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The matrix-weighted dyadic convex body maximal operator is not bounded

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ABSTRACT

The convex body maximal operator is a natural generalization of the Hardy–Littlewood maximal operator. In this paper we are considering its dyadic version in the presence of a matrix weight. To our surprise it turns out that this operator is not bounded. This is in a sharp contrast to a Doob's inequality in this context. At first, we show that the convex body Carleson Embedding Theorem with matrix weight fails. We then deduce the unboundedness of the matrix-weighted convex body maximal operator.

0. Notation

I_0	interval $[0,1]$;
$\mathcal{D}$	the dyadic lattice, i.e., the collection of all dyadic intervals;
I_+, I_-	left and right halves of the interval $I \in \mathcal{D}$;
$\mathcal{D}_\pm$	$\mathcal{D}_\pm = \{I_\pm : I \in \mathcal{D}\}$ so that $\mathcal{D} = \mathcal{D}_+ \dot\cup \mathcal{D}_- \dot\cup \{I_0\}$;

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$\mathcal{D}(K)$	$\mathcal{D}(K) = \{I \in \mathcal{D} : I \subset K\}$ for $K \in \mathcal{D}$;
$\lvert I\rvert$	the Lebesgue measure of the set $I \subset I_0$;
$\mathcal{D}^n$	$\mathcal{D}^n := \{I \in \mathcal{D} : \lvert I\rvert = 2^{-n}\}$;
$\mathcal{D}^{\leqslant n}$	$\mathcal{D}^{\leqslant n} := \bigcup_{k \leqslant n} \mathcal{D}^k = \{I \in \mathcal{D} : \lvert I\rvert \geqslant 2^{-n}\}$;
$\mathcal{C}_\pm$	for a collection $\mathcal{C} \subset \mathcal{D}$ of dyadic intervals, $\mathcal{C}_\pm = \mathcal{C} \cap \mathcal{D}_\pm$;
$(\mathcal{F}_n)_{n\geqslant 0}$	the dyadic filtration: $\mathcal{F}_n$ is the σ-algebra generated by $\mathcal{D}^n$;
$\langle f\rangle_I$	average of a scalar, vector or matrix function f over I: $\langle f\rangle_I := \lvert I\rvert^{-1} \int_I f(x)\mathrm{d}x$;
$f(I)$	for $I \subset I_0$ we denote by $f(I)$ the integral of a scalar, vector or matrix function f over I: $f(I) = \int_I f(x)\mathrm{d}x$;
$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbb{R}^d}$	the standard inner product in $\mathbb{R}^d$;
$\lvert\cdot\rvert_{\mathbb{R}^d}$	the norm induced by $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ with subscript omitted if $d = 1$;
$\lvert\cdot\rvert_{\mathbf{op}}$	the operator norm of a matrix;
$\lVert\,\cdot\,\rVert_X$	norm in the function space X;
$B(X)$	unit ball in the normed space X;
$\mathbf{1}_I$	the characteristic function of I.

Generally, since we are dealing with vector- and matrix-valued functions we will use the symbol $\|\cdot\|$ (usually with a subscript) for the norm in a functions space, while $|\cdot|$ is used for the norm in the underlying vector (matrix) space. Thus, for a vector valued function f, the symbol $\|f\|_{L^2}$ denotes its L^2-norm, but the symbol $|f|$ stands for the scalar valued function $x \mapsto |f(x)|$.

Finally, we will use the *linear algebra notation*, identifying vector a in a Hilbert space $\mathcal{H}$ with the operator $\alpha \mapsto \alpha a$ acting from scalars to $\mathcal{H}$. In this case the symbol a^* denotes the (bounded) linear functional $x \mapsto \langle x, a\rangle$. In our case $\mathcal{H}$ is the real Hilbert space $\mathbb{R}^d$, vectors in $\mathbb{R}^d$ are the column vectors, and a^* is just the transpose of the column a.

1. Introduction

The simple dyadic maximal function

$$\mathcal{M}f(x) = \sup_{I\in\mathcal{D}:x\in I} |\langle f\rangle_I|$$

and the dyadic Hardy-Littlewood maximal function

$$\mathcal{M}^{\mathrm{c}}f(x) = \sup_{I\in\mathcal{D}:x\in I} \langle |f|\rangle_I$$

together with their scalar weighted analogues

$$\mathcal{M}_w f(x) = \sup_{I\in\mathcal{D}:x\in I} \frac{1}{w(I)} \left| \int_I f(y)w(y)\mathrm{d}y \right|$$

and

$$\mathcal{M}_w^{\mathrm{c}} f(x) = \sup_{I \in \mathcal{D}: x \in I} \frac{1}{w(I)} \int\limits_I |f(y)| \, w(y) \mathrm{d}y$$

for positive $w \in L^1$ are classical objects in harmonic analysis. (The reason for the superscript c will become clear in the next section.)

The inequality

$$\|\mathcal{M}_w\|_{L^2_w(\mathbb{R}) \to L^2_w(\mathbb{R})} \leqslant 2$$

is a special case of Doob's inequality and the same inequality for $\mathcal{M}_w^{\mathrm{c}}$ follows immediately from the observations that $\mathcal{M}_w^{\mathrm{c}} f = \mathcal{M}_w |f|$ and $\| \, |f| \, \|_{L^2_w(\mathbb{R})} = \|f\|_{L^2_w(\mathbb{R})}$.

The boundedness of $\mathcal{M}_w$ or $\mathcal{M}_w^{\mathrm{c}}$ in $L^2_w(\mathbb{R})$ can be rewritten as the boundedness of the operators

$$M_w f(x) = \sup_{I \in \mathcal{D}: x \in I} \left| w^{1/2}(x) \frac{1}{w(I)} \int\limits_I f(y) w(y) \mathrm{d}y \right|$$

and

$$M_w^{\mathrm{c}} f(x) = \sup_{I \in \mathcal{D}: x \in I} w^{1/2}(x) \frac{1}{w(I)} \int\limits_I |f(y)| w(y) \mathrm{d}y$$

from $L^2_w(\mathbb{R})$ to the usual $L^2(\mathbb{R})$.

The natural counterpart (see the next section for details) of $M_w f$ and $M_w^{\mathrm{c}} f$ in the case when f is a vector-valued function and W a matrix weight (a positive definite matrix function) are

$$M_W f(x) = \sup_{I \in \mathcal{D}: x \in I} \left| W^{1/2}(x) \langle W \rangle_I^{-1} \langle W f \rangle_I \right|_{\mathbb{R}^d}$$

and

$$M_W^{\mathrm{c}} f(x) = \sup_{\substack{I \in \mathcal{D} \,:\, x \in I \\ \varphi_I : I \to [-1,1]}} \left| W^{1/2}(x) \langle W \rangle_I^{-1} \langle \varphi_I W f \rangle_I \right|_{\mathbb{R}^d}.$$

When $d = 1$ use the signum of Wf on I for φ_I to get the usual absolute value. This definition will be shown in the next section to be equivalent to a Christ-Goldberg type definition [1]. The boundedness of these operators from $L^2_W(\mathbb{R}^d)$ to $L^2(\mathbb{R})$ when $W = \mathrm{Id}_d$ remains a simple consequence of Doob's inequality, but is more complicated for general matrix weights. The positive result for $M_W f$ was established in [4] by reducing it to the weighted Carleson Embedding Theorem from [2].:

Theorem 1.1. *(Culiuc, Treil) Let W be a matrix weight and $(A_I)_{I \in \mathcal{D}}$ a sequence of positive definite matrices. Then the following are equivalent:*

(i) *There exists $c_{(\mathrm{i})} > 0$ such that for all $K \in \mathcal{D}$,*

$$\frac{1}{|K|} \sum_{I \in \mathcal{D}(K)} \langle W \rangle_I A_I \langle W \rangle_I \leqslant c_{(\mathrm{i})} \langle W \rangle_K.$$

(ii) *There exists $c_{(\mathrm{ii})} > 0$ such that for all $f \in L^2_W(\mathbb{R}^d)$,*

$$\sum_{I \in \mathcal{D}} \left| A_I^{1/2} \langle W f \rangle_I \right|^2_{\mathbb{R}^d} \leqslant c_{(\mathrm{ii})} \|f\|^2_{L^2_W(\mathbb{R}^d)}.$$

Moreover, the best possible constants $c_{(\mathrm{i})}$ and $c_{(\mathrm{ii})}$ in (i) *and* (ii) *satisfy $c_{(\mathrm{i})} \leqslant c_{(\mathrm{ii})} \leqslant C c_{(\mathrm{i})}$ with $C > 0$ only depending on the dimension d but not on W.*

The bound for the norm $\|M_W\|_{L^2_W(\mathbb{R}^d) \to L^2(\mathbb{R})}$ follows by the so-called linearization technique and the exact statement is as follows:

Theorem 1.2. *(Petermichl, Pott, Reguera) Let W be a matrix weight. Then*

$$\|M_W\|_{L^2_W(\mathbb{R}^d) \to L^2(\mathbb{R})} \leqslant C,$$

where C depends only on d.

These results raised hopes that the larger maximal function M_W^{c} may be bounded from $L^2_W(\mathbb{R}^d)$ to $L^2(\mathbb{R})$ as well, so it came as a surprise to us that in general it actually is not. The main purpose of this paper is to present and discuss the counter-example. Again, to construct it, we show that the corresponding convex body Carleson Embedding Theorem fails, which was also quite surprising for us and which may be of independent interest. Here are the precise formulations of our negative results:

Theorem 1.3. *For any fixed dimension $d \geqslant 2$, there exist a matrix weight W and a sequence $(A_I)_{I \in \mathcal{D}}$ of positive definite matrices such that there exists $c_{(\mathrm{i})} > 0$ such that for all $K \in \mathcal{D}$*

$$\frac{1}{|K|} \sum_{I \in \mathcal{D}(K)} \langle W \rangle_I A_I \langle W \rangle_I \leqslant c_{(\mathrm{i})} \langle W \rangle_K, \tag{1.1}$$

but there exist $f \in L^2_W(\mathbb{R}^d)$ and a sequence $(\varphi_I)_{I \in \mathcal{D}}$, $-1 \leqslant \varphi_I \leqslant 1$, such that

$$\sum_{I \in \mathcal{D}} \left| A_I^{1/2} \langle \varphi_I W f \rangle_I \right|^2_{\mathbb{R}^d} = \infty. \tag{1.2}$$

Theorem 1.4. *For any fixed dimension $d \geqslant 2$, there exists a matrix weight W such that M_W^{c} does not map $L^2_W(\mathbb{R}^d)$ to $L^2(\mathbb{R})$.*

2. Discussion of definitions

Let f be a vector-valued function with values in $\mathbb{R}^d$. A Hardy-Littlewood maximal function of f at x is a quantity allowing one to control all averages of f over intervals containing x. The most straightforward quantity of that type is just $\sup_{I\in\mathcal{D}:x\in I}\langle|f|_{\mathbb{R}^d}\rangle_I$. However, this quantity is often too crude when f is large in some directions and small in some other ones (all such information is completely lost here) and is too strongly tied to the particular choice of the Euclidean norm in $\mathbb{R}^d$ to yield useful bounds in the case when the size of f is measured in some other norm, especially in a norm depending on a point, as it is the case in matrix-weighted spaces.

A more reasonable idea would be to use the convex-body average $\langle\!\langle f\rangle\!\rangle_I$ of an L^1 function f over an interval I, defined as

$$\langle\!\langle f\rangle\!\rangle_I = \{\langle\varphi_I f\rangle_I : \quad \varphi_I : I \to [-1,1]\}.$$

See [3]. $\langle\!\langle f\rangle\!\rangle_I$ is always a symmetric compact convex set in $\mathbb{R}^d$ (the compactness follows from the weak-$*$ compactness of the unit ball in $L^\infty(I,\mathbb{R}) = L^1(I,\mathbb{R})^*$, where these spaces consist of functions defined on the interval I) but, in general, it may contain no inner points. However, contrary to the terminology of some convex body geometry books, we will still call it a convex body.

This convex body average makes sense even if $d = 1$, in which case it is just the interval $[-\langle|f|\rangle_I, \langle|f|\rangle_I]$, so there it carries just as much information as $\langle|f|\rangle_I$.

Let now ρ be any norm in $\mathbb{R}^d$. For a set $\Omega \subset \mathbb{R}^d$, define its norm as

$$\rho(\Omega) = \sup_{x\in\Omega}\rho(x).$$

If $\rho(x) = |x|_{\mathbb{R}^d}$ is the usual Euclidean norm, then we have

$$d^{-1}\langle|f|_{\mathbb{R}^d}\rangle_I \leqslant \rho(\langle\!\langle f\rangle\!\rangle_I) \leqslant \langle|f|_{\mathbb{R}^d}\rangle_I.$$

The right inequality is just the triangle inequality. To obtain the left one, just write $f = (f_1, \ldots, f_d)$ and note that for all k

$$\langle|f|_{\mathbb{R}^d}\rangle_I \leqslant \sum_{k=1}^{d}\langle|f_k|\rangle_I,$$

so there exists k with $\langle|f_k|\rangle_I \geqslant d^{-1}\langle|f|_{\mathbb{R}^d}\rangle_I$. On the other hand, choosing $\varphi_I = \operatorname{sign} f_k$, we get

$$\rho(\langle\!\langle f\rangle\!\rangle_I) \geqslant (\langle\varphi_I f\rangle_I)_k = \langle|f_k|\rangle_I$$

and the left inequality follows.

The advantage of the convex body approach is that this inequality is preserved if we change the standard Euclidean norm in $\mathbb{R}^d$ to any other Euclidean norm, i.e., if we take a positive definite matrix A and consider $\rho_A(x) = |A^{1/2}x|_{\mathbb{R}^d}$. The corresponding estimates

$$d^{-1}\langle\rho_A(f)\rangle_I \leqslant \rho_A(\langle\!\langle f\rangle\!\rangle_I) \leqslant \langle\rho_A(f)\rangle_I \tag{2.1}$$

immediately follow from the standard Euclidean ones by considering $A^{1/2}f$ instead of f.

Inspired by the idea of the convex body average, we can now define a convex body valued maximal function.

The simple maximal function $\mathbf{M}$ will be just

$$\mathbf{M}f(x) = \left\{ \sum_{I\in\mathcal{D}:x\in I} a_I\langle f\rangle_I : \;\; a_I \in \mathbb{R}, \sum_{I\in\mathcal{D}:x\in I} |a_I| \leqslant 1 \right\}$$

i.e., $\mathbf{M}f(x)$ is the absolute convex hull of averages of f over intervals containing x. The generalization of the Hardy-Littlewood maximal function will now be

$$\mathbf{M}^{\mathrm{c}}f(x) = \left\{ \sum_{I\in\mathcal{D}:x\in I} a_I\langle\!\langle f\rangle\!\rangle_I : \;\; a_I \in \mathbb{R}, \sum_{I\in\mathcal{D}:x\in I} |a_I| \leqslant 1 \right\},$$

where the sum of convex bodies is understood in the Minkowski sense: $A + B = \{a + b : a \in A, b \in B\}$. Plugging in the definition of $\langle\!\langle f\rangle\!\rangle_I$, we can also rewrite this as

$$\mathbf{M}^{\mathrm{c}}f(x) = \left\{ \sum_{I\in\mathcal{D}:x\in I} a_I\langle\varphi_I f\rangle_I : \;\; a_I \in \mathbb{R}, \sum_{I\in\mathcal{D}:x\in I} |a_I| \leqslant 1, \; \varphi_I : I \to [-1,1] \right\}.$$

For a matrix weight W, one can easily write the weighted analogues $\mathbf{M}_W$ and $\mathbf{M}^{\mathrm{c}}_W$ of $\mathbf{M}$ and $\mathbf{M}^{\mathrm{c}}$:

$$\mathbf{M}_W f(x) = \left\{ \sum_{I\in\mathcal{D}:x\in I} a_I\langle W\rangle_I^{-1}\langle Wf\rangle_I : \;\; a_I \in \mathbb{R}, \sum_{I\in\mathcal{D}:x\in I} |a_I| \leqslant 1 \right\}$$

and

$$\mathbf{M}^{\mathrm{c}}_W f(x) = \left\{ \sum_{I\in\mathcal{D}:x\in I} a_I\langle W\rangle_I^{-1}\langle\varphi_I Wf\rangle_I : \;\; a_I \in \mathbb{R}, \sum_{I\in\mathcal{D}:x\in I} |a_I| \leqslant 1, \; \varphi_I : I \to [-1,1] \right\}.$$

The values here are again convex bodies.

The weighted space $L^2_W(\mathbb{R}^d)$ is just the space of all measurable functions f with values in $\mathbb{R}^d$ for which

$$\|f\|^2_{L^2_W(\mathbb{R}^d)} = \int \left|W^{1/2}f\right|^2_{\mathbb{R}^d} = \int \langle Wf, f\rangle_{\mathbb{R}^d} < +\infty.$$

In other words, it is the L^2 space in which the size of f is measured not by the usual Euclidean norm in $\mathbb{R}^d$ but by the norm $\rho_{W(x)}$. In line with our definition at the beginning of this section, we can now write

$$\|\mathbf{M}_W f\|^2_{L^2_W(\mathbb{R}^d)} = \int \rho_{W(x)}(\mathbf{M}_W f(x))^2 \mathrm{d}x$$

and similarly for $\mathbf{M}^{\mathrm{c}}_W$, where, as before, the norm of a subset of $\mathbb{R}^d$ is understood as the supremum of the norms of the vectors contained in it. Since $\mathbf{M}_W f(x)$ is the absolute convex hull of $\langle W\rangle_I^{-1}\langle Wf\rangle_I$ with $I \in \mathcal{D}$ such that $x \in I$, by the convexity property of the norm,

$$\rho_{W(x)}(\mathbf{M}_W f(x)) = \sup_{I\in\mathcal{D}:x\in I} \rho_{W(x)}(\langle W\rangle_I^{-1}\langle Wf\rangle_I) = \sup_{I\in\mathcal{D}:x\in I} \left| W^{1/2}(x)\langle W\rangle_I^{-1}\langle Wf\rangle_I \right|_{\mathbb{R}^d}$$

and, by the same logic,

$$\rho_{W(x)}(\mathbf{M}^{\mathrm{c}}_W f(x)) = \sup_{\substack{I\in\mathcal{D}\,:\,x\in I\\ \varphi_I:I\to[-1,1]}} \left| W^{1/2}(x)\langle W\rangle_I^{-1}\langle \varphi_I Wf\rangle_I \right|_{\mathbb{R}^d}$$

leading to the maximal functions M_W and M^{c}_W used in the introduction so that

$$\|\mathbf{M}_W f\|_{L^2_W(\mathbb{R}^d)} = \|M_W f\|_{L^2(\mathbb{R})}$$

and similarly for $\mathbf{M}^{\mathrm{c}}_W$ and M^{c}_W.

The reader familiar with Christ-Goldberg definitions of maximal functions in the context of matrix weighted spaces may prefer to consider the quantity

$$\frac{1}{|I|}\int_I \left| W^{1/2}(x)\langle W\rangle_I^{-1} W(y)f(y) \right|_{\mathbb{R}^d} \mathrm{d}y$$

instead of our

$$\sup_{\varphi_I:I\to[-1,1]} \left| W^{1/2}(x)\langle W\rangle_I^{-1}\langle \varphi_I Wf\rangle_I \right|_{\mathbb{R}^d}.$$

However, our inequalities in (2.1) show that these quantities are the same up to a factor of d.

The final aspect we want to discuss is the measurability of $\mathbf{M}_W$ and $\mathbf{M}^{\mathrm{c}}_W$ and another natural way to define the $L^2_W(\mathbb{R}^d)$ norm of set-valued functions. We shall do this discussion for $\mathbf{M}^{\mathrm{c}}_W$. The case of $\mathbf{M}_W$ is the same but simpler.

Let us introduce the truncated maximal functions $M^{\mathrm{c}}_{W,n}$ and $\mathbf{M}^{\mathrm{c}}_{W,n}$ in which we take into consideration only the intervals $I \in \mathcal{D}^{\leqslant n}$, so

$$\mathbf{M}^{\mathrm{c}}_{W,n}f(x) = \left\{ \sum_{I\in\mathcal{D}^{\leqslant n}:x\in I} a_I \langle W\rangle_I^{-1}\langle \varphi_I W f\rangle_I : \quad a_I \in \mathbb{R}, \right.$$

$$\left. \sum_{I\in\mathcal{D}^{\leqslant n}:x\in I} |a_I| \leqslant 1,\ \varphi_I : I \to [-1,1] \right\}$$

and

$$M^{\mathrm{c}}_{W,n}f(x) = \sup_{v\in\mathbf{M}^{\mathrm{c}}_{W,n}f(x)} \left|W^{1/2}(x)v\right|_{\mathbb{R}^d} = \sup_{\substack{I\in\mathcal{D}\,:\,x\in I\\ \varphi_I:I\to[-1,1]}} \left|W^{1/2}(x)\langle W\rangle_I^{-1}\langle\varphi_I W f\rangle_I\right|_{\mathbb{R}^d}.$$

Notice that $\mathbf{M}^{\mathrm{c}}_{W,n}f(x)$ is a symmetric compact convex set. Moreover, this set depends only on the n-th level dyadic interval $I\in\mathcal{D}^n$ the point x belongs to.

Let now $K\subset\mathbb{R}^d$ be any convex compact set and let $\varepsilon>0$. Then there exists a finite set $\{v_1,\ldots,v_N\}\subset K$ such that $K\subset(1+\varepsilon)\operatorname{conv}(v_1,\ldots,v_N)$. In particular, it implies that for every $x\in I$,

$$\max_{1\leqslant j\leqslant N}\left|W^{1/2}(x)v_j\right|_{\mathbb{R}^d} \leqslant \sup_{v\in K}\left|W^{1/2}(x)v\right|_{\mathbb{R}^d} \leqslant (1+\varepsilon)\max_{1\leqslant j\leqslant N}\left|W^{1/2}(x)v_j\right|_{\mathbb{R}^d}.$$

For each $x\in I$, define $j(x)$ as the least index j for which $\left|W^{1/2}(x)v_j\right|_{\mathbb{R}^d}$ is maximal. In other words, $j(x)=J$ if and only if $\left|W^{1/2}(x)v_j\right|_{\mathbb{R}^d} < \left|W^{1/2}(x)v_J\right|_{\mathbb{R}^d}$ for $j<J$ and $\left|W^{1/2}(x)v_j\right|_{\mathbb{R}^d} \leqslant \left|W^{1/2}(x)v_J\right|_{\mathbb{R}^d}$ for $j>J$. Then $g(x)=v_{j(x)}$ is a measurable function on I such that $g(x)\in K$ and

$$\left|W^{1/2}(x)g(x)\right|_{\mathbb{R}^d} \geqslant (1+\varepsilon)^{-1}\sup_{v\in K}\left|W^{1/2}(x)v\right|_{\mathbb{R}^d}$$

for all $x\in I$.

Applying this construction to every $I\in\mathcal{D}^n$ with $K=\mathbf{M}^{\mathrm{c}}_{W,n}f|_I$, we get a measurable $g: I_0\to\mathbb{R}^d$ such that $g(x)\in\mathbf{M}^{\mathrm{c}}_{W,n}f(x)$ and

$$\left|W^{1/2}(x)g(x)\right|_{\mathbb{R}^d} > (1+\varepsilon)^{-1}M^{\mathrm{c}}_{W,n}f(x)$$

for all $x\in I_0$. Letting $\varepsilon\to0$ (along some sequence), we obtain that $M^{\mathrm{c}}_{W,n}f$ is the supremum of a countable family of measurable functions and, thereby, measurable. Finally, letting $n\to\infty$, we get that $M^{\mathrm{c}}_W f$ is measurable and, moreover,

$$\|M^{\mathrm{c}}_W f\|_{L^2(\mathbb{R})} = \sup\{\|g\|_{L^2_W(\mathbb{R}^d)} : \quad g: I_0\to\mathbb{R}^d,\ g \text{ is measurable},\ g\in\mathbf{M}^{\mathrm{c}}_W f\}.$$

3. Proofs

We first prove Theorem 1.3, which states that the convex body analogue of the weighted Carleson Embedding Theorem (i.e., of Theorem 1.1) fails.

Using this result we then conclude the failure of the maximal function estimate.

3.1. Failure of the convex body Carleson embedding theorem: proof of Theorem 1.3

3.1.1. Overview of the construction

We consider a counterexample in $\mathbb{R}^2$, which trivially extends to an example in all higher dimensions.

We will be constructing the weight W supported on the interval I_0 as a martingale, from the top down. This means at the nth step ($n = 0, 1, 2, \ldots$) we construct the averages W_I, $I \in \mathcal{D}^n$, which will be the averages of the constructed weight W. The averages $W_I = \langle W \rangle_I$ should satisfy the *martingale dynamics*,

$$W_I = \frac{1}{2}\left(W_{I_+} + W_{I_-}\right), \tag{3.1}$$

so the sequence of weights W_n,

$$W_n := \sum_{I \in \mathcal{D}^n} W_I \mathbf{1}_I$$

is a martingale with respect to the dyadic filtration. In our case the sequence W_n will be uniformly bounded, so we will have convergence[5] $W_n \to W$ (say, entry-wise in L^1),

$$W_I = \langle W \rangle_I \qquad \forall I \in \mathcal{D}.$$

We will work with the spectral decompositions of the averages W_I

$$W_I = \alpha_I a_I a_I^* + \beta_I b_I b_I^*$$

where α_I, β_I are the eigenvalues, and a_I, b_I are the corresponding normalized eigenvectors of W_I. This means, in particular, that $|a_I|_{\mathbb{R}^2} = |b_I|_{\mathbb{R}^2} = 1$ and $a_I \perp b_I$.

In our situation, the eigenvalues α_I, β_I will depend only on $|I|$, i.e., for $I \in \mathcal{D}^n$ we will have $\alpha_I = \alpha_n$, $\beta_I = \beta_n$. Note that this condition means that $\operatorname{trace} W_I = \operatorname{trace} W_J$ if $I, J \in \mathcal{D}^n$; then the martingale property (3.1) and linearity of the trace imply that $\operatorname{trace} W_I = \operatorname{trace} W_{I_0}$ for all $I \in \mathcal{D}$, which, in turn, implies the uniform boundedness of W_n, $|W_n(x)|_{\mathrm{op}} \leqslant \alpha_0 + \beta_0$.

In our construction, we will have the condition numbers (eccentricity) $\alpha_n/\beta_n \to \infty$ as $n \to \infty$. The operators A_I will be just appropriately renormalized projections onto b_I, so the largest term $\alpha_I a_I a_I^*$ of W_I disappears from the testing condition (1.1) in Theorem 1.3. (See Fig. 1.)

However, an appropriate (and very simple) choice of the functions φ_I in the conclusion (1.2) of Theorem 1.3 will bring this huge term $\alpha_I a_I a_I^*$ into play, and that will give us the desired blow-up

[5] For the convergence a much weaker condition on *uniform integrability* is sufficient.

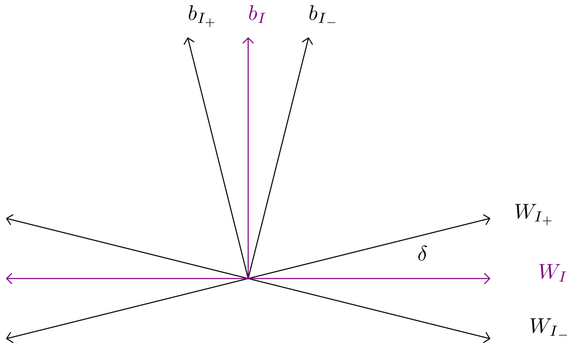

Fig. 1. In this picture we see the images of the unit ball under the weight W, sketched as arrows though they are thin ellipses. Further, we see the directions of the vectors b.

3.1.2. Gory details

Now let us assume that for all $I \in \mathcal{D}^n$, we constructed W_I, and its spectral decomposition is given by

$$W_I = \alpha_n a_I a_I^* + \beta_n b_I b_I^*.$$

Recall that a_I and b_I is an orthonormal pair of vectors. Let us define the averages $W_{I_\pm}$ as

$$W_{I_\pm} = \alpha_{n+1} a_{I_\pm} a_{I_\pm}^* + \beta_{n+1} b_{I_\pm} b_{I_\pm}^*,$$

where $a_{I_\pm}$ and $b_{I_\pm}$ are small rotations of the vectors a_I and b_I, namely, for some small $\delta_{n+1} > 0$

$$a_{I_\pm} = \left(\frac{1}{1+\delta_{n+1}^2}\right)^{1/2} (a_I \pm \delta_{n+1} b_I), \qquad b_{I_\pm} = \left(\frac{1}{1+\delta_{n+1}^2}\right)^{1/2} (b_I \mp \delta_{n+1} a_I); \tag{3.2}$$

note that a_{I_+}, b_{I_+} and a_{I_-}, b_{I_-} are orthonormal pairs.

Let us now find the relations between α_n, β_n, δ_{n+1}, α_{n+1}, and β_{n+1}, so that the martingale dynamics (3.1) holds. We have

$$\begin{aligned} W_I &= \frac{1}{2}\left(W_{I_-} + W_{I_+}\right) \\ &= \frac{1}{2} \cdot \frac{\alpha_{n+1}}{1+\delta_{n+1}^2} (a_I - \delta_{n+1} b_I)(a_I^* - \delta_{n+1} b_I^*) \\ &\quad + \frac{1}{2} \cdot \frac{\beta_{n+1}}{1+\delta_{n+1}^2} (\delta_{n+1} a_I + b_I)(\delta_{n+1} a_I^* + b_I^*) \\ &\quad + \frac{1}{2} \cdot \frac{\alpha_{n+1}}{1+\delta_{n+1}^2} (a_I + \delta_{n+1} b_I)(a_I^* + \delta_{n+1} b_I^*) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\cdot\frac{\beta_{n+1}}{1+\delta_{n+1}^2}(-\delta_{n+1}a_I+b_I)(-\delta_{n+1}a_I^*+b_I^*)\\
&=\frac{\alpha_{n+1}+\beta_{n+1}\delta_{n+1}^2}{1+\delta_{n+1}^2}a_Ia_I^*\\
&+\frac{\alpha_{n+1}\delta_{n+1}^2+\beta_{n+1}}{1+\delta_{n+1}^2}b_Ib_I^*.
\end{aligned}
$$

Thus, the martingale dynamics (3.1) holds if and only if

$$
\alpha_n=\frac{\alpha_{n+1}+\beta_{n+1}\delta_{n+1}^2}{1+\delta_{n+1}^2}\qquad\text{and}\qquad\beta_n=\frac{\alpha_{n+1}\delta_{n+1}^2+\beta_{n+1}}{1+\delta_{n+1}^2}. \tag{3.3}
$$

Note that it follows from relations (3.3) that

$$
\alpha_n+\beta_n=\alpha_{n+1}+\beta_{n+1}, \tag{3.4}
$$

as it should be, according to the martingale dynamics (3.1) and the linearity of the trace.

Now, given α_n, β_n, α_{n+1}, β_{n+1} satisfying (3.4), we can easily find δ_{n+1} by solving equations (3.3),

$$
\delta_{n+1}^2=\frac{\alpha_{n+1}-\alpha_n}{\alpha_n-\beta_{n+1}}=\frac{\beta_n-\beta_{n+1}}{\alpha_{n+1}-\beta_n}; \tag{3.5}
$$

the two expressions for δ_{n+1}^2 here coincide because of (3.4).

We now want the condition numbers α_n/β_n to increase exponentially, so let us take

$$
\frac{\beta_n}{\alpha_n}=\varepsilon^{2n+2},\qquad n=0,1,2,\dots\ , \tag{3.6}
$$

where a small $\varepsilon>0$ is to be chosen later. For the sake of convenience in the calculations, we want all δ_n to be small, and the extra ε^2 in (3.6) helps with that.

If we fix the sums in (3.4), $\alpha_n+\beta_n=1$, then we get from (3.6) that

$$
\alpha_n=\frac{1}{1+\varepsilon^{2n+2}},\qquad\text{and}\qquad\beta_n=\frac{\varepsilon^{2n+2}}{1+\varepsilon^{2n+2}}\,. \tag{3.7}
$$

Substituting these expressions into (3.5) (with n replaced by $n-1$), we get

$$
\delta_n^2=\frac{\varepsilon^{2n}(1-\varepsilon^2)}{1-\varepsilon^{4n+2}},\qquad n=1,2,3,\dots \tag{3.8}
$$

So, as we discussed above in Section 3.1.1, there exists a weight W with its averages given by $\langle W\rangle_I=W_I$ for all $I\in\mathcal{D}$.

3.1.3. A_I and the testing condition (1.1)

We define $A_I^{1/2}$ to be multiples of the projections onto b_I, i.e.,

$$A_I^{1/2} = |I|^{1/2} r_I b_I b_I^*,$$

where for $I \in \mathcal{D}^n$

$$r_I = r_n = \frac{1}{\varepsilon^{n+1}}.$$

First, we will prove that the testing condition (1.1) holds, i.e., that for every dyadic interval $K \in \mathcal{D}$ and every vector $e \in \mathbb{R}^2$ we have

$$\sum_{I \in \mathcal{D}(K)} \left| A_I^{1/2} \langle We \rangle_I \right|_{\mathbb{R}^2}^2 \leqslant C|K| \left\langle \langle W \rangle_K e, e \right\rangle_{\mathbb{R}^2}. \tag{3.9}$$

Let $K \in \mathcal{D}$ be such that $|K| = 2^{-n_0}$ for some $n_0 \geqslant 0$. Let us denote the sum on the left hand side of (3.9) by Σ_1. Observe that

$$A_I^{1/2} \langle W \rangle_I = |I|^{1/2} \beta_I r_I b_I b_I^* = \beta_I A_I^{1/2},$$

where we recall that $\beta_I = \beta_n$ for $I \in \mathcal{D}^n$. So we get

$$\begin{aligned}
\Sigma_1 &= \sum_{I \in \mathcal{D}(K)} \left| A_I^{1/2} \langle We \rangle_I \right|_{\mathbb{R}^2}^2 = \sum_{I \in \mathcal{D}(K)} \left| A_I^{1/2} \langle W \rangle_I e \right|_{\mathbb{R}^2}^2 \\
&= \sum_{I \in \mathcal{D}(K)} |I| (\beta_I r_I)^2 \, |b_I b_I^* e|_{\mathbb{R}^2}^2 \\
&= \sum_{I \in \mathcal{D}(K)} |I| (\beta_I r_I)^2 \langle e, b_I \rangle_{\mathbb{R}^2}^2.
\end{aligned}$$

Decomposing

$$e = e_1 a_K + e_2 b_K, \qquad e_1,\, e_2 \in \mathbb{R},$$

we see that

$$\langle e, b_I \rangle_{\mathbb{R}^2}^2 \leqslant |e|_{\mathbb{R}^2}^2 = e_1^2 + e_2^2.$$

Using the fact that for $I \in \mathcal{D}^n$ we have $\beta_I = \beta_n \leqslant \varepsilon^{2n+2}$ and $r_I = r_n = \frac{1}{\varepsilon^{n+1}}$, we estimate

$$\begin{aligned}
\Sigma_1 &\leqslant |e|_{\mathbb{R}^2}^2 \sum_{I \in \mathcal{D}(K)} |I| (\beta_I r_I)^2 \leqslant |e|_{\mathbb{R}^2}^2 \sum_{n=n_0}^{\infty} 2^{-n_0} \left(\varepsilon^{2n+2} \cdot \varepsilon^{-n-1} \right)^2 \\
&= |e|_{\mathbb{R}^2}^2 \, 2^{-n_0} \varepsilon^{2n_0+2} \sum_{n=0}^{\infty} \varepsilon^{2n} = 2^{-n_0} \frac{\varepsilon^{2n_0+2}}{1-\varepsilon^2} \left(e_1^2 + e_2^2 \right).
\end{aligned}$$

The right-hand side of (3.9) can be estimated from below

$$\begin{aligned}|K|\,\langle\langle W\rangle_K e, e\rangle_{\mathbb{R}^2} &= 2^{-n_0}(\alpha_{n_0}e_1^2+\beta_{n_0}e_2^2)\\ &\geqslant 2^{-n_0}(1-\varepsilon^2)(e_1^2+\varepsilon^{2n_0+2}e_2^2)\\ &\geqslant 2^{-n_0}(1-\varepsilon^2)\varepsilon^{2n_0+2}(e_1^2+e_2^2).\end{aligned}$$

In the second inequality we used the fact that $\alpha_{n_0}\geqslant 1-\varepsilon^2$ and $\beta_{n_0}\geqslant(1-\varepsilon^2)\varepsilon^{2n_0+2}$, derived from the equations in (3.7).

Comparing this estimate with the above upper bound for Σ_1, we see that for a constant C such that $C(1-\varepsilon^2)^2\geqslant 1$ we have

$$\Sigma_1\leqslant C|K|\,\langle\langle W\rangle_K e, e\rangle_{\mathbb{R}^2},$$

so the testing condition (3.9) holds.

3.1.4. The blow-up

Now we want to show that there exist a vector $e\in\mathbb{R}^2$ and scalar functions φ_I supported on I with $-1\leqslant\varphi_I\leqslant 1$ such that for $f=\mathbf{1}_{I_0}e$,

$$\Sigma_2:=\sum_{I\in\mathcal{D}}\left|A_I^{1/2}\langle\varphi_I Wf\rangle_I\right|_{\mathbb{R}^2}^2=\infty. \tag{3.10}$$

Let us chose $e=a=a_{I_0}$. Note that $\|\mathbf{1}_{I_0}e\|^2_{L^2_W(\mathbb{R}^2)}=\langle W_{I_0}a,a\rangle_{\mathbb{R}^2}=\frac{1}{1+\varepsilon^2}<\infty$.

We will choose $\varphi_I=\mathbf{1}_{I_+}$ for all intervals I. Therefore for $f=\mathbf{1}_{I_0}a$ we have

$$\langle\varphi_I Wf\rangle_I=\frac{1}{|I|}\int_I\varphi_I(x)W(x)a\,\mathrm{d}x=\frac{1}{|I|}\int_{I_+}W(x)a\,\mathrm{d}x=\frac{|I_+|}{|I|}\langle W\rangle_{I_+}a=\frac{1}{2}\langle W\rangle_{I_+}a. \tag{3.11}$$

Hence we can expand the sum in (3.10) as

$$\begin{aligned}\sum_{I\in\mathcal{D}}\left|A_I^{1/2}\langle\varphi_I Wa\rangle_I\right|^2_{\mathbb{R}^2}&=\frac{1}{4}\sum_{I\in\mathcal{D}}r_I^2|I|\left|b_Ib_I^*\langle W\rangle_{I_+}a\right|^2_{\mathbb{R}^2}\\ &=\frac{1}{4}\sum_{I\in\mathcal{D}}r_I^2|I|\left|\langle a,\langle W\rangle_{I_+}b_I\rangle_{\mathbb{R}^2}\right|^2.\end{aligned} \tag{3.12}$$

For $I\in\mathcal{D}^n$ denote

$$\gamma_I=\gamma_{n+1}=\arctan\delta_{n+1},$$

so the relations (3.2) can be rewritten as

$$a_{I_\pm}=(\cos\gamma_I)a_I\pm(\sin\gamma_I)b_I,\qquad b_{I_\pm}=(\cos\gamma_I)b_I\mp(\sin\gamma_I)a_I. \tag{3.13}$$

Then, since $\langle W\rangle_{I_+} = \alpha_{I_+} a_{I_+} a_{I_+}^* + \beta_{I_+} b_{I_+} b_{I_+}^*$, we can see from (3.13) that

$$\langle W\rangle_{I_+} b_I = \alpha_{I_+} \sin\gamma_I a_{I_+} + \beta_{I_+} \cos\gamma_I b_{I_+}.$$

Then we can rewrite Σ_2 from (3.12) as

$$\frac{1}{4}\sum_{I\in\mathcal{D}} r_I^2|I| \left(\alpha_{I_+}(\sin\gamma_I)\langle a, a_{I_+}\rangle_{\mathbb{R}^2} + \beta_{I_+}(\cos\gamma_I)\langle a, b_{I_+}\rangle_{\mathbb{R}^2}\right)^2 = \frac{1}{4}\sum_{I\in\mathcal{D}} r_I^2|I|\left(D_I + F_I\right)^2, \tag{3.14}$$

where

$$D_I := \alpha_{I_+}(\sin\gamma_I)\langle a, a_{I_+}\rangle_{\mathbb{R}^2}, \qquad F_I := \beta_{I_+}(\cos\gamma_I)\langle a, b_{I_+}\rangle_{\mathbb{R}^2}.$$

We will show that if ε is sufficiently small, then

$$D_I \geqslant 2|F_I| \tag{3.15}$$

and

$$D_I \geqslant (8r_I)^{-1}. \tag{3.16}$$

Then we can ignore terms F_I in (3.14), and estimate Σ_2 (with some $c > 0$)

$$\Sigma_2 \geqslant c\sum_{I\in\mathcal{D}} r_I^2|I|r_I^{-2} = c\sum_{n=0}^{\infty}\sum_{I\in\mathcal{D}^n}|I| = c\sum_{n=0}^{\infty} 1 = \infty, \tag{3.17}$$

thus proving (3.11) modulo estimates (3.15) and (3.16).

So, let us prove estimates (3.15) and (3.16). Trivially $|F_I| \leqslant \beta_{I_+}$, and since for $I \in \mathcal{D}^n$ we have $\beta_{I_+} = \beta_{n+1} \leqslant \varepsilon^{2n+4}$, it follows that

$$|F_I| \leqslant \beta_{I_+} \leqslant \varepsilon^{2n+4}. \tag{3.18}$$

Now let us look at D_I. Since

$$0 < \gamma_n = \arctan\delta_n \leqslant \delta_n \leqslant \varepsilon^n$$

(the last inequality follows immediately from (3.8)), the angle γ between a and a_{I_+} can be bounded as

$$0 \leqslant \gamma \leqslant \sum_{n=1}^{\infty}\gamma_n \leqslant \sum_{n=1}^{\infty}\varepsilon^n = \frac{\varepsilon}{1-\varepsilon}. \tag{3.19}$$

Recalling that $|a|_{\mathbb{R}^2} = |a_{I_+}|_{\mathbb{R}^2} = 1$, we therefore can see that $\langle a, a_{I_+}\rangle_{\mathbb{R}^2} \geqslant 1/2$ for sufficiently small ε. Also, we have for sufficiently small ε

$$
\begin{aligned}
\sin\gamma_I &= \sin\gamma_{n+1} = \delta_{n+1}(1+\delta_{n+1}^2)^{-1/2} \geqslant \frac{\varepsilon^{n+1}}{2},\\
\alpha_{I_+} &= \alpha_{n+1} \geqslant 1/2.
\end{aligned}
$$

Combining the above three estimates, we get that

$$
D_I \geqslant \varepsilon^{n+1}/8 = (8r_I)^{-1}, \tag{3.20}
$$

i.e., that the estimate (3.16) holds (for sufficiently small ε). Comparing the above bound (3.20) with (3.18), we can immediately see that (3.15) holds for $0 < \varepsilon \leqslant 1/2$. □

Remark 3.1. Notice that the left-hand side of (3.10) blows up even if we do not sum over all $I \in \mathcal{D}$ but only over right half intervals, such as when $\varphi_I = \mathbf{1}_{I_+}$ if $I \in \mathcal{D}_+$ and $\varphi_I = 0$ otherwise. In that case the estimates (3.15), (3.16) still hold, so in the same way as before we get for $f = \mathbf{1}_{I_0} a$

$$
\sum_{I\in\mathcal{D}_+} \left| A_I^{1/2}\langle \varphi_I W f\rangle_I \right|_{\mathbb{R}^2}^2 \geqslant c \sum_{I\in\mathcal{D}_+} r_I^2 |I| r_I^{-2} = c\sum_{n=0}^{\infty}\sum_{I\in\mathcal{D}_+^n} |I| = c\sum_{n=0}^{\infty} 1/2 = \infty. \tag{3.21}
$$

3.2. Failure of convex body weighted maximal theorem: proof of Theorem 1.4

3.2.1. Construction

Let us take the sequence A_I and W from the above example. We set, using the constant from inequality (3.9)

$$
\widetilde{A}_I = C^{-1}\langle W\rangle_I A_I \langle W\rangle_I, \tag{3.22}
$$

where due to the Carleson property in inequality (3.9) we have

$$
\frac{1}{|I|}\sum_{J\in\mathcal{D}(I)} \widetilde{A}_J = \frac{1}{C|I|}\sum_{J\in\mathcal{D}(I)} \langle W\rangle_J A_J \langle W\rangle_J \leqslant \langle W\rangle_I. \tag{3.23}
$$

Fix n. Consider the weight W_n,

$$
W_n := \sum_{I\in\mathcal{D}^n} \langle W\rangle_I \mathbf{1}_I;
$$

so W_n is just the martingale defining W at time n.

For an interval $I \in \mathcal{D}_+^{\leqslant n}$ denote by S_I the leftmost interval of size 2^{-n-1} contained in I. That is, the interval of size 2^{-n-1} reached from I via sign tosses to the left. Denote by $\mathcal{S}^n$ the collection of all such intervals, $\mathcal{S}^n := \{S_I : I \in \mathcal{D}_+^{\leqslant n}\}$. (See Fig. 2.) Observe that the intervals in $\mathcal{S}^n$ are pairwise disjoint. Indeed, since they are all of equal length they are either disjoint or identical. But any interval J in $\mathcal{S}^n$ cannot arise from both I, I' in

$\mathcal{D}_{+}^{\leqslant n}$ with, say, $J \subsetneq I' \subsetneq I$. By construction, the path from I to J consists only of sign tosses to the left. Since $I' \in \mathcal{D}_{+}^{\leqslant n}$, the last sign toss from the path from I to I' was to the right, so it cannot lie on the path from I to J.

Now we define a family of weights $W_{n,s}$, $0 \leqslant s$,

$$W_{n,s} := W_n + s \sum_{I \in \mathcal{D}_{+}^{\leqslant n}} |S_I|^{-1} \mathbf{1}_{S_I} \widetilde{A}_I,$$

where the matrices $\widetilde{A}_I$ are defined above by (3.22). Note that the weights $W_{n,s}$ are measurable in $\mathcal{F}_{n+1}$, meaning that they are constant on intervals $I \in \mathcal{D}^{n+1}$.

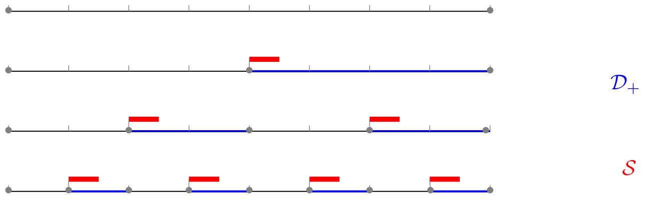

Fig. 2. In this picture we present the construction of the intervals S_I for $n = 3$. The intervals $I \in \mathcal{D}_{+}^{\leqslant n}$ are marked in blue, and the corresponding intervals S_I are marked in red. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

To prove Theorem 1.4 we will show by contradiction that for the family of weights $W_{n,s}$ we *do not have* the uniform estimate

$$\left\| M^{\mathrm{c}}_{W_{n,s}} f \right\|_{L^2(\mathbb{R})} \leqslant C \|f\|_{L^2_{W_{n,s}}(\mathbb{R}^2)} \qquad \forall f \in L^2_{W_{n,s}}(\mathbb{R}^2) \tag{3.24}$$

with C not depending on n and s. An elementary reasoning then gives us a weight $\widetilde{W}$ such that

$$\left\| M^{\mathrm{c}}_{\widetilde{W}} \tilde{f} \right\|_{L^2(\mathbb{R})} = \infty$$

for some function $\tilde{f} \in L^2_{\widetilde{W}}(\mathbb{R}^2)$.

Indeed, let s_k, n_k be such that for the weights $W_k := W_{n_k,s_k}$ on I_0 there exist non-zero $f_k \in L^2_{W_k}(\mathbb{R}^2)$ supported on I_0 such that

$$\left\| M^{\mathrm{c}}_{W_k} f \right\|^2_{L^2} > 4^k \|f\|^2_{L^2_{W_k}(\mathbb{R}^2)}. \tag{3.25}$$

Note, that this inequality is invariant under rescaling, meaning that it does not change if we multiply W_k and f_k by some non-zero constants. It is also easy to see that it does not change under an affine change of variables (applied to all objects simultaneously).

To construct the weight $\widetilde{W}$ on I_0 let us represent I_0 as a union of disjoint intervals $I_k \in \mathcal{D}$, $k \geqslant 1$ (for example, take $I_k := [2^{-k}, 2^{-k+1})$). Let $\widetilde{W}_k$ be the weight W_k transplanted via an affine change of variables to the interval I_k and normalized (by multiplying by a non-zero constant) in such a way that $\langle W_k\rangle_{I_k} \leqslant \mathbf{I}$ (the identity matrix).

Let also $\tilde{f}_k$ be the function f_k transplanted by the same affine change as W_k to the interval I_k and normalized by $\|\tilde{f}_k\|^2_{\widetilde{W}_k} = 2^{-k}$.

Defining

$$\widetilde{W}(x) := \sum_{k=0}^{\infty} \mathbf{1}_{I_k}(x)\widetilde{W}_k(x), \qquad \tilde{f}(x) := \sum_{k=0}^{\infty} \mathbf{1}_{I_k}(x)\tilde{f}_k(x),$$

we immediately see that $\|\tilde{f}\|^2_{\widetilde{W}} = 1$, and the estimates (3.25) imply that

$$\left\|\mathbf{1}_{I_k} M^{\mathrm{c}}_{\widetilde{W}} \tilde{f}\right\|^2_{L^2} \geqslant \left\|\mathbf{1}_{I_k} M^{\mathrm{c}}_{\widetilde{W}_k} \tilde{f}_k\right\|^2_{L^2} > 4^k \|\tilde{f}_k\|^2_{\widetilde{W}_k} \geqslant 2^k,$$

which gives the desired blow-up.

3.2.2. An a priori estimate

Now, let us assume that we have the uniform estimate (3.24) just for one function $f = \mathbf{1}_{I_0} a$, where $a \in \mathbb{R}^2$ is the same as in inequality (3.21) in Remark 3.1.

Our goal is to arrive at a contradiction to (3.21). From our assumption we first deduce some weaker estimate, from which using a trick from [2] we will get the desired conclusion.

To get to the final contradiction we need to estimate the weighted maximal function from below. We start with the trivial observation that for a weight W and a function $f \in L^2_W(\mathbb{R}^2)$ and any collection of disjoint measurable sets $S_I \subset I$ and measurable functions $\varphi_I : I \to [-1, 1]$ parametrized by $I \in \mathcal{D}$, the function $F = F[f, W, S_I, \varphi_I]$,

$$F(x) := \sum_{I\in\mathcal{D}} \left|\mathbf{1}_{S_I}(x) W(x)^{1/2} \langle W\rangle_I^{-1} \langle \varphi_I W f\rangle_I\right|_{\mathbb{R}^2}$$

is pointwise estimated as $F(x) \leqslant M^{\mathrm{c}}_W f(x)$. Indeed, if $x \notin \cup_{I\in\mathcal{D}} S_I$, then $F(x) = 0$. Otherwise the sum defining $F(x)$ consists of exactly one term

$$F(x) = \left|W(x)^{1/2} \langle W\rangle_I^{-1} \langle \varphi_I W f\rangle_I\right|_{\mathbb{R}^2} \tag{3.26}$$

where $I = I(x)$ is the unique $I \in \mathcal{D}$ such that $x \in S_I$; uniqueness of I follows from the disjointness of S_I. To get the maximal function $M^{\mathrm{c}}_W f(x)$, we need to take the supremum of the right hand side of (3.26) over all $I \in \mathcal{D}$, $I \ni x$ and over all φ_I, so no particular choice of I and φ_I can give us more than the maximal function.

It is easy to compute the norm of F and rewrite the trivial inequality $\|F\|^2_{L^2(\mathbb{R})} \leqslant \|M^{\mathrm{c}}_W f\|^2_{L^2(\mathbb{R})}$ as

$$\sum_{I\in\mathcal{D}} |S_I| \left|\langle W\rangle_{S_I}^{1/2} \langle W\rangle_I^{-1} \langle \varphi_I W f\rangle_I\right|^2_{\mathbb{R}^2} \leqslant \|M^{\mathrm{c}}_W f\|^2_{L^2(\mathbb{R})}. \tag{3.27}$$

The above inequality (3.27) holds for any collection of disjoint measurable sets $S_I \subset I$ and functions $\varphi_I : I \to [-1, 1]$.

We remark that if we take in the left hand side of (3.27) the supremum over all such collections, we will get exactly $\|M_W^{\mathrm{c}} f\|^2_{L^2(\mathbb{R})}$. This is well known to the experts as the *linearization* for the maximal function, that reduces without loss of generality the estimates of a nonlinear operator (the maximal function) to the estimates of the (linear) embedding operator. Since for the current paper we only need the trivial estimate (3.27), we will leave the proof of the details of this *linearization* as an exercise for a curious reader.

We specify the estimate (3.27) to the case of $W = W_{n,s}$ constructed above and $f = \mathbf{1}_{I_0} a$, with S_I defined in Section 3.2.1 and $\varphi_I = \mathbf{1}_{I_+}$ for all $I \in \mathcal{D}_+^{\leqslant n}$ and $\varphi_I = 0$ else, similarly as in Section 3.1.4. By noticing that for $I \in \mathcal{D}_+^{\leqslant n}$ we have

$$W_{n,s}(S_I) = W(S_I) + s\widetilde{A}_I \geqslant s\widetilde{A}_I,$$

we get from the estimate (3.27) that

$$\sum_{I \in \mathcal{D}_+^{\leqslant n}} s \left| \widetilde{A}_I^{1/2} \langle W_{n,s} \rangle_I^{-1} \langle \varphi_I W_{n,s} f \rangle_I \right|^2_{\mathbb{R}^2} \leqslant \left\| M_{W_{n,s}}^{\mathrm{c}} f \right\|^2_{L^2(\mathbb{R})}. \tag{3.28}$$

But for $f = \mathbf{1}_{I_0} a$, $a \in \mathbb{R}^2$, we have

$$\|f\|^2_{L^2_{W_{n,s}}} = \langle \langle W \rangle_{I_0} a, a \rangle_{\mathbb{R}^2} + s \sum_{I \in \mathcal{D}_+} \left\langle \widetilde{A}_I a, a \right\rangle_{\mathbb{R}^2} \leqslant (1+s) \left\langle \langle W \rangle_{I_0} a, a \right\rangle_{\mathbb{R}^2},$$

so we get from (3.28) that

$$s \sum_{I \in \mathcal{D}_+^{\leqslant n}} \left| \widetilde{A}_I^{1/2} \langle W_{n,s} \rangle_I^{-1} \langle \varphi_I W_{n,s} f \rangle_I \right|^2_{\mathbb{R}^2} \leqslant (1+s) C \left\langle \langle W \rangle_{I_0} a, a \right\rangle_{\mathbb{R}^2}, \tag{3.29}$$

which is our a priori estimate.

3.2.3. From estimate (3.29) to a contradiction

For $I \in \mathcal{D}_+^{\leqslant n}$ define

$$R_{n,I}(s) := \left| \widetilde{A}_I^{1/2} \langle W_{n,s} \rangle_I^{-1} \langle \varphi_I W_{n,s} f \rangle_I \right|^2_{\mathbb{R}^2},$$

so (3.29) can be rewritten as

$$s \sum_{I \in \mathcal{D}_+^{\leqslant n}} R_{n,I}(s) \leqslant (1+s) C \left\langle \langle W \rangle_{I_0} a, a \right\rangle_{\mathbb{R}^2}. \tag{3.30}$$

By the cofactor inversion formula, the entries of the matrix $\langle W_{n,s}\rangle_I^{-1}$ are rational functions of s of the form $p_{n,I}(s)/Q_{n,I}(s)$ where $p_{n,I}(s)$ is affine in s and $Q_{n,I}(s) = \det\big(\langle W_{n,s}\rangle_I\big)$ has degree 2.

The components of the vector $\langle W_{n,s}\rangle_{I_+} a$ are polynomials of degree 1, so we can write

$$R_{n,I}(s) = \frac{P_{n,I}}{Q_{n,I}^2}(s), \quad Q_{n,I}(s) = \det\big(\langle W_{n,s}\rangle_I\big)$$

and $P_{n,I}$ is a polynomial of degree at most 4.

Recall that for $I \in \mathcal{D}_+^{\leqslant n}$

$$\langle W_{n,s}\rangle_I = \langle W\rangle_I + s\frac{1}{|I|}\sum_{J\in\mathcal{D}_+^{\leqslant n}(I)} \tilde{A}_J \leqslant (1+s)\langle W\rangle_I$$

and thus for all $s \geqslant 0$,

$$Q_{n,I}(s) \leqslant (1+s)^2 \det\left(\langle W\rangle_I\right) = (1+s)^2 Q_{n,I}(0).$$

Therefore, the estimate (3.30) implies that

$$\sum_{I\in\mathcal{D}_+^{\leqslant n}} \frac{P_{n,I}(s)}{Q_{n,I}(0)^2} \leqslant \frac{(1+s)^5}{s} C\left\langle \langle W\rangle_{I_0} a, a\right\rangle_{\mathbb{R}^2} \tag{3.31}$$

We need the following Lemma, see [2, Lemma 2.2]:

Lemma 3.2. *If p is a polynomial such that*

$$|p(s)| \leqslant \frac{(1+s)^N}{s} \qquad \forall s > 0,$$

then $|p(0)| \leqslant e^2 N^2$.

We apply this lemma to the polynomial p,

$$p(s) := \left(C\left\langle \langle W\rangle_{I_0} a, a\right\rangle_{\mathbb{R}^2}\right)^{-1} \sum_{I\in\mathcal{D}_+^{\leqslant n}} \frac{P_{n,I}(s)}{Q_{n,I}(0)^2}.$$

Note that $P_{n,I}(s)$ are non-negative for $s \geqslant 0$. Estimate (3.31) means that p satisfies the assumption of Lemma 3.2 with $N = 5$, therefore

$$\sum_{I\in\mathcal{D}_+^{\leqslant n}} R_{n,I}(0) \leqslant Ce^2 5^2 \left\langle \langle W\rangle_{I_0} a, a\right\rangle_{\mathbb{R}^2}. \tag{3.32}$$

Recall that for $I \in \mathcal{D}_+^{\leqslant n}$

$$R_{n,I}(0) = \left| \widetilde{A}_I^{1/2} \langle W_n \rangle_I^{-1} \langle \varphi_I W_n f \rangle_I \right|_{\mathbb{R}^2}^2 ,$$

where $f = \mathbf{1}_{I_0} a$ and $\varphi_I = \mathbf{1}_{I_+}$. Noticing that for $I \in \mathcal{D}_+^{\leqslant n-1}$

$$\langle W_n \rangle_I = \langle W \rangle_I, \qquad \langle \varphi_I W_n f \rangle_I = \langle \varphi_I W f \rangle_I,$$

we can deduce from (3.32) that

$$\sum_{I \in \mathcal{D}_+^{\leqslant n-1}} \left| \widetilde{A}_I^{1/2} \langle W \rangle_I^{-1} \langle \varphi_I W f \rangle_I \right|_{\mathbb{R}^2}^2 \leqslant Ce^2 5^2 \left\langle \langle W \rangle_{I_0} a, a \right\rangle_{\mathbb{R}^2} .$$

Letting $n \to \infty$ we get that

$$\sum_{I \in \mathcal{D}_+} \left| \widetilde{A}_I^{1/2} \langle W \rangle_I^{-1} \langle \varphi_I W f \rangle_I \right|_{\mathbb{R}^2}^2 \leqslant Ce^2 5^2 \left\langle \langle W \rangle_{I_0} a, a \right\rangle_{\mathbb{R}^2} . \tag{3.33}$$

Recall that by (3.22) we have $\widetilde{A}_I = C^{-1} \langle W \rangle_I A_I \langle W \rangle_I$, so we rewrite the above estimate (3.33) as

$$\sum_{I \in \mathcal{D}_+} \left| A_I^{1/2} \langle \varphi_I W f \rangle_I \right|_{\mathbb{R}^2}^2 \leqslant Ce^2 5^2 \left\langle \langle W \rangle_{I_0} a, a \right\rangle_{\mathbb{R}^2} < \infty,$$

which contradicts the blow-up estimate (3.21) obtained before. □

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