely related category for which the Harish-Chandra approach yields a complete description.

1.2.2. *Springer theory and character sheaves.* Let us turn to the category $\mathcal{D}(G/G)$ of "class sheaves" on G, i.e. conjugation equivariant constructible sheaves or D-modules on the group G. Analogous to the Eisenstein and constant term functors above, one has functors of parabolic induction and restriction:

$$\mathrm{Ind}^G_{P,L} : \mathcal{D}(L/L) \leftrightarrows \mathcal{D}(G/G) : \mathrm{Res}^G_{P,L}$$

defined by pull-push along the diagram:

[2]More precisely, one must consider a fiberwise compactification of $\mathrm{Bun}_P(C)$ over $\mathrm{Bun}_G(C)$. One must also specify which functors (star or shriek) are employed for this process. We will ignore these distinctions for the purposes of this informal discussion.

$$
\begin{tikzcd}
& P/P \arrow[dl] \arrow[dr] & \\
L/L & & G/G.
\end{tikzcd}
\tag{1.2}
$$

Just as in Section 1.2.1 one defines the cuspidal objects as those whose parabolic restriction to any proper Levi is zero. Again, one aims to describe the category $\mathcal{D}(G/G)$ in terms of parabolic induction from cuspidal objects in $\mathcal{D}(L/L)$ as L ranges over Levi subgroups.

In his seminal paper [Lus84], Lusztig obtained a block decomposition of the category of equivariant perverse sheaves on the unipotent cone $G^{\mathsf{uni}} \subseteq G$:

$$
\mathrm{Perv}_G(G^{\mathsf{uni}}) = \bigoplus_{(L,\mathcal{O},\mathcal{E})} \mathrm{Rep}(W_{G,L}). \tag{1.3}
$$

Here, the blocks are indexed by *cuspidal data*, consisting of a pair of a Levi subgroup L together with a simple cuspidal local system $\mathcal{E}$ on the unipotent orbit $\mathcal{O}$ of the unipotent cone of L. The classical Springer correspondence for representations of the Weyl group $W = W_{G,T}$ is recovered inside of (1.3) as one of these blocks, corresponding to the unique cuspidal datum with $L = T$, a maximal torus.

More recently, more general forms of the decomposition (1.3) have been obtained for the derived category of nilpotent orbital sheaves by Rider–Russell [RR16], the category of equivariant D-modules on the Lie algebra $\mathfrak{g}$ by Gunningham [Gun18, Gun17] and the derived category of character sheaves by Li [Li18].

In the setting of D-modules on $\mathfrak{g}/G$ or sheaves supported on the unipotent cone of G, the set of cuspidal data indexing the decomposition is the same as in (1.3). We will call this set the *unipotent* cuspidal data for G. In the case of character sheaves on G/G, however, this set must be expanded to account for unipotent cuspidal data for pseudo-Levi subgroups H of G. More generally, one can consider E-cuspidal data for any arithmetic genus 1 curve as we explain further now.

1.2.3. *Elliptic Springer theory.* The main object of study in this paper is the stack $\underline{G}_E \subseteq \mathrm{Bun}_G(E)$ of semistable G-bundles on a curve of arithmetic genus 1.

The stack $\underline{G}_E$ which we study in the present paper forms a bridge between the situations described in Section 1.2.1 and Section 1.2.2. On the one hand $\underline{G}_E$ sits inside $\mathrm{Bun}_G(E)$ as the locus of degree 0 semistable bundles. On the other hand, when E is taken to be a nodal curve one has an isomorphism $\underline{G}_E \cong G/G$.

Viewing the category $\mathcal{D}(\underline{G}_E)$ through the lens of Section 1.2.2, one defines functors of parabolic induction and restriction using the correspondence

$$
\underline{L}_E \leftarrow \underline{P}_E \rightarrow \underline{G}_E
$$

and study the corresponding Harish-Chandra (or generalized Springer decomposition). The ordinary Springer correspondence in the elliptic setting was studied by Ben-Zvi and Nadler [BZN15].

One can also formulate a generalized Springer correspondence for the (various flavours of) category $\mathcal{D}(\underline{G}_E)$ which recovers the standard patterns for orbital and character sheaves in the cuspidal and nodal cases. This will be expanded on in future work; for now, let us just note that the indexing set of the generalized Springer decomposition (which we are calling E-cuspidal data) involves a choice of an E-pseudo-Levi subgroup H of G, together with a $W_{G,H}$-equivariant simple unipotent cuspidal local system for H.

In this way Theorem 1.4 may be thought of as a geometric antecedent to the generalized Springer correspondence for $\mathcal{D}(\underline{G}_E)$: it expresses the geometry of the stack $\underline{G}_E$ in terms of unipotent orbits for E-pseudo-Levi subgroups. We note that the work of Li with Nadler [LN21] is another expression of the idea that $\underline{G}_E$ is glued together from data indexed by E-pseudo-Levi subgroups. However, the techniques of *loc. cit.* involve an analytic uniformization of $\underline{G}_E$, whereas the present paper stays within the realm of algebraic geometry.

1.2.4. *Elliptic geometric Langlands.* We may view the category $\mathcal{D}(\underline{G}_E)$ as a subcategory (via extension by zero from the semistable locus, say) of the automorphic geometric Langlands category $\mathcal{D}(\mathrm{Bun}_G(E))$ for an elliptic curve E. In this way, (generalized) Springer theory is embedded into the geometric Langlands correspondence. This perspective is exposited in [LN21, Section 1.3.1.2].

As a cautionary remark: there are now two different notions of cuspidal for an object of $\mathcal{D}(\underline{G}_E)$: one using the constant term functor via diagram (1.1) (we will call this Langlands cuspidal) and the other using the parabolic restriction functor via diagram (1.2) (we will call this Springer cuspidal). If an object of $\mathcal{D}(\underline{G}_E)$ is Langlands cuspidal it is necessarily Springer cuspidal, but it is not clear if the converse holds: the constant term may be supported on non-zero components of $\mathrm{Bun}_T(E)$.

Despite these difficulties, our results in this paper can be used to obtain strong restrictions on the existence and support of Langlands cuspidals on Bun_G. As a simple example, it can be shown that any Langlands cuspidal Hecke-eigensheaf in $\mathcal{D}(\mathrm{Bun}_{SL_2})$ must restrict to one of the four Springer cuspidal objects in $\mathcal{D}((\underline{SL}_2)_E)$.

1.2.5. *Quantum geometric Langlands.* However, the relationship between Springer theory and geometric Langlands is the most direct when one considers the quantum deformation. Namely, one studies the category $\mathcal{D}_\kappa(\mathrm{Bun}_G)$ of $\mathcal{L}^{\otimes\kappa}$-twisted D-modules on Bun_G where $\mathcal{L}$ is the determinant line bundle (for this discussion let us restrict to the case where the ground field k has characteristic zero and the group G is simple). Here κ can be taken to be any complex number. The quantum geometric Langlands conjecture posits an equivalence (assuming κ is not the critical level which we normalize to $\kappa = 0$)

$$\mathcal{D}_\kappa(\mathrm{Bun}_G) \simeq \mathcal{D}_{-1/\kappa}(\mathrm{Bun}_{G^\vee}).$$

When κ is irrational it can be shown that any non-zero object of $\mathcal{D}_\kappa$ is cleanly supported on the semistable locus. In this way, the quantum geometric Langlands equivalence at irrational level κ reduces to a statement about cuspidal data for G and ${}^LG$. We plan to return to this in future work.

There is also a Betti formulation of quantum geometric Langlands (see [BZN18]). For an elliptic curve E, this involves[3] the category $\mathcal{D}_q(G/G)$ of quantum D-modules on G/G (see [BZBJ18]). (Here, the parameter q is roughly an exponential of the $\kappa^\vee$ appearing in the de Rham formulation above.) There is a conjectural generalized Springer correspondence for $\mathcal{D}_q(G/G)$. Interestingly, though the categories are very

[3]The Betti formulation of quantum geometric Langlands is not as symmetric as in the de Rham setting; here we are describing the category that naturally lives on the spectral side of geometric Langlands correspondence; the automorphic version will be described in terms of certain twisted sheaves on Bun_G.

different, one expects the same (discrete) cuspidal data to appear for the generalized Springer decomposition of $\mathcal{D}(\underline{G_E})$ and $\mathcal{D}_q(G/G)$. S.G. hopes to expand on this point in forthcoming work with David Jordan and Monica Vazirani. We note that the "Springer block" of such a quantum generalized Springer would involve W-equivariant modules for the algebra of q-difference modules on the torus T, which may explain the connection with the work of Baranovsky, Evens and Ginzburg [BEG03].

1.2.6. *Eisenstein sheaves and elliptic Hall categories.* The main motivation for D.F. to study the stratification described in this paper concerns not cuspidal sheaves but rather Eisenstein sheaves. More precisely, one can define the category $\mathcal{Q}_G(E)$ of principal spherical Eisenstein sheaves on an elliptic curve E as the category generated by $\mathrm{Eis}^G_{T,B}(\underline{\mathbb{Q}_l})$ inside $\mathcal{D}^b(\mathrm{Bun}_G(E))$. This can be thought of as a categorical version of the space of spherical automorphic functions for the field of functions on a smooth projective curve over a finite field. The latter, at least for the groups GL_n, is nothing else but the degree n part of the spherical Hall algebra of the curve. See [Lau90, SV11, Sch12, Fra13] for more details about the relationship to automorphic functions. Therefore, one can think of $\mathcal{Q}_G(E)$ for the groups GL_n as a categorical version of the spherical Hall algebra. It can be actually proved [Sch12] that one obtains in this way a categorification of the elliptic Hall algebra. The situation for higher genus curves is not well understood.

A sensible question to ask (for elliptic curves) is the classification of simple objects of the spherical category $\mathcal{Q}_G(E)$. Actually the proof of the above mentioned result on categorification goes by first establishing such a classification. The main result of [BZN15] implies that there is an injection $\mathrm{Irr}(W) \hookrightarrow \mathrm{Irr}(\mathcal{Q}_G(E))$, where W is the Weyl group of G. Previously, in [Sch12] this was shown to be a bijection for GL_n. The precise expectation for simply connected groups is that the above map is a bijection.

Our main results in this article allow us to rule out some of the sheaves that could appear in $\mathrm{Irr}(\mathcal{Q}_G(E))$ and that do not arise from $\mathrm{Irr}(W)$. In the case of SL_n and some groups of small rank this is enough to confirm the above sought for bijection. However, at the moment, we do not know if it is true for all simply connected groups.

1.2.7. *Affine character sheaves and local geometric Langlands.* The local geometric Langlands program provides yet another interpretation of the category $\mathcal{D}(\underline{G_E})$. Roughly speaking, one is motivated by local Langlands to study the category $\mathcal{D}(LG/LG)$ of class sheaves for the loop group LG of G. This may be considered as a natural home for what one might call affine character sheaves. This category is technically very difficult to study (or even define). However, a slight deformation $LG/_qLG$ of the stack LG/LG is closely related to the moduli stack $\mathrm{Bun}_G(E_q)$ for a certain elliptic curve E_q. Thus one may consider $\mathcal{D}(\mathrm{Bun}_G(E_q))$ as an avatar of affine class sheaves (this idea appears in [BNP13]).

The most natural formulation of the relationship between LG/LG and Bun_G is analytic. In (unpublished) work of Looijenga, it was shown that there is an equivalence of complex analytic stacks

$$L^{hol}G/_qL^{hol}G \simeq \mathrm{Bun}^{an}_G(E_q).$$

Here $q \in \mathbb{C}^\times$, $|q| \neq 1$, $E_q = \mathbb{C}^\times/q^{\mathbb{Z}}$ is the corresponding elliptic curve, and $L^{hol}G = \mathrm{Map}_{hol}(\mathbb{C}^\times, G)$ is the holomorphic loop group which acts on itself by q twisted conjugation:

$$\mathsf{Ad}_q(g(z))h(z) = g(qz)h(z)g(z)^{-1}.$$

This idea of analytic uniformization was generalized by Li with Nadler [LN21], leading to analytic proofs of results closely related to those in this paper.

2. Preliminaries

2.1. **Notation and generalities on G-bundles.** Let X be a scheme and G an affine algebraic group over an algebraically closed field k. By a G-bundle (or principal G-bundle, or G-torsor) over a scheme X we mean a scheme $\pi\colon \mathcal{P} \to X$ over X with a π-invariant right G-action such that, étale locally on X, $\pi\colon \mathcal{P} \to X$ is G-equivariantly isomorphic to $\pi_1 : X \times G \to X$. In other words, there exists $X' \to X$ étale and surjective such that the pullback of $\mathcal{P}$ to X' is G-equivariantly isomorphic to $X' \times G$

$$\begin{tikzcd} X' \times G \arrow[r, "\simeq"] \arrow[dr, "\pi_1"'] & X' \times_X \mathcal{P} \arrow[d, "\pi'"] \\ & X' \end{tikzcd}$$

When there's no danger of confusion we will simply write $\mathcal{P}$ for a G-bundle and omit the mention of $\pi\colon \mathcal{P} \to X$.

If $\mathcal{P}$ is a G-bundle over a scheme X and Y is a quasi-projective variety with a left G-action, then we can form the associated fiber space over X with fiber Y as $Y_{\mathcal{P}} = \mathcal{P} \times^G Y := (\mathcal{P} \times Y)/G$. If moreover Y has a right H-action for some group H, then $Y_{\mathcal{P}}$ is naturally endowed with an H-action.

We will apply the above construction in two particular cases:

- if $\rho\colon H \to G$ is a morphism of groups, then to an H-bundle $\mathcal{P}$ we associate a G-bundle: $G_{\mathcal{P}} = \mathcal{P} \overset{H}{\times} G$. We'll also denote it by $\rho_*(\mathcal{P})$.
- if V is a representation of G (viewed as an affine space with a left G-action) then to a G-bundle $\mathcal{P}$ we associate the vector bundle $V_{\mathcal{P}} = (\mathcal{P} \times V)/G$.

Example 2.1. A particularly important case of the first situation above is for the group morphism $m\colon Z(G) \times G \to G$. If $\mathcal{P}'$ is a $Z(G)$-bundle and $\mathcal{P}$ is a G-bundle we denote the induced G-bundle $m_*(\mathcal{P}' \times \mathcal{P})$ by $\mathcal{P}' \cdot \mathcal{P}$.

Example 2.2.

(1) If $G = \mathbb{G}_m^2$ then a G-bundle is simply a pair of line bundles $(\mathcal{L}, \mathcal{M})$. If $\alpha\colon \mathbb{G}_m^2 \to \mathbb{G}_m$ is given by $\alpha(t,s) = ts^{-2}$ then $\alpha_*(\mathcal{L}, \mathcal{M}) = \mathcal{L} \otimes \mathcal{M}^{-2}$.

(2) If $G = \mathrm{GL}_n$ then the category of G-bundles is equivalent to the category of vector bundles of rank n. The correspondence is given by associating to a GL_n-bundle $\mathcal{P}$ the vector bundle $\mathcal{P} \times^{\mathrm{GL}_n} \mathbb{A}^n$. We will use this correspondence tacitly especially in the case $G = \mathbb{G}_m$ where we think of a $\mathbb{G}_m$-bundle as a line bundle using the natural representation of $\mathbb{G}_m$ on $\mathbb{A}^1$. In this situation, Example 2.1 corresponds to tensoring a vector bundle by a line bundle.

(3) Consider the Lie algebra $\mathfrak{g}$ with the adjoint action of G. For a G-bundle $\mathcal{P}$ we have its adjoint bundle $\mathcal{P} \times^G \mathfrak{g} =: \mathfrak{g}_{\mathcal{P}}$ that will play an important role in the text. If $G = \mathrm{GL}_n$ then the adjoint bundle is none other than the vector bundle of endomorphisms.

2.2. **Semistability.** First let us recall the definition of slope: for a vector bundle $\mathcal{V}$ on a smooth curve X/k we put $\mu(\mathcal{V}) := \deg(\mathcal{V})/\operatorname{rank}(\mathcal{V}) \in \mathbb{Q}$ and call it the slope of $\mathcal{V}$.

Definition 2.3.

- A vector bundle $\mathcal{V}$ on X is semistable (respectively, stable) if for all proper subvector bundles $\mathcal{W} \leq \mathcal{V}$ we have $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$ (respectively, $\mu(\mathcal{W}) < \mu(\mathcal{V})$).
- A G-bundle $\mathcal{P}$ is semistable if for any parabolic subgroup $P \leq G$ and any reduction of structure group of $\mathcal{P}$ to P, say $\mathcal{P}_P$, and any dominant character $\chi \colon P \to \mathbb{G}_m$, the line bundle $\chi_{\mathcal{P}_P}$ is of degree ≤ 0.
- If H is a non-reductive group then an H-bundle $\mathcal{P}$ is called semistable if the induced $H/R_u(H)$-bundle is semistable, where we denoted by $R_u(H)$ the unipotent radical of H.

Remark 2.4. One can also define semistability of a G-bundle through the adjoint representation: $\mathcal{P}$ is (ad-)semistable if $\mathfrak{g}_{\mathcal{P}}$ is a semistable vector bundle. In characteristic 0 these two definitions are equivalent, however in characteristic p they are not. See also [Ram96] for the definitions of semistability or [Sch14] for a definition through a slope map closer in spirit to the one for vector bundles.

The situation for elliptic curves is as in characteristic 0:

Proposition 2.5. *A G-bundle $\mathcal{P}$ of degree 0 on an elliptic curve is semistable if and only if all its associated vector bundles $V_{\mathcal{P}}$ for $V \in \operatorname{Rep}_k(G)$ are semistable.*

Proof. In characteristic 0, this is true in arbitrary genus, see [Ram96, Prop. 3.17]. The proof hinges on the Narasimhan–Seshadri theorem [NS65].

We now treat the positive characteristic case where genus 1 is necessary. Assume that $\mathcal{P}$ is semistable. We appeal to a result of Sun [Sun99, Thm 2.1]: a G-bundle on an elliptic curve is semistable if and only if it is strongly-semistable (i.e., all its Frobenius twists are semistable). A result of Ramanan–Ramanathan [RR84, Thm 3.23] (see also [Sun99, Cor 1.1]) implies that for a strongly semistable G-bundle of degree 0 all the associated vector bundles $V_{\mathcal{P}}$ for V a highest weight representation are semistable of degree 0. Since any representation of G has a filtration with associated quotients of highest weight, we deduce that $V_{\mathcal{P}}$ is semistable of degree 0.

Conversely, suppose that $V_{\mathcal{P}}$ is a semistable vector bundle for all representations V of G. Let $P \leq G$ be a parabolic subgroup and $\mathcal{P}_P$ a reduction of $\mathcal{P}$ to P. Let $\chi \colon P \to \mathbb{G}_m$ be a dominant character. We need to show that the line bundle $\chi_{\mathcal{P}_P}$ is of degree ≤ 0. Since χ is dominant, there exists a highest weight G-module, say $V(\chi)$, such that P acts on the highest weight line through χ. We deduce that the line bundle $\chi_{\mathcal{P}_P}$ is a subsheaf of the vector bundle $V(\chi)_{\mathcal{P}'_P} = V(\chi)_{\mathcal{P}}$ which is semistable of degree 0. Therefore $\deg(\chi_{\mathcal{P}_P}) \leq 0$ as we wanted. □

Remark 2.6. The above proof extends to the case of G-bundles of arbitrary degree as long as we only use highest weight representations. The issue for arbitrary representations is that if $V_{\mathcal{P}}$ and $W_{\mathcal{P}}$ are semistable vector bundles of different degrees then their direct sum, i.e., $(V \oplus W)_{\mathcal{P}}$, is not semistable. Another way to remedy this is to consider only those representations of G that factor through the adjoint group (for these, all the associated vector bundles are of degree 0).

It will also be useful to consider a notion of semistability for G-bundles on certain singular curves, namely the nodal and cuspidal curve E_{node} and E_{cusp} considered in Section 1.1.[4] We caution the reader that this is only a working definition for us and we do not pretend it is the good notion of semistability over a singular curve.

Definition 2.7. A G-bundle on a singular curve X is said to be semistable if its pullback under the normalization map $\widetilde{X} \to X$ is a semistable G-bundle on $\widetilde{X}$.

Remark 2.8. See [Bho01, Section 2] and [Bal19, Sch05] for a more in depth discussion of semistability for G-bundles on singular curves.

Remark 2.9. It follows from results of [FM01, Thm 3.3.1] that over a cuspidal or nodal curve the following two conditions are equivalent for a G-bundle $\mathcal{P}$:

(1) $\mathcal{P}$ is trivial when pulled back to the normalization $\mathbb{P}^1$,
(2) for any representation V of G the associated vector bundle $V_{\mathcal{P}}$ is slope semistable.

This justifies our choice of calling a G-bundle on a nodal/cuspidal semistable provided its pullback to the normalization is trivial.

2.3. **Moduli spaces and stacks.** Let X/k be a projective curve. For each test scheme S, we write $\mathrm{Bun}_G(X)(S)$ for the groupoid of G-bundles on $X_S = X \times S$. An excellent account of basic properties of this moduli stack can be found in [Wan11].

Consider the substack $\mathrm{Bun}_G^{\mathsf{ss}}(X)$ whose k-points consist of semistable G-bundles. It is proved in [Sch14, Prop. 6.1] that it is an open substack.

We write $\mathrm{Bun}_G^0(X)$ for the connected component containing the trivial bundle; such bundles are said to have degree 0. For connected groups G and smooth projective curves X, the stack $\mathrm{Bun}_G(X)$ decomposes into connected components indexed by the algebraic fundamental group $\pi_1(G)$ (the quotient of the cocharacter lattice by the coroot lattice), see [Hof10]. The parametrisation can be viewed as a generalization of the usual Chern class of a vector bundle on a curve.

Now let E be a genus 1 curve over k with one marked smooth point which may either have a simple node or a cusp singularity. Thus, E is either a smooth elliptic curve E, a genus 1 curve with a single node E_{node} or a genus 1 curve with a single cusp E_{cusp}.

We write $\underline{G}_E := \mathrm{Bun}_G^0(E)^{\mathsf{ss}}$ for the moduli stack of degree 0 semistable G-bundles on E. It is an open dense substack of the degree 0 component $\mathrm{Bun}_G^0(E)$. The case when E is an elliptic curve is the main object of study in this paper. As noted in Section 1.1, the nodal and cuspidal cases may be expressed in more concrete terms as follows.

Proposition 2.10 ([FM01, Thm 3.1.5, Thm 3.2.4])**.** *There is an equivalence of stacks*

$$\underline{G}_{E_{\mathrm{node}}} \cong G/_{ad}G$$

and

$$\underline{G}_{E_{\mathrm{cusp}}} \cong \mathfrak{g}/_{ad}G.$$

Remark 2.11. In loc. cit. the proofs are over the complex numbers and for simply connected groups. However, the same proof works in arbitrary characteristic and for any reductive group.

[4]For such curves, the normalization is isomorphic to $\mathbb{P}^1$, where the only degree 0-semistable bundle is the trivial bundle.

2.4. **Reductions of the structure group.** Suppose we are given $\rho : H \to G$ a morphism of groups and a G-bundle $\mathcal{P}$ on a scheme E. A reduction of $\mathcal{P}$ to H is a pair $(\mathcal{P}', \phi)$ of an H-bundle together with an isomorphism of G-bundles $\phi\colon \mathcal{P}' \times^H G \simeq \mathcal{P}$. We say that two reductions $(\mathcal{P}_1, \phi_1)$ and $(\mathcal{P}_2, \phi_2)$ of $\mathcal{P}$ are isomorphic if there exists an isomorphism of H-bundles $\mathcal{P}_1 \simeq \mathcal{P}_2$ that intertwines ϕ_1 and ϕ_2.

The collection of possible reductions of a G-bundle $\mathcal{P}$ to an H-bundle forms naturally a groupoid. If moreover ρ is injective, this groupoid is equivalent to a set as can be easily checked.

An alternative way of giving a reduction of $\mathcal{P}$ to H is to give a section $s\colon E \to \mathcal{P}/H$ of the bundle[5] $\mathcal{P}/H \to E$. Indeed, to such a section we associate the pullback $\mathcal{P}'$ together with $\phi\colon \mathcal{P}' \to \mathcal{P}$:

$$\begin{tikzcd} \mathcal{P}' \arrow[r, "\phi"] \arrow[d] & \mathcal{P} \arrow[d] \\ E \arrow[r, "s"] & \mathcal{P}/H \end{tikzcd}$$

Since $\mathcal{P} \to \mathcal{P}/H$ is an H-bundle, $\mathcal{P}' \to E$ is one as well. Moreover, the map ϕ is H-equivariant and induces a G-equivariant isomorphism $\mathcal{P}' \times^H G \to \mathcal{P}$.

To say that two such sections s_1 and s_2 are isomorphic we need to look at E and $\mathcal{P}/H$ as groupoids (the first one is discrete but the second one is non-discrete when ρ is not injective). Then s_1 and s_2 are isomorphic if there exists a natural transformation $\eta\colon s_1 \Rightarrow s_2$ that is an isomorphism. In the case of a subgroup $H \leq G$ the groupoids are discrete and saying that s_1 and s_2 are isomorphic is the same as saying they are equal (as functions).

Conversely, if $(\mathcal{P}', \phi)$ is a reduction of $\mathcal{P}$ to H then quotienting out by H the composition $\mathcal{P}' \overset{f \mapsto (f \times 1)}{\longrightarrow} \mathcal{P}' \times^H G \overset{\phi}{\simeq} \mathcal{P}$ we get a section $E \to \mathcal{P}/H$.

The above correspondence is an equivalence of categories and we will use freely either way of looking at a reduction.

The most important cases for us are $H \leq G$ and $B \twoheadrightarrow T$.

Remark 2.12. From the above discussion we have that $\mathrm{Bun}_H(E) \to \mathrm{Bun}_G(E)$ is a representable morphism of stacks when H is a subgroup of G. This is used silently throughout the text.

2.5. **Framed bundles.** We start by recalling some basic notions for framed bundles over a pointed projective curve (E, x_0) and then consider the moduli stack G_E of framed degree 0 semistable G-bundles. We show, through Lemma 2.13, that it is a variety that is a G-bundle over the corresponding non-framed moduli stack. Then we proceed to the main point of this section, namely we show that for a closed subgroup $H \leq G$ we have a closed embedding $H_E \hookrightarrow G_E$ (see Proposition 2.19).

A framed G-bundle $p = (\mathcal{P}_G, \theta)$ is a G-bundle on E together with a G-equivariant isomorphism $\theta\colon \mathcal{P}_G|_{x_0} \simeq G$ of the fiber of $\mathcal{P}_G$ over x_0 with the group G. The degree and semistability are defined in terms of the underlying G-bundle.

Sometimes, for convenience, we will omit the mention of θ and simply say that p is a framed G-bundle.

It is clear that the moduli stack of framed G-bundles $\mathrm{Bun}_G^{\mathsf{fr}}(E)$ is a G-torsor over the moduli stack of G-bundles $\mathrm{Bun}_G(E)$.

[5] If ρ is not injective this bundle is actually a gerbe with fiber $B(\ker \rho)$.

Let $\mathcal{P}_G$ be a G-bundle and θ, θ' two framings. Then there exists a unique $g \in G$ such that $\theta' = g \cdot \theta$, more precisely such that the following diagram commutes

$$\begin{tikzcd} \mathcal{P}_G|_{x_0} \arrow[r, "\theta"] \arrow[dr, "\theta'"'] & G \arrow[d, "g\cdot"] \\ & G \end{tikzcd}$$

For $(\mathcal{P}_G, \theta)$ a framed G-bundle we have an induced map $i_\theta\colon \operatorname{Aut}(\mathcal{P}_G) \to G$ defined by restricting the automorphism to the fiber $\mathcal{P}_G|_{x_0}$ and using the basic fact recalled above that any two framings differ by an element of G. If $(\mathcal{P}_G, \theta')$ is another framing on $\mathcal{P}_G$, then the induced map $i_{\theta'}\colon \operatorname{Aut}(\mathcal{P}_G) \to G$ satisfies $i_{\theta'} = g i_\theta g^{-1}$, where $g \in G$ is such that $\theta' = g \cdot \theta$.

In the case that interests us we have moreover

Lemma 2.13. *Let $p := (\mathcal{P}_G, \theta)$ be a degree 0, semistable, framed G-bundle over an elliptic curve E. Then the induced morphism $i_\theta\colon \operatorname{Aut}(\mathcal{P}_G) \to G$ is injective and its image is $\operatorname{Stab}_G(p)$.*

To prove Lemma 2.13 we will make use of a property of degree 0 semistable vector bundles that holds in any genus. Recall that for the purposes of this paper, a vector bundle on a singular curve is defined to be semistable of degree 0 if and only if its pullback to the normalization is semistable of degree 0.

Lemma 2.14. *Let X be a projective curve, and $\mathcal{V}$, $\mathcal{W}$ semistable vector bundles of degree 0. Then any map $f : \mathcal{V} \to \mathcal{W}$ is of constant rank.*

Proof. First let us note that we may reduce to the case when X is smooth. Indeed, given such a map $f : \mathcal{V} \to \mathcal{W}$ on X, we may pullback to a map $\widetilde{f} : \widetilde{\mathcal{V}} \to \widetilde{\mathcal{W}}$ on the normalization $\eta : \widetilde{X} \to X$, and restricting to the fiber of $\widetilde{f}$ at $\widetilde{x} \in \widetilde{X}$ identifies with the fiber of η at $x = \eta(x)$.

Now suppose that X is smooth. Then $\ker(f)$ and $\operatorname{im}(f)$ must both be semistable of degree 0 because $\mathcal{V}$ and $\mathcal{W}$ are semistable of degree 0. Now, if $\operatorname{coker}(f)$ had torsion then its preimage would be a positive degree subbundle in $\mathcal{V}$ contradicting semistability. Thus we must have that f is of constant rank as required. ☐

Proof of Lemma 2.13. Let us first start with $G = \mathrm{GL}_n$, so $\mathcal{P}_G$ is equivalent to a semistable vector bundle, say $\mathcal{V}$, of degree 0.

We will show that restriction to the fiber at x_0 together with the framing gives an inclusion $\operatorname{Aut}(\mathcal{V}) \hookrightarrow \mathrm{GL}_n$. Suppose there is an automorphism ψ in the kernel. Then $\psi - \mathsf{Id}$ is an endomorphism of $\mathcal{V}$ which is zero at x_0. Thus by Lemma 2.14, $\psi - \mathsf{Id}$ must be identically zero as required.

Back to the general case. Consider $G \hookrightarrow \mathrm{GL}(V)$ a faithful highest weight representation. Then we know from [Sun99, Cor 1.1, Thm 2.1] that $V_{\mathcal{P}}$ is a semistable, degree 0 vector bundle.

Therefore by the above, its automorphisms lie inside $\mathrm{GL}(V)$.

We have the following commutative diagram of varieties

$$\begin{tikzcd} \operatorname{Aut}(\mathcal{P}) \arrow[r, "\simeq"] \arrow[d] & \operatorname{Hom}_{G\text{-}\mathsf{eq}}(\mathcal{P}, G) \arrow[r, hook] & \operatorname{Hom}_{G\text{-}\mathsf{eq}}(\mathcal{P}, \mathrm{GL}(V)) \arrow[r, "\simeq"] \arrow[d, hook] & \operatorname{Aut}(V_{\mathcal{P}}), \\ G \arrow[rr, hook] & & \mathrm{GL}(V) & \end{tikzcd}$$

where the marked $\simeq$ are the canonical isomorphisms which are instances of

$$H^0(\mathcal{P} \overset{G}{\times} Y) \simeq \mathrm{Hom}_{G\text{-}\mathsf{eq}}(\mathcal{P}, Y)$$

for any G-variety Y.

From the injectivity of the three marked maps, we deduce that the first vertical morphism is injective as well, which is what we wanted.

Moving on to the second statement, it is a general fact and fairly easy to show that if an algebraic group G acts on a scheme X then the morphism $\pi\colon X \to [X/G]$ is a G-bundle, where $[X/G]$ is the quotient stack. Moreover, if $x \in X$ then $\mathrm{Aut}(\pi(x)) = \mathrm{Stab}_G(x)$.

In our situation this can also be shown directly. Put $p = (\mathcal{P}, \theta)$ and denote by $i_\theta\colon \mathrm{Aut}(\mathcal{P}) \to G$ the induced map. If $\phi \in \mathrm{Aut}(\mathcal{P})$ then check that $\phi\colon (\mathcal{P}, i_\theta(\phi) \cdot \theta) \simeq (\mathcal{P}, \theta)$, i.e., $p = g \cdot p$ as we have already showed above that $\mathrm{Aut}(p) = 1$. In other words, $i_\theta(\phi) \in \mathrm{Stab}_G(p)$. Conversely, let $g \in \mathrm{Stab}_G(p)$. This means that $(\mathcal{P}, g \cdot \theta) = (\mathcal{P}, \theta)$. Explicitly, there exists an isomorphism $\phi\colon \mathcal{P} \to \mathcal{P}$ compatible with $g \cdot \theta$ and θ. One checks routinely that $i_\theta(\phi) = g$. We have shown that $i_\theta\colon \mathrm{Aut}(\mathcal{P}) \simeq \mathrm{Stab}_G(p)$. □

Remark 2.15. Lemma 2.13 is also true for higher genus curves. Given a semistable G-bundle $\mathcal{P}$ of degree 0 one needs to pick up a faithful representation V of G such that $V_{\mathcal{P}}$ is semistable . By results of [IMP03], this is the case for V a low-height representation. However, we will never use this result and so we do not provide the details.

Let now $H \leq G$ be a closed subgroup of G and consider the induction map from H-bundles to G-bundles. First we show that the induction preserves semistability:

Lemma 2.16. *The map* $\mathrm{Bun}_H^0(E) \to \mathrm{Bun}_G^0(E)$ *sends semistable bundles to semistable bundles.*

Proof. First we treat the case when H is reductive. Let $\mathcal{P}$ be a semistable H-bundle of degree 0. In genus 1 we can use the definition of semistability through associated vector bundles (see Proposition 2.5) and this simplifies the argument.

Let V be a representation of G and restrict it to H. Then $(\mathcal{P} \overset{H}{\times} G) \overset{G}{\times} V = \mathcal{P} \overset{H}{\times} V$ is a semistable vector bundle of degree 0. Hence $\mathcal{P} \overset{H}{\times} G$ is a semistable G-bundle of degree 0.

Suppose now that H is not reductive and consider a Levi decomposition $H = LU$ where L is isomorphic to H/U and $U := R_u(H)$ is the unipotent radical of H. By definition (see Definition 2.3), an H-bundle $\mathcal{P}$ is reductive if the induced L-bundle $\mathcal{P}/U$ is reductive. Let $\mathcal{P}$ be a semistable H-bundle. We want to show that $\mathcal{P} \overset{H}{\times} G$ is semistable. We will use the characterization of semistability through the associated vector bundles, see Proposition 2.5. Let V be a representation of G. We have to show that $\mathcal{P} \overset{H}{\times} G \overset{G}{\times} V$ is a semistable vector bundle. Notice first that we can rewrite it as $\mathcal{P} \overset{H}{\times} V$ where V is viewed as a representation of H. There exists a filtration of V such that the unipotent group U acts trivially on the associated graded. Call this associated graded V'. Hence the vector bundle $V_{\mathcal{P}}$ has a filtration such that its associated graded is $V'_{\mathcal{P}}$. But $V'_{\mathcal{P}}$ is isomorphic to $\mathcal{P}/U \overset{L}{\times} V'$ which is semistable of degree 0 (by Proposition 2.5). It follows that $V_{\mathcal{P}}$ is also semistable of degree 0. □

Remark 2.17.

(1) Note that Lemma 2.16 is false without assuming degree 0 for the H-bundle. For example, for $T = \mathbb{G}_m^2 \hookrightarrow \mathrm{GL}(2)$ we have that $(\mathcal{O}(-1), \mathcal{O}(1))$ is a semistable T-bundle of degree $(-1, 1)$ but the induced $\mathrm{GL}(2)$-bundle (a rank 2 vector bundle) $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ is not semistable, although it is of degree 0.

(2) There is a way to extend Lemma 2.16 to bundles of arbitrary degree. In order to have a correct statement we could use the slope map of Schieder [Sch14]: we require that the degree of the H-bundle has the same slope as the degree of the induced G-bundle. This strategy has been used in [Fră15]. However, this generalization is never used in this paper.

When (E, x_0) is a pointed, irreducible projective curve of arithmetic genus 1 we denote by G_E the moduli stack of degree 0, semistable framed G-bundles over E where the framing is at $x_0 \in E$. Lemma 2.13 shows that G_E is an algebraic space that is a G-torsor over the stack $\underline{G}_E$. Actually we can be more precise:

Proposition 2.18. *For any linear algebraic group G, the moduli stack G_E is a separated, reduced scheme of finite type, i.e., a variety.*

Proof. (Sketch) The case of cuspidal and nodal curves is dealt with by Proposition 2.10.

If E is smooth, the moduli stack $\mathrm{Bun}_G^0(E)$ is algebraic, smooth and locally of finite presentation (see [Wan11, Thm 1.0.1]). In the proof of loc. cit. it is shown that Bun_G is covered by quasi-compact open substacks that admit a smooth map from a Quot scheme. Assume for a moment that G_E is representable by a scheme. Then we immediately deduce that it is reduced, separated and of finite type since Quot schemes are separated and of finite type.

We are left to showing that G_E is representable by a scheme. The case $G = \mathrm{GL}_n$ follows from [Sim94, Thm. 4.10].

For a general linear algebraic group, embed $G \hookrightarrow \mathrm{GL}(V)$ and use Lemma 2.16 to deduce a morphism

$$\mathrm{Bun}_G^{0,\mathsf{ss},\mathsf{fr}}(E) \to \mathrm{Bun}_{\mathrm{GL}(V)}^{0,\mathsf{ss},\mathsf{fr}}(E).$$

Since $\mathrm{GL}(V)_E$ is a scheme and $\mathrm{Bun}_G(E) \to \mathrm{Bun}_{\mathrm{GL}(V)}(E)$ is schematic (valid more generally, see for example [Wan11, Cor. 3.2.4]) we deduce that G_E is a scheme as well. $\square$

Proposition 2.19, used silently in the sequel, is conceptually important in understanding the partition of the moduli stack G_E in terms of subgroups and it paves the way to the Jordan–Chevalley decomposition as formulated in Theorem 1.3.

Proposition 2.19. *For any closed subgroup $H \leq G$ the induction map between moduli spaces of framed bundles $H_E \to G_E$ is a closed embedding.*

To prove Proposition 2.19, we will make use of Lemma 2.20 of Lemma 2.14 and of an additional technical lemma on equivariant embeddings:

Lemma 2.20. *Suppose X is a projective curve and $\mathcal{V}$ a semistable vector bundle of degree* 0.

(1) *Suppose $\mathcal{L}$ is a degree 0 line bundle on X. Then any injective map of sheaves $\mathcal{L} \to \mathcal{V}$ is necessarily the inclusion of a subbundle.*

(2) *Suppose* $\mathcal{L}_1, \mathcal{L}_2$ *are line subbundles of* $\mathcal{V}$ *such that* $\mathcal{L}_1|_x = \mathcal{L}_2|_x$ *for some* $x \in X$. *Then* $\mathcal{L}_1 = \mathcal{L}_2$.

Proof. Part (1) follows immediately from Lemma 2.14, and part (2) follows by considering the canonical morphism $\mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathcal{V}$. ☐

Lemma 2.21. [6] *Let* $H \leq G$ *be a closed subgroup. Then we can find a representation* V *of* G *and a line* $L \subset V$ *such that*

(1) $\mathrm{Stab}_G(L) = H$,
(2) $G/H \hookrightarrow \mathbb{P}(V)$ *equivariantly,*
(3) *the complement* $\overline{G/H} \backslash (G/H)$ *is empty or the support of a Cartier divisor.*

Proof. By a theorem of Chevalley (see, for example, [Bor91, II.5.1]) we can find a representation W and a line $L \subset W$ with properties (1), (2) above. If H is a parabolic subgroup then we're done. If not, put $X := \overline{G/H}$ and denote by Z the complement $X \setminus (G/H)$. Let $\tilde{X}$ be the normalization of the blow-up of X along Z. The G-action on X extends by universal properties to $\tilde{X}$ and the complement of G/H in $\tilde{X}$ is, by construction, the support of a Cartier divisor.

Now we use Sumihiro's theorem [Sum74] to embed $\tilde{X} \hookrightarrow \mathbb{P}(V)$ equivariantly in the projectivization of a representation V of G. ☐

Proof of Proposition 2.19. By Lemma 2.21 we may pick a representation V of G and a line $L \subseteq V$ such that the morphism $g \mapsto g \cdot L$ defines an equivariant embedding into the projective space $G/H \hookrightarrow \mathbb{P}(V)$ and such that the complement of G/H in its closure is the support of a Cartier divisor.

Now suppose $\mathcal{P}_G$ is a G-bundle on E and put $\mathcal{V} := V_{\mathcal{P}_G}$. We have an embedding of associated bundles:

$$\mathcal{P}_G/H = \mathcal{P}_G \times^G (G/H) \hookrightarrow \mathcal{P}_G \times^G \mathbb{P}(V) = \mathbb{P}(\mathcal{V}).$$

The data of an H-reduction $\mathcal{P}_H$ of $\mathcal{P}_G$ is equivalent to a section $s : E \to \mathcal{P}_G/H$ which we will consider as a section of the projective bundle $\mathbb{P}(\mathcal{V})$ via the embedding above. In other words, an H-reduction corresponds to a certain line subbundle $\mathcal{L} = \mathcal{P}_H \times^H L \hookrightarrow \mathcal{V}$. Note that if $\mathcal{P}_G$ and $\mathcal{P}_H$ are semistable and of degree 0, then so are $\mathcal{V}$ and $\mathcal{L}$.

Let us first show that $H_E \to G_E$ is injective. Given a framed G-bundle $p = (\mathcal{P}_G, \theta)$, we suppose it has two reductions to a framed H-bundle, corresponding to two line subbundles $\mathcal{L}_1, \mathcal{L}_2$ as explained above. Under the framing isomorphism

$$\mathcal{V}|_{x_0} = \mathcal{P}_G|_{x_0} \times^G V \cong G \times^G V = V,$$

we have that $\mathcal{L}_1|_x$ and $\mathcal{L}_2|_x$ both correspond to the line $L \subseteq V$. Thus, by Lemma 2.20, we must have $\mathcal{L}_1 = \mathcal{L}_2$ as required. Since an injective map of schemes is automatically separated (see [Bos12, Cor 7.4.10]) we deduce that $H_E \to G_E$ is separated.

Now let us show that $H_E \to G_E$ is proper. We use the valuative criterion of properness and since we already noted the morphism is separated we only need the existence part. Thus, let S denote the spectrum of a valuation ring, U its generic point, Z its closed point, and let E_S, E_U, E_Z be the base change of E to S, U, and Z respectively.

[6]We thank M. Brion for providing us the proof.

Given a family $(\mathcal{P}_{G,S}, \theta)$ of framed, degree 0, semistable G-bundles on E_S together with a compatibly framed H-reduction on E_U, we must show that it extends to a compatibly framed H-reduction on E_S.

As before, we have the associated vector bundle $\mathcal{V}_S$ over E_S, and we record the data of the H-reduction as a line subbundle $\mathcal{L}_U \hookrightarrow \mathcal{V}_U$. We must show that

(1) $\mathcal{L}_U$ extends to a line subbundle $\mathcal{L}_S \subseteq \mathcal{V}_S$ over E_S and
(2) The corresponding section $E_S \to \mathbb{P}(\mathcal{V}_S)$ lands inside the subbundle $\mathcal{P}_{G,S}/H$.

For the first part, we note that the line subbundle extends to a sub*sheaf* $\mathcal{L}_S \to \mathcal{V}_S$. Indeed, by the properness of $\mathbb{P}(\mathcal{V}_S) \to E_S$, we may extend the section $s : E_U \to \mathbb{P}(\mathcal{V}_U)$ over the generic point of E_Z to define a line subbundle $\mathcal{L}'$ on E'_S, where E'_S is an open subset of E_S whose complement is of codimension 2. This line subbundle is the restriction of a unique line bundle $\mathcal{L}_S$ on E_S (e.g. by taking the closure of a Weil divisor representing it), and the map $\mathcal{L}' \to \mathcal{V}_S|_{E'_S}$ necessarily extends over the codimension 2 locus.

Now we observe that the restriction $\mathcal{L}_Z$ of $\mathcal{L}_S$ to E_Z is a degree 0 line bundle (as it deforms to the degree 0 line bundle $\mathcal{L}_U$) with a non-zero map to $\mathcal{V}_Z$. It follows from Lemma 2.20 that $\mathcal{L}_Z \to \mathcal{V}_Z$ must be a subbundle as required.

We move on to prove the second part.

We will show that the section $s : E_S \to \mathbb{P}(\mathcal{V}_S)$ defining the line subbundle $\mathcal{L}_S$ above lands inside $\mathcal{P}_{G,S}/H$. If G/H is closed in $\mathbb{P}(V)$ then $\mathcal{P}_{G,S}/H$ is closed in $\mathbb{P}(\mathcal{V}_S)$ and so the image of the section is contained entirely in it.

In case G/H is not projective, remember that $D := \overline{G/H} \setminus G/H$ is the support of a Cartier divisor (see Lemma 2.21).

The section s lands inside $\mathcal{P}_{G,S} \overset{G}{\times} (\overline{G/H})$ which is the closure of $\mathcal{P}_{G,S}/H$ in $\mathbb{P}(\mathcal{V}_S)$. The complement of $\mathcal{P}_{G,S}/H$ in its closure is the support of a Cartier divisor, namely $C_S := \mathcal{P}_{G,S} \overset{G}{\times} D$. Thus the set of points $x \in E_S$ for which $s(x) \notin \mathcal{P}_{G,S}/H$ is $Q := s^{-1}(C_S)$ which is either empty or a divisor (since the image of the section s is not contained in C_S). As $s(x) \in \mathcal{P}_{G,S}/H$ for all $x \in E_U$, we must have that $Q \subseteq E_Z$. Since E_Z is irreducible we either have $Q = E_Z$ or Q is empty. But now note that the framing on $\mathcal{P}_{G,S}$ defines a trivialization

$$\mathbb{P}(\mathcal{V}_S)|_{\{x_0\}\times S} \cong \mathbb{P}(V \otimes \mathcal{O}_S)$$

and the section s takes the constant value $[L]$ along the entire slice $\{x_0\} \times S$ of E_S. In particular, $Q \neq E_Z$ and so it must be empty. $\square$

2.6. **The coarse moduli space and the characteristic polynomial map.** The coarse moduli *space* of degree 0, semistable G-bundles, which is none other than the GIT quotient $G_E/\!\!/G$, was identified by Laszlo [Las98] to be isomorphic to the GIT quotient $T_E/\!\!/W$ for a maximal torus T (see also [Fra21] for a proof in arbitrary characteristic). There is a natural G-invariant morphism from the framed moduli stack to the moduli space of G-bundles that we think of as the characteristic polynomial map (in analogy to Lie theory)

$$\chi = \chi_G \colon G_E \to \mathcal{M}_E(G).$$

Notice that the G-invariance of χ is equivalent to χ factorizing through the moduli stack

$$\underline{\chi} \colon \underline{G}_E \to \mathcal{M}_E(G).$$

It is a good moment to revisit the notion of a semisimple G-bundle.

Lemma 2.22. *Let $\mathcal{P} \in \underline{G}_E$. The following are equivalent:*

(1) *$\mathcal{P}$ has a reduction to any maximal torus;*
(2) *$\mathcal{P}$ has a reduction to some maximal torus;*
(3) *Every framed lift $p \in G_E$ of $\mathcal{P}$ has a closed G-orbit;*
(4) *$\mathcal{P}$ is a closed point in the stack $\underline{G}_E$.*

Definition 2.23. A G-bundle $\mathcal{P} \in \underline{G}_E$ is called semisimple if it satisfies any of the above properties.

Proof. The equivalence of (1) and (2) follows from the fact that all maximal tori are conjugate. The equivalence of (3) and (4) follows readily by considering the map $G_E \to \underline{G}_E$ which is a G-bundle. The orbits of G are mapped to points of $\underline{G}_E$ and an orbit is closed if and only if the point in the stack quotient is closed.

The proof of (3)⇒(2) follows from Lemma 2.24.

Now let us prove (2)⇒(3). We suppose that $\mathcal{P}$ has a reduction to a maximal torus T. Consider the following commutative diagram

$$\begin{tikzcd} G_E \arrow[r, "\chi"] & \mathcal{M}_{G,E} \\ T_E \arrow[u, "\iota"] \arrow[r, "\chi'"] & T_E/\!\!/W. \arrow[u, "\simeq"'] \end{tikzcd}$$

By assumption, there exists $s \in T_E$ such that $\iota(s)$ is a lift of $\mathcal{P}$ to the framed moduli space G_E. We need to show that the orbit $G \cdot \iota(s)$ is closed in G_E. Since $G_E \to \mathcal{M}_{G,E}$ is a GIT quotient we know that in any fiber there is a unique closed G-orbit. In particular, the fiber $\chi^{-1}(\chi(\iota(s))$ contains a closed orbit, say $G \cdot p'$. By the implication (3)⇒(2), there exists $s' \in T_E$ such that $\iota(s') = p'$. Since $\chi'(s) = \chi'(s')$ we must have $Ws = Ws'$ which implies $G \cdot \iota(s) = G \cdot p'$. We conclude that the orbit $G \cdot \iota(s)$ is closed, namely that $\mathcal{P}$ is a semisimple G-bundle. ☐

Lemma 2.24. *For every $\mathcal{P} \in \underline{G}_E$ there is a bundle $\mathcal{P}^{\mathsf{ss}}$ in its closure which admits a reduction to a maximal torus.*

Proof. We will construct a family $\widetilde{\mathcal{P}}$ of bundles over $\mathbb{A}^1/\mathbb{G}_m$ whose generic fiber is $\mathcal{P}$ and special fiber admits a reduction to T. First note that by [BZN15, Thm. 2.2(1)][7] every such bundle $\mathcal{P}$ admits a degree 0 reduction $\mathcal{P}_B$ to a Borel subgroup B (note that this fact is very particular to degree 0 semistable bundles in genus 1). Let T denote a maximal torus contained in B, and choose a cocharacter $h\colon \mathbb{G}_m \to T$ such that the conjugation action on B has strictly positive weights on the unipotent radical N. In particular, the family of homomorphisms

$$f\colon B \times (\mathbb{A}^1 \setminus \{0\}) \to G$$

defined by

$$f_t(b) := f(b,t) = h(t)bh(t)^{-1}$$

extends uniquely to $t = 0 \in \mathbb{A}^1$, where f_0 projects onto $T \subseteq G$. We consider this family as a homomorphism of group schemes $f\colon B_{\mathbb{A}^1} \to G_{\mathbb{A}^1}$ and consider the associated bundle (over $E \times \mathbb{A}^1$):

$$\widetilde{\mathcal{P}} = \mathcal{P}_{B_{\mathbb{A}^1}} \times^{B_{\mathbb{A}^1},f} G_{\mathbb{A}^1}.$$

[7]In loc. cit. the derived group is assumed to be simply connected and they work in characteristic zero. However, their proof works without these assumptions. See aldo [Fra21, Remark 2.13] and [Fră15] for a slightly different proof of a more general result in this generality.

We note that the fiber over $1 \in \mathbb{A}^1$ is $\mathcal{P}_B \times^B G \cong \mathcal{P}$ and the fiber over $0 \in \mathbb{A}^1$ the semisimple bundle $\mathcal{P}_B \times^{B, f_0} G$. Moreover, the action of $\mathbb{G}_m$ via the cocharacter h provides an equivariant structure, descending $\widetilde{\mathcal{P}}$ to a family over $\mathbb{A}^1/\mathbb{G}_m$ with the desired properties. □

As there is a unique closed orbit in the fibers of $\chi\colon G_E \to \mathcal{M}_E(G)$, we see that the bundle $\mathcal{P}^{\mathsf{ss}}$ constructed in Lemma 2.24 is unique (up to isomorphism). We record this fact here for later reference.

Proposition 2.25. *Given $\mathcal{P} \in \underline{G}_E$, there is a unique-up-to-isomorphism semisimple bundle $\mathcal{P}^{\mathsf{ss}} \in \underline{G}_E$ with the property that $\chi(\mathcal{P}^{\mathsf{ss}}) = \chi(\mathcal{P})$.*

We refer to the bundle $\mathcal{P}^{\mathsf{ss}}$ as "the" *semisimplification* of $\mathcal{P}$.

Remark 2.26. The maps χ and $\underline{\chi}$ (see beginning of Section 2.6) map a semisimple bundle $\mathcal{P}$ into its equivalence class in $T_E /\!\!/ W$.

Definition 2.27. The moduli stacks/varieties of unipotent G-bundles are defined to be the preimages of $\mathbf{1} \in \mathcal{M}_E(G)$:

$$\underline{G}_E^{\mathsf{uni}} := \underline{\chi}^{-1}(\mathbf{1}),$$
$$G_E^{\mathsf{uni}} := \chi^{-1}(\mathbf{1}).$$

Remark 2.28. Notice that for $p \in G_E$ we have $\chi(p) = \mathbf{1}$ if and only if the closure $\{p\}^-$ contains the trivial bundle. This is due to the fact that GIT moduli spaces parameterize closed orbits and an arbitrary orbit is sent to the unique closed orbit contained in its closure. Hence the informal definition of unipotent bundles from Section 1.1.1 coincides with the above one.

3. The partition of semisimple bundles by Lusztig type

In this section we construct and study a certain locally closed partition of the coarse moduli space $\mathcal{M}_E(G)$.

Recall that the points of the coarse moduli space $\mathcal{M}_E(G)$ are in bijection with isomorphism classes of semisimple objects of $\underline{G}_E$. Let $\mathcal{P}_G \in \underline{G}_E$ be a semisimple object and let $p \in G_E$ be a framed lift. Then the automorphism group $\mathrm{Aut}(\mathcal{P}_G)$ is identified with the (possibly disconnected) reductive subgroup $\mathrm{Stab}_G(p)$ of G. Different choices of framing define conjugate subgroups of G, and thus the conjugacy class of $\mathrm{Stab}_G(p)$ in G is a well-defined invariant of the bundle $\mathcal{P}_G$.

In this way, we may partition $\mathcal{M}_E(G)$ according to the corresponding conjugacy class in G of its automorphism group. It will be convenient to encode the data of $\mathrm{Stab}_G(p)$ in two stages:

- The neutral component $H = \mathrm{Stab}_G(p)^\circ$, which is a connected reductive subgroup of G.
- The component group $\pi_0 \mathrm{Stab}_G(p)$, which is a subgroup of the finite group $N_G(H)/H$.

The goal of this section is to study this partition of $\mathcal{M}_E(G)$, and express it in combinatorial/root-theoretic terms.

3.1. **E-pseudo-Levi subgroups.** Denote by $\mathcal{E}$ the set of conjugacy classes of connected reductive subgroups.[8] Consider the map (of sets)

$$h\colon \mathcal{M}_E(G) \to \mathcal{E}$$

which takes the isomorphism class of a semisimple bundle $\mathcal{P}_G$ to the G-conjugacy class of $\mathrm{Stab}_G(p)^\circ$, where p is a framed lift of $\mathcal{P}_G$. We denote by

$$\mathcal{M}_E(G)_{[H]} := h^{-1}([H])$$

the fibers of the map h.

Definition 3.1. We say that a connected reductive subgroup H of G is an *E-pseudo-Levi subgroup* if it is of the form $\mathrm{Stab}_G(p)^\circ$ for some semisimple $p \in G_E$. We write $\mathcal{E}_E \subset \mathcal{E}$ for the set of conjugacy classes of E-pseudo-Levi subgroups of E.

Remark 3.2. As mentioned in Section 1 (see also Proposition 2.10), if E is cuspidal (respectively, nodal) then $G_E \cong \mathfrak{g}$ (respectively, $G_E \cong G$). Thus E-pseudo-Levi subgroups correspond to connected centralizers of semisimple elements of $\mathfrak{g}$ (respectively, G). These are precisely the Levi (respectively, pseudo-Levi) subgroups of G (see Appendix A).

In Section 3.6 we will give an alternative description of this partition, which allows us to understand the closure relations. In particular, at the end of Section 3.4 we will establish the following result (which may also be deduced from the theory of Luna stratifications; see Remark 3.4).

Proposition 3.3. *For each E-pseudo-Levi H, the subset $\mathcal{M}_E(G)_{[H]}$ is a locally closed subvariety of $\mathcal{M}_E(G)$.*

We may record the finer partition according to the conjugacy class of the full automorphism group as follows. If $\mathcal{P}_G$ is a representative of a point in $\mathcal{M}_E(G)_{[H]}$ and p a framed lift, then $\mathrm{Stab}_G(p) \subseteq N_G(H)$ (as every algebraic group normalizes its own neutral component). Thus the component group of $\mathrm{Stab}_G(p)$ is naturally a subgroup of the relative Weyl group:

$$\pi_0\mathrm{Stab}_G(p) = \mathrm{Stab}_G(p)/H \subseteq W_{G,H} := N_G(H)/H.$$

Given a subgroup A of $W_{G,H}$, we write $\mathcal{M}_E(G)_{[H,A]}$ for the subset of $\mathcal{M}_E(G)_{[H]}$ corresponding to semisimple bundles $\mathcal{P}_G$ such that the component group of $\mathrm{Aut}(\mathcal{P}_G)$ is identified with a conjugate of A as above. (The pair $[H, A]$ is defined up to simultaneous conjugation in G.)

In particular, the subset $\mathcal{M}_E(G)_{[H,1]}$ consists of isomorphism classes of semisimple bundles $\mathcal{P}_G$ whose full automorphism group $\mathrm{Aut}(\mathcal{P}_G)$ is identified with the connected group H.

Remark 3.4. Suppose we are given a G-variety Y such that every point has a G-invariant affine chart. Then we have the categorical quotient $Y/\!\!/G$ whose points are in bijection with closed G-orbits on Y. We may partition $Y/\!\!/G$ according to the conjugacy class in G of the (necessarily reductive) stabilizer of the corresponding closed orbits. One can show that this defines a locally closed partition ([Lun73], see e.g. [KR08] for an overview). In the case of G acting on the framed moduli

[8] There is a slight subtlety in characteristic 2 or 3 where we further need to impose the condition (3.1), see remarks in Section 3.2. For the sake of readability, we do not introduce a further adornment for $\mathcal{E}$.

space G_E, this reproduces the partition $\mathcal{M}_E(G)_{[H,A]}$ as described above. For our purposes it is convenient to focus mainly on the coarser partition according to connected reductive subgroups H, hence the choice of notation.

3.2. **Borel–de Siebenthal theory.** Fix a maximal torus T of G, and let Φ denote the corresponding set of roots. A subset $\Sigma \subseteq \Phi$ is called *closed* if $\mathbb{Z}\Sigma \cap \Phi = \Sigma$. We denote by $\mathcal{A}$ the set of closed subsets of Φ. To a closed subset Σ we associate the following subgroup of T

$$Z(\Sigma) := \bigcap_{\alpha \in \Sigma} \ker(\alpha).$$

We will also consider reductive subgroups H of G containing the maximal torus T that moreover satisfy an additional condition:

$$H = C_G(Z(H))^\circ. \tag{3.1}$$

It is proved in [Gil10] that if the characteristic of the field is not 2[9] or 3 then all reductive subgroups containing a maximal torus satisfy condition (3.1).

The connection between closed subsets of roots and connected reductive subgroups is given by Theorem 3.5 of Borel–de Siebenthal, extended to positive characteristic in [Gil10, Leh12]:

Theorem 3.5 ([BDS49, Gil10, Leh12])**.** *The collection of connected reductive subgroups $H \leq G$ that contain the maximal torus T, and in characteristic 2 or 3 satisfy condition* (3.1)*, is in bijection with $\mathcal{A}$. The correspondence is given by associating to H its root system and, conversely, to $\Sigma \in \mathcal{A}$ the subgroup $C_G(Z(\Sigma))^\circ$.*

Moreover, under this correspondence we have $G(\Sigma) = C_G(Z(\Sigma))^\circ$ and

$$Z(G(\Sigma)) = Z(\Sigma) = Z(C_G(Z(\Sigma))). \tag{3.2}$$

Proof. Apart from the last equality in Eq. (3.2), all is part of Borel–de Siebenthal theory. See for example [BDS49, Théorème 4 and Section 6, page 213] and [Gil10, Leh12].

For the last equality, it is enough to show that $Z(C_G(Z(\Sigma)))$ is contained in the neutral component $C_G(Z(\Sigma))^\circ$. If h is in $Z(C_G(Z(\Sigma)))$, then $h \in C_G(T) = T \subset C_G(Z(\Sigma))^\circ$. Since $Z(C_G(Z(\Sigma))^\circ) = Z(\Sigma)$ we deduce $Z(C_G(Z(\Sigma))) \subset C_G(Z(\Sigma))^\circ$ which moreover implies $Z(C_G(Z(\Sigma))) \subset Z(\Sigma)$. ☐

Remark 3.6. In this paper all reductive subgroups that will appear satisfy condition (3.1), essentially by construction. See Lemma 3.10 for the group of automorphisms of a semisimple bundle. See for example [Gil10, Lemma 0.1] or [Leh12, Proposition 1.1] for centralizers of subgroups of the maximal torus.

3.3. ***E*-root subsystems.** In this subsection, we will present an alternative approach to the theory of E-pseudo-Levi subgroups in terms of their associated root data.

Recall that T_E denotes the algebraic group parameterizing framed T-bundles on the fixed curve E. Thus T_E is isomorphic to either $\mathfrak{t}$ or T in the cuspidal and nodal cases respectively. In general

$$T_E \cong \operatorname{Hom}(\mathbb{X}^*(T), J(E)), \tag{3.3}$$

[9]For example, in the Lie algebra of type B_2 the vector space spanned by the root spaces corresponding to short roots is a Lie subalgebra because the structure constants are divisible by 2. This yields a sub-Lie algebra of $\mathfrak{so}(5)$ that does not exist in other characteristics and its root system is not closed.

where $J(E)$ denotes the Jacobian variety of E, and $\mathbb{X}^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ the character lattice.

Note that each character $\alpha \in \mathbb{X}^*(T)$ gives rise to a homomorphism $\alpha_* : T_E \to J(E)$, taking a T-bundle to its associated line bundle. For $p \in T_E$, we set

$$\Sigma_p := \{\alpha \in \Phi \mid p \in \ker(\alpha_*)\}\,. \tag{3.4}$$

We write $\mathcal{A}_E$ for the subset of $\mathcal{A}$ consisting of subsets $\Sigma_p \subset \Phi$ which occur in this way. The elements of $\mathcal{A}_E$ are called E-root subsystems of Φ.

Thus we have a map

$$\kappa : T_E \to \mathcal{A}$$

which assigns to a point $p \in T_E$ the set Σ_p. The image is $\mathcal{A}_E$ by definition and for $\Sigma \in \mathcal{A}_E$ we put

$$(T_E)_\Sigma := \kappa^{-1}(\Sigma)$$

with its reduced induced structure. Equivalently, by unwinding the definition of κ, one can describe this subvariety as

$$(T_E)_\Sigma = \{p \in T_E \mid \alpha \in \Phi,\ \alpha_*(p) = \mathbf{1}_{J(E)} \text{ if and only if } \alpha \in \Sigma\}. \tag{3.5}$$

Lemma 3.7. *The map $\kappa : T_E \to \mathcal{A}_E$ is continuous with respect to the topology on $\mathcal{A}_E$ induced by the partial order given by inclusion. In other words the partition*

$$T_E = \bigsqcup_{\Sigma \in \mathcal{A}_E} (T_E)_\Sigma$$

is locally closed.

Proof. It suffices to show that the preimage

$$(T_E)_{\geq \Sigma} = \kappa^{-1}(\{\Sigma' \mid \Sigma' \supseteq \Sigma\})$$

is closed. But this subset is just the intersection of root hyperplanes $\ker(\alpha_*) \subseteq T_E$ for $\alpha \in \Sigma$. □

As we will see in Section 3.4, the partition of T_E according to E-root subsystems records the conjugacy class of the neutral component the stabilizer of G acting on T_E. We may also record the component group of the stabilizer as follows.

Let $W = W_{G,T}$ be the Weyl group. Let $\Sigma \subseteq \Phi$ be an E-root subsystem. We write $N_W(\Sigma) \subseteq W$ for the normalizer of Σ. Let W_Σ denote the Weyl group of Σ (considered as a root system in its own right). We have that $N_W(\Sigma) = N_W(W_\Sigma)$ and so W_Σ is a normal subgroup in $N_W(\Sigma)$.

Lemma 3.8. *Let $p \in T_E$ and set $\Sigma = \Sigma_p$. Then*

$$W_\Sigma \subseteq \mathrm{Stab}_W(p) \subseteq N_W(\Sigma).$$

We write A_p for the corresponding subgroup of $W_{G,\Sigma} := N_W(\Sigma)/W_\Sigma$.

Thus to each $p \in T_E$ we have a pair (Σ_p, A_p) consisting of an E-root subsystem Σ_p and a subgroup A_p of $W_{G,\Sigma}$.

Proof. First note that, by definition of Σ, p is contained in each of the root hyperplanes corresponding to roots in Σ. Thus p is fixed by the corresponding reflections in W_Σ and thus by all of W_Σ. This proves the inclusion on the left.

Now let $w \in \mathrm{Stab}_G(p)$, and suppose $\alpha \in \Sigma$. Then

$$w(\alpha)_*(p) = \alpha_*(w^{-1}p) = \alpha_*(p) = \mathbf{1}_{J(E)}.$$

Thus $w \in N_W(\Sigma)$, establishing the inclusion on the right. □

3.4. **Connecting E-pseudo-Levis and E-root subsystems of roots.** Recall that we have fixed a maximal torus $T \subseteq G$, with associated roots Φ and Weyl group W.

It follows from Borel-de-Siebenthal theory (see Section 3.2) that the assignment $\Sigma \mapsto [G(\Sigma)]$ defines an order preserving bijection between $\mathcal{E}$ and $\mathcal{A}/W$. The following result states that Σ is an E-root subsystem if and only if $G(\Sigma)$ is an E-pseudo-Levi subgroup.

Proposition 3.9. *The assignment $\Sigma \mapsto G(\Sigma)$ defines an order-preserving bijection*

$$\mathcal{A}_E/W \xrightarrow{\sim} \mathcal{E}_E.$$

This follows immediately from the following result:

Lemma 3.10. *Let $p \in T_E$ and view it inside G_E. Then $\mathrm{Stab}_G(p)^\circ = G(\Sigma_p)$.*

Proof. To show that these two connected subgroups of G are equal, it suffices to show that their corresponding Lie algebras are equal inside $\mathfrak{g}$. Put $p = (\mathcal{P}_G, \theta)$. The Lie algebra of $\mathrm{Stab}_G(p)^\circ \cong \mathrm{Aut}(\mathcal{P}_G)$ is given by the global sections of the adjoint bundle $H^0(E; \mathfrak{g}_{\mathcal{P}})$. As $\mathcal{P}_G$ is induced from a T-bundle $\mathcal{P}$, the adjoint bundle $\mathfrak{g}_{\mathcal{P}_G}$ splits as a direct sum

$$\mathfrak{g}_{\mathcal{P}_G} = \mathfrak{g}_{\mathcal{P}} = \mathfrak{t}_{\mathcal{P}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha,\mathcal{P}}.$$

For each $\alpha \in \Phi$, the line bundle $\mathfrak{g}_{\alpha,\mathcal{P}}$ is trivial precisely when $\alpha \in \Sigma_p$. In this case, the framing provides a canonical identification

$$H^0(E; \mathfrak{g}_{\alpha,\mathcal{P}}) = \mathfrak{g}_\alpha.$$

Similarly, we have $H^0(E; \mathfrak{t}_{\mathcal{P}}) = \mathfrak{t}$. On the other hand, if $\alpha \notin \Sigma_p$, then $H^0(E; \mathfrak{g}_{\alpha,\mathcal{P}}) = 0$ because a line bundle of degree 0 on a curve has a section if and only if it is trivial. Thus we have that

$$H^0(E; \mathfrak{g}_{\mathcal{P}}) = \mathfrak{g}(\Sigma_p)$$

as required. □

Now recall that the map

$$q \colon T_E \to \mathcal{M}_E(G)$$

identifies $\mathcal{M}_E(G)$ with the categorical quotient $T_E /\!\!/ W$. Putting all this together, we have

Lemma 3.11. *The following diagram commutes:*

$$\begin{tikzcd} T_E \arrow[r, "\kappa"] \arrow[d, "q"'] & \mathcal{A}_E \arrow[d] \\ \mathcal{M}_E(G) \arrow[r, "h"'] & \mathcal{E}_E \end{tikzcd}$$

This proves Proposition 3.3 in view of Lemma 3.7.

3.5. **The component group.** Given a semisimple bundle in $\mathcal{P}_G \in \underline{G_E}$, we have shown that the neutral component $\mathrm{Aut}(\mathcal{P}_G)^\circ$ may be identified with $G(\Sigma_p)$ where Σ_p is the set of roots which annihilate a given framed lift $p \in T_E$ of $\mathcal{P}_G$.

We will now refine this to give a combinatorial description (i.e. in terms of the W-action on T_E) of the component group of $\mathrm{Stab}_G(p)$ ($\cong \mathrm{Aut}(\mathcal{P}_G)$).

Recall that for a connected reductive subgroup $T \subseteq H \subseteq G$ with root subsystem $\Sigma \subseteq \Phi$, we have an isomorphism

$$W_{G,H} := N_G(H)/H \cong N_W(\Sigma)/W_\Sigma := W_{G,\Sigma}.$$

This finite group is referred to as the relative Weyl group of H (or of Σ) in G. It naturally acts on the algebraic group $Z(H)_E$ (also written $Z(\Sigma)_E$). The following result identifies the component group of a semisimple bundle in $\mathcal{M}_E(G)_{[H]}$ with the stabilizer in the relative Weyl group of a corresponding lift to $p \in Z(H)_\Sigma$.

Lemma 3.12. *Let $p \in T_E$, and let $H = G(\Sigma_p)$. Then there are compatible identifications*

$$\begin{tikzcd}
 & \mathrm{Stab}_{W_{G,\Sigma}}(p) \arrow[r, hook] \arrow[d, "\wr"'] & N_W(\Sigma)/W_\Sigma \arrow[d, "\wr"] \\
\pi_0 \mathrm{Stab}_G(p) & \mathrm{Stab}_G(p)/H \arrow[l, "\sim"'] \arrow[r, hook] & N_G(H)/H.
\end{tikzcd}$$

Proof. Note that we have a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & N_H(T) \arrow[r] \arrow[d] & \mathrm{Stab}_{N_G(H)\cap N_G(T)}(p) \arrow[r] \arrow[d] & \mathrm{Stab}_{N_W(\Sigma)}(p)/W_\Sigma \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & H \arrow[r] & \mathrm{Stab}_G(p)/H \arrow[r] & \mathrm{Stab}_G(p)/H \arrow[r] & 1
\end{tikzcd}$$

It is a straightforward diagram chase to show that the right most vertical morphism is an isomorphism. Similarly, one checks that this isomorphism is compatible with the embeddings as required. $\square$

3.6. **Closure relations for the partition.** We shall see presently that the closure of the variety $(T_E)_\Sigma$ is a union of varieties $(T_E)_{\Sigma'}$. However, the Σ' that appear in the closure relation are determined by a slightly modified partial order relation which we determine below.

Let us first establish some more notation. Given a diagonalizable affine group scheme Z, we define[10]

$$Z_E := \mathrm{Hom}(\mathbb{X}^*(Z), J(E)),$$

where $\mathbb{X}^*(Z) = \mathrm{Hom}(Z, \mathbb{G}_m)$ is the character group. In this case, Z_E is itself a commutative group scheme. More precisely, we may write Z (non-canonically) as a product $(\mathbb{G}_m)^r \times K$ where r is a non-negative integer and K is a finite abelian group. In that case, Z_E is isomorphic to the product $J(E)^r \times K_E$, where $K_E := \mathrm{Hom}(K, J(E))$ is a finite group scheme (possibly non-reduced and disconnected).

[10]The notation G_E was previously defined only in the case when G is a connected reductive group. By (3.3), this definition is compatible with the previous one in their common domain of definition (i.e., when Z is a torus). One can define G_E more generally for possibly disconnected reductive groups, but we will not need this for the present paper.

With these conventions, we note that there is a natural isomorphism of algebraic groups

$$(Z_E)^{\circ,\mathsf{red}} \simeq (Z^{\circ,\mathsf{red}})_E. \tag{3.6}$$

Given $\Sigma \in \mathcal{A}_E$ we consider the subgroup $L(\Sigma) := C_G(Z(\Sigma)^{\circ,\mathsf{red}})$ where we recall that $Z(\Sigma) \leq T$ is defined by

$$Z(\Sigma) = \bigcap_{\alpha\in\Sigma} \ker(\alpha).$$

The group $L(\Sigma)$ is a Levi subgroup of G (as it is the centralizer of a torus) which clearly contains $G(\Sigma)$ (as every element of $G(\Sigma)$ centralizes $Z(\Sigma)^\circ$). In fact, it is the smallest such Levi subgroup. We will write Σ° for the set of roots of $L(\Sigma)$. We are ready now to define a new partial order on $\mathcal{A}$.

Remark 3.13. We are interested in the sets $(T_E)_\Sigma$ as subvarieties of T_E and so it makes sense to discard the nilpotents that can appear in $Z(\Sigma)_E$ or $(T_E)_\Sigma$ (relevant only for bad[11] primes for the group G).

Definition 3.14. Given $\Sigma_1, \Sigma_2 \in \mathcal{A}$, we define the partial order $\succeq$ by

$$\Sigma_1 \succeq \Sigma_2 \text{ if } \Sigma_1 \supseteq \Sigma_2 \text{ and } \Sigma_2 = \Sigma_1 \cap \Sigma_2^\circ.$$

We say that a closed subset is *isolated* if it is maximal with respect to this partial order.

Note that maximal subsets Σ are characterized by the fact that $Z(\Sigma)^{\circ,\mathsf{red}} = Z(G)^{\circ,\mathsf{red}}$. This notion is useful in order to relate our partition/stratification to the one of Lusztig [Lus84, 3.1]:

Proposition 3.15. *We have the closure relation*

$$\overline{(T_E)_\Sigma} = \bigsqcup_{\Sigma'\succeq\Sigma} (T_E)_{\Sigma'}.$$

Proof. As per Remark 3.13 we do not have to consider the non-reduced scheme structure that could appear on $(T_E)_\Sigma$ and on its closure.

First we claim that we have the following description of the closure:

$$\{(T_E)_\Sigma\}^- = (Z(\Sigma)_E)^{\circ,\mathsf{red}} \cdot (T_E)_\Sigma. \tag{3.7}$$

Indeed, note that $(T_E)_\Sigma$ is open in $Z(\Sigma)_E^{\mathsf{red}} = (T_E)_{\geq\Sigma}$, and thus its closure is necessarily a union of connected components of $Z(\Sigma)_E^{\mathsf{red}}$. In particular, the closure must be a union of orbits for the neutral component $(Z(\Sigma)_E)^{\circ,\mathsf{red}}$.

Now suppose $p \in T_E$ is in the closure of $(T_E)_\Sigma$. We must show that

$$\Sigma = \Sigma_p \cap \Sigma^\circ.$$

As $(T_E)_{\geq\Sigma}$ is closed and contains $(T_E)_\Sigma$, it must also contain $\overline{(T_E)_\Sigma}$ and thus p. In other words, we have $\Sigma \subseteq \Sigma_p$. By construction we have $\Sigma \subseteq \Sigma^\circ$ and hence $\Sigma \subseteq \Sigma_p \cap \Sigma^\circ$.

It remains to show the other inclusion. Let $\alpha \in \Sigma_p \cap \Sigma^\circ$. By (3.7) we may write

$$p = q \cdot r,$$

[11] A prime p is bad for the reductive group G if $\mathbb{X}^*(T)/\mathbb{Z}\Sigma$ has p-torsion for some $\Sigma \subset \Phi$. Only a handful of primes are concerned for each reductive group G.

where $q \in (Z(\Sigma)_E)^{\circ,\mathsf{red}}$ and $r \in (T_E)_\Sigma$. As $\alpha \in \Sigma_p$ we get $\alpha_*(p) = \mathbf{1}_{J(E)}$ (see (3.4)); as $\alpha \in \Sigma^\circ$ we have $Z(\Sigma)^{\circ,\mathsf{red}} \leq \ker \alpha$ which coupled with (3.6) gives $\alpha_*(q) = \mathbf{1}_{J(E)}$. Thus we deduce that $\alpha_*(r) = \mathbf{1}_{J(E)}$ which, by (3.5), implies $\alpha \in \Sigma$ as required. □

3.7. **Summary of section.** Recall that we have defined a locally closed partition in two ways

$$\mathcal{M}_E(G) = \bigsqcup_{[\Sigma]\in\mathcal{A}_E/W} \mathcal{M}_E(G)_{[\Sigma]} = \bigsqcup_{[H]\in\mathcal{E}_E} \mathcal{M}_E(G)_{[H]}. \tag{3.8}$$

We thus obtain a locally closed partition

$$\underline{G}_E = \bigsqcup_{[\Sigma]\in\mathcal{A}_E/W} (\underline{G}_E)_{[\Sigma]} = \bigsqcup_{[H]\in\mathcal{E}_E} (\underline{G}_E)_{[H]} \tag{3.9}$$

by pulling back via the characteristic polynomial map $\underline{\chi}\colon \underline{G}_E \to \mathcal{M}_E(G)$. Similarly for the framed version G_E using $\chi\colon G_E \to \mathcal{M}_E(G)$.

A stratum that plays a special role for us is the one corresponding to $H = G$. We put

$$(\underline{G}_E)_\heartsuit := (\underline{G}_E)_{[G]}.$$

One should think of this locus as those G-bundles whose semisimplification "is central" (i.e. has a reduction to the center $Z(G)$ of G).

Note that the set $\mathcal{E}$ carries partial orders $\leq, \preceq$ induced from the same named orders on $\mathcal{A}$ (see Definition 3.14).

Proposition 3.16. *The partitions from* (3.8), (3.9) *of* $\mathcal{M}_E(G)$ *and* G_E *have the closure relations determined by the partial order* $\preceq$.

Proof. Follows from Lemma 3.11 and Proposition 3.15. □

Remark 3.17. The upshot of this section is that there are two approaches to determining the type of a semisimple bundle: either compute its automorphism group or choose a reduction to T and compute the subset of roots on which the bundle vanishes.

4. The Jordan–Chevalley theorem

In this section we will prove Theorem 1.1 and Theorem 1.4 from Section 1. A key concept here is the notion of *regularity* which we define in Section 4.1. Then we will establish the equivalence of the two main theorems in Section 4.3. Finally we will prove Theorem 1.4 over Section 4.4 and Section 4.5.

4.1. **The regular locus.** Fix H an E-pseudo-Levi subgroup of G. Recall that this means that there exists a semisimple framed G-bundle $q \in G_E$ such that $\mathrm{Stab}_G(q)^\circ = H$.

Definition 4.1. We say that $p \in H_E$ is

(1) *G-regular* if $\mathrm{Stab}_G(p)^\circ \subseteq H$,
(2) *strongly G-regular* if $\mathrm{Stab}_G(p) \subseteq N_G(H)$,

(3) *maximally G-regular* if $\mathrm{Stab}_G(p) \subseteq H$.

One may immediately check that for a given $p \in H_E$ we have implications

$$\text{maximally } G\text{-regular} \implies \text{strongly } G\text{-regular} \implies G\text{-regular}$$

(the second implication uses that $N_G(H)^\circ = H$).

If the stabilizers of semisimple elements of G_E are always connected, then all three notions above coincide because the regularity only depends on the semisimplification (see Proposition 4.4). For example, this happens when E is cuspidal (so $G_E = \mathfrak{g}$) or when E is nodal and $G = G_E$ is simply connected. However, in general, the three notions are all distinct as illustrated by Example 4.2.

Example 4.2. Consider the case where E is a nodal curve, and thus we may identify $G_E = G$. Assume also that $char(k) = 0$.

(1) Let $G = PGL_2$ and T the maximal torus represented by the classes of diagonal matrices. Consider the matrix

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One can check that $\mathrm{Stab}_G([A]) = N_G(T)$. Thus $[X]$ is strongly G-regular (and thus regular) element of T, but it is not maximally G-regular .

(2) Now let $G = PGL_3$ and H the Levi subgroup consisting of classes of block matrices of the form:

$$\left(\begin{array}{cc|c} * & * & 0 \\ * & * & 0 \\ \hline 0 & 0 & * \end{array}\right).$$

We write T for the diagonal maximal torus again.

Consider the matrix

$$Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix},$$

where ζ is a primitive third root of unity. Then $\mathrm{Stab}_G([Y])$ is the preimage in $N_G(T)$ of the cyclic subgroup group $A_3 \subseteq S_3 \cong N_G(T)/T$.

In particular

$$\mathrm{Stab}_G([Y])^\circ = T \subseteq H$$

and thus $[Y]$ is a G-regular element of H. However,

$$\mathrm{Stab}_G([Y]) \not\subseteq H = N_G(H),$$

so $[Y]$ is not a strongly G-regular element of H (it is however a strongly but not maximally G-regular element of T).

We write H_E^{reg} (respectively, $H_E^{\mathsf{str\text{-}reg}}$, respectively, $H_E^{\mathsf{max\text{-}reg}}$) for the locus of G-regular (respectively, strongly G-regular, respectively, maximally G-regular) elements. As these loci are manifestly H-invariant (in fact, $N_G(H)$-invariant) we have corresponding loci $\underline{H}_E^{\mathsf{reg}}$, $\underline{H}_E^{\mathsf{str\text{-}reg}}$, $\underline{H}_E^{\mathsf{max\text{-}reg}}$ in the stack $\underline{H}_E$.

Remark 4.3. Whereas the loci $(\underline{G}_E)_{[H]}$ and $(\underline{G}_E)_{[H,A]}$ introduced in Section 3 are intrinsic to G, the G-regular locus $(\underline{H}_E)^{\mathsf{reg}}$ and its relatives are defined in terms of how H sits as a subgroup of G (i.e. are not intrinsic to H). In what follows, we will often need to consider both the intrinsic loci of $\underline{H}_E$ (such as $(\underline{H}_E)_{[K]}$ for some E-pseudo-Levi K of H) and the G-regular loci. To keep track of these notions, we will always label intrinsic loci in the subscript and regularity conditions in the superscript.

It will be useful to have a few other characterizations of the regularity condition. To state the result we will need to recall some notation.

Let $\mathcal{P}_H \in \underline{H}_E$. Choose an element $p \in T_E$ lifting $\chi(\mathcal{P}_H)$ under the map

$$T_E \to \mathcal{M}_E(H) = T_E /\!\!/ W_H.$$

This element determines a closed root subsystem $\Sigma_p \subseteq \Phi$ (see Section 3.3). Note that we consider Σ_p as a subset of the roots Φ of G and it might not be contained in the root subsystem Σ_H corresponding to H.

Recall that by Proposition 2.25 there is a unique-up-to-isomorphism semisimple bundle $\mathcal{P}_H^{\mathsf{ss}}$ with $\chi(\mathcal{P}_H^{\mathsf{ss}}) = \chi(\mathcal{P}_H)$.

Proposition 4.4. *Let $\mathcal{P}_H \in \underline{H}_E$, and choose $p \in T_E$ and $\mathcal{P}_H^{\mathsf{ss}}$ as described above. The following are equivalent:*

(1) *$\mathcal{P}_H$ is regular,*
(2) *$\mathcal{P}_H^{\mathsf{ss}}$ is regular,*
(3) *The morphism $\pi\colon \underline{H}_E \to \underline{G}_E$ is étale at $\mathcal{P}_H$,*
(4) *$\mathrm{H}^\bullet(E; (\mathfrak{g}/\mathfrak{h})_{\mathcal{P}_H}) = 0$,*
(5) *$\Sigma_p \subseteq \Sigma_H$.*

Remark 4.5. The equivalence of (2) and (1) means that the condition of $\mathcal{P}_H$ being G-regular depends only on the "characteristic polynomial" $\chi(\mathcal{P}_H)$. Thus we have a locus

$$\mathcal{M}_E(H)^{\mathsf{reg}} \subseteq \mathcal{M}_E(H)$$

such that $\mathcal{P}_H$ is G-regular if and only if $\chi(\mathcal{P}_H) \in \mathcal{M}_E(H)^{\mathsf{reg}}$. According to (5), the locus $\mathcal{M}_E(H)^{\mathsf{reg}}$ is equal to the image of $(T_E)_{\leq \Sigma_H}$ under the map $T_E \to \mathcal{M}_E(H)$. In particular, it follows from Proposition 4.4 that $\mathcal{M}_E(H)^{\mathsf{reg}}$ (respectively, $\underline{H}_E^{\mathsf{reg}}$) is open and dense in $\mathcal{M}_E(H)$ (respectively, $\underline{H}_E$).

Proof of Proposition 4.4. First, let us show that conditions (4) and (3) are equivalent. The map π is étale precisely at the points where its differential is a quasi-isomorphism of tangent complexes. Recall that the cohomology of the tangent complex of $\underline{H}_E$ at a bundle $\mathcal{P}_H$ is given by the cohomology of the adjoint bundle of $\mathcal{P}_H$, that is, by $\mathrm{H}^i(E; \mathfrak{h}_{\mathcal{P}_H})$.

The differential of π at $\mathcal{P}_H \in \underline{H}_E$ is the map

$$\mathbb{T}_{\underline{H}_E, \mathcal{P}_H} \to \pi^* \mathbb{T}_{\underline{G}_E, \pi(\mathcal{P}_H)}$$

which upon taking cohomology groups becomes

$$\mathrm{H}^i(E; \mathfrak{h}_{\mathcal{P}_H}) \to \mathrm{H}^i(E; \mathfrak{g}_{\mathcal{P}_G}), i = 0, 1.$$

The cone of this map of complexes is given by $\mathrm{H}^\bullet(E; (\mathfrak{g}/\mathfrak{h})_{\mathcal{P}_H})$. Thus we have that $\mathrm{H}^\bullet(E; (\mathfrak{g}/\mathfrak{h})_{\mathcal{P}_H}) = 0$ if and only if π is étale at $\mathcal{P}_H$ as required.

Let us show the equivalence of (4) and (5). Fix $\mathcal{P}_{B_H}$ a reduction of $\mathcal{P}_H$ to a Borel subgroup of H such that the induced T-bundle, call it $\mathcal{P}_T$, satisfies $\mathcal{P}_T \overset{T}{\times} H = \mathcal{P}_H^{\mathsf{ss}}$. (See paragraph above Proposition 4.4.) The vector bundle $(\mathfrak{g}/\mathfrak{h})_{\mathcal{P}_H} = (\mathfrak{g}/\mathfrak{h})_{\mathcal{P}_{B_H}}$ carries a filtration whose associated graded is $(\mathfrak{g}/\mathfrak{h})_{\mathcal{P}_T}$. This latter bundle is a direct sum of line bundles $(\mathfrak{g}_\alpha)_{\mathcal{P}_T}$ corresponding to roots $\alpha \in \Phi \backslash \Sigma_H$. By definition, the bundle $(\mathfrak{g}_\alpha)_{\mathcal{P}_T}$ is trivial if and only if $\alpha \in \Sigma_{\mathcal{P}_T} = \Sigma_p$. Noting that the cohomology of a degree 0 line bundle on E vanishes if and only if it is non-trivial, we deduce that (4) is equivalent to (5).

To see the equivalence of (2) and (5) we first note that, by Lemma 3.10,

$$G(\Sigma_p) = \mathrm{Stab}_G(p)^\circ.$$

Condition (2) means that the left hand side is contained in H, whereas condition (5) means that the right hand side is contained in H.

By Lemma 4.6, the bundle $\mathcal{P}_H$ is G-regular if and only if the homomorphism $\mathrm{Aut}(\mathcal{P}_H) \to \mathrm{Aut}(\mathcal{P}_G)$ induces an isomorphism $\mathrm{Aut}(\mathcal{P}_H)^\circ \cong \mathrm{Aut}(\mathcal{P}_H)^\circ$ (where $\mathcal{P}_G := \mathcal{P}_H \overset{H}{\times} G$). Taking the corresponding Lie algebras, we see that $\mathcal{P}_H$ is G-regular if and only if the map

$$H^0(E;\mathfrak{h}_{\mathcal{P}_H}) \to H^0(E;\mathfrak{g}_{\mathcal{P}_H})$$

is an isomorphism. Using the long exact sequence associated to the short exact sequence of H-modules

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$$

and Serre duality (the canonical bundle is trivial), this is equivalent to condition (4). We've shown that (4) is equivalent to (1). ☐

Lemma 4.6. *An element $p \in H_E$ is G-regular if and only if* $\mathrm{Stab}_G(p)^\circ = \mathrm{Stab}_H(p)^\circ$.

Proof. First note that $\mathrm{Stab}_H(p)^\circ$ is a connected subgroup of $\mathrm{Stab}_G(p)$ so it is necessarily contained in the neutral component: $\mathrm{Stab}_H(p)^\circ \subseteq \mathrm{Stab}_G(p)^\circ$. If $\mathrm{Stab}_G(p)^\circ = \mathrm{Stab}_H(p)^\circ$, then certainly $\mathrm{Stab}_G(p)^\circ \subseteq H$ so p is regular. Conversely, if p is regular, then $\mathrm{Stab}_G(p)^\circ \subseteq H \cap \mathrm{Stab}_G(p) = \mathrm{Stab}_H(p)$ is a connected subgroup and thus it must be contained in the neutral component. So $\mathrm{Stab}_G(p)^\circ = \mathrm{Stab}_H(p)^\circ$ as required. ☐

We write $Z(H)_E^{\mathsf{reg}}$ for the intersection $Z(H)_E \cap H_E^{\mathsf{reg}}$ (similarly for $Z(H)_E^{\mathsf{str\text{-}reg}}$, $Z_H(E)^{\mathsf{max\text{-}reg}}$). The various notions of regularity are somewhat simpler here.

Lemma 4.7. *Fix an element $p \in Z(H)_E$.*

1. *The element p is G-regular if and only if it is strongly G-regular.*
2. *Assume p is regular. It is maximally G-regular if and only if the relative Weyl group $W_{G,H}$ acts freely on the orbit of p.*

Proof. (1) Suppose p is regular, i.e. $\mathrm{Stab}_G(p)^\circ \subseteq H$. But $p \in Z(H)_E$, so $H \subseteq \mathrm{Stab}_G(p)$ and thus $\mathrm{Stab}_G(p)^\circ = H$. As the neutral component of any algebraic group is a normal subgroup, we have $\mathrm{Stab}_G(p) \subseteq N_G(H)$ as required.

(2) By regularity of p, $\mathrm{Stab}_G(p)^\circ = H$ as explained above. Thus p is maximally G-regular if and only if $\mathrm{Stab}_G(p)$ is connected. According to Lemma 3.12, the component group of $\mathrm{Stab}_G(p)$ is precisely the stabilizer of p in $W_{G,H}$, hence the claimed result. ☐

4.2. **The main results.** For convenience, we remind the reader of the statements of Theorem 1.1 and Theorem 1.4, to be proved in this section.

To state the first result, recall that for an E-pseudo-Levi subgroup H of G, we defined the G-regular locus $(H_E)^{\mathsf{reg}}$ in Section 4.1. We denote by $Z(H)^{\mathsf{reg}}$ the intersection $Z(H)_E \cap H_E^{\mathsf{reg}}$. The locus of unipotent bundles H_E^{uni} is defined to be fiber $\chi_H^{-1}(\mathbf{1})$ of the characteristic polynomial map $\chi_H : H_E \to \mathcal{M}_E(H)$.

Theorem 4.8 (Jordan decomposition)**.** *Given a semistable, degree* 0*, framed G-bundle $p \in G_E$, there is a unique triple (H, p_s, p_u) where H is an E-pseudo-Levi, $p_s \in Z(H)^{\mathsf{reg}}_E$, $p_u \in H^{\mathsf{uni}}_E$, and*

$$p = p_s \cdot p_u,$$

where the multiplication is defined via the group morphism $m\colon Z(H) \times H \to H$ (see Example 2.1*).*

To state the next result recall that we have defined a partition of $\underline{G}_E$ (and also of G_E) in Section 3.7

$$\underline{G}_E = \bigsqcup_{[H] \in \mathcal{E}_E} (\underline{G}_E)_{[H]}$$

and we defined the heart locus to be $(\underline{G}_E)_{\heartsuit} := (\underline{G}_E)_{[G]}$.

Theorem 4.9 (Galois theorem)**.** *For H an E-pseudo-Levi subgroup of G the morphism*

$$(\underline{H}_E)^{\mathsf{reg}}_{\heartsuit} \to \underline{G}_E$$

is a $W_{G,H}$-Galois covering onto $(\underline{G}_E)_{[H]}$.

To begin with we will establish the following result, which expresses the Jordan–Chevalley decomposition on the heart locus (where, in fact, it is simply a direct product). This may be thought of as a baby version of Theorem 4.8.

Proposition 4.10. *There is an equivalence of stacks*

$$(\underline{G}_E)_{\heartsuit} \simeq Z(G)_E \times \underline{G}^{\mathsf{uni}}_E.$$

Proof. The following is the cartesian diagram defining $(\underline{G}_E)_{\heartsuit}$ and $\underline{G}^{\mathsf{uni}}_E$:

$$\begin{tikzcd}
\underline{G}^{\mathsf{uni}}_E \arrow[r, hook] \arrow[d] & (\underline{G}_E)_{\heartsuit} \arrow[r, hook] \arrow[d] & \underline{G}_E \arrow[d, "\chi"] \\
\{\mathbf{1}\} \arrow[r, hook] & Z(G)_E \arrow[r, hook] & \mathcal{M}_E(G)
\end{tikzcd}$$

The group $Z(G)_E$ acts on both $\underline{G}_E$ and $\mathcal{M}_E(G)$ and the map χ is equivariant. This readily implies the required isomorphism from the statement.

One could also argue as follows: the natural product map

$$Z(G)_E \times \underline{G}^{\mathsf{uni}}_E \to (\underline{G}_E)_{\heartsuit}$$

is $Z(G)_E$-equivariant and this enables us to define its inverse by the following formula

$$\mathcal{P} \mapsto (\chi(\mathcal{P}), \chi(\mathcal{P})^{-1} \cdot \mathcal{P}). \qquad \square$$

Corollary 4.11. *We have an equivalences of stacks*

$$(\underline{H}_E)^{\mathsf{reg}}_{\heartsuit} \simeq Z(H)^{\mathsf{reg}}_E \times \underline{H}^{\mathsf{uni}}_E \simeq (\underline{H}_E)^{\mathsf{str\text{-}reg}}_{\heartsuit}.$$

Proof. The first equivalence follows from Proposition 4.10 and from the fact that regularity of an H-bundle is governed by its semisimple part, i.e. by the part in $Z(H)_E$ (see Proposition 4.4). The analogous equivalence also holds for the strongly G-regular locus. The second equivalence then follows from the fact that $Z(H)^{\mathsf{reg}}_E = Z(H)^{\mathsf{str\text{-}reg}}_E$ (Lemma 4.7). $\square$

4.3. **Equivalence of Theorems 4.8 and 4.9.** Our next step will be to show that the two main results are mutually equivalent.

Let us first recast Theorem 4.9 in terms of framed bundles. It states that the map

$$\widetilde{\pi}_{\heartsuit}^{\mathsf{reg}}\colon G \times^{N_G(H)} (H_E)_{\heartsuit}^{\mathsf{reg}} \to G_E \tag{4.1}$$

is a (G-equivariant) isomorphism onto $(G_E)_{[H]}$.

To prove Theorem 4.9, we must show the following two statements:

(1) ("Surjectivity") The image of $\widetilde{\pi}_{\heartsuit}^{\mathsf{reg}}$ is precisely $(G_E)_{[H]}$.
(2) ("Injectivity") The map $\widetilde{\pi}_{\heartsuit}^{\mathsf{reg}}$ is injective.

Analogously, to prove Theorem 4.8, we must show the following two statements:

(1) ("Existence") Every $p \in G_E$ has a Jordan datum (H, p_s, p_u) with $p = p_s \cdot p_u \in (H_E)_{\heartsuit}^{\mathsf{reg}}$.
(2) ("Uniqueness") Given $p \in G_E$ with two Jordan data (H, p_s, p_u) and (H', p_s', p_u') we have $(H, p_s, p_u) = (H', p_s', p_u')$.

We will show that the "existence" (respectively, "uniqueness") part of Theorem 4.8 is equivalent to the "surjectivity" (respectively, "injectivity") in Theorem 4.9.

4.3.1. *Existence implies surjectivity.* Suppose $p \in (G_E)_{[H]}$ with associated Jordan data H', p_s, p_u. It follows that $p \in (H'_E)_{\heartsuit}^{\mathsf{reg}} \subseteq (G_E)_{[H]}$ and thus $H' = \mathsf{Ad}(g)H$ for some $g \in G$. Then p is the image of

$$(g, g^{-1} \cdot p) \in G \times^{N_G(H)} (H_E)_{\heartsuit}^{\mathsf{reg}}$$

as required.

4.3.2. *Surjectivity implies existence.* Given $p \in (G_E)$, we let $[H]$ denote the unique conjugacy class of E-pseudo-Levi subgroups such that p is in the locus $(G_E)_{[H]}$. Surjectivity means that there exists

$$(g, p') \in G \times (H_E)_{\heartsuit}^{\mathsf{reg}}$$

such that $g \cdot p' = p$. By replacing p' with $g^{-1} \cdot p'$ and H with $\mathsf{Ad}(g^{-1})H$, we may assume that $p \in (H_E)_{\heartsuit}^{\mathsf{reg}}$. Recall (Corollary 4.11) that $(H_E)_{\heartsuit}^{\mathsf{reg}} \cong Z(H)_E^{\mathsf{reg}} \times H_E^{\mathsf{uni}}$. Thus p has a Jordan decomposition $p = p_s \cdot p_u$ as required.

4.3.3. *Uniqueness implies injectivity.* We must show that if $p, p' \in (H_E)_{\heartsuit}^{\mathsf{reg}}$ and $g \in G$ such that $g \cdot p = p'$, then $g \in N_G(H)$. As we have assumed $p, p' \in (H_E)_{\heartsuit}^{\mathsf{reg}}$, we have Jordan decompositions $p = p_s \cdot p_u$ and $p' = p_s' \cdot p_u'$. Note that $H = \mathrm{Stab}_G(p_s)^\circ = \mathrm{Stab}_G(p_s')^\circ$. By the uniqueness of the Jordan decomposition, we must have that $g \cdot p_s = p_s'$. Thus $\mathsf{Ad}(g)H = H$, so $g \in N_G(H)$ as required.

4.3.4. *Injectivity implies uniqueness.* Let $p \in G_E$ and suppose we have two Jordan data (H, p_s, p_u) and (H', p_s', p_u'). To prove the uniqueness of the Jordan decomposition it suffices to show that $H = H'$ (as within $(H_E)_{\heartsuit}^{\mathsf{reg}} = Z(H)_E^{\mathsf{reg}} \times (H_E)^{\mathsf{uni}}$, every element has a unique Jordan decomposition).

First observe that the Lusztig type of p is well-defined, so $[H] = [H']$, i.e. there exists $g \in G$ such that $\mathsf{Ad}(g)H = H'$.

Now we have $p \in (H'_E)_{\heartsuit}^{\mathsf{reg}}$, and thus $g^{-1} \cdot p \in (H_E)_{\heartsuit}^{\mathsf{reg}}$. It follows that p is the image of both $(1, p)$ and $(g, g^{-1} \cdot p)$ under the map

$$\widetilde{\pi}_{\heartsuit}^{\mathsf{reg}}\colon G \times^{N_G(H)} (H_E)_{\heartsuit}^{\mathsf{reg}} \to G_E.$$

By injectivity, we must have that $g \in N_G(H)$. But then $H' = H$ as required.

The remainder of this section will be taken up with the proof of Theorem 4.9 (and thus Theorem 4.8).

4.4. **Proof of surjectivity.** In this subsection we give a proof of the surjectivity part of Theorem 4.9. Specifically, we show the following:

Proposition 4.12. *The restriction*

$$\pi_{\heartsuit}^{\mathsf{reg}} \colon (\underline{H}_E)_{\heartsuit}^{\mathsf{reg}} \to \underline{G}_E$$

maps surjectively onto $(\underline{G}_E)_{[H]}$.

The statement splits into two parts:

(1) The image of

$$\pi_{\heartsuit}^{\mathsf{reg}} \colon (\underline{H}_E)_{\heartsuit}^{\mathsf{reg}} \to \underline{G}_E$$

is contained in $(\underline{G}_E)_{[H]}$.

(2) The image of $\pi_{\heartsuit}^{\mathsf{reg}}$ contains all of $(\underline{G}_E)_{[H]}$.

4.4.1. *Proof of* (1). Let $\mathcal{P}_H \in \underline{H}_E$ and let $\mathcal{P}_G = \pi(\mathcal{P}_H)$ be the induced bundle. Then we will show

(a) if $\mathcal{P}_H \in \underline{H}_E^{\mathsf{reg}}$ then $\mathcal{P}_G \in (\underline{G}_E)_{\leq [H]}$;

(b) $\mathcal{P}_H \in (\underline{H}_E)_{\heartsuit}$ if and only if $\mathcal{P}_G \in (\underline{G}_E)_{\geq [H]}$.

Remark 4.13. The converse of part (a) above is false. For example, suppose we are in the group case $E = E_{\mathrm{node}}$ with $G = G_E = GL_3$. Let $H \subseteq G$ denote the subgroup consisting of matrices of the form

$$\left(\begin{array}{cc|c} * & * & 0 \\ * & * & 0 \\ \hline 0 & 0 & * \end{array}\right).$$

Now let $p \in H$ denote the matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then $p \in G_{[H]}$ as $\mathrm{Stab}_G(p)$ is conjugate to H, but $p \notin H^{\mathsf{reg}}$ as $\mathrm{Stab}_G(p) \nsubseteq H$.

Proof of (a). First suppose that $\mathcal{P}_H \in \underline{H}_E$ is semisimple. Let $p \in H_E$ denote a lift to a framed bundle.

Then $\mathcal{P}_H$ is G-regular if and only if $\mathrm{Aut}(\mathcal{P}_G)^\circ \cong \mathrm{Stab}_G(p)^\circ \subseteq H$. On the other hand, $\mathcal{P}_G$ is contained in $(\underline{G}_E)_{\leq [H]}$ if and only if $\mathrm{Stab}_G(p)$ is contained in some G-conjugate of H.

Thus we have the required implication in case $\mathcal{P}_H$ is semisimple. In general, the result follows from the fact that the conditions $\mathcal{P}_H \in \underline{H}_E^{\mathsf{reg}}$ and $\mathcal{P}_G \in (\underline{G}_E)_{\leq [H]}$ only depend on the characteristic polynomial of $\mathcal{P}_H$ (respectively, $\mathcal{P}_G$) and thus only depend on the isomorphism class of the semisimplification (see Section 2.6). ☐

Proof of (b). Again, suppose $\mathcal{P}_H$ is semisimple. Then $\mathcal{P}_H$ is contained in $(\underline{H}_E)_{\heartsuit}$ if and only if $H = \mathrm{Stab}_H(p) = \mathrm{Stab}_G(p) \cap H$. This in turn is equivalent to $\mathcal{P}_G \in (\underline{G}_E)_{\geq [H]}$. As before, the general case follows from the fact that the conditions only depend on the characteristic polynomial. ☐

4.4.2. *Proof of* (2). Suppose $\mathcal{P}_G$ is contained in $(\underline{G}_E)_{\leq[H]}$ (respectively, $(\underline{G}_E)_{[H]}$). Then we will show that there exists $\mathcal{P}_H$ in $(\underline{H}_E)^{\mathsf{reg}}$ (respectively, $(\underline{H}_E)^{\mathsf{reg}}_{\heartsuit}$) such that $\pi(\mathcal{P}_H) = \mathcal{P}_G$.

Recall that by Proposition 4.4, the substack $\underline{H}_E^{\mathsf{reg}}$ is precisely the locus on which the map π is étale. In particular, the image $\pi(\underline{H}_E^{\mathsf{reg}})$ is an open substack of $\underline{G}_E$. We must show that this open substack contains all of $(\underline{G}_E)_{\leq[H]}$.

First suppose $\mathcal{P}_G \in (\underline{G}_E)_{\leq[H]}$ is semisimple. Let T be a maximal torus of H (and thus of G). Then, by Lemma 3.10, there is a framed reduction $p \in T_E$ such that $\mathrm{Stab}_G(p)^\circ \subseteq H$. Thus the induced H-bundle $\mathcal{P}_H$ is a G-regular H-reduction of $\mathcal{P}_G$ as required.

Now let us drop the assumption that $\mathcal{P}_G$ is semisimple, and let $\mathcal{P}_G^{\mathsf{ss}}$ denote a semisimplification of $\mathcal{P}_G$. Then $\mathcal{P}_G^{\mathsf{ss}}$ is semisimple and contained in $(\underline{G}_E)_{\leq[H]}$, thus by the above argument $\mathcal{P}_G^{\mathsf{ss}}$ is contained in $\pi(\underline{H}_E^{\mathsf{reg}})$. But $\mathcal{P}_G^{\mathsf{ss}}$ is contained in the closure of the point $\mathcal{P}_G$; thus any open neighbourhood of $\mathcal{P}_G^{\mathsf{ss}}$ in $\underline{G}_E$ contains $\mathcal{P}_G$. As $\pi(\underline{G}_E^{\mathsf{reg}})$ is such an open neighbourhood, we must have $\mathcal{P}_G = \pi(\mathcal{P}_H)$ for some $\mathcal{P}_H \in \underline{H}_E^{\mathsf{reg}}$ as required.

It remains to show that if moreover $\mathcal{P}_G \in (\underline{G}_E)_{[H]}$ then the above constructed $\mathcal{P}_H$ belongs to $(\underline{H}_E)_{\heartsuit}$. If p is a framed lift of $\mathcal{P}_H$ such that $\mathrm{Stab}_G(p)^\circ \subset H$ then the condition $\mathcal{P}_G \in (\underline{G}_E)_{\geq[H]}$ means $H \subset \mathrm{Stab}_G(p)^\circ$. We deduce that $H = \mathrm{Stab}_G(p)^\circ$ which in turn implies $\mathrm{Stab}_H(p) = H$, or in other words $\mathcal{P}_H \in (\underline{H}_E)_{\heartsuit}$.

4.5. **Proof of injectivity.** In this section we prove the injectivity required (see (4.1)) in the proof of Theorem 4.9. In other words, we must show that

$$\overline{\pi}^{\mathsf{reg}}_{\heartsuit} \colon (\underline{H}_E)^{\mathsf{reg}}_{\heartsuit}/W_{G,H} \to \underline{G}_E \tag{4.2}$$

is an embedding.

Let us first sketch an outline of the strategy of proof. We wish to apply the following general principle:

Proposition 4.14. *If an étale morphism of schemes $X \to Y$ is an embedding over a dense open subset of Y and X is separated, then it is an open embedding.*

Proof. We reduce it to [Gro67, Thm 17.9.1]. Namely, according to loc. cit., a morphism of schemes $U \to V$ is an open immersion if and only if it is flat, locally of finite presentation and a monomorphism in the category of schemes.

In our case we only need to check that the morphism is a monomorphism which follows at once from birationality and separatedness. ☐

Remark 4.15. Since being an open immersion is a property that is smooth-local on the target, Proposition 4.14 can be applied to a morphism of finite type stacks that is representable by separated schemes and this is how it will be used below.

While $\overline{\pi}$ is étale over the locus $(\underline{H}_E)^{\mathsf{reg}}$, we cannot apply Proposition 4.14 directly to the morphism $\overline{\pi}$ as it is not generically an embedding - its generic fiber has cardinality $|W|/|N_W(W_H))|$ (see Lemma 4.23).

However, the failure of $\overline{\pi}$ to be generically an embedding is precisely accounted for by the corresponding map

$$\overline{\rho} : \mathcal{M}_E(H)/\!\!/W_{G,H} \to \mathcal{M}_E(G)$$

on the level of coarse moduli spaces, which also has generic degree $|W|/|N_W(W_H)|$ (Lemma 4.23). The idea is thus to replace the target $\underline{G}_E$ with the base-change $\widetilde{\underline{G}_E}$:

$$\begin{tikzcd}
\underline{H}_E/W_{G,H} \arrow[r, "\overline{\nu}"] \arrow[rr, bend left, "\overline{\pi}"] \arrow[dr, "\overline{\chi}_H"'] & \widetilde{\underline{G}_E} \arrow[r, "\widetilde{\rho}"] \arrow[d] \arrow[dr, phantom, "\square"] & \underline{G}_E \arrow[d, "\underline{\chi}_G"] \\
& \mathcal{M}_E(H)/\!\!/W_{G,H} \arrow[r, "\overline{\rho}"'] & \mathcal{M}_E(G)
\end{tikzcd}$$

In this way, we obtain a morphism $\overline{\nu}$ which is generically an embedding (Lemma 4.17). Moreover, we will show that $\overline{\nu}$ is étale when restricted to the locus $\underline{H}_E^{\mathsf{str\text{-}reg}}/W_{G,H}$ (see Lemma 4.16), and thus an open embedding on this locus by Proposition 4.14 (we can apply it in this situation since $\overline{\nu}$ comes from an obvious G-equivariant morphism between varieties). As this locus contains the desired substack $(\underline{H}_E)^{\mathsf{reg}}_{\heartsuit}/W_{G,H}$ (recall that $(H_E)^{\mathsf{str\text{-}reg}}_{\heartsuit} = (H_E)^{\mathsf{reg}}_{\heartsuit}$ by Corollary 4.11), the morphism

$$\overline{\nu}^{\mathsf{reg}}_{\heartsuit} \colon (\underline{H}_E)^{\mathsf{reg}}_{\heartsuit}/W_{G,H} \to \widetilde{\underline{G}_E}$$

is an embedding.

Finally, we show that the restriction of the base change morphism

$$\widetilde{\rho}^{\mathsf{reg}}_{\heartsuit} : (\widetilde{\underline{G}_E})^{\mathsf{reg}}_{\heartsuit} \to \underline{G}_E$$

is an embedding (Corollary 4.20). Thus the composite

$$(\underline{H}_E)^{\mathsf{reg}}_{\heartsuit}/W_{G,H} \hookrightarrow (\widetilde{\underline{G}_E})^{\mathsf{reg}}_{\heartsuit} \hookrightarrow \underline{G}_E$$

is an embedding as required in (4.2).

Now let us go through the steps in the proof one by one. We start with Lemmas 4.16 and 4.17, whose proof will be given in Section 4.6.

Lemma 4.16. *The following restriction of $\overline{\nu}$ is étale*

$$\overline{\nu}^{\mathsf{str\text{-}reg}} \colon (\underline{H}_E)^{\mathsf{str\text{-}reg}}/W_{G,H} \to \widetilde{\underline{G}_E}.$$

Lemma 4.17. *There is an open dense subset of $\underline{H}_E/W_{G,H}$ on which $\overline{\nu}$ is an open embedding.*

Assuming Lemma 4.16 and Lemma 4.17, we may now establish:

Lemma 4.18. *The restriction*

$$\overline{\nu}^{\mathsf{reg}}_{\heartsuit} \colon (\underline{H}_E)^{\mathsf{reg}}_{\heartsuit}/W_{G,H} \to \widetilde{\underline{G}_E}$$

is an embedding.

Proof. By Lemma 4.16 $\overline{\nu}^{\mathsf{str\text{-}reg}}$ is étale, and by Lemma 4.17 it is generically an embedding. Thus by Proposition 4.14, $\overline{\nu}^{\mathsf{str\text{-}reg}}$ is an open embedding. The claim then follows immediately from the fact that $(\underline{H}_E)^{\mathsf{reg}}_{\heartsuit} = (\underline{H}_E)^{\mathsf{str\text{-}reg}}_{\heartsuit}$ (Corollary 4.11). $\square$

Lemma 4.19 is an analogue on the level of coarse moduli spaces of the desired injectivity part of Theorem 4.9. Its proof is given in Section 4.6.

Lemma 4.19. *The morphism*

$$\overline{\rho}^{\mathsf{reg}}_{\heartsuit} \colon \mathcal{M}_E(H)^{\mathsf{reg}}_{\heartsuit}/\!\!/W_{G,H} \to \mathcal{M}_E(G)$$

is an embedding.

It follows immediately that the base change is also an embedding:

Corollary 4.20. *The morphism*

$$\widetilde{\rho}^{\mathsf{reg}}_{\heartsuit} \colon (\widetilde{\underline{G}_E})^{\mathsf{reg}}_{\heartsuit} \to \underline{G}_E$$

is an embedding.

Putting Lemma 4.18 and Corollary 4.20 together we obtain that the composite

$$(\underline{H}_E)^{\mathsf{reg}}_{\heartsuit}/W_{G,H} \hookrightarrow (\widetilde{\underline{G}_E})^{\mathsf{reg}}_{\heartsuit} \hookrightarrow \underline{G}_E$$

is an embedding. This completes the proof of Theorem 4.9 (modulo the proofs in the following subsection).

4.6. **Proofs of Lemma 4.17, Lemma 4.16, and Lemma 4.19.** For this subsection we will fix a maximal torus T of our fixed E-pseudo-Levi subgroup H (and thus T is also a maximal torus of G).

Proposition 4.21 (which is essentially Theorem 4.9 for the special case $H = T$) forms a key step in the proof of Lemma 4.17.

Proposition 4.21. *The induction map*

$$\underline{T}^{\mathsf{reg}}_E \to \underline{G}_E$$

is an unramified W-Galois cover onto $(\underline{G}_E)_{[T]}$.

Proof. Notice first that by Proposition 4.12 the image lands inside $(\underline{G}_E)_{[T]}$. Now the claim is equivalent to the statement that

$$G \times^{N_G(T)} T^{\mathsf{reg}}_E \to (G_E)_{[T]}$$

is an isomorphism. We have already seen that the map is onto and étale (Propositions 4.4 and 4.12). For injectivity, we must show that for all $p \in T^{\mathsf{reg}}_E$ we have $\mathrm{Stab}_G(p) \subseteq N_G(T)$. But by definition $\mathrm{Stab}_G(p)^\circ = T$, so the required statement follows from the fact that $\mathrm{Stab}_G(T)$ normalizes its own neutral component. $\square$

Definition 4.22. We say that a morphism $f : X \to Y$ is *generically a covering of degree* d if there are dense open subsets U of X and $V = f(U)$ of Y such that $f|_U : U \to V$ is a covering (i.e. finite and étale) of degree d.

Lemma 4.23. *The following morphisms are generically covering maps of degree $|W|/|N_W(W_H)|$:*

(1) $\pi : \underline{H}_E/W_{G,H} \to \underline{G}_E$,
(2) $\rho : \mathcal{M}_E(H)/\!\!/W_{G,H} \to \mathcal{M}_E(G)$.

Proof. (1) Consider the following diagram:

$$\begin{tikzcd}
\underline{T}_E \arrow[r, "\beta"] \arrow[rr, bend left, "\alpha"] \arrow[dr] & \underline{H}_E \arrow[r, "\gamma"] \arrow[d, "\delta"] & \underline{G}_E \\
& \underline{H}_E/W_{G,H} \arrow[ur, "\pi"'] &
\end{tikzcd}$$

By Proposition 4.21, we have that α is generically a covering of degree $|W|$ and β is generically a covering of degree $|W_H|$. Thus γ is generically a covering of degree

$|W|/|W_H|$. By construction, δ is a covering of degree $|W_{G,H}|$. Thus π is generically a covering of degree $|W|/|W_H||W_{G,H}| = |W|/|N_W(W_H)|$ as required.

(2) Now consider the diagram:

$$\begin{tikzcd}
T_E \arrow[r, "\beta'"] \arrow[rr, bend left, "\alpha'"] & T_E /\!\!/ N_W(W_H) \arrow[r] \arrow[d, "\wr"] & T_E /\!\!/ W \arrow[d, "\wr"] \\
& \mathcal{M}_E(H) /\!\!/ W_{G,H} \arrow[r, "\rho"'] & \mathcal{M}_E(G)
\end{tikzcd}$$

As W acts freely on a dense open subset of T_E (namely $T_E^{\mathsf{max\text{-}reg}}$; see Definition 4.1), we have that α' is generically a covering of degree $|W|$ and β' is generically a covering of degree $|N_W(W_H)|$. Thus ρ is generically a covering of degree $|W|/|N_W(W_H)|$ as required. ☐

Proof of Lemma 4.17. Consider again the diagram:

$$\begin{tikzcd}
\underline{H}_E/W_{G,H} \arrow[r, "\overline{\nu}"] \arrow[rr, bend left, "\overline{\pi}"] \arrow[dr, "\overline{\chi}_H"'] & \widetilde{\underline{G}_E} \arrow[r, "\widetilde{\rho}"] \arrow[d] \arrow[dr, phantom, "\square"] & \underline{G}_E \arrow[d, "\underline{\chi}_G"'] \\
& \mathcal{M}_E(H) /\!\!/ W_{G,H} \arrow[r, "\overline{\rho}"'] & \mathcal{M}_E(G)
\end{tikzcd}$$

By Lemma 4.23, $\overline{\pi}$ and $\overline{\rho}$ are both generically coverings of degree $|W|/|N_W(W_H)|$. Thus, by base change, $\widetilde{\rho}$ is generically a covering of degree $|W|/|N_W(W_H)|$. But as $\overline{\pi} = \widetilde{\rho} \circ \overline{\nu}$ we must have that $\overline{\nu}$ is generically a covering of degree 1, i.e. generically an embedding as required. ☐

Lemma 4.24. *The morphism*

$$\mathcal{M}_E(H) /\!\!/ W_{G,H} \to \mathcal{M}_E(G)$$

is étale on the locus $\mathcal{M}_E(H)^{\mathsf{str\text{-}reg}} /\!\!/ W_{G,H}$.

Proof. Recall that

$$\mathcal{M}_E(G) \simeq T_E /\!\!/ W$$

and

$$\mathcal{M}_E(H) /\!\!/ W_{G,H} \simeq T_E /\!\!/ N_W(\Sigma),$$

where Σ is the root system of H. Using [Gro71, Prop V.2.2] we deduce that if $N_W(\Sigma)$ contains the stabilizer $\mathrm{Stab}_W(p)$, then the map

$$T_E /\!\!/ N_W(\Sigma) \to T_E /\!\!/ W$$

is étale at p. Thus the claim reduces to Lemma 4.25. ☐

Lemma 4.25. *Let* $p \in T_E \cap H_E^{\mathsf{str\text{-}reg}}$. *Then* $\mathrm{Stab}_W(p) \subseteq N_W(\Sigma)$.

Proof. By assumption $\mathrm{Stab}_G(p) \subseteq N_G(H)$. Suppose $w \in \mathrm{Stab}_W(p)$ and choose a lift to $\widetilde{w} \in N_G(T)$. Then $\widetilde{w} \in N_G(H)$ by assumption, and thus w preserves the root system Σ of H as required. ☐

We may now proceed with:

Proof of Lemma 4.16. Consider once more the diagram:

$$\begin{tikzcd} \underline{H}_E/W_{G,H} \arrow[r, "\overline{\nu}"] \arrow[rr, bend left, "\overline{\pi}"] \arrow[dr, "\overline{\chi}_H"'] & \widetilde{\underline{G}_E} \arrow[r, "\widetilde{\rho}"] \arrow[d] \arrow[dr, phantom, "\square"] & \underline{G}_E \arrow[d, "\underline{\chi}_G"] \\ & \mathcal{M}_E(H)/\!\!/W_{G,H} \arrow[r, "\overline{\rho}"'] & \mathcal{M}_E(G) \end{tikzcd}$$

By Lemma 4.24, $\overline{\rho}$ is étale on $\mathcal{M}_E(H)^{\mathsf{str\text{-}reg}}/\!\!/W_{G,H}$. Therefore its base change $\widetilde{\rho}$ is étale on $\widetilde{\underline{G}_E}^{\mathsf{str\text{-}reg}}$. We also know that $\underline{H}_E^{\mathsf{str\text{-}reg}} \subseteq \underline{H}_E^{\mathsf{reg}}$ is étale over $\underline{G}_E$ by Proposition 4.4. Thus both the source and target of $\overline{\nu}^{\mathsf{str\text{-}reg}}$ are étale over $\underline{G}_E$, and hence $\overline{\nu}^{\mathsf{str\text{-}reg}}$ is itself étale as required. $\square$

Finally we come to

Proof of Lemma 4.19. We have a commutative diagram

$$\begin{tikzcd} & T_E \arrow[dl] \arrow[dr] & \\ \mathcal{M}_E(H) \arrow[rr] & & \mathcal{M}_E(G) \end{tikzcd}$$

which exhibits $\mathcal{M}_E(G)$ as $T_E/\!\!/W$ and $\mathcal{M}_E(H)$ as $T_E/\!\!/W_H$. The further quotient $\mathcal{M}_E(H)/\!\!/W_{G,H}$ is thus identified with $T_E/\!\!/N_W(W_H)$ (recall that $W_{G,H} \cong N_W(W_H)/W_H$).

Let us denote by $\Sigma = \Sigma_H \subseteq \Phi$ the roots of H with respect to T. The locus $\mathcal{M}_E(G)_{[H]}$ (respectively, $\mathcal{M}_E(H)^{\mathsf{reg}}_{\heartsuit}$) is precisely the image of $(T_E)_\Sigma$ (see Lemma 3.11).

Thus $\mathcal{M}_E(G)_{[H]}$ is identified with the quotient of $(T_E)_\Sigma$ by the subgroup of W which preserves the locus $(T_E)_\Sigma$. But this subgroup is precisely $N_W(W_H)$. Thus both $\mathcal{M}_E(G)_{[H]}$ and $\mathcal{M}_E(H)^{\mathsf{reg}}_{\heartsuit}/\!\!/W_{G,H}$ are identified with $(T_E)_\Sigma/\!\!/N_W(W_H)$. In particular, the map

$$\mathcal{M}_E(H)^{\mathsf{reg}}_{\heartsuit}/\!\!/W_{G,H} \to \mathcal{M}_E(G)_{[H]}$$

is an isomorphism as required. $\square$

5. The Tannakian approach and unipotent bundles

The goal of this section is to understand the geometry of the locus of unipotent bundles G_E^{uni} in G_E. Let $(G_E)^\wedge_{\mathsf{uni}}$ denote the formal neighbourhood of the unipotent locus. Similarly, we have the unipotent cone G^{uni} in G and its formal neighbourhood $G^\wedge_{\mathsf{uni}}$. We will prove:

Theorem 5.1. *Let E, E' be two pointed curves of arithmetic genus 1. Then any isomorphism of formal groups $\widehat{J(E)} \cong \widehat{J(E')}$ defines G-equivariant isomorphisms*

$$\begin{tikzcd} (G_{E'})^\wedge_{\mathsf{uni}} \arrow[r, "\sim"] & (G_E)^\wedge_{\mathsf{uni}} \\ G_{E'}^{\mathsf{uni}} \arrow[r, "\sim"] \arrow[u, hook] & G_E^{\mathsf{uni}} \arrow[u, hook] \end{tikzcd}$$

Corollary 5.2. *If E is an ordinary elliptic curve over k (in particular, if char(k) $= 0$) then we have isomorphisms*

$$\begin{tikzcd} G^{\wedge}_{\mathsf{uni}} \arrow[r, "\sim"] & (G_E)^{\wedge}_{\mathsf{uni}} \\ G^{\mathsf{uni}} \arrow[u, hook] \arrow[r, "\sim"] & G_E^{\mathsf{uni}} \arrow[u, hook] \end{tikzcd}$$

In order to prove Theorem 5.1 we will use the following result.

Theorem 5.3. *Given a genus 1 marked curve E as above, there is an equivalence of stacks*

$$\underline{G}_E \simeq \mathrm{Fun}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Tor}(J(E))).$$

Theorem 5.3 is a combination of a Tannaka duality statement, expressing G-bundles in terms of their associated vector bundles, and a Fourier-Mukai duality, relating vector bundles on E and torsion sheaves on $J(E)$.

The idea of the proof of Theorem 5.1 is to use the isomorphism of formal groups $\widehat{E'} \cong \widehat{E}$ to identify torsion sheaves on E and E' which are supported in a formal neighbourhood of the identity.

Remark 5.4. Note that in characteristic zero, we can identify $\widehat{E} \cong \widehat{\mathbb{G}_m} \cong \widehat{\mathbb{G}_a}$. Thus, under these conditions we obtain isomorphisms between the nilpotent, unipotent, and elliptic unipotent cones (and their formal neighbourhoods). These isomorphisms arise from exponential maps in characteristic 0.

In characteristic $p > 0$, there is no isomorphism of formal groups $\widehat{\mathbb{G}_a} \cong \widehat{\mathbb{G}_m}$. Nevertheless, under very mild conditions on the characteristic, there are G-equivariant isomorphisms (the so-called Springer isomorphisms) between the nilpotent cone in $\mathfrak{g}$ and the unipotent cone in G. It is natural to conjecture that there are also Springer isomorphisms between G_E^{uni} and G^{uni} (and $\mathfrak{g}^{\mathsf{uni}}$). We plan to return to this in future work.

5.1. **Tannaka duality.** In this subsection, we will use Tannaka duality to express the stack $\underline{G}_E$ in terms of its associated vector bundles. The primary reference will be the paper [Lur04], however see also [Nor76] for the original approach.

Let S be a k-scheme. We denote by $\mathrm{Rep}_k(G)$ the symmetric monoidal category of finite dimensional representations of G over k. Given a G-bundle $\mathcal{P}_G$ on S, the associated vector bundle construction affords a symmetric monoidal functor

$$\begin{aligned} \mathrm{ass}(\mathcal{P}_G)\colon\ \mathrm{Rep}(G) &\to \mathrm{Vect}(S), \\ V &\mapsto \mathcal{P}_G \times^G V, \end{aligned}$$

where the right hand side denotes the exact category of vector bundles on S (with monoidal structure given by tensor product). Moreover, $\mathrm{ass}(\mathcal{P}_G)$ is continuous (meaning it preserves all small colimits), exact, and sends finite dimensional representations to vector bundles on S.

The basic idea of Tannakian reconstruction is that the bundle $\mathcal{P}_G$ can be recovered from the functor $\mathrm{ass}(\mathcal{P}_G)$.

More precisely, let $\mathrm{Fun}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Vect}(S))$ denote the groupoid of exact tensor functors (note that $\mathrm{Vect}(S)$ is an exact category, so this makes sense).

Theorem 5.5. *The associated bundle construction defines an equivalence of groupoids*

$$\mathrm{ass}\colon \mathrm{Bun}_G(S) \xrightarrow{\sim} \mathrm{Fun}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Vect}(S)).$$

Proof. Recall that we can think of $\mathrm{Bun}_G(S)$ as the mapping stack $\mathcal{H}om(S, BG)$. By [Lur04, Theorem 5.11] the associated bundle construction defines an equivalence of groupoids

$$\mathrm{ass}\colon \mathrm{Bun}_G(S) \xrightarrow{\sim} \mathrm{Fun}^{\otimes}_{tame}(\mathrm{QC}(BG), \mathrm{QC}(S)), \tag{5.1}$$

where the right hand side denotes the groupoid of continuous (i.e., colimit preserving), tame tensor functors. Note that it is immediate to see that for a test scheme S' the map ass corresponds to the pullback

$$(f\colon S' \times S \to BG) \mapsto (f^*\colon \mathrm{QCoh}(BG) \to \mathrm{QCoh}(S \times S')).$$

By definition, a functor is *tame* (see [Lur04, Definition 5.9]) if it preserves flat objects and short exact sequences of flat objects.

As every object of $\mathrm{QC}(BG)$ (which is identified with the category of $\mathcal{O}(G)$-comodules) is flat, the right hand side consists of exact functors which take values in flat objects of $\mathrm{QC}(S)$. Moreover, $\mathrm{QC}(BG)$ is the Ind-completion of $\mathrm{Rep}_k(G)$, and thus the data of an exact, continuous functor from $\mathrm{QC}(BG)$ is equivalent to specifying an exact functor from $\mathrm{Rep}_k(G)$. By continuity of the tensor product, this equivalence preserves symmetric monoidal structures. Finally, by construction, every $\mathrm{ass}(\mathcal{P}_G)(V)$ is a vector bundle for every finite dimensional representation V. Thus we can identify the groupoid of tensor functors in (5.1) with those in the statement of the theorem, as required. ☐

Let E be a curve of arithmetic genus 1 as usual, and let $E_S = E \times S$ denote the base-change to an arbitrary test scheme S.

Note that a G-bundle on E_S is semistable if and only if all its associated vector bundles are semistable and of degree 0 (see Remark 2.4). Thus we may identify the sublocus $\underline{G}_E = \mathrm{Bun}^0_G(E)^{\mathsf{ss}}$ in terms of Tannaka duality:

Corollary 5.6. *For each test scheme S, the associated bundle construction defines an equivalence of groupoids*

$$\underline{G}_E(S) \xrightarrow{\sim} \mathrm{Fun}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Vect}^{\mathsf{ss},0}(E_S)),$$

where the right hand side denotes the groupoid of exact tensor functors.

5.2. **Fourier-Mukai transform.** Let J be a smooth one-dimensional commutative group scheme, e.g. $J(E)$ for E elliptic curve, $\mathbb{G}_m$, or $\mathbb{G}_a$. Given a test scheme S, we define the category of S-families of torsion sheaves on J:

$$\mathrm{Tor}(J)(S) := \left\{ \mathcal{P} \in \mathrm{Coh}(J_S) \,\middle|\, \begin{array}{l} \mathcal{P} \text{ flat over } S \\ \mathrm{Supp}(\mathcal{P}) \to S \text{ is finite} \end{array} \right\}.$$

The category $\mathrm{Tor}(J)(S)$ carries a monoidal structure given by convolution:

$$\mathcal{P}_1 * \mathcal{P}_2 = m_*(\mathcal{P}_1 \boxtimes \mathcal{P}_2),$$

where $m\colon J_S \times_S J_S \to J_S$ is the multiplication map. This construction defines a presheaf of tensor categories on Sch_k.

Theorem 5.7. *Let E be an irreducible projective curve of arithmetic genus one. The assignment*

$$\mathcal{L} \mapsto \mathcal{O}_{[\mathcal{L}]}$$

taking a degree 0 *line bundle on E to the corresponding skyscraper sheaf on $J(E)$ extends to an equivalence of symmetric monoidal categories*

$$\left(\mathrm{Vect}^{0,\mathsf{ss}}(E_S), \otimes\right) \simeq \left(\mathrm{Tor}(J(E))(S), *\right).$$

Proof. The equivalence of abelian categories is classical in the case of a smooth elliptic curve; in our situation where E is an integral curve of arithmetic genus 1, it may be deduced from [Teo99, Theorem 1.3 (for the absolute case), Theorem 1.9 (for the relative case)]. More generally, it is shown in *loc. cit.* that there is a canonical equivalence between degree 0 semistable torsion-free sheaves on E and torsion sheaves on E. It follows readily from the construction (see Proposition 1.8 of *loc. cit.*) that when the torsion-free sheaf happens to be a vector bundle, the corresponding torsion sheaf is supported on the smooth locus of E, which is identified with $J(E)$ (using the section x_0).

In fact, this construction is an example of a Fourier-Mukai transform. We have a canonical Poincaré line bundle $\mathcal{P}$ on $E \times J(E)$ (normalized at $x_0 \in E$) with the defining property that $\mathcal{P}|_{E\times\{\mathcal{L}\}} \cong \mathcal{L}$ for any degree zero line bundle $\mathcal{L} \in J(E)$. This kernel extends to a torsion-free sheaf $\widetilde{\mathcal{P}}$ on $E \times E$. Taking $\widetilde{\mathcal{P}}$ as a Fourier-Mukai kernel defines a functor

$$\mathbb{F}\colon D^b(E) \to D^b(E)$$

which according to Burban–Kreussler [BK05, Theorem 2.21] agrees with the functor of [Teo99] when restricted to semistable torsion-free sheaves of degree zero, and thus further restricts to the desired functor on semistable degree 0 vector bundles.

From this perspective, the claim about symmetric monoidal structures is a special case of a more general claim that the Fourier-Mukai transform $D^b(E) \to D^b(J(E))$ induced by the Poincaré bundle intertwines the tensor product on E with convolution (also known as Pontryagin product) induced by the group operation on $J(E)$. See e.g. [BBR09] for the case of abelian varieties, which applies to our setting when E is a smooth elliptic curve.

Alternatively, in the cuspidal and nodal cases, one may directly apply the results of [FM01, Corollary 2.1.4, 2.2.4] which give an equivalence between degree 0 semistable vector bundles on E of rank n and conjugacy classes of $n \times n$ matrices (respectively, invertible matrices). It may be readily checked that, taking all ranks n together, these equivalences define a symmetric monoidal equivalence between all degree zero semistable vector bundles on E and torsion modules for $k[t]$ (respectively, $k[t, t^{-1}]$). □

In particular, we obtain a description of the moduli stack of GL_n-bundles in terms of torsion sheaves. Each S-family of torsion sheaves has a well-defined length: the degree of $\mathrm{Supp}(\mathcal{P}) \to S$. For each test scheme S we let $\mathrm{Tor}_n(J)(S)$ denote the maximal subgroupoid of $\mathrm{Tor}(J)(S)$ whose objects are torsion sheaves of length n.

Corollary 5.8. *There are equivalences of stacks*

$$\underline{\mathrm{GL}}_{n,E} \cong \mathrm{Vect}_n^{0,\mathsf{ss}}(E) \cong \mathrm{Tor}_n(J(E)).$$

Putting together the Tannakian statement Corollary 5.6 with the Fourier-Mukai statement Theorem 5.7 we obtain:

Corollary 5.9. *There is an equivalence of stacks*

$$\underline{G}_E \simeq \mathrm{Fun}^{\otimes}(\mathrm{Rep}_k(G), \mathrm{Tor}(J(E))).$$

5.3. **Torsion sheaves and effective divisors.** We wish to identify the loci of unipotent bundles in terms of their associated torsion sheaves. This will be expressed in terms of the *Norm map* which associates a Cartier divisor to a torsion sheaf.

First let us recall that for a smooth curve J, we have isomorphisms

$$\mathrm{Div}_n(J) \cong \mathrm{Hilb}_n(J) \cong \mathrm{Sym}^n(J),$$

where:

- $\mathrm{Div}_n(J)$ denotes the moduli of effective Cartier divisors of degree n,
- $\mathrm{Hilb}_n(J)$ is the Hilbert scheme of points of length n on J,
- $\mathrm{Sym}^n(J)$ is the quotient $J^n/\mathfrak{S}_n$.

(See e.g. Milne [Mil86, Theorem 3.13].) Moreover, these are all smooth varieties of dimension n.

Given a test scheme S and an object $\mathcal{P}$ of $\mathrm{Tor}_n(J)(S)$ we will associate a relative effective Cartier divisor $Div_n(\mathcal{P})$ on J_S which measures the support of $\mathcal{P}$ with multiplicity.

We will sketch one construction of $Div(\mathcal{P})$ from [MFK94, Section 5.3] (see also [KM76]). Suppose we are given $\mathcal{P}$ an S-family of torsion sheaves on J. In particular, $\mathcal{P}$ is a coherent sheaf on J_S which is flat over S and with no support in depth 0. We consider the determinant line $\det(\mathcal{P})$, which may be defined locally in terms of a free resolution. As $\mathcal{P}$ has no generic support, $\det(\mathcal{P})$ carries a canonical section $\mathcal{O}_{J_S} \to \det(\mathcal{P})$ defined away from the support of $\mathcal{P}$. This rational section defines the Cartier divisor $Div(\mathcal{P})$.

This construction gives rise to a morphism of stacks:

$$Div\colon \mathrm{Tor}_n(J) \to \mathrm{Div}_n(J) \cong \mathrm{Sym}^n(J).$$

Note that $\mathrm{Sym}^n(J(E))$ is isomorphic to the base of the characteristic polynomial map $\mathcal{M}_E(GL_n)$. In fact, Lemma 5.10 explains that the morphism Div is a realization of the characteristic polynomial map via the equivalence of GL_n-bundles and torsion sheaves of length n.

Lemma 5.10. *The equivalence of Corollary* 5.8 *fits into a commutative square:*

$$\begin{tikzcd} \underline{GL}_{n,E} \arrow[r, "\sim"] \arrow[d] & \mathrm{Tor}_n(J(E)) \arrow[d, "Div"] \\ \mathcal{M}_E(GL_n) \arrow[r, "\sim"] & \mathrm{Sym}^n(J(E)) \end{tikzcd}$$

Proof. We have already constructed all the arrows in the diagram. To see that the diagram commutes, it is sufficient to check the commutativity on the dense substacks consisting of semisimple objects. This in turn reduces to checking the assertion for $n = 1$ where it is clear. $\square$

5.4. **Unipotent bundles.** Recall that $\mathbf{1} \in J$ denotes the unit for the group structure. Let $\mathbf{1}_n \in \mathrm{Sym}^n(J)$ correspond to the effective Cartier divisor on J given by the unique length n subscheme of J whose underlying reduced scheme is $\mathbf{1} \times S$. We will also use the notation $\mathbf{1}_n$ to denote the corresponding length n subscheme of J.

The space of unipotent torsion sheaves on J, denoted $\mathrm{Tor}_n(J)^{\mathsf{uni}}$, is defined to be the fiber of $\mathrm{Tor}_n(J)$ over the point $\mathbf{1}_n$ in $\mathrm{Sym}^n(J)$. In other words, an S-family of torsion sheaves is called unipotent if the corresponding divisor is equal to $\mathbf{1}_{n,T}$.

We write $\mathrm{Tor}_n(J)^{\wedge}_{\mathsf{uni}}$ for the completion of $\mathrm{Tor}_n(J)$ along the substack $\mathrm{Tor}_n(J)^{\mathsf{uni}}$. The goal of the remainder of this subsection is to show that these stacks only depend on the formal neighbourhood of $\mathbf{1} \in J$.

Just as above, we may define the presheaf of categories $\mathrm{Tor}_n(\widehat{J})$ and the presheaf of sets $\mathrm{Sym}^n(\widehat{J})$ parameterizing S-families of torsion sheaves (respectively, effective Cartier divisors) in $\widehat{J}$. Again, there is an associated divisor map denoted abusively also by Div:

$$Div\colon \mathrm{Tor}_n(\widehat{J}) \to \mathrm{Sym}^n(\widehat{J}).$$

We denote by $\mathrm{Tor}_n(\widehat{J})^{\mathsf{uni}}$ the fiber over $\mathbf{1}_n$.

Lemma 5.11. *We have natural isomorphisms giving rise to a commutative diagram*

$$\begin{tikzcd} \mathrm{Tor}_n(\widehat{J}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Tor}_n(J)^{\wedge}_{\mathsf{uni}} \arrow[d] \\ \mathrm{Sym}^n(\widehat{J}) \arrow[r, "\sim"] & \mathrm{Sym}^n(J)^{\wedge}_{\mathbf{1}_n}. \end{tikzcd}$$

In particular, there is an equivalence

$$\mathrm{Tor}_n(J)^{\mathsf{uni}} \cong \mathrm{Tor}_n(\widehat{J})^{\mathsf{uni}}.$$

Proof. First, we claim that for a test scheme S, the image of $\mathrm{Tor}_n(\widehat{J})(S)$ in $\mathrm{Tor}_n(J)(S)$ consists of S-families of torsion sheaves on J which are set-theoretically supported on the closed subset $\mathbf{1}_S \subseteq J_S$. Indeed, note that an S-family of torsion sheaves on $\widehat{J}$ is, by definition, given by an S-family of torsion sheaves on some infinitesimal thickening of $\mathbf{1} \in J$. Giving such a family is indeed equivalent to an S-family of torsion sheaves on J which is set-theoretically supported on $\mathbf{1}_S \subseteq J_S$. Similarly, one shows that the image of $\mathrm{Sym}^n(\widehat{J})(S)$ in $\mathrm{Sym}^n(J)(S)$ consists of S-families of divisors which are set-theoretically supported in $\mathbf{1}_S$.

To show that $\mathrm{Sym}^n(\widehat{J}) \cong \mathrm{Sym}^n(J)^{\wedge}_{\mathbf{1}_n}$ observe that an S-point of the completion $\mathrm{Sym}^n(J)^{\wedge}_{\mathbf{1}_n}$ corresponds to an S-family of degree n effective divisors in $\mathrm{Sym}^n(J)(S)$ whose restriction to S_{red} is equal to the divisor $\mathbf{1}_{n,S_{\mathsf{red}}}$. This is equivalent to saying that the set-theoretic support of the divisor is contained in $\mathbf{1}_S$ as required.

Finally, to show that $\mathrm{Tor}_n(\widehat{J}) \cong \mathrm{Tor}_n(J)^{\wedge}_{\mathsf{uni}}$, it suffices to prove that the diagram

$$\begin{tikzcd} \mathrm{Tor}_n(\widehat{J}) \arrow[r] \arrow[d] & \mathrm{Tor}_n(J) \arrow[d] \\ \mathrm{Sym}^n(\widehat{J}) \arrow[r] & \mathrm{Sym}^n(J) \end{tikzcd}$$

is cartesian. This follows from the observation that an S-family of torsion sheaves on J is set-theoretically supported on $\mathbf{1}_S$ if and only if the associated divisor is set-theoretically supported on $\mathbf{1}_S$. $\square$

Remark 5.12. In fact, an S-family of torsion sheaves is unipotent if and only if its (scheme-theoretic) support is contained in the subscheme $\mathbf{1}_{n,S}$. This is a consequence of the Cayley-Hamilton theorem (see Example 5.13).

Example 5.13. Let $J = \mathbb{G}_a = \mathrm{Spec}(k[t])$, and let $S = \mathrm{Spec}(R)$ for a Noetherian local ring R. Then the groupoid $\mathrm{Tor}_n(\mathbb{G}_a)(R)$ consists of $R[t]$-modules M which are free over R of rank n.

Given such a module M and choosing an R-basis, the action of t is expressed by a matrix A_M. Then the associated divisor is given by the function $\det(t - A_M)$ (a polynomial of degree n with coefficients in R). The coefficients of $\chi_A(t) = \det(t - A_M)$ define the corresponding element of $R^n = \mathbb{A}^n(R) \cong \mathrm{Sym}^n(\mathbb{G}_a)(R)$. Note that, by the Cayley-Hamilton theorem, M_A is scheme-theoretically supported in the divisor $\chi_A(t)$. An R-point is nilpotent (respectively, formally nilpotent) if $\chi_A(t) = t^n$ (respectively, the non-leading coefficients of $\chi_A(t)$ are nilpotent in R).

5.5. **Unipotent cones.** The next result explains that the subfunctors of $\underline{G}_E$ corresponding to unipotent (respectively, infinitesimally unipotent) bundles can be understood via the equivalence of Corollary 5.9 as those functors which factor through the subcategories of unipotent (respectively, infinitesimally unipotent) torsion sheaves.

Proposition 5.14. *Let $\mathcal{P}$ denote an object of $\underline{G}_E(S)$ with associated functor*

$$F\colon \mathrm{Rep}(G) \to \mathrm{QC}(J(E) \times S).$$

Then $\mathcal{P}$ is contained in the subgroupoid of unipotent (respectively, infinitesimally unipotent) bundles if and only if, for each $V \in \mathrm{Rep}(G)$, $F(V)$ is a unipotent (respectively, infinitesimally unipotent) family of torsion sheaves.

Proof. First note that for any morphism of groups $G \to G'$, there is a commutative diagram:

$$\begin{tikzcd} \underline{G}_E \arrow[r] \arrow[d] & \underline{G'}_E \arrow[d] \\ \mathcal{M}_E(G) \arrow[r] & \mathcal{M}_E(G') \end{tikzcd}$$

It follows that if an object of $\underline{G}_E(S)$ is unipotent (respectively, infinitesimally unipotent) then its image in $\underline{G'}_E(S)$ is unipotent (respectively, infinitesimally unipotent).

In particular, we obtain such a diagram with $G' = GL(V)$ for each representation V of G. Thus (noting Lemma 5.10) it follows that if $\mathcal{P}_G$ is (infinitesimally) unipotent then all the associated torsion sheaves $F(V)$ are (infinitesimally) unipotent.

Note that if V is a faithful representation, then the map

$$\mathcal{M}_E(G) \to \mathcal{M}_E(GL_n) \cong \mathrm{Sym}^n(J(E))$$

has the property that the set-theoretic fiber of the basepoint $\mathbf{1}_n$ consists only of the basepoint $\mathbf{1}_G$ of $\mathcal{M}_E(G)$ (i.e. a semisimple G-bundle is trivial if and only if the associated vector bundle is trivial, at a set-theoretic level). It follows then that if $F(V)$ is infinitesimally unipotent, then $\mathcal{P}$ is infinitesimally unipotent.

This argument does not quite work for the non-infinitesimal case, as the map

$$\mathcal{M}_E(G) \to \mathcal{M}_E(GL_n) \cong \mathrm{Sym}^n(J(E))$$

is not injective on S-points for a general (possibly non-reduced) scheme S.

However, Lemma 5.15 implies that it is enough to take a sufficiently large collection of representations $V_1, \ldots, V_m$ (for example, the fundamental representations) to obtain that the map

$$\mathcal{M}_E(G) \to \prod_i \mathcal{M}_E(GL_{d_i}) \cong \mathrm{Sym}^{d_1}(J(E)) \times \ldots \times \mathrm{Sym}^{d_m}(J(E))$$

is a closed embedding in a formal neighbourhood of the basepoint (and thus injective on S-points set-theoretically supported on the basepoint). □

Lemma 5.15. *Let $V_1, \ldots, V_m$ be such that their classes generate $R(G)$ as a ring. For example, we can take the collection of fundamental representations. Then the corresponding map*

$$\mathcal{M}_E(G) \to \mathrm{Sym}^{d_1}(J(E)) \times \ldots \times \mathrm{Sym}^{d_m}(J(E)) \tag{5.2}$$

is a closed embedding in a formal neighbourhood of the basepoint $\mathbf{1}_G \in \mathcal{M}_E(G)$.

Proof. By assumption, the map of representation rings

$$R(GL(V_1)) \otimes \ldots \otimes R(GL(V_r)) \to R(G) \tag{5.3}$$

is surjective. Thus the corresponding map of varieties

$$T /\!\!/ W \to \mathrm{Sym}^{d_1}(\mathbb{G}_m) \times \ldots \times \mathrm{Sym}^{d_r}(\mathbb{G}_m)$$

is a closed embedding. In particular it is a closed embedding after completing at the basepoint of $T /\!\!/ W$.

Choosing an isomorphism of formal schemes (not necessarily respecting the group structure) $\widehat{\mathbb{G}_m} \cong \widehat{J(E)}$ gives an identification of the maps in (5.2) and (5.3) in a formal neighbourhood of the basepoint as required. □

We are now ready to prove Theorem 5.1 (and consequently Theorem 1.6 from Section 1).

Theorem 5.16. *Any equivalence of formal groups $\widehat{J(E_1)} \cong \widehat{J(E_2)}$ defines equivalences:*

$$\begin{tikzcd} (\underline{G}_{E_1})^\wedge_{\mathsf{uni}} \arrow[r, "\sim"] & (\underline{G}_{E_2})^\wedge_{\mathsf{uni}} \\ \underline{G}^{\mathsf{uni}}_{E_1} \arrow[u] \arrow[r, "\sim"] & \underline{G}^{\mathsf{uni}}_{E_2}. \arrow[u] \end{tikzcd}$$

Proof. The first part of the theorem says that there is an equivalence of stacks of infinitesimally unipotent bundles:

$$(\underline{G}_{E_1})^\wedge_{\mathsf{uni}} \xrightarrow{\sim} (\underline{G}_{E_2})^\wedge_{\mathsf{uni}}.$$

By Corollary 5.9 and Proposition 5.14, we have an identification

$$(\underline{G}_{E_i})^\wedge_{\mathsf{uni}} \cong \mathrm{Fun}^{\otimes}(\mathrm{Rep}(G), \mathrm{Tor}(\widehat{J(E_i)})).$$

The identification of formal groups $\widehat{J(E_1)} \cong \widehat{J(E_2)}$ defines, for each test scheme S, an equivalence of symmetric monoidal categories $\mathrm{Tor}(\widehat{J(E_1)})(S) \simeq \mathrm{Tor}(\widehat{J(E_2)})(S)$, and thus we obtain the required equivalence.

The second part of the theorem means that this equivalence preserves the substacks of unipotent bundles. But according to Proposition 5.14, the unipotent bundles may be recognized as those functors whose corresponding tensor functors factor through the subcategory of unipotent torsion sheaves. As the subcategory of unipotent torsion sheaves is preserved under the equivalence

$$\mathrm{Tor}(\widehat{J(E_1)}) \simeq \mathrm{Tor}(\widehat{J(E_2)}),$$

we obtain the required result. □

6. Examples

In this section we compute (partially) some examples. The reference and notation for root systems that we used is from the appendix of [Bou81].

We recall that $\underline{G}_E$ stands for the stack of *semistable* G-bundles of degree 0 on E. Denote by J the Jacobian of E.

Example 6.1. Take $G = \mathrm{PGL}(2)$. We have two possible closed sets: empty set, full set. The Weyl group acts on $(\mathbb{G}_m)_E = J$ by $\mathcal{L} \mapsto \mathcal{L}^{-1}$.

We have a decomposition

$$\underline{G}_E = \mathcal{N}/G \bigsqcup (J \setminus \{\mathcal{O}\}) /(T \rtimes \mathfrak{S}_2).$$

Observe that the PGL(2)-bundles $\mathcal{O} \oplus \mathcal{L}$, where $\mathcal{L} \in J[2]$ and $\mathcal{L} \neq \mathcal{O}$, have automorphism group $T \rtimes \mathfrak{S}_2$ which is disconnected. This is to be expected because the centralizer of a semisimple element in a non-simply-connected group does not have to be connected.

Example 6.2. Take $G = \mathrm{SL}(2)$. Then we have

$$\underline{G}_E = (J[2] \times \mathcal{N}/G) \bigsqcup (J \setminus J[2])/(T \rtimes \mathfrak{S}_2)$$

and here all the bundles have connected automorphism groups.

Example 6.3. Take $G = \mathrm{GL}(2)$, $T = \mathbb{G}_m^2$. Then we have

$$\underline{G}_E = J \times \mathcal{N}/G \bigsqcup (T_E \setminus \mathsf{diag})/(T \rtimes \mathfrak{S}_2)$$

and here also all the automorphism groups are connected.

Example 6.4. Let G be a group of type G_2. The root system is

$$\Phi = \pm\{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\},$$

where α is the short root.

The closed subsets not contained in any proper Levi are Φ and

$$\Sigma = \pm\{\beta, 3\alpha+\beta, 3\alpha+2\beta\}.$$

We have $G(\Sigma) = \mathrm{SL}(3)$ which is a pseudo-Levi subgroup. The other closed sets (up to conjugation by W) are $\{\alpha\}$, $\{\beta\}$ and $\emptyset$ and one can easily see that $G(\alpha) \simeq \mathrm{GL}(2) \simeq G(\beta)$ and $G(\emptyset) = T \simeq \mathbb{G}_m^2$. It is also an exercise to check that $N_G(G(\alpha)) = G(\alpha)$ and similarly for $G(\beta)$.

The roots give us an isomorphism $T_E \simeq J^2$ where the first coordinate corresponds to α and the second to β. The partition of $\underline{G}_E$ is therefore

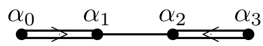
$$\begin{aligned}\underline{G}_E = \mathcal{N}_G/G &\sqcup \left(J[3] \times \mathcal{N}_{\mathrm{SL}(3)}\right) / \mathrm{SL}(3)\\ &\sqcup (J - J[3]) \times \mathcal{N}_{G(\alpha)}/G(\alpha)\\ &\sqcup J \times \mathcal{N}_{G(\beta)}/G(\beta)\\ &\sqcup (T_E - \mathsf{coord})/T \rtimes W,\end{aligned}$$

where $\mathsf{coord} \simeq J \times \{\mathcal{O}_E\} \cup \{\mathcal{O}_E\} \times J \subset J \times J$ corresponds to the coordinate axes.

Now a more involved example:

Example 6.5. Take $G = \mathrm{Sp}(6)$ which is a group of type C_3. The simple roots are $\{\alpha_1, \alpha_2, \alpha_3\}$ with α_3 the long root. The longest root is $2\alpha_1 + 2\alpha_2 + \alpha_3$.

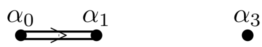
We put $\alpha_0 := -(2\alpha_1 + 2\alpha_2 + \alpha_3)$. The affine Dynkin diagram is



We use the Borel-de Siebenthal algorithm to produce closed sets (see Section 3.2). If we remove only α_0 or only α_3 we get back the group G. If we remove the affine vertex and some other vertices we get all the Levi subgroups of G.

If we remove one vertex different from α_0 or α_3 we get

$$\Sigma_1 = \pm\left\{\begin{array}{l}\alpha_2, \alpha_3, \alpha_2+\alpha_3, 2\alpha_2+\alpha_3,\\ 2\alpha_1 + 2\alpha_2 + \alpha_3\end{array}\right\} \rightsquigarrow \mathrm{SL}(2)\times\mathrm{Sp}(4),$$



$$\Sigma_2 = \pm\left\{\begin{array}{l}\alpha_3, \alpha_1, 2\alpha_1+2\alpha_2+\alpha_3,\\ \alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3\end{array}\right\} \rightsquigarrow \mathrm{Sp}(4)\times\mathrm{SL}(2).$$

One can check easily that $s_2 s_1$ takes Σ_1 into Σ_2, hence also the group $G(\Sigma_1)$ into $G(\Sigma_2)$.

By applying once more the above algorithm we get (discarding the Levi subgroups) the following root system:

$$\Sigma_3 = \pm\{\alpha_3, 2\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_2 + \alpha_3\} \rightsquigarrow \mathrm{SL}(2)^3,$$

where $\alpha_0' = -(2\alpha_1 + \alpha_0) = 2\alpha_2 + \alpha_3$.

The groups $G(\Sigma)$ are computed by inspecting the root/coroot system and by looking at the center and the fundamental group.

We can also compute the relative Weyl groups and find $W_{G,\Sigma_1} = 1$ and $W_{G,\Sigma_3} = \mathfrak{S}_3$. Actually the Weyl group of $\mathrm{Sp}(6)$ is $W = (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathfrak{S}_3$ and one can check (or see from the diagram) that the Weyl group of Σ_3 is $W_{\Sigma_3} \simeq \mathbb{Z}^3$.

If we iterate once more the algorithm we only get Levi subgroups of Φ, Σ_1 or Σ_3.

For the partial order $\succeq$ the closed sets Φ, Σ_1, Σ_3 are the maximal ones (up to permutation by W).

Hence the closed pieces in the partition of $\underline{G}_E$ are

$$J[2] \times \mathcal{N}_{\mathrm{Sp}(6)} / \mathrm{Sp}(6),$$

$$\big(J[2]^2 - \mathsf{diag}\big) \times \mathcal{N}_{\mathrm{SL}(2)\times\mathrm{Sp}(4)} / \mathrm{SL}(2) \times \mathrm{Sp}(4),$$

$$\big((J[2]^3 - \mathsf{diags}) \times \mathcal{N}_{\mathrm{SL}(2)^3} / \mathrm{SL}(2)^3\big) / \mathfrak{S}_3.$$

Here also, all the bundles have connected automorphism group even though we quotient by $\mathfrak{S}_3$ because its action is free on $J[2]^3 \setminus \mathsf{diags}$.

Example 6.6. The last example we compute (in detail) is a simply connected group of type D_4. We would like to provide an example to show that the automorphism group of a semisimple bundle can be disconnected.

Let $G = \mathrm{Spin}(8)$ be the simply connected group of type D_4 and denote by T a maximal torus. The simple roots are denoted by $\alpha_i, i = 1, 2, 3, 4$ and the center of G is isomorphic to $\mu_2 \times \mu_2$.

For convenience, let us spell out the root datum that we used for the computations (for more details one should consult [Bou81, Planche IV, p.256]):

- the character lattice and the root lattice are

$$X^*(T) = \langle \omega_1, \omega_2, \omega_3, \omega_4 \rangle \supset \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle,$$

  where the roots are given in terms of fundamental characters as:

$$\begin{aligned} \alpha_1 &= -2\omega_1 + \omega_2, \\ \alpha_2 &= -\omega_1 + 2\omega_2 - \omega_3 - \omega_4, \\ \alpha_3 &= -\omega_2 + 2\omega_3, \\ \alpha_4 &= -\omega_2 + 2\omega_4, \end{aligned}$$

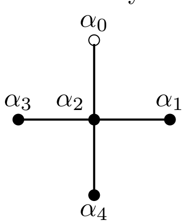
- the cocharacter lattice which equals the coroot lattice (because simply connected) is

$$X_*(T) = \langle \check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_3, \check{\alpha}_4 \rangle,$$

- the longest root is $\alpha_0 := \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$.

Notice that the simple coroots give us an isomorphism $(\check{\alpha}_1, \check{\alpha}_2, \check{\alpha}_3, \check{\alpha}_4) \colon \mathbb{G}_m^4 \to T$. It is useful to think of the simple coroots (and fundamental characters) as being the coordinates of T.

The affine Dynkin diagram is



where $\alpha_0 = -(\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4)$.

If we remove a vertex different from α_2 we get the whole Φ. Removing α_2 we obtain the diagram of a group of type $A_1 \times A_1 \times A_1 \times A_1$.

Therefore (using again Borel–de Siebenthal algorithm, Section 3.2), up to conjugacy, there are only two maximal closed sets, namely Φ and

$$\Sigma := \pm\{\alpha_1, \alpha_3, \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}$$

which is a root system of type A_1^4. All the other closed subsets that we can obtain iterating the algorithm are Levi subgroups of G or of $G(\Sigma)$.

The group $G(\Sigma)$ can be computed to be

$$(\mathrm{SL}(2) \times \mathrm{SL}(2) \times \mathrm{SL}(2) \times \mathrm{SL}(2))/\mu_2,$$

where μ_2 is the diagonal central subgroup. (This is achieved by computing the root datum for $G(\Sigma)$.) Notice that this is not simply connected!

Let us recall that the Weyl group of G is isomorphic to $\mathfrak{S}_4 \ltimes P$ where $P \leq (\mathbb{Z}/2\mathbb{Z})^4$ is the hyperplane $\prod x_i = 1$ and where the symmetric group acts on it by permutations. One sees best the action of $\mathfrak{S}_4 \ltimes P$ on characters/cocharacters by introducing additional variables $\varepsilon_i, i = 1, 2, 3, 4$ such that $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$, $\omega_3 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$ and $\omega_4 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)$. Using these coordinates the action of $\mathfrak{S}_4$ is by permuting the ε_i and the action of P is by multiplication (i.e. changing signs).

The Weyl group W_Σ of $G(\Sigma)$ is generated by the permutations (12), (34) together with $(-1, -1, 1, 1)$, $(1, 1, -1, -1) \in P$ where we think of $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ multiplicatively.

Another computation shows that the normalizer of W_Σ in W is generated by W_Σ, P and the permutation $(13)(24)$. The relative Weyl group is

$$W_{G,\Sigma} \simeq \langle (13)(24) \rangle \times \langle (-1,1,-1,1) \rangle \simeq \mathfrak{S}_2 \times \mathbb{Z}/2\mathbb{Z}.$$

There are two closed strata in $\underline{G}_E$, namely

$$\begin{aligned} &J[2]^2 \times \mathcal{N}_G/G, \\ &\left((J[2]^3 - J[2]^2) \times \mathcal{N}_{G(\Sigma)}\right)/G(\Sigma) \rtimes W_{G,\Sigma}. \end{aligned} \tag{6.1}$$

Let us be more explicit about the semisimple parts $J[2]^2$ and $J[2]^3$.

The center of Spin(8) is $\{t \in T \mid \alpha_i(t) = 1\}$ which using the identification $\mathbb{G}_m^4 \simeq T$ given by cocharacters becomes

$$\{(z_1, z_2, z_3, z_4) \in T \mid z_2 = z_1^2 = z_3^2 = z_4^2 \text{ and } z_2^2 = z_1 z_3 z_4\}$$

which can be rewritten as

$$\{(z_1, z_2, z_3, z_4) \in T \mid 1 = z_2 = z_1^2 = z_3^2 = z_4^2 = z_1 z_3 z_4\} \simeq \mu_2^2.$$

Similarly, the center of $G(\Sigma)$ is

$$Z(G(\Sigma)) = \{(z_1, z_2, z_3, z_4) \in T \mid 1 = z_2 = z_1^2 = z_3^2 = z_4^2\} \simeq \mu_2^3.$$

So in (6.1) the $J[2]^2$ corresponds to $Z(\mathrm{Spin}(8)) \simeq \mu_2^2$ and $J[2]^3$ corresponds to $Z(G(\Sigma)) \simeq \mu_2^3$. The complement can also be made explicit

$$J[2]^3 - J[2]^2 = \{(\mathcal{L}_1, \mathcal{O}, \mathcal{L}_3, \mathcal{L}_4) \mid \mathcal{L}_i \in J[2] \text{ and } \mathcal{L}_1\mathcal{L}_3\mathcal{L}_4 \not\simeq \mathcal{O}\}.$$

One can check that the relative Weyl group $W_{G,\Sigma}$ acts on the above locus without fixed points. Hence we will not find a semisimple bundle whose automorphism group (as a Spin(8)-bundle) contains $G(\Sigma)$ and is disconnected.

However, we'll produce a semisimple bundle whose automorphism group (as a Spin(8)-bundle) is disconnected with connected component precisely the maximal torus. Using the identification $\mathbb{G}_m^4 \simeq T$ given by the simple coroots, a T-bundle is a quadruple of line bundles $\mathcal{P}_T = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4)$.

To simplify the analysis we impose furthermore $\mathcal{L}_i^2 \simeq \mathcal{O}$ for $i = 1, 2, 3, 4$ (this will ensure later on that some element of the Weyl group stabilizes it). In order for $\mathrm{Aut}(\mathcal{P}_T \overset{T}{\times} \mathrm{Spin}(8))^\circ$ to be T we must have $\alpha_*(\mathcal{P}_T) \not\simeq \mathcal{O}$ for all roots α (there are 12 positive roots). Given the simplifying assumption we've made $\mathcal{L}_i^2 \simeq \mathcal{O}$, $i = 1, 2, 3, 4$, this boils down to the following conditions

$$\mathcal{L}_2 \not\simeq \mathcal{O} \qquad \text{and} \qquad \mathcal{O} \not\simeq \mathcal{L}_1\mathcal{L}_3\mathcal{L}_4 \not\simeq \mathcal{L}_2.$$

Let $\mathcal{L}_2, \mathcal{L}_4 \in J[2] \setminus \{\mathcal{O}\}$ be two non-isomorphic line bundles (this is possible if we're not in characteristic 2). Then the T-bundle $\mathcal{P}_T := (\mathcal{O}, \mathcal{L}_2, \mathcal{O}, \mathcal{L}_4)$ satisfies the above conditions and hence the automorphism group of $\mathcal{P} := \mathcal{P}_T \overset{T}{\times} \mathrm{Spin}(8)$ has connected component equal to T.

The following element in the Weyl group $\sigma := \sigma_{\alpha_0}\sigma_{\alpha_1}\sigma_{\alpha_3}\sigma_{\alpha_4}$ stabilizes $\mathcal{P}_T$ (as a T-bundle!). More precisely, in general we have $\sigma(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4) = (\mathcal{L}_1^{-1}, \mathcal{L}_2^{-1}, \mathcal{L}_3^{-1}, \mathcal{L}_4^{-1})$. Hence $\sigma \in \mathrm{Aut}(\mathcal{P})$ which implies that $\mathrm{Aut}(\mathcal{P})$ is a disconnected group (with connected component equal to T).

Appendix A. Classification of elliptic closed subsets

The focus of this section is on the case when E is an elliptic curve but we'll quickly review the cusp (rational) and nodal (trigonometric) cases. In Section 3.3 we gave a general recipe to produce closed subsets of the root system Φ of G as centralizers of elements in $\mathfrak{t}$, T and T_E. More precisely, we put

$$x \in \mathfrak{t} \rightsquigarrow \Sigma_x := \{\alpha \in \Phi \mid \alpha(x) = 0\}, \tag{A.1}$$

$$t \in T \rightsquigarrow \Sigma_t := \{\alpha \in \Phi \mid \alpha(t) = 1\}, \tag{A.2}$$

$$p \in T_E \rightsquigarrow \Sigma_p := \{\alpha \in \Phi \mid \alpha_*(\mathcal{P}) = \mathsf{triv}\}. \tag{A.3}$$

The collection of these subsets will be denoted (in this section) by $\mathcal{A}_{\mathsf{rat}}$, $\mathcal{A}_{\mathsf{trig}}$ respectively, $\mathcal{A}_{\mathsf{ell}}$.

Over the complex numbers, using the analytic uniformizations $(\mathbb{X}_*(T)\otimes_{\mathbb{Z}}\mathbb{C})/\mathbb{X}_*(T) \simeq T$ and $(\mathbb{X}_*(T) \otimes_{\mathbb{Z}} (\mathbb{R}^2))/\mathbb{X}_*(T)^{\oplus 2} \simeq E$, the proof of Proposition A.6 is all that is needed. However, to deal also with the positive characteristic, one needs to do a little combinatorics of root systems and diagonalizable groups. The key ingredient is a Lemma from [Ste68, 5.1] that we record as Lemma A.1 and its elliptic analogue Lemma A.3.

Let $a \in \mathfrak{t}$ and consider $\Sigma_a \in \mathcal{A}_{\mathsf{rat}}$. Then Σ_a is the root system of the Levi subgroup $C_G(k \cdot a)$ (the centralizer of a subtorus is always a Levi subgroup, see for example [DM91, Proposition 1.22]). If L is a Levi subgroup of G then for a generic element a in the center of the Lie algebra $\mathfrak{l}$, the closed set Σ_a is the set of roots of L.

The trigonometric situation is a bit more complicated. For $t \in T$ the closed subset Σ_t is the root system of the connected centralizer $C_G(t)^\circ$ which might not be a Levi subgroup if the order of t is finite. The reductive subgroups of G thus obtained are called pseudo-Levi subgroups and they are well known in the theory of reductive groups.

Going further to the elliptic case, it turns out that the subsets Σ_p are the root systems of intersections of two pseudo-Levi subgroups of G. We will sketch the proofs below after fixing some notation.

Fix a Borel subgroup B containing the maximal torus $T \subset B \subset G$, and let $\Phi \supset \Delta$ be the set of roots and of simple roots respectively.

A prime number p is said to be *good* for G if p does not divide any coefficient of the highest root[12] (w.r.t. Δ.) of Φ. This is a very tiny restriction on p and G. For example, $p > 5$ is good for every group and $p \geq 3$ is good for any classical group.

Denote by $\mathbb{X} = \mathbb{X}^*(T)$ the character lattice of T and by $\mathbb{Y}$ its dual lattice. One can construct, in a functorial way, the *compact* abelian Lie group $T_c = (\mathbb{Y} \otimes_{\mathbb{Z}} \mathbb{R})/\mathbb{Y}$ such that $\mathrm{Hom}_{\mathsf{gr}}(T_c, \mathbb{C}^\times) = \mathbb{X}$.

For $t \in T$ we put $\mathbb{X}_t := \{\lambda \in \mathbb{X} \mid \lambda(t) = 1\}$ and similarly for $x \in T_c$ we put $\mathbb{X}_x = \{\lambda \in \mathbb{X} \mid \lambda(x) = 1\}$. We define analogously Σ_t and Σ_x.

We start with a preparation lemma from [Ste68, 5.1] that is needed to pass from an arbitrary field to $\mathbb{R}$.

Lemma A.1.

(1) *For any $t \in T$ there exists $x \in T_c$ such that $\mathbb{X}_t = \mathbb{X}_x$.*
(2) *If $x \in T_c$ is of finite order prime to $p = \mathrm{char}(k)$ then there exists $t \in T$ such that $\mathbb{X}_t = \mathbb{X}_x$.*

[12] If the root system is not irreducible, consider all the highest roots.

Proof. We sketch the idea of the proof.

The abelian group $\mathbb{X}/\mathbb{X}_t$ is finitely generated and injects into $k^\times$ through the map $\lambda \mapsto \lambda(t)$. Since a finite subgroup of $k^\times$ is cyclic we deduce that the torsion part of the abelian group $\mathbb{X}/\mathbb{X}_t$ is cyclic. To $\mathbb{X}/\mathbb{X}_t$ corresponds a sublattice of $\mathbb{Y}$ and hence a subgroup $T'_c \subset T_c$ which is the product of a compact torus and a cyclic group. As such it has a topological generator, say $x \in T'_c$. By construction we have $\mathbb{X}_x = \mathbb{X}_t$.

Conversely, the finite cyclic group $\mathbb{X}/\mathbb{X}_x$ corresponds to a cyclic subgroup μ of T of order prime to p. Hence μ has a generator t of the same order as x and by construction we have $\mathbb{X}_t = \mathbb{X}_x$. □

Remark A.2. The reason we need the order to be prime to $p = \mathrm{char}(k)$ is that in $\mathbb{G}_m$ there are no points of order p, i.e., μ_p is infinitesimal.

If the field k has elements of infinite order, then $\mathbb{G}_m$ has a Zariski generator and hence any torus has a Zariski generator. Therefore in (2) above we can replace the assumption "x of finite order prime to p" by "the component group of $\overline{\langle x\rangle}$ has order prime to p."

For $\mathcal{P} \in T_E$ put $\mathbb{X}_{\mathcal{P}} = \{\lambda \in \mathbb{X} \mid \lambda_*(\mathcal{P}) \simeq \mathcal{O}\}$. Similarly one can prove

Lemma A.3. *For any $\mathcal{P} \in T_E$ there exist $x_1, x_2 \in T_c$ such that $\mathbb{X}_{\mathcal{P}} = \mathbb{X}_{x_1} \cap \mathbb{X}_{x_2}$. Conversely, for $x_1, x_2 \in T_c$ of finite order prime to p there exists $\mathcal{P} \in T_E$ such that $\mathbb{X}_{\mathcal{P}} = \mathbb{X}_{x_1} \cap \mathbb{X}_{x_2}$.*

Proof. In the proof of Lemma A.1 we used that a finite subgroup of $k^\times$ must be cyclic. We also used that $k^\times$ has a primitive rth root of unity if and only if r is prime to $\mathrm{char}(k)$.

The analogue for an elliptic curve is: a finite subgroup of $J(E)$ is a product of two cyclic groups. The group of torsion points $J(E)[r]$ is isomorphic to $\mathbb{Z}/r \times \mathbb{Z}/r$ if and only if r is prime to $\mathrm{char}(k)$.

Hence the quotient $\mathbb{X}/\mathbb{X}_{\mathcal{P}}$ is a product of a free abelian group and two cyclic groups. The rest of the proof is the same. □

We can now easily deduce

Corollary A.4.

(1) *For any $t \in T$ there exists $x \in T_c$ such that $\Sigma_x = \Sigma_t$.*
(2) *For any $\mathcal{P} \in T_E$ there exist $x_1, x_2 \in T_c$ such that $\Sigma_{\mathcal{P}} = \Sigma_{x_1} \cap \Sigma_{x_2}$.*

Conversely, by [MS03, Prop. 30,32] we have

Proposition A.5. *Assume* $\mathrm{char}(k)$ *is good for* G.

(1) *For any $x \in T_c$ there exists $t \in T$ such that $\Sigma_t = \Sigma_x$.*
(2) *For any $x_1, x_2 \in T_c$ there exists $\mathcal{P} \in T_E$ such that $\Sigma_{\mathcal{P}} = \Sigma_{x_1} \cap \Sigma_{x_2}$.*

Just as Levi subgroups correspond, up to conjugation, to subsets of the simple roots, pseudo-Levi subgroups admit a similar characterization. In order to state the result, we introduce some notation. Let

$$\widetilde{\Delta} := \Delta \sqcup \{\alpha_0^{(l)} : l \text{ a connected component of } \Phi\}$$

be the set of simple roots of the corresponding affine root system, where $\alpha_0^{(l)}$ is the negative of the longest root in the corresponding connected component $\Phi_+^{(l)}$. (If the

Dynkin diagram is not connected there are several longest roots corresponding to each connected component and we want to add the negative of each of them to Δ.)

Given a subset $S \subset \widetilde{\Delta}$ put $\Sigma_S := \mathbb{Z}S \cap \Phi$. By construction, Σ_S is a closed subset of Φ.

Proposition A.6 ([MS03, Lemma 29], [Lus95, Lemma 5.4]). *The set $\{\Sigma_x \mid x \in T_c\}/W$ consists of subsets of the form Σ_S for some proper subset S of $\widetilde{\Delta}$ as defined above.*

Proof. We recall the proof from loc. cit. for the convenience of the reader. Put $\pi\colon \mathfrak{t}_{\mathbb{R}} = \mathbb{Y} \otimes_{\mathbb{Z}} \mathbb{R} \to T_c$ the projection.

Let $x \in T_c$. If the closure of the subgroup generated by x is a torus, its centralizer is a Levi subgroup and we're done.

Otherwise, write $x = x's$ with s of finite order and x' that generates a torus. We have $\Sigma_{x's} = \Sigma_{x'} \cap \Sigma_s$ and since $\Sigma_{x'}$ corresponds to a Levi subgroup we are left to deal with Σ_s. So we can suppose x is of finite order in T_c.

The affine Weyl group $\mathbb{X}_*(T) \ltimes W$ acts on $\mathfrak{t}_{\mathbb{R}}$ and, up to conjugating x by some element of W, we can assume x lies in the image of the fundamental alcove through the map $\pi\colon \mathfrak{t}_{\mathbb{R}} \to T_c$. Hence we can take $\tilde{x} \in \pi^{-1}(x)$ in the fundamental alcove.

Looking at the roots as linear functions on $\mathfrak{t}_{\mathbb{R}}$ we have

$$\Sigma_x = \{\alpha \in \Phi \mid \alpha(\tilde{x}) \in \mathbb{Z}\}.$$

Define $S := \{\alpha \in \widetilde{\Delta}\} \mid \alpha(\tilde{x}) \in \{0, 1\}\}$. By construction $\Sigma_S \subset \Sigma_x$. Let us prove that $\Sigma_S = \Sigma_x$.

Recall that the fundamental alcove in $\mathfrak{t}_{\mathbb{R}}$ is defined by the inequalities

$$0 \leq \alpha(y) \leq 1, \text{ for all } \alpha \in \Phi^+.$$

Denoting by $\gamma^{(l)} = -\alpha_0^{(l)}$ the highest root of $\Phi^{(l)}$, the fundamental alcove can be rewritten as

$$\begin{aligned} 0 \leq \alpha_i(y) \leq 1 &\text{ for all } \alpha_i \in \Delta, \\ 0 \leq \gamma^{(l)}(y) \leq 1 &\text{ for all connected components } l \text{ of } \Phi. \end{aligned}$$

For $\alpha \in \Sigma_x^{(l),+}$ we have $0 \leq \alpha(\tilde{x}) \leq \gamma^{(l)}(\tilde{x}) \leq 1$, hence $\alpha(\tilde{x}) \in \{0, 1\}$.

If $\alpha(\tilde{x}) = 0$, then α, being a positive sum of simple roots, is a sum of simple roots all of which must belong to S, hence $\alpha \in \Sigma_S$.

If $\alpha(\tilde{x}) = 1$ then $\gamma(\tilde{x}) - \alpha(\tilde{x}) = 0$, hence all the simple roots appearing in $\gamma^{(l)} - \alpha$ must be in S. Thus $\gamma^{(l)} - \alpha \in \Sigma_S$ and since $\gamma^{(l)}(\tilde{x}) = 1$ we also have $\gamma^{(l)} \in \Sigma_S$. We deduce $\alpha \in \Sigma_S$ and the proof is finished. $\square$

Remark A.7. One can think of the above construction in the following way. Take the extended Dynkin diagram whose nodes are indexed by $\widetilde{\Delta}$ and remove some non-zero number of nodes (at least one from each connected component). This will then be the (non-extended) Dynkin diagram corresponding to some pseudo-Levi subgroup of G. In fact it is known that the Dynkin diagrams corresponding to closed subsets $\Sigma \subset \Phi$ can be obtained by repeatedly applying this procedure, see [BDS49] for more details.

Proposition A.8. *Assume that* $\operatorname{char}(k)$ *is good for G. The collection of closed subsets $\mathcal{A}_{\mathsf{ell}}$ consists precisely of intersections of two elements of $\mathcal{A}_{\mathsf{trig}}$. In other* *words, the $G(\Sigma)$ for $\Sigma \in \mathcal{A}_{\mathsf{ell}}$ are precisely the neutral component of intersections of two pseudo-Levi subgroups.*

Proof. It follows from Proposition A.6 together with Proposition A.5 □

Remark A.9. The elliptic closed subsets are obtained, up to W-conjugation, by applying two times the Borel–de Siebenthal algorithm: take the affine Dynkin diagram and remove some vertices and then consider the closed subset of roots generated by it (which is a root system again).

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References

[Bal19] V. Balaji, *Torsors on semistable curves and degenerations*, Proc. Indian Acad. Sci. Math. Sci. **132** (2022), no. 1, Paper No. 27, 63, DOI 10.1007/s12044-021-00651-6. MR4421927

[BBR09] Claudio Bartocci, Ugo Bruzzo, and Daniel Hernández Ruipérez, *Fourier-Mukai and Nahm transforms in geometry and mathematical physics*, Progress in Mathematics, vol. 276, Birkhäuser Boston, Inc., Boston, MA, 2009, DOI 10.1007/b11801. MR2511017

[BDS49] A. Borel and J. De Siebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos* (French), Comment. Math. Helv. **23** (1949), 200–221, DOI 10.1007/BF02565599. MR32659

[BEG03] Vladimir Baranovsky, Sam Evens, and Victor Ginzburg, *Representations of quantum tori and G-bundles on elliptic curves*, The orbit method in geometry and physics (Marseille, 2000), Progr. Math., vol. 213, Birkhäuser Boston, Boston, MA, 2003, pp. 29–48. MR1995373

[BG96] Vladimir Baranovsky and Victor Ginzburg, *Conjugacy classes in loop groups and G-bundles on elliptic curves*, Internat. Math. Res. Notices **15** (1996), 733–751, DOI 10.1155/S1073792896000463. MR1413870

[Bho01] Usha N. Bhosle, *Principal G-bundles on nodal curves*, Proc. Indian Acad. Sci. Math. Sci. **111** (2001), no. 3, 271–291, DOI 10.1007/BF02829597. MR1851092

[BK05] Igor Burban and Bernd Kreussler, *Fourier-Mukai transforms and semi-stable sheaves on nodal Weierstraßcubics*, J. Reine Angew. Math. **584** (2005), 45–82, DOI 10.1515/crll.2005.2005.584.45. MR2155085

[BNP13] David Ben-Zvi, David Nadler, and Anatoly Preygel, *A spectral incarnation of affine character sheaves*, Compos. Math. **153** (2017), no. 9, 1908–1944, DOI 10.1112/S0010437X17007278. MR3705281

[Bor91] Armand Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991, DOI 10.1007/978-1-4612-0941-6. MR1102012

[Bos12] Siegfried Bosch, *Algebraic geometry and commutative algebra*, Universitext, Springer, London, 2013, DOI 10.1007/978-1-4471-4829-6. MR3012586

[Bou81] Nicolas Bourbaki, *Éléments de mathématique* (French), Masson, Paris, 1981. Groupes et algèbres de Lie. Chapitres 4, 5 et 6. [Lie groups and Lie algebras. Chapters 4, 5 and 6]. MR647314

[BZBJ18] David Ben-Zvi, Adrien Brochier, and David Jordan, *Integrating quantum groups over surfaces*, J. Topol. **11** (2018), no. 4, 874–917, DOI 10.1112/topo.12072. MR3847209

[BZN15] David Ben-Zvi and David Nadler, *Elliptic Springer theory*, Compos. Math. **151** (2015), no. 8, 1568–1584, DOI 10.1112/S0010437X14008021. MR3383168

[BZN18] David Ben-Zvi and David Nadler, *Betti geometric Langlands*, Algebraic geometry: Salt Lake City 2015, Proc. Sympos. Pure Math., vol. 97, Amer. Math. Soc., Providence, RI, 2018, pp. 3–41. MR3821166

[DM91] François Digne and Jean Michel, *Representations of finite groups of Lie type*, London Mathematical Society Student Texts, vol. 21, Cambridge University Press, Cambridge, 1991, DOI 10.1017/CBO9781139172417. MR1118841

[FM01] R. Glenn Friedman and J. W. Morgan, *Holomorphic principal bundles over elliptic curves III: singular curves and fibrations*, 2001.

[Fra13] Dragoş Frăţilă, *Cusp eigenforms and the Hall algebra of an elliptic curve*, Compos. Math. **149** (2013), no. 6, 914–958, DOI 10.1112/S0010437X12000784. MR3077657

[Fră15] Dragoş Frăţilă, *On the stack of semistable G-bundles over an elliptic curve*, Math. Ann., 2015.

[Fra21] Dragoş Frăţilă, *Revisiting the moduli space of semistable G-bundles over elliptic curves*, Ann. Inst. Fourier (Grenoble) **71** (2021), no. 2, 615–641. MR4353915

[Gil10] P. Gille, *Borel-de Siebenthal's theorem*, Preprint, 2010, `http://math.univ-lyon1.fr/homes-www/gille/prenotes/bds.pdf`.

[Gro67] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II* (French), Inst. Hautes Études Sci. Publ. Math. **24** (1965), 231. MR199181

[Gro71] A. Grothendieck, *Revêtement étales et groupe fondamental (sga1)*, Lecture Note in Math., 224, 1971.

[GSB19] Ian Grojnowski and Nicholas Shepherd-Barron, *Del Pezzo surfaces as Springer fibres for exceptional groups*, Proc. Lond. Math. Soc. (3) **122** (2021), no. 1, 1–41, DOI 10.1112/plms.12290. MR4210255

[Gun17] S. Gunningham, *A derived decomposition for equivariant D-modules*, Preprint, `arXiv:1705.04297`, 2017.

[Gun18] Sam Gunningham, *Generalized Springer theory for D-modules on a reductive Lie algebra*, Selecta Math. (N.S.) **24** (2018), no. 5, 4223–4277, DOI 10.1007/s00029-018-0443-x. MR3874694

[Hof10] Norbert Hoffmann, *On moduli stacks of G-bundles over a curve*, Affine flag manifolds and principal bundles, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2010, pp. 155–163, DOI 10.1007/978-3-0346-0288-4_5. MR3013030

[IMP03] S. Ilangovan, V. B. Mehta, and A. J. Parameswaran, *Semistability and semisimplicity in representations of low height in positive characteristic*, A tribute to C. S. Seshadri (Chennai, 2002), Trends Math., Birkhäuser, Basel, 2003, pp. 271–282. MR2017588

[KM76] Finn Faye Knudsen and David Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div"*, Math. Scand. **39** (1976), no. 1, 19–55, DOI 10.7146/math.scand.a-11642. MR437541

[KR08] Jochen Kuttler and Zinovy Reichstein, *Is the Luna stratification intrinsic?* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **58** (2008), no. 2, 689–721. MR2410387

[Las98] Yves Laszlo, *About G-bundles over elliptic curves*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 2, 413–424. MR1625614

[Lau90] G. Laumon, *Faisceaux automorphes liés aux séries d'Eisenstein* (French), Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 227–281. MR1044822

[Leh12] Simon Pepin Lehalleur, *Subgroups of maximal rank of reductive groups* (English, with English and French summaries), Autour des schémas en groupes. Vol. III, Panor. Synthèses, vol. 47, Soc. Math. France, Paris, 2015, pp. 147–172. MR3525844

[Li18] Penghui Li, *Derived categories of character sheaves*, Preprint, `arXiv:1803.04289`, 2018.

[LN21] Penghui Li and David Nadler, *Uniformization of semistable bundles on elliptic curves*, Adv. Math. **380** (2021), Paper No. 107572, 67, DOI 10.1016/j.aim.2021.107572. MR4200472

[Lun73] Domingo Luna, *Slices étales* (French), Sur les groupes algébriques, Bull. Soc. Math. France, Paris, Mémoire 33, Soc. Math. France, Paris, 1973, pp. 81–105, DOI 10.24033/msmf.110. MR0342523

[Lur04] J. Lurie, *Tannaka duality for geometric stacks*, Preprint, `arXiv:math/0412266`, 2004.

[Lus84] G. Lusztig, *Intersection cohomology complexes on a reductive group*, Invent. Math. **75** (1984), no. 2, 205–272, DOI 10.1007/BF01388564. MR732546

[Lus95] George Lusztig, *Classification of unipotent representations of simple p-adic groups*, Internat. Math. Res. Notices **11** (1995), 517–589, DOI 10.1155/S1073792895000353. MR1369407

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994, DOI 10.1007/978-3-642-57916-5. MR1304906

[Mil86] J. S. Milne, *Jacobian varieties*, Arithmetic geometry (Storrs, Conn., 1984), Springer, New York, 1986, pp. 167–212. MR861976

[MS03] George J. McNinch and Eric Sommers, *Component groups of unipotent centralizers in good characteristic*, J. Algebra **260** (2003), no. 1, 323–337, DOI 10.1016/S0021-8693(02)00661-0. Special issue celebrating the 80th birthday of Robert Steinberg. MR1976698

[Nor76] Madhav V. Nori, *On the representations of the fundamental group*, Compositio Math. **33** (1976), no. 1, 29–41. MR417179

[NS65] M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. (2) **82** (1965), 540–567, DOI 10.2307/1970710. MR184252

[Ram96] A. Ramanathan, *Moduli for principal bundles over algebraic curves. I*, Proc. Indian Acad. Sci. Math. Sci. **106** (1996), no. 3, 301–328, DOI 10.1007/BF02867438. MR1420170

[RR84] S. Ramanan and A. Ramanathan, *Some remarks on the instability flag*, Tohoku Math. J. (2) **36** (1984), no. 2, 269–291, DOI 10.2748/tmj/1178228852. MR742599

[RR16] Laura Rider and Amber Russell, *Perverse sheaves on the nilpotent cone and Lusztig's generalized Springer correspondence*, Lie algebras, Lie superalgebras, vertex algebras and related topics, Proc. Sympos. Pure Math., vol. 92, Amer. Math. Soc., Providence, RI, 2016, pp. 273–292, DOI 10.1007/s10468-020-09966-w. MR3644235

[Sch05] Alexander H. W. Schmitt, *Singular principal G-bundles on nodal curves*, J. Eur. Math. Soc. (JEMS) **7** (2005), no. 2, 215–251, DOI 10.4171/JEMS/27. MR2127994

[Sch12] Olivier Schiffmann, *On the Hall algebra of an elliptic curve, II*, Duke Math. J. **161** (2012), no. 9, 1711–1750, DOI 10.1215/00127094-1593362. MR2942792

[Sch14] Simon Schieder, *The Harder-Narasimhan stratification of the moduli stack of G-bundles via Drinfeld's compactifications*, Selecta Math. (N.S.) **21** (2015), no. 3, 763–831, DOI 10.1007/s00029-014-0161-y. MR3366920

[Sim94] Carlos T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. **79** (1994), 47–129. MR1307297

[Ste68] Robert Steinberg, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968. MR0230728

[Sum74] Hideyasu Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28, DOI 10.1215/kjm/1250523277. MR337963

[Sun99] Xiaotao Sun, *Remarks on semistability of G-bundles in positive characteristic*, Compositio Math. **119** (1999), no. 1, 41–52, DOI 10.1023/A:1001512029096. MR1711507

[SV11] O. Schiffmann and E. Vasserot, *The elliptic Hall algebra, Cherednik Hecke algebras and Macdonald polynomials*, Compos. Math. **147** (2011), no. 1, 188–234, DOI 10.1112/S0010437X10004872. MR2771130

[Teo99] Titus Teodorescu, *Semistable torsion-free sheaves over curves of arithmetic genus one*, ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–Columbia University. MR2699956

[Wan11] J. Wang, *The moduli stack of G-bundles*, Preprint, `arXiv:1104.4828`, 2011.

IRMA 7 rue René Descartes, Strasbourg, France
Email address: fratila@math.unistra.fr

Department of Mathematical Sciences, Montana State University, Bozeman, Montana 59717
Email address: sam.gunningham@montana.edu

YMSC, Tsinghua University, Beijing, People's Republic of China
Email address: lipenghui@mail.tsinghua.edu.cn