CLARK MEASURES FOR RATIONAL INNER FUNCTIONS

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Abstract. We analyze the fine structure of Clark measures and Clark isometries associated with two-variable rational inner functions on the bidisk. In the degree $(n, 1)$ case, we give a complete description of supports and weights for both generic and exceptional Clark measures, characterize when the associated embedding operators are unitary, and give a formula for those embedding operators. We also highlight connections between our results and both the structure of Agler decompositions and study of extreme points for the set of positive pluriharmonic measures on 2-torus.

1. Introduction

A bounded analytic function $\phi\colon \mathbb{D}^d \to \mathbb{C}$ is said to be *inner* if $|\phi(\zeta)| = 1$ for almost every $\zeta \in \mathbb{T}^d$, where $\mathbb{D}$ is the unit disk and $\mathbb{T}$ is the unit circle. In the one-variable case, each inner function ψ defines a class of positive Borel measures $\{\sigma_\alpha\}_{\alpha\in\mathbb{T}}$ on $\mathbb{T}$ that satisfy

$$\frac{1 - |\psi(z)|^2}{|\alpha - \psi(z)|^2} = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\sigma_\alpha(\zeta), \qquad \text{for } z \in \mathbb{D}.$$

These measures have a number of important applications and properties; among other results, they are the spectral representing measures for rank 1 unitary perturbations of certain compressed shift operators and via Alexandrov's theorem, they disintegrate Lebesgue measure, see [11, 16] for comprehensive introductions to this classical theory. Generalizations of these measures to the polydisk $\mathbb{D}^d$ were recently studied by E. Doubtsov in [12]; other multivariate generalizations of Clark theory can be found in [3, 18]. In this paper, we obtain precise information about both the two-variable Clark measures on the bidisk defined in [12] and associated isometries, in the setting of two-variable rational inner functions.

1.1. Notation and Setup.

To define Clark measures on the bidisk, we need some notation. Denote the Poisson kernel on $\mathbb{D}^2$ by

$$P_z(\zeta) = P(z, \zeta) := \frac{(1 - |z_1|^2)(1 - |z_2|^2)}{|\zeta_1 - z_1|^2 |\zeta_2 - z_2|^2}, \qquad \text{for } z \in \mathbb{D}^2, \zeta \in \mathbb{T}^2,$$

and the Cauchy kernel for the bidisk by

$$C_w(z) = C(z, w) = \frac{1}{(1 - z_1\overline{w}_1)(1 - z_2\overline{w}_2)}, \qquad \text{for } z \in \mathbb{D}^2, w \in \overline{\mathbb{D}}^2.$$

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Recall that $C(\cdot,\cdot)$ acts as the reproducing kernel for the Hardy space $H^2(\mathbb{D}^2)$, which consists of all analytic functions $f\colon \mathbb{D}^2 \to \mathbb{C}$ satisfying the norm boundedness condition

$$\|f\|_{H^2}^2 := \sup_{0<r<1} \int_{\mathbb{T}^2} |f(r\zeta)|^2 dm_2(\zeta) < \infty.$$

Here, m_2 denotes normalized Lebesgue measure on $\mathbb{T}^2$ and later, we will use m to denote normalized Lebesgue measure on $\mathbb{T}$. For $f \in H^2(\mathbb{D}^2)$ and $\zeta \in \mathbb{T}^2$, we let $f^*(\zeta)$ denote the non-tangential value of f at ζ. Recall that $f^*(\zeta)$ exists for a.e. $\zeta \in \mathbb{T}^2$, see Chapter XVII, Theorem 4.8 in [35]. Also, throughout this paper, we will slightly abuse notation by using z, ζ to refer to points from both $\mathbb{C}$ and $\mathbb{C}^2$, but the meaning should be clear from the context.

Let ϕ be a non-constant inner function on $\mathbb{D}^2$ and let $\alpha \in \mathbb{T}$. Since $z \mapsto \Re[(\alpha + \phi(z))/(\alpha - \phi(z))]$ is a positive pluriharmonic function on the bidisk, there exists a unique positive Borel measure σ_α on $\mathbb{T}^2$ called a *Clark measure* such that

$$\Re\left(\frac{\alpha + \phi(z)}{\alpha - \phi(z)}\right) = \frac{1 - |\phi(z)|^2}{|\alpha - \phi(z)|^2} = \int_{\mathbb{T}^2} P_z(\zeta) d\sigma_\alpha(\zeta),$$

for all $z \in \mathbb{D}^2$. Observe that

$$\int_{\mathbb{T}^2} d\sigma_\alpha(\zeta) = \int_{\mathbb{T}^2} P(0,\zeta) d\sigma_\alpha(\zeta) = \Re\left(\frac{\alpha + \phi(0)}{\alpha - \phi(0)}\right) = \frac{1 - |\phi(0)|^2}{|\alpha - \phi(0)|^2} < \infty,$$

so σ_α is a finite measure. Since σ_α is a finite Borel measure on $\mathbb{T}^2$, it is actually a Radon measure and basic measure theory (see for example [14, Proposition 7.9]) implies that $C(\mathbb{T}^2)$ is dense in $L^2(\sigma_\alpha)$. Furthermore, as linear combinations of the Poisson kernels $\{P_z\}_{z\in\mathbb{D}^2}$ are dense in $C(\mathbb{T}^2)$, they are also dense in $L^2(\sigma_\alpha)$. Finally, as asserted in [12], the support of each σ_α should be contained in the closure of the set $\{\zeta \in \mathbb{T}^2 : \phi^*(\zeta) = \alpha\}$. This support condition can also be verified directly in the case when ϕ is a rational inner function.

There are close connections between the Clark measures σ_α and the *model space* associated with the function ϕ defined by

$$K_\phi := H^2(\mathbb{D}^2) \ominus \phi H^2(\mathbb{D}^2).$$

Then the reproducing kernel for K_ϕ is given by

$$k(z,w) = k_w(z) := (1 - \overline{\phi(w)}\phi(z))C_w(z), \text{ for } z, w \in \mathbb{D}^2.$$

In [12], Doubtsov defined an embedding map $J_\alpha\colon K_\phi \to L^2(\sigma_\alpha)$ by first specifying it on reproducing kernels as

$$J_\alpha[k_w](\zeta) := (1 - \alpha\overline{\phi(w)})C_w(\zeta), \quad \text{for } w \in \mathbb{D}^2, \zeta \in \mathbb{T}^2,$$

then showing this definition preserves inner products on linear combinations of reproducing kernels, and finally extending it to all of K_ϕ using density.

In one variable, it is a classical fact (see [11, Chapter 9] or [16, Chapter 11]) that the analogous embedding is in fact a unitary for each inner function and each $\alpha \in \mathbb{T}$. In the higher-dimensional setting, Doubtsov [12, Theorem 3.2] shows that J_α is a unitary operator if and only if the bidisk algebra $A(\mathbb{D}^2)$ is dense in $L^2(\sigma_\alpha)$. He also gives examples showing that the two-variable embeddings J_α can *fail* to be unitary. In this paper, we investigate this phenomenon as part of our detailed study of two-variable Clark measures associated with rational inner functions, i.e. with functions that are both rational and inner on $\mathbb{D}^2$. It is worth noting that we restrict to this two-variable situation (rather than a more general d-variable setting) because many of our key tools, which include Agler decompositions, certain model space properties, and well-understood unimodular level set behaviors, do not extend to even the three-variable setting, see [10, 20].

To describe the structure of two-variable rational inner functions, or RIFs, we require some notation. For a polynomial $p \in \mathbb{C}[z_1, z_2]$, let $\deg_i p$ denote the degree of p in z_i and set $\deg p = (\deg_1 p, \deg_2 p)$. Then if a pair of nonnegative integers m satisfies $m = (m_1, m_2) \geq \deg p$, which means $m_1 \geq \deg_1 p$ and $m_2 \geq \deg_2 p$, we can define the m-reflection of p as

$$\tilde{p}(z) = z_1^{m_1} z_2^{m_2} \overline{p\left(1/\bar{z}_1, 1/\bar{z}_2\right)}.$$

Rudin and Stout [31, 33] showed that each RIF ϕ is of the form

$$\phi(z) = \gamma \frac{\tilde{p}(z)}{p(z)},$$

where $\gamma \in \mathbb{T}$, p has no zeros on $\mathbb{D}^2$, $\tilde{p}$ is some m-reflection of p, and $\tilde{p}, p$ share no common factors. We define $\deg \phi = \deg \tilde{p} \geq \deg p$. To simplify our notation, we will assume $\gamma = 1$ for the duration of the paper. It is worth noting that, in the one-variable setting, each RIF is a finite Blaschke product and extends analytically to some disk containing $\overline{\mathbb{D}}$ in its interior.

Unlike this one-variable situation, two-variable rational inner functions can possess boundary singularities. For example,

$$\phi(z) = \frac{2z_1 z_2 - z_1 - z_2}{2 - z_1 - z_2} \tag{1}$$

has a singularity at $(1, 1) \in \mathbb{T}^2$. More generally, if $\phi = \tilde{p}/p$ and $p(\tau) = 0$ for some $\tau \in \mathbb{T}^2$, then τ is a *singularity* of ϕ in the sense that ϕ cannot be extended continuously to a neighborhood of τ; see [29, Corollary 1.7]. Moreover, if $p(\tau) = 0$, then $\tilde{p}(\tau) = 0$ and so Bézout's theorem gives a bound on the number of zeros p (or equivalently, the number of singularities ϕ) can have on $\mathbb{T}^2$. Specifically, if $\deg \tilde{p} = (m_1, m_2)$ and $\deg p = (n_1, n_2)$, Bézout's theorem implies that p and $\tilde{p}$ have exactly $n_1 m_2 + n_2 m_1$ common zeros in $\mathbb{C}_\infty \times \mathbb{C}_\infty$ counted according to intersection multiplicity, where $\mathbb{C}_\infty$ denotes the Riemann sphere; see p. 1287 in [21]. If all such common zeros of $\tilde{p}$ and p occur on $\mathbb{T}^2$, we say p is $\mathbb{T}^2$-*saturated*. As the intersection multiplicity of such common zeros on $\mathbb{T}^2$ must be even, ϕ can have at most $m_1 m_2$ distinct singularities on $\mathbb{T}^2$, see [21].

Still, these RIF singularities are somewhat mild. Indeed, if ϕ is a RIF, then [21, Corollary 14.6] states that for each $\zeta \in \mathbb{T}^2$, including any points ζ where $p(\zeta) = 0$, the non-tangential value $\phi^*(\zeta)$ exists and is unimodular. For more information about the zero set of p, denoted $\mathcal{Z}_p$, see [1, 21].

1.2. **Overview of Results.** The body of this paper begins with Section 2, which provides some information about the Clark measures σ_α associated to a general RIF ϕ. Specifically, Theorem 2.1 gives a simple proof that σ_α cannot possess any point-masses (a fact noted earlier in [28]), and the section also gives further information about the closed set

$$\mathcal{C}_\alpha := \{\zeta \in \mathbb{T}^2 \colon \tilde{p}(\zeta) = \alpha p(\zeta)\}, \tag{2}$$

which contains the support of σ_α.

From Section 3 onward, we study RIFs $\phi = \frac{\tilde{p}}{p}$ with $\deg \phi = (n, 1)$. In a sense, these are the simplest two-variable RIFs, but the constructions from [8, 9] and the examples in our Section 5 show that they can still be quite complicated. First, note that for these RIFs,

$$p(z) = p_1(z_1) + z_2 p_2(z_1) \tag{3}$$

is a polynomial of degree at most $(n, 1)$ that does not vanish on $\mathbb{D}^2$,

$$\tilde{p}(z) := z_2 \tilde{p}_1(z_1) + \tilde{p}_2(z_1), \text{ where each } \tilde{p}_i(z_1) = z_1^n \overline{p_i(1/\bar{z}_1)},$$

the polynomials p, $\tilde{p}$ share no common factors, and p has at most n distinct zeros on $\mathbb{T}^2$. In Subsection 3.1, we recall some important properties about the model spaces and formulas associated to such RIFs. For example, such RIFs possess a specific *Agler decomposition* or *sums of squares formula* of the form

$$p(z)\overline{p(w)} - \tilde{p}(z)\overline{\tilde{p}(w)} = (1 - z_1\bar{w}_1) \sum_{j=1}^{n} R_j(z)\overline{R_j(w)} + (1 - z_2\bar{w}_2)Q(z)\overline{Q(w)} \tag{4}$$

where $R_1, \ldots, R_n, Q \in \mathbb{C}[z_1, z_2]$, $\deg R_j \leq (n-1, 1)$, and $\deg Q \leq (n, 0)$.

In Subsection 3.2, we study some preliminary objects, which are key in analyzing both the Clark measures σ_α and the isometric operators J_α associated to ϕ. Those objects are detailed in the following definition:

Definition 1.1. *Fix $\phi = \tilde{p}/p$ with $\deg \phi = (n, 1)$ and $\alpha \in \mathbb{T}$. Define the following:*

- *The points $(\tau_1, \lambda_1), \ldots, (\tau_m, \lambda_m)$ are the zeros of p on $\mathbb{T}^2$. Here, $0 \leq m \leq n$.*
- *B_α is the rational function*
$$B_\alpha(z) := \frac{\tilde{p}_1(z) - \alpha p_2(z)}{\alpha p_1(z) - \tilde{p}_2(z)}, \tag{5}$$
where any common factors of the numerator and denominator have been cancelled.
- *E_α and L_k are the sets in $\mathbb{T}^2$ defined by $E_\alpha := \{(\zeta, \overline{B_\alpha(\zeta)}) : \zeta \in \mathbb{T}\}$ and $L_k = \{\tau_k\} \times \mathbb{T}$ for $k = 1, \ldots, m$.*
- *W_α is the function on $\mathbb{T}$ defined by*
$$W_\alpha(\zeta) := \frac{|p_1(\zeta)|^2 - |p_2(\zeta)|^2}{|\tilde{p}_1(\zeta) - \alpha p_2(\zeta)|^2}.$$

Lastly, we say $\alpha \in \mathbb{T}$ is an exceptional value *for ϕ if there is a k such that $\phi^*(\tau_k, \lambda_k) = \alpha$ and $\alpha \in \mathbb{T}$ is a generic value for ϕ otherwise.*

Subsection 3.3 contains our first main result, the following complete characterization of the Clark measures σ_α associated to a given degree $(n, 1)$ RIF ϕ:

Theorem 1.2. *For $\alpha \in \mathbb{T}$, the Clark measure σ_α satisfies*

$$\int_{\mathbb{T}^2} f(\zeta)\, d\sigma_\alpha(\zeta) = \int_{\mathbb{T}} f(\zeta, \overline{B_\alpha(\zeta)})\, d\nu_\alpha(\zeta) + \sum_{k=1}^{m} c_k^\alpha \int_{\mathbb{T}} f(\tau_k, \zeta)\, dm(\zeta)$$

for all $f \in L^1(\sigma_\alpha)$, where $d\nu_\alpha = W_\alpha dm$, the functions B_α, W_α are from Definition 1.1, and the constants c_k^α are nonzero (and positive) if and only if $\phi^*(\tau_k, \lambda_k) = \alpha$.

When they are non-zero, the constants c_k^α can be obtained from the formula in (26). We should note that although Clark measures can often be computed in the one-variable case, very little is known in this two-variable setting. Indeed, this theorem can be viewed as a significant generalization of Example 4.3 in [12], which described σ_α for the specific degree $(1,1)$ RIF given in (1), and of Example 3 in [25], which includes the case where $\phi(z) = z_2 b(z_1)$, for b a finite Blaschke product with $b(0) \in \mathbb{R}$. Because of its length, the proof of Theorem 1.2 is broken into two pieces, Propositions 3.8 and 3.9.

In Section 3.4, we prove our other main result, a formula for the isometry $J_\alpha : K_\phi \to L^2(\sigma_\alpha)$ and an exact characterization of when it is unitary:

Theorem 1.3. *Fix $\alpha \in \mathbb{T}$.*

i. *For each each $f \in K_\phi$, $(J_\alpha f)(\zeta) = f^*(\zeta)$ for σ_α-a.e. $\zeta \in \mathbb{T}^2$.*
ii. *$J_\alpha : K_\phi \to L^2(\sigma_\alpha)$ is unitary if and only if α is a generic value for ϕ.*

This result is in contrast to the one-variable case, where J_α is always unitary. Here, by computing the non-tangential values of ϕ at its finite number of singularities, this theorem allows us to easily identify whether a given J_α is unitary. Part (i) is true in the one-variable setting and follows from a famous (and more general) result of Poltoratski about normalized Cauchy transforms, see [30] and [11, Theorem 10.3.1]. Thus, our result can be viewed as a partial two-variable analogue of Poltoratski's result. Again, due to length, we break the proof into two pieces, Propositions 3.10 and 3.12.

Section 4 connects our $(n,1)$ results to two related areas of study. First, in Theorem 4.1, we connect Theorem 1.2 to the theory of Agler decompositions and use our results about σ_α and J_α to establish formulas for some of the polynomials R_j in (4). Then, we observe that this study of Clark measures can be put into a more general context. Specifically, let $P(\mathbb{T}^2)$ denote the set of Borel probability measures on $\mathbb{T}^2$ equipped with the topology of weak-$\star$ convergence and define

$$\mathcal{P}_2 = \{f \in \mathrm{Hol}(\mathbb{D}^2) : \Re f(z) > 0 \text{ and } f(0,0) = 1\},$$

which is compact in the topology of uniform convergence on compact subsets of $\mathbb{D}^2$. Let $M : \mathcal{P}_2 \to P(\mathbb{T}^2)$ denote the map that takes each $f \in \mathcal{P}_2$ to the unique Borel probability measure μ_f with

$$f(z) = \int_{\mathbb{T}^2} P_z(\zeta)\, d\mu_f(\zeta) \quad \text{for } z \in \mathbb{D}^2.$$

Then both $\mathcal{P}_2$ and its image $M(\mathcal{P}_2)$ are compact convex sets and by the Krein-Milman theorem, equal the closed, convex hull of their extreme points. It is also easy to show that f is an extreme point of $\mathcal{P}_2$ if and only if μ_f is an extreme point of $M(\mathcal{P}_2)$. In [32], Rudin posed the question

"What are the extreme points of $\mathcal{P}_2$ (or equivalently, of $M(\mathcal{P}_2)$)?"

While this question is still open, a number of interesting examples and related results (often in the n-variable situation) have been proved by Forelli [15], Knese [22], and McDonald [25, 26, 27, 28]. As the Clark measures σ_α are trivially in $M(\mathcal{P}_2)$ when $\phi(0) = 0$, it makes sense to consider our investigations in the context of Rudin's question and these subsequent results. In particular, the following is a quick corollary of Theorem 1.2 and a theorem from [22]:

Corollary 1.4. *Let $\phi = \frac{\tilde{p}}{p}$ be a degree $(n,1)$ RIF with $\tilde{p}(0,0) = 0$. If $\alpha \in \mathbb{T}$, then:*

i. *If* α *is an exceptional value for* ϕ*, then* σ_α *is not an extreme point of* $M(\mathcal{P}_2)$*.*
ii. *If* p *is saturated,* $\deg p = \deg \tilde{p}$*, and* α *is generic for* ϕ*, then* σ_α *is an extreme point of* $M(\mathcal{P}_2)$*.*

Part (i) of this corollary shows that in our setting, the more complicated Clark measures cannot be extreme points of $M(\mathcal{P}_2)$. Meanwhile part (ii) coupled with our earlier characterizations provide explicit formulas for some extreme points of $M(\mathcal{P}_2)$. We should mention that these appear somewhat related to the measures studied in [25, Example 3].

In the last section, we use our results to compute the Clark measures and study the J_α isometries associated to several degree $(n, 1)$ RIFs. First, in Example 5.1, we use our results to recover Example 4.3 in [12]. Then in Example 5.2, we apply our results to a more complicated degree $(2, 1)$ RIF with a single singularity and in Example 5.4, we study a degree $(3, 1)$ RIF with two singularities. Finally, in Example 5.6, we investigate a degree $(3, 3)$ RIF ϕ with a singularity at $(1, 1)$. In particular, we show that although $\phi^*(1, 1) = -1$ and hence $\alpha = -1$ is an exceptional value for ϕ, the operator J_{-1} is unitary. This demonstrates that, in its current form, Theorem 1.3(ii) does not extend to general RIFs.

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2. Clark measures for RIFs

We begin with some remarks about Clark measures associated with general RIFs on the bidisk, before focusing on the degree $(n, 1)$ case. First, the following result appears to be known, see for instance [28, p.732] and [23], but here, we give a simple proof in the RIF case by adapting some of the arguments from the one-variable proof of [11, Theorem 9.2.1].

Theorem 2.1. *If* ϕ *is a nonconstant RIF on* $\mathbb{D}^2$ *and* $\alpha \in \mathbb{T}$*, then* σ_α *does not possess any point masses.*

Proof. Without loss of generality, we will show that σ_α does not possess a point mass at $(1, 1)$. First, applying [12, Proposition 2.6] with $w = (0, 0)$ yields

$$\int_{\mathbb{T}^2} \frac{1}{(1 - z_1\bar{\zeta}_1)(1 - z_2\bar{\zeta}_2)} d\sigma_\alpha(\zeta) = \frac{1 - \phi(z)\overline{\phi(0,0)}}{(1 - \bar{\alpha}\phi(z))(1 - \alpha\overline{\phi(0,0)})}. \tag{6}$$

Observe that $\theta(z) := \phi(z, z)$ is a nonconstant finite Blaschke product. Then for $0 < r < 1$, set $z = r$ and multiply both sides of (6) by $(1 - r)^2$ to get

$$\int_{\mathbb{T}^2} \frac{(1 - r)^2}{(1 - r\bar{\zeta}_1)(1 - r\bar{\zeta}_2)} d\sigma_\alpha(\zeta) = \frac{(1 - r)^2(1 - \theta(r)\overline{\theta(0)})}{(1 - \bar{\alpha}\theta(r))(1 - \alpha\overline{\theta(0)})}. \tag{7}$$

Observe that

$$\lim_{r \nearrow 1} \frac{(1 - r)^2}{(1 - r\bar{\zeta}_1)(1 - r\bar{\zeta}_2)} = \begin{cases} 1 & \text{if } \zeta = (1, 1), \\ 0 & \text{otherwise.} \end{cases}$$

Since σ_α is a finite measure, the dominated convergence theorem implies

$$\lim_{r\nearrow 1}\int_{\mathbb{T}^2}\frac{(1-r)^2}{(1-r\bar{\zeta}_1)(1-r\bar{\zeta}_2)}d\sigma_\alpha(\zeta)=\sigma_\alpha\{(1,1)\}.$$

As θ is a nonconstant finite Blaschke product, $\theta(1)$ exists and equals some $\lambda\in\mathbb{T}$ and its derivative $\theta'(1)$ exists and is nonzero, see [17, Lemma 7.5]. If $\lambda\neq\alpha$, then

$$\lim_{r\nearrow 1}\frac{(1-r)^2(1-\theta(r)\overline{\theta(0)})}{(1-\bar{\alpha}\theta(r))(1-\alpha\overline{\theta(0)})}=\lim_{r\nearrow 1}(1-r)^2\frac{1-\lambda\overline{\theta(0)}}{(1-\bar{\alpha}\lambda)(1-\alpha\overline{\theta(0)})}=0.$$

If $\lambda=\alpha$, then

$$\begin{aligned}\lim_{r\nearrow 1}\frac{(1-r)^2(1-\theta(r)\overline{\theta(0)})}{(1-\bar{\alpha}\theta(r))(1-\alpha\overline{\theta(0)})}&=\frac{1-\alpha\overline{\theta(0)}}{\bar{\alpha}-\overline{\theta(0)}}\cdot\lim_{r\nearrow 1}\frac{1-r}{\theta(1)-\theta(r)}\cdot\lim_{r\nearrow 1}(1-r)\\&=\alpha\cdot\frac{1}{\theta'(1)}\cdot 0=0.\end{aligned}$$

Equating the two sides in (7) implies that $\sigma_\alpha\{(1,1)\}=0$. □

We can say a little bit more about Clark measures associated with RIFs. The papers [8, 9] include several results concerning boundary behavior of two-variable RIFs, and in particular, the structure of their unimodular level sets. Recall from (2) that for $\alpha\in\mathbb{T}$ and $\phi=\tilde{p}/p$,

$$\mathcal{C}_\alpha=\{\zeta\in\mathbb{T}^2\colon \tilde{p}(\zeta)=\alpha p(\zeta)\}.$$

Then each $\mathcal{C}_\alpha$ satisifes $\mathcal{C}_\alpha=\{\zeta\in\mathbb{T}^2\colon \phi^*(\zeta)=\alpha\}\cup(\mathcal{Z}_p\cap\mathbb{T}^2)$, and as was shown in [9, Theorem 2.8], the components of $\mathcal{C}_\alpha$ can be locally parametrized using one-variable analytic functions. Intuitively speaking, this implies that, for each α, the Clark measure σ_α of any two-variable RIF has support contained in a one-dimensional subset of $\mathbb{T}^2$. (We should mention that, technically speaking, the result in [9] was proved for $\phi=\frac{\tilde{p}}{p}$ with $\deg p=\deg\tilde{p}$, but that assumption does not appear to materially affect the conclusions.) As an aside, we also note that general pluriharmonic measures on $\mathbb{T}^2$ can have substantially larger, and even two-dimensional support, viz. [28].

It should be noted that knowing that σ_α is a Clark measure associated with *some* RIF and is supported on *some* set of the form $\{\zeta\in\mathbb{T}^2\colon \tilde{p}(\zeta)=\alpha p(\zeta)\}$ does not suffice to determine that measure (or its associated RIF) uniquely. Indeed, one can exhibit (see Example 5.2) two different RIFs $\phi_1=\frac{\tilde{p}_1}{p_1}$ and $\phi_2=\frac{\tilde{p}_2}{p_2}$ whose Clark measures are not multiples of each other but are both supported on the same set $\{\zeta\in\mathbb{T}^2\colon \tilde{p}_1(\zeta)=\alpha p_1(\zeta)\}=\{\zeta\in\mathbb{T}^2\colon \tilde{p}_2(\zeta)=\alpha p_2(\zeta)\}$ for some $\alpha\in\mathbb{T}$. Thus, in order to study Clark measures and isometries for two-variable RIFs, we will need to perform a structural analysis of the measures σ_α that goes beyond determining their supports.

3. Clark measures for Degree $(n,1)$ RIFs

Throughout the rest of this paper, we let $\phi=\tilde{p}/p$ denote a fixed degree $(n,1)$ rational inner function for some $n\geq 1$. Recall that we can decompose p as in (3). Then the polynomials p, $\tilde{p}$ share no common factors. Moreover, p has no zeros on $\mathbb{D}^2\cup(\mathbb{D}\times\mathbb{T})\cup(\mathbb{T}\times\mathbb{D})$ and at most n distinct zeros on $\mathbb{T}^2$. See for example, Lemma 10.1, the proof of Corollary 13.5, and Appendix C in [21].

Remark 3.1. For a given degree $(n, 1)$ RIF function ϕ and $\alpha \in \mathbb{T}$, recall the objects from Definition 1.1 and define the following additional objects:

- ν_α is the measure on $\mathbb{T}$ defined by $d\nu_\alpha := W_\alpha dm$.
- Q and $R_1, \dots, R_n$ are the polynomials given in Theorem 3.2.
- $b_1, \dots, b_n$ are the rational functions in the disk algebra $A(\mathbb{D})$ from Proposition 3.5.
- For $f \in K_\phi$, $h, g_1, \dots, g_n$ are the $H^2(\mathbb{D})$ functions given in Theorem 3.2.

Finally, recall that $\alpha \in \mathbb{T}$ is an *exceptional value* for ϕ if there is a k such that $\phi^*(\tau_k, \lambda_k) = \alpha$ and $\alpha \in \mathbb{T}$ is a *generic value* for ϕ otherwise.

3.1. **Model Space Preliminaries.** Clark measures are closely related to the model space K_ϕ and so, we pause to record some known facts about K_ϕ in the degree $(n, 1)$ case.

Theorem 3.2. *There are polynomials $Q, R_1, \dots, R_n \in \mathbb{C}[z_1, z_2]$ such that $\deg R_j \leq (n-1, 1)$, $\deg Q \leq (n, 0)$ and for $z, w \in \mathbb{C}^2$,*

$$p(z)\overline{p(w)} - \tilde{p}(z)\overline{\tilde{p}(w)} = (1 - z_1\bar{w}_1)\sum_{j=1}^{n} R_j(z)\overline{R_j(w)} + (1 - z_2\bar{w}_2)Q(z)\overline{Q(w)}. \tag{8}$$

Furthermore, each R_j and Q vanish at each (τ_k, λ_k) and a function $f \in K_\phi$ if and only if there exist $g_1, \dots, g_n, h \in H^2(\mathbb{D})$ such that

$$f(z) = \tfrac{Q}{p}(z)h(z_1) + \sum_{j=1}^{n} \tfrac{R_j}{p}(z)g_j(z_2) \quad \text{for } z \in \mathbb{D}^2. \tag{9}$$

Finally, if $f \in K_\phi$ is written as in (9), then

$$\|f\|^2_{K_\phi} = \|f\|^2_{H^2(\mathbb{D}^2)} = \|h\|^2_{H^2(\mathbb{D})} + \sum_{j=1}^{n} \|g_j\|^2_{H^2(\mathbb{D})}.$$

Proof. As this result is not new, we just give some intuition and references for the different components of the theorem. First, note that on $H^2(\mathbb{D}^2)$, there are two shift operators, M_{z_1} and M_{z_2}, defined by $(M_{z_i}f)(z) = z_i f(z)$ for $i = 1, 2$. Let $S_1^{\max}$ be the maximal M_{z_1}-invariant subspace of K_ϕ, where M_{z_1} is multiplication by z_1. Then, while not obvious, it is true that $S_2^{\min} := K_\phi \ominus S_1^{\max}$ is invariant under M_{z_2}, see [4, 5]. Let K_1, K_2 denote the reproducing kernels of the two Hilbert spaces

$$S_1^{\max} \ominus M_{z_1}S_1^{\max} := \mathcal{H}(K_1) \quad \text{and} \quad S_2^{\min} \ominus M_{z_2}S_2^{\min} := \mathcal{H}(K_2), \tag{10}$$

respectively. This yields the Agler decomposition

$$1 - \phi(z)\overline{\phi(w)} = (1 - z_1\bar{w}_1)K_2(z, w) + (1 - z_2\bar{w}_2)K_1(z, w).$$

Since ϕ is a degree $(n, 1)$ RIF, one can show that $\dim \mathcal{H}(K_1) = 1$ and $\dim \mathcal{H}(K_2) = n$. Let Q/p be an orthonormal basis for $\mathcal{H}(K_1)$ and $R_1/p, \dots, R_n/p$ be an orthonormal basis for

$\mathcal{H}(K_2)$. One can show that the Q, R_j are polynomials with $\deg Q \leq (n, 0)$ and $\deg R_j \leq (n-1, 1)$ and each Q, R_j vanishes at each (τ_k, λ_k). Furthermore,

$$K_1(z,w) = \frac{Q(z)\overline{Q(w)}}{p(z)\overline{p(w)}} \text{ and } K_2(z,w) = \sum_{j=1}^{n} \frac{R_j(z)\overline{R_j(w)}}{p(z)\overline{p(w)}},$$

and substituting the formulas into the Agler decomposition and multiplying through by the denominator gives (8). For the details, see for example [4, 6, 19] and the references within.

The characterization of functions in K_ϕ from (9) follows from the fact that the reproducing kernel $k(z, w)$ of K_ϕ satisfies

$$k(z,w) = \frac{1}{1 - z_2\bar{w}_2} \sum_{j=1}^{n} \frac{R_j(z)\overline{R_j(w)}}{p(z)\overline{p(w)}} + \frac{1}{1 - z_1\bar{w}_1} \frac{Q(z)\overline{Q(w)}}{p(z)\overline{p(w)}}$$

and from standard properties of reproducing kernels. A proof of the formula for the norm of functions in K_ϕ can be found, for example, in Remark 2.3 in [7]. □

3.2. **Support Sets and Consequences.** In this subsection, we obtain some information about the objects from Definition 1.1 and Remark 3.1. First, recall that $\mathcal{C}_\alpha$ from (2) contains the support of σ_α. The following lemma gives some crucial insight into when $\mathcal{C}_\alpha$ contains a line.

Lemma 3.3. *For $\gamma \in \mathbb{T}$, the set $\mathcal{C}_\alpha$ contains $\{\gamma\} \times \mathbb{T}$ if and only if $\gamma = \tau_k$ for some k and $\phi^*(\tau_k, \lambda_k) = \alpha$.*

Proof. Observe that $\{\gamma\} \times \mathbb{T} \subseteq \mathcal{C}_\alpha$ if and only if $\phi^*(\gamma, \zeta_2) \equiv \alpha$ for all $\zeta_2 \in \mathbb{T}$, except maybe at one ζ_2 where $p(\gamma, \zeta_2) = 0$.

Thus, for the forward direction, we can assume $\phi^*(\gamma, \zeta_2) \equiv \alpha$ (except maybe at one ζ_2). As our assumptions imply $\deg \tilde{p}(\gamma, \zeta_2) = 1$, the polynomials $p(\gamma, \cdot), \tilde{p}(\gamma, \cdot)$ must share a common factor with a zero on $\mathbb{T}$, say $(z_2 - \beta)$. This implies that $p(\gamma, \beta) = \tilde{p}(\gamma, \beta) = 0$. Thus, $(\gamma, \beta) = (\tau_k, \lambda_k)$ for some k.

Set $\tau := (\tau_k, \lambda_k)$ and write

$$p(z) = \sum_{j=M}^{n+1} P_j(\tau - z) \;\text{ and }\; \tilde{p}(z) = \sum_{j=M}^{n+1} Q_j(\tau - z),$$

where P_j and Q_j are homogeneous polynomials in z_1, z_2 of degree j and $M \geq 1$. Define $\lambda = \phi^*(\tau)$. By [21, Propositions 14.3, 14.5], $Q_M = \lambda P_M$ and since $\tilde{p}, p$ share no common factors, we can conclude that $M = 1$ and P_1 contains a term cz_2 with $c \neq 0$. Then

$$\alpha \equiv \phi(\tau_k, \zeta_2) = \frac{\lambda c(\lambda_k - \zeta_2)}{c(\lambda_k - \zeta_2)} = \lambda,$$

so $\alpha = \phi^*(\tau)$. Similarly, if $\alpha = \phi^*(\tau)$ for some $\tau = (\tau_k, \lambda_k)$, the above equality and arguments imply that we also have $\alpha \equiv \phi^*(\tau_k, \zeta_2)$ for all $\zeta_2 \in \mathbb{T} \setminus \{\lambda_k\}$ and so $\{\tau_k\} \times \mathbb{T}$ is in $\mathcal{C}_\alpha$. □

The following theorem gives additional information about the behavior of the function B_α and set $\mathcal{C}_\alpha$.

Theorem 3.4. *Let $\alpha \in \mathbb{T}$. Then the following statements hold.*

i. *The function* B_α *is a finite Blaschke product.*
ii. $\tilde{p}-\alpha p$ *and* $\tilde{p}_1-\alpha p_2$ *do not possess any repeated linear factors* $(z_1-\gamma)^2$ *with* $\gamma\in\mathbb{T}$.
iii. *Let* α *be a generic value of* ϕ. *Then* $\deg B_\alpha = n$ *and* $\mathcal{C}_\alpha$ *equals* E_α.
iv. *Let* α *be an exceptional value of* ϕ *and (after reordering if necessary), assume* $\phi^*(\tau_k,\lambda_k)=\alpha$ *for* $k=1,\dots,\ell$. *Then* $\deg B_\alpha = n-\ell$ *and* $\mathcal{C}_\alpha$ *is* $E_\alpha\cup(\cup_{k=1}^{\ell}L_k)$.
v. $\overline{B_\alpha(\tau_k)}=\lambda_k$ *for* $k=1,\dots,m$.
vi. *If* $\mathcal{C}_\alpha$ *contains* L_k*, then for all* $z_2\in\overline{\mathbb{D}}$ *with* $z_2\neq\lambda_k$, $\frac{\partial\phi}{\partial z_1}(\tau_k,z_2)$ *equals a fixed nonzero constant* C.

Proof. Fix $\alpha\in\mathbb{T}$ and recall the definition of B_α from (5). To establish (i), observe that if

$$r=\alpha p_1-\tilde{p}_2,\quad \text{then}\quad \tilde{r}=\bar{\alpha}\tilde{p}_1-p_2,$$

where $\tilde{r}(z_1)=z_1^n\overline{r(\frac{1}{\bar{z}_1})}$. As $|r(\zeta_1)|=|\tilde{r}(\zeta_1)|$ on $\mathbb{T}$ and

$$B_\alpha(z_1)=\alpha\frac{\tilde{r}(z_1)}{r(z_1)},$$

this shows that $|B_\alpha(\zeta_1)|=1$ a.e. on $\mathbb{T}$. As ϕ is nonconstant, $|\phi(z_1,0)|=|\frac{\tilde{p}_2}{p_1}(z_1)|<1$ on $\mathbb{D}$ and so, $\alpha p_1-\tilde{p}_2$ is nonvanishing on $\mathbb{D}$. Thus, after cancelling any common factors from its numerator and denominator, B_α is a finite Blaschke product. Moreover, these cancelled factors are always of the form $(z_1-\gamma)$, where $\gamma\in\mathbb{T}$.

To establish (ii), we first show that $\tilde{p}-\alpha p$ has no repeated linear factor with a zero from $\mathbb{T}$. By way of contradiction, assume $\tilde{p}-\alpha p$ is divisible by some $(z_1-\gamma)^2$ with $\gamma\in\mathbb{T}$. Then, $\{\gamma\}\times\mathbb{T}\subseteq\mathcal{C}_\alpha$ and for $\zeta_2\notin\{\lambda_1,\dots,\lambda_m\}$, we have $\phi(\gamma,\zeta_2)\equiv\alpha$ and

$$\tfrac{\partial\phi}{\partial z_1}(\gamma,\zeta_2)=\frac{p\frac{\partial\tilde{p}}{\partial z_1}-\tilde{p}\frac{\partial p}{\partial z_1}}{p^2}(\gamma,\zeta_2)=\frac{\frac{\partial\tilde{p}}{\partial z_1}-\alpha\frac{\partial p}{\partial z_1}}{p}(\gamma,\zeta_2)=0.$$

Fix any $\lambda\in\mathbb{T}$ with $\lambda\notin\{\lambda_1,\dots,\lambda_m\}$. Then $\phi_\lambda(z_1):=\phi(z_1,\lambda)$ is a nonconstant finite Blaschke product and so, $\phi'_\lambda(\gamma)\neq 0$. See, for example, Lemma 7.5 in [17]. Since

$$0\neq\phi'_\lambda(\gamma)=\tfrac{\partial\phi}{\partial z_1}(\gamma,\lambda)=0$$

by the above argument, we obtain the requisite contradiction. To prove the result for $\tilde{p}_1-\alpha p_2$, proceed by way of contradiction and assume that it has a factor $(z_1-\gamma)^2$ with $\gamma\in\mathbb{T}$. Then $\tilde{p}_2-\alpha p_1$ possesses the same factor. Then the equality

$$\tilde{p}(z)-\alpha p(z)=z_2\left(\tilde{p}_1(z_1)-\alpha p_2(z_1)\right)+\left(\tilde{p}_2(z_1)-\alpha p_1(z_1)\right)\tag{11}$$

implies that $\tilde{p}-\alpha p$ is also divisible by $(z_1-\gamma)^2$, a contradiction.

To establish (iii) and (iv), let $\gamma_1,\dots,\gamma_\ell$ denote the zeros of $\tilde{p}_1-\alpha p_2$ in $\mathbb{T}$. Part (ii) implies that they are distinct. Then (11) gives

$$\tilde{p}(z)-\alpha p(z)=q(z_1)\left(z_2-1/B_\alpha(z_1)\right)\prod_{k=1}^{\ell}(z_1-\gamma_k),\tag{12}$$

where $q\in\mathbb{C}[z]$ does not vanish on $\mathbb{T}$ since it is the numerator of B_α once the common terms between the numerator and denominator have been cancelled. Equation (12) immediately shows that $\mathcal{C}_\alpha$ is the union of E_α and the set of lines $\{\gamma_k\}\times\mathbb{T}$ for $k=1,\dots,\ell$. To compute $\deg B_\alpha$, note that $\deg(\tilde{p}_1-\alpha p_2)=n$. This occurs because since $|\phi|<1$ on $\mathbb{D}^2$, we have $|p_1(0)|>|\tilde{p}_2(0)|$ and thus, the coefficient of the degree n term in $\tilde{p}_1-\alpha p_2$ is nonzero. Thus,

$\deg B_\alpha = n - \ell$, its original degree minus the number of cancelled terms or equivalently, the number of lines of the form $\{\gamma\} \times \mathbb{T}$ in $\mathcal{C}_\alpha$.

To finish (iii), let α be generic. By Lemma 3.3, $\mathcal{C}_\alpha$ cannot contain any lines of the form $\{\gamma\} \times \mathbb{T}$ and so, our arguments give $\mathcal{C}_\alpha = E_\alpha$ and $\deg B_\alpha = n$.

To finish (iv), let α be exceptional and (after reordering if necessary), assume $\phi^*(\tau_k, \lambda_k) = \alpha$ for $k = 1, \dots, \ell$. Then Lemma 3.3 implies that $L_1, \dots, L_\ell$ are exactly the lines of the form $\{\gamma\} \times \mathbb{T}$ in $\mathcal{C}_\alpha$. Then the above arguments imply $\mathcal{C}_\alpha = E_\alpha \cup (\cup_{k=1}^{\ell} L_k)$ and $\deg B_\alpha = n - \ell$.

To prove (v), let $\tau := (\tau_k, \lambda_k)$ and note that since $\tilde{p}(\tau) = p(\tau) = 0$, it follows by definition that $\tau \in \mathcal{C}_\alpha$. First assume that $\phi^*(\tau) \neq \alpha$. Then $\mathcal{C}_\alpha = E_\alpha$ and so it must be the case that

$$(\tau_k, \lambda_k) = (\tau_k, \frac{1}{B_\alpha(\tau_k)}) = (\tau_k, \overline{B_\alpha(\tau_k)}),$$

as needed. Now if $\phi^*(\tau) = \alpha$, it follows that $\{\tau_k\} \times \mathbb{T} \subseteq \mathcal{C}_\alpha$. Then as in the proof of Lemma 3.3, we can write:

$$(\tilde{p} - \alpha p)(z) = \sum_{j=2}^{n+1} (Q_j - \alpha P_j)(z - \tau) = (\tau_k - z_1) G(z),$$

where Q_j, P_j are homogeneous polynomials of degree j and G is a polynomial. Here, we used the fact that $Q_1 = \alpha P_1$ and $\deg(\tilde{p} - \alpha p) \leq (n, 1)$. From this, it is clear that $G(\tau) = 0$. Since $(z_1 - \tau_k)$ divides $\tilde{p} - \alpha p$, (11) implies that it also divides $\tilde{p}_1 - \alpha p_2$. Thus,

$$(\tau_k - z_1) G(z) = (\tilde{p}_1 - \alpha p_2)(z_1) \left(z_2 - 1/B_\alpha(z_1)\right) = r(z_1)(\tau_k - z_1) \left(z_2 - 1/B_\alpha(z_1)\right),$$

for some $r \in \mathbb{C}[z]$. By (ii), we know that $r(\tau_k) \neq 0$. Dividing through by $(\tau_k - z_1)$ and plugging in τ implies $\lambda_k = 1/B_\alpha(\tau_k) = \overline{B_\alpha(\tau_k)}$.

To prove (vi), assume $L_k = \{\tau_k\} \times \mathbb{T} \subseteq \mathcal{C}_\alpha$. Then the same arguments as in (v) imply that

$$(\tilde{p} - \alpha p)(z) = r(z_1)(z_1 - \tau_k)(z_2 - 1/B_\alpha(z_1)), \tag{13}$$

for some $r \in \mathbb{C}[z]$ with $r(\tau_k) \neq 0$. Then for $z_2 \in \mathbb{C} \setminus \{\lambda_k\}$, we have $\phi(\tau_k, z_2) \equiv \alpha$ and we can use (13) to conclude

$$\tfrac{\partial \phi}{\partial z_1}(\tau_k, z_2) = \frac{\frac{\partial \tilde{p}}{\partial z_1} - \alpha \frac{\partial p}{\partial z_1}}{p}(\tau_k, z_2) = \frac{r(\tau_k)(z_2 - \lambda_k)}{c(z_2 - \lambda_k)} = \frac{r(\tau_k)}{c},$$

for some $c \neq 0$. □

We can also use the set $\mathcal{C}_\alpha$ to refine our understanding of the polynomials $R_1, \dots, R_n, Q$ from Theorem 3.2 as follows:

Proposition 3.5. *$R_1, \dots, R_n, Q$ satisfy the following properties:*

i. *For $\zeta_1 \in \mathbb{T}$, $|Q(\zeta_1)|^2 = |p_1(\zeta_1)|^2 - |p_2(\zeta_1)|^2$.*

ii. *For $z \in \overline{\mathbb{D}}^2$, $R_j(z) = r_j(z_1)\left(1 - B_\alpha(z_1) z_2\right) + z_2 Q(z_1) b_j(z_1)$, for some unique $r_j \in \mathbb{C}[z]$ with $\deg r_j \leq (n-1)$ and rational $b_j \in A(\mathbb{D})$.*

Proof. Part (i) follows from some algebra; substituting $z = w = (\zeta_1, z_2)$ into (8) gives

$$|p(\zeta_1, z_2)|^2 - |\tilde{p}(\zeta_1, z_2)|^2 = (1 - |z_2|^2)|Q(\zeta_1)|^2.$$

The left-hand-side becomes

$$|p_1(\zeta_1) + z_2 p_2(\zeta_1)|^2 - |z_2 \overline{p_1(\zeta_1)} + \overline{p_2(\zeta_1)}|^2 = (1 - |z_2|^2)\left(|p_1(\zeta_1)|^2 - |p_2(\zeta_1)|^2\right),$$

and dividing by $(1-|z_2|^2)$ gives the desired formula.

For (ii), recall that $\tilde{p}(z)=\alpha p(z)$ whenever $z_2 = 1/B_\alpha(z_1)$. Substituting $z_2=1/B_\alpha(z_1)$ and $w_2 = 1/B_\alpha(w_1)$ into (8) gives

$$0=(1-z_1\bar{w}_1)\sum_{j=1}^{n} R_j(z_1,1/B_\alpha(z_1))\overline{R_j(w_1,1/B_\alpha(w_1))}+(1-1/B_\alpha(z_1)\overline{1/B_\alpha(w_1)})Q(z_1)\overline{Q(w_1)},$$

for all z_1,w_1 where $B_\alpha(z_1)\neq 0$. We can rewrite this as

$$B_\alpha(z_1)\overline{B_\alpha(w_1)}\sum_{j=1}^{n} R_j(z_1,1/B_\alpha(z_1))\overline{R_j(w_1,1/B_\alpha(w_1))}=\frac{1-B_\alpha(z_1)\overline{B_\alpha(w_1)}}{1-z_1\bar{w}_1}Q(z_1)\overline{Q(w_1)},$$

for $z_1,w_1\in\mathbb{D}$. Then the right-hand-side is the reproducing kernel of the one-variable model space $\hat{K}_{B_\alpha}:=H^2(\mathbb{D})\ominus B_\alpha H^2(\mathbb{D})$ (which is composed of rational functions in $A(\mathbb{D})$, see [16, Chapter 5]) times the Q term. To finish the proof, fix R_j and by Theorem 3.2, write

$$R_j(z)=r_j(z_1)+z_2q_j(z_1),$$

for $r_j,q_j\in\mathbb{C}[z]$. Then standard properties of reproducing kernels imply that for some $b_j\in\hat{K}_{B_\alpha}$,

$$B_\alpha(z_1)r_j(z_1)+q_j(z_1)=b_j(z_1)Q(z_1)$$

for $z_1\in\overline{\mathbb{D}}$. Solving this for q_j and substituting back into the formula for R_j yields the desired result. □

In what follows, we will also require information about the weight function W

$$W(x,\zeta)=W_x(\zeta)=\frac{|p_1(\zeta)|^2-|p_2(\zeta)|^2}{|\tilde{p}_1(\zeta)-xp_2(\zeta)|^2}\text{ for }(x,\zeta)\in\mathbb{T}^2. \tag{14}$$

Lemma 3.6. *The function W from (14) satisfies the following properties:*

i. *W is well defined and continuous on $\mathbb{T}^2$, except possibly at the finite set of points (α,τ_k) where α is exceptional for ϕ and $L_k\subseteq\mathcal{C}_\alpha$.*
ii. *For $\alpha\in\mathbb{T}$, W_α has at most a finite number of discontinuities on $\mathbb{T}$, all of which are removable. So, W_α equals a bounded, continuous function m-a.e. on $\mathbb{T}$.*

Proof. For (i), assume that $\tilde{p}_1(\zeta)-xp_2(\zeta)$ vanishes at some $(\zeta,x)\in\mathbb{T}^2$. Then (11) implies that $\tilde{p}(\zeta,\cdot)-xp(\zeta,\cdot)\equiv 0$, so $\{\zeta\}\times\mathbb{T}\subseteq\mathcal{C}_x$. Then Lemma 3.3 implies that $\zeta=\tau_k$ for some k, x is exceptional, and $L_k\subseteq\mathcal{C}_x$.

For (ii), first observe that if α is generic, then (i) implies W_α is continuous, and hence bounded, on $\mathbb{T}$. If α is exceptional, after reordering the singularities of ϕ, assume $L_1,\dots,L_\ell$ are exactly the lines in $\mathcal{C}_\alpha$. Then by the proof of Theorem 3.4,

$$(\tilde{p}_1-\alpha p_2)(z)=q(z)\prod_{k=1}^{\ell}(z-\tau_k),$$

for some $q\in\mathbb{C}[z]$ that is non-vanishing on $\mathbb{T}$. Similarly, Theorem 3.2 and Proposition 3.5 imply that for $\zeta\in\mathbb{T}$,

$$|p_1(\zeta)|^2-|p_2(\zeta)|^2=|Q(\zeta)|^2=\prod_{k=1}^{\ell}|\zeta-\tau_k|^2|r(\zeta)|^2,$$

for some $r \in \mathbb{C}[z]$. Thus, for $\zeta \neq \tau_1, \dots \tau_\ell$,

$$W_\alpha(\zeta) = \left| \tfrac{r}{q}(\zeta) \right|^2.$$

This shows that the only possible singularities W_α could have on $\mathbb{T}$ are at $\tau_1, \dots, \tau_\ell$ and any such singularities must be removable. □

3.3. **Clark Measure Formulas.** Recall that the Clark measures associated to ϕ are characterized via Theorem 1.2. In the proof we will require the following lemma, which is a consequence of standard measure-theory facts. We include its proof here for the ease of the reader.

Lemma 3.7. *Let σ be a Borel measure on $\mathbb{T}^2$ and let $\zeta_2 = g(\zeta_1)$ be a continuous curve in $\mathbb{T}^2$. If W is a continuous function defined on $\mathbb{T}$ such that*

$$\int_{\mathbb{T}^2} f(\zeta) d\sigma(\zeta) = \int_{\mathbb{T}} f(\zeta, g(\zeta)) W(\zeta) dm(\zeta) \quad \text{for all } f \in C(\mathbb{T}^2), \tag{15}$$

then (15) holds for all $f \in L^1(\sigma)$.

Proof. For ease of notation, set $d\nu = W dm$. Then ν is a Borel measure on $\mathbb{T}$. Furthermore, if $E \subseteq \mathbb{T}^2$ is a Borel set, then

$$E_g := \{\zeta_1 \in \mathbb{T} : \text{there exists } \zeta_2 \in \mathbb{T} \text{ with } (\zeta_1, \zeta_2) \in E \cap \{(\zeta, g(\zeta)) : \zeta \in \mathbb{T}\}\} \tag{16}$$

is the projection of a Borel set in $\mathbb{T}^2$ onto its first coordinate. This implies that E_g is an analytic set and its characteristic function is Lesbesgue measurable and hence, ν-measurable, see for example Chapter 13 in [13]. We will use E_g frequently because $\chi_E(\zeta, g(\zeta)) = \chi_{E_g}(\zeta)$ for $\zeta \in \mathbb{T}$. The following proof has three steps.

Step 1: Establish (15) for $f = \chi_U$, where U is an arbitrary open set in $\mathbb{T}^2$. Let $\{K_n\}_{n=1}^\infty$ be a sequence of nested compact sets with $U = \bigcup_n K_n$. Then, by Urysohn's lemma, there exists a sequence $\{f_n\}_{n=1}^\infty$ of continuous functions on $\mathbb{T}^2$ having $0 \leq f_n \leq 1$ on $\mathbb{T}^2$ and

$$f_n = 1 \quad \text{on} \quad K_n \quad \text{and} \quad f_n = 0 \quad \text{on} \quad U^c.$$

Then $\lim_{n\to\infty} f_n(\zeta) = \chi_U(\zeta)$ for every $\zeta \in \mathbb{T}^2$, and since ν is a finite measure, we can apply the dominated convergence theorem to obtain

$$\sigma(U) = \lim_{n\to\infty} \int_{\mathbb{T}^2} f_n(\zeta) d\sigma(\zeta) = \lim_{n\to\infty} \int_{\mathbb{T}} f_n(\zeta, g(\zeta)) d\nu(\zeta) = \int_{\mathbb{T}} \chi_U(\zeta, g(\zeta)) d\nu(\zeta),$$

as desired.

Step 2: Establish (15) for $f = \chi_E$, where E is an arbitrary Borel set in $\mathbb{T}^2$. Since σ is a finite Borel measure on $\mathbb{T}^2$, and hence is Radon, for each $n \in \mathbb{N}$ there exists a compact set K_n and an open set U_n such that $K_n \subseteq E \subseteq U_n$ and $\sigma(U_n \setminus K_n) < 1/n$. Urysohn's lemma again guarantees the existence of a sequence $\{f_n\}_{n=1}^\infty$ of continuous functions with $0 \leq f_n \leq 1$, $f_n = 1$ on K_n, and $f_n = 0$ on U_n^c. Since

$$\int_{\mathbb{T}^2} |(f_n - \chi_E)(\zeta)| \, d\sigma(\zeta) \leq \int_{\mathbb{T}^2} \chi_{U_n \setminus K_n}(\zeta) d\sigma(\zeta) = \sigma(U_n \setminus K_n) < \tfrac{1}{n},$$

we have $\|f - f_n\|_{L^1(\sigma)} < 1/n$ and $f_n \to f$ in $L^1(\sigma)$ as $n \to \infty$. Since the f_n are continuous, this implies

$$\int_{\mathbb{T}^2} \chi_E(\zeta) d\sigma(\zeta) = \lim_{n\to\infty} \int_{\mathbb{T}^2} f_n(\zeta) d\sigma(\zeta) = \lim_{n\to\infty} \int_{\mathbb{T}} f_n(\zeta, g(\zeta)) d\nu(\zeta).$$

Because $\chi_{K_n} \le f_n \le \chi_{U_n}$ on $\mathbb{T}^2$, we then obtain

$$\int_{\mathbb{T}} \chi_{K_n}(\zeta, g(\zeta)) d\nu(\zeta) \le \int_{\mathbb{T}^2} f_n(\zeta) d\sigma(\zeta) \le \int_{\mathbb{T}} \chi_{U_n}(\zeta, g(\zeta)) d\nu(\zeta) \tag{17}$$

and as $\chi_{K_n} \le \chi_E \le \chi_{U_n}$ on $\mathbb{T}^2$,

$$\int_{\mathbb{T}} \chi_{K_n}(\zeta, g(\zeta)) d\nu(\zeta) \le \int_{\mathbb{T}} \chi_E(\zeta, g(\zeta)) d\nu(\zeta) \le \int_{\mathbb{T}} \chi_{U_n}(\zeta, g(\zeta)) d\nu(\zeta). \tag{18}$$

Combining (17) and (18) gives

$$\begin{aligned} \left| \int_{\mathbb{T}^2} \chi_E(\zeta) d\sigma(\zeta) - \int_{\mathbb{T}} \chi_E(\zeta, g(\zeta)) d\nu(\zeta) \right| &\le \|\chi_E - f_n\|_{L^1(\sigma)} \\ &\quad + \left| \int_{\mathbb{T}^2} f_n(\zeta) d\sigma(\zeta) - \int_{\mathbb{T}} \chi_E(\zeta, g(\zeta)) d\nu(\zeta) \right| \\ &\le \tfrac{1}{n} + \int_{\mathbb{T}} (\chi_{U_n} - \chi_{K_n})(\zeta, g(\zeta)) d\nu(\zeta) \\ &= \tfrac{1}{n} + \sigma(U_n \setminus K_n) < \tfrac{2}{n} \end{aligned}$$

for all n, where we used Step 1 applied to $U_n \setminus K_n = U_n \cap K_n^c$. Letting $n \to \infty$ gives (15) for $f = \chi_E$.

Step 3: Establish (15) for a general $f \in L^1(\sigma)$. Pick a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions on $\mathbb{T}^2$ such that $f_n \to f$ in $L^1(\sigma)$ and pointwise σ-almost everywhere on $\mathbb{T}^2$. Then there exists some Borel set $E \subset \mathbb{T}^2$ with $\sigma(E) = 0$ such that if $f_n(\zeta) \not\to f(\zeta)$, $n \to \infty$, then $\zeta \in E$. Then, if $f_n(\zeta, g(\zeta)) \not\to f(\zeta, g(\zeta))$ then $\zeta \in E_g$, where E_g is defined in (16). By Step 2, $\nu(E_g) = \sigma(E) = 0$. Hence

$$f_n(\zeta, g(\zeta)) \to f(\zeta, g(\zeta)) \quad \text{for} \quad \nu\text{-a.e } \zeta \in \mathbb{T}. \tag{19}$$

Since the f_n are continuous, we have

$$\int_{\mathbb{T}^2} |f_n(\zeta) - f_m(\zeta)| d\sigma(\zeta) = \int_{\mathbb{T}} |f_n(\zeta, g(\zeta)) - f_m(\zeta, g(\zeta))| d\nu(\zeta).$$

This implies that $\{f_n(\zeta, g(\zeta))\}_{n=1}^{\infty}$ is a Cauchy sequence and hence has a limit F in $L^1(\nu)$, and by (19) we must have $F = f$ in $L^1(\nu)$. Then L^1-convergence gives

$$\int_{\mathbb{T}^2} f(\zeta) d\sigma(\zeta) = \lim_{n\to\infty} \int_{\mathbb{T}^2} f_n(\zeta) d\sigma(\zeta) = \lim_{n\to\infty} \int_{\mathbb{T}} f_n(\zeta, g(\zeta)) d\nu(\zeta) = \int_{\mathbb{T}} f(\zeta, g(\zeta)) d\nu(\zeta),$$

which gives (15) for f. □

Now we proceed to the proof of Theorem 1.2. We split the proof into two propositions; the first considers generic values for ϕ and the second considers exceptional values for ϕ.

Proposition 3.8. *Let $\alpha \in \mathbb{T}$ be a generic value for ϕ. Then for all $f \in L^1(\sigma_\alpha)$,*

$$\int_{\mathbb{T}^2} f(\zeta)\, d\sigma_\alpha(\zeta) = \int_{\mathbb{T}} f(\zeta, \overline{B_\alpha(\zeta)})\, d\nu_\alpha(\zeta), \tag{20}$$

where $d\nu_\alpha = W_\alpha dm$ and B_α, W_α are from Definition 1.1.

Proof. For each fixed $z_2 \in \mathbb{D}$, define the function

$$\psi^\alpha_{z_2}(z_1) := \frac{\alpha + \phi(z_1, z_2)}{\alpha - \phi(z_1, z_2)} \text{ for } z_1 \in \mathbb{D}.$$

Recall that ϕ has no singularities on $\mathbb{T}\times\mathbb{D}$. Furthermore, since α is generic, if $\phi(\zeta_1, z_2) = \alpha$ for $\zeta_1 \in \mathbb{T}$, then $z_2 = 1/B_\alpha(\zeta_1) \in \mathbb{T}$. Thus, $\psi^\alpha_{z_2} \in A(\mathbb{D})$ and for all $z_1 \in \mathbb{D}$,

$$\Re\left(\psi^\alpha_{z_2}(z_1)\right) = \int_{\mathbb{T}} \frac{1-|z_1|^2}{|z_1-\zeta|^2} \Re\left(\psi^\alpha_{z_2}(\zeta)\right) dm(\zeta).$$

If $z_1 \in \mathbb{D}$ and $\zeta \in \mathbb{T}$, then the computation in the proof of Proposition 3.5(i) gives

$$|p(\zeta, z_2)|^2 - |\tilde{p}(\zeta, z_2)|^2 = (1-|z_2|^2)\left(|p_1(\zeta)|^2 - |p_2(\zeta)|^2\right),$$

which one can use to obtain

$$\Re\left(\psi^\alpha_{z_2}(\zeta)\right) = \frac{1-|\phi(\zeta, z_2)|^2}{|\alpha - \phi(\zeta, z_2)|^2} = \frac{|p(\zeta, z_2)|^2 - |\tilde{p}(\zeta, z_2)|^2}{|\alpha p(\zeta, z_2) - \tilde{p}(\zeta, z_2)|^2} = \frac{1-|z_2|^2}{|z_2 - \overline{B_\alpha(\zeta)}|^2} \frac{|p_1(\zeta)|^2 - |p_2(\zeta)|^2}{|\alpha p_2(\zeta) - \tilde{p}_1(\zeta)|^2}.$$

This implies that

$$\begin{aligned}\Re\left(\psi^\alpha_{z_2}(z_1)\right) &= \int_{\mathbb{T}} \frac{1-|z_1|^2}{|z_1-\zeta|^2} \frac{1-|z_2|^2}{|z_2 - \overline{B_\alpha(\zeta)}|^2} \frac{|p_1(\zeta)|^2 - |p_2(\zeta)|^2}{|\alpha p_2(\zeta) - \tilde{p}_1(\zeta)|^2} dm(\zeta) \\ &= \int_{\mathbb{T}} P_z(\zeta, \overline{B_\alpha(\zeta)}) W_\alpha(\zeta) dm(\zeta).\end{aligned}$$

Thus, by the definition of σ_α, we can conclude that

$$\int_{\mathbb{T}^2} P_z(\zeta) d\sigma_\alpha(\zeta) = \int_{\mathbb{T}} P_z(\zeta, \overline{B_\alpha(\zeta)}) d\nu_\alpha(\zeta).$$

Since finite linear combinations of Poisson functions P_z are dense in $C(\mathbb{T}^2)$, this formula extends to all functions in $C(\mathbb{T}^2)$. Since W_α is bounded by Proposition 3.6, the formula extends to $f \in L^1(\sigma_\alpha)$ by Lemma 3.7. □

Let us now consider the exceptional α values for ϕ.

Proposition 3.9. *Let $\alpha \in \mathbb{T}$ be an exceptional value for ϕ and (after re-ordering the zeros of p on $\mathbb{T}^2$ if necessary and applying Theorem 3.4 (iv)), assume $\mathcal{C}_\alpha = E_\alpha \cup (\cup_{k=1}^{\ell} L_k)$. Then for all $f \in L^1(\sigma_\alpha)$,*

$$\int_{\mathbb{T}^2} f(\zeta) d\sigma_\alpha(\zeta) = \int_{\mathbb{T}} f(\zeta, \overline{B_\alpha(\zeta)}) d\nu_\alpha(\zeta) + \sum_{k=1}^{\ell} c_k^\alpha \int_{\mathbb{T}} f(\tau_k, \zeta)\, dm(\zeta), \tag{21}$$

where $d\nu_\alpha = W_\alpha dm$, B_α, W_α are from Definition 1.1, and $c_1^\alpha, \dots, c_\ell^\alpha > 0$.

Proof. Recall that σ_α is supported on $\mathcal{C}_\alpha = E_\alpha \cup (\cup_{k=1}^{\ell} L_k)$. Since the L_k are disjoint and $E_\alpha \cap L_k = \{(\tau_k, \overline{B_\alpha(\tau_k)})\}$, which has σ_α measure 0, we only need to show

$$\int_{\mathbb{T}^2} f(\zeta)\chi_{E_\alpha}(\zeta)d\sigma_\alpha(\zeta) = \int_{\mathbb{T}} f(\zeta, \overline{B_\alpha(\zeta)})W_\alpha(\zeta)dm(\zeta) \tag{22}$$

$$\int_{\mathbb{T}^2} f(\zeta)\chi_{L_k}(\zeta)d\sigma_\alpha(\zeta) = c_k^\alpha \int_{\mathbb{T}} f(\tau_k, \zeta)dm(\zeta) \tag{23}$$

for $f \in L^1(\sigma_\alpha)$ and $k = 1, \dots, \ell$, where χ_E denotes the characteristic function of a set $E \subseteq \mathbb{T}^2$. Then Lemma 3.6 implies W_α is bounded on $\mathbb{T}$ (with at most ℓ removable singularities) and by Lemma 3.7, we need only establish (22) and (23) for $f \in C(\mathbb{T}^2)$.

To ease notation, throughout this proof, for any F defined on $\mathbb{T}^2$, we will use F_α to denote $F_\alpha(\zeta) := F(\zeta, \overline{B_\alpha(\zeta)})$ and F_k to denote $F_k(\zeta) := F(\tau_k, \zeta)$.

Part 1. We first prove (22). To that end, fix a small $\epsilon > 0$ and define

$$S_\epsilon = \{\zeta \in \mathbb{T} : \min_k |\zeta - \tau_k| < \epsilon\}$$

and define $S_{\epsilon/2}$ analogously. By Lemma 3.6 and the definition of B_α, we can find a small arc $A_\alpha \subseteq \mathbb{T}$ centered at α such that both $W(x, \zeta)$ and $B(x, \zeta) := B_x(\zeta)$ are uniformly continuous on $A_\alpha \times (\mathbb{T} \setminus S_{\epsilon/2})$. Choose $(\alpha_n) \subseteq \mathbb{T}$ such that each α_n is generic and $(\alpha_n) \to \alpha$. Then by [12, Corollary 2.2], (σ_{α_n}) converges weak-$\star$ to σ_α.

To exploit that fact, let Ψ_ϵ be a continuous function on $\mathbb{T}$ such that

$$\Psi_\epsilon \equiv 1 \text{ on } \mathbb{T} \setminus S_\epsilon, \quad \Psi_\epsilon \equiv 0 \text{ on } S_{\epsilon/2}, \quad 0 \le \Psi_\epsilon \le 1 \text{ on } S_\epsilon \setminus S_{\epsilon/2}.$$

Fix $f \in C(\mathbb{T}^2)$. By our assumptions and by Proposition 3.8,

$$\begin{aligned} \int_{\mathbb{T}^2} f(\zeta)\Psi_\epsilon(\zeta_1)d\sigma_\alpha(\zeta) &= \lim_{n\to\infty} \int_{\mathbb{T}^2} f(\zeta)\Psi_\epsilon(\zeta_1)d\sigma_{\alpha_n}(\zeta) \\ &= \lim_{n\to\infty} \int_{\mathbb{T}} f_{\alpha_n}(\zeta)\ \Psi_\epsilon(\zeta)\ d\nu_{\alpha_n}(\zeta) = \int_{\mathbb{T}} f_\alpha(\zeta)\ \Psi_\epsilon(\zeta)\ d\nu_\alpha(\zeta), \end{aligned} \tag{24}$$

where the last equality follows because

$$\int_{\mathbb{T}} |f_{\alpha_n}(\zeta)W_{\alpha_n}(\zeta) - f_\alpha(\zeta)W_\alpha(\zeta)|\,\Psi_\epsilon(\zeta)dm(\zeta) \le \sup_{\zeta\in\mathbb{T}\setminus S_{\epsilon/2}} |(f_{\alpha_n}W_{\alpha_n} - f_\alpha W_\alpha)(\zeta)| \to 0,$$

as $n \to \infty$ because $f_x(\zeta)W(x, \zeta)$ is uniformly continuous on $A_\alpha \times (\mathbb{T} \setminus S_{\epsilon/2})$. Furthermore, observe that since $\Psi_\epsilon(\zeta_1) \equiv 0$ on each L_k, we have

$$\begin{aligned} \left| \int_{\mathbb{T}^2} f(\zeta)\chi_{E_\alpha}(\zeta)d\sigma_\alpha(\zeta) - \int_{\mathbb{T}^2} f(\zeta)\Psi_\epsilon(\zeta_1)d\sigma_\alpha(\zeta) \right| &\le \int_{\mathbb{T}^2} |f(\zeta)|(1 - \Psi_\epsilon(\zeta_1))\chi_{E_\alpha}(\zeta)d\sigma_\alpha(\zeta) \\ &\le \|f\|_{L^\infty(\mathbb{T}^2)}\sigma_\alpha((S_\epsilon \times \mathbb{T}) \cap E_\alpha). \end{aligned}$$

Here, $(S_\epsilon \times \mathbb{T}) \cap E_\alpha$ is the intersection of the curve E_α with thin strips in $\mathbb{T}^2$, see Figure 1. Letting $\epsilon \searrow 0$ and using the dominated convergence theorem gives

$$\lim_{\epsilon\searrow 0} \sigma_\alpha((S_\epsilon \times \mathbb{T}) \cap E_\alpha) = \sigma_\alpha\left(\cup_{k=1}^{\ell}(\tau_k, \overline{B_\alpha(\tau_k)})\right) = 0.$$

This in turn implies that

$$\int_{\mathbb{T}^2} f(\zeta)\chi_{E_\alpha}(\zeta)d\sigma_\alpha(\zeta) = \lim_{\epsilon\searrow 0} \int_{\mathbb{T}^2} f(\zeta)\Psi_\epsilon(\zeta_1)d\sigma_\alpha(\zeta).$$

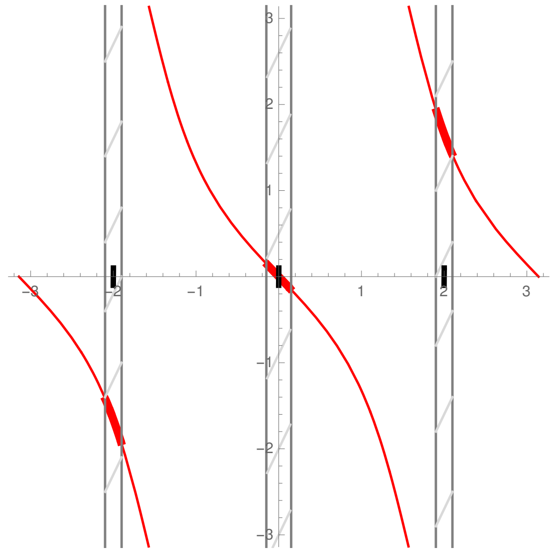


FIGURE 1. *This shows the set $(S_\epsilon \times \mathbb{T}) \cap E_\alpha$ graphed on $[-\pi,\pi]^2$. E_α is the red curve, τ_1, τ_2, τ_3 are the black tics on the horizontal axis, and $S_\epsilon \times \mathbb{T}$ is the union of thin gray strips.*

As f_α and W_α are both bounded, we can also conclude that

$$\lim_{\epsilon \searrow 0} \int_{\mathbb{T}} f_\alpha(\zeta)\Psi_\epsilon(\zeta)\ d\nu_\alpha(\zeta) = \int_{\mathbb{T}} f_\alpha(\zeta)\ d\nu_\alpha(\zeta).$$

Combining these last two equalities with (24) yields (22).

Part 2. Now we prove (23). We will show that for $z \in \mathbb{D}^2$, (23) holds for P_z. Since linear combinations of these are dense in $C(\mathbb{T}^2)$, the result will follow. To that end, fix $0 < r < 1$. Then the definition of σ_α gives

$$\int_{\mathbb{T}^2} P_{(r\tau_k, z_2)}(\zeta) d\sigma_\alpha(\zeta) = \Re\left(\frac{\alpha + \phi(r\tau_k, z_2)}{\alpha - \phi(r\tau_k, z_2)}\right). \tag{25}$$

We will multiply both sides by $(1-r)$ and let $r \nearrow 1$. First, observe that for $\zeta \in \mathbb{T}^2$,

$$\lim_{r\nearrow 1}(1-r)P_{(r\tau_k,z_2)}(\zeta) = \begin{cases} 0 & \text{if } \zeta_1 \neq \tau_k, \\ 2P_{z_2}(\zeta_2) & \text{if } \zeta_1 = \tau_k. \end{cases}$$

Then by the dominated convergence theorem,

$$\lim_{r\nearrow 1}\int_{\mathbb{T}^2}(1-r)P_{(r\tau_k,z_2)}(\zeta)d\sigma_\alpha(\zeta) = \int_{\mathbb{T}^2} 2P_{z_2}(\zeta_2)\chi_{L_k}(\zeta)d\sigma_\alpha(\zeta).$$

Observe that $L_k \subseteq \mathcal{C}_\alpha$ actually implies that $\phi(\tau_k, z_2) = \alpha$ for all $z_2 \in \mathbb{D}$. Furthermore, since ϕ is analytic at each (τ_k, z_2), we have

$$\lim_{z_1 \to \tau_k} \phi(z_1, z_2) = \alpha \text{ and } \lim_{z_1\to\tau_k} \frac{\phi(z_1,z_2)-\alpha}{z_1 - \tau_k} = \tfrac{\partial\phi}{\partial z_1}(\tau_k, z_2) := C \neq 0,$$

by Theorem 3.4 (vi). Then Carathéodory's theorem, see (VI-3) in [34], implies

$$\lim_{r\nearrow 1}\frac{1-|\phi(r\tau_k,z_2)|}{1-r} = C\tau_k\bar{\alpha} = |C|$$

and so

$$\lim_{r\nearrow 1}\Re\left(\frac{(1-r)(\alpha+\phi(r\tau_k,z_2))}{\alpha-\phi(r\tau_k,z_2)}\right)=\lim_{r\nearrow 1}(1-r)\frac{1-|\phi(r\tau_k,z_2)|^2}{|\alpha-\phi(r\tau_k,z_2)|^2}$$

$$=\lim_{r\nearrow 1}2\left|\frac{\tau_k-r\tau_k}{\alpha-\phi(r\tau_k,z_2)}\right|^2\frac{1-|\phi(r\tau_k,z_2)|}{1-r}=\frac{2}{|C|}.$$

Now set

$$c_k^\alpha=\frac{1}{|C|}=\frac{1}{|\frac{\partial\phi}{\partial z_1}(\tau_k,z_2)|}>0. \tag{26}$$

Then (25) and our subsequent computations combine to give

$$\int_{\mathbb{T}^2}P_{z_2}(\zeta_2)\chi_{L_k}(\zeta)d\sigma_\alpha(\zeta)=c_k^\alpha=c_k^\alpha\int_{\mathbb{T}}P_{z_2}(\zeta)dm(\zeta).$$

Multiplying both sides by $P_{z_1}(\tau_k)$ establishes (23) for $f=P_z$ and completes the proof. □

3.4. **Properties of J_α.** Recall that the isometry $J_\alpha\colon K_\phi\to L^2(\sigma_\alpha)$ is obtained by first defining the operator on reproducing kernels k_w as

$$J_\alpha[k_w](\zeta):=(1-\alpha\overline{\phi(w)})C_w(\zeta),\quad \text{for } w\in\mathbb{D}^2,\zeta\in\mathbb{T}^2,$$

and then extending it to the rest of K_ϕ. Theorem 1.3 details our main results about J_α, which are proved below in two propositions.

First, unlike the one-variable case, these isometries J_α need not be unitary. The exact situation in our setting is encoded in the following result:

Proposition 3.10. *The isometric embedding $J_\alpha : K_\phi\to L^2(\sigma_\alpha)$ is unitary if and only if α is a generic value for ϕ.*

Proof. ($\Rightarrow$) Assume that α is generic. By Theorem 3.2 in [12], we need only show that $A(\mathbb{D}^2)$ is dense in $L^2(\sigma_\alpha)$. Since σ_α is a finite Radon measure, $C(\mathbb{T}^2)$ is dense in $L^2(\sigma_\alpha)$ and by the Stone-Weierstrass theorem, the set of two-variable trigonometric polynomials is dense in $C(\mathbb{T}^2)$ and hence, in $L^2(\sigma_\alpha)$. Thus, to show J_α is unitary, we need only show that each two-variable trigonometric polynomial agrees with some function in $A(\mathbb{D}^2)$ on E_α, which contains the support of σ_α.

Let $h(\zeta)=\zeta_1^m\zeta_2^n$ be an arbitrary trigonometric monomial. To construct a function in $A(\mathbb{D}^2)$ that agrees with h on E_α, first define $t_1,t_2\in A(\mathbb{D}^2)$ by $t_1(z)=z_1$ and $t_2(z)=z_2$. Then, recall from Theorem 3.4 (iii) that $\deg B_\alpha=n$. Write $B_\alpha=\gamma\prod_{j=1}^n b_{a_j}$, where $\gamma\in\mathbb{T}$ and each $b_{a_j}=(z-a_j)/(1-\bar a_j z)$ is the Blaschke factor with zero $a_j\in\mathbb{D}$. Let Let $b_{\overline{a}_1}^{-1}$ denote the inverse function of $b_{\overline{a}_1}$ and define $s_1,s_2\in A(\mathbb{D}^2)$ by

$$s_1(z)=b_{\overline{a}_1}^{-1}\Big[\Big(\gamma\prod_{j=2}^n b_{a_j}(z_1)\Big)z_2\Big],$$

and $s_2(z)=B_\alpha(z_1)$. Then, restricting to E_α, we have

$$s_1(\zeta,\overline{B_\alpha(\zeta)})=b_{\overline{a}_1}^{-1}\Big[\Big(\gamma\prod_{j=2}^n b_{a_j}(\zeta)\Big)\overline{\Big(\gamma\prod_{j=1}^n b_{a_j}(\zeta)\Big)}\Big]=b_{\overline{a}_1}^{-1}\left[b_{\overline{a}_1}(\bar\zeta)\right]=\bar\zeta$$

and $s_2(\zeta, \overline{B_\alpha(\zeta)}) = B_\alpha(\zeta)$. As

$$h(\zeta, \overline{B_\alpha(\zeta)}) = \zeta^m \overline{B_\alpha(\zeta)}^n,$$

h agrees with one of $t_1^{|m|}t_2^{|n|}$, $t_1^{|m|}s_2^{|n|}$, $s_1^{|m|}t_2^{|n|}$, $s_1^{|m|}s_2^{|n|}$ on E_α. Taking linear combinations of these shows that every two-variable trigonometric polynomial agrees with some $F \in A(\mathbb{D}^2)$ on E_α and completes the proof of this forward direction.

($\Leftarrow$) Assume that α is exceptional and $\phi^*(\tau_k, \lambda_k) = \alpha$. By way of contradiction, assume that $A(\mathbb{D}^2)$ is dense in $L^2(\sigma_\alpha)$. Let $f(\zeta) = \bar{\zeta}_2$. By assumption, there is a sequence $(f_n) \subseteq A(\mathbb{D}^2)$ that converges to f in $L^2(\sigma_\alpha)$. Then by Theorem 1.2, there is a $c_k^\alpha > 0$ such that

$$\int_{\mathbb{T}} |f(\tau_k, \zeta) - f_n(\tau_k, \zeta)|^2 dm(\zeta) \leq \tfrac{1}{c_k^\alpha} \|f - f_n\|^2_{L^2(\sigma_\alpha)} \to 0,$$

as $n \to \infty$. Since each $f_n(\tau_k, \cdot)$ is in $H^2(\mathbb{D})$, so is the limit function $f(\tau_k, \cdot)$. Since $f(\tau_k, \zeta) = \bar{\zeta}$, it is clearly not in $H^2(\mathbb{D})$ and so, we obtain the needed contradiction. □

We can also identify a formula giving the isometric operator J_α. To do that, we require the following lemma describing the behavior of the non-tangential values of functions from the model space K_ϕ.

Lemma 3.11. *If $f \in K_\phi$, then f^* exists and agrees with a Borel measurable function σ_α-a.e. on $\mathbb{T}^2$ and*

$$\int_{\mathbb{T}^2} |f^*(\zeta)|^2 d\sigma_\alpha(\zeta) = \int_{\mathbb{T}} |f^*(\zeta, \overline{B_\alpha(\zeta)})|^2 d\nu_\alpha(\zeta) + \sum_{k=1}^m c_k^\alpha \int_{\mathbb{T}} |f^*(\tau_k, \zeta)|^2 dm(\zeta),$$

where $d\nu_\alpha = W_\alpha dm$, the functions B_α, W_α are from Definition 1.1, and the c_k^α are from Theorem 1.2.

Proof. By Theorem 3.2, there exist $g_1, \dots, g_n, h \in H^2(\mathbb{D})$ such that

$$f(z) = \tfrac{Q}{p}(z) h(z_1) + \sum_{j=1}^n \tfrac{R_j}{p}(z) g_j(z_2), \qquad \text{for } z \in \mathbb{D}^2.$$

Let $A \subseteq \mathbb{T}$ be a Borel set with $m(A) = 0$ such that $h, g_1, \dots, g_n$ have non-tangential limits at all $\zeta \in \mathbb{T} \setminus A$. To finish the set-up, assume $(z^n) = (z_1^n, z_2^n) \to (\tau_k, \zeta_2) \in \mathbb{T}^2$ non-tangentially, where $\zeta_2 \neq \lambda_k$. Then since $(z_1 - \tau_k)$ is a factor of Q and Q/p is continuous near (τ_k, ζ_2),

$$\lim_{n\to\infty} \left| \tfrac{Q}{p}(z^n) h(z_1^n) \right| \lesssim \lim_{n\to\infty} |z_1^n - \tau_k| \|h\|_{H^2} \tfrac{1}{\sqrt{1-|z_1^n|^2}} \lesssim \lim_{n\to\infty} \sqrt{1 - |z_1^n|} = 0, \tag{27}$$

where we also used the reproducing property of H^2 and the non-tangential property of (z^n). Similarly, if B_α equals some constant $\gamma \in \mathbb{T}$, then each $\lambda_k = \bar{\gamma}$ and in Proposition 3.5, each $b_j \equiv 0$ and $(z_2 - \overline{\gamma})$ divides R_j. Thus, arguments analogous to those in (27) imply that if $(z^n) = (z_1^n, z_2^n) \to (\zeta_1, \overline{\gamma}) \in \mathbb{T}^2$ non-tangentially with $\zeta_1 \in \mathbb{T} \setminus \{\tau_1, \dots, \tau_m\}$, then

$$\lim_{n\to\infty} \left| \tfrac{R_j}{p}(z^n) g_j(z_2^n) \right| = 0, \text{ for } j = 1, \dots, n.$$

This implies that $f^*(\zeta)$ exists for all $\zeta \in \mathbb{T}^2 \setminus \hat{A}$, where

$$\hat{A} := \{(\tau_k, \lambda_k) : k = 1, \dots, m\} \cup ((A \setminus \{\tau_1, \dots, \tau_m\}) \times \mathbb{T}) \cup (\mathbb{T} \times A \setminus \{\bar{\gamma}\}),$$

where we only include $\bar{\gamma}$ if B_α is constant. By definition, $\hat{A}$ is a Borel set and we claim $\sigma_\alpha(\hat{A}) = 0$. Since σ_α has no point masses, it is immediate that

$$\sigma_\alpha(\{(\tau_k, \lambda_k) : k = 1, \dots, m\}) = 0.$$

Set $A_1 = (A \setminus \{\tau_1, \dots, \tau_m\}) \times \mathbb{T}$. Then as $A_1 \cap L_k = \emptyset$ for each k and Lemma 3.6 shows W_α is bounded, we can use Theorem 1.2 to compute

$$\sigma_\alpha(A_1) = \int_{\mathbb{T}} \chi_{A_1}(\zeta, \overline{B_\alpha(\zeta)}) d\nu_\alpha(\zeta) \lesssim \int_{\mathbb{T}} \chi_A(\zeta) dm(\zeta) = 0.$$

If B_α is non-constant, set $A_2 = \mathbb{T} \times A$. Again by Theorem 1.2, there are constants c_k^α such that

$$\begin{aligned} \sigma_\alpha(A_2) &= \int_{\mathbb{T}} \chi_{A_2}(\zeta, \overline{B_\alpha(\zeta)}) d\nu_\alpha(\zeta) + \sum_{k=1}^m c_k^\alpha \int_{\mathbb{T}} \chi_{A_2}(\tau_k, \zeta) dm(\zeta) \\ &\lesssim m(\{\zeta \in \mathbb{T} : \overline{B_\alpha(\zeta)} \in A\}) + \sum_{k=1}^m c_k^\alpha m(A) = 0, \end{aligned}$$

where the first set has Lebesgue measure 0 because non-constant finite Blaschke products are smooth, have non-zero derivatives on $\mathbb{T}$, and are locally invertible on $\mathbb{T}$. Hence, the preimage $\bar{B}_\alpha^{-1}(A)$ must have measure 0 because A does. If $B_\alpha = \gamma$ is constant, set $A_2 = \mathbb{T} \times (A \setminus \{\bar{\gamma}\})$. Then

$$\sigma_\alpha(A_2) = \int_{\mathbb{T}} \chi_{A_2}(\zeta, \bar{\gamma}) d\nu_\alpha(\zeta) + \sum_{k=1}^m c_k^\alpha \int_{\mathbb{T}} \chi_{A_2}(\tau_k, \zeta) dm(\zeta) = 0$$

by the definition of A_2. Thus, f^* exists σ_α-a.e. on $\mathbb{T}^2$. Finally, observe that

$$F(\zeta) = \limsup_{r \nearrow 1} \Re(f(r\zeta)) + i \limsup_{r \nearrow 1} \Im(f(r\zeta))$$

is Borel measurable since each $f_r(\zeta) := f(r\zeta)$ is continuous on $\mathbb{T}^2$ and $F = f^*$ on $\mathbb{T}^2 \setminus \hat{A}$ and hence σ_α-a.e. Our prior arguments also imply $f^*(\zeta, \overline{B_\alpha(\zeta)}) = F(\zeta, \overline{B_\alpha(\zeta)})$ for ν_α-a.e. $\zeta \in \mathbb{T}$ and $f^*(\tau_k, \zeta) = F(\tau_k, \zeta)$ for m-a.e. $\zeta \in \mathbb{T}$. To finish the proof, for each $n \in \mathbb{N}$, define the Borel set

$$D_n = \{\zeta \in \mathbb{T}^2 : |F(\zeta)| < n\}.$$

Then Theorem 1.2 combined with the monotone convergence theorem gives

$$\begin{aligned} \int_{\mathbb{T}^2} |f^*(\zeta)|^2 d\sigma_\alpha(\zeta) &= \lim_{n\to\infty} \int_{\mathbb{T}^2} |F(\zeta)|^2 \chi_{D_n}(\zeta) d\sigma_\alpha(\zeta) \\ &= \lim_{n\to\infty} \left(\int_{\mathbb{T}} |(F\chi_{D_n}(\zeta, \overline{B_\alpha(\zeta)})|^2 \, d\nu_\alpha(\zeta) + \sum_{k=1}^m c_k^\alpha \int_{\mathbb{T}} |(F\chi_{D_n})(\tau_k, \zeta)|^2 dm(\zeta) \right) \\ &= \int_{\mathbb{T}} |f^*(\zeta, \overline{B_\alpha(\zeta)})|^2 \, d\nu_\alpha(\zeta) + \sum_{k=1}^m c_k^\alpha \int_{\mathbb{T}} |f^*(\tau_k, \zeta)|^2 dm(\zeta), \end{aligned}$$

which is what we needed to show. □

Using that lemma, we can now identify a formula for J_α.

Proposition 3.12. *For each each $f \in K_\phi$, the isometry $J_\alpha : K_\phi \to L^2(\sigma_\alpha)$ satisfies*

$$(J_\alpha f)(\zeta) = f^*(\zeta) \text{ for } \sigma_\alpha\text{-a.e. } \zeta \in \mathbb{T}^2.$$

Proof. Fix $f \in K_\phi$. By Lemma 3.11, f^* exists and equals a Borel-measurable function σ_α-a.e. on $\mathbb{T}^2$. We claim that $f^* \in L^2(\sigma_\alpha)$ and

$$\|f^*\|_{L^2(\sigma_\alpha)} \lesssim \|f\|_{K_\phi}, \tag{28}$$

where the implied constant does not depend on f. To see this, use Theorem 3.2 to write

$$f(z) = \tfrac{Q}{p}(z)h(z_1) + \sum_{j=1}^{n} \tfrac{R_j}{p}(z)g_j(z_2) \quad \text{for } z \in \mathbb{D}^2$$

and $g_1, \dots, g_n, h \in H^2(\mathbb{D})$. By the proof of Lemma 3.11, this formula extends to $\mathbb{T}^2$ via non-tangential limits both Lebesgue and σ_α-a.e. By Proposition 3.5, there is a $b_j \in A(\mathbb{D})$ such that

$$\left|(\tfrac{R_j}{p})(\zeta, \overline{B_\alpha(\zeta)})\right| = \left|(\tfrac{Q}{p})(\zeta, \overline{B_\alpha(\zeta)})\right| |b_j(\zeta)|,$$

for all $\zeta \in \mathbb{T}$. Working through the definitions and applying Proposition 3.5 give

$$|(\tfrac{Q}{p})(\zeta, \overline{B_\alpha(\zeta)})|^2 W_\alpha(\zeta) = \frac{|p_1(\zeta)|^2 - |p_2(\zeta)|^2}{(|p_1(\zeta)|^2 - |p_2(\zeta)|^2)^2} |(\tilde{p}_1 - \alpha p_2)(\zeta)|^2 \frac{|p_1(\zeta)|^2 - |p_2(\zeta)|^2}{|(\tilde{p}_1 - \alpha p_2)(\zeta)|^2} = 1$$

for all $\zeta \in \mathbb{T} \setminus \{\tau_1, \dots, \tau_m\}$. If B_α is non-constant, this immediately implies that

$$\begin{aligned}
&\int_{\mathbb{T}} |f^*(\zeta, \overline{B_\alpha(\zeta)})|^2 d\nu_\alpha(\zeta) \\
&\quad\lesssim \int_{\mathbb{T}} |(\tfrac{Q}{p})(\zeta, \overline{B_\alpha(\zeta)})|^2 \Big(|h^*(\zeta)|^2 + \sum_{j=1}^{n} |b_j(\zeta)|^2 |g_j^*(\overline{B_\alpha(\zeta)})|^2\Big) W_\alpha(\zeta) dm(\zeta) \\
&\quad= \int_{\mathbb{T}} |h^*(\zeta)|^2 + \sum_{j=1}^{n} |b_j(\zeta)|^2 |\overline{g}_j^*(B_\alpha(\zeta))|^2 dm(\zeta) \\
&\quad\lesssim \Big(\|h\|_{H^2}^2 + \sum_{j=1}^{n} \|\overline{g}_j \circ B_\alpha\|_{H^2}^2\Big) \lesssim \|f\|_{K_\phi}^2,
\end{aligned}$$

where $\bar{g}$ is the function in $H^2(\mathbb{D})$ whose Taylor coefficients are the complex conjugates of those of g. In this computation, we used Theorem 3.2 and the well-known fact that composition by a non-constant finite Blaschke product B_α induces a bounded operator on $H^2(\mathbb{D})$, see Theorem 5.1.5 in [24]. Here, the implied constant does not depend on f. If B_α is constant, then the one-variable model space $\hat{K}_{B_\alpha} = \{0\}$, so each $b_j \equiv 0$ and

$$\int_{\mathbb{T}} |f^*(\zeta, \overline{B_\alpha(\zeta)}))|^2 W_\alpha(\zeta) dm(\zeta) = \|h\|_{H^2}^2 \leq \|f\|_{K_\phi}^2.$$

Similarly, for each $1 \leq j \leq n$ and $1 \leq k \leq m$, Proposition 3.5 gives constants M_{jk} and d_{jk}^α such that for $\zeta \neq \lambda_k$,

$$\tfrac{Q}{p}(\tau_k, \zeta) \equiv 0 \text{ and } \tfrac{R_j}{p}(\tau_k, \zeta) = M_{jk} \frac{1 - B_\alpha(\tau_k)\zeta}{p(\tau_k, \zeta)} =: d_{jk}^\alpha,$$

since both the numerator and denominator are linear and by Theorem 3.4 (v), vanish at λ_k. This shows

$$c_k^\alpha \int_{\mathbb{T}} |f^*(\tau_k, \zeta)|^2 dm(\zeta) \lesssim \sum_{j=1}^{n} (d_{jk}^\alpha)^2 \int_{\mathbb{T}} |g_j(\zeta)|^2 dm(\zeta) \lesssim \|f\|_{K_\phi}^2.$$

By Lemma 3.11, this shows $f^* \in L^2(\sigma_\alpha)$. Furthermore, if we define a linear map $T_\alpha : K_\phi \to L^2(\sigma_\alpha)$ by $(T_\alpha f) = f^*$, then T_α is bounded. Moreover, observe that for $\zeta \in \mathcal{C}_\alpha \setminus \{(\tau_1, \lambda_1), \dots, (\tau_m, \lambda_m)\}$, we have

$$T_\alpha[k_w](\zeta) = (1 - \alpha\overline{\phi(w)})C_w(\zeta) = J_\alpha[k_w](\zeta).$$

Thus, these functions are equal in $L^2(\sigma_\alpha)$. Since T_α and J_α agree on a dense set of functions in K_ϕ, it follows that $T_\alpha = J_\alpha$, which completes the proof. □

4. Applications

The results from Section 3 have implications for the structure of Agler decompositions and connections to the study of extreme measures from [22, 25, 28] and the references therein. In this section, we again assume $\phi = \frac{\tilde{p}}{p}$ is a degree $(n, 1)$ rational inner function and throughout, will use the notation denoted earlier in Definition 1.1 and Remark 3.1.

4.1. Agler Decompositions.

Recall that each such ϕ possesses an Agler decomposition from Theorem 3.2 arising from a particular orthonormal list in K_ϕ. Moreover, the polynomial Q in that decomposition can be computed directly on $\mathbb{T}$ via Proposition 3.5. In the case of exceptional α, we can apply Theorem 1.2 to specify some of the remaining polynomials $R_1, \dots, R_n$ from (8).

Theorem 4.1. *Let $\alpha \in \mathbb{T}$ be exceptional for ϕ and (after reordering if necessary) assume $\phi^*(\tau_k, \lambda_k) = \alpha$ for $k = 1, \dots, \ell$. Using Theorem 3.4 (iv), write $B_\alpha = b^1_\alpha / b^2_\alpha$, where $\deg b^1_\alpha = n - \ell$. Then in (8), we can take*

$$R_j(z) = d^\alpha_j \Big(b^2_\alpha(z_1) - z_2 b^1_\alpha(z_1) \Big) \prod_{\substack{1 \le k \le \ell \\ k \ne j}} (z_1 - \tau_k), \tag{29}$$

for $j = 1, \dots, \ell$, where each $d^\alpha_j > 0$ is chosen so $c^\alpha_j \| R_j / p(\tau_j, \cdot) \|^2_{H^2(\mathbb{D})} = 1$ and c^α_j is from Proposition 3.9.

Proof. To begin, we need to recall several facts discussed in the proof of Theorem 3.2. Namely, there exist reproducing kernels $K_1, K_2 : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ defined in (10) such that

$$1 - \phi(z)\overline{\phi(w)} = (1 - z_1 \bar{w}_1) K_2(z, w) + (1 - z_2 \bar{w}_2) K_1(z, w).$$

Moreover, the associated Hilbert spaces $\mathcal{H}(K_1)$ and $\mathcal{H}(K_2)$ are subspaces of $K_\phi := H^2(\mathbb{D})^2 \ominus \phi H^2(\mathbb{D}^2)$. Then the polynomials $R_1, \dots, R_n$ from (8) are exactly obtained by specifying that the list $R_1/p, \dots, R_n/p$ be an orthonormal basis for $\mathcal{H}(K_2)$. Then to establish our desired result, we need only show that for the R_j defined in (29), the functions $R_1/p, \dots, R_\ell/p$ are in $\mathcal{H}(K_2)$ and they form an orthonormal set there (or equivalently are orthonormal as elements in K_ϕ).

To that end, as in Theorem 3.2, let $\hat{R}_1/p, \dots, \hat{R}_n/p$ be some orthonormal basis for $\mathcal{H}(K_2)$. Recall that $\hat{K}_{B_\alpha} := H^2(\mathbb{D}) \ominus B_\alpha H^2(\mathbb{D})$ denotes the one variable model space associated to B_α. Then Proposition 3.5 implies that for each j, there is a unique polynomial $\hat{r}_j$ with $\deg \hat{r}_j \le n - 1$ and function $b_j \in \hat{K}_{B_\alpha}$ such that

$$\hat{R}_j(z) = \hat{r}_j(z_1) \Big(1 - B_\alpha(z_1) z_2 \Big) + z_2 Q(z_1) b_j(z_1).$$

Define a linear map $T\colon \mathrm{Span}\{\hat{R}_1,\dots,\hat{R}_n\} \to \hat{K}_{B_\alpha}$ by first specifying $T(\hat{R}_j) = b_j$ and then extending by linearity. As $\dim \hat{K}_{B_\alpha} = n-\ell$, it follows that $\dim(\ker T) \geq \ell$. If $R \in \ker(T)$, then for some r with $\deg r < n$,

$$R(z) = r(z_1)(1-B_\alpha(z_1)z_2) = \frac{r(z_1)}{b^2_\alpha(z_1)}\Big(b^2_\alpha(z_1) - z_2 b^1_\alpha(z_1)\Big) = q(z_1)\Big(b^2_\alpha(z_1) - z_2 b^1_\alpha(z_1)\Big), \tag{30}$$

where $q \in \mathbb{C}[z]$ with $\deg q < \ell$. Note that the set of such R has dimension ℓ. By comparing dimensions, each R given in (30) must be in $\ker(T)$ and hence, each R given in (30) satifies $R/p \in \mathcal{H}(K_2)$. In particular, this implies that each R_j from (29) satisfies $R_j/p \in \mathcal{H}(K_2)$.

To show $R_1/p,\dots,R_\ell/p$ are orthonormal in K_ϕ, we use Proposition 3.9 and Theorem 3.12. First, observe that those two results combine to imply that $R_j/p(\tau_j,\cdot) \in H^2(\mathbb{D})\setminus\{0\}$, so d^α_j is well defined. Then, one can use the fact that each R_j vanishes on E_α and each L_k with $1 \leq k \leq \ell$ and $k \neq j$ to conclude:

$$\begin{aligned}
\left\langle \tfrac{R_i}{p}, \tfrac{R_j}{p}\right\rangle_{K_\phi} &= \left\langle J_\alpha\left(\tfrac{R_i}{p}\right), J_\alpha\left(\tfrac{R_j}{p}\right)\right\rangle_{L^2(\sigma_\alpha)}\\
&= \int_{\mathbb{T}} \tfrac{R_i}{p}(\zeta,\overline{B_\alpha(\zeta)})\overline{\tfrac{R_j}{p}(\zeta,\overline{B_\alpha(\zeta)})}d\nu_\alpha(\zeta) + \sum_{k=1}^{\ell} c^\alpha_k \int_{\mathbb{T}} \tfrac{R_i}{p}(\tau_k,\zeta)\overline{\tfrac{R_j}{p}(\tau_k,\zeta)}\, dm(\zeta)\\
&= 0 + \sum_{k=i \text{ or } k=j} c^\alpha_k \int_{\mathbb{T}} \tfrac{R_i}{p}(\tau_k,\zeta)\overline{\tfrac{R_j}{p}(\tau_k,\zeta)}\, dm(\zeta)\\
&= \begin{cases} 1 & \text{if } i = j\\ 0 & \text{if } i \neq j.\end{cases}
\end{aligned}$$

Thus, $\{R_1/p,\dots,R_\ell/p\}$ is an orthonormal set in K_ϕ and hence in $\mathcal{H}(K_2)$, which completes the proof. □

4.2. Extreme Points.

Recall that $\mathcal{P}_2 = \{f \in \mathrm{Hol}(\mathbb{D}^2) : \Re f(z) > 0 \text{ and } f(0,0) = 1\}$ and $M : \mathcal{P}_2 \to P(\mathbb{T}^2)$ is the map that takes $f \in \mathcal{P}_2$ to the unique Borel probability measure μ_f on $\mathbb{T}^2$ with

$$f(z) = \int_{\mathbb{T}^2} P_z(\zeta)d\mu_f(\zeta) \quad \text{for } z \in \mathbb{D}^2.$$

for some $f \in \mathcal{P}_2$ and f is an extreme point of $\mathcal{P}_2$ if and only if μ_f is an extreme point of $M(\mathcal{P}_2)$. As mentioned in the introduction, Forelli, McDonald, and Knese have proved a number of interesting results related to such extreme points. For example, Knese proved the following result in [22, Theorem 1.5]:

Theorem 4.2. *Let q be a polynomial with no zeros on $\mathbb{D}^2$ and let $\tilde{q}$ be the reflection of q with $\deg \tilde{q} = \deg q$. Assume that q is $\mathbb{T}^2$-saturated, $\tilde{q}, q$ share no common factors, $\tilde{q}(0,0) = 0$, and $q - \tilde{q}$ is irreducible. Then $f := \frac{q+\tilde{q}}{q-\tilde{q}}$ is an extreme point of $\mathcal{P}_2$.*

As mentioned in the introduction, our results in the $(n,1)$ setting coupled with Theorem 4.2 yield Corollary 1.4, which we restate here for convenience.

Corollary. ***1.4.*** *Assume $\tilde{p}(0,0) = 0$ and let $\alpha \in \mathbb{T}$. Then*

i. *If α is an exceptional value for ϕ, then σ_α is not an extreme point of $M(\mathcal{P}_2)$.*
ii. *If $\deg p = \deg \tilde{p}$, p is $\mathbb{T}^2$-saturated, and α is generic for ϕ, then σ_α is an extreme point of $M(\mathcal{P}_2)$.*

Proof. For (i), without loss of generality, assume $\phi^*(\tau_k, \lambda_k) = \alpha$ for $k = 1, \dots, \ell$. By Proposition 3.9, we can write

$$\sigma_\alpha(\zeta) = \mu_\alpha(\zeta) + c_1^\alpha \left(\delta_{\tau_1}(\zeta_1) \otimes m(\zeta_2)\right),$$

for a positive Borel measure μ_α on $\mathbb{T}^2$ and $c_1^\alpha > 0$. As $\phi(0,0) = 0$, we have

$$1 = \sigma_\alpha(\mathbb{T}^2) = \mu_\alpha(\mathbb{T}^2) + c_1^\alpha,$$

and as $\mu_\alpha(\mathbb{T}^2) > 0$, we have $c_1^\alpha < 1$. Then $\hat{\mu}_\alpha := \frac{1}{1-c_1^\alpha}\mu_\alpha$ is a probability measure and

$$\sigma_\alpha(\zeta) = (1 - c_1^\alpha)\hat{\mu}_\alpha(\zeta) + c_1^\alpha \left(\delta_{\tau_1}(\zeta_1) \otimes m(\zeta_2)\right), \tag{31}$$

so σ_α is a convex combination of two probability measures on $\mathbb{T}^2$. Clearly, the second one is in $M(\mathcal{P}_2)$, as

$$\Re\left(\frac{\tau_1 + z_1}{\tau_1 - z_1}\right) = \frac{1 - |z_1|^2}{|z_1 - \tau_1|^2} = \int_{\mathbb{T}^2} P_z(\zeta)\, d\left(\delta_{\tau_1}(\zeta_1) \otimes m(\zeta_2)\right).$$

For the first, observe that for each $z \in \mathbb{D}^2$,

$$\tfrac{1}{1-c_1^\alpha}\Re\left(\frac{\alpha + \phi(z)}{\alpha - \phi(z)} - c_1^\alpha \frac{\tau_1 + z_1}{\tau_1 - z_1}\right) = \int_{\mathbb{T}^2} P_z(\zeta) d\hat{\mu}_\alpha(\zeta) > 0.$$

This implies that $\hat{\mu}_\alpha \in M(\mathcal{P}_2)$ and by (31), σ_α is not an extreme point in $M(\mathcal{P}_2)$.

For (ii), choose $\lambda \in \mathbb{T}$ with $\lambda^2 = \alpha$, define $q = \lambda p$, and set

$$f := \frac{\alpha + \phi}{\alpha - \phi} = \frac{\alpha p + \tilde{p}}{\alpha p - \tilde{p}} = \frac{q + \tilde{q}}{q - \tilde{q}}.$$

Then $\tilde{q} - q = \bar{\lambda}(\tilde{p} - \alpha p)$ must be irreducible. To see this, assume that $\tilde{q} - q$ is not irreducible. Then (11) implies that $(\tilde{p} - \alpha p)(z_1, z_2) = q(z_1) r(z_1, z_2)$, for some nonconstant polynomials q, r and furthermore q must divide both the numerator and denominator of B_α given in (5), before common factors are cancelled. As $\alpha p_1 - \tilde{p}_2$ is nonvanishing on $\mathbb{D}$, the structure of B_α implies that each zero γ of q must satisfy $\gamma \in \mathbb{T}$. Since q is nonconstant, it has at least one such zero γ and then $\{\gamma\} \times \mathbb{T} \subseteq \mathcal{C}_\alpha$. By Lemma 3.3, this implies α is exceptional and gives the needed contradiction.

By Theorem 4.2, f is an extreme point of $\mathcal{P}_2$ and so σ_α from Theorem 1.2 is extreme in $M(\mathcal{P}_2)$. □

5. Examples

We illustrate our results by examining some specific RIFs and their associated Clark measures in detail. For the first example, we can confirm our general findings at exceptional values α via direct computation.

Example 5.1. Let

$$\phi(z) = \frac{\tilde{p}(z)}{p(z)} = \frac{2z_1 z_2 - z_1 - z_2}{2 - z_1 - z_2},$$

essentially the example considered in [12]. We have the sums of squares decomposition

$$|p(z)|^2 - |\tilde{p}(z)|^2 = (1 - |z_2|^2)2|1 - z_1|^2 + (1 - |z_1|^2)2|1 - z_2|^2$$

and for each $\alpha \in \mathbb{T}$, the associated B_α is

$$B_\alpha(z_1) = \frac{2z_1 - 1 + \alpha}{2\alpha - \alpha z_1 + z_1}.$$

Note that if $\alpha = -1$, then $B_{-1} \equiv 1$. If $\alpha \neq -1$, then $2\alpha - \alpha z_1 + z_1$ does not vanish on $\mathbb{T}$. Thus if $\alpha \neq -1$, then by Proposition 3.8, for all $f \in L^2(\sigma_\alpha)$, we have

$$\int_{\mathbb{T}^2} f(\zeta) d\sigma_\alpha(\zeta) = \int_{\mathbb{T}} f(\zeta, \overline{B_\alpha(\zeta)}) \frac{2|1-\zeta|^2}{|2\zeta - 1 + \alpha|^2}\, dm(\zeta),$$

and by Theorem 3.10, the isometric embedding $J_\alpha : K_\phi \to L^2(\sigma_\alpha)$ is unitary. Finally, if $\alpha = -1$, then $|2z_1 - 1 + \alpha|^2 = 4|z_1 - 1|^2$. By the given sums of squares decomposition,

$$\begin{aligned}\frac{1-|\phi(z)|^2}{|\alpha - \phi(z)|^2} &= \frac{|p(z)|^2 - |\tilde{p}(z)|^2}{|\alpha p(z) - \tilde{p}(z)|^2} \\ &= \frac{|p(z)|^2 - |\tilde{p}(z)|^2}{|z_2 - 1|^2| - p_2(z_1) - \tilde{p}_1(z_1)|^2} \\ &= \frac{(1-|z_2|^2)2|1-z_1|^2 + (1-|z_1|^2)2|1-z_2|^2}{|z_2-1|^2 \cdot 4|z_1-1|^2} \\ &= \frac{1}{2}\left(\frac{1-|z_2|^2}{|z_2-1|^2} + \frac{1-|z_1|^2}{|z_1-1|^2}\right),\end{aligned}$$

which shows $\sigma_\alpha = \frac{1}{2}(\delta_1(\zeta_1) \otimes m(\zeta_2) + m(\zeta_1) \otimes \delta_1(\zeta_2))$. This was observed in [12], and confirms the contents of Theorem 1.2. Note in particular that $\frac{\partial \phi}{\partial z_1}(z_1, z_2) = -2\frac{(z_2-1)^2}{(2-z_1-z_2)^2}$, so that $\frac{\partial \phi}{\partial z_1}(1, z_2) = -2$ independent of z_2.

See Figure 2(a) for a visual representation of the sets $\mathcal{C}_\alpha$. ♦

Now let us consider a RIF that was not studied in [12], and again illustrate how the exceptional measure σ_α can be identified using both our results and concrete Agler decompositions.

Example 5.2. Let $\phi = \frac{\tilde{p}}{p}$, where

$$p(z) = 4 - z_2 - 3z_1 - z_1 z_2 + z_1^2 \text{ and } \tilde{p}(z) = 4z_1^2 z_2 - z_1^2 - 3z_1 z_2 - z_1 + z_2.$$

This example was introduced by Agler-McCarthy-Young in [2]. In [21, Section 15], Knese provides the following sums of squares decomposition:

$$|p(z)|^2 - |\tilde{p}(z)|^2 = 4(1-|z_2|^2)|1-z_1|^4 + 4(1-|z_1|^2)\left(|1-z_1|^2|1-z_2|^2 + 2|1-z_1 z_2|^2\right).$$

The only singularity of ϕ occurs at $(1,1)$. For each $\alpha \in \mathbb{T}$, setting $\phi(z) = \alpha$ and solving for z_2 yields $z_2 = 1/B_\alpha(z_1)$, where

$$B_\alpha(z_1) = \frac{4z_1^2 - 3z_1 + 1 + \alpha + \alpha z_1}{4\alpha - 3z_1\alpha + z_1^2\alpha + z_1^2 + z_1}.$$

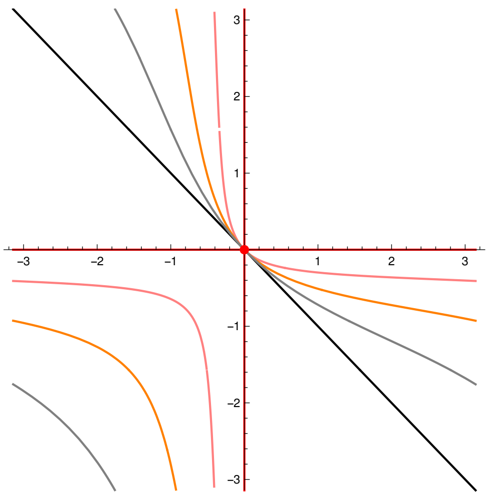
As ϕ has only one singularity, by previous discussions, the denominator of B_α can vanish at a point on $\mathbb{T}$ for at most one α. This occurs at $\alpha = -1$, where B_{-1} reduces to $B_{-1}(z) = z$ and $\phi = -1$ has the additional solution $z_1 = 1$. Thus, we can apply Proposition 3.8 to $\alpha \neq -1$ to obtain: for all $f \in L^1(\sigma_\alpha)$,

$$\int_{\mathbb{T}^2} f(\zeta) d\sigma_\alpha(\zeta) = \int_{\mathbb{T}} f(\zeta, \overline{B_\alpha(\zeta)})\, \frac{4|\zeta - 1|^4}{|4\zeta^2 - 3\zeta + 1 + \alpha + \alpha\zeta|^2}\, dm(\zeta).$$

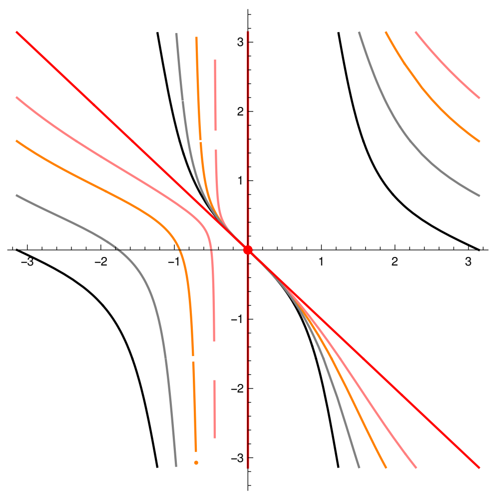
By Theorem 3.10, the isometric embedding $J_\alpha : K_\phi \to L^2(\sigma_\alpha)$ is unitary for every $\alpha \neq -1$.

Let us now examine the exceptional value $\alpha = -1 = \phi^*(1,1)$. A computation shows

$$p(z) + \tilde{p}(z) = 4(1-z_1)(1-z_1 z_2),$$



(a) Level curves for $\phi = (2z_1z_2 - z_1 - z_2)/(2 - z_1 - z_2)$ corresponding to $\alpha = 1$ (black), $\alpha = e^{i\pi/4}$ (gray), $\alpha = e^{i\pi/2}$ (orange), and $\alpha = e^{3i\pi/4}$ (pink). Level set corresponding to exceptional value $\alpha = -1$ marked in red.



(b) Level curves for $\phi = (4z_1^2z_2 - z_1^2 - 3z_1z_2 - z_1 + z_2)/(4 - z_2 - 3z_1 - z_1z_2 + z_1^2)$ corresponding to $\alpha = 1$ (black), $\alpha = e^{i\pi/4}$ (gray), $\alpha = e^{i\pi/2}$ (orange), and $\alpha = e^{3i\pi/4}$ (pink). Level set corresponding to exceptional value $\alpha = -1$ marked in red.

FIGURE 2. *Supports of σ_α for two different RIFs.*

and hence, for $\alpha = -1$, we have

$$\frac{1 - |\phi(z)|^2}{|\alpha - \phi(z)|^2} = \frac{|p(z)|^2 - |\tilde{p}(z)|^2}{16|1 - z_1|^2|1 - z_1z_2|^2}.$$

By the sums of squares formula above, we then obtain

$$\begin{aligned}\frac{1 - |\phi(z)|^2}{|\alpha - \phi(z)|^2} &= \frac{1}{4}|1 - z_1|^2\frac{1 - |z_2|^2}{|1 - z_1z_2|^2} + \frac{1}{4}|1 - z_2|^2\frac{1 - |z_1|^2}{|1 - z_1z_2|^2} + \frac{1}{2}\frac{1 - |z_1|^2}{|1 - z_1|^2} \\ &= \frac{1}{4}\frac{(1 - |z_1|^2)|1 - z_2|^2 + (1 - |z_2|^2)|1 - z_1|^2}{|1 - z_1z_2|^2} + \frac{1}{2}\frac{1 - |z_1|^2}{|1 - z_1|^2}.\end{aligned}$$

The second term is evidently the Poisson integral of the measure $\sigma_{-1}^{(2)} = \frac{1}{2}(\delta_1(\zeta_1) \otimes m_1(\zeta_2))$, which matches what we get from computing $\frac{\partial \phi}{\partial z_1}(1, z_2) = -2$ and taking the reciprocal of its absolute value.

The first term arises from the measure $\sigma_{-1}^{(1)}$ on $\mathbb{T}^2$ having

$$\int_{\mathbb{T}^2} f(\zeta) d\sigma_{-1}^{(1)}(\zeta) = \tfrac{1}{4}\int_{\mathbb{T}} f(\zeta, \overline{\zeta})|1 - \zeta|^2 dm(\zeta),$$

as can be seen by examining the Fourier coefficients

$$\widehat{\sigma_{-1}^{(1)}}(k, l) = \begin{cases} \frac{1}{2}, & k = l \\ -\frac{1}{4}, & k = l + 1 \\ -\frac{1}{4}, & l = k + 1 \\ 0 & \text{otherwise} \end{cases}$$

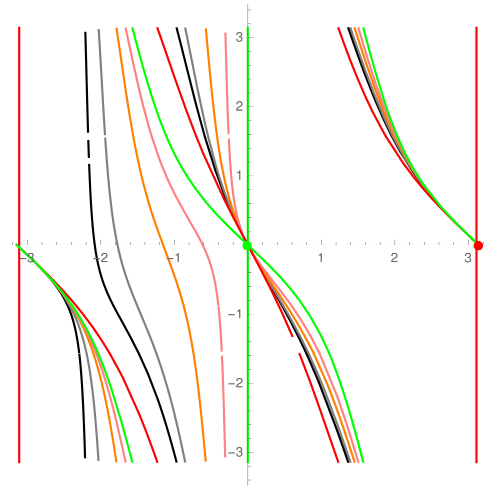


FIGURE 3. *Generic level curves for* $(4z_1^3z_2 - z_1^3 + z_1^2 - 3z_1 - 1)/(4 - z_2 + z_1z_2 - 3z_1^2z_2 - z_1^3z_2)$ *corresponding to several values of* α *(black, gray, orange, pink). Level sets corresponding to exceptional values* $\alpha = -1$ *and* $\alpha = 1$ *marked in green and red, respectively.*

and computing the Poisson integral of $\sigma_{-1}^{(1)}$ explicitly. The specific form of $\sigma_{-1}^{(1)}$ of course agrees with Theorem 1.2 once we set $\alpha = -1$ to get $W_{-1}(\zeta) = \frac{4|\zeta-1|^2}{|4\zeta^2-4\zeta|^2} = \frac{1}{4}|\zeta - 1|^2$.

Level curves $\mathcal{C}_\alpha$ for several values of α are displayed in Figure 2(b). ♦

Remark 5.3. The RIF $\phi = \frac{\tilde{p}}{p}$ with

$$p(z) = 2 - z_1z_2 - z_1^2z_2 \quad \text{and} \quad \tilde{p}(z) = 2z_1^2z_2 - z_1 - 1$$

has a singularity at $(1,1)$, and $\phi^*(1,1) = -1$ so that $\alpha = -1$ is an exceptional value.

One verifies that the associated $B_{-1}(z_1) = z_1$ so that σ_{-1} for this example is supported on the same set as the exceptional Clark measure in Example 5.2. However, we have

$$W_\alpha(\zeta) = \frac{|\zeta - 1|^2}{|(2+\alpha)\zeta + \alpha|^2},$$

which collapses to $W_{-1}(\zeta) = 1$ at the exceptional value, meaning that the two Clark measures do not coincide.

Our next example is a degree $(3,1)$ RIF with two different singularities on $\mathbb{T}^2$. Here, we are able to observe qualitative differences in W_α for the two corresponding exceptional values of α that reflect the finer distinctions between the two singularities.

Example 5.4. Let

$$p(z) = 4 - z_2 + z_1z_2 - 3z_1^2z_2 - z_1^3z_2 \quad \text{and} \quad \tilde{p}(z) = 4z_1^3z_2 - z_1^3 + z_1^2 - 3z_1 - 1$$

and set $\phi = \frac{\tilde{p}}{p}$. This function has singularities at $(1,1)$ and $(-1,1)$, and the associated exceptional α-values are $\phi^*(1,1) = -1$ and $\phi^*(-1,1) = 1$. Level sets for this example are displayed in Figure 3; see also [9, Example 7.4].

For $\alpha \neq 1, -1$, we have

$$B_\alpha(z_1) = \frac{\alpha - \alpha z_1 + 3\alpha z_1^2 + 4z_1^3 + \alpha z_1^3}{1 + 4\alpha + 3z_1 - z_1^2 + z_1^3}.$$

Note that for $\alpha = -1$, we get

$$B_{-1}(z_1) = \frac{z_1 - 1}{z_1 - 1}\frac{3z_1^2 + 1}{3 + z_1^2} = \frac{3z_1^2 + 1}{3 + z_1^2},$$

a Blaschke product of degree 2, while for $\alpha = 1$,

$$B_1(z_1) = \frac{z_1 + 1}{z_1 + 1}\frac{5z_1^2 - 2z_1 + 1}{z_1^2 - 2z_1 + 5} = \frac{5z_1^2 - 2z_1 + 1}{z_1^2 - 2z_1 + 5},$$

another degree 2 Blaschke product. The graphs $\{(\zeta, \overline{B_{-1}(\zeta)})$ and $\{(\zeta, \overline{B_1(\zeta)})\}$ together with vertical lines at $\zeta_1 = 1$ and $\zeta_1 = -1$ constitute $\operatorname{supp}(\sigma_{-1})$ and $\operatorname{supp}(\sigma_1)$, respectively.

We further read off that

$$p_1(z_1) = 4 \quad \text{and} \quad p_2(z_1) = -1 + z_1 - 3z_1^2 - z_1^3$$

so that, with W_α as in Remark 3.1,

$$W_\alpha(\zeta) = \frac{16 - |1 - \zeta + 3\zeta^2 + \zeta^3|^2}{|4\zeta^3 + \alpha\zeta^3 + 3\alpha\zeta^2 - \alpha\zeta + \alpha|^2}.$$

After some simplifications, we find that

$$W_\alpha(z) = \frac{|\zeta - 1|^2|\zeta + 1|^4}{|4\zeta^3 + \alpha\zeta^3 + 3\alpha\zeta^2 - \alpha\zeta + \alpha|^2}.$$

For the exceptional values $\alpha = \pm 1$, the weights in the point mass parts of $\sigma_{\pm 1}$ can be obtained by computing

$$\frac{\partial \phi}{\partial z_1}(1, z_2) = -1 \quad \text{and} \quad \frac{\partial \phi}{\partial z_1}(-1, z_2) = -2,$$

which imply

$$c_1^{-1} = \frac{1}{|\frac{\partial \phi}{\partial z_1}(1, z_2)|} = 1 \quad \text{and} \quad c_1^1 = \frac{1}{|\frac{\partial \phi}{\partial z_1}(-1, z_2)|} = \frac{1}{2}.$$

(Note that $\phi(0,0) = -\frac{1}{4}$ so that the Clark measures $\sigma_{\pm 1}$ are not probability measures in this example.) Putting $\alpha = \pm 1$ in W_α, we have cancellation in numerator and denominator, and we obtain

$$W_{-1}(\zeta) = \frac{|\zeta + 1|^4}{|3\zeta^2 + 1|^2}$$

and

$$W_1(\zeta) = \frac{|\zeta - 1|^2|\zeta + 1|^2}{|5\zeta^2 - 2\zeta^2 + 1|^2}.$$

This gives us a complete description of the exceptional Clark measures.

Furthermore, observe that, $W_{-1}(1) \neq 0$ and so, W_{-1} does not vanish at the z_1-coordinate of the singularity with non-tangential value -1. In contrast, $W_1(-1) = 0$, so W_1 does vanish at the z_1-coordinate of the singularity with non-tangential value 1. This mirrors the singular behavior in Example 5.2, where function W_{-1} vanishes at $\zeta = 1$, the z_1-coordinate of the singularity where $\phi^*(1,1) = -1$. This pattern suggests a connection with contact order, which was studied in [8] and governs the integrability of partial derivatives of a RIF ϕ; in that sense, higher contact order indicates a stronger singularity. In our computations, the singularities at $(1,1)$ in Example 5.2 and at $(-1,1)$ in this example (where the exceptional W_α vanish at the z_1-coordinate of the associated singularity) are instances of singularities where ϕ exhibits contact order 4; the singularities in Example 5.1

and at $(1,1)$ in this example (where the exceptional W_α do not vanish at the z_1-coordinate of the associated singularity) are singularities where ϕ exhibits contact order 2, the lowest possible contact order. ♦

Remark 5.5. It would interesting to investigate how the exact nature of a singularity $\tau \in \mathbb{T}^2$ (contact order, number of branches of p coming together at τ, etc) of a RIF is reflected in the associated exceptional Clark measure. For example, if $\phi = \frac{\tilde{p}}{p}$ is a general degree (m,n) RIF having contact order at least 4, does the corresponding exceptional Clark measure have a density along $\mathcal{C}_\alpha$ that vanishes at τ?

Our final example is a rational inner function having bidegree $(3,3)$, and is not covered by our general results. It serves as a counterexample showing that Proposition 3.10 fails for higher-degree RIFs, and illustrates some complexities that arise from the fact that for RIFs of bidegree (m,n) with $m,n \geq 2$, a general α-level set is not necessarily parametrized by a single function.

Example 5.6. Let $\phi(z) = \frac{\tilde{p}}{p}(z)$ where

$$p(z) = 2 - z_1^2 z_2 - z_1 z_2^2 \quad \text{and} \quad \tilde{p}(z) = z_1 z_2(2z_1^2 z_2^2 - z_1 - z_2).$$

This example is obtained by applying the level line embedding construction described in [9, Section 6.1] to the essentially $\mathbb{T}^2$-symmetric polynomial

$$r(z) = (1 - z_1^2 z_2)(1 - z_1 z_2^2).$$

As is guaranteed by the embedding construction, we have $p(1,1) = 0 = \tilde{p}(1,1)$ and $\phi^*(1,1) = -1$, as well as

$$\tilde{p} + p = 2(1 - z_1^2 z_2)(1 - z_1 z_2^2).$$

These facts can also be checked directly. We also note that p and $\tilde{p}$, and hence ϕ, are invariant under the simultaneous coordinatewise rotations $z_j \mapsto e^{2i\pi/3} z_j$ and $z_j \mapsto e^{-2i\pi/3} z_j$. Some level sets of ϕ are displayed in Figure 4.

Recall from Example 5.1 that $|2 - x - y|^2 - |2xy - x - y|^2 = (1 - |x|^2)2|1 - y|^2 + (1 - |y|^2)2|1 - x|^2$. Substituting $x = z_1^2 z_2$ and $y = z_1 z_2^2$ into this formula, we get the decomposition

$$|2 - z_1^2 z_2 - z_1 z_2^2|^2 - |2z_1^3 z_2^3 - z_1^2 z_2 - z_1 z_2^2|^2 = (1 - |z_1^2 z_2|^2)2|1 - z_1 z_2^2|^2 + (1 - |z_1 z_2^2|^2)2|1 - z_1^2 z_2|.$$

It follows that

$$\frac{1 - |\phi(z)|^2}{|1 + \phi(z)|^2} = \frac{1}{2}\frac{1 - |z_1^2 z_2|^2}{|1 - z_1^2 z_2|^2} + \frac{1}{2}\frac{1 - |z_1 z_2^2|^2}{|1 - z_1 z_2^2|^2},$$

and by computing Fourier coefficients, one can show that the two expressions on the right are the Poisson integrals of the measures having

$$\int_{\mathbb{T}^2} f(\zeta) d\sigma_{-1}^{(1)} = \int_{\mathbb{T}} f(\zeta, \bar{\zeta}^2) dm(\zeta) \quad \text{and} \quad \int_{\mathbb{T}^2} f(\zeta) d\sigma_{-1}^{(2)} = \int_{\mathbb{T}} f(\bar{\zeta}^2, \zeta) dm(\zeta)$$

respectively. By Doubtsov's Theorem 3.2 in [12], which applies to general RIFs, J_{-1} is unitary if and only if the bidisk algebra is dense in $L^2(\sigma_{-1})$.

Let us show that this is indeed the case. By definition, $h_1(z) = z_1$ and $h_2(z) = z_2$ are elements of $A(\mathbb{D}^2)$. Next consider the function $g_1(z) = \bar{z}_1$ and the function

$$f_1(z) = z_1 z_2 + (1 - z_1^2 z_2) z_2^2 \in A(\mathbb{D}^2).$$

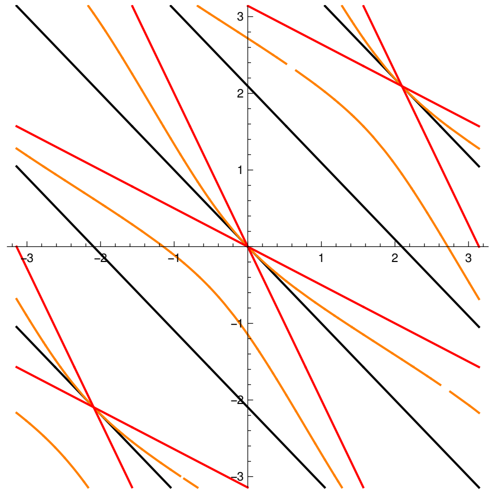


FIGURE 4. *Level curves for* $(2z_1^3z_2^3 - z_1^2z_2 - z_1z_2^2)/(2 - z_1^2z_2 - z_1z_2^2)$ *corresponding to* $\alpha = 1$ *(black) and* $\alpha = e^{\pi i/2}$ *(orange). Level set corresponding to exceptional value* $\alpha = -1$ *marked in red.*

Since

$$f_1(\zeta, \bar{\zeta}^2) = \zeta\bar{\zeta}^2 + (1 - \bar{\zeta}^2\zeta^2)\bar{\zeta}^4 = \bar{\zeta} = g_1(\zeta, \bar{\zeta}^2)$$

and

$$f_1(\bar{\zeta}^2, \zeta) = \bar{\zeta}^2\zeta + (1 - \zeta\bar{\zeta}^4)\zeta^2 = \zeta^2 = g_1(\bar{\zeta}^2, \zeta)$$

we have $g_1 = f_1$ on the support of σ_{-1}. A similar computation shows that the bidisk algebra function

$$f_2(z) = z_1z_2 + (1 - z_1z_2^2)z_1^2$$

coincides with $g_2(z) = \bar{z}_2$ on $\mathrm{supp}(\sigma_{-1})$. Thus, if $g(\zeta) = \zeta_1^m\zeta_2^n$ is any trigonometric polynomial, then, on the support of σ_{-1}, g coincides with one of functions $h_1^{|m|}h_2^{|n|}$, $h_1^{|m|}f_2^{|n|}$, $f_1^{|m|}h_2^{|n|}$, and $f_1^{|m|}f_2^{|n|}$, which are all in $A(\mathbb{D}^2)$. Since the trigonometric polynomials are dense in $C(\mathbb{T}^2)$, which in turn is dense in $L^2(\sigma_{-1})$, $A(\mathbb{D}^2)$ is also dense. Thus, J_{-1} is unitary even though $\alpha = -1$ is the non-tangential value of ϕ at a singularity. ♦

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