

Commutators in the two scalar and matrix weighted setting

Joshua Isralowitz¹ | Sandra Pott² | Sergei Treil³

¹Department of Mathematics and Statistics, SUNY Albany, Albany, New York, USA

²Centre for Mathematical Sciences, Lund University, Lund, Sweden

³Department of Mathematics, Brown University, Providence, Rhode Island, USA

Correspondence

Joshua Isralowitz, Department of Mathematics and Statistics, SUNY Albany, 1400 Washington Ave., Albany, NY 12222, USA.

Email: jisralowitz@albany.edu

Abstract

In this paper, we approach the two weighted boundedness of commutators via matrix weights. This approach provides both a sufficient and a necessary condition for the two weighted boundedness of commutators with an arbitrary linear operator in terms of one matrix weighted norm inequalities for this operator. Furthermore, using this approach, we surprisingly provide conditions that almost characterize the two matrix weighted boundedness of commutators with CZOs and completely arbitrary matrix weights, which is even new in the fully scalar one weighted setting. Finally, our method allows us to extend the two weighted Holmes/Lacey/Wick results to the fully matrix setting (two matrix weights and a matrix symbol), completing a line of research initiated by the first two authors.

MSC 2020

42B20, 42B35 (primary), 47A30 (secondary)

1 | INTRODUCTION AND MAIN RESULTS

Let w be a weight on $\mathbb{R}^d$ and let $L^p(w)$ be the standard weighted Lebesgue space with respect to the norm

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^d} |f(x)|^p w(x) dx \right)^{\frac{1}{p}}.$$

Furthermore, let A_p be the Muckenhoupt class of weights w satisfying

$$\sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} \left(\int_Q w(x) dx \right) \left(\int_Q w^{-\frac{1}{p-1}}(x) dx \right)^{p-1} < \infty$$

where f_Q is the unweighted average over Q (which will also occasionally be denoted by m_Q).

Given a weight ν , we say $b \in \text{BMO}_\nu$ if

$$\|b\|_{\text{BMO}_\nu} = \sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} \frac{1}{\nu(Q)} \int_Q |b(x) - m_Q b| dx < \infty$$

(where $\nu(Q) = \int_Q \nu$) so that clearly $\text{BMO} = \text{BMO}_\nu$ when $\nu \equiv 1$. Further, given a linear operator T , define the commutator $[M_b, T] = M_b T - T M_b$ with M_b being multiplication by b . In the papers [9, 10], the authors extended earlier work of Bloom [1] and proved that if $u, v \in A_p$ and T is any Calderón–Zygmund operator (CZO), then

$$\|[M_b, T]\|_{L^p(u) \rightarrow L^p(v)} \lesssim \|b\|_{\text{BMO}_\nu} \quad (1.1)$$

where $\nu = (uv^{-1})^{\frac{1}{p}}$ and it was proved in [10] that if R_s is the s th Riesz transform, then

$$\|b\|_{\text{BMO}_\nu} \lesssim \max_{1 \leq s \leq d} \|[M_b, R_s]\|_{L^p(u) \rightarrow L^p(v)}. \quad (1.2)$$

The purpose of this paper is to give largely self contained proofs of (1.1) and (1.2) and to extend both to the case of two matrix A_p weights and a matrix symbol B by using arguments inspired by the matrix weighted techniques developed in [6]. Furthermore, as byproducts of some of our results, we will provide both a sufficient and a necessary condition for the two weight boundedness of commutators with an arbitrary linear operator in terms of matrix weighted norm inequalities for this operator. Furthermore, we will provide conditions that almost characterize the two matrix weighted boundedness of commutators with CZOs and completely arbitrary matrix weights, which is even new in the fully scalar one weighted setting.

In particular, let $W : \mathbb{R}^d \rightarrow \mathbb{M}_{n \times n}$ be an $n \times n$ matrix weight (a positive definite almost everywhere $\mathbb{M}_{n \times n}$ valued function on $\mathbb{R}^d$) and let $L^p(W)$ be the space of $\mathbb{C}^n$ valued functions $\vec{f}$ such that

$$\|\vec{f}\|_{L^p(W)} = \left(\int_{\mathbb{R}^d} |W^{\frac{1}{p}}(x) \vec{f}(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

Furthermore, we will say that a matrix weight W is a matrix A_p weight (see [22]) if it satisfies

$$[W]_{A_p} = \sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} \int_Q \left(\int_Q \|W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y)\|^{p'} dy \right)^{\frac{p}{p'}} dx < \infty.$$

Before we state our results, let us rewrite Bloom's BMO condition in a way that naturally extends to the matrix weighted setting. First, by multiple uses of the A_p property and Hölder's inequality,

it is easy to see that

$$m_Q v \approx (m_Q u)^{\frac{1}{p}} (m_Q v^{-\frac{p'}{p}})^{\frac{1}{p'}} \approx (m_Q u)^{\frac{1}{p}} (m_Q v)^{-\frac{1}{p}} \approx (m_Q u^{\frac{1}{p}}) (m_Q v^{\frac{1}{p}})^{-1}$$

(where again, m_Q denotes unweighted average) so that $b \in \text{BMO}_v$ when u and v are A_p weights if and only if

$$\sup_{\substack{Q \subseteq \mathbb{R} \\ Q \text{ is a cube}}} \int_Q (m_Q v^{\frac{1}{p}}) (m_Q u^{\frac{1}{p}})^{-1} |b(x) - m_Q b| dx < \infty.$$

Now if U, V are matrix A_p weights, then we define $\text{BMO}_{V,U}^p$ to be the space of $n \times n$ locally integrable matrix functions B where

$$\|B\|_{\text{BMO}_{V,U}^p} = \sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} \left(\int_Q \|(m_Q V^{\frac{1}{p}})(B(x) - m_Q B)(m_Q U^{\frac{1}{p}})^{-1}\| dx \right)^{\frac{1}{p}} < \infty$$

so that $\|b\|_{\text{BMO}_{V,U}^p} \approx \|b\|_{\text{BMO}_v}$ if U, V are scalar weights and b is a scalar function. Note that the $\text{BMO}_{V,U}^p$ condition is much more naturally defined in terms of reducing matrices, which will be discussed in the next section.

In this paper, we will prove the following two theorems, the first of which is a generalization of a similar but much weaker result proved in [11].

Theorem 1.1. *Let T be any linear operator defined on scalar valued function where its canonical vector-valued extension $T \otimes \mathbf{I}_n$ is bounded on $L^p(W)$ for all $n \times n$ matrix A_p weights W and all $n \in \mathbb{N}$ with bound depending on T, n, d, p , and $[W]_{A_p}$ (which is known to be true for all CZOs, see [2] for a very easy proof). If U, V are $m \times m$ matrix A_p weights and B is an $m \times m$ locally integrable matrix function for some $m \in \mathbb{N}$, then*

$$\|[M_B, T \otimes \mathbf{I}_m]\|_{L^p(U) \rightarrow L^p(V)} \lesssim \|B\|_{\text{BMO}_{V,U}^p}$$

with bounds depending on $T, m, d, p, [U]_{A_p}$ and $[V]_{A_p}$.

In particular, let $\mathbf{I}_n$ denote the $n \times n$ identity matrix. In the case when u, v , and b are scalar valued (that is, $m = 1$), we have that (1.1) holds for any linear operator T such that $T \otimes \mathbf{I}_n$ is bounded on $L^p(W)$ for all $n \times n$ matrix A_p weights W and all $n \in \mathbb{N}$ (and in particular we have (1.1) for all CZOs).

We will need one more definition before we state our second main result. Given a finite collection $R = \{R_s\}_{s=1}^N$ of linear operators defined on scalar valued functions, we say that R is a lower bound collection if for any $n \in \mathbb{N}$ and any $n \times n$ matrix weight W we have

$$[W]_{A_p}^{\frac{1}{p}} \lesssim \max_{1 \leq s \leq N} \|R_s \otimes \mathbf{I}_n\|_{L^p(W) \rightarrow L^p(W)} \quad (1.3)$$

with the bound independent of W (but not necessarily independent of n), and each $R_s \otimes \mathbf{I}_n$ is bounded on $L^p(W)$ if W is a matrix A_p weight. It should be noted that, as one would expect, the Hilbert transform itself and more generally the collection $\{R_\ell\}_{\ell=1}^d$ of Riesz transforms are lower bound collections (which will be proved in Lemma 3.5.)

Theorem 1.2. If $R = \{R_s\}_{s=1}^N$ is a lower bound collection, then for any $m \times m$ matrix A_p weights U, V and any $m \times m$ locally integrable matrix symbol B we have

$$\|B\|_{BMO_{V,U}^p} \lesssim \max_{1 \leq s \leq N} \|[M_B, R_s \otimes \mathbf{I}_m]\|_{L^p(U) \rightarrow L^p(V)}.$$

Here, the bound depends possibly on m, R, p, d and R but is independent of U and V .

Let us briefly outline the strategy for proving Theorems 1.1 and 1.2. In the next section, we will use matrix weighted arguments inspired by [6] to prove Theorems 1.1 and 1.2 in terms of a weighted BMO quantity $\|B\|_{\widetilde{BMO}_{V,U}^p}$ that is equivalent to $\|B\|_{BMO_{V,U}^p}$ when U and V are matrix A_p weights (see Corollary 4.7) but is much more natural for more arbitrary matrix weights U and V . More precisely, define

$$\|B\|_{\widetilde{BMO}_{V,U}^p}^p = \sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} \int_Q \left(\int_Q \left\| V^{\frac{1}{p}}(x)(B(x) - B(y))U^{-\frac{1}{p}}(y) \right\|^{p'} dy \right)^{\frac{p}{p'}} dx.$$

In particular, in the case of two scalar weights u, v and a scalar symbol b , note that

$$\|b\|_{\widetilde{BMO}_{u,v}^p}^p = \sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} \int_Q \left(\int_Q |b(x) - b(y)|^{p'} u^{-\frac{p'}{p}}(y) dy \right)^{\frac{p}{p'}} v(x) dx$$

which has a particularly simple and appealing appearance when $p = 2$, namely

$$\|b\|_{\widetilde{BMO}_{u,v}^2}^2 = \sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} \int_Q \int_Q v(x) |b(x) - b(y)|^2 u^{-1}(y) dy dx$$

We will then give relatively short proofs of the following two results in Section 2.

Lemma 1.3. Let T be any linear operator defined on scalar valued functions where its canonical vector-valued extension $T \otimes \mathbf{I}_n$ satisfies

$$\|T \otimes \mathbf{I}_n\|_{L^p(W) \rightarrow L^p(W)} \leq \phi([W]_{A_p})$$

for some positive increasing function ϕ (possibly depending on T, d, n and p .) If U, V are $m \times m$ matrix A_p weights and B is a locally integrable $m \times m$ matrix valued function for some $m \in \mathbb{N}$, then

$$\|[M_B, T \otimes \mathbf{I}_m]\|_{L^p(U) \rightarrow L^p(V)} \leq \|B\|_{\widetilde{BMO}_{V,U}^p} \phi\left(3^{\frac{p}{p'}} ([U]_{A_p} + [V]_{A_p}) + 1\right).$$

Lemma 1.4. If $R = \{R_s\}_{s=1}^N$ is a lower bound collection of operators, then for any $m \times m$ matrix A_p weights U, V and an $m \times m$ matrix symbol B , we have

$$\|B\|_{\widetilde{BMO}_{V,U}^p} \lesssim \max_{1 \leq s \leq N} \|[M_B, R_s \otimes \mathbf{I}_n]\|_{L^p(U) \rightarrow L^p(V)}$$

where the bound depends possibly on n, p, d and R but is independent of U and V .

Recall that a scalar weight w on $\mathbb{R}^d$ is said to satisfy the A_∞ condition if we have

$$[w]_{A_\infty} = \sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} \frac{1}{w(Q)} \int_Q M(w \mathbb{1}_Q) < \infty,$$

where M is the ordinary Hardy–Littlewood maximal function on $\mathbb{R}^d$. Further, for a matrix weight U we define the ‘scalar A_∞ characteristic’ as in [2, 21] by

$$[U]_{A_{p,\infty}^{\text{sc}}} = \sup_{\vec{e} \in \mathbb{C}^n} \left[\left| U^{\frac{1}{p}} \vec{e} \right|^p \right]_{A_\infty}$$

which for any $1 < p < \infty$ obviously reduces to the ordinary A_∞ characteristic in the scalar setting.

At the end of Section 2, we will estimate $\|b\|_{\widetilde{\text{BMO}}_{U,U}^p}$ for a scalar function b and a matrix A_p weight U to give us the following quantitative version of Theorem 1.1.

Proposition 1.5. *Assume T satisfies the hypothesis of Lemma 1.3. Then there exists C independent of U, V, b and T where*

$$\|[M_b, T \otimes \mathbf{I}_m]\|_{L^p(U) \rightarrow L^p(U)} \leq \|b\|_{\text{BMO}} \left([U]_{A_{p,\infty}^{\text{sc}}} + \left[U^{-\frac{p'}{p}} \right]_{A_{p',\infty}^{\text{sc}}} \right) \phi(C[U]_{A_p}).$$

It is interesting to remark that Lemma 1.3 and Proposition 1.5 would provide new quantitative one and two weight commutator bounds in the scalar setting if the ‘matrix A_p conjecture’

$$\|T \otimes \mathbf{I}_m\|_{L^p(W) \rightarrow L^p(W)} \lesssim [W]_{A_p}^{\max\{1, \frac{1}{p-1}\}}$$

were to hold for all CZOs T , even in the case $p = 2$. Also, we will prove that the collection of Riesz transforms form a lower bound operator in Section 3 by utilizing the Schur multiplier/Wiener algebra ideas from [18], and thus recovering (1.2). In fact, we will show much more and prove the following surprising result.

Theorem 1.6. *Let $\{R_\ell\}_{\ell=1}^d$ be the collection of Riesz transforms, and let U and V be any (not necessarily A_p) matrix weights. If B is any locally integrable $m \times m$ matrix valued function, then*

$$\max \left\{ \|B\|_{\widetilde{\text{BMO}}_{V,U}^p}, \|B\|_{\widetilde{\text{BMO}}_{U',V'}^{p'}} \right\} \lesssim \max_{1 \leq \ell \leq d} \|[M_B, R_\ell \otimes \mathbf{I}_m]\|_{L^p(U) \rightarrow L^p(V)}. \quad (1.4)$$

Moreover, we will show that an Orlicz ‘bumped’ version of these conditions are sufficient for the general two matrix weighted boundedness of a CZO. In particular, we will prove the following result in Section 2, which is similar in statement and proof of [14, Lemma 4].

Proposition 1.7. *Let T be a CZO, U and V be any $m \times m$ matrix weights, and B be any locally integrable $m \times m$ matrix valued function. Let C and D be Young functions with $\overline{C} \in B_{p'}$ and $\overline{D} \in B_p$*

where $\overline{C}$ and $\overline{D}$ are the conjugate Young functions to C and D , respectively. Then

$$\| [M_B, T \otimes \mathbf{I}_m] \|_{L^p(U) \rightarrow L^p(V)} \lesssim \min\{\kappa_1, \kappa_2\},$$

where

$$\begin{aligned} \kappa_1 &= \sup_Q \| \| V^{\frac{1}{p}}(x)(B(x) - B(y))U^{-\frac{1}{p}}(y) \|_{C_{x,Q}} \|_{D_{y,Q}} \\ \kappa_2 &= \sup_Q \| \| V^{\frac{1}{p}}(x)(B(x) - B(y))U^{-\frac{1}{p}}(y) \|_{D_{y,Q}} \|_{C_{x,Q}}. \end{aligned}$$

We refer the reader to [14, Section 5.2] for the standard Orlicz space related definitions used in the statement of the Proposition 1.7 (and in particular the definition of the classes B_p and $B_{p'}$).

It is important to emphasize that Theorem 1.6 and Proposition 1.7 are new, even in the scalar $p = 2$ setting of a single weight. It is also interesting to note that formally ‘removing’ b from the condition $\|b\|_{\widetilde{\text{BMO}}_{v,u}^p}^p < \infty$ in the case of two scalar weights u and v reduces to the classical two weight A_p condition $(u, v) \in A_p$. From this perspective, $\|b\|_{\widetilde{\text{BMO}}_{v,u}^p}^p$ can be thought of as a first-order analogy of the ‘zero-order’ condition $(u, v) \in A_p$. In particular, it is well known (see [19]) that $(u, v) \in A_p$ is necessary for the two weighted norm boundedness of the Hilbert transform, and that an Orlicz bumped version of $(u, v) \in A_p$ is sufficient for the two weighted boundedness of any CZO T , see [15]. More precisely, either of the equivalent conditions in Proposition 1.7 when again b is ‘removed’ is sufficient for the two weighted boundedness of any CZO T . Thus, Theorem 1.6 and Proposition 1.7 should be thought of as a first-order commutator version of the well-known ‘zero order’ scalar results above.

A key tool for the proof of Proposition 1.7 is a new convex body domination theorem, which is interesting in its own right and therefore stated here. It was essentially proven in [14] (though not explicitly stated) and is more suitable for us here than the sparse convex body domination of commutators in [14, Theorem 4].

To state the result, we need some notation. Let $\mathcal{D}$ be a dyadic grid of cubes in $\mathbb{R}^d$. Recall that $S \subset \mathcal{D}$ is a sparse family if for every $Q \in S$ there exists a measurable subset $E_Q \subset Q$ such that:

- (1) $|Q| \leq 2|E_Q|$;
- (2) The sets E_Q are pairwise disjoint.

Theorem 1.8. *Let T be a CZO. For every $\mathbb{C}^m$ valued function $\vec{f}$ with compact support and every $m \times m$ valued matrix function B such that $B\vec{f} \in L^1$, there exists 3^d sparse collections S_j of dyadic cubes, a constant $c_{d,m,T}$, and for each $Q \in S_j$ a function $k_Q : Q \times Q \rightarrow \mathbb{R}$ with $\|k_Q\|_{L^\infty(Q \times Q)} \leq 1$ such that*

$$[M_B, T \otimes \mathbf{I}_m]\vec{f}(x) = c_{d,m,T} \sum_{j=1}^{3^d} \sum_{Q \in S_j} \mathbb{1}_Q(x) \int_Q k_Q(x, y)(B(x) - B(y))\vec{f}(y) dy \quad (x \in \mathbb{R}^d). \quad (1.5)$$

Note that this result is even new in the scalar case. It is important to remark that even in the scalar setting, it seems unclear whether the by now standard ideas from the proof of [16, Theorem 1.1] can be used to prove our sparse domination. A version of our sparse domination for iterated commutators will be the subject of a future paper.

In the last section, we will prove the equivalence of the quantities $\|B\|_{\widetilde{\text{BMO}}_{V,U}^p}$ and $\|B\|_{\text{BMO}_{V,U}^p}$ when U and V are matrix A_p weights, completing a line of work initiated in [11, 13]. Additionally, we will prove that the quantities $\|B\|_{\widetilde{\text{BMO}}_{V,U}^p}$ and $\|B\|_{\widetilde{\text{BMO}}_{U',V'}^{p'}}$ are equivalent again when U and V are matrix A_p weights. In particular, we will make use of the ideas and results from [11, 13] in conjunction with an ‘extrapolation of inverse Hölder inequality’ argument from [23]. For the sake of completion, however, we will reprove all relevant results from [11, 13], the proofs of which are more technical than those in Sections 2 and 3.

We will end this introduction with three remarks and an outline of the organization of the rest of the paper. First, it is an obvious question as to whether the techniques and results of this paper can be extended to the iterated commutator setting, and whether we can recover the more recent iterated commutator Bloom type results from [17] or the very recent unweighted two symbolled iterated commutator results of [8]. This will be pursued in a forthcoming paper. Second, for the reader who is either unfamiliar with matrix weighted norm inequalities or is primarily interested in the implications of our results in the scalar setting, we have attempted to make this paper almost entirely self-contained.

Third, if

$$\lambda_1 = \sup_Q \left(\int_Q \left\| V^{\frac{1}{p}}(x)(B(x) - m_Q B) \mathcal{U}_Q^{-1} \right\|^p dx \right)^{\frac{1}{p}}$$

$$\lambda_2 = \sup_Q \left(\int_Q \left\| U^{-\frac{1}{p}}(x)(B^*(x) - m_Q B^*) (\mathcal{V}'_Q)^{-1} \right\|^{p'} dx \right)^{\frac{1}{p'}}$$

where $\mathcal{U}_Q$ is an L^p reducing matrix for U on Q and $\mathcal{V}'_Q$ is an $L^{p'}$ reducing matrix for $V^{-\frac{1}{p}}$ on Q (again, see Section 2), then an easy use of Hölder’s inequality (see the proof of Corollary 4.7) says that $\lambda_1 \lesssim \|B\|_{\widetilde{\text{BMO}}_{V,U}^p}$ and $\lambda_2 \lesssim \|B\|_{\widetilde{\text{BMO}}_{U',V'}^{p'}}$ for arbitrary matrix weights U and V (and as previously mentioned, all four quantities are equivalent for matrix A_p weights U and V , see Corollary 4.7). Additionally, in the purely scalar two weighted setting, we have that

$$\lambda_1 = \sup_Q \left(\frac{1}{u(Q)} \int_Q |b(x) - m_Q b|^p v(x) dx \right)^{\frac{1}{p}}$$

$$\lambda_2 = \sup_Q \left(\frac{1}{v^{-\frac{p'}{p}}(Q)} \int_Q |b(x) - m_Q b|^{p'} u^{-\frac{p'}{p}}(x) dx \right)^{\frac{1}{p'}}$$

which proves very natural arbitrary two scalar weighted necessary conditions for the boundedness of commutators with all of the Riesz transforms.

Also, we can prove a version of Proposition 1.7 involving subtracted averages. Namely, arguing in a manner very similar to the proof of [14, Lemma 4] and the proof of Proposition 1.7 we have that if C, D, E, F are Young function with $\tilde{C}, \tilde{E} \in B_{p'}$ and $\tilde{D}, \tilde{F} \in B_p$, then

$$\|[M_B, T \otimes \mathbf{I}_m]\|_{L^p(U) \rightarrow L^p(V)} \lesssim \Lambda_1 + \Lambda_2$$

where $\Lambda_1 = \min\{\mu_1, \mu_2\}$ with

$$\mu_1 = \sup_Q \| \|V^{\frac{1}{p}}(x)(B(x) - m_Q B)U^{-\frac{1}{p}}(y)\|_{E_{x,Q}} \|_{F_{y,Q}}$$

$$\mu_2 = \sup_Q \| \|V^{\frac{1}{p}}(x)(B(x) - m_Q B)U^{-\frac{1}{p}}(y)\|_{F_{y,Q}} \|_{E_{x,Q}}$$

and $\Lambda_2 = \min\{\mu_3, \mu_4\}$ with

$$\mu_3 = \sup_Q \| \|V^{\frac{1}{p}}(x)(B(y) - m_Q B)U^{-\frac{1}{p}}(y)\|_{C_{x,Q}} \|_{D_{y,Q}}$$

$$\mu_4 = \sup_Q \| \|V^{\frac{1}{p}}(x)(B(y) - m_Q B)U^{-\frac{1}{p}}(y)\|_{D_{y,Q}} \|_{C_{x,Q}}$$

which in the unbumped (that is, when $C(x) = E(x) = x^p/p$ and $D(x) = F(x) = x^{p'}/p'$) scalar two weighted setting gives us

$$\Lambda_1 \approx \left(m_Q u^{-\frac{p'}{p}}\right)^{\frac{1}{p'}} \left(\int_Q |b(x) - m_Q b|^p v(x) dx\right)^{\frac{1}{p}}$$

$$\Lambda_2 \approx (m_Q v)^{\frac{1}{p}} \left(\int_Q |b(x) - m_Q b|^{p'} u^{-\frac{p'}{p}}(x) dx\right)^{\frac{1}{p'}}$$

which are natural joint BMO/A_p conditions. Further, by adding and subtracting $m_Q B$, it is trivial that in general $\kappa_1 \lesssim \mu_1 + \mu_2$ and $\kappa_2 \lesssim \mu_3 + \mu_4$ when $C = E$ and $D = F$. Despite all this, it seems unclear what the precise connection between all of these weighted (unbumped) BMO conditions are when dealing with not necessarily matrix A_p weights (even in the one weighted fully scalar setting.)

Finally, the paper is organized as follows. In Section 2, we will prove Lemma 1.3, Lemma 1.4, Proposition 1.5, Proposition 1.7, and Theorem 1.8. In Section 3, we will prove Theorem 1.6, and in the last section we will prove the equivalence of the quantities $\|B\|_{\widetilde{\text{BMO}}_{V,U}^p}$ and $\|B\|_{\text{BMO}_{V,U}^p}$ when U and V are matrix A_p weights, which will complete the proofs of Theorems 1.1 and 1.2.

2 | INTERMEDIATE UPPER AND LOWER BOUNDS

In this section, we will give a short proofs of Theorem 1.8, Lemma 1.3, Lemma 1.4, and Proposition 1.7, starting with Lemma 1.3.

2.1 | Proof of lemma 1.3

We first assume that $\|B\|_{\widetilde{\text{BMO}}_{V,U}^p} = 1$. Define the 2×2 block matrix function Φ by

$$\Phi = \begin{pmatrix} V^{\frac{1}{p}} & V^{\frac{1}{p}} B \\ 0 & U^{\frac{1}{p}} \end{pmatrix}$$

so that

$$\Phi^{-1} = \begin{pmatrix} V^{-\frac{1}{p}} & -BU^{-\frac{1}{p}} \\ 0 & U^{-\frac{1}{p}} \end{pmatrix}$$

and

$$\Phi(T \otimes \mathbf{I}_{2m})\Phi^{-1} = \Phi \begin{pmatrix} T \otimes \mathbf{I}_m & 0 \\ 0 & T \otimes \mathbf{I}_m \end{pmatrix} \Phi^{-1} = \begin{pmatrix} V^{\frac{1}{p}}(T \otimes \mathbf{I}_m)V^{-\frac{1}{p}} & V^{\frac{1}{p}}[M_B, T \otimes \mathbf{I}_m]U^{-\frac{1}{p}} \\ 0 & U^{\frac{1}{p}}(T \otimes \mathbf{I}_m)U^{-\frac{1}{p}} \end{pmatrix}.$$

Let $W = (\Phi^* \Phi)^{\frac{p}{2}}$. Then using the polar decomposition, we can write

$$\Phi = \mathcal{U}W^{\frac{1}{p}}$$

where $\mathcal{U}$ is unitary valued almost everywhere (a.e.). Supposing that W is a $2m \times 2m$ matrix A_p weight, we have by assumption that

$$\begin{aligned} \|[M_B, T \otimes \mathbf{I}_m]\|_{L^p(U) \rightarrow L^p(V)} &= \|V^{\frac{1}{p}}[M_B, T \otimes \mathbf{I}_m]U^{-\frac{1}{p}}\|_{L^p \rightarrow L^p} \\ &\leq \|\Phi(T \otimes \mathbf{I}_{2m})\Phi^{-1}\|_{L^p \rightarrow L^p} \\ &= \|W^{\frac{1}{p}}(T \otimes \mathbf{I}_{2m})W^{-\frac{1}{p}}\|_{L^p \rightarrow L^p} \\ &= \|T \otimes \mathbf{I}_{2m}\|_{L^p(W) \rightarrow L^p(W)} \\ &\leq \phi([W]_{A_p}). \end{aligned}$$

To finish the proof of Lemma 1.3, note that

$$\Phi(x)\Phi(y)^{-1} = \begin{pmatrix} V^{\frac{1}{p}}(x)V^{-\frac{1}{p}}(y) & V^{\frac{1}{p}}(x)(B(x) - B(y))U^{-\frac{1}{p}}(y) \\ 0 & U^{\frac{1}{p}}(x)U^{-\frac{1}{p}}(y) \end{pmatrix}$$

so that

$$\begin{aligned} &\int_Q \left(\int_Q \|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{p'} dy \right)^{\frac{p}{p'}} dx \\ &= \int_Q \left(\int_Q \|\Phi(x)\Phi^{-1}(y)\|^{p'} dy \right)^{\frac{p}{p'}} dx \\ &\leq 3^{\frac{p}{p'}} \left([U]_{A_p} + [V]_{A_p} + \int_Q \left(\int_Q \|V^{\frac{1}{p}}(x)(B(x) - B(y))U^{-\frac{1}{p}}(y)\|^{p'} dy \right)^{\frac{p}{p'}} dx \right) \end{aligned}$$

and thus

$$\|[M_B, T \otimes \mathbf{I}_m]\|_{L^p(U) \rightarrow L^p(V)} \leq \phi \left(3^{\frac{p}{p'}} ([U]_{A_p} + [V]_{A_p} + 1) \right).$$

Finally, if $\|B\|_{\widetilde{\text{BMO}}_{V,U}^p} \neq 1$, then we may assume that $0 < \|B\|_{\widetilde{\text{BMO}}_{V,U}^p} < \infty$. Rescaling with $B \mapsto B\|B\|_{\widetilde{\text{BMO}}_{V,U}^p}^{-1}$ then completes the proof.

2.2 | Proof of Theorem 1.8 and Proposition 1.7

We now prove Theorem 1.8.

Define the $\mathbb{C}^{2m}$ valued function $\tilde{f}$ by

$$\tilde{f}(x) = \begin{pmatrix} \vec{f}(x) \\ \vec{f}(x) \end{pmatrix}$$

and define the 2×2 block matrix $\Phi(x)$ by

$$\Phi(x) = \begin{pmatrix} \mathbf{I}_m & B(x) \\ 0 & \mathbf{I}_m \end{pmatrix}$$

so that

$$\Phi^{-1}(x) = \begin{pmatrix} \mathbf{I}_m & -B(x) \\ 0 & \mathbf{I}_m \end{pmatrix}.$$

Direct computation shows

$$\Phi(x)((T \otimes \mathbf{I}_m)\Phi^{-1}\tilde{f})(x) = \begin{pmatrix} (T \otimes \mathbf{I}_m)\vec{f}(x) + [M_B, T \otimes \mathbf{I}_m]\vec{f}(x) \\ (T \otimes \mathbf{I}_m)\vec{f}(x) \end{pmatrix}$$

and

$$\Phi^{-1}(y)\tilde{f}(y) = \begin{pmatrix} \mathbf{I}_m & -B(y) \\ 0 & \mathbf{I}_m \end{pmatrix} \begin{pmatrix} \vec{f}(y) \\ \vec{f}(y) \end{pmatrix} = \begin{pmatrix} \vec{f}(y) - B(y)\vec{f}(y) \\ \vec{f}(y) \end{pmatrix}.$$

Since $\Phi^{-1}\tilde{f} \in L_c^1$, Theorem 3.4 in [21] applied to $\Phi^{-1}\tilde{f}$ then says that there exists 3^d sparse collections S_j of dyadic cubes, a constant $c_{d,m,T}$, and for each $Q \in S_j$ a function $k_Q : Q \times Q \rightarrow \mathbb{R}$ with $\|k_Q\|_{L^\infty(Q \times Q)} \leq 1$ such that

$$\begin{aligned} & \begin{pmatrix} (T \otimes \mathbf{I}_m)\vec{f}(x) + [M_B, T \otimes \mathbf{I}_m]\vec{f}(x) \\ (T \otimes \mathbf{I}_m)\vec{f}(x) \end{pmatrix} \\ &= c_{d,m,T} \sum_{j=1}^{3^d} \sum_{Q \in S_j} \Phi(x) \begin{pmatrix} f_Q k_Q(x, y)(\vec{f}(y) - B(y)\vec{f}(y)) \\ f_Q k_Q(x, y)\vec{f}(y) \end{pmatrix} dy \mathbb{1}_Q(x) \\ &= c_{d,m,T} \sum_{j=1}^{3^d} \sum_{Q \in S_j} \begin{pmatrix} f_Q k_Q(x, y)(\vec{f}(y) - B(y)\vec{f}(y) + B(x)\vec{f}(y)) \\ f_Q k_Q(x, y)\vec{f}(y) \end{pmatrix} dy \mathbb{1}_Q(x). \end{aligned}$$

Subtracting

$$(T \otimes \mathbf{I}_m) \vec{f}(x) = c_{d,m,T} \sum_{j=1}^{3^d} \sum_{Q \in S_j} \int_Q k_Q(x, y) \vec{f}(y) dy$$

from the first column then completes the proof.

We now prove Proposition 1.7. The easy proof is similar to the proof of [14, Lemma 4]. We only prove that

$$\| [M_B, T \otimes \mathbf{I}_m] \|_{L^p(U) \rightarrow L^p(V)} \lesssim \sup_Q \| \| V^{\frac{1}{p}}(x)(B(x) - B(y))U^{-\frac{1}{p}}(y) \|_{C_{x,Q}} \|_{D_{y,Q}}$$

as the other estimate is virtually the same.

By the density of bounded functions with compact support in $L^p(W)$ for any matrix weight W (see [3, Proposition 3.6]), we can pick $\vec{f}, \vec{g}$ bounded with compact support and use (1.5) to get that (where for notational ease we suppress the summation over $j = 1$ to 3^d)

$$\begin{aligned} & \left| \left\langle [M_B, T \otimes \mathbf{I}_m] \vec{f}, \vec{g} \right\rangle_{L^2} \right| \\ & \leq \sum_{Q \in S} \int_Q \int_Q \left| \left\langle (B(x) - B(y)) \vec{f}(y), \vec{g}(x) \right\rangle \right| dy dx \\ & \leq \sum_{Q \in S} \int_Q \int_Q \left\| V^{\frac{1}{p}}(x)(B(x) - B(y))U^{-\frac{1}{p}}(y) \right\| \left\| U^{\frac{1}{p}}(y) \vec{f}(y) \right\| \left\| V^{-\frac{1}{p}}(x) \vec{g}(x) \right\| dx dy \\ & \leq 2 \left(\sup_Q \left\| \left\| V^{\frac{1}{p}}(x)(B(x) - B(y))U^{-\frac{1}{p}}(y) \right\|_{C_{x,Q}} \right\|_{D_{y,Q}} \right) \sum_{Q \in S} |E_Q| \left\| V^{-\frac{1}{p}} \vec{g} \right\|_{\overline{C},Q} \left\| U^{\frac{1}{p}} \vec{f} \right\|_{\overline{D},Q} \\ & \leq 2\kappa_1 \| M_{\overline{D}}(U^{\frac{1}{p}} \vec{f}) \|_{L^p} \| M_{\overline{C}}(V^{-\frac{1}{p}} \vec{g}) \|_{L^{p'}} \\ & \lesssim \kappa_1 \| \vec{f} \|_{L^p(U)} \| \vec{g} \|_{L^{p'}(V^{-\frac{p'}{p}})} \end{aligned}$$

where $M_{\overline{C}}(V^{-\frac{1}{p}} \vec{g})$ is the Orlicz maximal function defined by

$$M_{\overline{C}}(V^{-\frac{1}{p}} \vec{g})(x) = \sup_{\substack{\mathbb{R}^d \supseteq Q \ni x \\ Q \text{ is a cube}}} \left\| V^{-\frac{1}{p}} \vec{g} \right\|_{\overline{C},Q}$$

(with $M_{\overline{D}}(U^{\frac{1}{p}} \vec{f})$ defined similarly) and where last line follows from the fact that $\overline{C} \in B_{p'}$ and $\overline{D} \in B_p$. This completes the proof.

2.3 | Proof of lemma 1.4

We now give a short proof of Lemma 1.4. Defining W and Φ as before, we have by the previous computations and by assumption that

$$\begin{aligned} & \left([U]_{A_p} + [V]_{A_p} + \|B\|_{\widetilde{\text{BMO}}_{V,U}^p}^p \right)^{\frac{1}{p}} \\ & \approx [W]_{A_p}^{\frac{1}{p}} \\ & \lesssim \max_{1 \leq s \leq N} \|R_s \otimes \mathbf{I}_{2m}\|_{L^p(W) \rightarrow L^p(W)} \\ & \leq \max_{1 \leq s \leq N} (\| [M_B, R_s \otimes \mathbf{I}_m] \|_{L^p(U) \rightarrow L^p(V)} + \|R_s \otimes \mathbf{I}_m\|_{L^p(U) \rightarrow L^p(U)} + \|R_s \otimes \mathbf{I}_m\|_{L^p(V) \rightarrow L^p(V)}). \end{aligned}$$

Rescaling, and in particular letting $B \mapsto rB$ for $r > 0$ gives

$$\begin{aligned} & \left([U]_{A_p} + [V]_{A_p} + r^p \|B\|_{\widetilde{\text{BMO}}_{V,U}^p}^p \right)^{\frac{1}{p}} \\ & \lesssim \max_{1 \leq s \leq N} (r \| [M_B, R_s \otimes \mathbf{I}_m] \|_{L^p(U) \rightarrow L^p(V)} + \|R_s \otimes \mathbf{I}_m\|_{L^p(U) \rightarrow L^p(U)} + \|R_s \otimes \mathbf{I}_m\|_{L^p(V) \rightarrow L^p(V)}). \end{aligned}$$

Finally dividing both sides by r and letting $r \rightarrow \infty$ gives us that

$$\|B\|_{\widetilde{\text{BMO}}_{V,U}^p} \lesssim \max_{1 \leq s \leq N} \| [M_B, R_s \otimes \mathbf{I}_m] \|_{L^p(U) \rightarrow L^p(V)}.$$

2.4 | Proof of proposition 1.5

We finally give a very short proof of Proposition 1.5 by estimating $\|b\|_{\widetilde{\text{BMO}}_{U,U}^p}$. Namely, fix a cube Q . Then

$$\begin{aligned} & \int_Q \left(\int_Q \|U^{\frac{1}{p}}(x)U^{-\frac{1}{p}}(y)\|^{p'} |b(x) - b(y)|^{p'} dy \right)^{\frac{p}{p'}} dx \\ & \lesssim \int_Q \left(\int_Q \|U^{\frac{1}{p}}(x)U^{-\frac{1}{p}}(y)\|^{p'} |b(x) - m_Q b|^{p'} dy \right)^{\frac{p}{p'}} dx \\ & \quad + \int_Q \left(\int_Q \|U^{\frac{1}{p}}(x)U^{-\frac{1}{p}}(y)\|^{p'} |b(y) - m_Q b|^{p'} dy \right)^{\frac{p}{p'}} dx \\ & = (A) + (B). \end{aligned}$$

We only estimate (A) as (B) can be similarly estimated. By the classical scalar sharp reverse Hölder inequality, we can pick $\epsilon \approx [U]_{A_{p,\infty}^{\text{sc}}}^{-1}$ where for any $\vec{e} \in \mathbb{C}$, we have

$$\left(\int_Q \left| U^{\frac{1}{p}}(x) \mathcal{U}'_Q \vec{e} \right|^{\frac{p}{1-\epsilon}} \right)^{1-\epsilon} \lesssim \int_Q \left| U^{\frac{1}{p}}(x) \mathcal{U}'_Q \vec{e} \right|^p \approx [U]_{A_p}$$

and therefore

$$\begin{aligned}
 (A) &\leq \int_Q \left(\int_Q \|U^{\frac{1}{p}}(x) \mathcal{U}'_Q\|^{p'} \|(\mathcal{U}'_Q)^{-1} U^{-\frac{1}{p}}(y)\|^{p'} |b(x) - m_Q b|^{p'} dy \right)^{\frac{p}{p'}} dx \\
 &\lesssim \int_Q \|U^{\frac{1}{p}}(x) \mathcal{U}'_Q\|^p |b(x) - m_Q b|^p dx \\
 &\lesssim \left(\int_Q \|U^{\frac{1}{p}}(x) \mathcal{U}'_Q\|^{\frac{p}{1-\epsilon}} dx \right)^{1-\epsilon} \left(\int_Q |b(x) - m_Q b|^{\frac{p}{\epsilon}} dx \right)^{\epsilon} \\
 &\lesssim [U]_{A_p} [U]_{A_{p,\infty}^{sc}}^p \|b\|_{\text{BMO}}^p
 \end{aligned}$$

by the classical John-Nirenberg inequality. Here, $\mathcal{U}'_Q$ is a reducing matrix satisfying (3.5). Similarly we can estimate

$$(B) \lesssim [U]_{A_p} \left[U^{-\frac{p'}{p}} \right]_{A_{p',\infty}^{sc}}^p \|b\|_{\text{BMO}}^p$$

so by our assumption on T we have

$$\|[M_b, T \otimes \mathbf{I}_m]\|_{L^p(U) \rightarrow L^p(U)} \leq \phi \left(C[U]_{A_p} + C[U]_{A_p} \left(\left[U^{-\frac{p'}{p}} \right]_{A_{p',\infty}^{sc}}^p + [U]_{A_{p,\infty}^{sc}}^p \right) \|b\|_{\text{BMO}}^p \right).$$

Rescaling, setting $b \mapsto b \left(\left[U^{-\frac{p'}{p}} \right]_{A_{p',\infty}^{sc}} + [U]_{A_{p,\infty}^{sc}} \right)^{-1} \|b\|_{\text{BMO}}$ gives

$$\|[M_b, T \otimes \mathbf{I}_m]\|_{L^p(U) \rightarrow L^p(U)} \leq \phi \left(C[U]_{A_p} \right) \|b\|_{\text{BMO}} \left(\left[U^{-\frac{p'}{p}} \right]_{A_{p',\infty}^{sc}} + [U]_{A_{p,\infty}^{sc}} \right).$$

3 | LOWER BOUND FOR RIESZ TRANSFORMS

In this section, we prove Theorem 1.6. Clearly it is enough to prove Theorem 1.6 where $\|B\|_{\text{BMO}_{V,U}^p}$ is redefined by taking the supremum over balls instead of cubes, which will be more convenient for us. Recall that the Riesz transform R_ℓ is the Calderón-Zygmund operator with associated kernel

$$K_\ell(x, y) = \frac{x_\ell - y_\ell}{|x - y|^{d+1}}$$

in the usual sense.

Let W be an $m \times m$ matrix weight, and assume for the moment that $R_\ell \otimes \mathbf{I}_m$ is bounded on $L^p(W)$, so that in particular if $\vec{f} \in L^2 \cap L^p(W)$ and $\vec{g} \in L^2 \cap L^{p'}(W^{-\frac{p'}{p}})$ both have compact sup-

port with $\text{dist}(\text{supp } \vec{f}, \text{supp } \vec{g}) > 0$, and if E is a measurable subset of $\mathbb{R}^d$, then

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_{E \times E}(x, y) K_\ell(x, y) \left\langle \vec{f}(y), \vec{g}(x) \right\rangle_{\mathbb{C}^n} dy dx \right| \leq \| \mathbb{1}_E(R_\ell \otimes \mathbf{I}_m) \mathbb{1}_E \|_{L^p(W) \rightarrow L^p(W)} \| \vec{f} \|_{L^p(W)} \| \vec{g} \|_{L^{p'}(W^{1-p'})}. \quad (3.1)$$

As was stated in the introduction, we will need the Schur multiplier/Wiener algebra arguments in [18], which we quickly discuss now. In particular, we trivially have that the kernel $e^{-ia \cdot x} K_\ell(x, y) e^{ia \cdot y}$ satisfies (3.1) for any $a \in \mathbb{R}^d$. Thus, if $\rho \in L^1(\mathbb{R}^d)$, then Fubini's theorem says that the kernel

$$\hat{\rho}(x - y) K_\ell(x, y) = \int_{\mathbb{R}^d} \rho(a) K_\ell(x, y) e^{-ia \cdot (x - y)} da$$

satisfies (3.1) with $\| \mathbb{1}_E(R_\ell \otimes \mathbf{I}_n) \mathbb{1}_E \|_{L^p(W) \rightarrow L^p(W)}$ replaced with $\| \mathbb{1}_E(R_\ell \otimes \mathbf{I}_n) \mathbb{1}_E \|_{L^p(W) \rightarrow L^p(W)} \| \rho \|_{L^1(\mathbb{R}^d)}$ (where here $\hat{\rho}(s) = \int_{\mathbb{R}^d} \rho(a) e^{-is \cdot a} da$).

Let $W_0(\mathbb{R}^d)$ denote the Wiener algebra defined by $W_0(\mathbb{R}^d) = \{ \psi = \hat{\rho} : \rho \in L^1(\mathbb{R}^d) \}$. Then since $\hat{\rho}(\cdot/\varepsilon) = \varepsilon^d \hat{\rho}(\varepsilon \cdot)$ and

$$\int_{\mathbb{R}^d} |\varepsilon^d \rho(\varepsilon x)| dx = \| \rho \|_{L^1(\mathbb{R}^d)}$$

we have the following result which is similar to [18, Lemma 2.1].

Lemma 3.1. *If $\psi \in W_0(\mathbb{R}^d)$, $\varepsilon > 0$, and E is a measurable subset of $\mathbb{R}^d$, then*

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi\left(\frac{x - y}{\varepsilon}\right) \mathbb{1}_{E \times E}(x, y) K_\ell(x, y) \left\langle \vec{f}(y), \vec{g}(x) \right\rangle_{\mathbb{C}^n} dy dx \right| \leq \| \mathbb{1}_E(R_\ell \otimes \mathbf{I}_m) \mathbb{1}_E \|_{L^p(W) \rightarrow L^p(W)} \| \vec{f} \|_{L^p(W)} \| \vec{g} \|_{L^{p'}(W^{1-p'})} \quad (3.2)$$

for any $\vec{f} \in L^2 \cap L^p(W)$ and $\vec{g} \in L^2 \cap L^{p'}(W^{-\frac{p'}{p}})$ of compact support with $\text{dist}(\text{supp } \vec{f}, \text{supp } \vec{g}) > 0$.

We will need three more lemmas to show that the Riesz transforms satisfy (1.3), the first of which is probably known (though we provide a proof for the sake of completion) and the second being [18, Lemma 3.1].

Lemma 3.2. *If $\phi \in C_c^\infty(\mathbb{R}^d)$, then $|x| \phi(x) \in W_0(\mathbb{R}^d)$.*

Proof. The proof is similar the short proof of [4, Lemma 2]. Let $F(x) = |x| \phi(x)$ and pick $1 < \delta < \min\{1 + \frac{1}{d-1}, 2\}$. If $\alpha \in \{0, 1\}^d$ and

$$P_\alpha = \{x \in \mathbb{R}^d : (-1)^{\alpha_j} |x_{\alpha_j}| \leq (-1)^{\alpha_j}\},$$

then

$$\begin{aligned}\|\hat{F}\|_{L^1(\mathbb{R}^d)} &= \sum_{\alpha \in \{0,1\}^d} \int_{P_\alpha} |x^\alpha|^{-1} (|x^\alpha| |\hat{F}(x)|) dx \\ &\leq \sum_{\alpha \in \{0,1\}^d} \left(\int_{P_\alpha} |x^\alpha|^{-\delta} dx \right)^{\frac{1}{\delta}} \left(\int_{\mathbb{R}^d} |x^\alpha \hat{F}(x)|^{\delta'} dx \right)^{\frac{1}{\delta'}} \\ &\lesssim \left(\int_{\mathbb{R}^d} |\widehat{D^\alpha F}(x)|^{\delta'} dx \right)^{\frac{1}{\delta'}} \\ &\lesssim \left(\int_{\mathbb{R}^d} |D^\alpha F(x)|^\delta dx \right)^{\frac{1}{\delta}}\end{aligned}$$

where in the last inequality we used the classical Hausdorff–Young inequality. However, an elementary estimate using the Leibniz formula tells us that

$$|D^\alpha F(x)| \lesssim |x|^{1-|\alpha|}.$$

Thus, the fact that $1 < \delta < 1 + \frac{1}{d-1}$ gives us that $\|\hat{F}\|_{L^1(\mathbb{R}^d)} < \infty$, which by Fourier inversion completes the proof. $\square$

Lemma 3.3. *There exists Borel sets E_k^1 and E_k^2 such that:*

- (i) *for all $k \in \mathbb{N}$, we have $\text{dist}(E_k^1, E_k^2) > 0$;*
- (ii) *the operators defined by $P_k^\ell \vec{f} = \mathbb{1}_{E_k^\ell} \vec{f}$ for $\ell = 1, 2$ converge to $\frac{1}{2}Id$ in the $L^2(\mathbb{R}^d)$ weak operator topology;*
- (iii) *for any $1 \leq p < \infty$ and $\ell = 1, 2$, we have*

$$\lim_{k \rightarrow \infty} \|\mathbb{1}_{E_k^\ell} \vec{f}\|_{L^p(\mathbb{R}^d)} = 2^{-\frac{1}{p}} \|\vec{f}\|_{L^p(\mathbb{R}^d)}.$$

We now need to introduce the concept of a reducing matrix. Namely, for a set Q of finite nonzero measure, let $\mathcal{U}_Q, \mathcal{V}_Q, \mathcal{U}'_Q, \mathcal{V}'_Q$ respectively, be positive definite matrices where for any $\vec{e} \in \mathbb{C}^n$ we have

$$|\mathcal{U}_Q \vec{e}|^p \approx \int_Q |U^{\frac{1}{p}}(x) \vec{e}|^p dx, \quad |\mathcal{U}'_Q \vec{e}|^{p'} \approx \int_Q |U^{-\frac{1}{p}}(x) \vec{e}|^{p'} dx \quad (3.3)$$

and a similar statement holds for $\mathcal{V}_Q$ and $\mathcal{V}'_Q$ with respect to V . Despite its perhaps abstract appearance, the reader should think of $\mathcal{U}_Q$ as ‘the L^p average of $U^{\frac{1}{p}}$ over Q ’ and should similarly of $\mathcal{U}'_Q$ as an average. In fact, if U is a matrix A_p weight, then it can be shown (see [13, Lemma 2.2]) that for any $\vec{e} \in \mathbb{C}^n$

$$|\mathcal{U}_Q \vec{e}| \approx |m_Q(U^{\frac{1}{p}}) \vec{e}|$$

and a similar result holds for $\mathcal{U}'_Q$ since U is a matrix A_p weight if and only if $W^{-\frac{p'}{p}}$ is a matrix $A_{p'}$ weight. Also note that we can in fact rewrite the matrix A_p condition as

$$[U]_{A_p} \approx \sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} \|\mathcal{U}_Q \mathcal{U}'_Q\|^p$$

and thus since $\|\mathcal{U}_Q \mathcal{U}'_Q\| = \|\mathcal{U}'_Q \mathcal{U}_Q\|$, we can also write the matrix A_p condition out as

$$[W]_{A_p}^{\frac{p'}{p}} = \sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} \int_Q \left(\int_Q \|W^{\frac{1}{p}}(x) W^{-\frac{1}{p}}(y)\|^p dx \right)^{\frac{p'}{p}} dy < \infty.$$

Furthermore, note that we can rewrite the $\text{BMO}_{V,U}^p$ condition much more naturally as

$$\|B\|_{\text{BMO}_{V,U}^p} = \sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} \left(\int_Q \|\mathcal{V}_Q(B(x) - m_Q B) \mathcal{U}_Q^{-1}\| dx \right)^{\frac{1}{p}} < \infty.$$

The next proposition is implicitly proved in [7] (among other papers) but is not explicitly stated in the literature.

Proposition 3.4. *If $E \subseteq \mathbb{R}^d$ is a set of finite positive measure, then for any matrix weight W we have*

$$\|A_E\|_{L^p(W) \rightarrow L^p(W)} \approx \|\mathcal{W}'_E \mathcal{W}_E\|,$$

where A_E is the averaging operator

$$A_E \vec{f} = \mathbb{1}_E \int_E \vec{f}(x) dx$$

and $\mathcal{W}_E$ and $\mathcal{W}'_E$ are reducing matrices for W .

Proof. Let $\rho_{p,E}$ be the norm on $\mathbb{C}^n$ defined by

$$\rho_{p,E}(\vec{e}) = \left(\int_E \left| W^{\frac{1}{p}}(x) \vec{e} \right|^p dx \right)^{\frac{1}{p}} \approx |\mathcal{W}_E \vec{e}|.$$

A standard duality argument immediately says that

$$\rho_{p,E}(\vec{e}) = \sup_{\vec{v} \neq 0} \frac{\langle \vec{e}, \vec{v} \rangle_{\mathbb{C}^n}}{\rho_{p,E}^*(\vec{v})}$$

where

$$\rho_{p,E}^*(\vec{e}) = \sup_{\vec{v} \neq 0} \frac{\langle \vec{e}, \vec{v} \rangle_{\mathbb{C}^n}}{\rho_{p,E}(\vec{v})} \approx |\mathcal{W}_E^{-1} \vec{e}|.$$

Using these facts in conjunction with the fact that $(L^p(W))^* = L^{p'}(W^{-\frac{p'}{p}})$ under the unweighted L^2 inner product, we get that

$$\begin{aligned} \sup_{\|\vec{f}\|_{L^p(W)}=1} \left\| \mathbb{1}_E \int_E \vec{f}(x) dx \right\|_{L^p(W)} &= |E|^{-\frac{1}{p'}} \sup_{\|\vec{f}\|_{L^p(W)}=1} \sup_{\vec{e} \neq 0} \frac{\int_E \langle \vec{f}(x), \vec{e} \rangle_{\mathbb{C}^n}}{\rho_{p,E}^*(\vec{v})} \\ &= |E|^{-\frac{1}{p'}} \sup_{\vec{e} \neq 0} \frac{\|\mathbb{1}_E \vec{e}\|_{L^{p'}(W^{-\frac{p'}{p}})}}{\rho_{p,E}^*(\vec{v})} \approx \sup_{\vec{e} \neq 0} \frac{\|\mathcal{W}'_E \vec{e}\|}{\|\mathcal{W}_E^{-1} \vec{e}\|}. \end{aligned}$$

Replacing $\vec{e}$ by $\mathcal{W}_E \vec{e}$ completes the proof. $\square$

Putting together everything in this section gives us the following crucial Lemma

Lemma 3.5. *Let $\mathcal{B}$ be a ball and $E \subseteq \mathcal{B}$ have nonzero finite measure. Then*

$$\|\mathcal{W}'_E \mathcal{W}_E\| \lesssim \frac{|\mathcal{B}|}{|E|} \max_{1 \leq \ell \leq d} \|\mathbb{1}_E(R_\ell \otimes \mathbf{I}_m)\mathbb{1}_E\|_{L^p(W) \rightarrow L^p(W)}.$$

Proof. Let $\mathcal{B}$ be a ball with radius $\varepsilon > 0$. We will only consider the case that d is even, since the case that d is odd is much easier and does not require Lemma 3.2. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ satisfy $\phi(x) = 1$ if $|x| < 2$, so by Lemma 3.2 and the fact that $x_\ell |x|^{d-2} \phi(x) \in C_c^\infty(\mathbb{R}^d) \subseteq W_0(\mathbb{R}^d)$ we get that $x_\ell |x|^{d-1} \phi^2(x) \in W_0(\mathbb{R}^d)$. Using Lemma 3.1 and summing over ℓ then gives us that the kernel

$$\varepsilon^{-d} \phi^2\left(\frac{x-y}{\varepsilon}\right) \mathbb{1}_{\mathcal{B} \times \mathcal{B}}(x, y) = \frac{c_d}{|\mathcal{B}|} \mathbb{1}_{\mathcal{B} \times \mathcal{B}}(x, y)$$

replacing $\psi(\frac{x-y}{\varepsilon}) \mathbb{1}_{E \times E}(x, y) K_\ell(x, y)$ satisfies (3.2). Thus, if P_k^ℓ are the projections from the previous lemma, then for any $\vec{f} \in L^2 \cap L^p(W)$ and $\vec{g} \in L^2 \cap L^{p'}(W^{1-p'})$, we have

$$\begin{aligned} \left| \left\langle A_E \vec{f}, \vec{g} \right\rangle_{L^2} \right| &= \frac{|\mathcal{B}|}{|E|} \lim_{k \rightarrow \infty} \left| \left\langle \mathbb{1}_E A_{\mathcal{B}} \mathbb{1}_E P_k^1 \vec{f}, P_k^2 \vec{g} \right\rangle_{L^2} \right| \\ &\lesssim \frac{|\mathcal{B}|}{|E|} \max_{1 \leq \ell \leq d} \|\mathbb{1}_E(R_\ell \otimes \mathbf{I}_m)\mathbb{1}_E\|_{L^p(W) \rightarrow L^p(W)} \lim_{k \rightarrow \infty} \|P_k^1 \vec{f}\|_{L^p(W)} \|P_k^2 \vec{g}\|_{L^{p'}(W^{1-p'})} \\ &= \frac{|\mathcal{B}|}{|E|} \max_{1 \leq \ell \leq d} \|\mathbb{1}_E(R_\ell \otimes \mathbf{I}_m)\mathbb{1}_E\|_{L^p(W) \rightarrow L^p(W)} \|\vec{f}\|_{L^p(W)} \|\vec{g}\|_{L^{p'}(W^{1-p'})}. \end{aligned}$$

However, since bounded functions with compact support are dense in $L^p(W)$ and $L^{p'}(W^{1-p'})$, Proposition 3.4 then says that

$$\|\mathcal{W}'_E \mathcal{W}_E\| \approx \|A_E\|_{L^p(W) \rightarrow L^p(W)} \lesssim \frac{|\mathcal{B}|}{|E|} \max_{1 \leq \ell \leq d} \|\mathbb{1}_E(R_\ell \otimes \mathbf{I}_m)\mathbb{1}_E\|_{L^p(W) \rightarrow L^p(W)}. \quad \square$$

We now finish the proof of Theorem 1.6. Fix a ball $\mathcal{B}$ and define $E_M = \{x \in \mathcal{B} : \max\{\|U(x)\|, \|U^{-1}(x)\|, \|V(x)\|, \|V^{-1}(x)\|\} < M\}$ where $M > 0$ is large enough that $2|E_M| > |\mathcal{B}|$.

Defining

$$[W]_{A_p(E)} = \int_E \left(\int_E \|W^{\frac{1}{p}}(x)W^{-\frac{1}{p}}(y)\|^{p'} dy \right)^{\frac{p}{p'}} dx$$

and

$$\|B\|_{\widetilde{\text{MO}}_{V,U}^p(E)}^p = \int_E \left(\int_E \|V^{\frac{1}{p}}(x)(B(x) - B(y))U^{-\frac{1}{p}}(x)\|^{p'} dy \right)^{\frac{p}{p'}} dx$$

and also defining W and Φ as in the beginning of Section 2, we have using Lemma 3.5 with respect to $E = E_M$ that

$$\begin{aligned} & \left([U]_{A_p(E_M)} + [V]_{A_p(E_M)} + \|B\|_{\widetilde{\text{MO}}_{V,U}^p(E_M)}^p \right)^{\frac{1}{p}} \\ & \approx [W]_{A_p(E_M)}^{\frac{1}{p}} \\ & \approx \|\mathcal{W}'_{E_M} \mathcal{W}_{E_M}\| \\ & = \|\mathcal{W}_{E_M} \mathcal{W}'_{E_M}\| \\ & \lesssim \max_{1 \leq \ell \leq d} \|\mathbb{1}_{E_M}(R_\ell \otimes \mathbf{I}_{2m})\mathbb{1}_{E_M}\|_{L^p(W) \rightarrow L^p(W)} \\ & \lesssim \max_{1 \leq \ell \leq d} (\|[M_B, R_\ell \otimes \mathbf{I}_{2m}]\|_{L^p(U) \rightarrow L^p(V)} + \\ & \quad + \|\mathbb{1}_{E_M}(R_\ell \otimes \mathbf{I}_{2m})\mathbb{1}_{E_M}\|_{L^p(U) \rightarrow L^p(U)} + \|\mathbb{1}_{E_M}(R_\ell \otimes \mathbf{I}_{2m})\mathbb{1}_{E_M}\|_{L^p(V) \rightarrow L^p(V)}). \end{aligned}$$

Notice that all quantities above are bounded as all weights involved are pointwise bounded in norm and we assume $\|[M_B, R_\ell \otimes \mathbf{I}_{2m}]\|_{L^p(U) \rightarrow L^p(V)} < \infty$. Thus, as was done in the proof of Theorem 1.4, we can rescale and set $B \mapsto rB$, divide by r , and let $r \rightarrow \infty$ to get that

$$\|B\|_{\widetilde{\text{MO}}_{V,U}^p(E_M)} \lesssim \max_{1 \leq \ell \leq d} \|[M_B, R_\ell \otimes \mathbf{I}_m]\|_{L^p(U) \rightarrow L^p(V)}.$$

First letting $M \rightarrow \infty$ and using Fatou's lemma, and then taking the supremum over all balls $\mathcal{B}$ shows that

$$\|B\|_{\widetilde{\text{BMO}}_{V,U}^p} \lesssim \max_{1 \leq \ell \leq d} \|[M_B, R_\ell]\|_{L^p(U) \rightarrow L^p(V)}$$

as desired. To show the same estimate is true with $\|B\|_{\widetilde{\text{BMO}}_{U',V'}^{p'}}$ expand out $[W]_{A_p(E_M)}^{\frac{1}{p}} \approx \|\mathcal{W}'_{E_M} \mathcal{W}_{E_M}\|$ using the reducing matrix $\mathcal{W}_{E_M}$ first and repeat the arguments above, which completes the proof of (1.4).

4 | JOHN NIRENBERG THEOREMS

We will finish this paper by proving the equivalency between $\|B\|_{\widetilde{\text{BMO}}_{V,U}^p}$ and $\|B\|_{\text{BMO}_{V,U}^p}$ when U and V are matrix A_p weights. Note that we will not track the $[U]_{A_p}$ and $[V]_{A_p}$ dependence of our constants because we will need to use the lower matrix weighted Triebel–Lizorkin bounds from [20, 24] when $d = 1$ and $d > 1$ in [12], which are most likely far from sharp. We will need the following simple result that is a special case of [11, Theorem 2.2] and proved using a simple idea from [13]. Note that throughout this section $\mathcal{D}$ will refer to some dyadic lattice of cubes in $\mathbb{R}^d$.

Proposition 4.1. *Let U be a matrix A_p weight and let $A = \{a_Q\}_{Q \in \mathcal{D}}$ be a nonnegative Carleson sequence of scalars, meaning that*

$$\|A\|_*^2 = \sup_{J \in \mathcal{D}} \frac{1}{|J|} \sum_{Q \in \mathcal{D}(J)} a_Q^2 < \infty.$$

Then for any $\vec{f} \in L^p$ we have

$$\left(\int_{\mathbb{R}^d} \left(\sum_{Q \in \mathcal{D}} \frac{[a_Q m_Q |\mathcal{U}_Q U^{-\frac{1}{p}} \vec{f}|]^2}{|Q|} \mathbb{1}_Q(x) \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \lesssim \|A\|_* \|\vec{f}\|_{L^p}$$

where as before, $\mathcal{U}_Q$ is a reducing matrix satisfying (3.3).

Proof. Let

$$\tilde{A} = \sum_{\varepsilon \in \text{Sig}_d} \sum_{Q \in \mathcal{D}} a_Q h_Q^\varepsilon$$

where $\text{Sig}_d = \{1, 2, \dots, 2^d - 1\}$ and $\{h_Q^\varepsilon\}_{\{Q \in \mathcal{D}, \varepsilon \in \text{Sig}_d\}}$ is any Haar system on $\mathbb{R}^d$. Clearly for any $\mathcal{D} \ni Q \ni x$ we have that

$$m_Q |\mathcal{U}_Q U^{-\frac{1}{p}} \vec{f}| \leq m_Q M'_U \vec{f}$$

where $M'_U \vec{f}$ is the ‘Goldberg intermediary maximal function’ defined by

$$M'_U \vec{f}(x) = \sup_{\mathcal{D} \ni Q \ni x} m_Q |\mathcal{U}_Q U^{-\frac{1}{p}} \vec{f}|.$$

Thus, since $M'_U : L^p \rightarrow L^p$ for matrix A_p weights U (see [7, p.8]), we have that

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(\sum_{\varepsilon \in \text{Sig}_d} \sum_{Q \in \mathcal{D}} \frac{[a_Q m_Q M'_U \vec{f}]^2}{|Q|} \mathbb{1}_Q(x) \right)^{\frac{p}{2}} dx \\ & \lesssim \|\pi_A M'_U \vec{f}\|_{L^p}^p \lesssim \|A\|_*^p \|M'_U \vec{f}\|_{L^p}^p \lesssim \|A\|_*^p \|\vec{f}\|_{L^p}^p \end{aligned}$$

by unweighted dyadic Littlewood–Paley theory, where here $\pi_{\tilde{A}}$ is the paraproduct

$$\pi_{\tilde{A}}g(x) = \sum_{\varepsilon \in \text{Sig}_d} \sum_{Q \in D} m_Q g_{\tilde{A}_Q^\varepsilon} h_Q^\varepsilon(x).$$

□

The following is the key to proving the equivalence between $\|B\|_{\widetilde{\text{BMO}}_{V,U}^p}$ and $\|B\|_{\text{BMO}_{V,U}^p}$. Note that the lemma below was implicitly proved in [12] though not explicitly stated, and therefore for the sake of completion we will include the details.

Lemma 4.2. *If U and V are matrix A_p weights, then there exists $\epsilon > 0$ small enough where for any $0 < \epsilon' < \epsilon$ we have*

$$\sup_{I \in D(J)} \left(\int_I \|V^{\frac{1}{p}}(x)(B(x) - m_I B)\mathcal{U}_I^{-1}\|^p dx \right)^{\frac{1}{p}} \leq C \sup_{I \in D(J)} \left(\int_I \|\mathcal{V}_I(B(x) - m_I B)\mathcal{U}_I^{-1}\|^{1+\epsilon'} dx \right)^{\frac{1}{1+\epsilon'}}$$

and

$$\sup_{I \in D(J)} \left(\int_I \|\mathcal{V}_I(B(x) - m_I B)\mathcal{U}_I^{-1}\|^{1+\epsilon} dx \right)^{\frac{1}{1+\epsilon}} \leq C \sup_{I \in D(J)} \int_I \|V^{\frac{1}{p}}(x)(B(x) - m_I B)\mathcal{U}_I^{-1}\|^p dx.$$

where C is independent of B and J (but depends on ϵ' .)

We prove Lemma 4.2 through a series of lemmas.

Lemma 4.3. *If U, V are matrix A_p weights and B is locally integrable, then*

$$\sup_{I \in D(J)} \int_I \|V^{\frac{1}{p}}(x)(B(x) - m_I B)\mathcal{U}_I^{-1}\|^p dx \leq C \sup_{I \in D(J)} \left(\frac{1}{|I|} \sum_{\substack{Q \in D(I) \\ \varepsilon \in \text{Sig}_d}} \|\mathcal{V}_Q B_Q^\varepsilon \mathcal{U}_Q^{-1}\|^2 \right)^{\frac{1}{2}}$$

where C is independent of B and J .

Proof. Let $I \in D(J)$. By the Triebel–Lizorkin embedding (see [20, 24] for $d = 1$ and [12] for $d > 1$) we have that

$$\begin{aligned} & \int_I \|V^{\frac{1}{p}}(x)(B(x) - m_I B)\mathcal{U}_I^{-1}\|^p dx \\ & \lesssim \int_I \left(\sum_{\substack{Q \in D(I) \\ \varepsilon \in \text{Sig}_d}} \frac{\|\mathcal{V}_Q B_Q^\varepsilon \mathcal{U}_I^{-1}\|^2}{|Q|} \mathbb{1}_Q(t) \right)^{\frac{p}{2}} dt \\ & \leq \int_I \left(\sum_{\substack{Q \in D(I) \\ \varepsilon \in \text{Sig}_d}} \frac{\|\mathcal{V}_Q B_Q^\varepsilon \mathcal{U}_Q^{-1}\|^2}{|Q|} \|m_Q[(\mathcal{U}_Q U^{-\frac{1}{p}})U^{\frac{1}{p}}\mathcal{U}_I^{-1}\mathbb{1}_I]\|^2 \mathbb{1}_Q(t) \right)^{\frac{p}{2}} dt \end{aligned}$$

$$\lesssim \left(\sup_{I \in \mathcal{D}(J)} \frac{1}{|I|} \sum_{\substack{Q \in \mathcal{D}(I) \\ \varepsilon \in \text{Sig}_d}} \|\mathcal{V}_Q B_Q^\varepsilon \mathcal{U}_Q^{-1}\|^2 \right)^{\frac{p}{2}} \int_I \|U^{\frac{1}{p}} \mathcal{U}_I^{-1}\|^p dx$$

where in the last line we used Proposition 4.1. □

Lemma 4.4. *For $\varepsilon > 0$ small enough (independent of B) we have*

$$\left(\int_I \|\mathcal{V}_I(B(x) - m_I B) \mathcal{U}_I^{-1}\|^{1+\varepsilon} dx \right)^{\frac{1}{1+\varepsilon}} \leq C \left(\int_I \|V^{\frac{1}{p}}(x)(B(x) - m_I B) \mathcal{U}_I^{-1}\|^p dx \right)^{\frac{1}{p}}$$

where C is independent of I and B .

Proof.

$$\begin{aligned} & \left(\int_I \|\mathcal{V}_I(B(x) - m_I B) \mathcal{U}_I^{-1}\|^{1+\varepsilon} dx \right)^{\frac{1}{1+\varepsilon}} \\ & \leq \left(\int_I \|\mathcal{V}_I V^{-\frac{1}{p}}(x)\|^{1+\varepsilon} \|V^{\frac{1}{p}}(x)(B(x) - m_I B) \mathcal{U}_I^{-1}\|^{1+\varepsilon} dx \right)^{\frac{1}{1+\varepsilon}} \\ & \leq \left(\int_I \|\mathcal{V}_I V^{-\frac{1}{p}}(x)\|^{\frac{p(1+\varepsilon)}{p-1-\varepsilon}} dx \right)^{\frac{1}{1+\varepsilon} \left(\frac{p-1-\varepsilon}{p} \right)} \left(\int_I \|V^{\frac{1}{p}}(x)(B(x) - m_I B) \mathcal{U}_I^{-1}\|^p dx \right)^{\frac{1}{p}} \\ & \leq [V]_{A_p}^{\frac{1}{p}} \left(\int_I \|V^{\frac{1}{p}}(x)(B(x) - m_I B) \mathcal{U}_I^{-1}\|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

for $\varepsilon > 0$ small enough by the reverse Hölder inequality. □

We now recall the two matrix weighted stopping time from [12] which is a modification of the one matrix weighted stopping time from [13]. Finally assume that U, V are a matrix A_p weights and that λ is large. For any cube $I \in \mathcal{D}$, let $\mathcal{J}(I)$ be the collection of maximal $J \in \mathcal{D}(I)$ such that either of the two conditions

$$\|\mathcal{U}_I \mathcal{U}_J^{-1}\| > \lambda \text{ or } \|\mathcal{V}_I^{-1} \mathcal{V}_J\| > \lambda$$

are true. Also, let $\mathcal{F}(I)$ be the collection of dyadic subcubes of I not contained in any cube $J \in \mathcal{J}(I)$, so that clearly $J \in \mathcal{F}(J)$ for any $J \in \mathcal{D}$.

Let $\mathcal{J}^0(I) := \{I\}$ and inductively define $\mathcal{J}^j(I)$ and $\mathcal{F}^j(I)$ for $j \geq 1$ by

$$\mathcal{J}^j(I) := \{R \in \mathcal{J}(Q) : Q \in \mathcal{J}^{j-1}(I)\}$$

and $\mathcal{F}^j(I) = \{J' \in \mathcal{F}(J) : J \in \mathcal{J}^{j-1}(I)\}$. Clearly the cubes in $\mathcal{J}^j(I)$ for $j > 0$ are pairwise disjoint. Furthermore, since $J \in \mathcal{F}(J)$ for any $J \in \mathcal{D}(I)$, we have that $\mathcal{D}(I) = \bigcup_{j=1}^{\infty} \mathcal{F}^j(I)$ and that the collections $\mathcal{F}^j(I)$ are disjoint. We will slightly abuse notation and write $\bigcup \mathcal{J}(I)$ for the set $\bigcup_{J \in \mathcal{J}(I)} J$

and write $|\bigcup \mathcal{J}(I)|$ for $|\bigcup_{J \in \mathcal{J}(I)} J|$. By easy arguments (see [12]), we can pick λ depending on U and V so that

$$|\bigcup \mathcal{J}^j(I)| \leq 2^{-j} |I| \quad (8)$$

for every $I \in \mathcal{D}$.

Lemma 4.5. *If U, V are matrix A_p weights and $0 < \epsilon' \leq 1$, then there exists $C > 0$ independent of J and B where*

$$\sup_{I \in \mathcal{D}(J)} \left(\frac{1}{|I|} \sum_{\substack{Q \in \mathcal{D}(I) \\ \epsilon \in \text{Sig}_d}} \|\mathcal{V}_Q B_Q^\epsilon \mathcal{U}_Q^{-1}\|^2 \right)^{\frac{1}{2}} \leq C \sup_{I \in \mathcal{D}(J)} \left(\int_I \|\mathcal{V}_I(B(x) - m_I B) \mathcal{U}_I^{-1}\|^{1+\epsilon'} dx \right)^{\frac{1}{1+\epsilon'}}.$$

Proof. Fix $I \in \mathcal{D}(J)$. By the classical unweighted John–Nirenberg theorem and by unweighted dyadic Littlewood–Paley theory, it is enough to prove that

$$\int_I \left(\sum_{\substack{Q \in \mathcal{D}(I) \\ \epsilon \in \text{Sig}_d}} \frac{\|\mathcal{V}_Q B_Q^\epsilon \mathcal{U}_Q^{-1}\|^2}{|Q|} \mathbb{1}_Q(x) \right)^{\frac{1+\epsilon'}{2}} dx \leq C \sup_{I' \in \mathcal{D}(J)} \int_{I'} \|\mathcal{V}_{I'}(B(x) - m_{I'} B) \mathcal{U}_{I'}^{-1}\|^{1+\epsilon'} dx.$$

for $I \in \mathcal{D}(J)$ where C is independent of I, J and B . To that end

$$\begin{aligned} & \int_I \left(\sum_{\substack{Q \in \mathcal{D}(I) \\ \epsilon \in \text{Sig}_d}} \frac{\|\mathcal{V}_Q B_Q^\epsilon \mathcal{U}_Q^{-1}\|^2}{|Q|} \mathbb{1}_Q(x) \right)^{\frac{1+\epsilon'}{2}} dx \\ & \leq \int_I \left(\sum_{j=1}^{\infty} \sum_{K \in \mathcal{J}^{j-1}(I)} \sum_{\substack{Q \in \mathcal{F}(K) \\ \epsilon \in \text{Sig}_d}} \frac{(\|\mathcal{V}_Q \mathcal{V}_K^{-1}\| \|\mathcal{V}_K B_Q^\epsilon \mathcal{U}_K^{-1}\| \|\mathcal{U}_K \mathcal{U}_Q^{-1}\|)^2}{|Q|} \mathbb{1}_Q(x) \right)^{\frac{1+\epsilon'}{2}} dx \\ & \leq C \int_I \left(\sum_{j=1}^{\infty} \sum_{K \in \mathcal{J}^{j-1}(I)} \sum_{\substack{Q \in \mathcal{F}(K) \\ \epsilon \in \text{Sig}_d}} \frac{\|\mathcal{V}_K B_Q^\epsilon \mathcal{U}_K^{-1}\|^2}{|Q|} \mathbb{1}_Q(x) \right)^{\frac{1+\epsilon'}{2}} dx \\ & \leq \frac{C}{|I|} \sum_{j=1}^{\infty} \sum_{K \in \mathcal{J}^{j-1}(I)} \int_K \left(\sum_{\substack{Q \in \mathcal{D}(K) \\ \epsilon \in \text{Sig}_d}} \frac{\|\mathcal{V}_K B_Q^\epsilon \mathcal{U}_K^{-1}\|^2}{|Q|} \mathbb{1}_Q(x) \right)^{\frac{1+\epsilon'}{2}} dx \\ & \leq \frac{C}{|I|} \sum_{j=1}^{\infty} \sum_{K \in \mathcal{J}^{j-1}(I)} |K| \left(\int_K \|\mathcal{V}_K(B(x) - m_K B) \mathcal{U}_K^{-1}\|^{1+\epsilon'} dx \right) \end{aligned}$$

$$\begin{aligned} &\leq C \sup_{I' \in \mathcal{D}(J)} \left(\int_{I'} \|\mathcal{V}_{I'}(B(x) - m_{I'}B)\mathcal{U}_{I'}^{-1}\|^{1+\epsilon'} dx \right) |I|^{-1} \sum_{j=1}^{\infty} \sum_{K \in \mathcal{F}^{j-1}(I)} |K| \\ &\leq C \sup_{I' \in \mathcal{D}(J)} \int_{I'} \|\mathcal{V}_{I'}(B(x) - m_{I'}B)\mathcal{U}_{I'}^{-1}\|^{1+\epsilon'} dx. \end{aligned} \quad \square$$

The proof of Lemma 4.2 now follows immediately by combining Lemmas 4.3, 4.4, and 4.5. In particular, let $\epsilon > 0$ be from Lemma 4.4. Then for any $0 < \epsilon' < \epsilon$ we have

$$\begin{aligned} \sup_{I \in \mathcal{D}(J)} \left(\int_I \|\mathcal{V}_I(B(x) - m_I B)\mathcal{U}_I^{-1}\|^{1+\epsilon} dx \right)^{\frac{1}{1+\epsilon}} &\leq C \sup_{I \in \mathcal{D}(J)} \left(\int_I \|V^{\frac{1}{p}}(x)(B(x) - m_I B)\mathcal{U}_I^{-1}\|^p dx \right)^{\frac{1}{p}} \\ &\leq C \sup_{I \in \mathcal{D}(J)} \left(\frac{1}{|I|} \sum_{\substack{Q \in \mathcal{D}(I) \\ \varepsilon \in \text{Sig}_d}} \|\mathcal{V}_Q B_Q^\varepsilon \mathcal{U}_Q^{-1}\|^2 \right)^{\frac{1}{2}} \\ &\leq C \sup_{I \in \mathcal{D}(J)} \left(\int_I \|\mathcal{V}_I(B(x) - m_I B)\mathcal{U}_I^{-1}\|^{1+\epsilon'} dx \right)^{\frac{1}{1+\epsilon'}}. \end{aligned}$$

We now prove the following, □

Lemma 4.6. *If U, V are matrix A_p weights and $\epsilon' > 0$ is small enough, then*

$$\sup_{I \in \mathcal{D}} \left(\int_I \|\mathcal{V}_I(B(x) - m_I B)\mathcal{U}_I^{-1}\|^{1+\epsilon'} dx \right)^{\frac{1}{1+\epsilon'}} \lesssim \sup_{I \in \mathcal{D}} \int_I \|\mathcal{V}_I(B(x) - m_I B)\mathcal{U}_I^{-1}\| dx.$$

Proof. For fixed $R \in \mathbb{N}$, let P_R be the canonical projection operator

$$P_R B(x) = \sum_{\substack{I \in \mathcal{D} \\ |I| > 2^{-R}}} \sum_{\varepsilon \in \text{Sig}_d} B_I^\varepsilon h_I^\varepsilon = \sum_{\substack{I \in \mathcal{D} \\ |I| = 2^{-R}}} \mathbb{1}_I m_I B$$

which is trivially bounded on $L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$. For $I \in \mathcal{D}$, let

$$F_I(x) = \mathbb{1}_I(x) \mathcal{V}_I(B(x) - m_I B) \mathcal{U}_I^{-1},$$

let

$$F_I^R(x) = \mathbb{1}_I(x) \mathcal{V}_I(P_R B(x) - m_I(P_R B)) \mathcal{U}_I^{-1} = P_R [\mathbb{1}_I \mathcal{V}_I(B - m_I B) \mathcal{U}_I^{-1}],$$

and let $d\mu_I(x) = |I|^{-1} \mathbb{1}_I(x) dx$. Fix $J \in \mathcal{D}$ so trivially

$$\sup_{Q \in \mathcal{D}(J)} \|F_Q^R\|_{L^{1+\epsilon'}(d\mu_Q)} dx \leq \sup_{Q \in \mathcal{D}(J)} \|F_Q^R\|_{L^\infty} = C < \infty$$

where C possibly depends on J, R, B, U , and V . Also, clearly

$$\sup_{Q \in D} \|F_Q^R\|_{L^1(d\mu_Q)} \lesssim \|B\|_{\text{BMO}_{V,U}^p}$$

independent of $R > 0$ since P_R is bounded on $L^1(\mathbb{R}^d)$ independent of R .

Let $\epsilon > 0$ be from Lemma 4.2 (applied to $P_R B$) and let $0 < \epsilon' < \epsilon$. Let $p_1 = 1 + \epsilon'$, $p_2 = 1 + \epsilon$, $\alpha = \frac{p_2 - p_1}{p_2 - 1}$, $\beta = \frac{p_1 - \alpha}{\alpha}$, and let C be the constant in Lemma 4.2. Then by a use of Hölder's inequality with respect to the conjugate exponents α^{-1} and $(1 - \alpha)^{-1}$ we have

$$\begin{aligned} \sup_{Q \in D(J)} \|F_Q^R\|_{L^{p_1}(d\mu_Q)}^{p_1} &\leq \sup_{Q \in D(J)} \|F_Q^R\|_{L^1(d\mu_Q)}^\alpha \sup_{Q \in D(J)} \|F_Q^R\|_{L^{p_2}(d\mu_Q)}^{p_1 - \alpha} \\ &\leq \|B\|_{\text{BMO}_{V,U}^p}^\alpha C^{p_1 - \alpha} \sup_{Q \in D(J)} \|F_Q^R\|_{L^{p_1}(d\mu_Q)}^{p_1 - \alpha} \end{aligned}$$

which says that

$$\sup_{Q \in D(J)} \|F_Q^R\|_{L^{p_1}(d\mu_Q)} \leq \|B\|_{\text{BMO}_{V,U}^p} C^\beta.$$

Letting $R \rightarrow \infty$ first (using Fatou's lemma) and then taking the supremum over all $Q \in D(J)$ and then all $J \in D$ completes the proof. $\square$

Combining everything we have the following corollary, which finishes the proof that the quantities $\|B\|_{\widetilde{\text{BMO}}_{V,U}^p}$ and $\|B\|_{\text{BMO}_{V,U}^p}$ are equivalent.

Corollary 4.7. *If U, V are matrix weights A_p weights, then there exists $\epsilon > 0$ such that the following quantities are equivalent.*

- (a) $\sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} f_Q \| \mathcal{V}_Q(B(x) - m_Q B) \mathcal{U}_Q^{-1} \| dx$
- (b) $\sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} (f_Q \| V^{\frac{1}{p}}(x)(B(x) - m_Q B) \mathcal{U}_Q^{-1} \|^p dx)^{\frac{1}{p}}$
- (c) $\sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} (f_Q \| U^{-\frac{1}{p}}(x)(B^*(x) - m_Q B^*) (\mathcal{V}_Q')^{-1} \|^{p'} dx)^{\frac{1}{p'}}$
- (d) $\sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} (f_Q (f_Q \| V^{\frac{1}{p}}(x)(B(x) - B(y)) U^{-\frac{1}{p}}(y) \|^{p'} dy)^{\frac{p}{p'}} dx)^{\frac{1}{p}}$
- (e) $\sup_{\substack{Q \subseteq \mathbb{R}^d \\ Q \text{ is a cube}}} (f_Q (f_Q \| V^{\frac{1}{p}}(x)(B(x) - B(y)) U^{-\frac{1}{p}}(y) \|^p dx)^{\frac{p'}{p}} dy)^{\frac{1}{p'}}$

Proof. If $t \in \{0, \frac{1}{3}\}^d$ and $D^t = \{2^{-k}([0, 1]^d + m + (-1)^k t) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}$, then given any cube Q , there exists $t \in \{0, \frac{1}{3}\}^d$ and $Q_t \in D^t$ such that $Q \subset Q_t$ and $\ell(Q_t) \leq 6\ell(Q)$. Thus, by standard arguments, it is enough to prove the equivalence of (a)–(e) for any fixed dyadic grid.

With this in mind, the equivalence between the supremums in (a) and (b) follows immediately from Lemmas 4.2 and 4.6. As for (c), since U and V are matrix A_p weights,

$$\begin{aligned}\|\mathcal{V}_Q(B(x) - m_Q B)\mathcal{U}_Q^{-1}\| &= \|\mathcal{U}_Q^{-1}(B^*(x) - m_Q B^*)\mathcal{V}_Q'\| \\ &\approx \|\mathcal{U}_Q'(B^*(x) - m_Q B^*)(\mathcal{V}_Q')^{-1}\|\end{aligned}$$

and since clearly $\mathcal{U}_Q'$ is an $L^{p'}$ reducing matrix for $U^{-\frac{p'}{p}}$ and a similarly $\mathcal{V}_Q'$ is an $L^{p'}$ reducing operator for $V^{-\frac{p'}{p}}$, we get that the supremum in (a) is equivalent to the supremum in (c) by using the equivalence to (b) with respect to the pair $V^{-\frac{p'}{p}}, U^{-\frac{p'}{p}}$, and the exponent p' .

Also,

$$\begin{aligned}&\int_Q \|V^{\frac{1}{p}}(x)(B(x) - m_Q B)\mathcal{U}_Q^{-1}\|^p dx \\ &\leq \int_Q \left(\int_Q \|V^{\frac{1}{p}}(x)(B(x) - B(y))\mathcal{U}_Q^{-1}\| dy \right)^p dx \\ &\leq \int_Q \left(\int_Q \|V^{\frac{1}{p}}(x)(B(x) - B(y))U^{-\frac{1}{p}}(y)\| \|U^{\frac{1}{p}}(y)\mathcal{U}_Q^{-1}\| dy \right)^p dx \\ &\lesssim \int_Q \left(\int_Q \|V^{\frac{1}{p}}(x)(B(x) - B(y))U^{-\frac{1}{p}}(y)\|^{p'} dy \right)^{\frac{p}{p'}} dx\end{aligned}$$

which proves that d) implies b), and similarly (e) implies (c). Finally, adding and subtracting $m_Q B$ in both (d) and (e), respectively, shows that (b) and (c) together implies both (d) and (e), which completes the proof. $\square$

ACKNOWLEDGEMENT

The authors would like to thank the referees for spotting several typos and making several useful suggestions.

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. S. Bloom, *A commutator theorem and weighted BMO*, Trans. Amer. Math. Soc. **292** (1985), 103–122.
2. D. Cruz-Uribe OFS, J. Israelowitz, and K. Moen, *Two weight bump conditions for matrix weights*, Integral Equations Operator Theory **90** (2018), no. 3, 36.
3. D. Cruz-Uribe OFS, K. Moen, and S. Rodney, *Matrix A_p weights, degenerate Sobolev spaces, and mappings of finite distortion*, J. Geom. Anal. **26** (2016), no. 4, 2797–2830.

4. H. Dappa, and W. Trebel, *On L^1 - criteria for quasiradial Fourier multipliers with applications to some anisotropic function spaces*, Anal. Math. **9** (1983), no. 4, 275–289.
5. T. Gillespie, S. Pott, and J. Wilson, *Matrix BMO and matrix A_2 weights: a substitute for the classical $\exp -$ relationship*, Unpublished manuscript.
6. T. Gillespie, S. Pott, S. Treil, and A. Volberg, *Logarithmic growth for matrix martingale transforms*, J. Lond. Math. Soc. (2) **64** (2001), no. 3, 624–636.
7. M. Goldberg, *Matrix A_p weights via maximal functions*, Pacific J. Math. **211** (2003), no. 2, 201–220.
8. T. Hytönen, K. Li, and T. Oikari, *Iterated commutators under a joint condition on the tuple of multiplying functions*, <http://arxiv.org/abs/1910.00364>.
9. I. Holmes, M. Lacey, and B. Wick, *Bloom's inequality: commutators in a two-weight setting*, Arch. Math. (Basel) **106** (2016), no. 1, 53–63.
10. I. Holmes, M. Lacey, and B. Wick, *Commutators in the two-weight setting*, Math. Ann. **367** (2017), no. 1-2, 51–80.
11. J. Isralowitz, *Boundedness of commutators and H^1 -BMO duality in the two matrix weighted setting*, Integral Equations Operator Theory **89** (2017), no. 2, 257–287.
12. J. Isralowitz, *Matrix weighted Triebel-Lizorkin bounds: A simple stopping time proof*, <http://arxiv.org/abs/1507.06700>.
13. J. Isralowitz, H. K. Kwon, and S. Pott, *Matrix weighted norm inequalities for commutators and paraproducts with matrix symbols*, J. Lond. Math. Soc. (2) **96** (2017), no. 1, 243–270.
14. J. Isralowitz, I. P. Rivera-Ríos, and S. Pott, *Sharp A_1 weighted estimates for vector valued operators*, J. Geom. Anal. **31** (2021), no. 3, 3085–3116.
15. A. Lerner, *On an estimate of Calderón-Zygmund operators by dyadic positive operators*, J. Anal. Math. **121** (2013), no. 1, 141–161.
16. A. Lerner, S. Ombrosi, and I. P. Rivera-Ríos, *On pointwise and weighted estimates for commutators of Calderón-Zygmund operators*, Adv. Math. **319** (2017), 153–181.
17. A. Lerner, S. Ombrosi, and I. P. Rivera-Ríos, *Commutators of singular integrals revisited*, Bull. Lond. Math. Soc. **51** (2019) 107–119.
18. C. Liaw, and S. Treil, *Regularization of general singular integral operators*, Rev. Mat. Iberoam. **29** (2013), no. 1, 53–74.
19. B. Muckenhoupt, and R. Wheeden, *Two weight function norm inequalities for the Hardy-Littlewood maximal function and the Hilbert transform*, Studia Math. **55** (1976), no. 3, 279–294.
20. F. Nazarov, and S. Treil, *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis*, Algebra i Analiz **8** (1996), no. 5, 32–162.
21. F. Nazarov, S. Petermichl, S. Treil, and A. Volberg, *Convex body domination and weighted estimates with matrix weights*, Adv. Math. **318** (2017), 279–306.
22. S. Roudenko, *Matrix-weighted Besov spaces*, Trans. Amer. Math. Soc. **355** (2003), 273–314.
23. S. Treil, *Commutators, paraproducts and BMO in non-homogeneous martingale settings*, Rev. Mat. Iberoam. **29** (2013), no. 4, 1325–1372.
24. A. Volberg, *Matrix A_p weights via S -functions*, J. Amer. Math. Soc. **10** (1997), no. 2, 445–466.