On L_1-embeddability of unions of L_1-embeddable metric spaces and of twisted unions of hypercubes

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Abstract

We study properties of twisted unions of metric spaces introduced in [Johnson, Lindenstrauss, and Schechtman 1986], and in [Naor and Rabani 2017]. In particular, we prove that under certain natural mild assumptions twisted unions of L_1-embeddable metric spaces also embed in L_1 with distortions bounded above by constants that do not depend on the metric spaces themselves, or on their size, but only on certain general parameters. This answers a question stated in [Naor 2015] and in [Naor and Rabani 2017].

In the second part of the paper we give new simple examples of metric spaces such that their every embedding into L_p, $1 \le p < \infty$, has distortion at least 3, but which are a union of two subsets, each isometrically embeddable in L_p. This extends the result of [K. Makarychev and Y. Makarychev 2016] from Hilbert spaces to L_p-spaces, $1 \le p < \infty$.

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1 Introduction

One of natural general questions about metric spaces is the following:

Question 1.1. *Let a metric space (X, d) be a union of its metric subspaces A and B, disjoint or not. Assume that A and B have a certain metric property $\mathcal{P}$. Does this imply that X also has property $\mathcal{P}$, possibly in some weakened form?*

This question can be viewed as a part of a general theme of "local-global" properties, when one wants to analyze whether spaces (or other mathematical objects) that have certain properties "locally", i.e. on certain subspaces/subsets, also have related properties "globally", i.e. on the whole space. The study of the "local-global" theme is prevalent in many (if not all) areas of mathematics, including functional analysis, and of theoretical computer science. Questions in the "local-global" theme usually assume that *all* subspaces/subsets of a specified size satisfy the investigated property. Question 1.1 is different since it assumes that property $\mathcal{P}$ is satisfied by only one pair of subsets covering X.

We are particularly interested in the embeddability properties of metric spaces. We are aware of three embeddability properties for which the answers to Question 1.1 are positive, interesting, and useful. We state them below after recalling the necessary definitions.

Definition 1.2. Let (X, d_X) and (Y, d_Y) be metric spaces. An injective map $F : X \to Y$ is called a *bilipschitz embedding* if there exist constants $C_1, C_2 > 0$ so that for all $u, v \in X$

$$C_1 d_X(u, v) \le d_Y(F(u), F(v)) \le C_2 d_X(u, v).$$

The *distortion* of F is defined as $\mathrm{Lip}(F)\cdot\mathrm{Lip}(F^{-1}|_{F(X)})$, where $\mathrm{Lip}(\cdot)$ denotes the Lipschitz constant.

For $p \in [1,\infty]$, the *L_p-distortion* $c_p(X,d_X)$, or $c_p(X)$, if the metric d_X is clear, is defined as the infimum of distortions of all bilipschitz embeddings of (X,d_X) into any space $L_p(\Omega,\Sigma,\mu)$.

Definition 1.3. A map $f : (X,d_X) \to (Y,d_Y)$ between two metric spaces is called a *coarse embedding* if there exist non-decreasing functions $\rho_1,\rho_2 : [0,\infty) \to [0,\infty)$ such that $\lim_{t\to\infty}\rho_1(t) = \infty$ and

$$\forall u,v \in X \;\; \rho_1(d_X(u,v)) \le d_Y(f(u),f(v)) \le \rho_2(d_X(u,v)).$$

(Observe that this condition implies that ρ_2 has finite values, but that it does not imply that f is injective.)

Definition 1.4. A metric space (Y,d_Y) is called *ultrametric* if for any $u,v,w \in Y$

$$d_Y(u,w) \le \max\{d_Y(u,v),d_Y(v,w)\}.$$

Theorem 1.5 (Dadarlat, Guentner [7, Corollary 4.5])**.** *If a metric space X is a finite union of subsets each admitting a coarse embedding into a Hilbert space, then X also admits a coarse embedding into a Hilbert space.*

Theorem 1.6 (Mendel, Naor [21, Theorem 1.4])**.** *Let a metric space (X,d) be a union of its metric subspaces A and B. Assume that A and B embed into, possibly different, ultrametric spaces with distortions D_A and D_B, respectively. Then the metric space X embeds into an ultrametric space with distortion at most $(D_A+2)(D_B+2)-2$.*

The following theorem was proved by K. Makarychev and Y. Makarychev [19]. Fifteen years earlier, Lang and Plaut [17, Theorem 3.2] proved a weaker version of this theorem in which they estimated D in terms of D_A, D_B, a, and b (for finite a and b).

Theorem 1.7. *Suppose that a metric space (X,d) is the union of two metric subspaces A and B that embed into ℓ_2^a and ℓ_2^b (where a and b may be finite or infinite) with distortions D_A and D_B, respectively. Then X embeds into ℓ_2^{a+b+1} with distortion $D \le 7D_AD_B + 2(D_A+D_B)$.*
If $D_A = D_B = 1$, then X embeds into ℓ_2^{a+b+1} with distortion at most 8.93.

Remark 1.8. We note that there is an extensive literature on the property of L_1-embeddability within the "local-global" theme. For example, Arora, Lovász, Newman, Rabani, Rabinovich and Vempala [1] asked what is the least distortion with which one can embed the metric space X into L_1, given that every subset of X of cardinality k is embeddable into L_1 with distortion at most D. An answer to this question was given by Charikar, K. Makarychev, and Y. Makarychev [5], who proved, among other results, that if even a small fraction α (say 1%) of all subsets of size k of a metric space X, with $|X| = n$, embeds into ℓ_p with distortion at most D, then the entire space X embeds into ℓ_p, $1 \le p < \infty$, with distortion at most $D\cdot O(\log(n/k)+\log\log(1/\alpha)+\log p)$. In particular, if k is proportional to n, then one obtains a bounded distortion embedding of X into ℓ_p.

On the other hand, there exists an absolute constant $C > 0$ such that for every $n \in \mathbb{N}$ and $k < n$ there exists an n-point metric space X such that every subset of X of size at most k embeds isometrically into L_1, but every embedding of X into L_1 requires distortion at least $C(\log n/(\log k + \log\log n))$, [5, Theorem 3.13], see also the expositions in [14, Section 1.3] and [28, Section 4.4].

In connection with Theorem 1.7 it became natural to investigate the problem whether analogous results are valid for metric spaces embeddable into L_p, when $p \neq 2$, explicitly stated e.g. in [21, Remark 4.2], [19, Question 5], [24, Open Problem 9.6], and [25, Remark 18].

Problem 1.9. *Suppose (X, d) is a metric space and $X = A \cup B$, with $c_p(A)$ and $c_p(B)$ finite. Does this imply that $c_p(X)$ is finite? Can $c_p(X)$ be bounded from above only in terms of $c_p(A)$ and $c_p(B)$?*

It is easy to see that the answer is positive for $p = \infty$. Theorem 1.7 states that the answer is positive for $p = 2$. K. Makarychev and Y. Makarychev [19, Question 5] conjectured that the answer is negative for every $p \in [1, \infty]$ except 2 and ∞.

Problem 1.9 is particularly interesting in the case of $p = 1$. In this case, in addition to the Makarychev-Makarychev conjecture of the negative answer to Problem 1.9, since 2015 in the literature there were conjectures that a construction known as a twisted union of hypercubes might be a possible method of constructing a family of counterexamples.

Problem 1.10 (Naor [24, Open problem 3.3], Naor, Rabani [25, Remark 18])**.** *Must any embedding of a twisted union of hypercubes described in Examples 3.1 and 3.2 below into L_1 incur a bilipschitz distortion that tends to ∞ as the size of the hypercube tends to ∞? That is, does the twisted union of hypercubes give a negative answer to Problem 1.9 in the case $p = 1$?*

The idea of the construction of a twisted union of metric spaces can be traced back to [18] and has been used in [12] and [25] to provide examples that demonstrate that for $\alpha \in (1/2, 1]$, the α-extension constants from ℓ_∞ to ℓ_2 are not bounded. Variants of this construction were also used in [16, 4].

The general idea is explained in [25, Remark 19], and is as follows:

Definition 1.11. Suppose that (X, ϱ_X) and (Y, ϱ_Y) are metric spaces with X and Y disjoint as sets. Given mappings $\sigma : X \to Y$ and $r : X \to (0, \infty)$ (where σ can be any map, not necessarily injective nor surjective), we define the weighted graph structure on $X \cup Y$ by defining the following weighted edges: If $x_1, x_2 \in X$ then x_1 and x_2 are joined by an edge of weight $\varrho_X(x_1, x_2)$; if $y_1, y_2 \in Y$ then y_1 and y_2 are joined by an edge of weight $\varrho_Y(y_1, y_2)$. Also, for every $x \in X$, the elements x and $\sigma(x)$ are joined by an edge of weight $r(x)$. The space $X \cup Y$ endowed with the shortest-path metric induced by this weighted graph is called the *unrestricted twisted union of (X, ϱ_X) and (Y, ϱ_Y) with the joining mappings $\sigma : X \to Y$ and $r : X \to (0, \infty)$.*

Naor and Rabani point out that all metric spaces that they construct in [25] (specific twisted unions of hypercubes in [25, Section 4] and the magnification of a metric space in [25, Section 3.1]) to exhibit a maximal unbounded growth of certain extension constants can be described as subsets of this general construction, and they indicate that usefulness of this construction is probably yet to be fully explored.

One of the main goals of the present paper is to explore unrestricted twisted unions of metric spaces in several special cases. The cases we consider contain the examples used in Problem 1.10.

As in [24, 25], we restrict our attention to the case when $\sigma : X \to Y$ is a bijection between the sets X and Y. In this case, it is convenient to describe the set $X \cup Y$ as a Cartesian product $M \times \{0, 1\}$, where M is a set with two metrics, ϱ_0 and ϱ_1, and

$\sigma((x,0)) = (x,1)$. We define a metric ϱ_X on $X = M \times \{0\}$ by $\varrho_X((x,0),(y,0)) = \varrho_0(x,y)$, a metric ϱ_Y on $Y = M \times \{1\}$ by $\varrho_Y((x,1),(y,1)) = \varrho_1(x,y)$, and denote by d the metric obtained after applying the construction of Definition 1.11.

The first question that naturally arises is to find exact, or equivalent, formulas for the metric d restricted to each set $M \times \{i\}$, for $i = 0,1$ in terms of ϱ_0, ϱ_1, and the function $r(\cdot)$. We discuss this question in Section 3, where we show that if

$$d_i(x,y) \stackrel{\text{def}}{=} \min\{\varrho_i(x,y), \varrho_{1-i}(x,y) + r(x) + r(y)\}, \text{ for } i = 0,1, \text{ are metrics on } M \tag{1.1}$$

then

$$\forall x,y \in M, \quad d((x,a),(y,b)) = \begin{cases} d_0(x,y) & \text{if } a = b = 0, \\ d_1(x,y) & \text{if } a = b = 1. \end{cases}$$

Note that the definition of d_0, d_1 immediately implies

$$\forall x,y \in M, \quad |d_0(x,y) - d_1(x,y)| \leq r(x) + r(y). \tag{1.2}$$

If $r(x) = r > 0$ is constant, (1.2) becomes

$$\forall x,y \in M, \quad |d_0(x,y) - d_1(x,y)| \leq 2r. \tag{1.3}$$

Further, it is easy to see that

$$\forall x \in M, \quad d((x,0),(x,1)) = r(x)$$

if and only if

$$\forall x,y \in M, \quad |r(x) - r(y)| \leq d_0(x,y) + d_1(x,y), \tag{1.4}$$

which is satisfied trivially if r is the constant function on M.

Since our main interest is to use twisted unions to study Problem 1.9, we will restrict our attention to metrics that satisfy (1.1) (hence also (1.2)) and (1.4). This leads us to the following definition.

Definition 1.12 (Generalized Twisted Union and r-Twisted Union)**.** Let M be a set with two metrics, d_0 and d_1, and let $r : M \to (0,\infty)$ be a function such that (1.2) and (1.4) are satisfied:

Define the metric d on $M \times \mathbb{F}_2 = M \times \{0,1\}$ as the shortest path metric when $M \times \mathbb{F}_2$ is considered as a graph with the following edges and weights:

- for every $x, y \in M$ there is an edge with ends $(x,0)$ and $(y,0)$ of weight $d_0(x,y)$,
- for every $x, y \in M$ there is an edge with ends $(x,1)$ and $(y,1)$ of weight $d_1(x,y)$,
- for every $x \in M$ there is an edge with ends $(x,0)$ and $(x,1)$ of weight $r(x)$.

The space $(M \times \mathbb{F}_2, d)$ is called the *generalized twisted union of* (M, d_0) *and* (M, d_1) *with the joining function* $r : M \to (0,\infty)$.

If the function $r : M \to (0,\infty)$ is constant ($= r > 0$), the space $(M \times \mathbb{F}_2, d)$ is called the *twisted union of metric spaces* (M, d_0) *and* (M, d_1) *with the joining parameter* r, or the *r-twisted union*, for short.

In Section 3, under certain additional assumptions on d_0, d_1 and $r(\cdot)$, we prove that there exist constants $A, B > 0$ such that for all $x, y \in M$,

$$A(h(x,y) + r(x)) \leq d((x,0),(y,1)) \leq B(h(x,y) + r(x)),$$

where $h = \min\{d_0, d_1\}$, see Propositions 3.4, 3.5 and 3.7.

Our initial goal was to solve Problem 1.10. Somewhat to our surprise, we answered it negatively, see Corollary 5.2. After that, our next goal became to find situations in which the L_1-distortion of the twisted union can be estimated in terms of $c_1(M, d_0)$ and $c_1(M, d_1)$. We were not able to do this in full generality, without any additional restrictions on the twisted union (it is widely believed that the answer to Problem 1.9 is negative), however, we proved that for certain fairly large classes of metric spaces the answer to Problem 1.9 is affirmative, see Theorems 4.1, 4.3, 5.1, and Corollaries 5.2, 5.3. To avoid too many technical details in this Introduction, we state here only some of our results.

Theorem A (Corollary 5.2). *Let $n \in \mathbb{N}$, $\mathbb{F}_2^n$ be the n-dimensional Hamming cube, $r > 0$ be a constant, and ω_0 and ω_1 be concave non-decreasing continuous functions on $[0, \infty)$ vanishing at 0 and such that for all $t > 0$, $\omega_0(t) > 0$ and $\omega_1(t) > 0$. Let $X = \mathbb{F}_2^n \times \{0\}$, $\rho_X((x, 0), (y, 0)) = \omega_0(\|x-y\|_1)$ and $Y = \mathbb{F}_2^n \times \{1\}$, $\rho_Y((x, 1), (y, 1)) = \omega_1(\|x-y\|_1)$. Then the unrestricted twisted union of X and Y with the joining mappings $\sigma((x, 0)) = (x, 1)$ and $r(x) = r > 0$ embeds into L_1 with distortion bounded by an absolute constant $D < 26.6$.*

As we show in Section 5, Theorem A answers Problem 1.10.

The minimum of two metrics is not necessarily a metric. The inequality

$$c_1(M, \min\{d_0, d_1\}) \leq K$$

in the next statement uses the more general definition of c_1 for nonnegative functions of two variables given at the beginning of Section 2, see (2.1).

Theorem B (Theorem 5.1). *Let $r > 0$, M be a metric space with two metrics d_0 and d_1, such that (1.3) is satisfied, and for $i = 0, 1$, $c_1(M, d_i) \leq D_i$, and $c_1(M, \min\{d_0, d_1\}) \leq K$.*

Then the r-twisted union $(M \times \mathbb{F}_2, d)$ of (M, d_0) and (M, d_1) embeds into L_1 with distortion bounded by a constant D that depends only on D_0, D_1, and K.

Theorem C (Corollary 5.3). *Let $r > 0$, M be a metric space with two metrics d_0 and d_1, such that (1.3) is satisfied, and for $i = 0, 1$, $c_1(M, d_i) \leq D_i$. Suppose also that there exist a constant $C > 0$ such that for all $x, y \in M$*

$$d_1(x, y) \leq C d_0(x, y).$$

Then the r-twisted union $(M \times \mathbb{F}_2, d)$ of (M, d_0) and (M, d_1) embeds into L_1 with distortion bounded by a constant D that depends only on D_0, D_1 and C.

In Section 4 we prove our two main results on L_1-embeddability of generalized twisted unions (Theorems 4.1, 4.3). Their applications to L_1-embeddability of r-twisted unions, including examples considered in [24, 25], are presented in Section 5 (Theorem 5.1 and Corollaries 5.2 and 5.3).

In Section 6, we show that the lower bound on distortion of the embedding of a union of metric spaces found by K. Makarychev and Y. Makarychev [19, Theorem 1.2 and Section 3] for the Hilbert space, is also valid for all L_p with $1 \leq p < \infty$, and for many other Banach and metric spaces. Our proof uses the theory of stable metric spaces (see Definition 6.1) and our examples are infinite metric spaces. For spaces whose stability is known our proof is very simple (see Example 6.5).

2 Preliminary facts and notation

We use the standard terminology of the theories of Banach Spaces and Metric Embeddings, see [2, 26].

Suppose that $K \geq 1$, M is a set, and $f : M \times M \to [0, \infty)$ is an arbitrary function that is not necessarily a metric on M, such that there exists a map $\Psi : M \to L_1$ such that for all $x, y \in M$, we have

$$f(x, y) \leq \|\Psi(x) - \Psi(y)\|_1 \leq K f(x, y). \tag{2.1}$$

In this situation, even if the function f is not a metric (we even allow the function f to equal 0 on an arbitrary subset of $M \times M$), with a slight abuse of notation, we will say that (M, f) embeds in L_1 and write $c_1(M, f) \leq K < \infty$.

We need the following two results of Mendel and Naor [22].

Theorem 2.1. ([22, Lemma 5.4]) *For every constant $\lambda > 0$, L_1 with the truncated metric*

$$\varrho(x, y) = \min\{\lambda, \|x - y\|_1\}$$

embeds into L_1 with the standard norm with distortion not exceeding $e/(e-1)$.

This is a powerful result which, using the theory of concave functions of Brudnyĭ and Krugljak (see [3, Section 3.2] and [20, Remark 5.4]), gives an important general class of examples of metric spaces that embed into L_1, namely:

Corollary 2.2. ([22, Remark 5.5]) *There exists a universal constant $\Delta < \infty$, such that if $\omega : [0, \infty) \to [0, \infty)$ is any concave non-decreasing function with $\omega(0) = 0$ and $\omega(t) > 0$ for $t > 0$, then the metric space $(L_1, \omega(\|x - y\|_1))$ embeds into L_1 with distortion at most Δ. (No estimate for Δ is given in [22], but, using methods indicated there, it is easy to compute that $\Delta \leq (2\sqrt{2} + 3)e/(e-1) < 10$.)*

We will also use the following immediate corollary of Theorem 2.1.

Corollary 2.3. *Let M be a set, $K \geq 1$, and $f : M \times M \to [0, \infty)$ be a function such that $c_1(M, f) \leq K$. Then, for any constant $\lambda > 0$, $c_1(M, \min\{f(x, y), \lambda\}) \leq eK/(e-1)$.*

Proof. By Theorem 2.1, there exists a map $T : L_1 \to L_1$ such that for all $u, v \in L_1$

$$\min\{\|u - v\|_1, \lambda\} \leq \|Tu - Tv\|_1 \leq \frac{e}{e-1} \min\{\|u - v\|_1, \lambda\}.$$

Let $\Psi : M \to L_1$ be a map satisfying (2.1). Then, for all $x, y \in M$,

$$\begin{aligned} \|T\Psi(x) - T\Psi(y)\|_1 &\leq \frac{e}{e-1} \min\{\|\Psi(x) - \Psi(y)\|_1, \lambda\} \leq \frac{e}{e-1} \min\{K f(x, y), \lambda\} \\ &\leq \frac{eK}{e-1} \min\{f(x, y), \lambda\} \end{aligned}$$

and

$$\|T\Psi(x) - T\Psi(y)\|_1 \geq \min\{\|\Psi(x) - \Psi(y)\|_1, \lambda\} \geq \min\{f(x, y), \lambda\},$$

which ends the proof. □

If G and H are real valued quantities or functions, we use the notation $G \asymp H$ to mean that there exist $0 < \alpha \leq \beta < \infty$ such that $\alpha G \leq H \leq \beta G$. The numbers α and β can depend on the parameters mentioned in the statements of the results we are proving, but not on elements x, y of the considered metric space.

3 Twisted unions of hypercubes and of general metric spaces

We start by presenting the concrete examples of twisted unions of hypercubes from [12, p. 137], [24, p. 8], and [25, p. 144].

Example 3.1 ([12, 24, 25]). Let $n \in \mathbb{N}$ and $\mathbb{F}_2^n = \{0,1\}^n$ be the n-dimensional Hamming cube embedded into ℓ_1^n in a natural way. Let $r \in (0,\infty)$ and $\alpha \in (1/2, 1]$ be fixed constants.

Define the metric on $\mathbb{F}_2^n \times \mathbb{F}_2 = \{0,1\}^n \times \{0,1\}$ as the shortest path metric when $\mathbb{F}_2^n \times \mathbb{F}_2$ is considered as a graph with the following edges and weights:

- for every $x, y \in \mathbb{F}_2^n$ there is an edge with ends $(x,0)$ and $(y,0)$ of weight $\|x-y\|_1^{\frac{1}{2\alpha}}$,
- for every $x, y \in \mathbb{F}_2^n$ there is an edge with ends $(x,1)$ and $(y,1)$ of weight $\frac{\|x-y\|_1}{r^{2\alpha-1}}$,
- for every $x \in \mathbb{F}_2^n$ there is an edge with ends $(x,0)$ and $(x,1)$ of weight r.

The exact formula for the distance between any two points of $\mathbb{F}_2^n \times \mathbb{F}_2$ in this metric is computed in [25, Lemma 15], cf. also (3.1) and Remark 3.6 below.

$$d((x,a),(y,b)) = \begin{cases} \|x-y\|_1^{\frac{1}{2\alpha}}, & \text{if } \ a=b=0,\\ \min\{\frac{\|x-y\|_1}{r^{2\alpha-1}}, 2r + \|x-y\|_1^{\frac{1}{2\alpha}}\}, & \text{if } \ a=b=1,\\ r & \text{if } \ x=y, a\neq b,\\ r+\min\{\frac{\|x-y\|_1}{r^{2\alpha-1}}, \|x-y\|_1^{\frac{1}{2\alpha}}\} & \text{if } \ a\neq b. \end{cases}$$

Note that, by Corollary 2.2, $c_1(\mathbb{F}_2^n \times \{i\}, d) \le \Delta < 10$, for $i = 0,1$.

It is easy to see that Example 3.1 is a special case of the following more general example.

Example 3.2 ([25, Remark 18]). Let $n \in \mathbb{N}$, $\mathbb{F}_2^n$ be the n-dimensional Hamming cube, $r \in (0,\infty)$ be a constant, and ω_0 and ω_1 be concave non-decreasing continuous functions on $[0,\infty)$ vanishing at 0 and such that for all $t > 0$, $\omega_0(t) > 0$ and $\omega_1(t) > 0$. Define the metric d on $\mathbb{F}_2^n \times \mathbb{F}_2$ as the shortest path metric when $\mathbb{F}_2^n \times \mathbb{F}_2$ is considered as a graph with the following edges and weights:

- for every $x, y \in \mathbb{F}_2^n$ there is an edge with ends $(x,0)$ and $(y,0)$ of weight $\omega_0(\|x-y\|_1)$,
- for every $x, y \in \mathbb{F}_2^n$ there is an edge with ends $(x,1)$ and $(y,1)$ of weight $\omega_1(\|x-y\|_1)$,
- for every $x \in \mathbb{F}_2^n$ there is an edge with ends $(x,0)$ and $(x,1)$ of weight r.

For easy reference, we denote the metric space $(\mathbb{F}_2^n \times \mathbb{F}_2, d)$ by $TU(\mathbb{F}_2^n, \omega_0, \omega_1, r)$.

Since the functions ω_0 and ω_1 are concave, it is easy to see that the above weights imply that the shortest path metric on $\mathbb{F}_2^n \times \mathbb{F}_2$ satisfies

$$d((x,a),(y,b)) = \begin{cases} \min\{\omega_0(\|x-y\|_1), 2r+\omega_1(\|x-y\|_1)\}, & \text{if } \ a=b=0,\\ \min\{\omega_1(\|x-y\|_1), 2r+\omega_0(\|x-y\|_1)\}, & \text{if } \ a=b=1,\\ r & \text{if } \ x=y, a\neq b. \end{cases} \tag{3.1}$$

Indeed, first, by concavity of ω_0 and ω_1, we conclude that, for $i = 0,1$, the functions

$$f_i(x,y) \stackrel{\text{def}}{=} \min\{\omega_i(\|x-y\|_1), 2r + \omega_{1-i}(\|x-y\|_1)\}$$

are metrics on $\mathbb{F}_2^n$.

Next, by considering the paths: $(x,0),(y,0)$ and $(x,0),(x,1),(y,1),(y,0)$ we deduce that $d((x,0),(y,0)) \le f_0(x,y)$. Further, suppose that the shortest path connecting $(x,0)$ and $(y,0)$ is $(z_0,0),(z_1,0),(z_1,1),(z_2,1),(z_2,0),\dots,(z_{2k},1),(z_{2k},0),(z_{2k+1},0)$, where $k \in \mathbb{N}\cup\{0\}$, $z_0 = x, z_{2k+1} = y$, and the points $\{z_j\}_{j=0}^{2k+1}$ may allow repetitions. By the optimality of this path we have, for all $j \le k$,

$$\begin{aligned} \omega_0(\|z_{2j}-z_{2j+1}\|_1) &\le 2r + \omega_1(\|z_{2j}-z_{2j+1}\|_1),\\ 2r+\omega_1(\|z_{2j-1}-z_{2j}\|_1) &\le \omega_0(\|z_{2j-1}-z_{2j}\|_1). \end{aligned}$$

Thus the length of this path is equal to $\sum_{j=0}^{2k} f_0(z_j,z_{j+1}) \ge f_0(x,y)$, where the last inequality holds since f_0 is a metric. A similar argument works also for $d((x,1),(y,1))$.

Notice that if we define, for $i=0,1$,

$$\varpi_i(t) = \min\{\omega_i(t), 2r+\omega_{1-i}(t)\},$$

then the functions ϖ_0,ϖ_1 are concave, non-decreasing, continuous, vanishing only at $t=0$, and for all t

$$|\varpi_0(t)-\varpi_1(t)| \le 2r. \tag{3.2}$$

It is clear that if we use the functions $\varpi_0(\|x-y\|_1),\varpi_1(\|x-y\|_1)$ as weights in place of $\omega_0(\|x-y\|_1),\omega_1(\|x-y\|_1)$, respectively, this results in the same metric d on $\mathbb{F}_2^n\times\mathbb{F}_2$, that is $TU(\mathbb{F}_2^n,\omega_0,\omega_1,r) = TU(\mathbb{F}_2^n,\varpi_0,\varpi_1,r)$. Moreover, we have:

$$d((x,a),(y,b)) = \begin{cases} \varpi_0(\|x-y\|_1) & \text{if } \ a=b=0,\\ \varpi_1(\|x-y\|_1) & \text{if } \ a=b=1,\\ r & \text{if } \ x=y, a\ne b,\\ r+\min\{\varpi_0(\|x-y\|_1),\varpi_1(\|x-y\|_1)\} & \text{if } \ a\ne b, \end{cases} \tag{3.3}$$

where the last case follows by Proposition 3.5(b) below.

By Corollary 2.2, $c_1(\mathbb{F}_2^n\times\{i\},d)\le\Delta<10$, for $i=0,1$.

Remark 3.3. Spaces described in Examples 3.1 and 3.2 are unrestricted twisted unions (see Definition 1.11) of spaces $X=(M\times\{0\},\varrho_0)$ and $Y=(M\times\{1\},\varrho_1)$, with $\sigma((m,0)) = (m,1)$ for all $m\in M$, and metrics ϱ_0 and ϱ_1 are inherited from M, which is a set with two metrics ϱ_0 and ϱ_1.

Note, however, that for an arbitrary unrestricted twisted union, unlike the situation in Example 3.2, the functions defined by

$$d_i(x,y) \stackrel{\text{def}}{=} \min\{\varrho_i(x,y), \varrho_{1-i}(x,y)+r(x)+r(y)\}, \quad (i=0,1), \tag{3.4}$$

do not have to be metrics on M.

It is easy to see, following an argument similar to the justification of (3.1), that if the functions d_0 and d_1 are metrics on M, then the twisted union metric d satisfies:

$$d((x,a),(y,b)) = \begin{cases} d_0(x,y) & \text{if } \ a=b=0,\\ d_1(x,y) & \text{if } \ a=b=1. \end{cases} \tag{3.5}$$

Note that the definition of d_0, d_1 immediately implies

$$\forall x,y\in M, \quad |d_0(x,y)-d_1(x,y)| \le r(x)+r(y). \tag{3.6}$$

It is clear that

$$\forall x \in M, \quad d((x,0),(x,1)) = r(x) \tag{3.7}$$

if and only if

$$\forall x, y \in M, \quad |r(x) - r(y)| \le d_0(x,y) + d_1(x,y), \tag{3.8}$$

which by a straightforward case analysis is equivalent to

$$\forall x, y \in M, \quad |r(x) - r(y)| \le \varrho_0(x,y) + \varrho_1(x,y). \tag{3.9}$$

Clearly, if the function $r(\cdot)$ is constant ($= r > 0$) then (3.8) and (3.9) are satisfied trivially, and (3.6) becomes

$$\forall x, y \in M, \quad |d_0(x,y) - d_1(x,y)| \le 2r. \tag{3.10}$$

Since our main interest is to use twisted unions to study Problem 1.9, in the remainder of this paper we will restrict our considerations to twisted unions of metric spaces (M, ϱ_0) and (M, ϱ_1) with the joining function $r : M \to (0, \infty)$ such that (3.9) holds and the functions d_0, d_1 defined in (3.4) are both metrics on M. In this situation (3.6), (3.8), (3.7), and (3.5), all hold, and the twisted union of (M, ϱ_0) and (M, ϱ_1) with the joining function $r : M \to (0, \infty)$ coincides with the twisted union of (M, d_0) and (M, d_1) with the same joining function $r : M \to (0, \infty)$. Thus, by replacing, if necessary, the original metrics ϱ_0, ϱ_1 by metrics d_0, d_1, respectively, we study r-twisted unions and generalized twisted unions, as defined in Definition 1.12.

The first main goal of this paper is to compute expressions for the twisted union metric d between arbitrary points of $M \times \mathbb{F}_2$.

It follows from (3.5) and (3.7) that

$$d((x,a),(y,b)) = \begin{cases} d_0(x,y) & \text{if } \ a = b = 0, \\ d_1(x,y) & \text{if } \ a = b = 1, \\ r(x) & \text{if } \ x = y, a \ne b. \end{cases} \tag{3.11}$$

Computation of the formula for $d((x,0),(y,1))$ when $x \ne y$ is more delicate and is open in general. We prove, under two different natural assumptions that are independent of each other, that for all $x, y \in M$,

$$d((x,0),(y,1)) \asymp h(x,y) + r(x), \tag{3.12}$$

where

$$h(x,y) \stackrel{\text{def}}{=} \min\{d_0(x,y), d_1(x,y)\} \quad \forall x, y \in M.$$

Note that, in general, the minimum of two metrics does not need to be a metric, see also Remark 4.2.

In Proposition 3.4 we prove that (3.12) holds if the joining function $r(\cdot)$ is Lipschitz with respect to both metrics d_0 and d_1, and thus with respect to h, not just with respect to the sum of these metrics as required in (3.8).

In particular, (3.12) is satisfied when the joining function $r(\cdot)$ is constant (see Proposition 3.5).

Additionally, in Proposition 3.7 we prove that (3.12) holds if there exists a constant $C > 0$ such that for all $x, y \in M$,

$$d_1(x,y) \le C d_0(x,y).$$

This is true, for instance, in Example 3.1, since there, independent of the values of $r>0$ or $\alpha\in(1/2,1]$, for all $x,y\in M$, we have

$$\min\left\{\frac{\|x-y\|_1}{r^{2\alpha-1}}, 2r+\|x-y\|_1^{\frac{1}{2\alpha}}\right\}\le 2\|x-y\|_1^{\frac{1}{2\alpha}}. \tag{3.13}$$

The common setup for Propositions 3.4, 3.5 and 3.7 is:

Let M be a metric space with two metrics d_0 and d_1, $h=\min\{d_0,d_1\}$, $r:M\to(0,\infty)$ be a function such that (3.6) and (3.8) are satisfied, and $(M\times\mathbb{F}_2,d)$ be the generalized twisted union of (M,d_0) and (M,d_1) with the joining function $r(\cdot)$ (and thus (3.11) is valid for $(M\times\mathbb{F}_2,d)$). (3.14)

Proposition 3.4. *Suppose* (3.14) *and that the joining function* $r(\cdot)$ *is Lipschitz with respect to both metrics* d_0 *and* d_1*, that is, suppose that there exists a constant* $L\ge 0$*, such that for all* $x,y\in M$

$$|r(x)-r(y)|\le Lh(x,y). \tag{3.15}$$

Then for all $x,y\in M$

$$\frac{1}{A}\Big(h(x,y)+\max\{r(x),r(y)\}\Big)\le d((x,0),(y,1))\le h(x,y)+\max\{r(x),r(y)\}, \tag{3.16}$$

where $A=\max\{2L+1,3\}$*.*

(Note that (3.15) *implies that* $h(x,y)+\max\{r(x),r(y)\}\asymp h(x,y)+r(x)$*.)*

Proposition 3.5. *Suppose* (3.14) *and that the joining function* $r(\cdot)$ *is constant,* $r(x)=r>0$*, for all* $x\in M$*.*

(a) *Then for all* $x,y\in M$

$$\frac{1}{3}\Big(h(x,y)+r\Big)\le d((x,0),(y,1))\le h(x,y)+r. \tag{3.17}$$

(b) *If, in addition,* $h(x,y)$ *is a metric on* M*, then for all* $x,y\in M$

$$d((x,0),(y,1))=h(x,y)+r. \tag{3.18}$$

Remark 3.6. We included (3.18) above, because we are particularly interested in spaces described in Example 3.2, when M is a subset of L_1 and $d_i(x,y)=\varpi_i(\|x-y\|_1)$ for $i=0,1$, where functions ϖ_0,ϖ_1 are concave, non-decreasing, continuous, and vanishing only at $t=0$, see Corollary 5.2. In this situation $h=\min\{d_0,d_1\}$ is a metric on M.

We note that if h is a metric on M, then one can obtain slightly better constants also in (3.16) above.

Proof of Propositions 3.4 and 3.5. We fix $x,y\in M$. The condition (3.6) implies that on a shortest path from $(x,0)$ to $(y,1)$ we may avoid moving from $M\times\{0\}$ to $M\times\{1\}$ more than once. Thus

$$d((x,0),(y,1))=\inf_{z\in M}\left(d_0(x,z)+d_1(z,y)+r(z)\right).$$

If $h(x,y)=d_1(x,y)$, we pick $z=x$, otherwise we pick $z=y$, so that we get

$$d((x,0),(y,1))\le h(x,y)+\max\{r(x),r(y)\}.$$

which proves the upper estimates in (3.16), (3.17), and (3.18).

On the other hand, by (3.6), for all $u, v \in M$ we have $d_0(u,v) \le d_1(u,v) + r(u) + r(v)$. Thus, for every $x, y, z \in M$ we have

$$\begin{aligned} h(x,y) &\le d_0(x,y) \le d_0(x,z) + d_0(z,y) \le d_0(x,z) + d_1(z,y) + r(z) + r(y) \\ &\le d_0(x,z) + d_1(z,y) + r(z) + \max\{r(x), r(y)\}. \end{aligned} \tag{3.19}$$

Hence, for every $x, y, z \in M$ and $T > 1$ we have

$$\begin{aligned} &d_0(x,z) + d_1(z,y) + r(z) - \frac{1}{T}\Big(h(x,y) + \max\{r(x), r(y)\}\Big) \\ &\overset{(3.19)}{\ge} d_0(x,z) + d_1(z,y) + r(z) - \frac{1}{T}\Big(d_0(x,z) + d_1(z,y) + r(z) + 2\max\{r(x), r(y)\}\Big) \\ &\ge \Big(1 - \frac{1}{T}\Big)\Big(d_0(x,z) + d_1(z,y)\Big) + \Big(1 - \frac{3}{T}\Big) r(z) - \frac{2}{T}\Big|r(z) - \max\{r(x), r(y)\}\Big| \\ &\overset{(3.15)}{\ge} \Big(1 - \frac{1}{T}\Big)\Big(h(x,z) + h(z,y)\Big) + \Big(1 - \frac{3}{T}\Big) r(z) - \frac{2L}{T}\max\{h(x,z), h(z,y)\} \\ &\ge \Big(1 - \frac{2L+1}{T}\Big)\max\{h(x,z), h(z,y)\} + \Big(1 - \frac{3}{T}\Big) r(z). \end{aligned}$$

If $T \ge \max\{2L+1, 3\}$, the ultimate quantity is nonnegative, which proves (3.16).

Formula (3.17) immediately follows from (3.16), since when the function $r(\cdot)$ is constant then it is Lipschitz with the Lipschitz constant $L = 0$.

If $h(x,y)$ is a metric on M, then for all $z \in M$ we have

$$d_0(x,z) + d_1(z,y) + r \ge h(x,z) + h(z,y) + r \ge h(x,y) + r,$$

which proves (3.18). □

Proposition 3.7. *Suppose* (3.14) *and that*

$$\forall x, y \in M \quad d_1(x,y) \le C d_0(x,y). \tag{3.20}$$

Then

$$\frac{1}{2C+1}\Big(d_1(x,y) + r(x)\Big) \le d((x,0),(y,1)) \le d_1(x,y) + r(x). \tag{3.21}$$

Moreover, if $r(x) = r > 0$ *for all* $x \in M$*, then* (3.10) *and* (3.20) *imply that*

$$\frac{1}{\max\{C,1\}}\Big(d_1(x,y) + r\Big) \le d((x,0),(y,1)) \le d_1(x,y) + r. \tag{3.22}$$

Proof. The condition (3.6) implies that on a shortest path from $(x,0)$ to $(y,1)$ we may avoid moving from $M \times \{0\}$ to $M \times \{1\}$ more than once. Thus

$$d((x,0),(y,1)) = \inf_{z \in M}\left(d_0(x,z) + d_1(z,y) + r(z)\right).$$

If we pick $z = x$, we get

$$d((x,0),(y,1)) \le d_1(x,y) + r(x),$$

which proves the upper estimates in (3.21) and (3.22).

On the other hand, for every $z \in M$ and $T > 1$ we have

$$
\begin{aligned}
d_0(x,z) + d_1(z,y) + r(z) - \frac{1}{T}\Big(d_1(x,y) + r(x)\Big) \\
&\geq d_0(x,z) + d_1(z,y) - \frac{1}{T}d_1(x,y) - \frac{1}{T}\Big(r(x) - r(z)\Big) \\
&\overset{(3.8)}{\geq} d_0(x,z) + d_1(z,y) - \frac{1}{T}d_1(x,y) - \frac{1}{T}\Big(d_0(x,z) + d_1(x,z)\Big) \\
&\geq \Big(1 - \frac{1}{T}\Big)d_0(x,z) + d_1(z,y) - \frac{1}{T}d_1(x,y) - \frac{1}{T}d_1(x,z) \\
&\geq \Big(1 - \frac{1}{T}\Big)d_0(x,z) - \frac{1}{T}d_1(x,z) - \frac{1}{T}d_1(x,z) \\
&\overset{(3.20)}{\geq} \frac{1}{C}\Big(1 - \frac{1}{T}\Big)d_1(x,z) - \frac{2}{T}d_1(x,z) \\
&\geq \frac{T - 1 - 2C}{CT}d_1(x,z).
\end{aligned}
$$

The ultimate quantity is nonnegative if $T \geq 2C + 1$. This ends the proof of (3.21).

The lower estimate in (3.22) follows from the fact that (3.20) implies

$$
d_0(x,z) + d_1(z,y) + r \geq \frac{1}{C}d_1(x,z) + d_1(z,y) + r \geq \frac{1}{\max\{C,1\}}(d_1(x,y) + r). \qquad \square
$$

4 On L_1-embeddability of generalized twisted unions

In this section we present two general results (Theorems 4.1 and 4.3) on L_1-embeddability of generalized twisted unions which satisfy different natural restrictions, described in Propositions 3.4 and 3.7, for which we obtained an equivalent formula for the twisted union distance d between every pair of points of the union. Applications of Theorems 4.1 and 4.3 to L_1-embeddability of r-twisted unions are presented in Section 5.

Theorem 4.1. *Let M be a metric space endowed with two metrics d_0 and d_1, and let $r : M \to (0,\infty)$ be a function such that $\inf_{x\in M} r(x) > 0$ and (3.6) and (3.8) are satisfied. Suppose that the function $r(\cdot)$ is Lipschitz with respect to both metrics d_0 and d_1, that is, suppose that there exists a constant $L \geq 0$, such that for all $x, y \in M$*

$$
|r(x) - r(y)| \leq Lh(x,y). \tag{4.1}
$$

Denote for $i = 0, 1$,

$$
g_i(x,y) \overset{def}{=} \min\{d_i(x,y), r(x) + r(y)\}.
$$

If there exist constants C_0, C_1, C_2 such that for $i = 0, 1$, $c_1(M, g_i) \leq C_i$ and $c_1(M,h) \leq C_2$, where $h = \min\{d_0, d_1\}$, then the generalized twisted union $(M \times \mathbb{F}_2, d)$ of (M, d_0) and (M, d_1) with the joining function $r(\cdot)$ embeds into L_1 with distortion bounded above by a constant which depends only on C_0, C_1, C_2, and L.

(Note that we do not require that either of the functions g_0, g_1, or h, is a metric, however the following remark applies to each of them.)

Remark 4.2. In general, the minimum of two metrics does not need to be a metric. However, if $f = \min\{\varrho_0, \varrho_1\}$, where ϱ_0, ϱ_1 are metrics on M, and there exists a constant

$K < \infty$ such that $c_1(M, f) \leq K$, then there exists a metric γ on M, and a constant $\beta \in (0, 1]$ such that $2^{1/\beta} = 2K$, and for all $x, y \in M$,

$$\frac{1}{4}(f(x, y))^\beta \leq \gamma(x, y) \leq (f(x, y))^\beta.$$

This follows from a routine adjustment of a result of Kalton, Peck, and Roberts [13, Theorem 1.2], who studied properties of generalizations of F-norms that instead of the usual triangle inequality satisfy the ultimate inequality in

$$f(x, y) \leq K\big(f(x, z) + f(z, y)\big) \leq 2K \max\big\{f(x, z), f(z, y)\big\}. \tag{4.2}$$

Clearly, since $f = \min\{\varrho_0, \varrho_1\}$, then $c_1(M, f) \leq K$ implies (4.2) and f is separating ($f(x, y) = 0$ iff $x = y$) and symmetric ($f(x, y) = f(y, x)$ for all $x, y \in M$).

Such functions were studied already by Fréchet [8, 9], who called any symmetric, separating function $h : M \times M \to [0, \infty)$ satisfying a condition slightly weaker than (4.2) a *voisinage*. Chittenden [6] proved that any space with a voisinage is homeomorphic to a metric space. For modern theory of similar types of spaces see the monograph of Kalton, Peck, and Roberts [13].

Proof. First we observe, that for $i = 0, 1$, and all $x, y \in M$ we have

$$d_i(x, y) \leq g_i(x, y) + h(x, y) \leq 2d_i(x, y). \tag{4.3}$$

Indeed, by the definitions of h and g_i, the rightmost inequality is clear. If for some $x, y \in M$, either $g_i(x, y) = d_i(x, y)$ or $h(x, y) = d_i(x, y)$ then the leftmost inequality also holds. The remaining case is when $g_i(x, y) = r(x) + r(y)$ and $h(x, y) = d_{1-i}(x, y)$. In this case, by (3.6),

$$d_i(x, y) \leq d_{1-i}(x, y) + r(x) + r(y),$$

which proves (4.3).

Let $\psi, \varphi_0, \varphi_1$, be mappings from M into L_1 establishing upper bounds for $c_1(M, h)$, $c_1(M, g_0)$, and $c_1(M, g_1)$, respectively, see (2.1).

Let $m_0 \in M$ be such that

$$r(m_0) \leq 2 \inf_{x \in M} r(x). \tag{4.4}$$

We define an embedding of $(M \times \mathbb{F}_2, d)$ into $L_1 \oplus_1 L_1 \oplus_1 L_1 \oplus_1 \mathbb{R}$ by

$$\begin{aligned} G(x, 0) &= (\varphi_0(x), \varphi_1(m_0), \psi(x), r(x)) \\ G(x, 1) &= (\varphi_0(m_0), \varphi_1(x), \psi(x), 0). \end{aligned}$$

We have

$$\begin{aligned} \|G(x, 0) - G(y, 0)\| &= \|\varphi_0(x) - \varphi_0(y)\|_1 + \|\varphi_1(m_0) - \varphi_1(m_0)\|_1 + \|\psi(x) - \psi(y)\|_1 + |r(x) - r(y)| \\ &\overset{(4.1)}{\asymp} g_0(x, y) + 0 + h(x, y) \\ &\overset{(4.3)}{\asymp} d_0(x, y). \end{aligned}$$

$$\begin{aligned} \|G(x, 1) - G(y, 1)\| &= \|\varphi_0(m_0) - \varphi_0(m_0)\|_1 + \|\varphi_1(x) - \varphi_1(y)\|_1 + \|\psi(x) - \psi(y)\|_1 + |0| \\ &\asymp 0 + g_1(x, y) + h(x, y) + 0 \overset{(4.3)}{\asymp} d_1(x, y). \end{aligned}$$

By Proposition 3.4(a) we have

$$d((x,0),(y,1)) \asymp h(x,y) + \max\{r(x), r(y)\}.$$

We need to compare this with

$$\begin{aligned}
\|G(x,0)-&G(y,1)\| \\
&= \|\varphi_0(x) - \varphi_0(m_0)\|_1 + \|\varphi_1(m_0) - \varphi_1(y)\|_1 + \|\psi(x) - \psi(y)\|_1 \\
&+ |r(x) - 0| \\
&\asymp g_0(x, m_0) + g_1(m_0, y) + h(x,y) + r(x) \\
&\asymp \min\{d_0(x,m_0), r(x) + r(m_0)\} + \min\{d_1(y,m_0), r(y) + r(m_0)\} \\
&\quad + h(x,y) + r(x) \\
&\overset{(4.4)\&(4.1)}{\asymp} h(x,y) + \max\{r(x), r(y)\}.
\end{aligned} \tag{4.5}$$

The conclusion follows. □

Theorem 4.3. *Let M be a metric space endowed with two metrics d_0 and d_1, and let $r : M \to (0,\infty)$ be a function such that $\inf_{x\in M} r(x) > 0$ and (3.6) and (3.8) are satisfied.*

Suppose that there exists a function $f : M\times M \to [0,\infty)$ and constants $C_1, C_2, C_3, C_4 > 0$ such that $c_1(M, d_1) \le C_1$, $c_1(M, \min\{f(x,y), r(x) + r(y)\}) \le C_2$, and for all $x, y \in M$,

$$C_3 d_0(x,y) \le d_1(x,y) + f(x,y) \le C_4 d_0(x,y). \tag{4.6}$$

Then the generalized twisted union $(M \times \mathbb{F}_2, d)$ of (M, d_0) and (M, d_1) with the joining function $r(\cdot)$ embeds into L_1 with distortion bounded above by a constant which depends only on $C_1, C_2, C_3,$ and C_4.

Proof. Similarly as in the proof of (4.3), by a straightforward analysis of two cases, the conditions (4.6) and (3.6) imply that for all $x, y \in M$

$$C_3 d_0(x,y) \le d_1(x,y) + \min\{f(x,y), r(x) + r(y)\} \le C_4 d_0(x,y). \tag{4.7}$$

Let φ_1 and ψ be mappings from M into L_1 establishing upper bounds for $c_1(M, d_1)$ and $c_1(M, \min\{f(x,y), r(x) + r(y)\})$, respectively, see (2.1). Let $m_0 \in M$ be such that

$$r(m_0) \le 2 \inf_{x\in M} r(x). \tag{4.8}$$

We define the embedding of $(M \times \mathbb{F}_2, d)$ into $L_1 \oplus_1 L_1 \oplus_1 \mathbb{R}$ by

$$\begin{aligned}
F(x,0) &= (\varphi_1(x), \psi(x), r(x)) \\
F(x,1) &= (\varphi_1(x), \psi(m_0), 0).
\end{aligned} \tag{4.9}$$

The condition (4.6) implies that for all $x, y \in M$, $d_1(x,y) \le C_4 d_0(x,y)$, that is (3.20) holds. Thus we have

$$\begin{aligned}
\|F(x,0) - F(y,0)\| &= \|\varphi_1(x) - \varphi_1(y)\|_1 + \|\psi(x) - \psi(y)\|_1 + |r(x) - r(y)| \\
&\asymp d_1(x,y) + \min\{f(x,y), r(x) + r(y)\} + |r(x) - r(y)| \\
&\overset{(4.7),(3.8),\&(3.20)}{\asymp} d_0(x,y).
\end{aligned}$$

$$\|F(x,1) - F(y,1)\| = \|\varphi_1(x) - \varphi_1(y)\|_1 + \|\psi(m_0) - \psi(m_0)\|_1 + |0| \asymp d_1(x,y).$$

Further, by Proposition 3.7, we have

$$d((x,0),(y,1)) \asymp d_1(x,y) + r(x).$$

We need to compare this with

$$\begin{aligned}\|F(x,0) - F(y,1)\| &= \|\varphi_1(x) - \varphi_1(y)\|_1 + \|\psi(x) - \psi(m_0)\|_1 + |r(x)| \\ &\asymp d_1(x,y) + \min\{f(x,m_0), r(x) + r(m_0)\} + r(x) \\ &\overset{(4.8)}{\asymp} d_1(x,y) + r(x).\end{aligned}$$

□

5 On L_1-embeddability of r-twisted unions

In this section we apply results from Section 4 to study L_1-embeddability of r-twisted unions, including metric spaces defined in Examples 3.1 and 3.2. We prove Theorems A, B, and C, stated in the Introduction.

We start from an immediate consequence of Theorem 4.1.

Theorem 5.1. *Let $r > 0$, M be a metric space with two metrics d_0 and d_1, such that (3.10) is satisfied, and for $i = 0, 1$, $c_1(M, d_i) \le D_i$, and $c_1(M, h) \le K$, where $h = \min\{d_0, d_1\}$.*

Then the r-twisted union $(M \times \mathbb{F}_2, d)$ of (M, d_0) and (M, d_1) embeds into L_1 with distortion bounded by a constant D that depends only on D_0, D_1, and K.

Proof. We will use the same notation as in the statement of Theorem 4.1.

Since the parameter r is constant, (4.1) is trivially satisfied with $L = 0$, and, for $i = 0, 1$, $g_i(x,y) = \min\{d_i(x,y), 2r\}$. Since $c_1(M, d_i) \le D_i$, by Corollary 2.3, $c_1(M, g_i) \le eD_i/(e-1)$. Thus all assumptions of Theorem 4.1 are satisfied, which ends the proof. □

As an application of Theorem 5.1, we prove the L_1-embeddability of unrestricted twisted unions that are described in Example 3.2 and includes, in particular, the twisted union described in Example 3.1. Thus Corollary 5.2 answers Problem 1.10 in the negative (see also Corollary 5.3 below, for two additional proofs that the twisted union from Example 3.1 is L_1-embeddable).

Corollary 5.2. *Let $n \in \mathbb{N}$, $\mathbb{F}_2^n$ be the n-dimensional Hamming cube, $r > 0$ be a constant, and ω_0 and ω_1 be concave non-decreasing continuous functions on $[0, \infty)$ vanishing at 0 and such that for all $t > 0$, $\omega_0(t) > 0$ and $\omega_1(t) > 0$. Let $X = \mathbb{F}_2^n \times \{0\}$, $\rho_X((x,0),(y,0)) = \omega_0(\|x-y\|_1)$ and $Y = \mathbb{F}_2^n \times \{1\}$, $\rho_Y((x,1),(y,1)) = \omega_1(\|x-y\|_1)$. Then the unrestricted twisted union of X and Y with the joining mappings $\sigma((x,0)) = (x,1)$ and $r(x) = r > 0$ embeds into L_1 with distortion bounded by an absolute constant $D \le 1 + 2.776\Delta < 26.6$.*

Proof. We will use the same notation as in Example 3.2. By the discussion in Example 3.2, we have $TU(\mathbb{F}_2^n, \omega_0, \omega_1, r) = TU(\mathbb{F}_2^n, \varpi_0, \varpi_1, r)$. By (3.2), the metrics $\varpi_i(\|x-y\|_1)$, for $i = 0, 1$, satisfy (3.10). Define for all $t \ge 0$

$$\varpi_2(t) = \min\{\varpi_0(t), \varpi_1(t)\}.$$

Then ϖ_2 is a concave non-decreasing continuous function on $[0, \infty)$ vanishing only at 0. Thus, by Corollary 2.2, for $i = 0, 1, 2$, we have $c_1(\mathbb{F}_2^n, \varpi_i(\|x-y\|_1)) \le \Delta < 10$. Thus all assumptions of Theorem 5.1 are satisfied, and hence $c_1(TU(\mathbb{F}_2^n, \omega_0, \omega_1, r)) < \infty$.

To obtain the estimate in the parenthesis we use the same notation as in Theorem 4.1. Hence, for $i = 0, 1$, $g_i(x, y) = \gamma_i(\|x - y\|_1)$, where $\gamma_i(t) \stackrel{\text{def}}{=} \min\{\varpi_i(t), 2r\}$. Since the functions γ_i are concave, non-decreasing, continuous, and vanishing only at 0, by Corollary 2.2, we have $c_1(\mathbb{F}_2^n, g_i) \leq \Delta \leq (2\sqrt{2} + 3)e/(e - 1) < 10$.

Let $\tilde{\varphi}_i : \mathbb{F}_2^n \to L_1$, for $i = 0, 1$, be scaled bilipschitz embeddings that instead of being non-contractive, satisfy for all $x, y \in M$,

$$0.694 \cdot g_i(x, y) \leq \|\tilde{\varphi}_i(x) - \tilde{\varphi}_i(y)\|_1 \leq 0.694\Delta g_i(x, y).$$

Next we use (3.18) and the embeddings $\tilde{\varphi}_i$, for $i = 0, 1$, to carefully estimate lower and upper bounds in (4.5) from the proof of Theorem 4.1. The computation is easy, but a little tedious. We leave it to an interested reader. □

Our next result is a direct consequence of either Theorem 5.1 or 4.3.

We note that, by (3.13), the space described in Example 3.1 satisfies the assumptions of Corollary 5.3. Thus we obtain two additional proofs of L_1-embeddability of this space, cf. Corollary 5.2 above.

Corollary 5.3. *Let $r > 0$, M be a metric space with two metrics d_0 and d_1, such that* (3.10) *is satisfied, and $c_1(M, d_i) \leq D_i$, for $i = 0, 1$. Suppose also that there exist a constant $C > 0$ such that for all $x, y \in M$*

$$d_1(x, y) \leq C d_0(x, y). \tag{5.1}$$

Then the r-twisted union $(M \times \mathbb{F}_2, d)$ of (M, d_0) and (M, d_1) embeds into L_1 with distortion bounded by a constant D that depends only on D_0, D_1 and C.

Proof. We present two short proofs, each based on Theorems 5.1, or 4.3, respectively.

Note that, if $h = \min\{d_0, d_1\}$, then (5.1) implies that

$$\min\big\{1, 1/C\big\} d_1(x, y) \leq h(x, y) \leq d_1(x, y).$$

Therefore, since $c_1(M, d_1) \leq D_1$, we have $c_1(M, h) < \infty$, and thus Corollary 5.3 follows from Theorem 5.1.

Moreover, if, in the notation of Theorem 4.3, we define the function f to be equal to d_0, then, by (5.1), the inequality (4.6) is satisfied with $C_3 = 1$ and $C_4 = C + 1$. Since $c_1(M, d_0) \leq D_0$, by Corollary 2.3, $c_1(M, \min\{d_0(x, y), 2r\}) \leq eD_0/(e - 1)$. Thus Corollary 5.3 follows from Theorem 4.3. □

6 Lower bound on distortion

The goal of this section is to show that the lower bound on distortion of the union which was found in [19, Theorem 1.2 and Section 3] for Hilbert space is also valid for L_1 and many other Banach and metric spaces. Also, in some sense, our proof is simpler.

Definition 6.1. A metric space (X, d) is called *stable* if for any two bounded sequences $\{x_n\}$ and $\{y_m\}$ in X and for any two free ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$

$$\lim_{n,\mathcal{U}} \lim_{m,\mathcal{V}} d(x_n, y_m) = \lim_{m,\mathcal{V}} \lim_{n,\mathcal{U}} d(x_n, y_m).$$

This notion was introduced in the context of Banach spaces by Krivine and Maurey [15]. In the context of metric spaces this definition was introduced, in a slightly different, but equivalent form, in [10, p. 126]. (See [11] for an account on stable Banach spaces.)

To put our example into context we recall two simple well-known observations.

Observation 6.2 ([15])**.** *Hilbert space is stable.*

Proof.

$$\lim_{n,\mathcal{U}}\lim_{m,\mathcal{V}}\|x_n-y_m\|^2=\lim_{n,\mathcal{U}}\|x_n\|^2+\lim_{m,\mathcal{V}}\|y_m\|^2-2\lim_{n,\mathcal{U}}\lim_{m,\mathcal{V}}\langle x_n,y_m\rangle=\lim_{m,\mathcal{V}}\lim_{n,\mathcal{U}}\|x_n-y_m\|^2. \quad\square$$

Observation 6.3 ([27])**.** *The space $L_1(\mathbb{R})$ with the metric $\|x-y\|_{L_1(\mathbb{R})}^{\frac{1}{2}}$ is isometric to a subset of Hilbert space.*

Proof. ([23]) We define a map $T : L_1(\mathbb{R}) \to L_\infty(\mathbb{R}\times\mathbb{R})$ by:

$$T(f)(t,s) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } 0 < s \le f(t), \\ -1 & \text{if } f(t) < s < 0, \\ 0 & \text{otherwise.} \end{cases}$$

For all $f, g \in L_1(\mathbb{R})$ we have:

$$|T(f)(t,s) - T(g)(t,s)| = \begin{cases} 1 & \text{if } g(t) < s \le f(t) \text{ or } f(t) < s \le g(t), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned}\|T(f)-T(g)\|_{L_2(\mathbb{R}\times\mathbb{R})}^2 &= \int_{\mathbb{R}}\left(\int_{(g(t),f(t)] \text{ or } (f(t),g(t)]} 1\, ds\right) dt = \int_{\mathbb{R}} |f(t)-g(t)| dt \\ &= \|f-g\|_{L_1(\mathbb{R})}. \end{aligned}$$

$\square$

Corollary 6.4 ([15])**.** *The space L_1 is stable.*

Example 6.5. Consider the disjoint union of two copies of $\mathbb{N}$:

$$\{\bar{1},\bar{2},\dots,\bar{n},\dots\} \cup \{\underline{1},\underline{2},\dots,\underline{n},\dots\}.$$

Let d be the shortest path metric arising from the following graph structure: Each $\underline{j}$ is adjacent to $\bar{i}$ if and only if $j \le i$, and there are no other edges. Then

$$\lim_{i\to\infty} d(\underline{j},\bar{i}) = 1 \quad \text{and} \quad \lim_{j\to\infty} d(\underline{j},\bar{i}) = 3.$$

Observe that $d(\underline{i},\underline{j}) = 2$ and $d(\bar{i},\bar{j}) = 2$ for all $i \ne j$.

Therefore both copies of $\mathbb{N}$ are equilateral and thus embed isometrically into ℓ_1. On the other hand, since by Corollary 6.4, L_1 is stable, the distortion of any embedding of the set constructed in Example 6.5 into L_1 is at least 3.

Of course the same example can be used for any stable metric space containing an isometric copy of a countable equilateral set. Known theory [10, 11] implies that, for example, spaces L_p for $1 \le p < \infty$ satisfy this condition.

Finally we would like to mention that the distortion 3 in Example 6.5 cannot be increased using the same idea. Namely we prove:

Proposition 6.6. *Let $\{x_n\}$ and $\{y_m\}$ be two bounded sequences in a metric space X. Then*

$$\lim_{m,\mathcal{V}}\lim_{n,\mathcal{U}} d(x_n, y_m) \le 3 \lim_{n,\mathcal{U}}\lim_{m,\mathcal{V}} d(x_n, y_m)$$

for any free ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$.

Proof. Let $d_{nm} = d(x_n, y_m)$. Passing to subsequences we may assume that the following limits exist: $\lim_{n\to\infty} d_{nm} = S_m$, $\lim_{m\to\infty} S_m = S$, $\lim_{m\to\infty} d_{nm} = L_n$, $\lim_{n\to\infty} L_n = L$, where $S = \lim_{m,\mathcal{V}}\lim_{n,\mathcal{U}} d(x_n, y_m)$ and $L = \lim_{n,\mathcal{U}}\lim_{m,\mathcal{V}} d(x_n, y_m)$.

We need to show $S \le 3L$.

Given $\varepsilon > 0$, let $M \in \mathbb{N}$ be such that $S_m > S - \varepsilon$ for all $m \ge M$ and let $N \in \mathbb{N}$ be such that $L_n < L + \varepsilon$ for all $n \ge N$.

Let $m_N \in \mathbb{N}$ be such that $m_N \ge M$ and $d(x_N, y_{m_N}) < L + \varepsilon$

Let $n_M \in \mathbb{N}$ be such that $n_M \ge N$ and $d(x_{n_M}, y_{m_N}) > S - \varepsilon$.

Finally let $f \in \mathbb{N}$ be such that $d(x_N, y_f) < L + \varepsilon$ and $d(x_{n_M}, y_f) < L + \varepsilon$.

Using the triangle inequality we get

$$S - \varepsilon < d(x_{n_M}, y_{m_N}) \le d(x_{n_M}, y_f) + d(y_f, x_N) + d(x_N, y_{m_N}) < 3(L + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we get $S \le 3L$. □

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7 References

[1] S. Arora, L. Lovász, I. Newman, Y. Rabani, Y. Rabinovich, and S. Vempala, Local versus global properties of metric spaces, *SIAM J. Comput.* **41** (2012), no. 1, 250–271.
(Conference version in: *Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms*, ACM, New York, SIAM, Philadelphia, 2006, pp. 41–50.)

[2] Y. Benyamini, J. Lindenstrauss, *Geometric nonlinear functional analysis*. Vol. **1**. American Mathematical Society Colloquium Publications, **48**. American Mathematical Society, Providence, RI, 2000.

[3] Yu. A. Brudnyĭ, N. Ya. Krugljak, *Interpolation functors and interpolation spaces*. Vol. I. Translated from the Russian by Natalie Wadhwa. With a preface by Jaak Peetre. North-Holland Mathematical Library, 47. North-Holland Publishing Co., Amsterdam, 1991.

[4] G. Calinescu, H. Karloff, and Y. Rabani. Approximation algorithms for the 0-extension problem. *SIAM J. Comput.*, **34** (2004/05), 358–372.

[5] M. Charikar, K. Makarychev, Y. Makarychev, Local global tradeoffs in metric embeddings. *SIAM J. Comput.* **39** (2010), no. 6, 2487–2512.

[6] E. W. Chittenden, On the equivalence of Écart and voisinage. *Trans. Amer. Math. Soc.* **18** (1917), no. 2, 161–166.

[7] M. Dadarlat, E. Guentner, Uniform embeddability of relatively hyperbolic groups. *J. Reine Angew. Math.* **612** (2007), 1–15.

[8] M. Fréchet, Sur quelques points du calcul fonctionnel, *Rendiconti del Circolo Matematico di Palermo*, **22** (2dsemester, 1906), pp. 1-74.

[9] M. Fréchet, Sur les classes V normales. (French) [Normal V-classes] *Trans. Amer. Math. Soc.* **14** (1913), no. 3, 320–324.

[10] D. J. H. Garling, Stable Banach spaces, random measures and Orlicz function spaces, in: *Probability measures on groups* (Oberwolfach, 1981), pp. 121–175, *Lecture Notes in Math.*, **928**, Springer, Berlin-New York, 1982.

[11] S. Guerre-Delabrière, *Classical sequences in Banach spaces.* With a foreword by Haskell P. Rosenthal. Monographs and Textbooks in Pure and Applied Mathematics, **166**. Marcel Dekker, Inc., New York, 1992.

[12] W.B. Johnson, J. Lindenstrauss, G. Schechtman, Extensions of Lipschitz maps into Banach spaces. *Israel J. Math.* **54** (1986), no. 2, 129–138.

[13] N. J. Kalton, N. T. Peck, J. W. Roberts, *An F-space sampler.* London Mathematical Society Lecture Note Series, **89**. Cambridge University Press, Cambridge, 1984. xii+240 pp.

[14] S. A. Khot, N. K. Vishnoi, The unique games conjecture, integrability gap for cut problems and embeddability of negative-type metrics into ℓ_1. *J. ACM* **62** (2015), no. 1, Art. 8, 39 pp.
(Conference version: In *Proceedings of FOCS.* 2005, 53–62.)

[15] J.-L. Krivine, B. Maurey, Espaces de Banach stables, *Israel J. Math.*, **39** (1981), 273–295.

[16] U. Lang, Extendability of large-scale Lipschitz maps. *Trans. Amer. Math. Soc.*, **351** (1999), 3975–3988.

[17] U. Lang, C. Plaut, Bilipschitz embeddings of metric spaces into space forms. *Geom. Dedicata*, **87** (2001), no. 1-3, 285–307.

[18] J. Lindenstrauss, On nonlinear projections in Banach spaces, *Michigan Math. J.*, **11** (1964), 263–287.

[19] K. Makarychev, Y. Makarychev, Union of Euclidean metric spaces is Euclidean, *Discrete Analysis* 2016:14, 14pp.; `arXiv:1602.08426`.

[20] M. Mendel, A. Naor, Euclidean quotients of finite metric spaces. *Adv. Math.* **189** (2004), no. 2, 451–494.

[21] M. Mendel, A. Naor, Ultrametric skeletons. *Proc. Natl. Acad. Sci. USA* **110** (2013), no. 48, 19256–19262.

[22] M. Mendel, A. Naor, Expanders with respect to Hadamard spaces and random graphs. *Duke Math. J.* **164** (2015), no. 8, 1471–1548.

[23] A. Naor, L_1 embeddings of the Heisenberg group and fast estimation of graph isoperimetry, in: *Proceedings of the International Congress of Mathematicians*, 2010, Hyderabad India, vol. **III**, 2011, pp. 1549–1575.

[24] A. Naor, *Metric Embeddings and Lipschitz Extensions*, Lecture Notes prepared by Alexandros Eskenazis, Princeton, 2015; available at `http://web.math.princeton.edu/~naor/`

[25] A. Naor, Y. Rabani, On Lipschitz extension from finite subsets. *Israel J. Math.* **219** (2017), no. 1, 115–161.

[26] M. I. Ostrovskii, *Metric Embeddings: Bilipschitz and Coarse Embeddings into Banach Spaces*, de Gruyter Studies in Mathematics, **49**. Walter de Gruyter & Co., Berlin, 2013.

[27] I. J. Schoenberg, Metric spaces and positive definite functions, *Trans. Amer. Math. Soc.*, **44** (1938), 522–553.

[28] L. Trevisan, On Khot's unique games conjecture. *Bull. Amer. Math. Soc. (N.S.)* **49** (2012), no. 1, 91–111.

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