Toward Minimal Data Rate Enforcing Regular Safety Properties: An Invariance Entropy Approach

Mahendra Singh Tomar and Majid Zamani, *Senior Member, IEEE*

***Abstract*—The study of minimal data rate for control using some notions of entropy has been so far limited to classical control tasks such as set invariance, state-estimation, or stabilization. In this letter, for the first time, we present a study on sufficient data rates to enforce regular safety properties over uncertain systems with dynamics described by set valued maps. Every regular safety property has a set of bad prefixes which can be modelled by a deterministic finite automaton (DFA). The main idea is to construct a hybrid system by taking the product of the deterministic finite automata with the given system and studying the invariance feedback entropy (IFE) of controlled invariant sets of the hybrid system. If there exists a nonempty controlled invariant set for the hybrid system satisfying a certain property then there exists a coder-controller with a data rate not less than the IFE that can enforce the regular safety property over the original control system. We demonstrate the effectiveness of our results by designing a coder-controller enforcing a regular safety property over a linear control system.**

***Index Terms*—Data rate constrained feedback, entropy, networked control systems, regular safety properties.**

I. Introduction

A NETWORKED control system has a large number of devices distributed spatially. Many such devices exchange information over some digital communication channel that can only transmit a finite number of bits per unit of time. For efficient utilization of the channel's transmission capacity, each device should use a small number of bits/time (data rate). The smaller the data rate, the more devices can share the same channel. This gives rise to the study of smallest data rate needed in the feedback path that permits the satisfaction of a given control task. Consider the simple case of one system together with its associated coder and controller as shown in Fig. 1. Since only a finite number of bits can be transmitted at any given time, the exact value of information collected by sensors cannot be transmitted with complete accuracy. Therefore, only a finite precision of information is transmitted and thus there is an inexactness in the information received at the controller/decoder side. The smaller the number of bits used, the lower is the precision of information received at the controller side. For control tasks such as state estimation or set invariance, it has been shown that the feedback data rate cannot be smaller than a lower limit which can be identified in terms of some notions of entropy, see [1], [2].

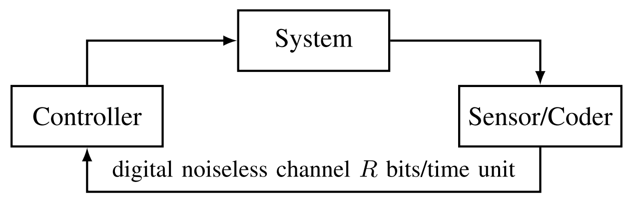

Fig. 1. Coder-controller feedback loop.

For linear control systems and specifications such as stabilization, observation, and set invariance, it has been shown that the minimum data rate above which those specification can be enforced is given by the unstable eigenvalues of the system matrix (see e.g., [1], [3], [4]). There is an extensive literature studying limited feedback data rate for linear control systems, see e.g., [5] and references therein. Comprehensive reviews of results on data rate limited control can be found, e.g., in [4], [6], [7], [8], [9], [10].

The control specifications that we consider in this letter are called regular safety properties [11]. A safety property is a set of infinite words over a finite alphabet, such that every infinite word that violates the safety property has a finite bad prefix. A safety property is called regular if its set of bad prefixes constitutes a regular language. This set of bad prefixes can be described via a deterministic finite automata (DFA) that involves a set of accepting-states [11]. If an accepting-state is reached in any finite trace of the DFA, then the safety property is violated. Verifying a regular safety property for a system can be reduced to an invariant checking on the product of the system and a DFA recognizing bad prefixes of the safety property [11]. By taking the product of the DFA with a given

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Mahendra Singh Tomar is with the Computer Science Department, University of Colorado Boulder, Boulder, CO 80309 USA (e-mail: mahendra.tomar@colorado.edu).

Majid Zamani is with the Computer Science Department, University of Colorado Boulder, Boulder, CO 80309 USA, and also with the Computer Science Department, LMU Munich, 80539 Munich, Germany.

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control system, one obtains a hybrid control system. Any trajectory of the hybrid system conforms to some trace of the DFA. If the hybrid system evolves over a hybrid domain that does not intersect states corresponding to the accepting-states of the DFA, then closed-loop trajectories satisfy the regular safety property. Hence, we focus on controlled invariant sets of the hybrid system that do not intersect the accepting states of the DFA, and that have non empty intersection with the initial set of states in the hybrid domain. Any coder-controller that renders these sets invariant also enforces the regular safety property over the original control system.

For uncertain control systems, the necessary state information required by any controller, to make a subset of the state space invariant, is quantified by invariance feedback entropy (IFE) [12]. In other words, the IFE characterizes the smallest asymptotic average data rate, from the coder to the controller, above which the subset can be made invariant over a digital noiseless channel. In this letter, we study IFE of hybrid controlled invariant sets of products of control systems and their DFAs. If the IFE is finite, then there exists a coder-controller with a data rate not less than the IFE such that the regular safety property can be enforced over the original control system.

We consider uncertain control systems with dynamics described by set valued maps. First, we show that the IFE of a given controlled invariant set of the hybrid system is lower bounded by the IFE of the projection of the hybrid controlled invariant set onto the original control system. For the case of discrete time linear control systems (dtLCS), this lower bound can be expressed in terms of the unstable eigenvalues of the systems matrices. Then we present the relationship between the invariance feedback entropy of the constructed hybrid system and the smallest feedback data rate enforcing the regular safety property over the original system. For the particular case of invariance as a regular safety property, the minimal data rate for the original system is equal to the invariance feedback entropy of the hybrid system. We also show that the lower bound is tight for the case of invariance as a regular safety property. This result can be potentially helpful to analyze hybrid systems by focussing on parts of the DFA describing the bad prefixes. Further, we discuss a class of regular safety properties (identified by the structure of the DFA describing bad prefixes) such that, for dtLCS, the smallest required data rate is upper bounded in terms of the unstable eigenvalues of the system matrix. Finally, we present a two dimensional linear system and describe a coder-controller that enforces a given regular safety property and operates at a data rate equal to the derived upper bound.

II. Preliminaries

A. Notation

We use $\mathbb{R}$ and $\mathbb{Z}$ to denote sets of real numbers and integers, respectively. Restriction of such a set is denoted with subscript annotation, e.g., $\mathbb{Z}_{\geq 0}$ denotes the non-negative integers. By $[k_1; k_2]$ we denote the set of integers $\{j \in \mathbb{Z} \mid k_1 \leq j \leq k_2\}$. For a finite set A, we use $\#A$ to denote the number of elements of A. We denote the closed and right half-open intervals in $\mathbb{Z}$ by $[a; b]$ and $[a; b)$, respectively. The restriction of a map $F : A \to B$ to a subset $M \subseteq A$ is denoted by $F|_M$. The notation $F : A \rightrightarrows B$ denotes a set-valued map, i.e., for $a \in A$, $F(a) \subseteq B$. We use B^A to denote the set of all functions $f : A \to B$. For a set B, a sequence $\alpha = \{\alpha(t)\}_{t=0}^{n} \in B^{[0;n]}$ and $\tau \leq n$, the concatenation of $A \subseteq B$ to a sub-sequence $\alpha|_{[0;\tau]}$ is denoted by $\alpha|_{[0;\tau]}A$. For $x \in \mathbb{R}^n$ and $W \subset \mathbb{R}^n$ by $x + W$ we denote the set $\{x + w \mid w \in W\}$. We use $\mathrm{spec}(\mathbf{A})$ to denote the multiset of eigenvalues of a matrix $\mathbf{A}$ and it is such that if any eigenvalue has algebraic multiplicity a then it appears a times in $\mathrm{spec}(\mathbf{A})$. A *cover* of a set Q is a collection of subsets of Q such that the union of the sets in the collection contains Q. An underline, e.g., $\underline{X}$, denotes that the quantity belongs to the hybrid domain.

First we recall the definition of invariance feedback entropy and the associated terms. Then, we describe deterministic finite automata (DFA) which are used together with a given control system to define hybrid systems.

B. Invariance Feedback Entropy

We define a system as a triple

$$\Sigma = (X, U, F),$$

where X and U are nonempty sets and $F : X \times U \rightrightarrows X$ is a set-valued transition map such that for all $x_t \in X$, $u_t \in U$ we have $x_{t+1} \in F(x_t, u_t) \neq \emptyset$.

Given a nonempty set $Q \subseteq X$, a cover $\mathcal{A}$ of Q and a map $G : \mathcal{A} \to U$, the tuple $(\mathcal{A}, G)$ is called an *invariant cover* $(\mathcal{A}, G)$ of Σ and Q if $\mathcal{A}$ is finite and for all $A \in \mathcal{A}$ we have $F(A, G(A)) \subseteq Q$. Here, $F(A, G(A))$ refers to $\cup_{x \in A} F(x, G(A))$. An invariant cover immediately provides a coder-controller scheme that keeps the trajectories starting in the set Q confined within it and operates at a data rate of $\log_2 \#\mathcal{A}$ bits/sec.

For $\tau \in \mathbb{Z}_{>0}$, let $\mathcal{S} \subseteq \mathcal{A}^{[0;\tau)}$ be a set of sequences in $\mathcal{A}$ of lengths τ. For $\alpha \in \mathcal{S}$ and $t \in [0; \tau - 1)$ define

$$P_{\mathcal{S}}(\alpha|_{[0;t]}) := \{A \in \mathcal{A} \mid \alpha|_{[0;t]}A = \hat{\alpha}|_{[0;t+1]}, \text{ for some } \hat{\alpha} \in \mathcal{S}\}, \tag{1}$$

as the set of immediate successor cover elements A of $\alpha|_{[0;t]}$ in $\mathcal{S}$ and for $t = \tau - 1$

$$P_{\mathcal{S}}(\alpha|_{[0;t]}) := \{A \in \mathcal{A} \mid A = \hat{\alpha}(0), \text{ for some } \hat{\alpha} \in \mathcal{S}\},$$

as the set of the first elements $\hat{\alpha}(0)$ of the members $\hat{\alpha}$ of $\mathcal{S}$. A set $\mathcal{S} \subseteq \mathcal{A}^{[0;\tau)}$ is called (τ, Q)-*spanning* in $(\mathcal{A}, G)$ if $P_{\mathcal{S}}(\alpha)$ with $\alpha \in \mathcal{S}$ covers Q and for every $\alpha \in \mathcal{S}$, $t \in [0; \tau - 1)$,

$$F(\alpha(t), G(\alpha(t))) \subseteq \bigcup_{A' \in P_{\mathcal{S}}(\alpha|_{[0;t]})} A'. \tag{2}$$

For every (τ, Q)-spanning set $\mathcal{S}$, we define an *expansion number* $N(\mathcal{S})$ as

$$N(\mathcal{S}) := \max_{\alpha \in \mathcal{S}} \prod_{t=0}^{\tau-1} \#P_{\mathcal{S}}(\alpha|_{[0;t]}).$$

Note that a (τ, Q)-spanning set also provides a coder-controller scheme to enforce invariance of the set Q. This scheme is τ periodic and requires a data rate $(1\backslash\tau)\log_2(N(\mathcal{S}))$ that can be lower than $\log_2 \#\mathcal{A}$.

For a given invariant cover $(\mathcal{A}, G)$, we denote by $r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma)$ the smallest expansion number possible for any (τ, Q)-spanning set in $(\mathcal{A}, G)$, i.e.,

$$r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma) := \min\{N(\mathcal{S}) \mid \mathcal{S} \text{ is } (\tau, Q)\text{-spanning in } (\mathcal{A}, G)\}.$$

Then the *entropy* of the invariant cover $(\mathcal{A}, G)$ is given by

$$h(\mathcal{A}, G) := \lim_{\tau\to\infty} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma),$$

where the existence of the limit follows from the subadditivity of $\log_2 r_{\text{inv}}(\cdot, \mathcal{A}, G, \Sigma)$ [12]. The *invariance feedback entropy (IFE)* of Σ and Q is defined as

$$h_{\text{inv}}(Q, \Sigma) := \inf_{(\mathcal{A},G)} h(\mathcal{A}, G),$$

where the infimum is taken over all invariant covers $(\mathcal{A}, G)$ of Σ and Q, using the convention that $\inf \emptyset = \infty$.

The data rate theorem in [13] shows that the invariance feedback entropy tightly lower bounds the data rate amongst the set of coder-controllers that can make the set Q invariant.

Next, we formally define dtLCS and DFA describing bad prefixes of some safety property. Then, we introduce the definition of a hybrid system and its controlled invariant sets called HCI sets. A nonempty HCI set identifies a set of initial states such that trajectories of the original control system, starting from this set, satisfy the regular safety property.

C. Some Definitions

Definition 1 (dtLCS): A discrete-time linear control system (dtLCS) is a system $\Sigma = (X, U, F)$ with $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, and F given as

$$\forall_{x\in X}\forall_{u\in U} \qquad F(x, u) = \mathbf{A}x + \mathbf{B}u + W, \tag{3}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and the disturbance set $W \subset \mathbb{R}^n$ is Lebesgue measurable.

Definition 2 (DFA): A deterministic finite automaton is a tuple $\mathfrak{A} = (\mathcal{Q}, q_0, \Pi, \delta, Acc)$ where $\mathcal{Q}$ is a finite set of states, $q_0 \in \mathcal{Q}$ is the initial state with $q_0 \notin Acc$, Π is a finite set of alphabet, $\delta : \mathcal{Q} \times \Pi \to \mathcal{Q}$ is a transition map, and $Acc \subseteq \mathcal{Q}$ denotes the accepting states.

In this letter we focus on regular safety properties whose set of bad prefixes can be described by a DFA.

We also consider a labeling function $L : X \to \Pi$ that assigns to every state in X an element of the set of alphabet of the DFA. The set of alphabet $\Pi = \{\sigma_1, \dots, \sigma_M\}$ together with the labeling function L provide a partition of the state set $X = \cup_{j=1}^{M} X_j$, where $X_j = L^{-1}(\sigma_j)$.

Definition 3 (Hybrid System): Consider a system $\Sigma = (X, U, F)$, a DFA $\mathfrak{A} = (\mathcal{Q}, q_0, \Pi, Acc)$, and a labeling function $L : X \to \Pi$. The product of Σ and $\mathfrak{A}$ is a hybrid system defined as

$$\Sigma_p := (\underline{X}, U, \underline{F}), \tag{4}$$

where $\underline{X} := \{(q, q', x) \in \mathcal{Q}\times\mathcal{Q}\times X \mid (q, L(x), q') \in \delta\}$, and for $(q, q', x) \in \underline{X}$, the transition map $\underline{F} : \underline{X} \times U \rightrightarrows \underline{X}$ is defined as

$$\underline{F}\big((q, q', x), u\big) := \{(q', \hat{q}, x') \in \underline{X} \mid x' \in F(x, u)\}. \tag{5}$$

We use π to denote the projection of $\underline{A} \subset \underline{X}$ on X: $\pi(\underline{A}) := \{x \in X \mid (q_1, q_2, x) \in \underline{A}, \text{ for some } q_1, q_2 \in \mathcal{Q}\}$. Next we define a set $\underline{Acc} := \{(q, q', x) \in \underline{X} \mid q' \in Acc\}$ using the set of accepting states Acc of the DFA. The set $\underline{Acc}$ is used to describe the satisfaction of the regular safety property by a state sequence of the hybrid system Σ_p as formalized below.

Consider a state sequence $\underline{\xi}_x = (\underline{x}(0), \dots, \underline{x}(k), \dots)$ of Σ_p with $\underline{x}(k) = (q_1(k), q_2(k), x(k))$. We say $\underline{\xi}_x$ satisfies a regular safety property $\mathfrak{A}_s$ (with a bad prefix DFA $\mathfrak{A}$), denoted by $\underline{\xi}_x \models \mathfrak{A}_s$, if $q_1(0) = q_0$ and $\underline{x}(k) \notin \underline{Acc}$ for all $k \in \mathbb{Z}_{\geq 0}$.

Note that if $\underline{\xi}_x \models \mathfrak{A}_s$, then the corresponding state sequence $(x(0), \dots, x(k), \dots)$ of Σ also satisfies the safety property $\mathfrak{A}_s$.

Definition 4 (HCI Set): A set $\underline{Q} \subseteq \underline{X}\backslash\underline{Acc}$ is a hybrid controlled invariant (HCI) set for Σ_p if $\forall \underline{x} \in \underline{Q}$, $\exists u \in U$ such that $\underline{F}(\underline{x}, u) \subseteq \underline{Q}$. By $\underline{I}^*$ we denote the maximal HCI set, i.e., $\underline{I}^* \supseteq \underline{Q}'$ if $\underline{Q}'$ is an HCI set for Σ_p.

In the next section we show that the invariance feedback entropy of any hybrid controlled invariant set is lower bounded by the IFE of the projection of the set onto the original control system.

III. Relation Between Entropies of Σ and Σ_p

The following theorem establishes that any finite average feedback data rate which can be used to make a hybrid set controlled invariant is also sufficient to render the projection of the set invariant.

Theorem 1: Consider a system $\Sigma = (X, U, F)$, a regular safety property $\mathfrak{A}_s$ with a bad prefix DFA $\mathfrak{A}$, the hybrid system Σ_p as in (4) and a nonempty compact HCI set $\underline{Q}$. The invariance feedback entropy of Σ_p and $\underline{Q}$ satisfies

$$h_{\text{inv}}(\underline{Q}, \Sigma_p) \geq h_{\text{inv}}(\pi(\underline{Q}), \Sigma). \tag{6}$$

Further, let Σ be a dtLCS, $\hat{Q} := \pi(\underline{Q})$, and $\mathbb{R}^n = \mathbb{E}_1 \oplus \mathbb{E}_2$, where $\mathbb{E}_1$ is an $\mathbf{A}$ invariant subspace of $\mathbb{R}^n$ with $\mathbb{E}_1 \neq \{0\}$, and $\oplus$ stands for the direct sum. Let $\pi_1 : \mathbb{R}^n \to \mathbb{E}_1$ be the projection onto $\mathbb{E}_1$ along $\mathbb{E}_2$, and $\mu_1(\pi_1(W)) < \mu_1(\pi_1(\hat{Q}))$, also let n_1 denote the dimension of the linear space $\mathbb{E}_1$ and μ_1 denote the n_1-dimensional Lebesgue measure. Then one gets

$$h_{\text{inv}}(\underline{Q}, \Sigma_p) \geq \log_2\left(|\det \mathbf{A}|_{\mathbb{E}_1}| \frac{\mu_1(\pi_1(\hat{Q}))}{(\mu_1(\pi_1(\hat{Q}))^{1/n_1} - \mu_1(\pi_1(W))^{1/n_1})^{n_1}}\right). \tag{7}$$

Proof: If $h_{\text{inv}}(\underline{Q}, \Sigma_p) = \infty$, the inequality (6) holds independently of the left-hand-side. Subsequently, we assume that $h_{\text{inv}}(\underline{Q}, \Sigma_p) < \infty$. From finiteness of $h_{\text{inv}}(\underline{Q}, \Sigma_p)$ and [13, Lemma 3] we know that an invariant cover of Σ_p and $\underline{Q}$ exists. We pick $\varepsilon \in \mathbb{R}_{>0}$ and an invariant cover $(\mathcal{A}_2, G_2)$ of Σ_p and $\underline{Q}$, so that $h(\mathcal{A}_2, G_2) \leq h_{\text{inv}}(\underline{Q}, \Sigma_p) + \varepsilon$.

Consider $\mathcal{A}_1 := \{\pi(A) \mid \underline{A} \in \mathcal{A}_2\}$. In [13, Lemma 9], let $M = \pi$, $Q_1 = \pi(\underline{Q})$, $Q_2 = \underline{Q}$, $X_1 = X = \pi(\underline{X})$, $X_2 = \underline{X}$, $r =$ identity map, $U_1 = U_2 = U$, $F_1 = F$, $F_2 = \underline{F}$, $\Sigma_1 = \Sigma$, and $\Sigma_2 = \Sigma_p$. We observe that conditions $1-3$ in [13, Lemma 9] hold, while condition 4 holds with the equality. Thus, there exists a map $G_1^* : \mathcal{A}_1 \to U$ such that $(\mathcal{A}_1, G_1^*)$ is an invariant cover of Σ and $\pi(\underline{Q})$, and $h(\mathcal{A}_1, G_1^*) \leq h(\mathcal{A}_2, G_2)$. By our choice of $(\mathcal{A}_2, G_2)$, we have $h(\mathcal{A}_2, G_2) \leq h_{\text{inv}}(\underline{Q}, \Sigma_p) + \varepsilon$ and by the definition of invariance feedback entropy we have $h_{\text{inv}}(\pi(\underline{Q}), \Sigma) \leq h(\mathcal{A}_1, G_1^*)$. Hence we get $h_{\text{inv}}(\pi(\underline{Q}), \Sigma) \leq h_{\text{inv}}(\underline{Q}, \Sigma_p) + \varepsilon$. Since ε is arbitrary, we obtain (6).

Since Q is compact, its projection $\pi(\underline{Q}) = \hat{Q}$ is also compact. Thus, [13, Th. 7] together with (6) satisfies the inequality in (7). ■

In the following section, we describe the coder-controller. After that we elaborate on the significance of the study of IFE of the hybrid system Σ_p in the context of the smallest data rate needed to enforce the regular safety property over the original system Σ.

IV. The Coder-Controller

We consider the definition of coder-controllers as introduced in [13, Sec. V.A]. Consider the Fig. 1 with the coder located at the sensor side. A coder-controller is a triple $\mathcal{C} = (S, \gamma, \delta)$ where S is the coding alphabet, and γ and δ are compatible coder and controller function, respectively. At any time step t the coder encodes the current state of the system and transmits a symbol $s_t \in S$ generated by the coder function $\gamma : \cup_{t\in\mathbb{Z}_{\geq 0}} X^{[0;t]} \rightarrow S$. The symbol is transmitted over a discrete noiseless channel to the controller which generates a control input determined by the controller function $\delta : \cup_{t\in\mathcal{S}_{\geq 0}} S^{[0;t]} \rightarrow U$. Let $\mathcal{Z}_\tau$ denote the set of all possible symbol sequences of length τ generated in closed loop by the coder-controller. For any symbol sequence $\zeta \in \mathcal{Z}_\tau$, by $Z(\zeta)$ we denote the set of all possible successor symbols, i.e., $Z(\zeta) := \{s \in S \mid \exists \hat{\zeta} \in \mathcal{Z}_{\tau+1}, \hat{\zeta} = \zeta s\}$. The transmission data rate of a coder-controller $\mathcal{C}$ is defined by

$$R(\mathcal{C}) := \limsup_{\tau\to\infty} \max_{\zeta\in\mathcal{Z}_\tau} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log_2 \#Z(\zeta|_{[0;t)}),$$

as the asymptotic worst-case average number of bits to identify a successor symbol.

From the data rate theorem [13], we have

$$h_{\text{inv}}(Q, \Sigma) = \inf_{\mathcal{C}\in\hat{\mathcal{C}}(Q)} R(\mathcal{C}),$$

where $\hat{\mathcal{C}}(Q)$ is the set of all such coder-controllers that can make a nonempty set $Q \subseteq X$ invariant for system Σ.

Significance of invariance feedback entropy of the hybrid system: Consider the maximal hybrid controlled invariant set $\underline{I}^*$ and an HCI set $\underline{Q} \subseteq \underline{I}^*$. Let $\underline{X}_0 := \{(q, q', x) \in \underline{X} \mid q = q_0\}$ denote the initial set of states in the hybrid domain and let $\underline{I}_0^* := \underline{I}^* \cap \underline{X}_0$. Since $\underline{I}^*$ is an HCI set, for all $x = \pi(\underline{x})$, $\underline{x} \in \underline{I}_0^*$, there exists a set of control sequences $\omega \in U^{[0;\infty)}$ that enforces the regular safety property on the system Σ. Note that the regular safety property is enforceable only for the states in the set $\pi(\underline{I}_0^*)$.

By $\underline{Q}_0 := \underline{Q} \cap \underline{X}_0$ we denote the subset of $\underline{Q}$ included in the initial set of states in the hybrid domain. Let $\tilde{\mathcal{C}}(\pi(\underline{Q}_0))$ be the set of all such coder-controllers that can enforce the given regular safety property for the set of initial states $\pi(\underline{Q}_0)$ for the system Σ. Further, let

$$R_m(\pi(\underline{Q}_0)) := \inf_{\mathcal{C}\in\tilde{\mathcal{C}}(\pi(\underline{Q}_0))} R(\mathcal{C}) \tag{8}$$

denote the smallest data rate amongst the coder-controllers in $\tilde{\mathcal{C}}(\pi(\underline{Q}_0))$. We observe that $R_m(\pi(\underline{Q}_0)) \leq R_m(\pi(\underline{I}_0^*))$ because $\pi(\underline{Q}_0) \subseteq \pi(\underline{I}_0^*)$.

Any coder-controller that renders the trajectories starting from $\underline{Q}_0$ invariant in $\underline{Q}$ also enforces the regular safety property over the system Σ for the set of initial states $\pi(\underline{Q}_0)$. Therefore, we have

$$R_m(\pi(\underline{Q}_0)) \leq h_{\text{inv}}(\underline{Q}, \Sigma_p), \tag{9}$$

i.e., the invariance feedback entropy of the HCI set $\underline{Q}$ for the hybrid system gives a value of the data rate which is sufficient to enforce the regular safety property for the set of initial states $\pi(\underline{Q}_0)$ of the system Σ.

Remark 1: Consider the case $\underline{Q} = \underline{Q}_0$. For any hybrid state $\underline{x}_0 = (q_0, q', x) \in \underline{Q}_0$ and any $u \in U$, if the next state $\underline{x}_1 \in \underline{F}(\underline{x}_0, u)$ is inside $\underline{Q}_0$ then from the definition of $\underline{F}$ in (5) we get $q' = q_0$. Thus, $\underline{Q} = \underline{Q}_0 \subseteq \{(q_0, q_0, x) \in \underline{X}_0\}$. This implies that enforcing the regular safety property is equivalent to enforcing invariance of $\underline{Q}$. Therefore

$$R_m(\pi(\underline{Q}_0)) = h_{\text{inv}}(\underline{Q}, \Sigma_p).$$

The condition $\underline{I}^* = \underline{I}_0^*$ holds for invariance as the particular case of regular safety property as discussed further in the next section.

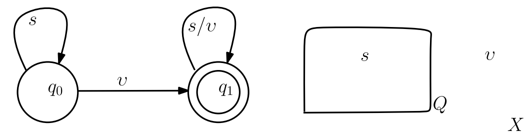


Fig. 2. Bad prefix DFA for invariance as a particular case of regular safety property. Here the labelling function is: $L(Q) = s$ and $L(X\backslash Q) = \upsilon$.

V. Invariants as Regular Safety Property

The following theorem shows that the lower bound in (6) is tight for invariance.

Theorem 2: Consider a system $\Sigma = (X, U, F)$, a subset Q of X, a labeling function $L : X \to \{s, \upsilon\}$, $L(Q) = s$, $L(X\backslash Q) = \upsilon$ and invariance as a regular safety property with the bad prefix DFA $\mathfrak{A} = (\mathcal{Q}, q_0, \Pi, \delta, Acc)$ as shown in Fig. 2. Here $\mathcal{Q} = \{q_0, q_1\}$, $\Pi = \{s, v\}$, and $Acc = \{q_1\}$. For any nonempty HCI set $\underline{Q}$ the following relation holds

$$h_{\text{inv}}(\underline{Q}, \Sigma_p) = h_{\text{inv}}(\pi(\underline{Q}), \Sigma). \tag{10}$$

Further, if Σ is a dtLCS as in (3) with $W = \{0\}$ and the set $\pi(\underline{Q})$ is compact then we have

$$h_{\text{inv}}(\underline{Q}, \Sigma_p) = H(\mathbf{A}),$$

where

$$H(\mathbf{A}) = \sum_{\substack{|\lambda| > 1, \\ \lambda \in \text{spec}(\mathbf{A})}} \log_2|\lambda|. \tag{11}$$

Proof: The set $\underline{X}$ can be partitioned into 3 subsets: $\underline{X} = \{(q_0, q_0, m) \mid m \in Q\} \cup \{(q_0, q_1, p) \mid p \in X\backslash Q\} \cup \{(q_1, q_1, p) \mid p \in X\}$. Let the maximal hybrid controlled invariant set be $\underline{I}^* \subseteq \{(q_0, q_0, m) \mid m \in Q\}$. Now consider an HCI set $\underline{Q} \subseteq \underline{I}^*$. From the controlled invariance of the set $\underline{Q}$ for Σ_p, we get the controlled invariance of $\pi(\underline{Q})$ for Σ. We assume $h_{\text{inv}}(\pi(\underline{Q}), \Sigma)$ to be finite and then from [13, Lemma 3] one gets the existence of an invariant cover of $(\Sigma, \pi(\underline{Q}))$. For $\varepsilon > 0$, consider an invariant cover $(\mathcal{A}, G)$ of $(\Sigma, \pi(\underline{Q}))$ such that $h(\mathcal{A}, G) \leq h_{\text{inv}}(\pi(\underline{Q}), \Sigma) + \varepsilon$. Let $\underline{\mathcal{A}} := \{\{q_0\} \times \{q_0\} \times A \mid A \in \mathcal{A}\}$ and for $\underline{A} \in \underline{\mathcal{A}}$ let $\underline{G}(\underline{A}) := G(\pi(\underline{A}))$. Since $\underline{Q} = \{q_0\} \times \{q_0\} \times \pi(\underline{Q})$, one gets that $\underline{\mathcal{A}}$ covers $\underline{Q}$. Now we show that $(\underline{\mathcal{A}}, \underline{G})$ is an invariant cover of $(\Sigma_p, \underline{Q})$. For $\underline{A} \in \underline{\mathcal{A}}$, from (5) we have $\underline{F}(\underline{A}, \underline{G}(\underline{A})) = \{(q_0, q, x') \in \underline{X} \mid x' \in F(\pi(\underline{A}), G(\pi(\underline{A})))\}$ and from $(\mathcal{A}, G)$ being an invariant cover we have $F(\pi(\underline{A}), G(\pi(\underline{A}))) \subseteq \pi(\underline{Q})$. Since $L(\pi(\underline{Q})) = s$ and $\delta(q_0, s) = q_0$, for every $(q_0, q, x) \in \underline{F}(\underline{A}, \underline{G}(\underline{A}))$ we get $q = q_0$. Thus $\underline{F}(\underline{A}, \underline{G}(\underline{A})) \subseteq \underline{Q}$ and $(\underline{\mathcal{A}}, \underline{G})$ is an invariant cover of $(\Sigma_p, \underline{Q})$.

Consider a $(\tau, \pi(\underline{Q}))$-spanning set $\mathcal{S}$ in $(\mathcal{A}, G)$ of Σ such that it has the smallest expansion number, i.e., $\mathcal{N}(\mathcal{S}) = r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma)$. Define a set $\underline{\mathcal{S}} \subseteq \underline{\mathcal{A}}^{[0;\tau)}$ as $\underline{\mathcal{S}} := \{\underline{\alpha} \in \underline{\mathcal{A}}^{[0;\tau)} \mid \exists \alpha \in \mathcal{S} \text{ s.t. } \underline{\alpha}(t) = \{q_0\} \times \{q_0\} \times \alpha(t) \forall t \in [0;\tau)\}$. We show that $\underline{\mathcal{S}}$ is $(\tau, \underline{Q})$-spanning in $(\underline{\mathcal{A}}, \underline{G})$ for Σ_p. Since $\underline{Q} = \{q_0\} \times \{q_0\} \times \pi(\underline{Q})$ and $\{\alpha(0) \mid \alpha \in \mathcal{S}\}$ covers $\pi(\underline{Q})$, we obtain that $\{\alpha(0) \mid \alpha \in \underline{\mathcal{S}}\}$ covers $\underline{Q}$.

To show (2), observe that for all $\alpha \in \mathcal{S}$ and $t \in [0; \tau - 2]$, we have $F(\alpha(t), G(\alpha(t))) \subseteq \pi(\underline{Q})$ and, hence, $L(F(\alpha(t), G(\alpha(t)))) = L(\pi(\underline{Q})) = s$ and $\delta(q_0, L(x)) = q_0$ for every $x \in F(\alpha(t), G(\alpha(t)))$. Now consider $\underline{\alpha} \in \underline{\mathcal{S}}$ and $\alpha \in \mathcal{S}$ such that $\underline{\alpha}(t) = \{q_0\} \times \{q_0\} \times \alpha(t)$ for all $t \in [0; \tau)$. Then

$$
\begin{aligned}
&\underline{F}(\underline{\alpha}(t), \underline{G}(\underline{\alpha}(t))) = \underline{F}(\underline{\alpha}(t), G(\alpha(t))) \\
&\quad = \{(q_0, \delta(q_0, L(x)), x) \in \underline{X} \mid x \in F(\alpha(t), G(\alpha(t)))\} \\
&\quad = \{(q_0, q_0, x) \in \underline{X} \mid x \in F(\alpha(t), G(\alpha(t)))\} \\
&\quad \overset{(2)}{\subseteq} \{(q_0, q_0, x) \in \underline{X} \mid x \in \cup_{A \in \mathcal{P}_{\mathcal{S}}(\alpha|_{[0;t]})} A)\} \\
&\quad = \cup_{\underline{A} \in \mathcal{P}_{\underline{\mathcal{S}}}(\underline{\alpha}|_{[0;t]})} \underline{A}.
\end{aligned}
$$

Thus, $\underline{\mathcal{S}}$ is a $(\tau, \underline{Q})$-spanning set in $(\underline{\mathcal{A}}, \underline{G})$ for Σ_p. Note that $\#\mathcal{P}_{\mathcal{S}}(\alpha|_{[0;t]}) = \#\mathcal{P}_{\underline{\mathcal{S}}}(\underline{\alpha}|_{[0;t]})$ for all $t \in [0; \tau - 1]$. Therefore,

$$
\begin{aligned}
\mathcal{N}(\mathcal{S}) &= \mathcal{N}(\underline{\mathcal{S}}) \\
r_{\text{inv}}(\tau, \underline{\mathcal{A}}, \underline{G}, \Sigma_p) &\leq \mathcal{N}(\mathcal{S}) = r_{\text{inv}}(\tau, \mathcal{A}, G, \Sigma) \\
h(\underline{\mathcal{A}}, \underline{G}) &\leq h(\mathcal{A}, G) \leq h_{\text{inv}}(\pi(\underline{Q}), \Sigma) + \varepsilon \\
h_{\text{inv}}(\underline{Q}, \Sigma_p) &\leq h_{\text{inv}}(\pi(\underline{Q}), \Sigma) + \varepsilon.
\end{aligned}
$$

Since ε is arbitrary, we get

$$h_{\text{inv}}(\underline{Q}, \Sigma_p) \leq h_{\text{inv}}(\pi(\underline{Q}), \Sigma).$$

This together with (6) gives $h_{\text{inv}}(\underline{Q}, \Sigma_p) = h_{\text{inv}}(\pi(\underline{Q}), \Sigma)$. For the case of Σ being a dtLCS with $W = \{0\}$ and the set $\pi(\underline{Q})$ being compact and controlled invariant, from the result on invariance entropy for deterministic systems (see [4, Th. 3.1]), we have $h_{\text{inv}}(\pi(\underline{Q}), \Sigma) = H(\mathbf{A})$. ∎

Corollary 1: Consider a system $\Sigma = (X, U, F)$ and a bad prefix DFA $\mathfrak{A} = (\mathcal{Q}, q_0, \Pi, \delta, Acc)$. Then for any HCI set $\underline{Q}$ of the form $\underline{Q} = \{(q, q, x) \in \underline{X}\}$ with a given $q \in \mathcal{Q} \backslash Acc$, the equality in (10) holds.

Proof: The proof is similar to that of Theorem 2 with q_0 replaced by q. ∎

Next we consider the bad prefix DFA shown in Fig. 3. For regular safety properties of this particular structure, we show that the smallest required feedback data rate for a dtLCS is upper bounded in terms of the unstable eigenvalues.

For the DFA, the set of alphabet Π is $\{a_i \mid 0 \leq i \leq N\} \cup \{b_i \mid 0 \leq i \leq N-1\} \cup \{c\}$ with $a_i \neq c$, $b_i \neq c$ and $a_i \neq b_i$. Each $\sigma \in \Pi$ denotes a subset of X and Π is a partition of X with

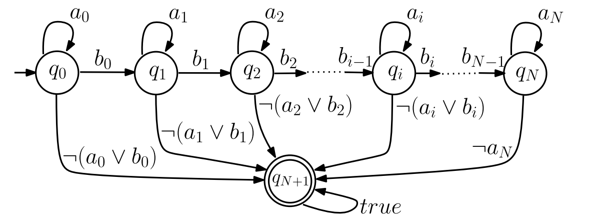


Fig. 3. Bad prefix DFA for a regular safety property. Here $\neg(a_i \vee b_i)$ denotes the set $\mathbb{R}^n \backslash (a_i \cup b_i)$.

compact a_i's and b_i's. The labelling function $L : X \to \Pi$ is given as $L(x) = \sigma$ if $x \in \sigma$. For a hybrid state set $\underline{X}$, let $\underline{X}_{i,i}$ and $\underline{X}_{i,i+1}$ denote the subsets of $\underline{X}$ that do not involve the accepting state q_{N+1}, i.e., $\underline{X}_{i,i} := \{(q_i, q_i, x) \mid x \in a_i\}, \forall i \in [0; N]$, and $\underline{X}_{i,i+1} := \{(q_i, q_{i+1}, x) \mid x \in b_i\}, \forall i \in [0; N-1]$.

Consider an HCI set $\underline{Q}$ and its subset $\underline{Q}_0 := \{(q, q', x) \in \underline{Q} \mid q = q_0\}$. Let $\mathcal{J}$ be the largest subset of $[0; N]$ such that for all $k \in \mathcal{J}$, we have $\underline{X}_{k,k} \cap \underline{Q} \neq \emptyset$. By $\underline{Q}^m_{k,k}$, we denote the maximal HCI subset of $\underline{X}_{k,k} \cap \underline{Q}$.

For the proof of the next result, we need the following lemma [4, Proposition 1.11].

Lemma 1: For a dtLCS Σ with $W = \{0\}$, if Q is a controlled invariant set, then also $\text{cl}(Q)$ is controlled invariant.

Proposition 1: Consider a system $\Sigma = (X, U, F)$, where F is single-valued and for which the regular safety property, corresponding to the bad prefix DFA in Figure 3, can be enforced. Let $\underline{Q}$ be an HCI set. Then

$$h_{\text{inv}}(\underline{Q}, \Sigma_p) \leq \max_{k \in \mathcal{J}} h_{\text{inv}}(\pi(\underline{Q}^m_{k,k}), \Sigma).$$

For a dtLCS Σ with $W = \{0\}$, we have $R_m(\pi(\underline{Q}_0)) \leq H(\mathbf{A})$, where $R_m(\pi(\underline{Q}_0))$ and $H(\mathbf{A})$ are defined in (8) and (11), respectively. Further if $\underline{I}^* \subseteq \underline{X}_{0,0}$, then we have $R_m(\pi(\underline{I}^*_0)) = H(\mathbf{A})$.

Proof: Let $\hat{k} = \max \mathcal{J}$. Then we have $\underline{Q}^m_{\hat{k},\hat{k}} = \underline{X}_{\hat{k},\hat{k}} \cap \underline{Q}$. Observe that since $\underline{Q}$ is controlled invariant, for all $\underline{x} \in \underline{X}_{\hat{k},\hat{k}} \cap \underline{Q}$ there exists $u \in U$ such that $\underline{F}(\underline{x}, u) \in \underline{Q}$. Now by the structure of $\underline{X}$ and $\hat{k}$ being the largest member of $\mathcal{J}$, we get $\underline{F}(\underline{x}, u) \in \underline{X}_{\hat{k},\hat{k}} \cap \underline{Q}$, i.e., $\underline{X}_{\hat{k},\hat{k}} \cap \underline{Q}$ is also an HCI set.

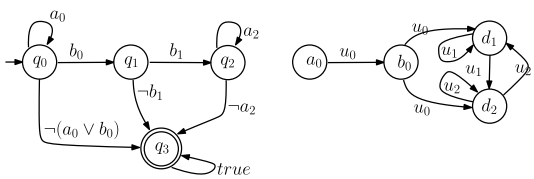
If $\hat{k} = 0$, then $\underline{Q} \subseteq \underline{X}_{0,0}$ and thus from Corollary 1 we have $h_{\text{inv}}(\underline{Q}, \Sigma_p) = h_{\text{inv}}(\pi(\underline{Q}), \Sigma)$.

Now we consider $\hat{k} > 0$. For $k \in \mathcal{J}$, $k < \hat{k}$, by the structure of $\underline{X}$ we know that for every $\underline{x} \in ((\underline{X}_{k,k} \cap \underline{Q}) \backslash \underline{Q}^m_{k,k})$ one has $\underline{F}(\underline{x}, u) \in \underline{X}_{k,k+1}$ for all $u \in U$. Also by the structure of $\underline{X}$ and for every $\underline{x} \in \underline{X}_{k,k+1}$, one has $\underline{F}(\underline{x}, u) \in (\underline{X}_{k+1,k+1} \cup \underline{X}_{k+1,k+2})$ for all $u \in U$. Thus we know that no trajectory stays in $((\underline{X}_{k,k} \cap \underline{Q}) \backslash \underline{Q}^m_{k,k})$ or $\underline{X}_{k,k+1}$ for more than one time instant. Therefore, these sets do not play any role in the invariance feedback entropy which involves a long horizon average of base 2 logarithm of the number of possible successors as identified by $P_{\mathcal{S}}(\cdot)$ which is defined in (1). This together with [14, Proposition 1] result in

$$h_{\text{inv}}(\underline{Q}, \Sigma_p) \leq \max_{k \in \mathcal{J}} h_{\text{inv}}(\underline{Q}^m_{k,k}, \Sigma_p).$$

This can be rewritten using Corollary 1 as

$$h_{\text{inv}}(\underline{Q}, \Sigma_p) \leq \max_{k \in \mathcal{J}} h_{\text{inv}}(\pi(\underline{Q}^m_{k,k}), \Sigma).$$



(1) (2)

Fig. 4. For the case study, the subfigures (1) and (2) show the DFA and the directed graph, respectively.

Now we consider Σ to be a dtLCS with $W = \{0\}$. Then from Lemma 1 and for all $k \in \mathcal{J}$, the subsets $\underline{Q}^m_{k,k}$ are closed and, thus, compact. Therefore, we obtain $h_{\text{inv}}(\pi(\underline{Q}^m_{k,k}), \Sigma) = H(\mathbf{A})$ for all $k \in \mathcal{J}$ and from (9) one gets $R_m(\pi(\underline{Q}_0)) \leq H(\mathbf{A})$.

If $\underline{I}^* \subseteq \underline{X}_{0,0}$, then $\underline{I}^* = \underline{I}^*_0$ and thus from Remark 1 and Corollary 1, we have $R_m(\pi(\underline{I}^*_0)) = h_{\text{inv}}(\underline{I}^*, \Sigma_p) = h_{\text{inv}}(\pi(\underline{I}^*), \Sigma) = H(\mathbf{A})$. ■

VI. Case Study

Consider a dtLCS

$$\Sigma: x_{t+1} \in F(x_t, u_t) = \mathbf{A}x_t + u_t + W, \quad \mathbf{A} = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$

with $x_t \in X = \mathbb{R}^2$, $u_t \in U = [-1, 1]^2$ and $W = [-0.1, 0.1]^2$. Also consider the bad prefix DFA as shown in Fig. 4 with $a_0 = [-0.25, 0.25] \times [5.5, 5.75]$, $b_0 = [-0.5, 0.5] \times [2.25, 3.25]$ and $b_1 = a_2 = [-1, 1] \times [-2, 2]$. The aforementioned safety property requires that all state sequences initiating in region a_0, should either stay in a_0 or reach and stay in region $a_2 = b_1$ after spending a single step in region b_0. Also state sequences starting in b_0 should enter in $a_2 = b_1$ in the next step and stay therein afterwards. We observe that the safety property is enforceable for the set of initial states $a_0 \cup b_0$. The hybrid system has an HCI set $\underline{Q} = \underline{X}_{0,0} \cup \underline{X}_{0,1} \cup \underline{X}_{1,2} \cup \underline{X}_{2,2}$ with $\underline{Q}^m_{0,0} = \emptyset$ and $\underline{Q}^m_{2,2} = \underline{X}_{2,2}$. From Proposition 1, we have $R_m(a_0 \cup b_0) \leq h_{\text{inv}}(\pi(\underline{Q}^m_{2,2}), \Sigma)$, and from [14, Th. 1] and [13, Th. 9] we have $h_{\text{inv}}(\pi(\underline{Q}^m_{2,2}), \Sigma) \leq 1$. Now we describe a coder-controller that enforces the regular safety property and operates at an average data rate $= 1$. We define sets $d_1 = [0, 1] \times [-2, 2]$, $d_2 = [-1, 0] \times [-2, 2]$, and $E = \mathbb{R}^2 \backslash (a_2 \cup b_0 \cup a_0)$. The coding alphabet is $S = \{a_0, b_0, d_1, d_2, E\}$. For any state sequence $\{x_k\}^t_{k=0}$, $t \geq 0$, the coder function is $\gamma(\{x_k\}^t_{k=0}) = \sigma$ where $\sigma \in S$ is such that $x_t \in \sigma$. For any symbol sequence $\{s_k\}^t_{k=0}$, the controller function is $\delta(\{s_k\}^t_{k=0}) = G(s_t)$ where the map $G: S \to U$ is given by $G(a_0) = u_0 = [0; 0]$, $G(b_0) = u_0$, $G(d_1) = u_1 = [-0.9; 0.7]$, $G(d_2) = u_2 = [0.9; 0.7]$, and $G(E) = u_0$. Now we construct a directed graph, with $\{a_0, b_0, d_1, d_2\}$ as the set of nodes, that captures all trajectories generated under the control input map $G : S \to U$, i.e., for every trajectory $\{x_t\}^\infty_{t=0}$, $x_0 \in \mathbb{R}^2 \backslash E$ with controls $u_t = G(s_t)$, $s_t \ni x_t$, there exists a path $\{s_t\}^\infty_{t=0}$ (here s_t denotes a node) in the graph such that $x_t \in s_t$ for all $t \geq 0$. The graph is shown in Fig. 4(2). For any node s the label for all its outgoing edges is $G(s)$ and the set of successor nodes is $\{\hat{s} \in S \backslash E \mid (\mathbf{A}s + \mathbf{B}G(s)) \cap \hat{s} \neq \emptyset\}$. From Fig. 4(2), we observe that after a finite amount of time the system state will trace the strongly connected component (SCC) constituted by nodes d_1 and d_2. Once the state arrives in the SCC, the coder transmits 1 bit at every time step so that the controller can identify the set that contains the current state out of the two possibilities of d_1 and d_2. Thus, the asymptotic average data rate of the coder-controller is 1 bit/unit-step.

VII. Conclusion

In this letter we described sufficient data rate to enforce a regular safety property over a limited (finite) data rate channel. The study involves construction of a hybrid system by taking a product of the given system with the bad prefix DFA of the regular safety property. For the hybrid system, the maximal controlled invariant set $\underline{I}^*$, that doesn't include any accepting states of the DFA, is of special interest. This set allows enforcing the regular safety property by a coder-controller that is designed to make the set invariant. The invariance feedback entropy of $\underline{I}^*$ and the hybrid system provides us with a data rate which is sufficient enough to enforce the regular safety property over the original system. As the first main result, we showed that the IFE of an HCI set is lower bounded by the IFE of the projection of the set on to the original system. Our second result establishes that this lower bound is tight for the case of invariance as a special case of regular safety property. Finally we also described a particular DFA structure such that, for a linear deterministic control system, the sum of the logarithm of the unstable eigenvalues is a sufficient data rate to enforce the property over a limited data rate channel.

A potential future direction is to study the gap between the described sufficient data rate and the minimal allowable data rate for any regular safety property. Another interesting direction is to study whether this analysis can be extended to omega regular properties [11].

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