



# Numerical over-approximation of invariance entropy via finite abstractions<sup>☆</sup>

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## ARTICLE INFO

### Article history:

Received 14 April 2022

Received in revised form 6 October 2022

Accepted 11 October 2022

Available online 27 October 2022

### Keywords:

Invariance entropy

Finite abstractions

Numerical methods

## ABSTRACT

For a closed-loop system with a digital channel between the sensor and controller, invariance entropy quantifies the smallest average rate of information above which a compact subset  $Q$  of the state set can be made invariant. There exist different versions of invariance entropy for deterministic and uncertain control systems, which are equivalent in the deterministic case. In this paper, we present the first numerical approaches to obtain rigorous upper bounds of these quantities. Our approaches are based on set-valued numerical analysis and graph-theoretic constructions. We combine existing algorithms from the literature to carry out our computations for several linear and nonlinear examples. A comparison with the theoretical values of the entropy shows that our bounds are of the same order of magnitude as the actual values.

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## 1. Introduction

In classical control theory, sensors and controllers are usually connected through point-to-point wiring. In networked control systems (NCS), sensors and controllers are often spatially distributed and involve wireless digital communication networks for data transfer. Compared to classical control systems, NCS provide many advantages such as reduced wiring, low installation and maintenance costs, greater system flexibility and ease of modification. NCS find applications in many areas such as car automation, intelligent buildings, and transportation networks. However, the use of communication networks in feedback control loops makes the analysis and design of NCS much more complex. In NCS, the use of digital channels for data transfer from sensors to controllers limits the amount of data that can be transferred per unit of time. This introduces quantization errors that can affect the control performance adversely.

The problem of stabilizing or observing a system over a communication channel with a limited bit rate has attracted a lot of attention in the past two decades. In this context, a classical result, often called the *data-rate theorem*, states that the minimal bit rate or channel capacity above which a linear system can

be stabilized or observed is given by the log-sum of the unstable eigenvalues. This result has been proved under various assumptions on the system model, channel model, communication protocol, and stabilization/estimation objectives. Comprehensive reviews of results on data-rate-limited control can be found, e.g., in the surveys [1–3] and the books [4,5].

For nonlinear systems, the smallest bit rate of a channel between the coder and the controller, to achieve some control task such as stabilization or invariance, can be characterized in terms of certain notions of *entropy* which are defined in terms of the open-loop system and are independent of the choice of the coder-controller. In spirit, they are similar to classical entropy notions used in the theory of dynamical systems to quantify the rate at which a system generates information, see [6].

In this paper, we first consider deterministic systems and focus on the notion of *invariance entropy* (IE) introduced in [7] as a measure for the smallest average data rate above which a compact controlled invariant subset  $Q$  of the state set can be made invariant. We present the first attempt to compute upper bounds of IE numerically. Our approach combines different algorithms. First, we compute a symbolic abstraction of the given control system over the set  $Q$  and the corresponding invariant controller using the tool SCOTS [8]. This results in a fine box partition of  $Q$  with a set of admissible control inputs assigned to each box for maintaining invariance of  $Q$ . In the second step, we use the tool dtControl [9] that converts the controller from a look-up table into a decision tree. Each leaf node of the tree represents a group of boxes to which the same single control input is assigned. The set of groups constitute a coarse partition of  $Q$ . Finally, in the third step, an algorithm that was proposed in [10] for estimation

<sup>☆</sup> This work is supported in part by the NSF, USA under Grant CMMI-2013969 and by the German Research Foundation (DFG) through grants RU 2229/1-1 and ZA 873/4-1.

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of topological entropy is adopted. Its output serves as an upper bound for the IE.

In addition, we also develop a method to approximate the IE of uncertain control systems, as introduced in [11,12], that generalizes the IE of deterministic systems. If the IE of a set  $Q$  (for an uncertain system) is finite [11, Sec. 4.2], an upper bound can be computed using a graph constructed from a finite abstraction of the system [11, Sec. 6]. However, the number of vertices in the graph is of the order of  $2^{2^n}$ , where  $n$  is the number of states in the finite abstraction. In this paper, we present an upper bound for the IE of uncertain systems that can be computed from a weighted directed graph constructed from an invariant partition (a pair of a finite partition of  $Q$  and a map that assigns a control input to every partition element). Our main result characterizes the entropy of the invariant partition in terms of the weights of the graph and establishes that it is the same as the maximum cycle mean of the graph. We should highlight that the number of vertices in this graph is not larger than  $n$ . Our proposed procedures still suffer from the curse of dimensionality due to the construction of finite abstractions of control systems. Moreover, at this point, we are not able to quantify the gap between the upper bounds and the actual values of the IE.

**Brief literature review.** The notion of invariance entropy for deterministic systems is equivalent to *topological feedback entropy* that has been introduced earlier in [13]; see [14] for a proof. Various notions of invariance entropy have been proposed to tackle different control problems or other classes of systems, see for instance [15] (exponential stabilization), [16] (invariance in networks of systems), [11] (invariance for uncertain systems), [17,18] (measure-theoretic versions of invariance entropy) and [19] (stochastic stabilization). An over-approximation of invariance entropy through a compositional approach, for networks of uncertain control systems, was discussed in [20]. Also the problem of state estimation over digital channels has been studied extensively by several groups of researchers. As it turns out, the classical notions of entropy used in dynamical systems, namely measure-theoretic and topological entropy (or variations of them), can be used to describe the smallest data rate or channel capacity above which the state of an autonomous dynamical system can be estimated with an arbitrarily small error, see [21–25]. Motivated by the observation that estimation schemes based on topological entropy suffer from a lack of robustness and are hard to implement, the authors of [26,27] introduce the notion of *restoration entropy* which characterizes the minimal data rate for so-called *regular* and *fine observability*. Finally, algorithms for state estimation over digital channels have been proposed in several works, particularly in [22,26,28,29].

**Related work:** In [30], the authors consider linear uncertain control systems and provide an algorithm to compute an invariant cover, the cardinality of which serves as an upper bound for the invariance entropy. In contrast, our proposed procedure here is applicable to nonlinear systems as well.

**Notation:** We write  $\mathbb{N} = \{1, 2, 3, \dots\}$  for the natural numbers,  $\mathbb{Z}$  for the integers and  $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ . By  $\mathbb{R}$ , we denote the set of real numbers and define  $\mathbb{R}_+ := \{r \in \mathbb{R} : r \geq 0\}$  and  $\mathbb{R}_{>0} := \mathbb{R}_+ \setminus \{0\}$ . By  $[a; b] = \mathbb{Z} \cap [a, b]$  and  $[a; b) = \mathbb{Z} \cap [a, b)$ , we denote closed and right-open discrete intervals. We write  $|A|$  for the cardinality of a set  $A$  and  $\rho(R)$  for the spectral radius of a square matrix  $R$ . The notation  $Y^X$  is used for the set of all functions  $f : X \rightarrow Y$ . For  $\tau \in \mathbb{Z}_+$ , we use  $X^\tau$  to denote  $X^{[0; \tau]}$ . By  $f : X \rightrightarrows Y$ , we denote a set-valued map from  $X$  to  $Y$ . A *cover*  $\mathcal{A}$  of set  $Q$  is a family of subsets of  $Q$  such that  $\bigcup_{A \in \mathcal{A}} A = Q$ . A cover  $\mathcal{A}$  is called a *partition* if for all  $A_1, A_2 \in \mathcal{A}$ ,  $A_1, A_2 \neq \emptyset$  and  $A_1 \cap A_2 = \emptyset$ . We write  $f|_M$  for the restriction of a map  $f$  to a subset  $M \subseteq X$ .

## 2. Background on invariance entropy

In this section, we provide the necessary theoretical background for our proposed numerical methods.

A *deterministic discrete-time control system* is given by

$$\Sigma : \quad x_{t+1} = f(x_t, u_t), \quad (1)$$

where  $f : X \times U \rightarrow X$ ,  $X \subseteq \mathbb{R}^n$ ,  $U \subseteq \mathbb{R}^m$ , is a (not necessarily continuous) map. The *transition map*  $\varphi : \mathbb{Z}_+ \times X \times U^{\mathbb{Z}_+} \rightarrow X$  of  $\Sigma$  is defined as

$$\varphi(t, x, \omega) := \begin{cases} x & \text{if } t = 0, \\ f(\varphi(t-1, x, \omega), \omega_{t-1}) & \text{if } t > 0. \end{cases}$$

Now, consider a compact set  $Q \subseteq X$  which is *controlled invariant*, i.e., for each  $x \in Q$  there is  $u \in U$  with  $f(x, u) \in Q$ . For any  $\tau \in \mathbb{N}$ , a set  $S \subset U^\tau$  is called  $(\tau, Q)$ -spanning if for each  $x \in Q$  there is  $\omega \in S$  with  $\varphi(t, x, \omega) \in Q$  for  $0 \leq t \leq \tau$ . We write  $r_{\text{inv}}(\tau, Q)$  for the minimal cardinality among all  $(\tau, Q)$ -spanning sets and define the *invariance entropy* (IE) of  $Q$  as

$$h_{\text{inv}}(Q) := \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log_2 r_{\text{inv}}(\tau, Q),$$

if  $r_{\text{inv}}(\tau, Q)$  is finite for all  $\tau$ ; otherwise,  $h_{\text{inv}}(Q) := \infty$ . The existence of the limit follows from the subadditivity of the sequence  $(\log_2 r_{\text{inv}}(\tau, Q))_{\tau \in \mathbb{N}}$ , using Fekete's subadditivity lemma (see [14, Lem. 2.1] for a proof).

The method we propose to estimate  $h_{\text{inv}}(Q)$  is based on an alternative characterization of this quantity that we will now describe. A triple  $(\mathcal{A}, \tau, G)$  is called an *invariant partition* of  $Q$  if  $\mathcal{A}$  is a finite partition of  $Q$ ,  $\tau \in \mathbb{N}$ , and  $G : \mathcal{A} \rightarrow U^\tau$  is a map satisfying<sup>1</sup>  $\varphi(t, A, G(A)) \subseteq Q$  for every  $A \in \mathcal{A}$  and  $0 \leq t \leq \tau$  (note that  $\varphi(t, x, \omega)$  only depends on  $\omega|_{[0; t]}$ ). For a given  $C = (\mathcal{A}, \tau, G)$ , we define

$$T_C : Q \rightarrow Q, \quad T_C(x) := \varphi(\tau, x, G(A_x)),$$

where  $A_x \in \mathcal{A}$  is chosen such that  $x \in A_x$ . Since  $\mathcal{A}$  is a partition of  $Q$ ,  $T_C$  is well-defined.

Now, let  $C = (\mathcal{A}, \tau, G)$  be an invariant partition. For each  $N \in \mathbb{N}$ , we introduce the set

$$\mathcal{W}_N(T_C) := \{\alpha \in \mathcal{A}^N : \exists x \in Q$$

$$\text{s.t. } T_C^i(x) \in \alpha_i, \quad 0 \leq i < N\},$$

which is constituted by all those  $N$ -length sequences of subsets of  $Q$  (i.e. elements of the partition  $\mathcal{A}$ ) such that there exists a trajectory of  $T_C$  that follows the sequence.

Next we define

$$h^*(T_C) := \lim_{N \rightarrow \infty} \frac{1}{N} \log_2 |\mathcal{W}_N(T_C)|.$$

Again, subadditivity guarantees the existence of the limit. Then, by [31, Thm. 2.3], the IE satisfies

$$h_{\text{inv}}(Q) = \inf_{C=(\mathcal{A},\tau,G)} \frac{1}{\tau} h^*(T_C), \quad (2)$$

where the infimum is taken over all invariant partitions of  $Q$ . In particular,  $h_{\text{inv}}(Q) < \infty$  if and only if an invariant partition of  $Q$  exists [31, Prop. 2.20, Lem. 2.3].

An *uncertain discrete-time control system* is given by

$$\Sigma : \quad x_{t+1} \in F(x_t, u_t), \quad (3)$$

where  $X \subseteq \mathbb{R}^n$ ,  $U \subseteq \mathbb{R}^m$ , and  $F : X \times U \rightrightarrows X$  is a set-valued map satisfying  $F(x, u) \neq \emptyset$  for all  $(x, u) \in X \times U$ .

Consider a compact set  $Q \subseteq X$  which is controlled invariant, i.e., for each  $x \in Q$  there is  $u \in U$  with  $F(x, u) \subseteq Q$ . We define

<sup>1</sup> For a set  $A \subset X$ , by  $\varphi(t, A, G(A))$  we mean  $\{\varphi(t, x, G(A)) \mid x \in A\}$ .

the invariance entropy of  $Q$  in a quite different manner as in the deterministic case. However, in the special case when  $F$  is single-valued, i.e., when  $\Sigma$  is deterministic, it coincides with the previous notion.

A pair  $(\mathcal{A}, G)$  is called an *invariant cover* of  $Q$  (w.r.t.  $\Sigma$ ) if  $\mathcal{A}$  is a finite cover of  $Q$  and  $F(A, G(A)) \subseteq Q$  for all  $A \in \mathcal{A}$ . In the case when  $\mathcal{A}$  is a partition, we call  $(\mathcal{A}, G)$  an *invariant partition*, analogously to the deterministic case.<sup>2</sup>

For  $\tau \in \mathbb{N}$ , let  $\mathcal{J} \subseteq \mathcal{A}^{[0:\tau]}$  be a set of sequences in  $\mathcal{A}$  of length  $\tau$ . For  $\alpha \in \mathcal{J}$  and  $t \in [0; \tau - 2]$ , define

$$P_{\mathcal{J}}(\alpha|_{[0:t]}) := \{A \in \mathcal{A} : \alpha|_{[0:t]} A = \hat{\alpha}|_{[0:t+1]} \text{ for some } \hat{\alpha} \in \mathcal{J}\} \quad (4)$$

as the set of immediate successor cover elements  $A$  of  $\alpha|_{[0:t]}$  in  $\mathcal{J}$ , and for  $t = \tau - 1$ , define

$$P_{\mathcal{J}}(\alpha|_{[0:t]}) = P_{\mathcal{J}}(\alpha) := \{A \in \mathcal{A} : A = \hat{\alpha}(0) \text{ for some } \hat{\alpha} \in \mathcal{J}\},$$

as the set of the first components of the sequences in  $\mathcal{J}$ . Although this set does not depend on  $\alpha$ , for consistency reasons, we still use the same notation as in (4). A set  $\mathcal{J} \subseteq \mathcal{A}^{[0:\tau]}$  is called  $(\tau, Q)$ -spanning in  $(\mathcal{A}, G)$  if  $P_{\mathcal{J}}(\alpha)$  covers  $Q$  and for all  $\alpha \in \mathcal{J}$  and  $t \in [0; \tau - 2]$

$$F(\alpha(t), G(\alpha(t))) \subseteq \bigcup_{A' \in P_{\mathcal{J}}(\alpha|_{[0:t]})} A'. \quad (5)$$

In this case, we associate to  $\mathcal{J}$  its *expansion number*

$$\mathcal{N}(\mathcal{J}) := \max_{\alpha \in \mathcal{J}} \prod_{t=0}^{\tau-1} |P_{\mathcal{J}}(\alpha|_{[0:t]})|, \quad (6)$$

and write  $\bar{r}_{\text{inv}}(\tau, Q, \mathcal{A}, G)$  for the smallest expansion number among all  $(\tau, Q)$ -spanning sets in  $(\mathcal{A}, G)$ , i.e.,  $\bar{r}_{\text{inv}}(\tau, Q, \mathcal{A}, G) := \min\{\mathcal{N}(\mathcal{J}) \mid \mathcal{J} \text{ is } (\tau, Q)\text{-spanning in } (\mathcal{A}, G)\}$ . The *entropy of an invariant cover*  $(\mathcal{A}, G)$  is then defined as

$$\bar{h}(\mathcal{A}, G) := \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \log_2 \bar{r}_{\text{inv}}(\tau, Q, \mathcal{A}, G).$$

The existence of the limit follows again by subadditivity. The invariance entropy of  $Q$  is now defined as

$$\bar{h}_{\text{inv}}(Q) := \inf_{(\mathcal{A}, G)} \bar{h}(\mathcal{A}, G),$$

where the infimum is taken over all invariant covers of  $Q$ . Although this definition does not seem to have much similarity with the definition(s) for deterministic systems,  $\bar{h}_{\text{inv}}(Q)$  reduces to  $h_{\text{inv}}(Q)$  in the case when  $F$  is single-valued, see [12, Thm. 4].

### 3. Upper bounds: Deterministic case

In this section, we explain how to obtain a computable upper bound for  $h_{\text{inv}}(Q)$ , based on (2). Suppose that we have an invariant partition  $\mathcal{C} = (\mathcal{A}, \tau, G)$  with  $\mathcal{A} = \{A_1, \dots, A_q\}$  at our disposal. Then any upper bound on  $h^*(T_{\mathcal{C}})$  will yield an upper bound on  $h_{\text{inv}}(Q)$ .

Let us first select a refinement  $\mathcal{B} = \{B_1, \dots, B_r\}$  of  $\mathcal{A}$ , i.e., a partition of  $Q$  such that each  $B \in \mathcal{B}$  is contained in some  $A \in \mathcal{A}$ . Now we define

$$\mathcal{W}_{\mathcal{N}}(\mathcal{B}, \mathcal{A}) := \{\alpha \in \mathcal{A}^N : \exists \beta \in \mathcal{B}^N \text{ with } T_{\mathcal{C}}(\beta_j) \cap \beta_{j+1} \neq \emptyset\}$$

<sup>2</sup> However, for uncertain systems, time steps larger than 1 should be avoided, so there is no number  $\tau$  here. Note that for an uncertain system, the image of any  $x \in X$ ,  $u \in U$  under the system dynamics  $F$  is a set. Under subsequent applications of control signals, the image may continue to expand and no longer be contained inside  $Q$ . Thus, even if a set is controlled invariant, there may not exist for every  $x \in Q$  any control sequence of length larger than 1 that can ensure that the subsequent states lie within  $Q$ .

$\forall j \in [0; N - 2] \text{ s.t. } \beta_i \subseteq \alpha_i \forall i \in [0; N - 1]$ .

From [10, Sec. 2.2], we have

$$h(\mathcal{B}, \mathcal{A}) := \lim_{N \rightarrow \infty} \frac{\log_2 |\mathcal{W}_{\mathcal{N}}(\mathcal{B}, \mathcal{A})|}{N} \geq h^*(T_{\mathcal{C}}).$$

Moreover, assuming compactness of the partition sets and continuity of the map, it can be shown that  $h(\mathcal{B}, \mathcal{A})$  converges to  $h^*(T_{\mathcal{C}})$  as the maximal diameter of the elements of  $\mathcal{B}$  tends to zero, see [10, Thm. 4].<sup>3</sup>

The paper [10] describes an algorithm for the exact computation of  $h(\mathcal{B}, \mathcal{A})$ , based on symbolic dynamics. First, we associate a transition matrix to  $\mathcal{B}$  via<sup>4</sup>

$$\Gamma_{i,j} := \begin{cases} 1 & \text{if } T_{\mathcal{C}}(B_i) \cap B_j \neq \emptyset \\ 0 & \text{otherwise} \end{cases}, \quad i, j = 1, \dots, r. \quad (7)$$

Then we construct a directed labeled graph  $\mathcal{G}$  from the transition matrix  $\Gamma$ . The set of nodes is  $\mathcal{B}$  and  $\Gamma_{i,j} = 1$  indicates that there is a directed edge from  $B_i$  to  $B_j$ . To this edge, we assign the edge label

$$L(B_i) := j, \text{ where } j \text{ is such that } B_i \subset A_j. \quad (8)$$

Elements of  $\mathcal{W}_{\mathcal{N}}(\mathcal{B}, \mathcal{A})$  are thus generated by concatenating labels along walks of length  $N$  on  $\mathcal{G}$ . To compute  $h(\mathcal{B}, \mathcal{A})$ , a *right-resolving graph*<sup>5</sup>  $\bar{\mathcal{G}}$  needs to be determined (see [32, §3.3]), such that the subset of  $\mathbb{N}^{\mathbb{Z}}$  generated by concatenation of edge labels along walks in the graph is same for both  $\mathcal{G}$  and  $\bar{\mathcal{G}}$ .<sup>6</sup> Each node in the right-resolving graph  $\bar{\mathcal{G}}$  is some subset of  $\mathcal{B}$ , while the set of edge labels is identical for both  $\mathcal{G}$  and  $\bar{\mathcal{G}}$ ; for details on its computation, see [10]. For any edge in  $\bar{\mathcal{G}}$ , with edge label  $e$ , directed from node  $n_1 \subset \mathcal{B}$  to  $n_2 \subset \mathcal{B}$ , we have  $n_2 = \{B_j \in \mathcal{B} \mid B_i \in n_1, L(B_i) = e, \Gamma_{i,j} = 1\}$ , i.e.,  $n_2$  is the set of those elements of  $\mathcal{B}$  which have an incoming edge from such members of  $n_1$  whose image under the map  $L$  equals  $e$ . Let  $\tilde{n}$  be the number of nodes in  $\bar{\mathcal{G}}$ . An associated  $\tilde{n} \times \tilde{n}$  adjacency matrix  $R$  is defined as

$$R_{i,j} := \# \text{ of edges from node } i \text{ to node } j \text{ in } \bar{\mathcal{G}}.$$

If  $\mathcal{G}$  is strongly connected (i.e., for every pair of nodes  $u$  and  $v$ , there exists a directed path from  $u$  to  $v$ ), then by [10, Prop. 7] we have  $h(\mathcal{B}, \mathcal{A}) = \log_2 \rho(R)$ . In general, we need to determine the strongly connected components  $\mathcal{G}_1, \dots, \mathcal{G}_p$  of  $\mathcal{G}$  and compute a right-resolving graph for each component separately, resulting in adjacency matrices  $R^1, \dots, R^p$ . Then (see [10, Rem. 9])

$$h_{\text{inv}}(Q) \leq \frac{1}{\tau} h(\mathcal{B}, \mathcal{A}) = \frac{1}{\tau} \max_{1 \leq k \leq p} \log_2 \rho(R^k),$$

where  $p$  is the number of strongly connected components of  $\mathcal{G}$  and  $R^k$  is the adjacency matrix of the  $k$ th strongly connected component  $\mathcal{G}_k$ . This leaves us with the problem of constructing an invariant partition  $\mathcal{C}$  with low entropy, in order to obtain an upper bound for  $h_{\text{inv}}(Q)$  which is not too conservative. We are not really able to minimize the entropy, but we can produce invariant partitions  $(\mathcal{A}, \tau, G)$  for a given input sequence length  $\tau$  with a (potentially) small number of partition elements. Due to the trivial inequality  $h^*(T_{\mathcal{C}}) \leq \log_2 |\mathcal{A}|$ , this is desirable.

<sup>3</sup> Of course, such an assumption can, in general, not be satisfied. We expect that the result still holds true if only a negligibly small amount of the exponential orbit complexity of the closed-loop dynamics is concentrated on the boundaries of the sets  $A_i$ . If  $T_{\mathcal{C}}$  was continuous on  $Q$ , this could be formalized by requiring that these boundaries have measure zero w.r.t. any  $T_{\mathcal{C}}$ -invariant Borel probability measure.

<sup>4</sup> In the language of symbolic dynamics, the matrix  $\Gamma$  defines a *subshift of finite type* over the alphabet  $\{1, \dots, r\}$ .

<sup>5</sup> A labeled graph is right-resolving if, for each vertex, all the outgoing edges have different labels.

<sup>6</sup> The subset of  $\mathbb{N}^{\mathbb{Z}}$  generated by concatenating edge labels along all walks on  $\mathcal{G}$  forms a *sofic shift* whose topological entropy equals  $h(\mathcal{B}, \mathcal{A})$ .

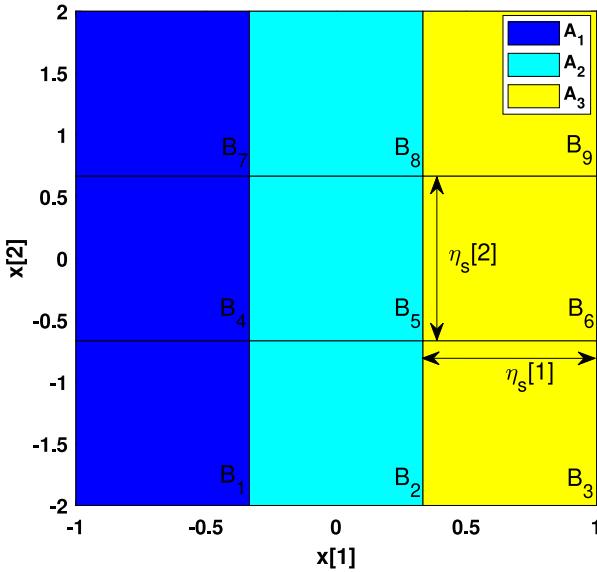


Fig. 1. The partitions  $\mathcal{A}$  and  $\mathcal{B}$  for Example 1.

We now explain step by step how we determine an invariant partition and compute its entropy. The following example is used to illustrate each step.

**Example 1.** Consider the linear control system

$$\Sigma : x_{t+1} = Ax_t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_t, \quad A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

with  $x_t \in \mathbb{R}^2$  and  $u_t \in [-1, 1]$ . For the compact controlled invariant set  $Q = [-1, 1] \times [-2, 2]$ , see [33, Ex. 21], we intend to compute an upper bound of  $h_{\text{inv}}(Q)$ .

Given a discrete-time system  $\Sigma$  as in (1) and a compact controlled invariant set  $Q \subseteq X$ , we proceed according to the following steps.

- (1) Compute a symbolic invariant controller for the set  $Q$ . Consider the smallest hyperrectangle  $Q_X$  that encloses  $Q$  and assume that  $Q_X \subseteq X$ . We use SCOTS to compute an invariant controller for  $\Sigma$  with  $Q_X$  as the state set and  $\eta_s, \eta_i$  as the grid parameters for the state and input sets, respectively. A smaller value for  $\eta_s$  results in a finer grid on the state set, which typically results in a better upper bound. We denote the set of boxes in the domain of the computed controller by  $\mathcal{B} = \{B_1, \dots, B_r\}$  and put

$$\bar{Q} := \bigcup_{i=1}^r B_i \subseteq Q.$$

The set  $\bar{Q}$  is our approximation of  $Q$ .

**Example 1 (continued).** We used SCOTS with the state set  $Q_X = Q$  and the state and input set grid parameters  $\eta_s = [2/3, 4/3]^\top$  and  $\eta_i = 1$ . This results in a state set grid with 9 boxes,  $\mathcal{B} = \{B_1, \dots, B_9\}$  and  $\bar{Q} = Q$  (see Fig. 1).

(2) The controller obtained in the previous step is, in general, non-deterministic, i.e., different control inputs are assigned to the same state. In this step, we determinize the obtained controller. We denote the closed-loop system ( $\Sigma$  with the determinized controller  $C$ ) by  $\Sigma_C$ . To determinize the controller, we used the

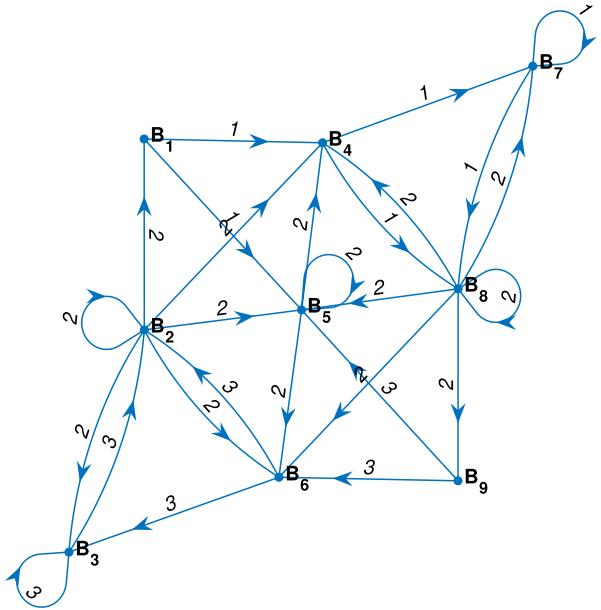


Fig. 2. The graph  $\mathcal{G}$  for Example 1.

state-of-the-art toolbox dtControl [9], which utilizes the *decision tree learning algorithm*. This also provides the required coarse partition  $\mathcal{A}$ , of which  $\mathcal{B}$  is a refinement.

**Example 1 (continued).** For the example, we used dtControl with parameters Classifier = 'cart' and Determinizer = 'maxfreq'. This results in an invariant partition  $(\mathcal{A}, 1, G)$  for the set  $\bar{Q} := \bigcup_{B \in \mathcal{B}} B$ , where  $\mathcal{A}$  is a partition of  $\bar{Q}$  such that every  $A \in \mathcal{A}$  is a union of some sets in  $\mathcal{B}$  and  $G(A) \in U$  is the control input assigned to the set  $A$  given by dtControl. Fig. 1 shows the obtained partitions  $\mathcal{A}$  and  $\mathcal{B}$ .

(3) For the dynamical system  $\Sigma_C$ , we obtain the transition matrix  $\Gamma$  (defined in (7)) for the boxes in  $\bar{Q}$ .

(4) We obtain the edge labels map  $L(B_i)$  as in (8).

**Example 1 (continued).** For any  $B_i \in \mathcal{B}$ ,

$$L(B_i) = \begin{cases} 1 & \text{if } i = 1 + 3t, 0 \leq t \leq 2, \\ 2 & \text{if } i = 2 + 3t, 0 \leq t \leq 2, \\ 3 & \text{if } i = 3 + 3t, 0 \leq t \leq 2. \end{cases}$$

(5) We construct a directed labeled graph  $\mathcal{G}$  with  $\mathcal{B}$  as the set of nodes. If  $\Gamma_{i,j} = 1$ , there is a directed edge from the node  $B_i$  to  $B_j$  with label  $L(B_i)$ .

(6) We determine the strongly connected components of  $\mathcal{G}$ .

**Example 1 (continued).**  $\mathcal{G}$  is strongly connected. Fig. 2 shows the constructed graph  $\mathcal{G}$ .

(7) For every strongly connected component  $\mathcal{G}_k$  of  $\mathcal{G}$ , we find a right-resolving graph  $\bar{\mathcal{G}}_k$ . The directed graph  $\bar{\mathcal{G}}_k$  is deterministic in the sense that for every node all the outgoing edges have a distinct label.

**Example 1 (continued).** Right-resolving graph of  $\mathcal{G}$  with nodes  $r_1 = \{B_i : i \in \{7, 8\}\}$ ,  $r_2 = \{B_i : 4 \leq i \leq 6\}$ ,  $r_3 = \{B_i : i \in \{2, 3\}\}$ ,  $r_4 = \{B_i : 4 \leq i \leq 9\}$ ,  $r_5 = \{B_i : i \in \{2, 3, 5, 6\}\}$ ,  $r_6 = \{B_i : 1 \leq i \leq 6\}$ , and  $r_7 = \{B_i : i \in \{4, 5, 7, 8\}\}$ . The constructed right-resolving graph  $\bar{\mathcal{G}}$  is shown in Fig. 3.

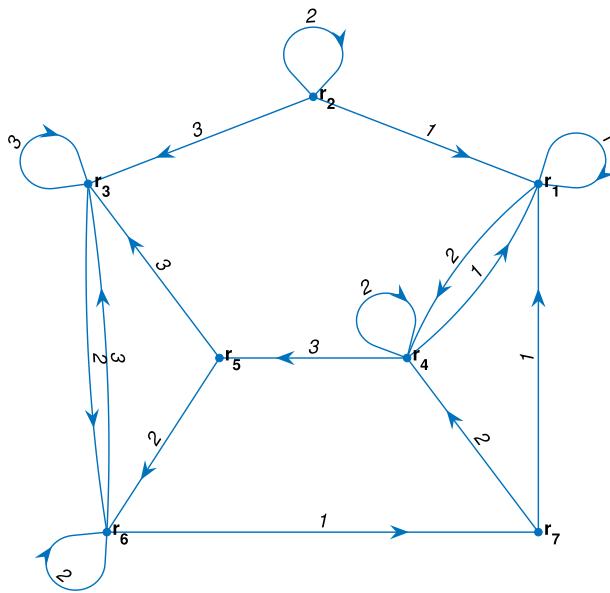


Fig. 3. The right-resolving graph  $\bar{G}$  for Example 1.

(8) Using  $\bar{G}_k$ , we construct an adjacency matrix  $R^k$  by  $R_{i,j}^k := l$ , where  $l$  is the number of edges from node  $i$  to node  $j$  in  $\bar{G}_k$ .

**Example 1 (continued).** From  $\bar{G}_k$ , we obtain

$$R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

with  $\rho(R) = 2.41421$  and  $\log_2(2.4142) = 1.2716$ .

#### 4. Upper bounds: Uncertain case

In this section, we explain how to obtain a computable upper bound for the IE of an uncertain system.

Suppose again that we know an invariant partition  $(\mathcal{A}, G)$  and recall that the time step  $\tau$  is always set to 1 for uncertain systems. We define a set-valued map  $T : Q \rightrightarrows Q$  by  $T(x) := F(x, G(A_x))$ , where  $x \in A_x \in \mathcal{A}$ . We also define a weighted directed graph  $\bar{G}$  with  $\mathcal{A}$  as its set of nodes. The graph  $\bar{G}$  contains an edge from  $A$  to  $A'$ , denoted by  $e_{AA'}$ , if  $T(A) \cap A' \neq \emptyset$ . We define maps  $D : \mathcal{A} \rightrightarrows \mathcal{A}$  and  $w : \mathcal{A} \rightarrow \mathbb{R}_+$  by

$$D(A) := \{A' \in \mathcal{A} : T(A) \cap A' \neq \emptyset\}, \quad (9)$$

$$w(A) := \log_2 |D(A)|.$$

The weight of the edge  $e_{AA'}$  is defined to be  $w(A)$ . We observe that

$$T(A) \subseteq \bigcup_{\hat{A} \in D(A)} \hat{A}. \quad (10)$$

Given  $\tau \in \mathbb{N} \cup \{\infty\}$ , we let  $W_\tau(\bar{G})$  denote the set of all (node) paths  $(A_i)_{i=0}^{\tau-1}$  in  $\bar{G}$  of length  $\tau$ .

Consider a cycle  $c = (e_{A_i A_{i+1}})_{i=1}^k$ ,  $A_{k+1} = A_1$ , in  $\bar{G}$ . The *mean cycle weight* of  $c$  is defined as

$$w_m(c) := \frac{1}{k} \sum_{i=1}^k w(A_i).$$

The *maximum cycle mean weight* is then defined as

$$w_m^*(\bar{G}) := \max_c w_m(c),$$

the maximum taken over all cycles in  $\bar{G}$  (the maximum exists because, due to the finiteness of the graph, it suffices to take the maximum over finitely many cycles).

Our algorithm is based on the following theorem, which yields a characterization of the entropy of an invariant partition in terms of the associated graph  $\bar{G}$ . Note that an invariant partition immediately provides a coder-controller for making the set  $Q$  invariant. Then, the number of bits needed to transfer from the coder/sensor to the controller is  $\log_2(\text{number of successor partition elements})$ . Since the graph has finite number of edges, the entropy of the invariant partition is equal to the worst case asymptotic average bit rate and equal to the maximum cycle mean weight. This is stated formally in the following theorem.

**Theorem 1.** For an uncertain control system  $\Sigma$  as in (3), a compact controlled invariant set  $Q \subseteq X$  and an invariant partition  $(\mathcal{A}, G)$ , we have

$$\bar{h}(\mathcal{A}, G) = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \max_{\alpha \in W_\infty(\bar{G})} \sum_{t=0}^{\tau-2} w(\alpha(t)) = w_m^*(\bar{G}). \quad (11)$$

**Remark 1.** In [34], the authors show that the logarithm of the joint spectral radius of a finite set  $\mathcal{A}$  of rank one matrices equals the maximum cycle mean in a directed complete graph. The set of nodes in the graph is  $\mathcal{A}$ , the number of edges is  $|\mathcal{A}|^2$  and the weight of any edge is defined using the matrices/nodes connected by the edge. The result of [34] can be used to establish that, for the case of a non-complete graph, the entropy of an invariant partition is upper bounded by the maximum cycle mean weight.

The rest of this section is devoted to the proof of the theorem. We start with two lemmas.

**Lemma 1.**  $W_\tau(\bar{G})$  is a  $(\tau, Q)$ -spanning set in  $(\mathcal{A}, G)$ .

**Proof.** Since  $(\mathcal{A}, G)$  is an invariant cover, we have  $D(A) \neq \emptyset$  for every  $A \in \mathcal{A}$ . Thus, for every node in  $\bar{G}$  there is at least one outgoing edge. Hence, for all  $A \in \mathcal{A}$  and  $\tau \in \mathbb{N}$ , there is at least one path of length  $\tau$  starting from  $A$ . It follows that

$$\{\alpha(0) : \alpha \in W_\tau(\bar{G})\} = \mathcal{A}.$$

Consider any  $\alpha \in W_\tau(\bar{G})$  and  $t \in [0; \tau-1]$ . By the definition of  $\bar{G}$ , we have an edge from  $\alpha(t)$  to every  $A \in D(\alpha(t))$ . Thus, for every  $t \in [0; \tau-2]$  we have

$$P_{W_\tau(\bar{G})}(\alpha|_{[0:t]}) = D(\alpha(t)). \quad (12)$$

Using (10), we conclude that  $W_\tau(\bar{G})$  satisfies (5), and hence is a  $(\tau, Q)$ -spanning set in  $(\mathcal{A}, G)$ .  $\square$

**Lemma 2.** For any  $(\tau, Q)$ -spanning set  $\mathcal{J}$  in  $(\mathcal{A}, G)$ ,  $W_\tau(\bar{G}) \subset \mathcal{J}$ .

**Proof.** Let  $\mathcal{J}$  be a  $(\tau, Q)$ -spanning set in  $(\mathcal{A}, G)$ . Then, since  $\mathcal{A}$  is a partition,  $\{\alpha(0) : \alpha \in \mathcal{J}\} = \mathcal{A}$ . If  $\alpha \in \mathcal{J}$  and  $t \in [0; \tau-1]$ , then from (5) it follows that  $P_{\mathcal{J}}(\alpha|_{[0:t]})$  covers  $F(\alpha(t), G(\alpha(t))) = T(\alpha(t))$ . Since  $\mathcal{A}$  is a partition,  $D(\alpha(t))$  must be contained in every subset of  $\mathcal{A}$  that covers  $T(\alpha(t))$ , thus  $P_{\mathcal{J}}(\alpha|_{[0:t]}) \supseteq D(\alpha(t))$ . Let  $\beta \in W_\tau(\bar{G})$ . Then  $\beta(0) \in \mathcal{A} = \{\alpha(0) : \alpha \in \mathcal{J}\}$ , implying  $\beta(0) = \alpha(0)$  for some  $\alpha \in \mathcal{J}$ . From (12), we have  $P_{W_\tau(\bar{G})}(\beta(0)) = D(\beta(0))$ . Similarly to the reasoning above, since  $\mathcal{A}$  is a partition,  $D(\beta(0))$  is contained in every subset of  $\mathcal{A}$  which covers  $T(\beta(0))$ . As  $\mathcal{J}$  is  $(\tau, Q)$ -spanning, from (5) we know that  $T(\alpha(0))$  is covered by  $P_{\mathcal{J}}(\alpha(0))$ , implying  $P_{\mathcal{J}}(\alpha(0)) \supseteq D(\beta(0))$ . From the definition of

$\mathcal{G}$ , we obtain  $\beta(1) \in D(\beta(0))$ , which leads to  $\beta(1) \in P_{\mathcal{J}}(\alpha(0))$ . Thus, there exists an  $\alpha \in \mathcal{J}$  with  $\alpha|_{[0,1]} = \beta|_{[0,1]}$ . Inductively, we obtain the existence of  $\alpha \in \mathcal{J}$  with  $\alpha = \beta$ , which concludes the proof.  $\square$

We can now prove **Theorem 1**.

**Proof of Theorem 1.** From (6) and **Lemma 2**, we conclude that for every  $(\tau, Q)$ -spanning set  $\mathcal{J}$  in  $(\mathcal{A}, G)$ , the inequality  $\mathcal{N}(W_{\tau}(\mathcal{G})) \leq \mathcal{N}(\mathcal{J})$  holds, implying that

$$\bar{r}_{\text{inv}}(\tau, Q, \mathcal{A}, G) = \mathcal{N}(W_{\tau}(\mathcal{G})) \quad \text{for all } \tau \in \mathbb{N}. \quad (13)$$

By taking logarithms on both sides of (6) and using (9) and (12), we obtain

$$\log_2 \mathcal{N}(W_{\tau}(\mathcal{G})) = \max_{\alpha \in W_{\tau}(\mathcal{G})} \sum_{t=0}^{\tau-2} w(\alpha(t)) + \log_2 |\mathcal{A}|. \quad (14)$$

Putting (13) and (14) together, it follows that

$$\bar{h}(\mathcal{A}, G) = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \max_{\alpha \in W_{\tau}(\mathcal{G})} \sum_{t=0}^{\tau-2} w(\alpha(t)).$$

Observing that the elements of  $W_{\tau}(\mathcal{G})$  are restrictions of elements of  $W_{\infty}(\mathcal{G})$  to  $[0; \tau - 1]$ , the first equality in (11) follows.

For the proof of the second equality in (11), let  $\mathcal{A} = \{A_1, \dots, A_q\}$  and consider an arbitrary  $\alpha \in W_{\infty}(\mathcal{G})$ . From [35, Lem. 3], we know that for each  $\tau$  we can write

$$\sum_{t=0}^{\tau-2} w(\alpha(t)) = \sum_{t=0}^a w(\beta(t)) + \sum_{i=1}^r l_i w_m(\sigma_i),$$

for some  $\beta \in W_{\infty}(\mathcal{G})$ ,  $a < n - 1$  and proper cycles  $\sigma_i$  of length  $l_i$  so that  $\tau - 1 = a + 1 + \sum_{i=1}^r l_i$ . It thus follows that

$$\begin{aligned} \sum_{t=0}^{\tau-2} w(\alpha(t)) &\leq n \max_{A_i \in \mathcal{A}} w(A_i) + w_m^* \sum_{i=1}^r l_i \\ &\leq n \log_2 n + \tau w_m^*, \end{aligned}$$

leading to

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \max_{\alpha \in W_{\infty}(\mathcal{G})} \sum_{t=0}^{\tau-2} w(\alpha(t)) \leq w_m^*.$$

To show the converse inequality, consider an  $\alpha \in W_{\infty}(\mathcal{G})$  that traces a proper cycle with mean weight equal to the maximum cycle mean  $w_m^*$ . Let  $l$  be the length of the cycle and write  $\tau - 1 = rl + a$  for any  $\tau > 1$ , where  $r \geq 0$  and  $0 \leq a < l$  are integers. This implies

$$\frac{1}{\tau} \sum_{t=0}^{\tau-2} w(\alpha(t)) \geq \frac{lr}{\tau} w_m^*,$$

and hence

$$\frac{1}{\tau} \max_{\alpha \in W_{\infty}(\mathcal{G})} \sum_{t=0}^{\tau-2} w(\alpha(t)) \geq \frac{1}{\tau} w_m^*(\tau - 1 - n).$$

It now easily follows that  $\bar{h}(\mathcal{A}, G) \geq w_m^*$ , which concludes the proof.  $\square$

## 5. Relationship between the upper bounds

In this section, we prove that in the deterministic case, where the obtained upper bound of the IE for deterministic systems and the one for uncertain ones both apply, these bounds are related by an inequality.

Consider a deterministic system  $\Sigma$  as in (1), a compact controlled invariant set  $Q \subseteq X$ , and an invariant partition  $(\mathcal{A}, G)$  with  $\mathcal{A} = \{A_1, \dots, A_q\}$ . Let  $\mathcal{B} = \{B_1, \dots, B_r\}$  be a refinement of  $\mathcal{A}$  and construct the weighted directed graph  $\mathcal{G}$  as described in Section 3. The sets  $\mathcal{W}_N(\mathcal{B}, \mathcal{A})$ , the quantity  $h(\mathcal{B}, \mathcal{A})$  and the transition matrix  $\Gamma$  are defined as in (7). For simplicity, we assume that  $\mathcal{G}$  is strongly connected, in which case we know that

$$h(\mathcal{B}, \mathcal{A}) = \lim_{N \rightarrow \infty} \frac{|\mathcal{W}_N(\mathcal{B}, \mathcal{A})|}{N} = \log_2 \rho(R),$$

where  $R$  is the adjacency matrix associated with a right-resolving graph.

**Proposition 1.** Given the invariant partition  $(\mathcal{A}, G)$ , for any refinement  $\mathcal{B}$  of  $\mathcal{A}$ , it holds that

$$\bar{h}(\mathcal{A}, G) \geq h(\mathcal{B}, \mathcal{A}) = \log_2 \rho(R).$$

**Proof.** We use the notation  $\mathcal{W}_{\tau}(\mathcal{A})$  to refer to the set  $\mathcal{W}_{\tau}(\mathcal{B}, \mathcal{A})$ , which is defined in Section 3, for the case when  $\mathcal{B} = \mathcal{A}$ , i.e.,  $\mathcal{W}_{\tau}(\mathcal{A}) := \{\alpha \in \mathcal{A}^{\tau} : T_c(\alpha_i) \cap \alpha_{i+1} \neq \emptyset \quad \forall i \in [0; \tau-2]\}$ . Constructing the graph  $\mathcal{G}$  associated with  $\mathcal{A}$  as in Section 4, leads to

$$\mathcal{W}_{\tau}(\mathcal{G}) = \mathcal{W}_{\tau}(\mathcal{A}) \quad \text{for all } \tau \in \mathbb{N}.$$

From [12, Lem. 2] and (13), we obtain

$$|\mathcal{W}_{\tau}(\mathcal{A})| = |\mathcal{W}_{\tau}(\mathcal{G})| \leq \mathcal{N}(W_{\tau}(\mathcal{G})) = \bar{r}_{\text{inv}}(\tau, Q, \mathcal{A}, G). \quad (15)$$

Then (15) yields

$$h(\mathcal{A}) := \lim_{\tau \rightarrow \infty} \frac{\log_2 |\mathcal{W}_{\tau}(\mathcal{A})|}{\tau} \leq \bar{h}(\mathcal{A}, G).$$

It is clear that  $h(\mathcal{B}, \mathcal{A}) \leq h(\mathcal{A})$ . Hence,

$$\log_2 \rho(R) = h(\mathcal{B}, \mathcal{A}) \leq h(\mathcal{A}) \leq \bar{h}(\mathcal{A}, G).$$

This concludes the proof.  $\square$

## 6. Examples

In this section, we illustrate the effectiveness of our results on some case studies.

### 6.1. A linear discrete-time system

Consider the following linear control system obtained from a similarity transformation applied to the system in **Example 1**:

$$x_{k+1} = Ax_k + \begin{bmatrix} 0.9463 \\ 1.051 \end{bmatrix} u_k, \quad A = \begin{bmatrix} 2 & 0.0784 \\ 0.0784 & 0.5041 \end{bmatrix},$$

with  $x_k \in \mathbb{R}^2$  and  $u_k \in U = [-1, 1]$ . Consider the set  $Q$  given by the inequality

$$\begin{bmatrix} 0.0261 & -0.4993 \\ 0.9986 & 0.0523 \\ -0.0261 & 0.4993 \\ -0.9986 & -0.0523 \end{bmatrix} x \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad x \in \mathbb{R}^2,$$

which is compact and controlled invariant.

To compute an upper bound on the IE of  $Q$ , we put  $Q_X := [-1.2, 1.2] \times [-2.1, 2.1]$ ,  $\eta_s := [0.04, 0.08]^T$  and  $\eta_i := 0.2$ . For these parameters, **Table 1** lists the values of  $h(\mathcal{B}, \mathcal{A})$  for different selections of the coarse partition  $\mathcal{A}$ . For the same values of  $\eta_s$  and  $\eta_i$ , the obtained value for the bound in **Theorem 1** is  $w_m^*(\mathcal{G}) = 2.5849$  (with computation time 0.048 s). For **dtControl** parameters **Classifier** = 'logreg' and **Determinizer** = 'maxfreq', **Table 2** shows the variation of the upper bound  $h(\mathcal{B}, \mathcal{A})/\tau$  with increasing control sequence length  $\tau$ .

**Table 1**

Entropy estimates for Example 6.1 with different choices of the determinization options in `dtControl`. Here,  $h_{\text{inv}}(Q) = 1.003$ .

Classifier	Determinizer	$ \mathcal{A} $	$h(\mathcal{B}, \mathcal{A})$	Time
cart	maxfreq	10	1.2133	10 s
logreg	maxfreq	9	1.1802	10 s
linsvm	maxfreq	10	1.2133	10 s
cart	minnorm	135	1.7848	9 s
logreg	minnorm	111	1.8015	11 s
linsvm	minnorm	143	1.8300	10 s

**Table 2**

Entropy estimates for Example 6.1 with control sequences of length  $\tau$ . Here,  $h_{\text{inv}}(Q) = 1.003$ .

$\tau$	$h(\mathcal{B}, \mathcal{A})/\tau$	Time
1	1.1802	9.6 s
2	1.0688	16.7 s
3	1.0588	1 min 11 s

## 6.2. A scalar continuous-time nonlinear control system

Consider the following scalar continuous-time control system discussed in [31, Ex. 7.2]:

$$\Sigma : \dot{x} = (-2b \sin x \cos x - \sin^2 x + \cos^2 x) + u \cos^2 x,$$

where  $u \in [-\rho, \rho]$ ,  $b > 0$  and  $0 < \rho < b^2 + 1 =: a$ . The equation describes the projectivized linearization of a controlled damped mathematical pendulum at the unstable position, where the control acts as a reset force. The following set is controlled invariant:

$$Q = [\arctan(-b - \sqrt{a + \rho}), \arctan(-b - \sqrt{a - \rho})].$$

In fact,  $Q$  is the closure of a maximal set of complete approximate controllability. With  $\mathcal{T}_s \in \mathbb{R}_{>0}$  as the sampling time, we first obtain a discrete-time system as in (1). Theory suggests that the following formula holds, see<sup>7</sup> [31, Ex. 7.2]:

$$h_{\text{inv}}(Q) = \frac{2}{\ln 2} \sqrt{a - \rho}.$$

Discretizing the given system with sampling time  $\mathcal{T}_s$  results in a discrete-time system  $\Sigma^{\mathcal{T}_s}$  that satisfies

$$h_{\text{inv}}(Q; \Sigma^{\mathcal{T}_s}) \geq \mathcal{T}_s \cdot h_{\text{inv}}(Q) = \frac{2\mathcal{T}_s}{\ln 2} \sqrt{a - \rho}.$$

The inequality is due to the fact that continuous-time open-loop control functions are lost due to the sampling (since only the piecewise constant control functions, constant on each interval of the form  $[k\mathcal{T}_s, (k+1)\mathcal{T}_s]$ ,  $k \in \mathbb{Z}_+$ , are preserved under sampling). Since  $Q$  can be made invariant by constant control inputs only,  $Q$  is also a controlled invariant set of  $\Sigma^{\mathcal{T}_s}$ . Tables 3 and 4 list the values of  $h(\mathcal{B}, \mathcal{A})/\mathcal{T}_s$  for different choices of the sampling time with the parameters  $(\rho = 1, b = 1, \eta_s = 10^{-6}, \eta_i = 0.2\rho)$  and  $(\rho = 50, b = 10, \eta_s = 10^{-6}, \eta_i = 0.2\rho)$ , respectively. In both tables, the `dtControl` parameters are `Classifier` = 'cart' and `Determinizer` = 'maxfreq'. Table 5 shows the values of  $h(\mathcal{B}, \mathcal{A})/\mathcal{T}_s$  for different selections of the coarse partition  $\mathcal{A}$  with the parameters  $\mathcal{T}_s = 0.01, \eta_s = 10^{-6}, \eta_i = 0.2\rho, \rho = 1, b = 1$ . For the same selection of parameters as in Table 3 with  $\mathcal{T}_s = 0.01$ , Table 6 presents the variation of the upper bound  $h(\mathcal{B}, \mathcal{A})/(\tau \mathcal{T}_s)$  with increasing length  $\tau$  of the control sequences.

<sup>7</sup> The factor  $\ln(2)$  appears due to the choice of the base-2 logarithm instead of the natural logarithm, which is typically used for continuous-time systems.

**Table 3**

Entropy estimates for Example 6.2 with  $\rho = 1, b = 1$  and different choices of the sampling time  $\mathcal{T}_s$ . Here,  $h_{\text{inv}}(Q) = 2.8854$ .

$\mathcal{T}_s$	$ \mathcal{A} $	$h(\mathcal{B}, \mathcal{A})/\mathcal{T}_s$	Time
0.8	11	4.0207	21.23 h
0.5	6	4.0847	2.98 h
0.1	2	4.744	3.33 min
0.01	2	5.1994	55 s
0.001	2	24.7	60 s

**Table 4**

Entropy estimates for Example 6.2 with  $\rho = 50, b = 10$  and different choices of the sampling time  $\mathcal{T}_s$ . Here,  $h_{\text{inv}}(Q) = 20.6058$ .

$\mathcal{T}_s$	$ \mathcal{A} $	$h(\mathcal{B}, \mathcal{A})/\mathcal{T}_s$	Time
0.11	15	28.5012	1.9 h
0.1	11	29.1723	1.35 h
0.01	2	34.4707	13 s
0.001	2	55.5067	12 s
0.0001	2	1.5635e+03	31 s

**Table 5**

Entropy estimates for Example 6.2 with different choices of `dtControl` parameters. Here,  $h_{\text{inv}}(Q) = 2.8854$ .

Classifier	Determinizer	$ \mathcal{A} $	$h(\mathcal{B}, \mathcal{A})/\mathcal{T}_s$	Time
cart	maxfreq	2	5.1994	55 s
logreg	maxfreq	2	5.1994	65 s
linsvm	maxfreq	2	5.1994	61 s
cart	minnorm	11	6.4475	57 s
logreg	minnorm	11	6.4475	74 s

**Table 6**

Upper bound  $h(\mathcal{B}, \mathcal{A})/(\tau \mathcal{T}_s)$  for Example 6.2 with control sequences of length  $\tau$ , Classifier = 'cart', and Determinizer = 'maxfreq' in `dtControl`. Here,  $h_{\text{inv}}(Q) = 2.8854$ .

$\tau$	$h(\mathcal{B}, \mathcal{A})/(\tau \mathcal{T}_s)$	Time
1	5.1994	57 s
2	5.0036	7.5 min
3	4.9547	1.91 h
4	4.9266	27.27 h

## 6.3. A 2d uniformly hyperbolic set

Consider the map

$$f(x, y) := (5 - 0.3y - x^2, x), \quad f : \mathbb{R}^2 \rightarrow \mathbb{R}^2,$$

from the Hénon family, one of the most-studied families of dynamical systems that exhibit chaotic behavior. We extend  $f$  to a control system with additive control:

$$\Sigma : \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 5 - 0.3y_t - x_t^2 + u_t \\ x_t + v_t \end{bmatrix}$$

with  $\max\{|u_t|, |v_t|\} \leq \varepsilon$ . It is known that  $f$  has a non-attracting uniformly hyperbolic set  $\Lambda$ , which is a topological horseshoe. This set is contained in the square centered at the origin with side length [36, Thm. 4.2]

$$r := 1.3 + \sqrt{(1.3)^2 + 20} \approx 5.9573.$$

If the size  $\varepsilon$  of the control range is chosen small enough, the set  $\Lambda$  is "blown up" to a compact controlled invariant set  $Q^\varepsilon$  with nonempty interior which is not much larger than  $\Lambda$ , see [37, Sec. 6]. Moreover, the theory suggests that as  $\varepsilon \downarrow 0$ ,  $h_{\text{inv}}(Q^\varepsilon)$  converges to the negative topological pressure of  $f|_\Lambda$  w.r.t. the negative unstable log-determinant on  $\Lambda$ ; see [38] for definitions. A numerical estimate for this quantity, obtained in [39, Table 2] via Ulam's method, is 0.696.

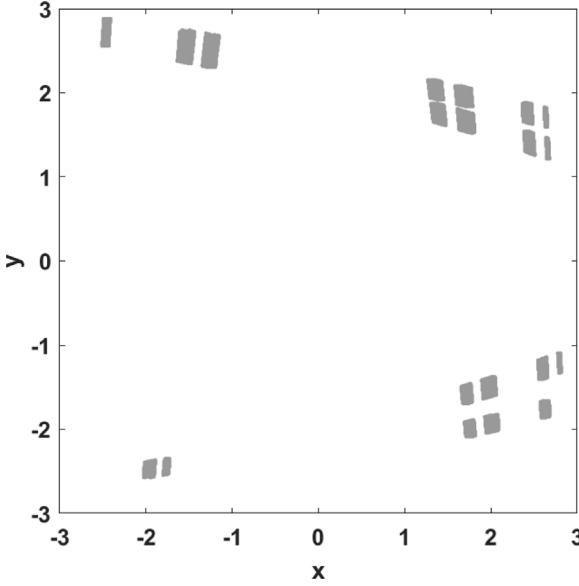
Fig. 4. The set  $Q$  for Section 6.3.

Table 7

Entropy estimates for Example 6.3 with different selections of dtControl options. Here,  $h_{\text{inv}}(Q) \approx 0.696$ .

Classifier	Determinizer	$ \mathcal{A} $	$h(\mathcal{B}, \mathcal{A})$	Time
cart	maxfreq	573	2.3884	0.95 min
linsvm	maxfreq	567	2.3956	1.82 min
logreg	maxfreq	454	2.3994	1.4 min
cart	minnorm	1921	2.9342	1 min
logreg	minnorm	1533	2.9215	2 min
linsvm	minnorm	1923	2.9376	2.15 min

We select  $\tilde{Q} = [-r/2, r/2]^2$ . For  $\varepsilon = 0.08$ , using SCOTS with parameter values  $\eta_s = [0.009, 0.009]^\top$  and  $\eta_i = [0.01, 0.01]^\top$ , through iteration, we obtain an all-time controlled invariant set  $Q \subset \tilde{Q}$ . In the iteration, we begin with the set  $\tilde{Q}$  and, as the first step, we compute an invariant controller for the system  $\Sigma$ . Let  $Q_1$  be the domain of the obtained controller. Consider the time-reversed system

$$\Sigma^- : \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} y_t - v_t \\ \frac{1}{0.3}(5 - y_t^2 + u_t - x_t) \end{bmatrix}.$$

In the second step, we compute an invariant controller for  $\Sigma^-$  in the set  $Q_1$ , and denote the controller domain by  $Q_2$ . In the third step, we compute an invariant controller for  $\Sigma$ , but in the set  $Q_2$ , and denote the controller domain by  $Q_3$ . The steps are repeated until  $Q_i = Q_{i+1} =: Q$ . In this way, we hope to approximate  $Q^\varepsilon$ .

Fig. 4 shows the set  $Q$ . For the parameter values  $\varepsilon = 0.08$ ,  $\eta_s = [0.009, 0.009]^\top$ ,  $\eta_i = [0.01, 0.01]^\top$ , Table 7 lists the values of  $h(\mathcal{B}, \mathcal{A})$  for different choices of the coarse partition  $\mathcal{A}$ . For the same values of  $\varepsilon$ ,  $\eta_s$  and  $\eta_i$ , the obtained value for the bound in Theorem 1 is  $w_m^*(\mathcal{G}) = 3.5646$  (with computation time 2.51 s).

#### 6.4. An uncertain linear system

We consider an uncertain linear control system

$$\Sigma : x_{t+1} \in \begin{bmatrix} 2 & 1 \\ -0.4 & 0.5 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_t + W$$

with  $x_t \in \mathbb{R}^2$ ,  $u_t \in U := [-1, 1]$ , and the disturbance set  $W := [-0.1, 0.1]^2$ . For a set  $Q \subseteq [-1, 1] \times [-2, 2]$ , we compute an upper and a lower bound of  $h_{\text{inv}}(Q)$ . We used SCOTS to obtain

Table 8

Entropy estimates for Example 6.4, with  $\eta_i = 0.05$ .

$\eta_s$	$w_m^*$	Time(sec)
0.03	6.4594	1.112
0.06	5	0.129
0.09	4.2811	0.051
0.1	4.3923	0.033
0.2	3.3219	0.027

an invariant controller for the state set  $[-1, 1] \times [-2, 2]$  with  $[0.2, 0.2]^\top$  and 0.05 as the state and input set grid parameters, respectively. The set  $Q$  is taken to be the domain of the obtained controller that consists of 109 state grid cells each of size  $0.2 \times 0.2$ . Fig. 5 shows the set  $Q$ .

**Computation of the lower bound:** We utilize [12, Thm. 7] to compute a lower bound. From [12, Rem. 2], we know that the lower bound in [12, Thm. 7] is invariant under coordinate transformations. After a similarity transformation which diagonalizes the dynamical matrix, we have

$$z_{t+1} = \begin{bmatrix} 1.6531 & 0 \\ 0 & 0.8469 \end{bmatrix} z_t + V^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + V^{-1} W,$$

where  $V = \begin{bmatrix} 0.9448 & -0.6552 \\ -0.3277 & 0.7555 \end{bmatrix}$ . For  $i = 1, 2$ , let  $\pi_i$  denote the canonical projection to the  $i$ th coordinate. Then,  $\pi_1 V^{-1} Q = [-2.1207, 2.1207]$ ,  $\pi_2 V^{-1} Q = [-3.4, 3.4]$ ,  $\pi_1 V^{-1} W = [-0.2827, 0.2827]$  and  $\pi_2 V^{-1} W = [-0.2550, 0.2550]$ . By [12, Thm. 7], one obtains

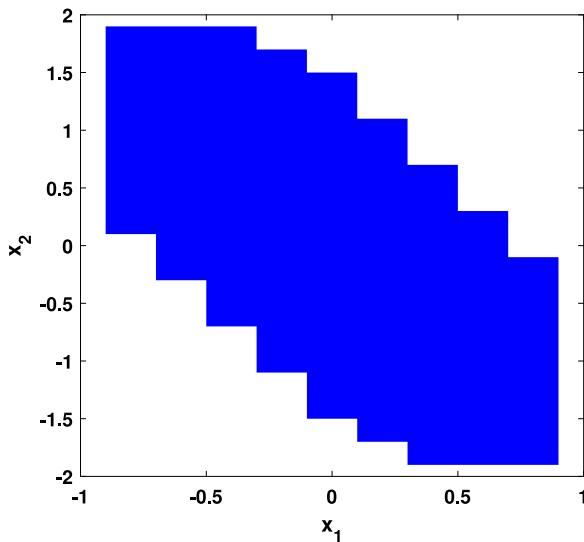
$$0.9316 \leq \bar{h}_{\text{inv}}(Q).$$

**Computation of the upper bound:** We construct an invariant partition  $(\mathcal{A}, G)$  of  $Q$  by selecting the set of grid cells in the domain of the controller obtained from SCOTS as the cover  $\mathcal{A}$ . Let  $C : \mathcal{A} \rightrightarrows U$  denote the controller from SCOTS. For  $A \in \mathcal{A}$ ,  $C(A)$  is the list of control inputs in the controller assigned to cell  $A$  such that each of the control inputs in the list ensures invariance of the states in  $A$  w.r.t. the set  $Q$ . For each  $A \in \mathcal{A}$ , we define  $G(A) := u \in C(A)$ , where  $u$  is chosen such that  $F(A, u)$  has nonempty intersection with a minimum number of elements of  $\mathcal{A}$ . If there are multiple such control values, then one of them is selected randomly. Using  $(\mathcal{A}, G)$  and the transition function  $F$  of the system, we construct a weighted directed graph  $\mathcal{G}$  as described in Section 4. We used the LEMON library to compute the maximum cycle mean weight for the graph  $\mathcal{G}$  and obtained  $w_m^*(\mathcal{G}) = 3.3219$  with computation time 0.027 s. Thus,  $\bar{h}_{\text{inv}}(Q) \leq 3.3219$ .

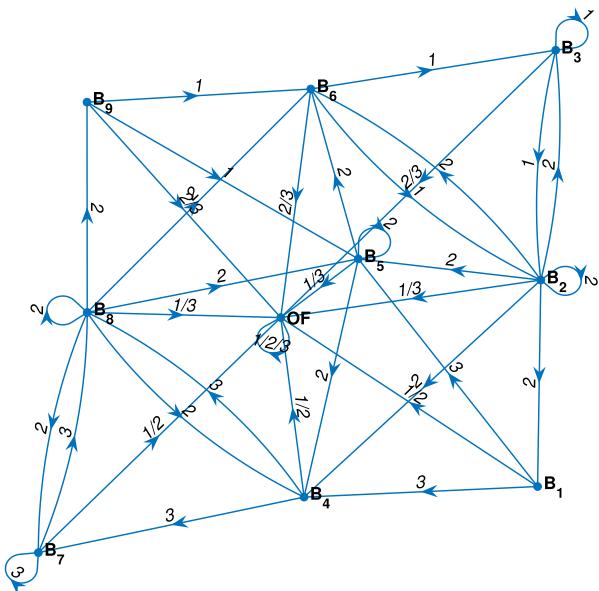
**Discussion on the selection of partition:** A better upper bound is expected when the number of outgoing edges, for every node in the graph, is made smaller. As a heuristic, gradually smaller values of the state grid parameter  $\eta_s$  can be tried. But very small  $\eta_s$  that make the width of the grid cells smaller than that of the disturbance set should be avoided, because in this case, the number of outgoing edges for any cell will begin to rise. This can also be observed in Table 8.

**Remark 2.** Please see the extended version of the paper [40, Sec. 7] for details on LEMON library, dtControl, pseudocode representation of the procedures for the computation of upper bounds and the complexity of the involved steps. The code is publicly accessible at [https://github.com/mahendrasinghtomar/Invariance\\_Entropy\\_upper\\_bounds](https://github.com/mahendrasinghtomar/Invariance_Entropy_upper_bounds).

**Remark 3.** To reduce the underlying computational complexity for the uncertain cases, one can leverage the proposed compositionality results in [20]. Particularly, one can compute an over-approximation of the invariance entropy for a large-scale



**Fig. 5.** The set  $Q$  in Section 6.4 which is defined as the domain of the invariant controller computed from SCOTS.



**Fig. 6.** Symbolic model for Example 1. Here edge labels refer to the indices of control inputs;  $u_1 = -1$ ,  $u_2 = 0$ , and  $u_3 = 1$ . The node  $OF$  is overflow and it denotes that the state may have gone outside the set  $Q$ .

interconnected system in a divide and conquer manner by computing over-approximations of entropies for subsystems individually using the method proposed here and then composing them to compute an over-approximation of the entropy for the overall system. In this case, the complexity breaks down to the level of subsystems.

In the next section we briefly describe the computation procedure for invariance controllers as implemented inside the tool SCOTS. Further, we elaborate on the design and operation of coder-controllers based on invariant partitions.

## 7. Controllers for invariance

The invariant partitions utilized in this work are derived from controllers for invariance computed via SCOTS. Given a system

$\Sigma_1$  and a set of discretization parameters, SCOTS first puts rectangular grids on both the state and the control input sets. For every pair of cells in the state and the input grid, an over-approximation of the reachable set is computed which is then utilized to build a discrete abstraction (a.k.a. symbolic model)  $\Sigma_2$  which is related via feedback refinement relation [41] to  $\Sigma_1$ ; see [41] for more details. The grid cells from the state set serve as the nodes while the grid cells from the input set serve as the edge labels in the symbolic model. The outgoing edges for any node are determined by the nonempty intersection of the over-approximations of reachable sets with the state grid. For Example 1, the symbolic model constructed in SCOTS is shown in Fig. 6. Next a maximal fixed-point problem is solved for the symbolic model to obtain a controller for invariance. Since  $\Sigma_1$  and  $\Sigma_2$  are related under feedback refinement relation, the controller designed for the symbolic model is applicable for the original system through the set membership relation; see [8,41,42] for more details.

The obtained controller is a set-valued map that assigns to each state of the invariant set  $Q$  a set of control values. The deterministic version (i.e., single control value assigned to each state) of this controller is an invariant partition. As noted above Theorem 1, an invariant partition  $(\mathcal{A}, G)$  immediately provides a coder-controller scheme to render  $Q$  invariant. Now, we briefly describe the operation of such a static coder-controller [12, Section VI.B]. At any time  $t$ , the coder identifies the partition element  $A_t$  that contains the current state and sends  $\lceil \log_2 |\mathcal{A}| \rceil$  bits information to the controller. This number of bits is sufficient enough for the controller to identify the state containing partition  $A_t$  out of  $|\mathcal{A}|$  possibilities; and it then applies the control input  $G(A_t)$  to the plant ensuring that the successor state lies within the set  $Q$ . The average data rate for such a coder-controller is  $\lceil \log_2 |\mathcal{A}| \rceil$  bits/sec.

Note that an alternative coder-controller scheme, in which instead of  $\lceil \log_2 |\mathcal{A}| \rceil$  bits only  $\log_2(\text{number of successor partition elements})$  is transferred for all time  $t \in \mathbb{Z}_+$  except at  $t = 0$ , can help to reduce the average bit requirement. For this, given an invariant partition, we can construct a directed graph  $\mathcal{G}$  as described in Section 4 but with edge weights  $w(A) = \lceil \log_2 |D(A_{t-1})| \rceil$  instead of (9). We assume this graph is known to both the coder and the controller. At time  $t = 0$ , the coder sends  $\lceil \log_2 |\mathcal{A}| \rceil$  bits so that the controller can identify the partition element  $A_0$  containing the current state  $x_0$ . While at all other times  $t > 0$ , the coder transmits  $\lceil \log_2 |D(A_{t-1})| \rceil$  bits to identify the partition element  $A_t \in D(A_{t-1})$  which contains the state  $x_t$ ; here  $D(A_{t-1})$  is defined in (9). The average data rate for such a coder-controller equals the maximum cycle mean of the graph  $\mathcal{G}$ , and it is not larger than  $\lceil \log_2 |\mathcal{A}| \rceil$ .

## 8. Conclusion and future work

All the computations in this work were performed on a computer with an Intel Core i5-8250U processor and 8 GB RAM. The 'time' listed in all the tables are the computation times for estimating the entropy.

Our first contribution is the combination of three different algorithms designed for different purposes to numerically compute an upper bound of the invariance entropy of deterministic control systems. The second contribution is a procedure to numerically compute an upper bound for the invariance entropy of uncertain control systems. We also describe the relationship between the two upper bounds and thus the need for the second bound. Finally, we illustrate the effectiveness of the proposed procedures on four examples. Open questions for future work include the selection of entropy-minimizing partitions and the computation of lower bounds of IE for uncertain nonlinear systems.

## CRediT authorship contribution statement

**M.S. Tomar:** Methodology, Software, Writing – original draft, Investigation. **C. Kawan:** Conceptualization, Methodology, Software, Supervision, Writing – original draft. **M. Zamani:** Funding acquisition, Supervision, Writing – review & editing.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

The link for code is available.

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