Chunking Tasks for Present-Biased Agents

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Everyone puts things off sometimes. How can we combat this tendency to procrastinate? A well-known technique used by instructors is to break up a large project into more manageable chunks. But how should this be done best? Here we study the process of chunking using the graph-theoretic model of present bias introduced by Kleinberg and Oren [2014]. We first analyze how to optimally chunk single edges within a task graph, given a limited number of chunks. We show that for edges on the shortest path, the optimal chunking makes initial chunks easy and later chunks progressively harder. For edges not on the shortest path, optimal chunking is significantly more complex, but we provide an efficient algorithm that chunks the edge optimally. We then use our optimal edge-chunking algorithm to optimally chunk task graphs. We show that with a linear number of chunks on each edge, the biased agent's cost can be exponentially lowered, to within a constant factor of the true cheapest path. Finally, we extend our model to the case where a task designer must chunk a graph for multiple types of agents simultaneously. The problem grows significantly more complex with even two types of agents, but we provide optimal graph chunking algorithms for two types. Our work highlights the efficacy of chunking as a means to combat present bias.

CCS Concepts: • **Theory of computation → Algorithmic mechanism design**; • **Applied computing → Economics**; **Psychology**.

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1 INTRODUCTION

Everyone puts things off sometimes. How can we combat this tendency to procrastinate? A well-known technique used by instructors is to break up a large project into more manageable chunks. But how should this be done best? Here we study the process of chunking using the graph-theoretic model of *present bias* introduced by Kleinberg and Oren [2014]. One of our main results confirms the intuition long held by teachers: in many cases, the best way to chunk a single task involves making the initial subtasks easy and then getting progressively harder. We also provide algorithms that can best "distribute" chunks across many tasks, which could be applied in an automated to-do list chunking app.

Present bias is the tendency of agents to overweight costs and rewards experienced in the current time period, which helps explain many irrational behaviors, from procrastination to task abandonment. Kleinberg and Oren [2014] had the crucial insight that this diverse behavior could be captured in a single graph-theoretic model. They represent tasks using a directed, acyclic graph G, with designated start s and end t. A path through this graph corresponds to a plan to complete the task; each edge represents one step of this plan. The weights on edges represent the costs of completing that step. While the model is simple, it is deceptively complex to analyze; it has been

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a popular starting point for present bias in the CS community (see, e.g., [Albers and Kraft 2017; Anagnostopoulos et al. 2020; Fomin et al. 2020; Gravin et al. 2016; Ma et al. 2019; Oren and Soker 2019]).

The goal of an agent is to complete the task while incurring the least cost. An optimal (unbiased) agent simply computes the shortest path and takes it. A *naive* present-biased agent with bias parameter $b > 1$ behaves as follows. At s, they compute their perceived cost of each path to t by scaling up the cost of the first edge on each path by b. Then they take one step along this path, say to vertex u, and then *recompute* their perceived costs, this time by scaling up the costs on the edges out of u. Notice that the agent may plan to take some path P at s, but then deviate from their plan after one step. This is because they (naively) do not take the future impact of their present bias into account when planning; see Figure 1 for an example.

We extend the Kleinberg-Oren model by giving a task designer the power to break up an edge into chunks. The agent completes the chunks one at a time, which reduces the impact of their present bias. We consider the chunks to be a mental feature – the designer does not actually check that the agent completes the task in chunks, but instead suggests a chunking to the agent. Our model is a good fit for many, but not necessarily all tasks. We now highlight three families of applications and consider the extent to which our results apply to them.

The first family of applications are personal tasks, such as in the example given by George Akerlof of repeatedly putting off an errand until the next day [Akerlof 1991]. In these examples, we believe that chunking can be an effective tool. Breaking even a simple task like "mailing books" down into smaller components like "gather the books", "package the books", and "drive to the post office" seems like a typical way to convince oneself to do an errand. However, there is no real task designer here. Further, our results assume a known bias, but agents in our model are not fully aware that they have present bias. Thus, personal tasks are not the main application we consider (though our overall takeaway that chunking is valuable still applies to these tasks).

Next we consider educational examples, where students procrastinate on course work (while not planning with this in mind). Our model applies well here, as the task designer (the instructor) really does have a vested interest in ensuring that students complete the course, and do so as efficiently as possible. As mentioned before, we do not model the teacher as actively enforcing the chunks, for example with grades or deadlines. Our model is better understood as the teacher suggesting chunks to the students. We discuss further at the end of Section 2.

Finally, another application with great potential is to automatically chunk to-do lists. Consider an app that automatically takes in a user's to-do list, which could have multiple dependencies, and

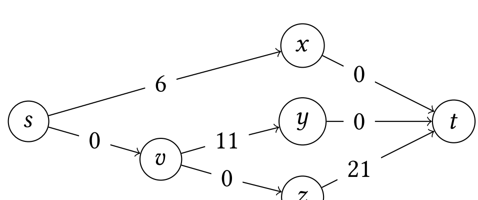


Fig. 1. Taken from Saraf et al. [2020]. The cheapest path is (s, x, t) with total cost 6. However, an agent with bias $b = 2$ will take path (s, v, z, t), with cost 21. Importantly, when the agent is deciding which vertex to move to from s, they evaluate the path starting with x as having total cost 12, while the path starting with v has total cost 11. This is because they assume they will behave optimally at v by taking path (v, y, t). However, they apply their bias at v and deviate to the most expensive path.

suggests ways to chunk some tasks. To avoid overwhelming the user, the app would not want to suggest too many chunks. [1]

We are not the first to consider ways of alleviating the harm caused by present bias (which can be quite significant—as shown by Kleinberg and Oren [2014] and Tang et al. [2017], the ratio of the optimal agent's cost to the biased agent's cost can be exponential in the size of the graph). Kleinberg and Oren [2014] propose a model where a reward is given after finishing the task, and where the agent will abandon the task if at any point, they perceive the remaining cost to be higher than the reward. Unlike an optimal agent, a biased agent may abandon a task partway through; see Figure 2 for an example. As a result, Kleinberg and Oren give the task designer the power to arbitrarily delete vertices and edges, which can model deadlines. They then investigate the structure of *minimally motivating subgraphs*, the smallest subgraph where the agent completes the task, for some fixed reward. Follow-up work of Tang et al. [2017] shows that finding *any* motivating subgraph is NP-hard. Instead of deleting edges, Albers and Kraft [2019] consider the problem of spreading a fixed reward onto arbitrary vertices to motivate an agent to complete a task, and find that this too is NP-hard (with a constrained budget).

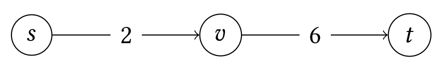


Fig. 2. Let (s, v) represent buying a gym membership and (v, t) represent working out regularly for a month [Roughgarden 2016]. At t, the agent receives a reward of 11 due to health benefits. With bias $b = 2$, the agent initially believes this task is worth completing, but due to his bias, abandons the task at vertex v, after having already purchased the membership.

These results focus on the problem of convincing an agent to complete a task, rather than redirecting agents to cheaper paths. Though these goals are related, it's natural to wonder how we might sway agents towards more optimal behavior, rather than merely settling for task completion. In other words, even if agents are willing to complete a task using a needlessly expensive path (perhaps because of a large reward), we should still consider how to make them behave more optimally. Kleinberg et al. [2016] partially investigate this question in a model involving *sophisticated* agents, who plan around their present bias. They consider several types of *commitment devices* – tools by which sophisticated agents can constrain their future selves. However, these tools may require more powerful agents or designers, and don't necessarily make sense for naive agents. Saraf et al. [2020] takes a different approach, arguing that task designers can induce optimal behavior by setting up a competition between biased agents. While they obtain strong results for several families of graphs, there are also graphs where their competitive model can offer no benefit to agents.

Finally, Kleinberg and Oren [2014] consider a restricted version of our chunking problem, which is close to a special case of our model. They focus on the single edge graph (s, t), and derive the optimal chunking in that setting. When considering general graphs, we obtain a similar result when chunking edges on the cheapest path; for other edges, the optimal chunking is more complex. Further, looking at general graphs allows us to ask how a fixed chunking budget should be best allocated across multiple edges, and, more broadly, how to convince agents to take a different (and cheaper) path.

The rest of the paper is organized as follows. In Section 2, we present a model for chunking and explain its simplifying features. In Section 3, we focus on chunking single edges within a graph.

[1] Interestingly, it seems that Google's acquisition of the startup Timeful has led to users of Gmail getting various "nudge" reminders, where the nudges chosen are based in part on research on present bias [J. Kleinberg, private communication, 2022].

We first describe how chunking an edge (u, v) can be thought of as lowering the agent's present bias towards only that edge. We then explore the structure of optimal edge chunkings, that is, chunkings that lower the agent's "selective bias" as much as possible. For edges on the shortest path, we provide a closed form for the optimal chunking. For other edges, optimal chunkings are considerably more complex, but we provide an efficient algorithm to compute them. In Section 4, we provide an algorithm to optimally distribute a fixed number of chunks across multiple edges within a graph. In Section 5, we provide a tight bound on the cost ratio for biased agents in terms of the number of chunks alloted to the task designer. Our bound implies that with a linear number of chunks alloted to each edge, the cost ratio can be reduced to a constant factor. Finally, in Section 6, we consider the problem of chunking a single task graph for two types of agents simultaneously, where an agent's type is their bias. As an example, consider an instructor who wants a good chunking for both rare and frequent procrastinators. We provide algorithms to chunk optimally under local and global budgets for two types of agents. We also show how to extend our result to m types of agents, if we add the (simplifying) constraint that all agents must take the same path through the graph.

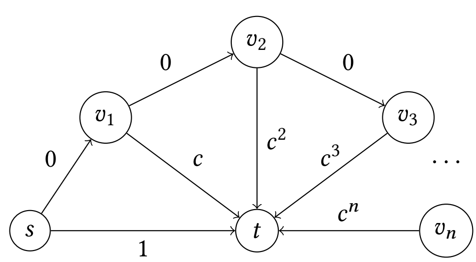
2 CHUNKING MODEL

We first explain the model of present bias in more detail. As mentioned before, we start with a weighted, directed, acyclic graph G that represents a task, with start s and end t. A present-biased agent with bias parameter b behaves as follows. Let $c(v \longrightarrow t)$ represent the cost of the shortest path from v to t, and let $c(u, v)$ represent the weight of edge (u, v). From node u, the agent goes to vertex $\operatorname{argmin}_{v:(u,v)\in E} bc(u, v) + c(v \longrightarrow t)$. We refer to $bc(u, v) + c(v \longrightarrow t)$ as the agent's *perceived cost of starting with edge* (u, v) *and then taking the shortest path to* t. We abbreviate this as *the agent's perceived cost of starting with* (u, v). At each vertex, they go to the neighbor that minimizes their perceived cost, continuing until they reach t.

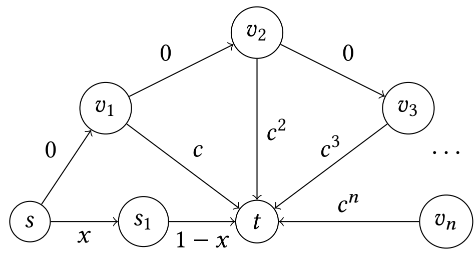
We next consider chunking. We distinguish two different settings where chunking helps:

(1) The task designer wants agents to take the cheapest path through the graph, rather than the more expensive path their bias would lead them to take.
(2) In a model where agents can abandon their path at any time (if the perceived cost is less than the reward), the task designer wants to prevent such abandonment.

We mainly focus on the first case in this paper, but our analysis easily extends to the abandonment setting. To investigate different models of chunking, consider the following graph, the n-fan (in which a biased agent can take an exponentially more expensive path than optimal [Kleinberg and Oren 2014]):

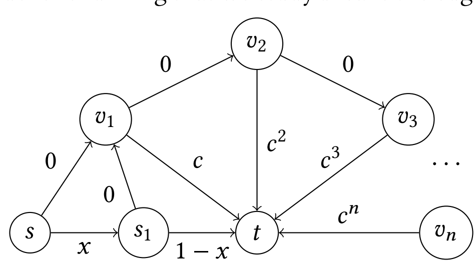


The task designer wants the agent to take the path (s, t) instead of the longer path around the fan that an agent will take when their bias $b > c$. The simplest model of chunking allows them to break the edge (s, t) into pieces as follows:



The designer gets to choose x (i.e., they get to choose how much work is done in the first and second chunk). Note that the intermediate node s_1 doesn't have any connections, except to t. It's easy to show that the best choice of x is 0, as this means that the agent's present bias does not play any role in their decision (all edges out of s have 0 cost). From a different perspective, this model seems to be taking advantage of the "lock-in" effect of s_1 – once the agent goes there, they cannot take an alternative path, even though they did not actually do any work to get there. But our intuition suggests that chunking a very difficult task into a cost 0 "task" followed by the same difficult task should not help much. So, this doesn't seem to be a good model for chunking. As an aside, even if we require that x is not too small, the obvious solution for the task designer is to make the first chunk as small as allowed – there's not much interesting in this model.[2]

The more interesting model of chunking that we study breaks the edge (s, t) into pieces as follows:



Here, the node s_1 keeps all the edges to other task nodes that s had. This reflects the fact that even after completing a chunk, an agent may decide to take another path to t – completing a chunk doesn't "lock" an agent into a particular path. Of course, they will be less likely to take another path if they finished a particularly difficult chunk. Thus the model has the necessary tension – the designer wants to set x high enough so that the agent actually still takes the (s, s_1, t) path, but not so high that they don't take edge (s, s_1) in the first place. Put another way, since the agent can deviate at s_1, the designer wants to ensure that the perceived costs of starting with (s, s_1) and with (s_1, t) are both low. While we have shown only 2-chunk examples, in our general model the task designer splits an edge into k chunks, whose costs sum to the original cost.[3]

It is worth discussing three simplifying features of our model. First, we assume that tasks can be arbitrarily split: each edge in the chunking can have any cost, so long as the total cost remains fixed. A more realistic model might constrain edges to have fixed chunking options. For example, when chunking an essay, it could be the case that each chunk must consist of some number of

[2]If we move to the abandonment setting, the task designer *is* incentivized to do a non-trivial split here; they would want to balance the perceived costs of starting with edges in their chunking in order to avoid abandonment. However, the model we investigate induces a similar balancing problem even without abandonment (and extends naturally to the abandonment setting).

[3]Note that in our formalization, we remove the original edge for simplicity. However, if we kept the original edge, the agent would never strictly prefer it, no matter what the chunking. So it's mathematically equivalent to think of the original edge still being there. Moreover, this interpretation maps better to our examples, where the task designer does not actually enforce the chunking.

paragraphs; essays cannot be chunked more finely. However, we believe that solving our continuous relaxation will provide reasonable insight into the discrete problem. Our informal argument is as follows: if the number of potential chunks in the discrete problem is high, then our optimal solution to the continuous version will be a good approximation. If the number of potential chunks is low, then solving the discrete problem is easy (there aren't many possible chunkings). Though we will not consider the discrete version further, it would be interesting to understand if there are fundamentally different challenges in that setting.

The second simplifying assumption is that the chunking "overhead" cost to the agent is zero. In other words, no matter how many chunks an edge is split into, the total cost of that edge remains fixed (notably, it does not increase). In reality, there is probably some cost to the agent per chunk. For instance, the agent might stop working between chunks, and then have some cost associated with getting back to work. We assume that this "restarting" cost is very low relative to the other costs, and thus ignore it. In any case, since each chunk gives the task designer (weakly) more power in our model, we typically assume that there is some given chunking budget k; if chunking instead had some fixed overhead, there would exist an optimal k, as additional chunks have diminishing returns but fixed overhead.

Lastly, we specify how agents break ties. If an agent at u views multiple neighbors as having the same perceived cost, the agent will pick the neighbor that is part of a chunked path if exactly one neighbor is part of a chunked path. Otherwise, they pick the first vertex in some lexicographical ordering. This tie-breaking behavior is mathematically convenient when constructing the optimal chunking, as we can simply ensure that the perceived cost starting with each step in the chunking matches the agent's otherwise best option. For a more thorough treatment of tie-breaking rules in the base model of present bias, see [Dementiev et al. 2021].

We also contrast our model with a model of "checkpoints". As we mentioned, we consider chunking to be a purely mental tool to combat present bias. One might consider a stronger model, where the task designer (e.g., an instructor) can incentivize agents to complete a task in chunks. For example, the instructor might set an earlier (graded) deadline for the thesis statement of an essay. We can model this as the task designer having the power to split up the final reward r onto intermediate vertices or edges, in addition to being able to chunk edges. Although we will not investigate this checkpoint model in this paper, we hope to investigate it in future work. While both the chunking model and checkpoint model are realistic choices to model classwork, we believe that the chunking model is a better fit for algorithmically chunking a user's to-do list; in that setting, the algorithm cannot enforce the chunks, but merely suggests them to the user.

3 OPTIMAL EDGE-CHUNKING

In this section, we consider how to optimally chunk a single edge. What do we mean by an *optimal* chunking? As mentioned earlier, we think of chunking as lowering an agent's selective bias towards the chunked edge. In other words, for any chunking, an agent with bias b will take the chunked path from u to v if and only if an agent with bias $b' < b$ towards edge (u, v) (and bias b otherwise) will take (u, v) in the original graph. We say that such a chunking *induces a selective bias of* b' *towards* (u, v).[4] So, by an *optimal* chunking, we mean one in which the agent's selective bias is brought as low as possible (given a fixed bound k on the total number of chunks).

Our results show that as the number of chunks tends to infinity, the selective bias tends to 1 (i.e., unbiased behavior). Thus, the number of chunks is a powerful parameter in our model; in the next section we answer the broader question of how to best chunk is an arbitrary task graph with a limited chunking budget.

[4]When it is clear from context, we often leave the edge unspecified.

3.1 Edges on the shortest path

The problem of optimally chunking is subtly different for edges on the shortest path (where "shortest" ignores bias) and edges on other paths. We first consider the simpler case of edges on the shortest path, and start with two chunks.

Lemma 1. *To optimally split an edge (u, v) that is on the shortest path into two chunks, the first chunk should be a $\frac{b-1}{2b-1}$ fraction of the work. With this split, the agent will behave with a selective bias of $\frac{b}{2-1/b}$.*

Proof. Suppose we chunk (u, v) into (u_1, u_2, v). First, note that, because (u, v) is on the shortest path in the original graph, no matter how the edge is chunked, the optimal behavior from u_2 will be to go to v – this can only be cheaper than (u, v) in the original graph. Thus, the perceived cost of starting with edge (u_1, u_2) while at vertex u_1 is $bc(u_1, u_2) + c(u_2, v) + c(v \longrightarrow t)$, as the agent naively believes they will behave optimally in the future. This is the only way that we use the fact that (u, v) is on the shortest path.

The designer wants to minimize the maximum of the perceived cost of starting with (u_1, u_2) and the perceived cost of starting with (u_2, v), to best ensure that the agent takes the chunked path. These perceived costs are $bc(u_1, u_2) + c(u_2, v) + c(v \longrightarrow t)$ and $bc(u_2, v) + c(v \longrightarrow t)$ respectively.

Let $x = c(u, v)$ represent the total amount of work to be chunked, and let x_1 and x_2 represent $c(u_1, u_2)$ and $c(u_2, v)$ respectively. Note that $x_2 = x - x_1$. We now plug the x's into the expressions above to get perceived costs of

$$bx_1 + x - x_1 + c(v \longrightarrow t) \text{ and}$$
$$b(x - x_1) + c(v \longrightarrow t).$$

We want to set x_1 to minimize the maximum of the two quantities. That is, we choose x so that

$$\begin{aligned}
&\operatorname*{argmin}_{0 \le x_1 \le x} \max(bx_1 + x - x_1 + c(v \longrightarrow t), b(x - x_1) + c(v \longrightarrow t)) \\
&= \operatorname*{argmin}_{0 \le x_1 \le x} \max(bx_1 + x - x_1, b(x - x_1)) \\
&= \operatorname*{argmin}_{0 \le x_1 \le x} \max((b - 1)x_1 + x, -bx_1 + bx).
\end{aligned}$$

Both expressions are linear functions of x_1, with the first increasing and the second decreasing. The minimum of the maximum is thus where they intersect, that is, when

$$(b - 1)x_1 + x = -bx_1 + bx.$$

Simple algebra then shows that

$$x_1 = \frac{b - 1}{2b - 1}x.$$

With this value of x_1, the perceived costs starting with (u_1, u_2) and with (u_2, v) are identical. The latter perceived cost is

$$\begin{aligned}
b(x - x_1) + c(v \longrightarrow t) &= bx \cdot \frac{b}{2b - 1} + c(v \longrightarrow t) \\
&= \frac{b}{2 - 1/b} \cdot c(u, v) + c(v \longrightarrow t). && \text{(since } x = c(u, v))
\end{aligned}$$

(It's easy to verify that the former perceived cost matches.) Thus, the agent with bias b takes the path (u_1, u_2, v) when an agent with bias $b^* = \frac{b}{2-1/b}$ would have taken (u, v) in the original graph. □

We now state the following theorem, which extends the above results to k chunks. We first state a more general version which will be helpful in the next section. The proof is in the appendix.

Theorem 1. *Suppose we partition an edge (u, v) of cost x into k chunks. Let $u_1, \ldots, u_k$ represent the vertices in this chunking, and let $c(u_i, u_{i+1}) = x_i$, where, for $1 \leq i \leq k$, the x_i's are defined below.*

$$1 \leq i \leq k : x_i = \frac{(b-1)^{k-i} b^{i-1}}{b^k - (b-1)^k} x.$$

With this chunking, the agent has selective bias $\frac{1}{1-\left(\frac{b-1}{b}\right)^k}$. If, with this chunking, the shortest path from u_i to t is through u_{i+1} for all $i > 1$, then this chunking is optimal.

The following corollary immediately follows from this theorem.

Corollary 1. *For an edge (u, v) on the shortest path, the chunking given in Theorem 1 is optimal.*

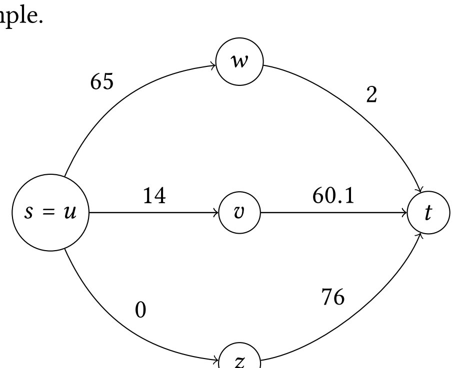
Proof. No matter how an edge on the shortest path is chunked, the shortest path from any chunk to t must be through the next chunk, as chunking does not increase the total cost of the edge. This satisfies the condition in the theorem to get optimality. □

The corollary says that the designer is not best served by evenly splitting the cost between the edges – the designer should lower the cost of earlier edges. When they do so, the agent will behave as if they had selective bias $\frac{1}{1-\left(\frac{b-1}{b}\right)^k}$ in the original graph towards edge (u, v) (while having bias b towards all other edges).

For a simple application of this corollary, suppose the agent's bias is 2. Then, splitting each edge on the shortest path once (so $k = 1$) causes the agent to behave as if they have bias $4/3$ on the shortest path in the unmodified graph (and they still perceive other edges with bias 2).

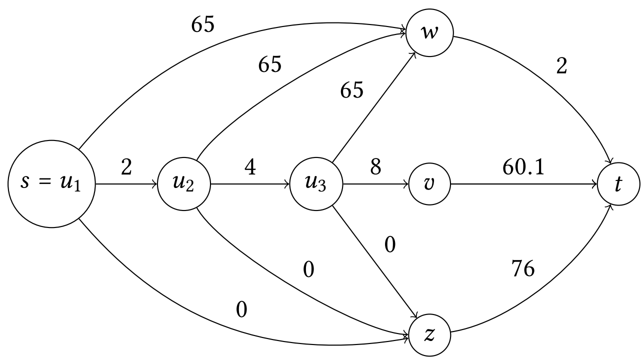
3.2 Edges not on the shortest path; a motivating example

We first motivate our results. For edges that are on the shortest path, it's clear why a designer would want to chunk them – they want to convince agents to incur as little cost as possible. However, in the next section we consider the natural problem where the designer has a fixed chunking budget k. In such cases, our earlier results imply that if the agent's bias is sufficiently high, it may not be possible to convince them to stick to the shortest path. However, the designer may be able to lower the agent's cost by chunking other edges, which are not on the shortest path. Consider the following graph as an example.



Suppose that the agent has bias 2. Let P_w, P_v, P_z represent the paths to t through w, v, or z respectively. The agent's bias causes them to take P_z, the most expensive path. How should we best use a fixed budget of 3 chunks to lower the agent's cost? First, note that by Theorem 1, the optimal chunking of (u, w) induces a selective bias of $8/7$. Even with this optimal chunking, the agent would still prefer P_z, as $8/7 \cdot 65 + 2 > 76$. So, we cannot lower the agent's cost by chunking (u, w). Will chunking (u, v) instead help?

Note that, for edges not on the shortest path (which we will sometimes abbreviate to "non short-path edges"), we could still apply the chunking from Theorem 1 to get the selective bias described in that theorem. For (u, v), Theorem 1 tells us to set $x_1 = 2, x_2 = 4$, and $x_3 = 8$, resulting in the following graph.



Under this chunking, the cheapest path from u_1, u_2, or u_3 to t all go through w. The agent's perceived costs of starting with the edges in the chunking are, in order, $71, 75$, and 76.1 (so the agent would take edge (u_3, z) instead of sticking to the chunking). If (u, v) was a shortest edge in the original graph (for example, if w did not exist), then the same chunking would have identical perceived costs of 76.1 starting with all edges. But when the cheapest path from a chunked vertex to t is through the external vertex w, the perceived cost of starting with early edges decreases. An optimal chunking should thus increase the cost of early edges and decrease the cost of later edges to result in more balanced perceived costs. In the example above, if we split the costs so that $c(u_1, u_2) = c(u_2, u_3) = 3.55$ and $c(u_3, v) = 6.9$, then the cheapest path from u_1 or u_2 to t is through w, while the cheapest path from u_3 to t is through v. Thus, the perceived costs of starting with the first edge and the second edge are both 74.1, and the perceived cost of starting with (u_3, v) is 73.9. This is the optimal chunking of (u, v), and it improves the agent's cost by convincing them to take P_v instead of P_z. Thus, this example shows that we have good reason to chunk non short-path edges, and our existing chunking results are insufficient for such edges.

3.3 Optimally chunking for edges not on the shortest path

As the example in the previous section suggests, it's important to keep track of the shortest path from chunking vertices to t. Note that if the shortest path from u_i to t is through w rather than u_{i+1}, then the shortest path from any u_j to t, where $j < i$, is also through w.

Thus, for any chunking, define u_τ as the *transition vertex*: the last vertex where the shortest path is through w, where w is the next vertex on the shortest path from u to t in the original graph. If the shortest path always follows the chunking, then define τ as 0. On the other hand, if the shortest path is always through external vertices, then $\tau = k$. For a shortest-path edge, all chunkings have $\tau = 0$ (and thus the optimal chunking is given by Theorem 1). But for non short-path edges, the optimal chunking may have a higher value of τ (in the previous example, the optimal chunking had transition vertex $\tau = 2$). Though the case where $\tau = 0$ admits a nice closed form, in general we provide an algorithm that determines the optimal chunking by trying all possible values of τ.

We can think of τ as the smallest value such that, for all neighbors w of u, we have $c(u, w) + c(w \longrightarrow t) \geq c(u_{\tau+1}, u_{\tau+2}) + c(u_{\tau+2} \longrightarrow t)$. We can rewrite this as follows, using the notation of Theorem 1:

$$\begin{aligned} c(u, w) + c(w \longrightarrow t) &\geq c(u_{\tau+1}, u_{\tau+2}) + c(u_{\tau+2} \longrightarrow t) \\ &= x_{\tau+1} + \textstyle\sum_{i=\tau+2}^{k} x_k + c(v \longrightarrow t) \\ &= x - \textstyle\sum_{i=1}^{\tau} x_i + c(v \longrightarrow t). \end{aligned} \tag{1}$$

Let $\delta = x + c(v \longrightarrow t) - (c(u,w) + c(w \longrightarrow t))$ represent the difference between the cost of the cheapest path from u to t through v and the cost of the cheapest path from u through w in the original graph (in the previous example, $\delta = 74.1 - 67 = 7.1$). Then Equation 1 is equivalent to $\sum_{i=1}^{\tau} x_i \geq \delta$. For an edge on the shortest path, δ is negative, which is why τ must be equal to 0 for those edges. Moreover, if $\delta \leq x$, then it is possible to split the costs among the edges to allow any choice of τ: we simply put at least δ of the cost on the first τ edges while ensuring that the sum of costs of the first $\tau - 1$ edges does not exceed δ. So, in addition to requiring that $\sum_{i=1}^{\tau} x_i \geq \delta$, we also need $\sum_{i=1}^{\tau-1} x_i < \delta$.

Before we get to our main result, we first introduce some more definitions and notation. Let $e_i = (u_i, u_{i+1})$ be the ith edge of a chunking, and let $p(e_i) = bx_i + c(u_{i+1} \longrightarrow t)$ represent the perceived cost of starting with edge e_i. Let the *bottleneck* of a chunking be the highest perceived cost starting with any edge on that chunking (i.e. $\max_i p(e_i)$). It's easy to see that the bottleneck of a chunking determines the selective bias the chunking will induce; any agent who will get past the bottleneck will complete the entire chunked path. So an optimal chunking is a chunking with the smallest bottleneck. Finally, let a k-chunking of an edge be any chunking that splits the edge into k chunks.

We now state some useful lemmas; their proofs can be found in the appendix.

Lemma 2. *Suppose that C is a chunking with bottleneck β. If another chunking O has bottleneck $\beta' < \beta$ and the same transition vertex τ, then O must lower the cost of all edges that are bottlenecks in C, and thus raise the cost of the remaining edges.*

Though the lemma seems obvious at first glance, it relies crucially on the fact that C and O have the same transition vertex τ. It's possible for O to not lower the cost of all edges that are bottlenecks in C but still get a lower bottleneck cost if O has a different transition point. But with τ fixed, the difference between the perceived costs starting with any edge in C compared to O depends only on the cost the chunkings assign to the edge.

Lemma 3. *If a chunking C has the same perceived cost starting with any edge in the chunking, then C is optimal.*

Lemma 3 guides the algorithm, which tries to ensure that the perceived costs starting with edges in C are as close as possible. At a high level, the algorithm enumerates over all values of $\tau \in \{1, \ldots, k\}$. We start with a chunking where the first τ edges are assigned cost δ/τ, which ensures that they all have the same perceived cost α. We then use Theorem 1 to distribute the remaining cost over the last $k - \tau$ edges, which also equalizes their perceived cost to some β. If $\alpha \geq \beta$, we argue that this chunking is optimal for the fixed τ. Otherwise, we make some local updates to the chunking, which brings β as close to α as possible while maintaining the invariant that $\beta \geq \alpha$. The full description of this algorithm, Algorithm 1, can be found in the appendix.

Theorem 2. *Given any edge (u, v), we can determine the optimal k-chunking in $O(k)$ time, assuming that the shortest paths from $u \longrightarrow t$ and $v \longrightarrow t$ have been precomputed.*

Proof Sketch. For a fixed τ, we start by setting $x_1 = x_2 = \cdots = x_\tau = \delta/\tau$, and chunk the remaining $x - \delta$ cost over the remaining $k - \tau$ edges according to Theorem 1. Doing so ensures that $p(e_i) = \alpha$ for all $i \leq \tau$ and that $p(e_i) = \beta$ for all $i > \tau$ (α and β are defined in the appendix). If $\alpha = \beta$, by Lemma 3 we're done. In the case where $\alpha > \beta$, we show that we're done for this fixed τ.

The case where $\beta > \alpha$ is the bulk of the proof. The key is that $p(e_\tau)$ can be grouped into *either* the earlier or later edges. Since $\beta > \alpha$, we carefully increase the cost of the first $\tau - 1$ edges and decrease the cost of the later edges to produce the optimal chunking for this value of τ. □

4 OPTIMAL CHUNKING IN TASK GRAPHS

In the previous section, we focused on optimally chunking a single edge. One reason *why* a task designer might want to do that is to convince agents to take much cheaper paths through the graph, by chunking the right edges. In this section, we assume that the designer can chunk any edge in the graph, but can place only a limited number of chunks (their chunking "budget"). Which edges should they chunk to ensure that the present-biased agent takes as cheap a path as possible, and how should they chunk those edges?

We first answer the latter question. Is lowering the agent's selective bias towards an edge as much as possible (i.e., optimally chunking that edge) always the best way to reduce their overall cost? Though this might seem obviously true, a surprising fact is that a present-biased agent's cost is not monotone in their bias; a smaller bias may sometimes increase their total cost [Kleinberg et al. 2016]. Despite this, when trying to minimize the agent's cost, the designer should optimally chunk any edge they want to chunk (e.g., by using Algorithm 1). The only challenge is in finding which edges to chunk.

To see why this is true, first note that chunking an edge (u, v) will not change its overall cost, and thus will not impact the agent's decisions unless they are at u. Second, it's easy to see that chunking cannot *increase* one's selective bias, as no edge in the chunking can have more cost than the original edge cost. Thus, any chunking of edge (u, v) serves to convince the agent to take (u, v). And the best way to accomplish that is to minimize the agent's perceived cost starting with that chunked edge, which is exactly what an optimal edge-chunking does.

4.1 Local Constraints

We consider two types of constraints on the designer. We call the first a *local* constraint; in this case the designer can break any set of edges into up to k chunks, for some parameter k. If we think of edges as representing relatively large subtasks, then this just says that any relatively large subtask can be split into up to k smaller subtasks. We call the second a *global* constraint: in this case, the designer gets a budget of k chunks, and can use no more than k chunks altogether.

In this section we consider local constraints. A naive approach would be to just optimally chunk every edge into k chunks, using our earlier results. But this wouldn't necessarily give the best overall chunking for the graph. Why not? The intuition is that we want the agent's perceived cost of the path that the designer actually wants the agent to use to be low. We are better served by *not* chunking edges away from this path, so that the agent is not tempted to deviate. So at a high level, the algorithm first figures out the cheapest *feasible* path for the agent (given k), and then uses the optimal edge-chunking algorithm to actually chunk this path.

Theorem 3. *Given any task graph $G = (V, E)$ and a local constraint k, we can optimally chunk G with at most $|E|$ applications of Algorithm 1, for a total runtime of $O(|E|k + |V|)$.*

Proof. First, we can use well-known algorithms to find the costs of the shortest path from any node to t in time $O(|E|+|V|)$, since G is a directed, acyclic graph [Cormen et al. 2009]. Given a vertex u, let $w = \text{argmin}_{v:(u,v)\in E}\, bc(u, v) + c(v \longrightarrow t)$ be the vertex that the present-biased agent would go to without any chunking. Further, let $\alpha_u = p(u, w)$ be the perceived cost of starting with edge (u, w). Let $v \neq w$ be an arbitrary out-neighbor of u (i.e., a vertex v such that there is an edge (u, v)). Algorithm 1 gives us the lowest possible bottleneck cost of a k-chunking of (u, v); denote this as $\beta_{u,v}$. If $\beta_{u,v} \leq \alpha_u$, the agent can be made to take (u, v). If not, then they won't take (u, v) under any k-chunking.

The algorithm is straightforward. At every vertex u, determine α_u as well as $\beta_{u,v}$ for all out-neighbors v of u. If $\beta_{u,v} > \alpha_u$, remove edge (u, v) from the graph. Call the resulting graph G'. Then, simply compute the shortest path in G', and chunk every edge on that path with Algorithm 1.

There will always be an s-t path in G', as the edges the agents would take without chunking can never be removed. By construction, the path in G' that we chunk is one that the agent will take in G after chunking. Finally, there can be no cheaper path, as we remove only edges that the agent cannot be convinced to take. □

We briefly discuss a different perspective on the algorithm above, which will be useful when comparing to the results of the next section. We can think of the algorithm as a dynamic program with the following recurrence:

$$cost[u] = \min_{v:(u,v)\in E, \beta_{u,v} \leq \alpha_u} c(u,v) + cost[v].$$

Here, $cost[u]$ is the cost of the cheapest u to t path we can convince the agent to take, and the base case is simply $cost[t] = 0$. This recurrence is exactly the recurrence that a shortest-path algorithm solves, except for the condition that $\beta_{u,v} \leq \alpha_u$. Thus, the first part of the algorithm simply removes edges that do not satisfy this condition, and then the solution to the shortest path problem will solve the above recurrence.

4.2 Global Chunking Budget

In this section we consider global constraints; the designer must consider where to best allocate chunks to have the most impact. As before, we can use the optimal edge-chunking algorithm to solve this problem; only marginally more computation is required.

Theorem 4. *Given any task graph $G = (V, E)$ and a global constraint k, we can determine the optimal chunking configuration with at most $O(|E| \log k)$ applications of Algorithm 1, for a total runtime of $O(|E|k \log k + |V|)$.*

Proof. As before, we first compute the cost of the shortest path from any node to t in time $O(|V|+|E|)$. For a local budget, we sorted edges into feasible and infeasible edges, where an edge was feasible if we could convince the agent to take it with at most k chunks. Here, we instead determine the minimum number of chunks that's necessary for an agent to take each edge (if the number is at most k). Since the optimal bottleneck cost is decreasing in the number of chunks k, we can simply use binary search to find this minimum number.

In more detail, let u be an arbitrary vertex and define α_u as above. For any out-neighbor v of u, let $\beta^l_{u,v}$ be the lowest possible bottleneck cost of any l-chunking of (u, v). Let $l_{u,v}$ be the smallest $l \leq k$ such that $\beta^l_{u,v} \leq \alpha_u$. If no such l exists, then $l_{u,v} = \infty$. $l_{u,v}$ can be computed in $O(\log k)$ applications of Algorithm 1 with binary search, since $\beta^l_{u,v}$ is decreasing in l.

Now let $cost[u, i]$ denote the cost of the cheapest path from u to t that we can convince the agent to take with at most i chunks. The base case is simply $cost[t, i] = 0$ for all $0 \leq i \leq k$. The recurrence is as follows.

$$cost[u, i] = \min_{v:(u,v)\in E, l_{u,v} \leq i} c(u,v) + cost[v, i - l_{u,v}].$$

The final solution is $cost[s, k]$. The correctness of this recurrence follows from the fact that $l_{u,v}$ is the smallest number of chunks needed to convince the agent to take edge (u, v). For the runtime, note that it takes $O(Ek \log k)$ to compute $l_{u,v}$ for all $(u, v) \in E$. For the recurrence, the min considers $|E|$ possibilities for each value of $i \in \{0, \ldots, k\}$, for a total runtime of $O(|E|k)$. Finally, to actually compute the recurrence, we can simply proceed backwards through some topological ordering of the graph. □

5 OPTIMIZING THE COST RATIO

Define the cost ratio of a present-biased agent to be $C_b(s \rightarrow t)/c(s \rightarrow t)$, where $C_b(s \rightarrow t)$ is the cost that a present-biased agent with bias parameter b incurs in the graph, and $c(s \rightarrow t)$ is the shortest path cost. The goal of this section is to understand how the cost ratio of the present-biased agent decreases as the task designer places more chunks in the graph. Put another way, in the previous section we provided algorithms that optimally chunked task graphs, given a fixed chunking budget k. Here, we prove performance guarantees on those algorithms, where the algorithm's "performance" is measured in how much it reduces the cost of the agent's path.

Existing results have characterized the worst-case cost ratio over all task graphs.

Theorem 5 (Adapted from Tang et al. [2017]). *The cost ratio for an agent with present bias b is at most b^n, over all task graphs. The n-fan (see Figure 3) can get arbitrarily close to this cost ratio as c approaches b from below.*

We want to characterize the worst-case cost ratio after chunking. More precisely, we consider the following question. Let G be arbitrary, and let G' denote an optimal k-chunking of G. What is the worst-case cost ratio for G'? We start by considering local constraints; thus, G' is the result of breaking an arbitrary number of edges in G into at most k chunks. Let $b_{\min}$ be the selective bias guaranteed by Theorem 1. That is, let:

$$b_{\min} = \frac{1}{1 - \left(\frac{b-1}{b}\right)^k}.$$

Theorem 6. *If G' is an optimal chunking of G with local constraint k, then the cost ratio for an agent with present bias b in G' is at most b^n_{min}.*

Proof. We simply chunk every edge into k chunks using the chunking given in Theorem 1, which results in the agent viewing every edge with a selective bias of $b_{\min}$. Call the resulting graph G''. By the definition of selective bias, for every edge $(u, v) \in G$, an agent with bias $b_{\min}$ would go from u to v if and only if the agent with bias b would traverse the chunking $(u_1, u_2, \ldots, u_k, v)$ in G. Since this holds for every edge, the agent will incur exactly the same cost as an agent with bias $b_{\min}$ would incur in G. So by Theorem 5, they incur cost at most $b^n_{\min}$ in G'', with bias b.

The theorem follows from the fact that G' is an optimal chunking of G, so the agent will only do better there as compared to G''. □

Corollary 2. *Given a local constraint $k = O(n)$, the optimal chunking G' of G has constant cost ratio.*

Proof. The proof involves only arithmetic after applying Theorem 6. Details can be found in the appendix. □

The corollary shows that we can get an exponential reduction in the agent's worst-case cost with only a linear number of chunks on every edge, demonstrating the power of chunking. However, from a different perspective, the bound in Theorem 6 seems weak. We showed earlier that it's never necessary to chunk two edges leading out of the same vertex, but here we chunk all edges. Further, we chunk every edge with Theorem 1, despite that chunking not being optimal for non short-path edges. Despite these concerns, the bound in the theorem is tight, as demonstrated by chunking the n-fan.

Lemma 4. *If G is an n-fan with $c < b_{min}$ and G' is an optimal chunking of G given local constraint k, then the cost ratio for an agent with present bias b is c^n in G'.*

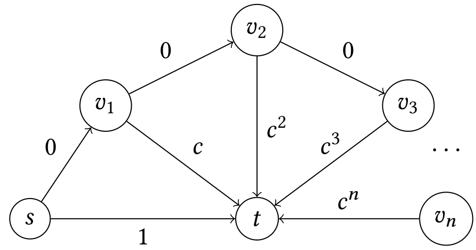


Fig. 3. This graph is the n-fan. If $c < b$, the agent will prefer edge (v_i, v_{i+1}) to (v_i, t) for all i. Thus, the agent goes all the way around the fan, and incurs cost c^n.

Proof. Let G' be constructed by chunking every edge in the n-fan via Theorem 1 (we can ignore the 0 cost edges, as chunking a 0 cost edge has no impact on the agent's decisions). In G', the agent acts as if they had bias $b_{\min}$ in G. And such an agent would incur cost c^n by going all the way around the fan, since $c < b_{\min}$. It remains to show that G' is an optimal chunking of G.

In fact, we show the stronger claim that any chunking of G with a local budget of k is (weakly) optimal, as no such chunking can cause the agent to take a cheaper path. To see this, suppose there is a chunking G^* of G such the agent goes from v_i to t, for $i < n$ (this is the only way they could take a cheaper path). Then, G^*'s chunking of edge (v_i, t) must have lower bottleneck cost than in G'. We claim that this is impossible, because Theorem 1 will give the optimal chunking for edge (v_i, t). To see this, notice that the shortest path from v_i to t is through edge (v_i, t), which is exactly when Theorem 1's chunking is optimal. As a result, no G^* exists, and so G' is an optimal chunking. □

We have provided a tight characterization for the worst-case cost ratio in terms of the number of chunks given a local constraint. We conjecture that a similar result extends to global constraints. Let k be the global chunking budget. Clearly, we could get an upper bound on the worst-case cost ratio similar to that of Theorem 6 by evenly splitting the chunks so that each edge satisfies a local constraint of k/m, where $m = |E|$. We conjecture that this would also be an asymptotically tight bound, as it seems that the optimal chunking in the n-fan would need to spread chunks evenly among half the edges (i.e., the edges (v_i, t)).

6 OPTIMAL CHUNKING FOR MULTIPLE AGENTS

We now consider the problem of chunking a task graph for two types of agent, where an agent's type is their bias. For example, an instructor might reasonably expect some students to procrastinate rarely and others to procrastinate frequently. Yet the instructor cannot chunk the task separately for different students (indeed, they may well not know a given student's type). How should they chunk the task while balancing the cost that both types of students incur? We answer this question in two settings. We first show how to optimally chunk the graph for two types of agents, A_1 and A_2 with $b_1 < b_2$. Second, we show how to optimally chunk the graph for m types of agents, with the additional constraint that all agents take the same path. Allowing agents to take different paths gives the designer more power but also makes the problem significantly more complex to analyze; removing this possibility allows us to design for m types, rather than 2.

Note that in the case of a single agent, there is an obvious way to define the "optimal" way to chunk an edge – it's the one that agent perceives as cheapest. This definition is also useful for chunking the task graph optimally, as it tells us which edges we can persuade the agent to take. With two agents, it's unclear what it would mean to "optimally" chunk an edge. An intuitive definition would be that the optimal chunking for an edge minimizes the average perceived cost of the two agents. But that is wholly unhelpful for graph chunking, as it doesn't tell us which edges we can persuade either agent to take. So, we instead consider two related problems: convincing agents to take the same path, and convincing agents to split up. Solving these two problems will allow us to chunk the task graph while minimizing the sum of the agents' costs.

6.1 Splitting Agents onto Separate Paths

In this section, we want to find the chunking C^* of (u, v) such that A_1 takes C^* and A_2 finds C^* "maximally unappealing": formally, C^* has the maximum perceived cost for A_2 over all chunkings A_1 would take. We can use such a chunking to split up two agents who are both at the same vertex. We start by defining some terms. Let $p(e; b_i)$ represent the perceived cost of edge e for the agent with bias b_i. Here, agent A_1 has bias b_1, and agent A_2 has bias b_2, where $b_1 < b_2$. Then, let $\alpha_u^{(i)}$ represent A_i's perceived cost of their best option at u (without chunking). So, $\alpha_u^{(i)} = p(u, w_i; b_i)$, where $w_i = \text{argmin}_{v:(u,v)\in E}\, b_i c(u, w_i) + c(w_i \rightarrow t)$.

We now describe the algorithm that solves this problem, Algorithm 2, at a high level; a full description can be found in the appendix. Algorithm 2 first computes C_1^*, the optimal chunking of (u, v) for A_1.[5] Then, the algorithm iterates over all choices of e_i and raises $p(e_i; b_2)$ as much as possible while ensuring that A_1 still takes the chunking. It does so by "siphoning" cost from other edges in the chunking onto e_i. It repeats this process for all choices of e_i. This siphoning has three phases.

In the first phase, we siphon from $x_{i-1}, \ldots, x_1$ to x_i.[6] In the second phase, we siphon from $x_{i+1}, \ldots, x_k$ to x_i. These phases are very straightforward, and terminate when $p(e_i; b_1) = \alpha_u^{(1)}$, where $\alpha_u^{(1)}$ is the perceived cost of the best alternative to (u, v) from A_1's perspective. In the third phase, we decrease $x_{>i}$ and increase $x_{\leq i}$; because $b_1 < b_2$, doing this results in increasing $p(e_i; b_2)$ without increasing $p(e_i; b_1)$.

Call the resulting chunking C_i. Note that A_1 will surely take C_i: A_1 took the original chunking, and all edges which were increased (potentially all $e_{\leq i}$) were not increased beyond $\alpha_u^{(1)}$. We first prove the following conditions of the algorithm.

Lemma 5. *Let $C_i = (e_1, \ldots, e_k)$ be the chunking produced by iteration i of Algorithm 2. Then:*

(a) $\sum_{j\neq i} x_j > 0 \implies p(e_i; b_1) = \alpha_u^{(1)}$

(b) $\sum_{j>i} x_j > 0 \implies \forall j \leq i, p(e_j; b_1) = \alpha_u^{(1)}$

Proof. For (a), if any $x_j > 0$, then the algorithm terminated early in phase 1 or phase 2, which implies that $p(e_i; b_1) = \alpha_u^{(1)}$. For (b), if more could be siphoned from $x_{>i}$, then the algorithm would siphon more in phase 3, unless no edges in $e_{<i}$ can be increased further. □

The following theorem says that A_2 finds edge e_i in C_i maximally unappealing over all chunkings A_1 would take; the proof is in the appendix.

Theorem 7. *If C_i is the output of the ith iteration of Algorithm 2 and C' is another chunking such that $p(e_i'; b_2) > p(e_i; b_2)$, then A_1 will not take C'.*

The theorem can be applied to show that our algorithm is correct. Let C^* be the chunking with the maximum perceived cost from A_2's perspective that A_1 will still take. Let i^* be the bottleneck of C^* for A_2. Then, the contrapositive of the theorem shows that our algorithm will find C^* (or a chunking with equivalent A_2-perceived cost) when $i = i^*$.

[5]The algorithm does not rely on starting with an *optimal* chunking; any chunking that A_1 takes would work.

[6]To ease exposition, we can think of "siphoning" as a continuous process where one cost is decreased as another increases. In practice, how much to siphon can be computed in $O(1)$ time; see the appendix for details.

Unfortunately, this problem is not symmetric with respect to A_1 and A_2. In other words, we still must solve the problem of chunking an edge such that A_2 takes it but A_1 finds it maximally unappealing. The only modification we need to make is to phase 3, where we instead *increase* $x_{>i}$ and decrease $x_{\leq i}$, which will increase $p(e_i; b_1)$ without increasing $p(e_i; b_2)$. More details can be found in the appendix.

6.2 Keeping Agents on the Same Path

In this section, we consider the problem of chunking a single edge (u, v) so that all agents take the chunking. This problem can solved greedily, even if we have m types of agents. This algorithm will

Algorithm 3: Greedily chunk edge (u, v) into k chunks for m agents

```
for i = k to 1 do
    maximize x_i such that p(e_i; b_j) ≤ α_u^(j) for all j ∈ [m]
    if x_i < 0 then
        return ⊥
    if ∑_i x_i ≥ x then
        lower x_i so that ∑_i x_i = x
        return chunking C
return ⊥                                          // ∑_i x_i < x
```

produce a chunking that the agents will all take, iff such a chunking exists. We use the following lemma, which is proven in the appendix. To introduce the lemma, we define a *partial chunking* as a chunking that does not assign all the cost of the original edge. Algorithm 3 can be viewed as building partial chunkings into a complete chunking.

Lemma 6. *Let C and C' be two (possibly partial) chunkings of the same edge. Suppose that $\sum_{i=l}^{k} x'_i > \sum_{i=l}^{k} x_i$. Then, there exists an $i \in [l, k]$ such that for all $b > 1$, $p(e'_i; b) > p(e_i; b)$.*

The lemma says that if a chunking C' assigns more cost to the last $k - l$ edges than C', then one of those last $k - l$ edges must have a higher perceived cost (for *any* present-biased agent). We now prove that the algorithm is correct.

Theorem 8. *Algorithm 3 runs in time $O(mk)$. Further:*

(a) If Algorithm 3 returns a chunking C, then all agents will take C.
(b) If Algorithm 3 returns $\perp$, then no chunking exists that all agents would take.

Proof. Statement (a) is obvious; if a chunking is returned, then it must be the case that $p(e_i; b_1) \leq \alpha_u^{(j)}$ for all i and for all j. Thus, every chunk is more appealing than every agent's best outside option, and so all agents take C. The runtime is also obvious: inside the loop, the only work being done is computing the maximum x_i such that $p(e_i; b_j) \leq \alpha_u^{(j)}$), for all $j \in [m]$.

We prove statement (b) by looking at two cases. For the first case, suppose the algorithm returns $\perp$ at iteration i. This means that when $x_i = 0$, $p(e_i; b_j) > \alpha_u^{(j)}$ for some agent j. However, note that if $x_i = 0$, then $p(e_i; b_j) = \min(c(u, w) + c(w \to t), \sum_{l>i} x_l + c(v \to t))$, where $\sum_{l>i} x_l \leq x$ (or the algorithm would have terminated at $i + 1$). Over all chunkings, the smallest perceived cost of the first edge is achieved when no weight is placed on it. Let $e_1^{\min}$ be the first edge in such a chunking. Then, $p(e_1^{\min}; b_j) = \min(c(u, w) + c(w \to t), x + c(v \to t))$. Since $x \geq \sum_{j>i} x_j$, we know that $p(e_1^{\min}; b_j) \geq p(e_i; b_j) > \alpha_u^{(j)}$. Thus, in any other chunking, the agent j would deviate at the first chunk.

In the second case, suppose the algorithm returns $\perp$ at the end. This means that, for all i, $p(e_i; b_j) = \alpha_u^{(j)}$ for some agent j and $\sum_i x_i < x$. In other words, the chunking C that the algorithm produces is a partial chunking, and a complete chunking must assign more cost. However, Lemma 6 says that if any chunking C' assigns more cost, then there would be some edge e' of C' which all agents would perceive as more expensive. So, some agent would abandon their path at e'. Thus, there is no complete chunking that all agents would take. □

6.3 Optimal Graph Chunking for Multiple Agents

We now revisit the problem of optimal graph chunking, with a local or global chunking budget, k.

6.3.1 Two Types. We first assume we have a local chunking budget of k chunks per edge, and try to minimize the sum of the two agents' (real) costs.[7] We first reformulate our solution to the single agent case to introduce the idea of "persuadable" edges. In that case, we used the recurrence $cost[u]$ to represent the minimum cost of any $u \to t$ path that we could persuade the agent to take. We computed the recurrence via $cost[u] = \min_{v \in \mathcal{P}(u)} c(u, v) + cost[v]$, where $\mathcal{P}(u) = \{v : (u, v) \in E, \beta_{u,v} \leq \alpha_u\}$ represents the set of vertices we can persuade the agent to take from u.

We can define a very similar recurrence for two agents. Say that two paths P and Q are (A_1, A_2)-compatible if we can chunk (some of) the edges along P and Q such that A_1 takes P and A_2 takes Q. Let $cost[u, y]$ represent the minimum sum of the costs of any $u \to t$ path P and a $y \to t$ path Q such that (P, Q) are (A_1, A_2)-compatible. Further, let $\mathcal{P}(u, y)$ be the set of all edges (v, z) such that (u, v) and (y, z) can be "compatibly-chunked". This means that, if $(u, v) = (y, z)$, then there exists a chunking of (u, v) that both agents take. Otherwise, there exist chunkings C_1, C_2 of (u, v) and (y, z) such that A_1 takes C_1 and A_2 takes C_2. If $u \neq y$ (i.e., the agents start at different vertices), then $\mathcal{P}(u, y)$ can be easily computed via the algorithms in Section 4. And $\mathcal{P}(u, u)$ can be computed via the algorithms in Section 6.1 and 6.2.

With these functions, the recurrence can be broken into three cases. The first case is when A_2 is about to go to the vertex, u, that A_1 is currently at. In this case, we need to ensure that our chunking of (u, v) for A_1 doesn't cause issues for A_2. This case can be represented as:

$$C_1(u, v, y) = \begin{cases} c(y, u) + cost[u, u] & \text{if } (v, u) \in \mathcal{P}(u, y) \\ \infty & \text{otherwise.} \end{cases}$$

The second case is similar, but with the agents flipped.

$$C_2(u, y, z) = \begin{cases} c(u, y) + cost[y, y] & \text{if } (y, z) \in \mathcal{P}(u, y) \\ \infty & \text{otherwise.} \end{cases}$$

Finally, if neither of the previous cases occur, the cost is:

$$C_3(u, v, y, z) = c(u, v) + c(y, z) + cost[v, z].$$

Putting it all together, the recurrence is:

$$cost[u, y] = \min_{(v,z) \in \mathcal{P}(u,y)} \min(C_1(u, v, y), C_2(u, y, z), C_3(u, v, y, z)).$$

We first prove the correctness of this recurrence.

Lemma 7. *The recurrence for $cost[u, y]$ above is the cost of the cheapest paths $P : u \to t$ and $Q : y \to t$ such that P and Q are (A_1, A_2)-compatible.*

[7] It's trivial to modify the recurrence to instead minimize the maximum of the two types' costs, a weighted average (useful if one type is much more common), or many other such functions.

Proof. Assume that $cost[v, z]$ have been correctly computed for all v (resp. z) that are out-neighbors of u (resp. y). We know that $u \neq v$ and $y \neq z$, because there are no self-loops in a DAG. We now proceed by cases.

Case 1: $u = y$. First, note that $v \neq u, z \neq u$ for all $(v, z) \in \mathcal{P}(u, u)$. So, we will only be in the first case of the min. In this case, $P(u, y) = P(u, u)$ will return all (v, z) such that there exist chunkings C_1 of (u, v) and C_2 of (u, z) such that A_1 takes C_1 and A_2 takes C_2, if both are at u. Further, if $v = z$, then $C_1 = C_2$ (i.e., (u, v) is chunked such that both agents take it). Recall that $cost[v, z]$ is the cheapest cost of (A_1, A_2) compatible paths $P' : v \rightarrow t$ and $Q' : z \rightarrow t$. Since A_1 going from $u \rightarrow v$ is compatible with A_2 going from $u \rightarrow z$, we get that the paths $P : (u, v) \cup P'$ and $Q : (u, z) \cup Q'$ are (A_1, A_2) compatible.

Case 2: $u \neq y$. When $u \neq y$, all three cases of the min are possible. Since $P(u, y)$ describes all possible ways to chunk for A_1 at u and A_2 at y, the min will be correct as long as all three cases lead to (A_1, A_2)-compatible paths, so that's what we'll prove.

In the first case, assume that $v \neq y, u \neq z$. From the correctness of $cost[v, z]$, and the fact that (u, v) and (y, z) share no endpoints, it immediately follows that the $u \rightarrow t$ and $y \rightarrow t$ paths are (A_1, A_2)-compatible.

In the second case, assume that $v = y$ (this implies that $z \neq u$, as otherwise u and y form a cycle). In other words, A_1 will go from u to y and meet A_2 there. Thus, we simply add the edge (u, y) to A_1's path and continue the traversal with both agents at y. So by the correctness of $cost[y, y]$, it follows that the $u \rightarrow t$ and $y \rightarrow t$ paths are (A_1, A_2)-compatible.

The third case, where $z = u$, is symmetric to the second case, but with the agents swapped. □

Suppose that there is a local budget of k chunks per edge.

Theorem 9. *Given any task graph $G = (V, E)$ and a local constraint k, we can optimally chunk G for two types of agents in time $O(|E|^2 k^2 + |V|)$.*

Proof Sketch. The runtime of the algorithm is dominated by determining when it's possible to split the agents onto separate paths. All together, this will take $O(|E|^2)$ applications of the algorithm in Section 6.1, for a total runtime of $O(|E|^2 k^2)$. The algorithm first computes $\mathcal{P}(u, y)$ for all $u, y \in V$, and then computes the *cost* recurrence. More details can be found in the appendix. □

Finally, suppose there is a global budget of k chunks.

Theorem 10. *Given any task graph $G = (V, E)$ and a global constraint k, we can optimally chunk G for two types of agents in time $O(|E|^2 k^3 \log k + |V|)$.*

Proof Sketch. Like in the single-agent global budget case, we first modify the function $\mathcal{P}$ to $\mathcal{P}'$, where $\mathcal{P}'(u, y)$ returns the set of (v, z, i) such that i is the minimum number of chunks to compatibly chunk (u, v) and (y, z) (where $i = \infty$ if no chunking is possible). The bottleneck is in computing the minimum number of chunks to split the agents from one vertex to two separate vertices. □

6.3.2 m Types of Agents Taking the Same Path. Assume that there are m types of agents but only chunkings where all m types take the same path are allowed. This easily reduces to the single agent case (found in Section 4), but we simply use Algorithm 3 to determine what edges we can persuade the group of agents to take. More detail can be found in the appendix; here, we simply state the main theorems.

Theorem 11. *Given any task graph $G = (V, E)$ and a local constraint k, we can find the optimal single-path chunking of G for m types of agents with at most $|E|$ applications of Algorithm 3, for a total runtime of $O(|E| mk + |V|)$.*

Theorem 12. *Given any task graph $G = (V, E)$ and a global constraint k, we can find the optimal single-path chunking of G for m types of agents with at most $|E| \log k$ applications of Algorithm 3, for a total runtime of $O(|E|mk \log k + |V|)$.*

7 CONCLUSION

We have supplemented a graph-theoretic model of present bias with a model of chunking, giving task designers the ability to chunk edges in order to reduce the impact of present bias. We found that the best way to chunk an edge is relatively straightforward for edges on the shortest path, but significantly more complicated for edges off the shortest path. We then used our optimal edge-chunking algorithm to optimally chunk task graphs. We provided tight theoretical guarantees on how much we can reduce an agent's cost ratio as a function of the number of chunks we place in the graph. Finally, we showed how to optimally chunk task graphs for two types of agents simultaneously. Overall, our work highlights the efficacy of chunking as a means to defeat the harms agents incur due to their present bias.

Our work raises several open questions. We highlight two interesting future directions. First, we saw that the problem grew significantly more complicated when designing for two types of agents. Can we extend our results to an arbitrary number of types? More generally, suppose the task designer was uncertain about the agents' present-bias and captured this uncertainty with a distribution over b. Our work can be seen as solving this problem when the support of this bias distribution is two. But can we chunk in the case where b is continuously distributed?

Second, as explained before, our model is best understood as the task designer suggesting a chunking to agents, rather than enforcing this chunking. In some situations, such as classroom settings, the task designer may want to place intermediate checkpoints to guarantee that agents make regular progress on the task. How should these checkpoints be modeled, and how much can they lower agents' costs compared to chunking?

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A OPTIMAL EDGE CHUNKING PROOFS

Theorem 1. *Suppose we partition an edge (u, v) of cost x into k chunks. Let $u_1, \ldots, u_k$ represent the vertices in this chunking, and let $c(u_i, u_{i+1}) = x_i$, where, for $1 \le i \le k$, the x_i's are defined below.*

$$1 \le i \le k : x_i = \frac{(b-1)^{k-i} b^{i-1}}{b^k - (b-1)^k} x.$$

With this chunking, the agent has selective bias $\frac{1}{1-\left(\frac{b-1}{b}\right)^k}$. If, with this chunking, the shortest path from u_i to t is through u_{i+1} for all $i > 1$, then this chunking is optimal.

Proof. Lemma 1 proves the case where $k = 2$. Suppose that the theorem holds for $k - 1$ chunks; we prove it for k chunks. For now, we assume that the shortest path from u_i to u is u_{i+1} for all $i > 1$. At the end, we'll consider when this is not true. Say we put cost x_1 on the first edge. Then, we apply the inductive hypothesis to the other $k - 1$ edges, now with a task of cost $x - x_1$. The costs $x_2, \ldots, x_k$ are thus:

$$x_i = \frac{(b-1)^{k-i-1} b^{i-2}}{b^{k-1} - (b-1)^{k-1}} (x - x_1).$$

Because the shortest path through u_i is u_{i+1} for all $i > 2$ as well, we know from the inductive hypothesis that this chunking is optimal (given that x_1 is on the first edge). Further, the perceived costs of starting with those edges are all

$$\frac{1}{1 - \left(\frac{b-1}{b}\right)^{k-1}} (x - x_1) + c(v \longrightarrow t).$$

We want to minimize the maximum of the perceived cost of starting with edge (u_1, u_2) and all the other edges. As before, we can do so by setting the perceived costs equal, as one side is decreasing in x_1 while the other is increasing in x_1. Because the shortest path from u_i is through u_{i+1} for $i > 1$, the perceived cost of starting with (u_1, u_2) is $bx_1 + c(u_2 \longrightarrow t) = bx_1 + x_2 + c(u_3 \longrightarrow t) = \cdots = bx_1 + \sum_{i=2}^{k} x_i + c(v \longrightarrow t) = bx_1 + x - x_1 + c(v \longrightarrow t) = (b-1)x_1 + x + c(v \longrightarrow t)$.

$$\begin{aligned}
(b-1)x_1 + x + c(v \longrightarrow t) &= \frac{1}{1 - \left(\frac{b-1}{b}\right)^{k-1}} (x - x_1) + c(v \longrightarrow t) \\
(b-1)x_1 + x &= \frac{b^{k-1}}{b^{k-1} - (b-1)^{k-1}} (x - x_1) \\
\left(b - 1 + \frac{b^{k-1}}{b^{k-1} - (b-1)^{k-1}}\right) x_1 &= \left(\frac{b^{k-1}}{b^{k-1} - (b-1)^{k-1}} - 1\right) x \\
\frac{(b-1)(b^{k-1} - (b-1)^{k-1}) + b^{k-1}}{b^{k-1} - (b-1)^{k-1}} x_1 &= \frac{b^{k-1} - b^{k-1} + (b-1)^{k-1}}{b^{k-1} - (b-1)^{k-1}} x
\end{aligned}$$

$$(b^k - b^{k-1} - (b-1)^k + b^{k-1})x_1 = (b-1)^{k-1}x$$
$$x_1 = \frac{(b-1)^{k-1}}{b^k - (b-1)^k}x.$$

Thus, x_1 matches the chunking in the theorem. We now verify that the perceived cost matches:

$$\begin{aligned}(b-1)x_1 + x + c(v \to t) &= (b-1) \cdot \frac{(b-1)^{k-1}}{b^k - (b-1)^k} \cdot x + x + c(v \to t) \\ &= \left(\frac{(b-1)^k}{b^k - (b-1)^k} + 1\right) x + c(v \to t) \\ &= \frac{(b-1)^k + b^k - (b-1)^k}{b^k - (b-1)^k} x + c(v \to t) \\ &= \frac{b^k}{b^k - (b-1)^k} x + c(v \to t) \\ &= \frac{1}{1 - \left(\frac{b-1}{b}\right)^k} x + c(v \to t).\end{aligned}$$

A similar calculation will show that all the perceived costs are the same:

$$\begin{aligned}&\frac{1}{1 - \left(\frac{b-1}{b}\right)^{k-1}}(x - x_1) + c(v \to t) \\ &= \frac{b^{k-1}}{b^{k-1} - (b-1)^{k-1}} \left(x - \frac{(b-1)^{k-1}}{b^k - (b-1)^k}x\right) + c(v \to t) \\ &= \frac{b^{k-1}}{b^{k-1} - (b-1)^{k-1}} \left(1 - \frac{(b-1)^{k-1}}{b^k - (b-1)^k}\right) x + c(v \to t) \\ &= \frac{b^{k-1}}{b^{k-1} - (b-1)^{k-1}} \left(\frac{b^k - (b-1)^k - (b-1)^{k-1}}{b^k - (b-1)^k}\right) x + c(v \to t) \\ &= \frac{b^{k-1}}{b^{k-1} - (b-1)^{k-1}} \left(\frac{b^k - (b-1)^{k-1}(b-1+1)}{b^k - (b-1)^k}\right) x + c(v \to t) \\ &= \frac{b^{k-1}}{b^{k-1} - (b-1)^{k-1}} \left(\frac{b(b^{k-1} - (b-1)^{k-1})}{b^k - (b-1)^k}\right) x + c(v \to t) \\ &= \frac{b^k}{b^k - (b-1)^k} x + c(v \to t) \\ &= \frac{1}{1 - \left(\frac{b-1}{b}\right)^k} x + c(v \to t).\end{aligned}$$

Finally, we can plug the value of x_1 into the formula for x_i:

$$\begin{aligned}x_i &= \frac{(b-1)^{k-i} b^{i-2}}{b^{k-1} - (b-1)^{k-1}}(x - x_1) \\ &= \frac{(b-1)^{k-i} b^{i-2}}{b^{k-1} - (b-1)^{k-1}} \left(x - \frac{(b-1)^{k-1}}{b^k - (b-1)^k}x\right) \\ &= \frac{(b-1)^{k-i} b^{i-2}}{b^{k-1} - (b-1)^{k-1}} \left(1 - \frac{(b-1)^{k-1}}{b^k - (b-1)^k}\right) x\end{aligned}$$

$$= \frac{(b-1)^{k-i}b^{i-2}}{b^{k-1}-(b-1)^{k-1}}\left(\frac{b(b^{k-1}-(b-1)^{k-1})}{b^k-(b-1)^k}\right)x \qquad \text{(see previous derivation)}$$
$$= \frac{(b-1)^{k-i}b^{i-1}}{b^k-(b-1)^k}x.$$

Thus, we've shown all components of the inductive statement. To summarize, under the assumption that the shortest path from u_i is through u_{i+1} for all $i > 1$, we've shown that the chunking in the theorem is optimal and produces the correct selective bias.

When the shortest path from u_i to t is through some external vertex w instead of u_{i+1}, we've overestimated the perceived cost at some edges. In our calculations, we assumed that all edges would have perceived cost $bx_i + \sum_{j>i} x_j + c(v \longrightarrow t)$, but actually some edges would have a *lower* perceived cost of $p(e_i) = bx_i + c(w \longrightarrow t)$. However, the final edge (u_k, v) would still have perceived cost $bx_k + c(v \longrightarrow t)$, as we assumed in the theorem, and thus the perceived cost of that edge in the chunking would be $\frac{1}{1-\left(\frac{b-1}{b}\right)^k}x + c(v \longrightarrow t)$. So, though optimality can no longer be guaranteed, the chunking in the theorem produces the expected selective bias regardless of whether the edge is on the shortest path. □

As a brief sanity check, we show that the x_i's defined in the theorem actually sum to x.

Proposition 1.

$$\forall k \geq 1, \sum_{i=1}^{k}(b-1)^{k-i}b^{i-1} = b^k - (b-1)^k.$$

Proof. When $k = 1$, the left side is $(b-1)^0 b^0 = 1$, while the right side is $b^1 - (b-1)^1 = 1$. Suppose that the statement holds for k. Then:

$$\begin{aligned}
\sum_{i=1}^{k+1}(b-1)^{k+1-i}b^{i-1} &= b^k + \sum_{i=1}^{k}(b-1)^{k+1-i}b^{i-1} \\
&= b^k + (b-1)\sum_{i=1}^{k}(b-1)^{k-i}b^{i-1} \\
&= b^k + (b-1)(b^k - (b-1)^k) && \text{by the inductive hypothesis} \\
&= b^k + b^{k+1} - b^k - (b-1)^{k+1} \\
&= b^{k+1} - (b-1)^{k+1}.
\end{aligned}$$

□

Lemma 2. *Suppose that C is a chunking with bottleneck β. If another chunking O has bottleneck $\beta' < \beta$ and the same transition vertex τ, then O must lower the cost of all edges that are bottlenecks in C, and thus raise the cost of the remaining edges.*

Proof. Let C have bottleneck β and O have bottleneck β', where both chunkings have the same transition vertex τ. Let $J = \{j : p(e_j^C) < \beta\}$ and $I = \{i : p(e_i^C) = \beta\}$ partition the indices. We will show that $\sum_{j\in J} x_j^O > \sum_{j\in J} x_j^C$ and that $x_i^O < x_i^C$ for all $i \in I$.

Since O has a lower bottleneck, it must be the case that $p(e_k^O) < \beta$ for all k. This implies that for all $i \in I$, we get that $p(e_i^O) < p(e_i^C)$ (since $p(e_i^C) = \beta$). Note that $c(u_i^O \longrightarrow t) = c(u_i^C \longrightarrow t)$, as both chunkings have the same transition vertex τ. Since $p(e_i^O) = bx_i^O + c(u_i^O \longrightarrow t)$ and $p(e_i^C) = bx_i^C + c(u_i^C \longrightarrow t)$, the fact that $p(e_i^O) < p(e_i^C)$ implies that $x_i^O < x_i^C$.

Clearly if $x_i^O < x_i^C$ for all $i \in I$, then $\sum_{j\in J} x_j^O > \sum_{j\in J} x_j^C$, as I and J partition the indices, and both chunkings must sum to x. □

Lemma 3. *If a chunking C has the same perceived cost starting with any edge in the chunking, then C is optimal.*

Proof. Let C have bottleneck β and transition vertex τ, and let O have bottleneck $\beta' < \beta$ (and an arbitrary transition vertex). We prove that $\sum_{i=1}^{j} x_i^C > \sum_{i=1}^{j} x_i^O$ for all j by induction. With this proven, we get our desired contradiction with $\sum_{i=1}^{k} x_i^C = x > \sum_{i=1}^{k} x_i^O$, which means that O does not assign all the cost.

For the base case of $j = 1$, note that $p(e_1^C) > p(e_1^O)$ (because the bottleneck is lower). Expanding the perceived cost equations:

$$p(e_1^C) > p(e_1^O)$$
$$bx_1^C + c(u_2^C \to t) > bx_1^O + c(u_2^O \to t)$$
$$bx_1^C + \min c(u,w) + c(w \to t), x - x_1^C + c(v \to t) > bx_1^O + \min c(u,w) + c(w \to t), x - x_1^O + c(v \to t).$$

If $x_1^O = x_1^C + \varepsilon$ for any positive ε, the first term would go up by $b\varepsilon$ and the min would decrease by at most ε (if the both mins were the second term). Because $b > 1$, this would never satisfy the above equation, and so $x_1^C > x_1^O$.

The inductive case is essentially analogous to the base case. The perceived cost equation for arbitrary j expands to:

$$bx_1^C + \min(c(u,w) + c(w \to t), x - \sum_{i=1}^{j-1} x_i^C - x_j^C + c(v \to t))$$
$$> bx_1^O + \min(c(u,w) + c(w \to t), x - \sum_{i=1}^{j-1} x_i^O - x_j^O + c(v \to t)).$$

The inductive hypothesis tells us that $\sum_{i=1}^{j-1} x_i^C > \sum_{i=1}^{j-1} x_i^O$, so these terms do not change the argument. The only way that the inequality can be satisfied is if $x_j^C < x_j^O$. Otherwise, if $x_1^O = x_1^C + \varepsilon$ for any positive ε, the first term would go up by $b\varepsilon$ and the min would decrease by at most ε (since the sum is greater on the left hand side). So by induction, we get the desired result. □

Theorem 2. *Given any edge (u,v), we can determine the optimal k-chunking in $O(k)$ time, assuming that the shortest paths from $u \to t$ and $v \to t$ have been precomputed.*

Proof. Let w denote the node following u on the shortest path from u to t. If $v = w$, we can simply apply Theorem 1 to immediately get the best partition. So assume $v \neq w$. This means that $\delta > 0$.

We first focus on the difficult case where $\delta \leq x$; the case where $\delta > x$ will be covered at the end. As mentioned earlier, this means that we can satisfy any value of τ, by placing at least δ cost on the first τ edges while ensuring that the total cost of the first $\tau - 1$ edges is less than δ. The case where $\tau = k$ is an edge case that will be handled at the end. So suppose that $\tau \in \{1, \ldots, k-1\}$. We explain how to optimally chunk (u,v) for this fixed value of τ; in other words, we produce the optimal chunking over all chunkings that satisfy $\sum_{i=1}^{\tau} x_i \geq \delta$ and $\sum_{i=1}^{\tau-1} x_i \leq \delta$.

We start by setting $x_1 = x_2 = \cdots = x_\tau = \delta/\tau$. Then for all $i < \tau$, $p(e_i) = bx_i + c(u,w) + c(w \to t) = \frac{b\delta}{\tau} + c(u,w) + c(w \to t)$. Further:

$$\begin{aligned}
p(e_\tau) &= bx_\tau + c(u_{\tau+1} \to t) \\
&= bx_\tau + \textstyle\sum_{i=\tau+1}^{k} x_i + c(v \to t) \text{(shortest path from } u_{\tau+1} \text{ follows the chunking)} \\
&= bx_\tau + x - \textstyle\sum_{i=1}^{\tau} x_i + c(v \to t) \\
&= b \cdot \tfrac{\delta}{\tau} - \tau \cdot \tfrac{\delta}{\tau} + x + c(v \to t) \qquad \text{(substituing } x_i = \delta/\tau \text{ for } i \leq \tau) \\
&= \tfrac{b\delta}{\tau} + c(u,w) + c(w \to t) \qquad \text{(since } \delta = x + c(v \to t) - c(u,w) - c(w \to t)).
\end{aligned}$$

Let $\alpha = \frac{b\delta}{\tau} + c(u, w) + c(w \longrightarrow t)$. Then $p(e_i) = \alpha$ for all $i \leq \tau$.

Now we can chunk the remaining $x - \delta$ cost over the remaining $k - \tau$ edges according to Theorem 1, which gives them perceived costs:

$$\frac{x - \delta}{1 - \left(\frac{b-1}{b}\right)^{k-\tau}} + c(v \longrightarrow t) \stackrel{\text{def}}{=} \beta.$$

From Lemma 3, we know that if $\alpha = \beta$, we have the optimal chunking (for *any* transition vertex τ, not just the current τ). In that case, we stop the algorithm and return this chunking. Otherwise, there are two cases:

Case 1: $\alpha > \beta$. In this case, we claim that our chunking is optimal among all chunkings with transition vertex τ. Notice that our chunking has bottleneck α. By Lemma 2, if another chunking, O with the same τ has bottleneck lower than α, it must assign lower cost to all of the first τ edges. But this means that $\sum_{i=1}^{\tau} x_i^O < \delta$, which means the transition vertex would be later than τ. Thus, if $\alpha > \beta$, our chunking is optimal (for this τ).

Case 2: $\beta > \alpha$. The key to this case is that the perceived cost of starting with e_τ can be understood in two ways, which allows us to group it into either the earlier or later set of edges. This isn't the case for any other edge, and using this fact will allow us to modify our original chunking to lower β. More specifically, the original chunking ensures that the perceived cost of starting with e_τ is equal to all previous edges; the first modification we do in this case is to set the perceived cost of starting with e_τ equal to all *later* edges instead.

We start by leaving x_i fixed at δ/τ for all $i < \tau$, but then chunking the remaining $x - \delta \cdot \frac{\tau-1}{\tau}$ work over the remaining $k - 1 + 1$ edges according to Theorem 1, which modifies x_τ. Because this assignment equalizes the perceived cost of starting with e_τ with that of later edges, it must have increased x_τ to be higher than δ/τ; by similar reasoning, all x_i where $i > \tau$ must have decreased. Thus, this chunking has $p(e_i) = \beta'$ for all $i \geq \tau$, where $\beta' < \beta$. Further, since the perceived cost of starting with e_τ was α, and x_τ increased, the new perceived cost of starting with e_τ, β' must still be higher than α.

We now increase $x_1, \ldots, x_{\tau-1}$ to raise α and lower β'. We do so by setting x_i to a placeholder y for all $i < \tau$ and then solving for the optimal y. Note setting all these values equal is (weakly) dominant, because the perceived costs of starting with these edges are all $bx_i + c(u, w) + c(w \longrightarrow t)$. Thus, if another chunking had $x_i \neq x_j$, where $i, j < \tau$, then setting x_i and x_j equal to their average would only decrease $\max_{i<\tau} p(e_i)$. This would either reduce the bottleneck (if the bottleneck is before τ) or keep it the same. So we can set them all equal to y without loss of generality.

With this, $p(e_i)$ for $i < \tau$ is $by + c(u, w) + c(w \longrightarrow t)$. We then use Theorem 1 to optimally split the remaining $x - y(\tau - 1)$ work over the remaining $k - \tau + 1$ edges. With that, for all $i \geq \tau$, we get

$$p(e_i) = \frac{x - y(\tau - 1)}{1 - \left(\frac{b-1}{b}\right)^{k+1-\tau}} + c(v \longrightarrow t).$$

We now set the two perceived costs equal and solve for the best y:

$$\frac{x - y(\tau - 1)}{1 - \left(\frac{b-1}{b}\right)^{k-\tau+1}} + c(v \longrightarrow t) = by + c(u, w) + c(w \longrightarrow t), \text{ so}$$

$$y\left(\frac{\tau - 1}{1 - \left(\frac{b-1}{b}\right)^{k-\tau+1}} + b\right) = \frac{x}{1 - \left(\frac{b-1}{b}\right)^{k-\tau+1}} + c(v \longrightarrow t) - c(w \longrightarrow t) - c(u, w).$$

To ease notation, let $z_\tau = 1 - \left(\frac{b-1}{b}\right)^{k-\tau+1}$. We can then simplify as follows:

$$
\begin{aligned}
y\left(\frac{\tau - 1 + z_\tau b}{z_\tau}\right) &= \frac{x}{z_\tau} + c(v \longrightarrow t) - c(u, w) - c(w \longrightarrow t), \text{ so} \\
y &= \frac{x + z_\tau(c(v \longrightarrow t) - c(w \longrightarrow t) - c(u, w))}{\tau - 1 + z_\tau b} \\
&= \frac{\delta z_\tau + (1 - z_\tau)x}{\tau - 1 + z_\tau b} \stackrel{\text{def}}{=} y^*.
\end{aligned}
$$

For our final chunking, C^*, we set $x_i = \min(y^*, \frac{\delta}{\tau-1})$ for $i < \tau$, and split the remaining work over the latter edges via Theorem 1. Under this chunking, let $\alpha^* = p(e_i)$ for $i < \tau$ and let $\beta^* = p(e_i)$ for $i \geq \tau$. We claim the following.

Claim 1. $\alpha^*, \beta^* > \alpha$

The intuition for this is that the chunking C^* increases the cost of early edges, while decreasing the cost of later edges. But we still ensure that the later edges have perceived cost at least as great as the early edges.

Proof. Note that with $y = \delta/l$, we got that $\alpha < \beta'$. Further, with $y = y^*$, the perceived costs starting with any edge would be equal, by definition of y^*. Thus, we know that $y^* > \delta/l$. It follows that $\min(y^*, \frac{\delta}{l-1}) > \delta/l$, and thus $\alpha^* > \alpha$.

Note that if $y = y^*$, then $\beta^* = \alpha^* > \alpha$, since choosing edge costs so that the perceived costs of starting with all edge in the chunking are equal means that the costs on the early edges increase. Further, β^* is decreasing in y. Since $y = \min(y^*, \frac{\delta}{l-1}) \leq y^*$, this implies that $\beta^* > \alpha$. □

We now show that C^* has transition vertex τ. By construction, we have that $\sum_{i<\tau} x_i \leq \delta$. Let $i \leq \tau$ be arbitrary. When x_i was δ/τ in the original chunking, we had that $p(e_i)$ was α. By Claim 1, we know that $p(e_i) > \alpha$, which means that $x_i > \delta/\tau$ (since perceived costs are strictly increasing in the actual cost). Thus, $\sum_{i\leq\tau} x_i \geq \delta$.

We claim that C^* is optimal (for the fixed transition vertex). First, note that if $y^* \leq \frac{\delta}{\tau-1}$ and thus $x_i = y^*$ for all $i < \tau$, then $\alpha^* = \beta^*$ and the chunking is optimal (over *all* transition vertices) by Lemma 3. Otherwise, suppose that $y^* > \frac{\delta}{\tau-1}$ and so $x_i = y = \frac{\delta}{\tau-1}$ for all $i < \tau$. Since $y < y^*$, and α^* is increasing in y, we know that $\beta^* > \alpha^*$. So the bottleneck of C^* is β^* in this case; by Lemma 2, any better chunking O with the same transition vertex must have $\sum_{i<\tau} x_i^O > \sum_{i<\tau} x_i^{C^*} = \delta$. Thus, O would have an earlier transition vertex, which is a contradiction.

Lastly, we discuss the runtime of the algorithm. In our analysis, for a fixed τ, we must compare the α and β values in two chunkings – the initial one where $x_i = \delta/t$ for all $i \leq \tau$, and the modified one where $x_i = y$ for all $i < \tau$. Since we have closed-form equations for the α and β values in each chunking, we do not need to construct them for each τ. We simply keep track of which value of τ produces the smallest perceived cost, and whether the best chunking for that τ was the initial chunking or the modified one. We can thus do only constant work for each τ, resulting in a runtime of $O(k)$. See Algorithm 1 for details.

Finally, we prove the remaining two edges cases.

The first is when $\tau = k$. In this case, the first chunking would set all costs equal to δ/k, which would not cover the full cost of the original edge. However, this case is also very simple, as all edges have the same perceived cost of $bx_i + c(u, w) + c(w \longrightarrow t)$ when $\tau = k$. So, this case proceeds as follows. First, we set all $x_i = x/k$. If $x/k < \frac{\delta}{k-1}$, this would satisfy the constraint that $\tau = k$, and since all edges would have the same perceived cost, this would be optimal. Otherwise, we would set $x_i = \frac{\delta}{k-1}$ for all $i < k$ and $x_k = x - \delta$, which would be optimal for $\tau = k$, as this would be as close as we could get to uniform costs.

Finally, we consider the case where $\delta > x$. We established earlier that the shortest path will switch from the chunking to the w vertices if at least δ work has been completed on the chunking. Since $\delta > x$, this can't happen, and so no matter how we chunk, the shortest path from any $u_{<k}$ is through w. This means that $p(e_i) = bx_i + c(u, w) + c(w \to t)$ for all $i < k$. Note that $e_k = (u_k, v)$; so, this final edge locks the agent into going to v. Thus, $p(e_k) = bx_k + c(v \to t) = b(x - \sum_{i<k} x_i) + c(v \to t)$. To optimally chunk, we set all $x_i = y$ for $i < k$ and then set the perceived cost of starting with the final edge equal to this to find the optimal y.

$$\begin{aligned} by + c(u, w) + c(w \to t) &= b(x - (k-1)y) + c(v \to t) \\ byk &= bx + c(v \to t) - c(u, w) - c(w \to t) = \delta + (b-1)x \\ y &= \frac{\delta + (b-1)x}{bk} \stackrel{\text{def}}{=} y^*. \end{aligned}$$

We now simply set $y = \min\left(y^*, \frac{x}{k-1}\right)$. If $y^* \le \frac{x}{k-1}$, then all perceived costs are equal, so this chunking is optimal by Lemma 3. If $y^* > \frac{x}{k-1}$, then the perceived cost of starting with the final edge is still higher, but the actual cost of that edge cannot be reduced below 0. Note that the case where $y^* > \frac{x}{k-1}$ (and $\delta > x$) is the only case where the optimal chunking might put a cost of 0 on any edge. □

B COST RATIO COROLLARY

Corollary 2. *Given a local constraint $k = O(n)$, the optimal chunking G' of G has constant cost ratio.*

Proof. Let c be a constant. By Theorem 6, we will get a cost ratio of $O(c)$ if $b_{\min} \le c^{1/n}$. We thus solve for the following equation for k:

$$\begin{aligned} \frac{1}{1 - \left(\frac{b-1}{b}\right)^k} &= c^{1/n} \\ \frac{1}{c^{1/n}} &= 1 - \left(\frac{b-1}{b}\right)^k \\ \left(\frac{b-1}{b}\right)^k &= 1 - \frac{1}{c^{1/n}} \\ \left(\frac{b-1}{b}\right)^k &= \frac{c^{1/n} - 1}{c^{1/n}} \\ k &= \frac{\log\left(\frac{c^{1/n}-1}{c^{1/n}}\right)}{\log\left(\frac{b-1}{b}\right)} \\ &= \frac{\log\left(\frac{c^{1/n}}{c^{1/n}-1}\right)}{\log\left(\frac{b}{b-1}\right)}. \end{aligned}$$

Since b is a constant, $\log\left(\frac{b}{b-1}\right)$ is constant, and k is thus dominated by the numerator. Similarly, $c^{1/n} < c$, and thus we are interested in the asymptotic behavior of $\log\left(\frac{1}{c^{1/n}-1}\right)$. The series expansion as $n \to \infty$ is $\frac{n}{\log c} - \frac{1}{2} + \frac{\log c}{12n} + O(\frac{1}{n^2}) = O(n)$. □

C NON-SHORT PATH EDGE CHUNKING ALGORITHM

Algorithm 1: Optimally chunk any edge. Uses Chunk-Shortest-Edge(k, x) as a subroutine, which returns the optimal k-chunking of a shortest edge of cost x, which is given by Theorem 1.

Input: A DAG G, edge (u, v) in G, bias factor b and chunking parameter k
Output: The optimal chunking for edge (u, v) and the associated bottleneck cost

$x \leftarrow c(u, v)$, $w \leftarrow$ next node in shortest $u \rightarrow t$ path
if $w = v$ **then** // edge case for when (u, v) is on the shortest path
  return Chunk-Shortest-Edge(k, x), $\frac{1}{1-\left(\frac{b-1}{b}\right)^k} x + c(v \rightarrow t)$
$\delta \leftarrow x + c(v \rightarrow t) - c(u, w) - c(w \rightarrow t)$
if $\delta > x$ **then**
  $y^* \leftarrow \frac{\delta+(b-1)x}{bk}$
  $C \leftarrow x_1, \ldots, x_{k-1} \mapsto \max(y^*, \frac{x}{k-1})$ and $x_k \mapsto x - (k-1)\max(y^*, \frac{x}{k-1})$
  $min\_bottleneck \leftarrow b\max(y^*, \frac{x}{k-1}) + c(u, w) + c(w \rightarrow t)$
  return $C, min\_bottleneck$
$min\_bottleneck \leftarrow \infty, \tau^* \leftarrow 0, opt\_chunk\_type \leftarrow 0$
for $\tau = 1$ **to** $k-1$ **do**
  $\alpha_0 \leftarrow \frac{b\delta}{\tau} + c(u, w) + c(w \rightarrow t)$, $\beta_0 \leftarrow \frac{x-\delta}{1-\left(\frac{b-1}{b}\right)^{k-\tau}} + c(v \rightarrow t)$
  if $\alpha_0 = \beta_0$ **then**
    $C \leftarrow x_1, \ldots, x_\tau \mapsto \delta/\tau$ and $x_{\tau+1}, \ldots, x_k \mapsto$ Chunk-Shortest-Edge$(k-\tau, x-\delta)$
    return C, α_0
  else if $\alpha_0 > \beta_0$ **then**
    if $\alpha_0 < min\_bottleneck$ **then**
      $min\_bottleneck \leftarrow \alpha_0, \tau^* \leftarrow \tau, opt\_chunk\_type \leftarrow 0$
  else
    if $\tau = 1$ **then** // edge case for $\tau = 1$
      return Chunk-Shortest-Edge(k, x), $\frac{1}{1-\left(\frac{b-1}{b}\right)^k} x + c(v \rightarrow t)$
    $z_\tau \leftarrow 1 - \left(\frac{b-1}{b}\right)^{k-\tau+1}$, $y^* \leftarrow \frac{\delta z_\tau + (1-z_\tau)x}{\tau-1+z_\tau b}$
    if $\frac{\delta}{\tau-1} > y^*$ **then**
      $C \leftarrow x_1, \ldots, x_{\tau-1} \mapsto y^*$ and $x_\tau, \ldots, x_k \mapsto$ Chunk-Shortest-Edge$(k-\tau+1, x-(\tau-1)y)$
      return $C, by^* + c(u, w) + c(w \rightarrow t)$
    else
      $\beta \leftarrow \frac{x-\delta}{z_\tau} + c(v \rightarrow t)$
      if $\beta < min\_bottleneck$ **then**
        $min\_bottleneck \leftarrow \beta, \tau^* \leftarrow \tau, opt\_chunk\_type \leftarrow 1$
if $\frac{x}{k} \leq \frac{\delta}{k-1}$ **then**
  $C \leftarrow x_1, \ldots, x_k \mapsto x/k$, $min\_bottleneck \leftarrow \frac{bx}{k} + c(u, w) + d(w)$
  return $C, min\_bottleneck$
else
  $\alpha \leftarrow \frac{b\delta}{k-1} + c(u, w) + c(w \rightarrow t)$, $\beta \leftarrow b(x-\delta) + c(v \rightarrow t)$
  if $\min(\alpha, \beta) \leq min\_bottleneck$ **then**
    $C \leftarrow x_1, \ldots, x_{k-1} \mapsto \frac{\delta}{k-1}$ and $x_k \mapsto x - \delta$, $min\_bottleneck \leftarrow \min(\alpha, \beta)$
    return $C, min\_bottleneck$
if $opt\_chunk\_type = 0$ **then**
  $C \leftarrow x_1, \ldots, x_{\tau^*} \mapsto \delta/\tau^*$ and $x_{\tau^*+1}, \ldots, x_k \mapsto$ Chunk-Shortest-Edge$(k-\tau^*, x-\delta)$
else
  $C \leftarrow x_1, \ldots, x_{\tau^*-1} \mapsto \frac{\delta}{\tau^*-1}$ and $x_{\tau^*}, \ldots, x_k \mapsto$ Chunk-Shortest-Edge$(k-\tau^*+1, x-\delta)$
return $C, min\_bottleneck$

D SPLITTING AGENTS ONTO SEPARATE PATHS

We first provide a full description of Algorithm 2. We now prove that this algorithm is correct via

Algorithm 2: Chunk (u, v) such that A_1 takes the chunking and A_2 doesn't, if possible.

maxBottleneck $\leftarrow 0$, $C^* \leftarrow \emptyset$
for $i = 1$ **to** k **do**
  $C_i = (x_1, \ldots, x_k) \leftarrow$ optimal chunking of (u, v) for A_1
  for $j = i - 1$ **to** 1 **do**
    if $p(e_i; b_1) < \alpha_u^{(1)}$ **then**
      $\delta = \min(x_j, (p(e_i; b_1) - \alpha_u^{(1)})/b_1)$
      $x_j \leftarrow x_j - \delta$
      $x_i \leftarrow x_i + \delta$
  for $j = i + 1$ **to** k **do**
    if $p(e_i; b_1) < \alpha_u^{(1)}$ **then**
      $\gamma_w \leftarrow c(u, w) + c(w \rightarrow t), \gamma_x \leftarrow \sum_{l>i} x_l + c(v \rightarrow t)$
      if $\gamma_x \leq \gamma_w$ **then**
        $\delta = \min(x_j, (p(e_i; b_1) - \alpha_u^{(1)})/(b_1 - 1))$
      else if $p(e_i; b_1) - \alpha_u^{(1)} \leq b_1(\gamma_x - \gamma_w)$ **then**
        $\delta = \min(x_j, (p(e_i; b_1) - \alpha_u^{(1)})/(b_1 - 1))$
      else
        $\delta' = \gamma_x - \gamma_w$
        $x_j \leftarrow x_j - \delta'$
        $x_i \leftarrow x_i + \delta'$
        $\delta = \min(x_j, (p(e_i; b_1) - \alpha_u^{(1)})/(b_1 - 1))$
      $x_j \leftarrow x_j - \delta$
      $x_i \leftarrow x_i + \delta$
  $\lambda \leftarrow \sum_{j=1}^{i-1} p(e_j; b_1) - \alpha_u^{(1)}$
  $\delta^* \leftarrow \min(\lambda/(b_1 - 1), \sum_{j>i} x_j / b_1$
  decrease $x_{>i}$ by $b_1 \delta^*$
  $x_i \leftarrow x_i + \delta^*$
  $j \leftarrow i - 1$
  while $\delta^* > 0$ **do**
    $\delta \leftarrow \min(x_j, \delta^*)$
    $x_j \leftarrow x_j - \min(x_j, \delta)$
    $\delta^* \leftarrow \delta^* - \delta$
    $j \leftarrow j - 1$
  bottleneck $\leftarrow \max_j p(e_j; b_2)$
  if *bottleneck > maxBottleneck* **then**
    maxBottleneck $\leftarrow$ bottleneck
    $C^* \leftarrow C$
return C^*

the following theorem.

Theorem 7. *If C_i is the output of the ith iteration of Algorithm 2 and C' is another chunking such that $p(e_i'; b_2) > p(e_i; b_2)$, then A_1 will not take C'.*

Proof. First, suppose that $\sum_{j \neq i} x_j = 0$, that is, all of the weight is on e_i in C_i. It's obvious that $p(e_i'; b_2) \leq p(e_i; b_2)$, as the perceived cost of any chunk cannot exceed $bc(u, v) + c(v \rightarrow t)$, and C_i achieves this cost on e_i. It follows that $\sum_{j \neq i} x_j > 0$, which implies that $p(e_i; b_1) = \alpha_u^{(1)}$ by Lemma 5(a). We now consider two cases.

Case 1: $x_i' > x_i$. Suppose that $\sum_{j>i} x_j = 0$. We must have $x_j' \geq x_j$,for all $j \geq i$, as costs must be non-negative. Thus, $p(e_i'; b_1) > p(e_i; b_1)$, and A_1 will deviate from C' at edge i.

So, suppose instead that $\sum_{j>i} x_j > 0$. By Lemma 5(b), we have that $\forall j \leq i, p(e_j; b_1) = \alpha_u^{(1)}$. A similar now argument applies: if $\sum_{j>i} x_j' < \sum_{j>i} x_j$, then more weight must be put on $x_{\leq i}$, and it's clear that doing so would cause A_1 to deviate before or at edge i (concretely, A_1 would deviate at the first edge with higher weight). But if $\sum_{j>i} x_j' \geq \sum_{j>i} x_j$, then $p(e_i'; b_1) > p(e_i; b_1)$, and A_1 will deviate from C' at edge i. Either way, A_1 will not take the chunking C'.

Case 2: $x_i' \leq x_i$. Recall that we can write $p(e_i; b_2)$ as $b_2 x_i + c(u_{i+1} \longrightarrow t)$, where $c(u_{i+1} \longrightarrow t)$, the cost of the cheapest path from u_{i+1} to t, is $\min(c(u, w) + c(w \longrightarrow t), \sum_{j>i} x_j + c(v \longrightarrow t))$.

$$
\begin{aligned}
& p(e_i'; b_2) > p(e_i; b_2) \\
\iff & b_2 x_i' + c(u_{i+1}' \longrightarrow t) > b_2 x_i + c(u_{i+1} \longrightarrow t) \\
\iff & c(u_{i+1}' \longrightarrow t) - c(u_{i+1} \longrightarrow t) > b_2(x_i - x_i') \\
\implies & c(u_{i+1}' \longrightarrow t) - c(u_{i+1} \longrightarrow t) > b_1(x_i - x_i') && \text{(since } b_2 > b_1 \text{ and } x_i - x_i' \geq 0\text{)} \\
\iff & b_1 x_i' + c(u_{i+1}' \longrightarrow t) > b_1 x_i + c(u_{i+1} \longrightarrow t) \\
\iff & p(e_i'; b_1) > p(e_i; b_1).
\end{aligned}
$$

Since $p(e_i; b_1) = \alpha_u^{(1)}$, A_1 won't take C' (they will deviate at e_i'). □

We now describe the flipped version of this problem, where we chunk (u, v) so that A_2 takes it but A_1 finds it maximally unappealing. The flipped algorithm has the same phase 1 and 2 as before.[8] Phase 3 is modified to:

3. Let λ be the total amount of cost that could be added to $x_{>i}$ while ensuring that $p(e_j; b_1) \leq \alpha_u^{(1)}$ for all $j > i$. Let $\delta = \min(\lambda/b_2, \sum_{j<i} x_j/(b_2 - 1), x_i)$. Decrease x_i by δ, decrease the cumulative cost of $x_{<i}$ by $(b_2 - 1)\delta$, and increase the cumulative cost of $x_{>i}$ by $b_2\delta$.

We also modify part (b) of the lemma.

Lemma 8. *Let $C = (e_1, \ldots, e_k)$ be the chunking produced by the algorithm above. Then:*

(a) $\sum_{j \neq i} x_j > 0 \implies p(e_i; b_2) = \alpha_u^{(2)}$
(b) $\sum_{j<i} x_j > 0$ *and* $x_i > 0 \implies \forall j > i, p(e_j; b_2) = \alpha_u^{(2)}$

Proof. The proof of (a) is identical to before. For (b), as before, if more could be siphoned from $\sum_{j<i} x_j$ and x_i, the algorithm would, unless no edges in $e_{>i}$ can be increased further. □

Theorem 13. *Let C be the output of the algorithm above. Let C' be another chunking such that $p(e_i'; b_1) > p(e_i; b_1)$. Then, A_2 will not take C'.*

Proof. First, suppose that $\sum_{j \neq i} x_j = 0$, i.e., all of the weight is on e_i in C. Then, it's obvious that $p(e_i'; b_1) \leq p(e_i; b_1)$, as the perceived cost of any chunk cannot exceed $bc(u, v) + c(v \longrightarrow t)$, and C achieves this cost on e_i.

So, we know that $\sum_{j \neq i} x_j > 0$, which implies that $p(e_i; b_1) = \alpha_u^{(1)}$ by Lemma 8(a). We now consider two cases.

[8] We omit the full pseudocode for the modified algorithm, as it's easy to modify the third phase of Algorithm 2.

Case 1: $\sum_{j>i} x'_j \leq \sum_{j>i} x_j$. Recall that $c(u_{i+1} \to t) = \min(c(u,w) + c(w \to t), \sum_{j>i} x_j + c(v \to t))$. Thus, $\sum_{j>i} x'_j \leq \sum_{j>i} x_j$ implies that $c(u_{i+1} \to t) \geq c(u'_{i+1} \to t)$.

$$\begin{aligned}
& p(e'_i; b_1) > p(e_i; b_1) \\
\iff & b_1 x'_i + c(u'_{i+1} \to t) > b_1 x_i + c(u_{i+1} \to t) \\
\iff & b_1(x'_i - x_i) > c(u_{i+1} \to t) - c(u'_{i+1} \to t) \\
\implies & b_2(x'_i - x_i) > c(u_{i+1} \to t) - c(u'_{i+1} \to t) \\
& \qquad \text{(since } b_2 > b_1 \text{ and } c(u_{i+1} \to t) - c(u'_{i+1} \to t) \geq 0) \\
\iff & b_2 x'_i + c(u'_{i+1} \to t) > b_2 x_i + c(u_{i+1} \to t) \\
\iff & p(e'_i; b_2) > p(e_i; b_2).
\end{aligned}$$

Since $p(e_i; b_2) = \alpha_u^{(2)}$, A_2 won't take C'.

Case 2: $\sum_{j>i} x'_j > \sum_{j>i} x_j$. Suppose, for the sake of contradiction, that $\sum_{j<i} x_j = 0$. Then, $\sum_{j \geq i} x_j = x$, i.e., all the weight is on edges $e_{\geq i}$. Now, $p(e'_i; b_1) > p(e_i; b_1)$ requires either that $x'_i > x_i$, or that C' assigns more cost to edges $e_{\geq i}$ than C. The latter is impossible because C assigns all the weight to edges $e_{\geq i}$, and thus $x'_i > x_i$. This implies that $\sum_{j>i} x'_j < \sum_{j>i} x_j$, which gives us a contradiction.

So it follows that $\sum_{j<i} x_j > 0$. We now consider two cases. First, suppose that $x_i > 0$. We apply Lemma 8(b), which says that $\forall j > i, p(e_j; b_2) = \alpha_u^{(2)}$. Since $\sum_{j>i} x'_j > \sum_{j>i} x_j$, by Lemma 6 there must be some edge e'_j such that $p(e'_j; b_2) > p(e_j; b_2) = \alpha_u^{(2)}$, and thus A_2 deviates from C'.

Second, suppose that $x_i = 0$. $x'_i \geq x_i$. This combined with the fact that $\sum_{j>i} x'_j > \sum_{j>i} x_j$ implies that $p(e'_i; b_2) > p(e_i; b_2) = \alpha_u^{(2)}$, and thus A_2 doesn't take C'. □

E KEEPING AGENTS ON THE SAME PATH

Lemma 6. *Let C and C' be two (possibly partial) chunkings of the same edge. Suppose that $\sum_{i=l}^k x'_i > \sum_{i=l}^k x_i$. Then, there exists an $i \in [l,k]$ such that for all $b > 1$, $p(e'_i; b) > p(e_i; b)$.*

Proof. We prove the contrapositive. That is, suppose that, for all $i \in [l,k]$, there exists some b_i such that $p(e_i; b_i) \geq p(e'_i; b_i)$. We show that $\sum_{i=j}^k x_i \geq \sum_{i=j}^k x'_i$ by induction from $j = k$ to l.

For the base case, suppose that $j = k$. Note that $p(e_k; b_k) \geq p(e'_k; b_k)$ if and only if $b_k x_k + c(v \to t) \geq b_k x'_k + c(v \to t)$, which implies that $x_k \geq x'_k$, as desired. For the inductive case, assume that $\sum_{i>j}^k x_i \geq \sum_{i>j}^k x'_i$. First, we expand $p(e_j; b_j) \geq p(e'_j; b_j)$:

$$bx_j + \min(c(u,w) + c(w \to t), \sum_{i>j}^k x_i + c(v \to t)) \geq bx'_j + \min(c(u,w) + c(w \to t), \sum_{i>j}^k x'_i + c(v \to t)). \tag{2}$$

We now proceed by cases.

Case 1. Suppose that $\min(c(u,w) + c(w \to t), \sum_{i>j}^k x'_i + c(v \to t)) = c(u,w) + c(w \to t)$. Since $\sum_{i>j}^k x'_i + c(v \to t) \leq \sum_{i>j}^k x_i + c(v \to t)$ by the inductive hypothesis, we also know that $\min(c(u,w) + c(w \to t), \sum_{i>j}^k x_i + c(v \to t)) = c(u,w) + c(w \to t)$. Thus, Equation 2 holds if and only if:

$$\begin{aligned}
& bx_j + c(u,w) + c(w \to t) \geq bx'_j + c(u,w) + c(w \to t) \\
\iff & bx_j \geq bx'_j \\
\iff & x_j \geq x'_j.
\end{aligned}$$

Combining this with the inductive hypothesis yields $\sum_{i \geq j} x_i \geq \sum_{i \geq j} x'_i$, as desired.

Case 2. Suppose that $\min(c(u, w) + c(w \longrightarrow t), \sum_{i>j}^{k} x'_i + c(v \longrightarrow t)) = \sum_{i>j}^{k} x'_i + c(v \longrightarrow t)$. Clearly $\min(c(u, w) + c(w \longrightarrow t), \sum_{i>j}^{k} x_i + c(v \longrightarrow t)) \leq \sum_{i>j}^{k} x_i + c(v \longrightarrow t)$. Thus, Equation 2 implies:

$$
\begin{aligned}
& bx_j + \sum_{i>j} x_i + c(v \longrightarrow t) \geq bx'_j + \sum_{i>j} x'_i + c(v \longrightarrow t) \\
\iff & bx_j + b\sum_{i>j} x_i \geq bx'_j + \sum_{i>j} x'_i + (b-1)\sum_{i>j} x_i && \text{(adding } (b-1)\textstyle\sum_{i>j} x_i \text{ to both sides)} \\
\implies & bx_j + b\sum_{i>j} x_i \geq bx'_j + b\sum_{i>j} x'_i && \text{(since } \textstyle\sum_{i>j} x_i \geq \sum_{i>j} x'_i \text{ by the IH)} \\
\iff & \sum_{i \geq j} x_i \geq \sum_{i \geq j} x'_i.
\end{aligned}
$$

The last line proves the inductive step, and thus completes the proof. □

F GRAPH-CHUNKING THEOREMS FOR MULTIPLE AGENTS

Theorem 9. *Given any task graph $G = (V, E)$ and a local constraint k, we can optimally chunk G for two types of agents in time $O(|E|^2 k^2 + |V|)$.*

Proof. The main computational bottleneck is computing $\mathcal{P}(u, y)$ for all $u, y \in V$. For $u \neq y$, this is very simple: we can chunk edges for each agent independently when they aren't at the same node. Doing so requires $2|E|$ applications of Algorithm 1 ($|E|$ applications for each agent), for a runtime of $O(2|E|k)$. For $\mathcal{P}(u, u)$, consider all $(v, z) \in N(u) \times N(u)$. There are a total of $|E|^2$ such pairs over all choices of u. When $v \neq z$, we apply Algorithm 2 (to (u, v) and (u, z)), and when $v = z$, we apply Algorithm 3. Algorithm 2 runs in $O(k^2)$ time and Algorithm 3 runs in $O(k)$ time (for $m = 2$ agents). Thus, the total runtime to compute $\mathcal{P}$ is $O(2|E|^2k^2 + |E|k + 2|E|k) = O(|E|^2k^2)$.

Once we have $\mathcal{P}$, we need to compute the cost recurrence. For each element in $\mathcal{P}$, we compute the min over the three constant time functions C_1, C_2, and C_3, for a total time of $O(|\mathcal{P}|) = O(|E|^2)$, since $\mathcal{P} \subseteq E \times E$. Thus, the cost recurrence takes $O(|E|^2)$ time to compute, which means that the total runtime is dominated by computing $\mathcal{P}$.

For correctness, $\mathcal{P}$ is correct by Theorem 13, Theorem 7, and Theorem 8. Given the correctness of $\mathcal{P}$, the cost recurrence is correct by Lemma 7. □

Theorem 10. *Given any task graph $G = (V, E)$ and a global constraint k, we can optimally chunk G for two types of agents in time $O(|E|^2 k^3 \log k + |V|)$.*

Proof. We slightly modify the definition of $\mathcal{P}'(u, y)$ to be the set of (v, z, i) such that i is the minimum number of chunks needed for (u, v) and (v, z) to be compatibly chunked. With this, we can modify the cost recurrence in the obvious way. The individual cases become:

$$
C_1(u, v, y, i) = \begin{cases} c(y, u) + cost[u, u, i] & \text{if } (v, u) \in \mathcal{P}'(u, y) \\ \infty & \text{otherwise} \end{cases}
$$

$$
C_2(u, y, z, i) = \begin{cases} c(u, y) + cost[y, y, i] & \text{if } (y, z) \in \mathcal{P}'(u, y) \\ \infty & \text{otherwise} \end{cases}
$$

$$
C_3(u, v, y, z, i) = c(u, v) + c(y, z) + cost[v, z, i].
$$

And the recurrence becomes:

$$
cost[u, y, i] = \min_{(v,z,l) \in \mathcal{P}'(u,y): l \leq i} \min(C_1(u, v, y, i-l), C_2(u, y, z, i-l), C_3(u, v, y, z, i-l)).
$$

Computing this recurrence will take time $O(|E|^2 k)$, but this will not be the bottleneck. The correctness of this recurrence follows simply from the correctness of $\mathcal{P}'$. It remains to show how to compute this new $\mathcal{P}'$.

For $\mathcal{P}'(u, y)$ where $u \neq y$, it is easy to return the minimum number of chunks needed to chunk the edges; we already solved this problem with binary search in the single-agent global budget case (Theorem 4). This takes $O(2|E| \log k)$ time for all $u \neq y$. Now suppose $u = y$. If $v = z$, and we're thus trying to keep agents on the same path, we can also use binary search with Algorithm 3 to find the minimum number of chunks to get both agents to stick to the path. This takes $O(|E| \log k)$ time in total.

The bottleneck is computing the minimum number of chunks to get A_1 to take (u, v) and A_2 to take (u, z). We can visualize the problem as searching through a two dimensional binary array, where $arr[i, j] = 1$ iff we can get a compatible chunking where A_1 takes an i-chunking of (u, v) and A_2 takes a j-chunking of (u, z). Luckily, the array is row-wise and column-wise sorted; that is, we can always simulate an i-chunking with an $(i + 1)$-chunking (e.g., set the first chunk to 0), so if A_1 can take an i-chunking of (u, v) and A_2 can take a j-chunking of (u, z), then it's true that A_1 can take an $(i + 1)$-chunking of (u, v) and A_2 can take a j-chunking of (u, z). Our goal is to find $\min_{i,j:arr[i,j]=1} i + j$. In the worst case, the matrix has dimensions $k \times k$.[9]

One solution is to run binary search on each column of the matrix; this involves looking at $O(k \log k)$ entries of the matrix. The minimum indices will clearly be found this way, as the minimum point will be the lowest 1 entry in some column. Evaluating each entry requires us to run Algorithm 2, which runs in $O(k^2)$. Thus, the total runtime over all edges in the graph is $O(|E|^2 k^3 \log k)$. This brings the total computation cost to $O(|E|^2 k^3 \log k + |V|)$. The correctness of $\mathcal{P}'$ follows obviously from the correctness of Algorithm 1, Algorithm 2, and Algorithm 3. □

Theorem 11. *Given any task graph $G = (V, E)$ and a local constraint k, we can find the optimal single-path chunking of G for m types of agents with at most $|E|$ applications of Algorithm 3, for a total runtime of $O(|E|mk + |V|)$.*

Proof. We simply use Algorithm 3 to determine which edges can be chunked such that all agents will take the chunking. Keep only those edges in the graph, and run a shortest-path algorithm. This is exactly analogous to Theorem 3, except that the runtime increases by a factor of m because Algorithm 3 runs in $O(mk)$ time. □

Theorem 12. *Given any task graph $G = (V, E)$ and a global constraint k, we can find the optimal single-path chunking of G for m types of agents with at most $|E| \log k$ applications of Algorithm 3, for a total runtime of $O(|E|mk \log k + |V|)$.*

Proof. This is exactly the same as the proof of Theorem 4, except that we use binary search to find the minimum number of chunks l_e such that *all* agents take the optimal l_e chunking of edge e. Thus, we run Algorithm 3 $\log k$ times for each edge, resulting in a total runtime of $O(|E|mk \log k + |V|)$. □

[9]Technically, we care only about the lower triangle (i.e., entries $arr[i, j]$ where $i + j \leq k$), but this doesn't affect the asymptotic runtime.