

GENERIC STABILIZERS FOR SIMPLE ALGEBRAIC GROUPS

SKIP GARIBALDI AND ROBERT M. GURALNICK

To Gopal Prasad, in honor of his 75th birthday

ABSTRACT. We prove a myriad of results related to the stabilizer in an algebraic group G of a generic vector in a representation V of G over an algebraically closed field k . Our results are on the level of group schemes, which carries more information than considering both the Lie algebra of G and the group $G(k)$ of k -points. For G simple and V faithful and irreducible, we prove the existence of a stabilizer in general position, sometimes called a principal orbit type. We determine those G and V for which the stabilizer in general position is smooth, or $\dim V/G < \dim G$, or there is a $v \in V$ whose stabilizer in G is trivial.

CONTENTS

1. Introduction	1
2. Notation and background	3
3. Adjoint representation	6
4. Summary of some recent results	11
5. Stabilizers in general position	15
6. θ -groups	18
7. Infinitesimal stabilizers	20
8. Existence of an s.g.p.: Proof of Theorem 1.1	23
9. The s.g.p. is commutative for large V	24
10. Smoothness: Proof of Theorem 1.2	24
11. Representations with “few” invariants: proof of Theorem 1.3	27
12. Regular orbits	29
13. Not-necessarily-semisimple representations	30
14. Representations with the same invariants	31
References	35

1. INTRODUCTION

The aim of this paper is to prove, for an algebraically closed field k of arbitrary characteristic, analogs of results that are known in the case $k = \mathbb{C}$ concerning an irreducible representation V of a simple linear algebraic group G .

The first such result concerns the existence of a *stabilizer in general position* (s.g.p.) also known as a *principal orbit type* for G acting on V . For $v \in V$, write G_v for the fixer

Date: Version of November 14, 2021.

2020 *Mathematics Subject Classification.* 20G05 (primary); 14L24 (secondary).

The second author (RG) was partially supported the NSF grant DMS-1901595 and a Simons Foundation Fellowship 609771.

of v in G ; it is a closed sub-group-scheme of G . One says the action of G on V has an s.g.p. if there is a closed sub-group-scheme G_* of G and a dense open subset U of V such that for all $u \in U(k)$, there is a $g \in G(k)$ such that $gG_u g^{-1} = G_*$. We prove in §§5–8:

Theorem 1.1. *Every irreducible representation V of a simple algebraic group G over an algebraically closed field k has an s.g.p. If $\dim V > \dim G$, the s.g.p. G_* is a finite group scheme. If $\dim V > \dim G$ and $\ker[G \rightarrow \mathrm{GL}(V)]$ is central in G , then the identity component of G_* is contained in a torus and $\mathrm{Lie}(G_*)$ is a toral subalgebra of $\mathrm{Lie}(G)$.*

In case $\mathrm{char} k = 0$, Richardson proved the existence of an s.g.p. under weaker hypotheses, e.g., in case G is merely assumed to be reductive [Ri 72, Th. A]. However, that claim can fail when $\mathrm{char} k \neq 0$, see Example 5.2. (There are additional, related results for $\mathbb{C}$ that do not hold for k of prime characteristic, see Example 3.14.) Because of this, it is no surprise that the arguments used here when $\mathrm{char} k \neq 0$ are of a fundamentally different nature. We rely on recent results proved in [GuL], [GaGu I], [GaGu II], and [GaGu III], see §4 for a summary.

The previous work showed that, apart from an explicit list of cases, the stabilizer G_v of a generic $v \in V$ is the trivial group scheme, in which case the s.g.p. trivially exists. The proof of Theorem 1.1 involves analyzing the many remaining cases. Along the way, we determine the s.g.p. as a group scheme in almost all cases.

We also prove a result about when G_* is commutative, see §9.

We mention that when an s.g.p. G_* exists, one obtains as a consequence that the natural map in fppf cohomology $H_{\mathrm{fppf}}^1(k, N_G(G_*)) \rightarrow H_{\mathrm{fppf}}^1(k, G)$ is surjective, see [LöM, Cor. 4.5]. This provides in turn an upper bound on the essential dimension of G . We do not pursue this avenue here.

Smoothness. Another feature that appears when $\mathrm{char} k$ is prime is that the group scheme G_v need not be smooth. We call out those cases where it happens in the following result, proved in §10. In the statement, the expression “for generic $v \in V$ ” means that there is a dense open subset U of V such that the statement holds for all $v \in U(k)$ (in this case, that G_v is smooth).

Theorem 1.2. *Let V be a faithful and irreducible representation of a simple algebraic group G over an algebraically closed field k . If it is not the case that G_v is smooth for generic $v \in V$, then up to graph automorphism $(G, \mathrm{char} k, V)$ appears in Table 4 or $(G, \mathrm{char} k, V) = (G_2, 2, L(\omega_2))$.*

We were surprised to find that there was an example with $V/G = \mathrm{Spec} k$ yet the generic stabilizer is non-smooth, namely the representation of G_2 mentioned in theorem, see Lemma 10.1 below.

One could weaken the hypothesis “faithful” in the theorem. Let N be the kernel of $G \rightarrow \mathrm{GL}(V)$. The quotient G/N acts faithfully on V (see §2) and is simple (Lemma 2.1), so Theorem 1.2 applies to it. If $(G/N)_v$ is smooth (as given by Theorem 1.2) and N is smooth (a hypothesis to replace “faithful”), then G_v is smooth because $G_v/N = (G/N)_v$ [Milne, Prop. 1.62].

Rings of invariants. For a representation V of G , the quotient V/G in the sense of Rosenlicht is defined to be $\mathrm{Spec} k[V]^G$, where $k[V]^G$ is the ring of G -invariant functions on V . In case G is reductive, $k[V]^G$ is a finitely generated k -algebra [Sesh, Th. 2], and it has dimension $\dim V - \dim G + \dim G_v$ for generic $v \in V$.

Combining the determination of $G_v(k)$ from [GuL] with information about the possibilities for $\dim V$ from [Lü], we can determine all cases where $\dim V/G$ is “small”. In

the following result, $L(\tilde{\alpha})$ is the irreducible representation of G with highest weight the highest root $\tilde{\alpha}$. This result is proved in §11.

Theorem 1.3. *Suppose V is a faithful and irreducible representation of a simple algebraic group G over an algebraically closed field k . If $\dim k[V]^G < \dim G$, then $V = L(\tilde{\alpha})$ or $(G, \text{char } k, V)$ belongs to Table 3, 4, or 5, up to graph automorphism.*

Regular orbits. We also consider when a simple algebraic group acting on an irreducible module has a regular orbit, i.e., when there exists a vector whose stabilizer is trivial. Note that a necessary condition for there to exist a regular orbit is that the stabilizer of a generic vector is finite, for otherwise the dimension of any stabilizer is positive. In characteristic zero, it turns out that this is also sufficient. The following result is proved in §12.

Theorem 1.4. *Suppose that V is a faithful and irreducible representation of a simple algebraic group G over a field k of characteristic $p \neq 2, 3, 5$. Then exactly one of the following possibilities occurs:*

- (a) *there is some $v \in V$ such that the stabilizer G_v is the trivial group scheme;*
- (b) $\dim V \leq \dim G$;
- (c) p is odd, $G = \text{SL}_4$, and up to graph automorphism $V = L(\omega_1 + p^e \omega_2)$ for some $e \geq 1$; or
- (d) $p \neq 0$, G is a quotient of $\text{SL}_{\ell+1}$, and up to graph automorphism $V = L(\omega_1 + p^e \omega_1)$ or $L(\omega_1 + p^e \omega_\ell)$ for some $e \geq 1$.

Note that $\dim V \geq \dim G$ is an obvious necessary condition for the existence of a regular orbit. The theorem says that the condition is also sufficient, apart from a few exceptions, namely the cases $\dim V = \dim G$, (c), and (d).

Other results. In addition to the results described so far, we also provide some other applications, such as a result relating the generic stabilizer for G on V with the generic stabilizer for G on a section of V (see §13) and a shorter proof of the classification of groups with the same invariants from [GaGu 15] (see Theorem 14.1).

Acknowledgements. The results in this paper weave together and rest on several recent papers, including [GuL]. Although Ross Lawther is not listed here as a co-author, this work would not have been possible without his contributions. We also thank David Stewart and the referee for their valuable comments on an earlier version of this article.

2. NOTATION AND BACKGROUND

Throughout this paper, we assume that k is an algebraically closed field. We consider algebraic groups over k in the sense of [Milne], i.e., as affine group schemes of finite type over k . Sometimes we write group scheme when it seems important to do so for clarity of exposition.

For an algebraic group G and any commutative k -algebra R , we put $G(R) := \text{Hom}_k(k[G], R)$, the set of R -points of G ; it is an “abstract” or “ordinary” group. The algebraic group G is *finite* if it is finite as a scheme over k , which holds if and only if $G(k)$ is a finite group. We say that G is *commutative* if $G(R)$ is abelian for every k -algebra R .

We also consider the Lie algebra of G , which we denote by $\text{Lie}(G)$ or $\mathfrak{g}$. Note that G is *smooth* if and only if $\dim G = \dim \mathfrak{g}$, that G is *étale* if and only if $\mathfrak{g} = 0$, and G is the trivial group scheme $\text{Spec } k$ if and only if $\mathfrak{g} = 0$ and $G(k) = 1$. We put G° for the connected component of the identity in G . It is itself an algebraic group, and $\text{Lie}(G^\circ) = \text{Lie}(G)$.

The algebraic group G is *semisimple* if it is smooth and connected and has no smooth connected solvable normal subgroups other than 1. It is *simple* (Milne says “almost-simple”) if it is semisimple and every proper normal algebraic subgroup is finite.

We say that an action $\rho: G \rightarrow \operatorname{Aut}(V)$ is *faithful* if $\ker \rho$ is the trivial group scheme. Our main results are for a faithful and irreducible representation V of a simple algebraic group G . We view the hypothesis “faithful” as harmless. Indeed, suppose N is a normal algebraic subgroup of G that is contained in the kernel of ρ . Then there is a quotient map $q: G \rightarrow G/N$ [Milne, Th. 5.14] and a unique morphism $\bar{\rho}: G/N \rightarrow \operatorname{Aut}(V)$ so that $\rho = q\bar{\rho}$ and $\ker \bar{\rho} = (\ker \rho)/N$ [Milne, Th. 5.13, Th. 5.39]. In particular, the induced action $G/\ker \rho \rightarrow \operatorname{Aut}(V)$ is faithful. Moreover, when G is simple, so is G/N :

Lemma 2.1. *If N is a proper normal sub-group-scheme of a simple algebraic group G , then G/N is also simple.*

Proof. The quotient G/N is smooth and connected because G is [Milne, Cor. 5.26, Prop. 5.59]. The inverse image H in G of a proper normal sub-group-scheme of G/N is a proper normal subgroup of G containing N , so H is finite and its image H/N in G/N is finite. $\square$

In the setting of group schemes, N can be “large”. For example, suppose a representation $\rho: G \rightarrow \operatorname{GL}(V)$ is obtained as a Frobenius twist of another representation. Then ρ is not faithful, because $d\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is zero; $\ker \rho$ contains the first Frobenius kernel G_1 . In this case, G/G_1 is isomorphic to G [Jan 03, I.9.5].

Near the end of this section, we provide another example, where $G = \operatorname{Sp}_{2\ell}$, $\operatorname{char} k = 2$, and ρ is the spin representation.

Stabilizers. For a representation $\rho: G \rightarrow \operatorname{GL}(V)$ and $v \in V$, we write G_v for the closed sub-group-scheme of G with R -points $G_v(R) = \{g \in G(R) \mid \rho(g)v = v\}$ for every commutative k -algebra R , the stabilizer of v in the ordinary group $G(R)$. The Lie algebra $\operatorname{Lie}(G_v)$ is the annihilator of v in $\mathfrak{g}$, denoted $\mathfrak{g}_v$:

$$\mathfrak{g}_v = \{x \in \mathfrak{g} \mid d\rho(x)v = 0\}.$$

The following is a well-known example in the special case $k = \mathbb{C}$, see for example [Gurevich, 23.1] or [AnP, §1].

Example 2.2 (binary cubics). Take $G = \operatorname{SL}_2$ and V the space of binary cubics, i.e., homogeneous polynomials of degree 3 in variables x, y , over a field k of characteristic $\neq 2$. A generic vector $v \in V$ is one that vanishes on three distinct lines in k^2 , such as $v = xy(x - y)$. One computes that G_v is the group scheme $\mathbb{Z}/3$. (Compare the case $\lambda = 3$ in Example 5.4 below.)

The element $w := x^2y$ is not in the orbit of v . One can compute directly that $G_w = 1$, i.e., the G -orbit of w is regular. (Compare Theorem 1.4 in the introduction.)

Focus now on the special case where $\operatorname{char} k = 3$. Then V is reducible with socle $L(3)$, consisting of cubes of linear forms, and head $L(1)$ spanned by the images of x^2y, xy^2 . The semisimplification V' of V is isomorphic to $L(1) \oplus L(1)^{[3]}$. The stabilizer in G of a generic vector in the natural module $L(1)$ is a 1-dimensional unipotent subgroup U , so the stabilizer in $G(k)$ of a generic $v' \in V'$ is an intersection $U(k) \cap U(k)^g$ for a generic $g \in G(k)$, i.e., $G_{v'}(k) = 1$. On the other hand, the Lie algebra $\mathfrak{g}$ acts trivially on $L(1)^{[3]}$, so $\mathfrak{g}_{v'} = \mathfrak{g}_v = \operatorname{Lie}(U)$. In summary, $G_{v'}$ is infinitesimal whereas G_v is étale.

Example 2.3 (Diagonalizable groups). If G is a diagonalizable group scheme, then the stabilizer of every generic $v \in V$ is the kernel of $G \rightarrow \operatorname{GL}(V)$. Therefore the s.g.p. exists trivially.

Suppose now merely that the identity component of G is a torus. In this case the s.g.p. need not be the kernel of the action. (This contradicts [PoV, §7.2, Prop.], whose statement is surely a typo.) Take for example V to be the natural representation of GL_2 and G to be the normalizer of the diagonal matrices, so $G \cong \mathbb{G}_m \wr \mathbb{Z}/2$. A vector $\begin{pmatrix} x \\ y \end{pmatrix} \in V$ with $xy \neq 0$ has stabilizer $\mathbb{Z}/2$ in G with non-identity element $\begin{pmatrix} 0 & x/y \\ y/x & 0 \end{pmatrix}$. Choosing another vector in V of the same type and not in the span of the first, we find a stabilizer that is distinct from but G -conjugate to the stabilizer of $\begin{pmatrix} x \\ y \end{pmatrix}$. In this case the s.g.p. exists, is not the kernel of the action, and is only determined by (G, V) up to conjugacy class.

Irreducible representations. Recall that every irreducible representation V of G has a highest weight λ . Write λ as a sum $\lambda = \sum_{\omega} c_{\omega} \omega$ where the sum runs over the fundamental dominant weights ω . One says that λ is *restricted* when $p := \mathrm{char} k \neq 0$ if $0 \leq c_{\omega} < p$ for all ω . (In case $\mathrm{char} k = 0$, all dominant weights are, by definition, restricted.)

Suppose now that $p \neq 0$. Write $\lambda = \lambda_0 + p^r \lambda_1$ for some $r \geq 1$, where $\lambda_0 = \sum_{\omega} c_{\omega} \omega$ and $0 \leq c_{\omega} < p^r$ for all ω . If λ_0 and $p^{r-1} \lambda_1$ belong to T^* (e.g., if G is simply connected), then $L(\lambda) \cong L(\lambda_0) \otimes L(p^{r-1} \lambda_1)^{[p]}$ [Jan 03, II.3.16], the tensor product of $L(\lambda_0)$ and a Frobenius twist of $L(p^{r-1} \lambda_1)$. As a representation of $\mathfrak{g}$ (forgetting about the action of $G(k)$), this is the direct sum of $\dim L(\lambda_1)$ copies of $L(\lambda_0)$.

We label the simple roots of G as in Table 1, which agrees with [Bour 02] as well as [GuL]. Note that our other references, [GaGu I], [GaGu II], and [GaGu III] follow the numbering of [Lü], which is different.

name	torsion primes	diagram
$A_{\ell} (\ell \geq 1)$	none	
$B_{\ell} (\ell \geq 3)$	2	
$C_{\ell} (\ell \geq 2)$	none	
$D_{\ell} (\ell \geq 4)$	2	
E_6	2, 3	
E_7	2, 3	
E_8	2, 3, 5	
F_4	2, 3	
G_2	2	

TABLE 1. Dynkin diagrams of simple root systems, with simple roots numbered as in [Bour 02], and their torsion primes from [Dem, p. 299].

Irreducible representations $L(\lambda)$, $L(\lambda')$ of G are equivalent *up to graph automorphism* if there is an automorphism ϕ of the Dynkin diagram (i.e., an automorphism of the root system that normalizes the set of simple roots) so that $\phi(\lambda) = \lambda'$; such representations are equivalent up to an automorphism of G . For example, the representations $\wedge^d k^n$ and $\wedge^{n-d} k^n$ of SL_n are equivalent up to graph automorphism. Our results, which are about stabilizers in general position and so on, are transparently the same for representations that are equivalent up to graph automorphism.

Special characteristic. For the remainder of this section, suppose that G is simple and simply connected. We say that $\mathrm{char} k$ is *special* for G if G has type G_2 and $\mathrm{char} k = 3$ or G has type B_ℓ ($\ell \geq 2$), C_ℓ ($\ell \geq 2$), or F_4 and $\mathrm{char} k = 2$. (This was written as “exceptionally bad characteristic” in the title of [GaGu III].) That is, $\mathrm{char} k$ is special if the Dynkin diagram of G has an edge with multiplicity $\mathrm{char} k$. When that holds, there is a *very special isogeny* $\pi : G \rightarrow \bar{G}$ where $\bar{G}$ is also simple simply connected and the root system of $\bar{G}$ is inverse to the root system of G , see [CoGP, §7.1] or [St 63, §10] for a concrete description.

Now suppose that $\mathrm{char} k$ is special for G , so in particular Δ has two root lengths. Write a dominant weight λ as $\lambda = \sum c_\delta \omega_\delta$, where $c_\delta \geq 0$ and ω_δ is the fundamental weight dual to $\delta^\vee$ for $\delta \in \Delta$. We write $\lambda = \lambda_s + \lambda_\ell$ where $\lambda_s = \sum_{\delta \text{ short}} c_\delta \omega_\delta$ and $\lambda_\ell = \sum_{\delta \text{ long}} c_\delta \omega_\delta$, i.e., $\langle \lambda_s, \delta^\vee \rangle = 0$ for δ long and $\langle \lambda_\ell, \delta^\vee \rangle = 0$ for δ short. Steinberg [St 63] shows that $L(\lambda) \cong L(\lambda_\ell) \otimes L(\lambda_s)$ and that furthermore the action of G on $L(\lambda_\ell)$ factors through the very special isogeny. For example, in case $G = \mathrm{Sp}_{2\ell}$ for some $\ell \geq 2$ and $\mathrm{char} k = 2$, $\bar{G} = \mathrm{Spin}_{2\ell+1}$ and the non-faithful representation $L(\omega_\ell)$ of G is obtained by composing the very special isogeny π with the spin representation of $\bar{G}$, which is irreducible and faithful.

In particular, if $L(\lambda)$ is faithful, then $\lambda_s \neq 0$.

3. ADJOINT REPRESENTATION

We record in this section various results about G acting on $\mathrm{Lie}(G)$ and the irreducible representation $L(\tilde{\alpha})$.

The generic stabilizer for simple G acting on $\mathrm{Lie}(\mathrm{Ad}(G))$ is determined in [GaGu 16, Prop. 9.2]; this representation has an s.g.p. whose identity component is a maximal torus in G . It follows that $\dim k[\mathrm{Lie}(\mathrm{Ad}(G))]^G = \mathrm{rank} G$. However, $\mathrm{Lie}(\mathrm{Ad}(G))$ agrees with $\mathfrak{g} = \mathrm{Lie}(G)$ if and only if the center of G , i.e., the kernel of $G \rightarrow \mathrm{Ad}(G)$, is étale.

The general statement is that the Lie algebra $\tilde{\mathfrak{g}}$ of the simply connected cover $\tilde{G}$ of G is the Weyl module $V(\tilde{\alpha})$ with highest weight the highest root $\tilde{\alpha}$ and the head of $V(\tilde{\alpha})$ is the irreducible representation $L(\tilde{\alpha})$, see for example [Ga 09b, 2.5].

For G simple, $\mathfrak{g}$ is an irreducible representation of G — i.e., $\mathrm{Lie}(G) = L(\tilde{\alpha})$ — if and only if the center $Z(\tilde{G})$ is étale and $\mathrm{char} k$ is not special for G , see [Hiss].

Lemma 3.1. *The irreducible representation $L(\tilde{\alpha})$ of a simple group G is faithful if and only if G is adjoint, $\mathrm{char} k$ is not special for G , and $(G, \mathrm{char} k) \neq (\mathrm{PGL}_2, 2)$. If those equivalent conditions hold, then the s.g.p. exists, its identity component is a maximal torus of G , and*

$$\dim k[L(\tilde{\alpha})]^G = \mathrm{rank} G - \dim \mathrm{Lie}(Z(\tilde{G})).$$

Proof. The kernel of the action of $\tilde{G}$ on $V(\tilde{\alpha})$ is $Z(\tilde{G})$ [Milne, Prop. 21.7], so the kernel of the action of G on $V(\tilde{\alpha})$ is $Z(\tilde{G})/\ker[\tilde{G} \rightarrow G] \cong Z(G)$. Thus, for $L(\tilde{\alpha})$ to be faithful it is necessary that G is adjoint.

If $\text{char } k$ is special for G , then $\mathfrak{n} := \ker d\pi$ is a nonzero proper ideal in $\tilde{\mathfrak{g}} = V(\tilde{\alpha})$, so its image in $L(\tilde{\alpha})$ is zero. In particular, the image of $\mathfrak{n}$ in $\mathfrak{g}$ acts trivially on $L(\tilde{\alpha})$. As $\mathfrak{n} \not\subseteq \text{Lie}(Z(\tilde{G}))$ [Hiss], its image in $\mathfrak{g}$ is not zero, so $\mathfrak{g}$ and the group scheme G do not act faithfully on $L(\tilde{\alpha})$.

Suppose for the remainder of the proof that G is adjoint and $\text{char } k$ is not special for G . Since $G(k)$ is simple as an ordinary group, it acts faithfully on $L(\tilde{\alpha})$, whence the kernel of the action of G is infinitesimal.

The socle of $\mathfrak{g}$ as a representation of G is irreducible [Hiss]; it is the subalgebra generated by the root subspaces, i.e., the image of the Lie algebra of the simply connected cover of $\tilde{G}$ of G , see for example [GaGu I, Lemma 3.1(1)]. That is, $L(\tilde{\alpha})$ is not faithful if and only if the composition $\tilde{\mathfrak{g}} \rightarrow \mathfrak{g} \rightarrow \mathfrak{gl}(L(\tilde{\alpha}))$ is zero if and only if $(G, \text{char } k) = (\text{PGL}_2, 2)$ by [GaGu I, Lemma 3.1(2)]. This concludes the proof of the first sentence of the lemma. For the remainder of the proof we suppose additionally that $(G, \text{char } k) \neq (\text{PGL}_2, 2)$.

Pick a maximal torus $\tilde{T}$ in $\tilde{G}$. The orbit of a generic element in $V(\tilde{\alpha})$ meets $\text{Lie}(\tilde{T})$ by [DemG, XIII.5.1, XIV.3.18], compare Lemma 5.8 below. A generic element $t \in \text{Lie}(\tilde{T})$ has image a generic element $v \in L(\tilde{\alpha})$. The arguments of [GaGu I, Example 3.4] apply and the stabilizer G_v of v has identity component the image T of $\tilde{T}$ and in particular $T \subseteq G_v \subseteq N_G(T)$.

An element $n \in N_G(T)(k)$ belongs to G_v if and only if $\text{Ad}(n)t - t$ is in the kernel of the map $V(\tilde{\alpha}) \rightarrow L(\tilde{\alpha})$, i.e., is in $\text{Lie}(Z(\tilde{G}))$. Equivalently, if and only if every root α vanishes on $\text{Ad}(n)t - t$. Since t is generic, this is equivalent to $\langle \alpha \circ \text{Ad}(n) - \alpha, \beta^\vee \rangle = 0$ in k for every root α and coroot $\beta^\vee$, equivalently, the element $\alpha \circ \text{Ad}(n) - \alpha$ in the weight lattice is divisible by $\text{char } k$ for every root α . This only depends on the image of n in the Weyl group $N_G(T)/T$ and not on the choice of t , so G_v depends only on the choice of $\tilde{T}$ (equivalently, T), verifying that G_v is an s.g.p. for the action of G on $L(\tilde{\alpha})$. $\square$

Example 3.2. The s.g.p. appearing in the statement of Lemma 3.1 need not be connected. Take G of type E_8 , in which case the irreducible representation $L(\tilde{\alpha})$ is $\mathfrak{g}$ itself and G is both simply connected and adjoint. In the notation of the proof, G_v is connected if and only if $\text{char } k \neq 2$, see [GaGu 16, Prop. 9.2]. (Note that Steinberg's result [St 75, Th. 0.2], which shows in some cases that semisimple elements in $\mathfrak{g}$ have connected centralizers, assumes $\text{char } k$ is not a torsion prime, so it does not apply here when $\text{char } k$ is 2, 3, or 5.) When $\text{char } k = 2$, $G_v/G_v^\circ \cong \mathbb{Z}/2$, where the nontrivial element acts on the torus G_v° by inversion.

Example 3.3. Suppose $G = \text{PGL}_3$ and $\text{char } k = 3$. Then $L(\tilde{\alpha})$ is $\mathfrak{sl}_3/k$ as both a G -module and a Lie algebra. In the notation of the proof of Lemma 3.1, the image of $\text{Lie}(\tilde{T})$ in $L(\tilde{\alpha})$ is a 1-dimensional maximal toral subalgebra kv . The Weyl group $N_G(T)/T$ is the symmetric group on 3 letters, which acts on kv as ± 1 , whence the component group G_v/G_v° is $\mathbb{Z}/3$. We thank a referee for suggesting this example.

Example 3.4. We apply Lemma 3.1 to construct Table 2, where we list the cases where simple G acts faithfully on $L(\tilde{\alpha})$ and $\dim k[L(\tilde{\alpha})]^G \leq 2$. If $\dim \text{Lie}(Z(\tilde{G})) = 0$, i.e., $Z(\tilde{G})$ is étale, then (type of $G, \text{char } k$) is one of $(A_1, \neq 2)$, $(A_2, \neq 3)$, $(B_2, \neq 2)$, or $(G_2, \neq 3)$. If $\dim \text{Lie}(Z(\tilde{G})) = 1$, then $\text{rank } G \leq 3$ and (type of $G, \text{char } k$) is $(A_2, 3)$ or $(A_3, 2)$. Finally, if $\dim \text{Lie}(Z(\tilde{G})) = 2$, then G has type D_{2m} for some $m \geq 2$, so (type of $G, \text{char } k$) is $(D_4, 2)$.

We will use the following in the proof of Theorem 14.1 at the end of the paper, but we put it here because it only concerns $L(\tilde{\alpha})$ as a representation.

type of G	char k	$\dim L(\tilde{\alpha})$	$\dim k[L(\tilde{\alpha})]^G$
A_1	$\neq 2$	3	1
A_2	$\neq 3$	8	2
A_2	3	7	1
A_3	2	14	2
B_2	$\neq 2$	10	2
D_4	2	26	2
G_2	$\neq 3$	14	2

TABLE 2. Simple G that act faithfully on $L(\tilde{\alpha})$ such that $\dim k[L(\tilde{\alpha})]^G \leq 2$.

Lemma 3.5. *Let G be a simple algebraic group. If there is a proper, connected, and smooth algebraic subgroup of G that acts irreducibly on $L(\tilde{\alpha})$, then char k is special for G .*

Proof. Let G' be a connected, proper, and smooth subgroup of G that acts irreducibly on $L(\tilde{\alpha})$. Since the kernel of the action by G on $L(\tilde{\alpha})$ is finite and G' acts irreducibly, we conclude that G' has trivial radical and so is semisimple.

For sake of contradiction, suppose char k is not special. We may assume that G is adjoint, so $L(\tilde{\alpha})$ is the socle of $\mathfrak{g}$ as a representation of G and $\mathfrak{g}/L(\tilde{\alpha})$ is isomorphic to the Lie algebra of the center of the simply connected cover of G . As the Lie algebra $\mathfrak{g}'$ is a G' -invariant subspace of $\mathfrak{g}$, it contains $L(\tilde{\alpha})$. In particular, G and G' have the same number of roots and the same unipotent radicals for the Borel subgroups. As these unipotent radicals of a semisimple group generate the group, we find $G = G'$. $\square$

For the sake of completeness, we provide the following converse to Lemma 3.5. In the statement, the hypotheses “connected” and “smooth” are redundant because they are included in the definition of simple (see §2). We have included them here to emphasize that this is a converse to the preceding.

Lemma 3.6. *Let G be a simple algebraic group. If char k is special for G , then there is a proper, connected, smooth, and simple subgroup of G that acts irreducibly on $L(\tilde{\alpha})$.*

Proof. We may assume that G is simply connected, so $L(\tilde{\alpha})$ is the head of $\mathfrak{g}$.

If G is of type C_n for $n \geq 2$, the root subgroups for the short roots generate a subgroup G' of type D_n . The restriction of $L(\tilde{\alpha})$ to G' is a Frobenius twist of the natural representation of dimension $2n$, so G' acts irreducibly on $L(\tilde{\alpha})$.

If G has type B_n for $n \geq 3$, F_4 , or G_2 , then the root subgroups for the long roots generate a subgroup G' of type D_n , D_4 , or A_2 respectively. The description of $\mathfrak{g}$ in [Hiss] or [Hog] shows that $\text{Lie}(G')$ maps onto $L(\tilde{\alpha})$ and indeed $L(\tilde{\alpha})$ is also the irreducible part of the adjoint representation of G' . $\square$

Invariant polynomial functions. We now study the rings of invariant polynomial functions $k[\mathfrak{g}]^G$ and $k[L(\tilde{\alpha})]^G$, especially in the case where $\mathfrak{g} \neq L(\tilde{\alpha})$.

Suppose G is simple and let T be a maximal torus in G . For W the Weyl group $N_G(T)/T$, the natural restriction map

$$(3.7) \quad k[\mathfrak{g}]^G \rightarrow k[t]^W$$

is injective. Furthermore, it is an isomorphism if and only if $(G, \text{char } k) \neq (\text{Sp}_{2\ell}, 2)$ for $\ell \geq 1$, see [SpSt, II.3.17'], [Jan 04, p. 82, 7.12], or [ChaR]. This is sometimes called the *Chevalley Restriction Theorem*.

If G is simply connected, then $k[\mathfrak{t}]$ from the previous paragraph is the symmetric algebra on the weights with coefficients in k . If $\text{char } k$ is not a torsion prime for (the root system of) G as in Table 1, then $k[\mathfrak{t}]^W$ is a polynomial algebra [Dem, p. 297, Cor.]. If additionally $(G, \text{char } k) \neq (\text{Sp}_{2\ell}, 2)$ for $\ell \geq 1$, then the generators of $k[\mathfrak{t}]^W$ have the same degrees as they do for the analogous group in case $k = \mathbb{C}$ [Dem, p. 296, Cor., Th. 3].

Example 3.8. Suppose G has type G_2 . The only torsion prime for G is 2, so when $\text{char } k \neq 2$, $k[\mathfrak{g}]^G$ is a polynomial ring with generators of degree 2 and 6 by the above and [Bour 05, §VIII.8.3].

In case $\text{char } k = 2$, we apply the Chevalley Restriction Theorem by hand. The element -1 in the Weyl group acts trivially on $\mathfrak{t}$, so $k[\mathfrak{t}]^W$ is the ring of invariant functions of the symmetric group on 3 letters on its irreducible 2-dimensional representation. We conclude that $k[\mathfrak{g}]^G$ is a polynomial ring with generators of degrees 2 and 3. (Alternatively, one can apply the next two examples to an A_2 subgroup containing T to see that $k[\mathfrak{g}]^G = k[\mathfrak{t}]^W = k[\mathfrak{t}]^{S_3} = k[\mathfrak{sl}_3]^{\text{SL}_3}$ to draw the same conclusion.)

Example 3.9. For $G = \text{SL}_n$, the ring of G -invariant functions on the space of n -by- n matrices is polynomial with generators the coefficients of the characteristic polynomial. Tracking the proof of [N, Prop. 4.1], one finds that the ring of invariants $k[\mathfrak{g}]^G$ is also polynomial, with the same generators except for the trace. Away from the case $n = 2$ and $\text{char } k = 2$, the Chevalley Restriction Theorem and Demazure's result provides the same conclusion.

In case $n = 2$ and $\text{char } k = 2$, the Weyl group W acts trivially on the Lie algebra $\mathfrak{t}$ of a maximal torus in G , so $k[\mathfrak{t}]^W$ is a polynomial ring in one variable, generated by a linear function. The image of the restriction map $k[\mathfrak{g}]^G \rightarrow k[\mathfrak{t}]^W$ is $(k[\mathfrak{t}]^W)^{[2]}$.

We now address $k[L(\tilde{\alpha})]^G$ in case $\mathfrak{g}$ is a reducible representation.

Example 3.10. Suppose G is simple, $\text{char } k$ is not special for G , and $(\text{type of } G, \text{char } k) \neq (A_1, 2)$. In particular, the Chevalley Restriction Theorem applies. As the action of G on $L(\tilde{\alpha})$ factors through the adjoint group $\bar{G}$ of G and our goal is to calculate $k[L(\tilde{\alpha})]^G$, we are free to choose G to be simply connected.

Under the hypotheses, $L(\tilde{\alpha})$ is the image of $\mathfrak{g}$ in $\bar{\mathfrak{g}}$. As $Z(G)$ is contained in T [Milne, Prop. 21.7], its Lie algebra $\mathfrak{z}$ is contained in $\mathfrak{t}$. The natural maps $k[\mathfrak{t}/\mathfrak{z}]^G \hookrightarrow k[\mathfrak{t}]^G$ and $k[\mathfrak{g}/\mathfrak{z}]^G \hookrightarrow k[\mathfrak{g}]^G$ are compatible with the isomorphism (3.7) in the sense that it induces an isomorphism $k[L(\tilde{\alpha})]^G \xrightarrow{\sim} k[\mathfrak{t}_0]^W$, where $\mathfrak{t}_0 := \mathfrak{t}/\mathfrak{z}$ is the image of $\mathfrak{t}$ in the Lie algebra of the image $\bar{T}$ of T . It is the subspace of $\text{Lie}(\bar{T})$ spanned by the elements h_α in a Chevalley basis where α is a root. (See [GaGu 15, Example 8.3] for a concrete illustration in the case G is isogenous to SL_4 and $\text{char } k = 2$, where $L(\tilde{\alpha})$ has a polynomial ring of invariants with generators of degrees 2 and 3.)

Corollary 3.11. *Suppose G is simple and $L(\tilde{\alpha})$ is faithful. If $\dim k[L(\tilde{\alpha})]^G \leq 2$, then $k[L(\tilde{\alpha})]^G$ is a polynomial ring.*

Proof. The type of G and $\text{char } k$ appear as a row in Table 2. If $\dim k[L(\tilde{\alpha})]^G = 1$, then $k[L(\tilde{\alpha})]^G$ is a polynomial ring for dimension reasons [BeGuL, Prop. 6.1], so assume $\dim k[L(\tilde{\alpha})]^G = 2$.

If $Z(\bar{G})$ is étale, we apply the Chevalley Restriction Theorem. Otherwise (type of $G, \text{char } k) = (A_3, 2)$ or $(D_4, 2)$ and we apply Example 3.10. In either case, $k[L(\tilde{\alpha})]^G$

is isomorphic to the ring of invariant polynomials of a pseudoreflection group (the Weyl group) acting on a 2-dimensional space, so it is a polynomial ring by [N, Th. 5.1] or [KeM, Prop. 7.1]. $\square$

See Proposition 11.3 below for a stronger version of Corollary 3.11.

Proposition 3.12. *Let $G = \mathrm{PGL}_n$ for some $n \geq 2$ over a field k . The following are equivalent:*

- (1) $k[L(\tilde{\alpha})]^G$ is a polynomial ring.
- (2) $k[\mathfrak{g}]^G$ is a polynomial ring.
- (3) $n \leq 4$ or $\mathrm{char} k$ does not divide n .

Proof. Suppose first that $\mathrm{char} k$ does not divide n . Then the representations $\mathfrak{sl}_n$, $\mathfrak{pgl}_n$, and $L(\tilde{\alpha})$ are all naturally identified with each other and $k[\mathfrak{sl}_n]^{\mathrm{SL}_n}$ is a polynomial ring as in Example 3.9.

Therefore, we restrict our attention to the case where $\mathrm{char} k$ does divide n , applying Example 3.10 for $k[L(\tilde{\alpha})]^G$ and the Chevalley Restriction Theorem for $k[\mathfrak{g}]^G$. As the roots of G all have the same length, we may identify the root system with its dual and so identify the toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ with the weight lattice tensored with k and $\mathfrak{t}_0$ with the subspace spanned by the roots.

If we identify the Weyl group with the symmetric group S_n and consider its permutation representation X with basis $x_1, \dots, x_n$, then the ring $k[\mathfrak{t}]$ of functions on $\mathfrak{t}$ with the action by the Weyl group is identified with the symmetric algebra A_Y on the subspace Y of X of elements whose coordinates sum to zero. Moreover, the space of characters on T that vanish on $\mathfrak{t}_0$ are a 1-dimensional subspace of Y and contain $\sum x_i$, so the functions on $k[\mathfrak{t}_0]$ are identified with the symmetric algebra A_Z on $Z := Y/k \sum x_i$. If $n > 4$, then $A_Y^{S_n}$ and $A_Z^{S_n}$ are not polynomial rings, see [N, §4] and [KeM, §5], proving the claim in that case.

If $n \leq 4$, then $\dim k[L(\tilde{\alpha})]^G \leq 2$ and we have already observed that $k[L(\tilde{\alpha})]^G$ is a polynomial ring in Corollary 3.11. If $n = 2$ or 3 , then $k[\mathfrak{g}]^G$ is by the Chevalley Restriction Theorem the functions in 1 or 2 variables that are invariant under a finite reflection group, so for the same reason we conclude that $k[\mathfrak{g}]^G$ is a polynomial ring.

Finally, consider $k[\mathfrak{g}]^G$ in the case $n = 4$ and $k = \mathbb{F}_2$. The ring A_Y has generators

$$(3.13) \quad y_1 = x_1 - x_2, \quad y_2 = x_2 - x_3, \quad y_3 = x_3 - x_4.$$

Set

$$\begin{aligned} f_1 &= y_1 + y_3, \\ f_2 &= y_1^2 y_2^2 + y_2^4 + y_1^2 y_2 y_3 + y_1 y_2^2 y_3 + y_1^2 y_3^2 + y_1 y_2 y_3^2 + y_2^2 y_3^2, \quad \text{and} \\ f_3 &= y_1^4 y_2^2 + y_1^2 y_2^4 + y_1^4 y_2 y_3 + y_1 y_2^4 y_3 + y_1^4 y_3^2 + y_1^2 y_2^2 y_3^2 + y_2^4 y_3^2 + y_1^2 y_3^4 + y_1 y_2 y_3^4 + y_2^2 y_3^4 \end{aligned}$$

in A_Y . Rewriting these using (3.13), we find that $f_1 = \sum x_i$ is in $A_Y^{S_4}$ and similarly for f_2 and f_3 . The determinant of the Jacobian matrix with (i, j) entry $\partial f_i / \partial y_j$ is not zero (e.g., the term $y_1^5 y_2^2 y_3$ appears) and $\prod \deg f_i = |S_4|$, so the f_i are algebraically independent and $k[\mathfrak{g}]^G \cong A_Y^{S_4} = k[f_1, f_2, f_3]$ is a polynomial ring by the criterion from [DerK, Th. 3.9.4]. $\square$

Example 3.14. Let V be an irreducible and faithful representation of a simple algebraic group G over k . One can ask whether the property that $G_v \neq 1$ for generic $v \in V$ (denoted by (ST) in [PoV, §8]) is equivalent to the property that $k[V]^G$ is a polynomial ring (denoted by (FA) in ibid.). If $\mathrm{char} k = 0$, then the two properties are equivalent, see [PoV, Th. 8.8] and [KaPV].

If $G = \mathrm{PGL}_n$ such that $\mathrm{char} k$ divides n and $n \geq 5$ and $V = L(\tilde{\alpha})$, then (G, V) satisfies (ST) (Lemma 3.1) but not (FA) (Proposition 3.12).

Alternatively, take $G = \mathrm{PSp}_{2n}$ with the same hypotheses on n . The representation $V = L(\omega_2)$ is faithful and irreducible and $G_v \neq 1$ for generic $v \in V$, see Table 3. Example 8.5 in [GaGu 15] shows that $k[V]^G$ is isomorphic to the ring of invariants in the preceding paragraph, so it too is not a polynomial ring.

4. SUMMARY OF SOME RECENT RESULTS

Recent results of the authors, which rely on Lübeck's paper [Lü], combine with that paper to give the following results.

Theorem 4.1. *Let V be an irreducible and faithful representation of a simple linear algebraic group G over an algebraically closed field k .*

- (1) *Suppose $\dim V > \dim G$. Then G_v is finite for generic $v \in V$, and $\dim k[V]^G = \dim V - \dim G$. Moreover, $G_v \neq 1$ for generic $v \in V$ if and only if $(G, \mathrm{char} k, V)$ is listed in Table 4 up to graph automorphism.*
- (2) *If $\dim V = \dim G$, then G is adjoint, V is the adjoint representation $\mathrm{Lie}(G) = L(\tilde{\alpha})$, $\mathrm{char} k$ is not special for G , there is a s.g.p. (whose identity component is a maximal torus), and $\dim k[V]^G = \mathrm{rank} G$.*
- (3) *If $\dim V < \dim G$, then $(G, \mathrm{char} k, V)$ is listed in Table 3 up to graph automorphism or $V = L(\tilde{\alpha})$.*

In particular, G_v is finite for generic $v \in V$ if and only if $\dim V > \dim G$.

We now justify the columns in Tables 3 and 4 concerning G_v . The k -points $G(k)_v = G_v(k)$ for generic $v \in V$ are from [GuL].

In Table 4, the Lie algebra $\mathfrak{g}_v$ was computed in [GaGu II] and [GaGu III]. Since $G(k)_v$ is finite in all cases, if additionally $\mathfrak{g}_v = 0$ then G_v is étale (in particular, smooth) and so completely described by $G(k)_v$. In particular, G_v is commutative if and only if $G(k)_v$ is an abelian group. If $\mathfrak{g}_v \neq 0$, then G_v is not smooth, and we study G_v in §6 and §7.

Proof. (1): Assume $\dim V > \dim G$. The rows in Table 4 are a union of the rows in Table 1 in each of [GaGu III] (those with $\mathfrak{g}_v \neq 0$) and [GuL] (those with $G_v(k) \neq 1$), although we have omitted those entries corresponding to non-faithful representations, such as spin representations of $\mathrm{Sp}_{2\ell}$ when $\mathrm{char} k = 2$. Conversely, if $\mathfrak{g}_v = 0$ and $G(k)_v = 1$, then $G_v = 1$.

Note that in each row of Table 4, $G_v(k)$ is finite. This gives the claim on $\dim k[V]^G$.

For the remainder of the proof, we assume that $\dim V \leq \dim G$, so the highest weight of V is restricted by an easy dimension argument as in [GaGu II, Lemma 1.1]. That is, V is among the representations enumerated in [Lü].

If $\dim V = \dim G$, checking the tables in [Lü] verifies that V is the adjoint representation and $\mathrm{char} k$ is not special. Moreover, as $\dim V = \dim G$, the center of the simply connected cover of G is étale and the generic stabilizer is computed in [GaGu 16, Prop. 9.2]. This verifies (2).

For (3), the list in Table 3 is somewhat shorter than in [Lü], because we have omitted those representations that factor through the very special isogeny, i.e., those λ that vanish on the short simple roots such as the spin representations of type C when $\mathrm{char} k = 2$.

For the final claim, note that for $V = L(\tilde{\alpha})$, G_v is not finite for generic $v \in V$ by Lemma 3.1. $\square$

G	char k	V	$G_v(k)$	$\dim k[V]^G$
A_ℓ ($\ell \geq 1$)	any	$k^{\ell+1}$	$A_{\ell-1}U_\ell$	0
A_ℓ ($\ell \geq 1$)	$\neq 2$	$S^2(k^{\ell+1})$	$D_{(\ell+1)/2}$ or $B_{\ell/2}$	1
A_ℓ (odd $\ell \geq 3$)	any	$\wedge^2(k^{\ell+1})$	$C_{(\ell+1)/2}$	1
A_ℓ (even $\ell \geq 4$)	$\cdots$ same as line above $\cdots$	$\cdots$	$C_{\ell/2}U_\ell$	0
SL_6/μ_3	any	$\wedge^3 k^6$	$A_2^2.\mathbb{Z}/(2, p)$	1
SL_7	any	$\wedge^3 k^7$	G_2	1
SL_8	any	$\wedge^3 k^8$	$A_2.\mathbb{Z}/(2, p)$	1
$\mathrm{SO}_{2\ell+1}$ ($\ell \geq 2$)	$\neq 2$	$L(\omega_1)$ (natural)	D_ℓ	1
Spin_7	any	$L(\omega_3)$ (spin)	G_2	1
Spin_9	any	$L(\omega_4)$ (spin)	B_3	1
Spin_{11}	any	$L(\omega_5)$ (spin)	$A_4.\mathbb{Z}/(2, p)$	1
Spin_{13}	any	$L(\omega_6)$ (spin)	$A_2^2.(\mathbb{Z}/(2, p))^2$	2
$\mathrm{Sp}_{2\ell}$ ($\ell \geq 2$)	any	$L(\omega_1)$ (natural)	$C_{\ell-1}U_{2\ell-1}$	0
Sp_6	$\neq 2$	$L(\omega_3)$ (“spin”)	$\tilde{A}_2$	1
PSp_6	any	$L(\omega_2)$	$C_1^3.\mathbb{Z}/(3, p)$	$2 - \varepsilon$
PSp_8	$\cdots$ same as line above $\cdots$	$\cdots$	$C_1^4.(\mathbb{Z}/(2, p))^2$	$3 - \varepsilon$
$\mathrm{PSp}_{2\ell}$ ($\ell \geq 5$)	$\cdots$ same as line above $\cdots$	$\cdots$	C_1^ℓ	$\ell - 1 - \varepsilon$
$\mathrm{SO}_{2\ell}$ ($\ell \geq 4$)	any	$L(\omega_1)$ (natural)	$B_{\ell-1}$	1
Spin_{10}	any	spin	B_3U_8	0
HSpin_{12}	any	half-spin	$A_5.\mathbb{Z}/(2, p)$	1
Spin_{14}	any	spin	$G_2^2.\mathbb{Z}/(2, p)$	1
G_2	$\neq 2$	$L(\omega_1)$ (natural)	A_2	1
G_2	2	same as line above	A_1U_5	0
F_4	any	$L(\omega_4)$ (natural)	D_4	$2 - \varepsilon$
E_6	any	$L(\omega_1)$ (minuscule)	F_4	1
E_7	any	$L(\omega_7)$ (minuscule)	$E_6.\mathbb{Z}/(2, p)$	1

TABLE 3. Irreducible faithful representations V for a simple algebraic group G over an algebraically closed field k of characteristic p such that $\dim V \leq \dim G$, except for $L(\bar{\alpha})$ and up to graph automorphism. The symbol ε represents 0 or 1, where the value is determined by ℓ and char k .

Remark 4.2. In Table 4, the stabilizer G_v in Spin_{13} when char $k = 2$ is $(\mathrm{SL}_3 \times \mathrm{SL}_3) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/2)$, as described in [GuL, Prop. 5.2.9]. The $\mathbb{Z}/2$ ’s are generated by an element that acts as an outer automorphism on each SL_3 and an element that interchanges the two SL_3 ’s. This corrects a mistake in [GaGu 17, Prop. 9.2], where one of the $\mathbb{Z}/2$ factors was omitted.

The following is an analogue for group schemes of a result proved for $G(k)$ in [GuL, Cor. 11]. Recall that an algebraic group G is said to act *generically freely* on V if $G_v = 1$ for generic $v \in V$.

Corollary 4.3. *Let G be a simple linear algebraic group acting faithfully and irreducibly on a representation V . If V has a nonzero weight space with multiplicity > 1 , then G acts generically freely on V .*

G	char k	V	$\dim V$	G_v	G_v smooth?	G_v commutative?	$\dim k[V]^G$
SL_2	$\neq 2, 3$	$S^3 k^{2,2}$	4	$\mathbb{Z}/3$	yes	yes	1
SL_2/μ_2	$\neq 2, 3$	$S^4 k^{2,2}$	5	$(\mathbb{Z}/2)^2$	yes	yes	2
SL_3/μ_3	$\neq 2, 3$	$S^3 k^3$	10	$(\mathbb{Z}/3)^2$	yes	yes	2
SL_4	3	$L(\omega_1 + \omega_2)$	16	Alt_5	yes	no	1
SL_4/μ_4	$\neq 2$	$L(2\omega_2)$	$20 - \varepsilon$	$(\mathbb{Z}/2)^4$	yes	yes	$5 - \varepsilon$
SL_4	p odd	$L(p^e \omega_1 + \omega_2), e \geq 1$	24	μ_{p^e}	no	yes	9
SL_4/μ_2	2	$L(2^e \omega_1 + \omega_2), e \geq 2$	24	μ_{2^e}	no	yes	9
SL_8/μ_4	$\neq 2$	$\wedge^4 k^8$	70	$(\mathbb{Z}/2)^6$	yes	yes	7
SL_8/μ_4	2	... same as line above ...	...	$(\mathbb{Z}/2)^3 \times (\mu_2)^3$	no	yes	7
SL_9/μ_3	$\neq 2, 3$	$\wedge^3 k^9$	84	$(\mathbb{Z}/3)^4$	yes	yes	4
SL_9/μ_3	2	... same as line above ...	...	$(\mathbb{Z}/3)^4 \cdot (\mathbb{Z}/2)$	yes	no	4
SL_9/μ_3	3	... same as line above ...	...	$(\mathbb{Z}/3)^2 \times (\mu_3)^2$	no	yes	4
A_ℓ	$p \neq 0$	$L(\omega_1 + p^e \omega_\ell), e \geq 1$	$(\ell + 1)^2$	$\mathrm{PSU}_{\ell+1}(p^e)$	yes	no	1
A_ℓ	$p \neq 0$	$L(\omega_1 + p^e \omega_1), e \geq 1$	$(\ell + 1)^2$	$\mathrm{PSL}_{\ell+1}(p^e)$	yes	no	1
Spin_5	5	$L(\omega_1 + \omega_2)$	12	μ_5	no	yes	2
$\mathrm{SO}_{2\ell+1} (\ell \geq 2)$	$\neq 2$	$L(2\omega_1)$ (“ $S^{2''}$ ”)	$2\ell^2 + 3\ell - \varepsilon$	$(\mathbb{Z}/2)^{2\ell}$	yes	yes	$2\ell - \varepsilon$
Sp_8	3	$L(\omega_3)$ (“ $\wedge^3$ ”)	40	$(\mu_3)^2$	no	yes	4
PSp_8	$\neq 2$	$L(\omega_4)$ (“spin”)	$42 - \varepsilon$	$(\mathbb{Z}/2)^6$	yes	yes	$6 - \varepsilon$
$\mathrm{SO}_{2\ell}/\mu_2 (\ell \geq 4)$	$\neq 2$	$L(2\omega_1)$ (“ $S^{2''}$ ”)	$2\ell^2 + \ell - 1 - \varepsilon$	$(\mathbb{Z}/2)^{2\ell-2}$	yes	yes	$2\ell - 1 - \varepsilon$
HSpin_{16}	$\neq 2$	half-spin	128	$(\mathbb{Z}/2)^8$	yes	yes	8
HSpin_{16}	2	... same as line above ...	...	$(\mathbb{Z}/2)^4 \times (\mu_2)^4$	no	yes	8

TABLE 4. Irreducible faithful representations V for a simple algebraic group G over an algebraically closed field k such that $\dim V > \dim G$ and $G_v \neq 1$, up to graph automorphism. In the dimensions, ε represents 0 or 1; the value it takes depends on ℓ and char k .

G	char k	V	$\dim V$	$\dim k[V]^G$
SL_3	$\neq 2, 3$	$S^4 k^3$	15	7
SL_3	$\neq 2$	$L(\omega_1 + 2\omega_2)$	15	7
A_3	$p \neq 0$	$L(p^e \omega_1 + \omega_2)$ or $L(\omega_1 + p^e \omega_2)$ for $e \geq 1$	24	9
SL_5	3	$L(2\omega_2)$	45	21
SL_4	$\neq 3$	$L(\omega_1 + \omega_2)$	20	5
SL_5	$\neq 3$	same as line above	40	16
$\mathrm{SL}_n / \mu_{(n,3)} (5 \leq n \leq 9)$	3	same as line above	$(n-1)n(n+4)/6$	$(n^3 - 3n^2 - 4n + 6)/6$
SL_5	$\neq 2$	$L(\omega_1 + \omega_3)$	45	21
SL_5	2	same as line above	40	16
SL_9	all	$\wedge^4 k^9$	126	46
$\mathrm{SL}_n / \mu_{(n,3)} (4 \leq n \leq 8)$	$\neq 2, 3$	$S^3 k^n$	$\binom{n+2}{3}$	$\binom{n+2}{3} - n^2 + 1$
$\mathrm{SL}_n / \mu_{(n,3)} (10 \leq n \leq 14)$	all	$\wedge^3 k^n$	$\binom{n}{3}$	$\binom{n}{3} - n^2 + 1$
Spin_5	$\neq 5$	$L(\omega_1 + \omega_2)$	16	6
Spin_7	$\neq 2$	$L(2\omega_3)$	35	14
Spin_7	7	$L(\omega_1 + \omega_3)$	40	19
Spin_{15}	all	$L(\omega_7) (\text{spin})$	128	23
Spin_{17}	all	$L(\omega_8) (\text{spin})$	256	120
Sp_4	$p \neq 0$	$L(\omega_1 + p^e \omega_1)$ for $e \geq 1$	16	6
Sp_8	$\neq 3$	$L(\omega_3) ("^{\wedge 3} ")$	48	12
Sp_{10}	2	same as line above	100	45
D_4	2	$L(\omega_1 + \omega_2)$	48	20
G_2	$\neq 2$	$L(2\omega_2)$	$27 - \varepsilon$	$13 - \varepsilon$

TABLE 5. Some irreducible faithful representations V for a simple algebraic group G over an algebraically closed field k such that $\dim G < \dim V < 2 \dim G$. Combining with Table 4 provides a full list up to graph automorphism. The minimum value for $\dim k[V]^G$ in this table is 5, occurring for $(\mathrm{SL}_4, \neq 3, L(\omega_1 + \omega_2))$ and for $(\mathrm{SL}_4, \neq 2, 3, S^3 k^4)$.

Proof. We apply Theorem 4.1 and verify that the nonzero eigenvalues have multiplicity 1 in each faithful irreducible representation $V = L(\lambda)$ that is not generically free.

The list of $L(\lambda)$ where the nonzero eigenvalues have multiplicity 1 for k of prime characteristic has been obtained in [TZ, Prop. 8], leveraging [Seitz, 6.1] and [ZS].

Alternatively, for each λ , if the irreducible representation over $\mathbb{C}$ with the same highest weight has nonzero eigenvalues of multiplicity 1 (the list of such is known from [Howe, Th. 4.6.3]), then we are done. The remaining λ can be treated in an ad hoc manner.

Yet another way to phrase the proof is to suppose that V has a nonzero weight space with multiplicity 1 and apply [GuL, Cor. 11] to deduce that $G_v(k) = 1$ for generic $v \in V$. This reduces the proof to verifying that nonzero eigenvalues have multiplicity 1 in the faithful irreducible representations that have nonzero but infinitesimal generic stabilizer. There are just four of these, appearing in Table 4. (While the generic stabilizers for these cases are determined later in this paper, at this point we only need to know which representations have infinitesimal generic stabilizers, which is known from combining the results of [GuL] and [GaGu III].) $\square$

Corollary 4.4. *Let V be an irreducible representation of a simple algebraic group G . Then either G has an open orbit in V or there is a dense open subset of V consisting of closed G -orbits.*

Proof. As the G -orbits in V are not changed by replacing G by a quotient, we may assume that G acts faithfully. Suppose G does not have an open orbit in V , i.e., $\dim k[V]^G > 0$. Then by the theorem and Tables 3 and 4, for generic $v \in V$, the group $G_v(k)$ is the k -points of a reductive group, i.e., $(G_v)_{\text{red}}$ is reductive. Thus the quotient $G/(G_v)_{\text{red}}$ is affine [Ri 77], whence the claim by [Po 72]. $\square$

5. STABILIZERS IN GENERAL POSITION

For a vector v in a representation V of an algebraic group G , one can consider separately (1) G_v , the stabilizer of v in G as a closed sub-group-scheme of G ; (2) $G(k)_v$, the stabilizer of v in the abstract group $G(k)$ of k -points of G ; or (3) $\mathfrak{g}_v$, the annihilator of v in the Lie algebra $\mathfrak{g}$ of G . The first carries at least as much information as the other two, in the sense that

$$G_v(k) = G(k)_v \quad \text{and} \quad \text{Lie}(G_v) = \mathfrak{g}_v.$$

So far, we have focused on the notion of stabilizer in general position in the sense of group schemes, i.e., (1). One can also consider versions for (2) and (3), namely:

Definition 5.1. For the group of k -points, $G(k)$, we say that a stabilizer in general position (s.g.p.) exists if there is a subgroup $G(k)_*$ of $G(k)$ and a dense open subset U of V so that for every $u \in U(k)$, there is a $g \in G(k)$ such that $gG(k)_u g^{-1} = G(k)_*$.

An s.g.p. for the Lie algebra $\mathfrak{g}$ of G is a subalgebra $\mathfrak{g}_*$ of $\mathfrak{g}$ such that for every $u \in U(k)$, there is a $g \in G(k)$ such that $(\text{Ad } g)\mathfrak{g}_u = \mathfrak{g}_*$.

Recall that, if $\text{char } k = 0$ and G is reductive, then a stabilizer in general position G_* exists by [Ri 72], see [PoV, §7] for a survey. Indeed, an s.g.p. exists even for the action of reductive G on a smooth affine variety X when $\text{char } k = 0$. However, the same claim does not hold in prime characteristic, even on the level of k -points, as the following example demonstrates.

Example 5.2. The generic stabilizer need not exist when G is semisimple. For example, take $G = \text{Spin}_7 \times \text{SL}_4$, $\text{char } k = 2$, and $V = (\text{spin}) \otimes k^4$. Combining Prop. 6.2.9 and the proof of Lemma 4.6.1 in [GuL] shows that for generic $y \in \mathbb{P}(V)$, the stabilizer $G_y(k)$ is

semisimple with k -points of type $B_1 \times B_1$, but that there is no s.g.p. for the action of $G(k)$ on $\mathbb{P}(V)$. (That is, there is a dense open subset of $\mathbb{P}(V)$ on which the stabilizers in $G(k)$ are all isomorphic, but there is not one where they are conjugate under $G(k)$.) As $G_y(k)$ is semisimple, it follows that the stabilizer G_v of a generic $v \in V$ has the same k -points. Therefore, if the action of $G(k)$ on V has an s.g.p., then so does the action of $G(k)$ on $\mathbb{P}(V)$ (and it would be the same s.g.p.), a contradiction.

The following lemma allows us to leverage the results of Guralnick-Lawther (where the s.g.p. was computed for the abstract group $G(k)$) and Garibaldi-Guralnick (where the generic stabilizer was computed for the Lie algebra $\mathfrak{g}$).

Lemma 5.3. *Let G be a group scheme over an algebraically closed field k acting on an irreducible variety X so that*

- (1) *there is an s.g.p. $G(k)_*$ for the action by the abstract group of points $G(k)$ and*
- (2) *there is an $x \in X(k)$ such that $\dim \mathfrak{g}_x = \dim G(k)_*$.*

Then there is an s.g.p. for the action of the group scheme G on X and it is smooth.

Proof. The set $\{u \in X \mid \dim \mathfrak{g}_u \leq \dim G(k)_*\}$ is open by upper semicontinuity of dimension and it is nonempty by (2). Put U for its intersection with a nonempty open subset of X consisting of u such that $G(k)_u$ is conjugate to $G(k)_*$. For any $u \in U(k)$, we have

$$\dim G(k)_u \leq \dim \mathfrak{g}_u \leq \dim G(k)_*$$

where the first inequality holds because G_u is an algebraic group and the second is by construction of U . As $G(k)_u$ and $G(k)_*$ are conjugate, they have the same dimension, and we have verified that G_u is smooth.

For $u, u' \in U(k)$, there is a $g \in G(k)$ so that $gG_u g^{-1}$ and $G_{u'}$ have the same k -points, by construction of U . As both group schemes are smooth, they agree [Milne, Prop. 3.16]. $\square$

Example 5.4 (Type A_1). Suppose G has type A_1 and V is a faithful irreducible representation with highest weight λ , a natural number. The hypothesis that G is faithful says that $\text{char } k$ does not divide λ (for otherwise V is a Frobenius twist of another representation) and that $G = \text{SL}_2$ if and only if λ is odd.

If $\lambda = 1$, then V is the tautological representation of SL_2 , which has an open orbit ($\dim k[V]^G = 0$), and therefore there is an s.g.p.

If $\lambda = 2$ (so $\text{char } k \neq 2$), then V is the adjoint representation and the stabilizer of a generic element is a maximal torus. We find $\dim k[V]^G = 1$.

If $\lambda \geq 3$, then $\mathfrak{g}_v = 0$ for generic v by [GaGu I, Examples 1.8 and 3.3]. As in [GuL, Th. 2], $G_v(k)$ is finite, and $\neq 1$ only for the cases in Table 4. By Lemma 5.3, there is an s.g.p. for the action of G on V . In this case, $\dim k[V]^G = \dim V - 3$.

In summary, there is an s.g.p. for the action of G on V .

Here is another application of Lemma 5.3.

Lemma 5.5. *For $G = \text{PSp}_{2\ell}$ with $\ell \geq 3$ and V the Weyl module $V(\omega_2)$ or the irreducible module $L(\omega_2)$, an s.g.p. exists and it is smooth.*

Proof. Proposition 5.2.5 of [GuL] shows that an s.g.p. exists for the action of $G(k)$ on $V(\omega_2)$ and on $L(\omega_2)$, and it has dimension 3ℓ . Therefore, in view of Lemma 5.3, it suffices to verify that $\dim \mathfrak{g}_v = 3\ell$ for v generic in $V(\omega_2)$ or $L(\omega_2)$.

View the simply connected cover $\text{Sp}_{2\ell}$ of G as the subgroup of $\text{GL}_{2\ell}$ preserving the alternating bilinear form $s(m, m') := m^\top J m'$ where $J = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$. The group $\text{GSp}_{2\ell}$ of

similarities of s is generated by $\mathrm{Sp}_{2\ell}$ and scalar transformations. Its Lie algebra consists of those $x \in \mathfrak{gl}_{2\ell}$ of the form $\begin{pmatrix} A & B \\ C & \mu I_\ell - A^\top \end{pmatrix}$ for $A, B, C \in \mathfrak{gl}_\ell$ and $\mu \in k$ such that $B^\top = B$ and $C^\top = C$ as in [GaGu III, Example 8.1]. There is a natural exact sequence $1 \rightarrow \mathbb{G}_m \rightarrow \mathrm{GSp}_{2\ell} \rightarrow G \rightarrow 1$ and the corresponding map $\mathfrak{gsp}_{2\ell} \rightarrow \mathfrak{g}$ is surjective. Therefore, it suffices to prove that $\dim(\mathfrak{gsp}_{2\ell})_v = 3\ell + 1$ for generic $v \in V$.

We first treat the case $V = V(\omega_2)$. Let Y be the space of self-adjoint operators with respect to s , i.e., those transformations y of $k^{2\ell}$ such that $s(ym, m) = 0$ for all $m \in k^{2\ell}$. It is a representation of G under the action $\rho(g)y = gyg^{-1}$. The ring of G -invariant polynomial functions $k[Y]^G$ has a linear generator that sends $y \in Y$ to half the trace of Jy , see [GaGu 15, Example 8.5] and the kernel of this linear map is $V(\omega_2)$. Take V_1 to be the subspace of $V(\omega_2)$ consisting of diagonal matrices. The map $G \times V_1 \rightarrow V$ is dominant [GoGu, Cor. 2.10], so we may take for generic $v \in V$ a diagonal matrix $v = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$ where $L \in \mathfrak{gl}_\ell$ is diagonal with entries $(\lambda_1, \dots, \lambda_\ell)$ such that $\sum_{i=1}^\ell \lambda_i = 0$. Note that for $A \in \mathfrak{gl}_\ell$, $[A, L]$ has (i, j) -entry $a_{ij}(\lambda_j - \lambda_i)$, which is zero when $i = j$ and is a nonzero multiple of a_{ij} when $i \neq j$ (because $\ell \geq 3$). In particular, $[A, L] = 0$ if and only if A is diagonal. For $x \in \mathfrak{gsp}_{2\ell}$, we have

$$\mathrm{d}\rho(x)v = [x, v] = \begin{pmatrix} [A, L] & [B, L] \\ [C, L] & [-A^\top, L] \end{pmatrix},$$

whence x is in $(\mathfrak{gsp}_{2\ell})_v$ if and only if A, B , and C are diagonal, proving the claim for $V = V(\omega_2)$.

If $\mathrm{char} k$ does not divide ℓ , then $L(\omega_2) = V(\omega_2)$ and the proof is complete. So assume $\mathrm{char} k$ divides ℓ , in which case the subspace Y_2 of scalar matrices is a G -invariant submodule of $V(\omega_2)$ and $L(\omega_2) = V(\omega_2)/Y_2$. (For this and the previous sentence, compare for example [PrS, esp. Th. 2(iv)].) Re-reading the previous paragraph, we note that we actually proved that $[A, L]$ is scalar if and only if A is diagonal, and therefore we find that the stabilizer in $\mathfrak{gsp}_{2\ell}$ of a generic vector $v \in V(\omega_2)$ is the same as the stabilizer of its image in $L(\omega_2)$, completing the proof. $\square$

When aiming to prove the existence of an s.g.p., the following lemma allows us to focus on representations that are faithful. A shadow of it already appeared in the second paragraph of the proof of Lemma 5.5.

Lemma 5.6. *Let G be a group scheme acting on an irreducible variety X such that a normal closed sub-group-scheme N of G acts trivially on X . If there is an s.g.p. $(G/N)_*$ for the action of G/N on X , then the inverse image of $(G/N)_*$ in G is an s.g.p. for G acting on X .*

Proof. The usual correspondence theorem between closed sub-group-schemes of G containing N and closed sub-group-schemes of G/N as in [Milne, Th. 5.55] shows that, for $x \in X(k)$, $(G/N)_x = G_x/N$. For u in the open subset U of X on which the s.g.p. is defined and $\bar{g} \in (G/N)(k)$ satisfies $\bar{g}(G/N)_u \bar{g}^{-1} = (G/N)_*$, pick $g \in G(k)$ mapping to $\bar{g}$. Then $g^{-1}G_*g$ is a closed sub-group-scheme of G containing N with image $(G/N)_u$, so it is G_u . $\square$

Reducing to a smaller problem. Suppose V is a representation of an algebraic group G and suppose it has an s.g.p. G_* . Set $V_1 := V^{G_*}$, the subspace of elements fixed by G_* . Then, because G_* is an s.g.p., the map

$$(5.7) \quad \psi: G \times V_1 \rightarrow V \quad \text{defined by } \psi(g, v) := gv$$

is dominant, i.e., for generic $v \in V$, the orbit $G(k)v$ meets V_1 . In case $G_* = 1$ (i.e., G acts generically freely, which is from some points of view the typical case), then $V_1 = V$

and these statements are trivial. We remark that more sophisticated results are available, see [LuR] or [Po94, §2.1] for the case $\text{char } k = 0$ and [LöM] for results in arbitrary characteristic.

Note that the same argument works if we replace the role of G_* by an s.g.p. $G(k)_*$ for the group of k -points, or $\mathfrak{g}_*$ for the Lie algebra of G . The same argument also shows that (5.7) is dominant for $V_1 := V^{\mathfrak{h}}$, where $\mathfrak{h}$ is a subspace of $\mathfrak{g}$ with the property that there is an open subset $U \subset V$ such that for every $u \in U$ there is a $g \in G(k)$ such that $(\text{Ad } g)\mathfrak{g}_u \supseteq \mathfrak{h}$.

Roughly speaking, one can “reverse” the observation in the previous two paragraphs to find a subspace V_1 such that a generic $v \in V_1$ is a proxy for a generic element of V . We formalize this observation in a lemma, where we write $\text{Trans}_G(v, V_1)$ for the closed subscheme of G whose R -points are those $g \in G(R)$ such that gv is in $V_1 \otimes R$.

Lemma 5.8. *Let V be a representation of an algebraic group G . If a subspace V_1 of V satisfies*

$$\dim \text{Trans}_G(v_1, V_1) \leq \dim G + \dim V_1 - \dim V$$

for generic $v_1 \in V_1$, then for generic $v \in V$ the orbit Gv meets V_1 . If additionally there is a subgroup H of G such that $G_{v_1} = H$ for generic $v_1 \in V_1$, then H is an s.g.p. for the action of G on V .

Proof. Suppose that v' is also a generic element of V_1 . Then $\text{Trans}_G(v', v_1) \subseteq \text{Trans}_G(v', V_1)$. For the map ψ defined in (5.7), the fiber over v_1 has dimension at most $\dim G + \dim V_1 - \dim V$, i.e., the dimension of $\text{im } \psi$ is at least $\dim V$, whence ψ is dominant. $\square$

6. θ -GROUPS

The pairs $(G, V) = (\text{SL}_8 / \mu_4, \wedge^4 k^8)$, $(\text{HSpin}_{16}, \text{half-spin})$, or $(\text{SL}_9 / \mu_3, \wedge^3 k^9)$ from Table 4 are examples of “ θ -groups” or “Vinberg representations”. They are constructed as follows. Take $\tilde{G}$ to be a split adjoint group of type E_7 , E_8 , or E_8 respectively and set $m = 2, 2$, or 3 . Pick a maximal torus T in $\tilde{G}$ and a set of simple roots $\alpha_1, \dots, \alpha_\ell$ of $\tilde{G}$ relative to T . We define a $\mathbb{Z}/m$ -grading on $\tilde{\mathfrak{g}}$ by setting $\tilde{\mathfrak{g}}_0$ to contain $\mathfrak{t} := \text{Lie}(T)$ and $\tilde{\mathfrak{g}}_i$ to contain those root subalgebras $\tilde{\mathfrak{g}}_\alpha$ such that the height of the root α is congruent to $i \pmod m$. We find that G is a subgroup of $\tilde{G}$ such that $\mathfrak{g}$ is identified with $\tilde{\mathfrak{g}}_0$ and the adjoint action of G on $\tilde{\mathfrak{g}}_1$ is equivalent to the representation V . See [V], [AzBS], [PoV, §8.5], [Le], and [ReLYG] for more on this general family of representations.

If $\text{char } k \neq m$, then it was verified in [GaGu II, §7] that $\mathfrak{g}_v = 0$ for generic $v \in V$. As the s.g.p. exists for the action by the group of k -points $G(k)$ on V and G_v is smooth (because $\dim \mathfrak{g}_v = \dim G(k)_v$), the s.g.p. exists for the action of the group scheme G .

The rest of this section concerns the case $\text{char } k = m$.

Proposition 6.1. *Suppose $\text{char } k = 2$ and (G, V) is either $(\text{SL}_8 / \mu_4, \wedge^4 k^8)$ or $(\text{HSpin}_{16}, \text{half-spin})$. Then the s.g.p. exists for the action of G on V and is $(\mathbb{Z}/2)^r \times \mu_2^r$ for $r = 3$ or 4 , respectively.*

The HSpin_{16} case was proved in Premet’s appendix to [GaGu 17]. We adapt his method to encompass both cases and present it here in a side-by-side proof in order to highlight the similarities between the two cases. The case $G = \text{SL}_8 / \mu_4$ has two minor extra complications, namely that the adjoint group $\tilde{G}$ is not simply connected and that there is an outer automorphism of G arising from conjugation by an element of $\tilde{G}$.

Proof. The E_7 root system is contained in E_8 in the span of $\alpha_1, \dots, \alpha_7$. In the root system of $\tilde{G}$, we find a subsystem of type A_1^ℓ with simple roots $\gamma_1, \dots, \gamma_\ell$. Specifically, using the

notation

$$\begin{matrix} acdefg \\ b \end{matrix}$$

to denote the root $a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 + e\alpha_5 + f\alpha_6 + g\alpha_7$ of E_7 and similarly for E_8 , as in [Bour02, Pl. VI and VII]. We take $\gamma_1, \dots, \gamma_7$ to be

$$\begin{matrix} 010000 & 000000 & 000100 & 012100 & 000001 & 012221 & 234321 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 \end{matrix}$$

and in the case of E_8 we set also $\gamma_8 = \frac{2465432}{3}$. (These choices agree with the ones made in the proof of Prop. 5.1.1 in [GuL].) The γ_i generate a sublattice of T^* , corresponding to a quotient $\overline{T}$. We put $H := \ker[T \rightarrow \overline{T}]$, a finite group scheme that is the Cartier dual of $T^*/\overline{T}^*$, i.e., $H \cong \mu_2^r$ for r as in the statement.

We can describe H explicitly as follows. As the root system of $\tilde{G}$ is simply laced, we identify it with its inverse root system and roots with coroots. The coroot γ_i defines a cocharacter $\omega_i: \mathbb{G}_m \rightarrow T$ with differential $d\omega_i: k \rightarrow \mathfrak{t}$ such that $d\omega(1) = h_{\gamma_i}$, an element of the Cartan basis of $\tilde{\mathfrak{g}}$ corresponding to the coroot γ_i . The composition $\gamma_j \circ \omega_i$ is either trivial ($i \neq j$) or the squaring map ($i = j$), so ω_i embeds a copy of μ_2 in H . Varying i , we find all of H . On the level of Lie algebras, $\mathfrak{t}$ is identified with $T^* \otimes k$ and $\mathfrak{h} := \text{Lie}(H)$ is the span of the h_{γ_i} with $\dim \mathfrak{h} = r$. (Note that when $\tilde{G}$ has type E_7 , T^* is the weight lattice, and $h_{\alpha_2} + h_{\alpha_5} + h_{\alpha_7} = 0$ in $\mathfrak{t}$.)

To determine the centralizers $C_G(H)$ and $C_G(\mathfrak{h})$, we consider instead the bilinear pairing on the roots. For M the Cartan matrix and η the matrix whose i th row is the coefficients of γ_i , the product ηM viewed as a matrix with entries in k has right kernel spanned by the rows of η . That is, if α is in the root lattice Q and $\langle \alpha, \gamma_i \rangle$ is divisible by 2 for all γ_i , then α is in the span of the γ_i 's and $2Q$. One finds that the roots α of even height do not lie in this span, i.e., have odd inner product with some γ_i . Therefore, the 1-parameter subgroup G_α does not belong to either centralizer, because there is some γ_i such that α and γ_i have odd inner product. It follows that T is the identity component of $C_G(H)$ and $C_G(\mathfrak{h})$.

Each γ_i has odd height, and we set V_1 to be the 2ℓ -dimensional subspace of $\tilde{\mathfrak{g}}_1$ spanned by $\tilde{\mathfrak{g}}_{\gamma_i}$ and $\tilde{\mathfrak{g}}_{-\gamma_i}$ for all i . Let $v \in V_1$ be generic. (We remark that at this point we have observed that $\mathfrak{h} = \mathfrak{g}_v$.) For any 2-by-2 matrix $A = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$, A^2 is the scalar matrix with xy on the diagonal. It follows that the elements $v^{[2]^e} \in \mathfrak{h}$ for $e \geq 1$ span $\mathfrak{h}$, so the stabilizer G_v centralizes $\mathfrak{h}$ and normalizes $C_G(\mathfrak{h})^\circ$, i.e., G_v is contained in $N_G(T)$.

For $v' \in V_1$ also generic, the transporter $\{g \in G \mid gv = v'\}$ consists of elements normalizing T . It follows that $\text{Trans}_G(v_1, V_1) \subseteq N_G(T)$, so the transporter has dimension at most ℓ . For both choices of (G, V) we have $\dim V = \dim G + \dim V_1 - \ell$, so Lemma 5.8 applies.

We now compute the generic stabilizer G_v . As $\text{char } k = 2$, the simple reflections in the Weyl group are elements of $N_{\tilde{G}}(T)$ of order 2, giving an expression for $N_{\tilde{G}}(T)$ as a semidirect product of T and the Weyl group. An elementary argument as in [GaGu17, p. 552] shows that for each i $\tilde{G}_v$ contain an element whose image in the Weyl group $N_{\tilde{G}}(T)/T$ is the reflection in the root γ_i and that these elements account for all the cosets of T in $\tilde{G}_v$. That is, $\tilde{G}_v$ is a semi-direct product $H \rtimes (\mathbb{Z}/2)^\ell$.

The centralizer of G in $\tilde{G}$ is G itself— G is a maximal rank subgroup by construction. If $\tilde{G}$ has type E_8 , then there is no element of $\tilde{G}$ that normalizes G and such that conjugation is an outer automorphism of G because in that case we would find inside the representation $\tilde{\mathfrak{g}}$ of G both half-spin representations but there is only one; in this case $N_{\tilde{G}}(G) = G$. If $\tilde{G}$ has type E_7 , then $N_{\tilde{G}}(G) = G \rtimes \mathbb{Z}/2$, because -1 is an element of the Weyl group of E_7 (so is given by conjugation by an element of $N_{\tilde{G}}(T)$) but not the Weyl group of A_7 .

For each of the elements of $(\mathbb{Z}/2)^\ell$ in the Weyl group of $\tilde{G}$ generated by the reflections in the γ_i , one checks whether it normalizes the roots of G . In both cases, one finds a subgroup of order 16, i.e., $(\mathbb{Z}/2)^4$. In case $\tilde{G}$ has type E_8 , this subgroup belongs to G by the previous paragraph, proving the claim that G_v is $(\mathbb{Z}/2)^4 \times \mu_2^4$. In case $\tilde{G}$ has type E_7 , again by the preceding paragraph, one finds that G_v is $(\mathbb{Z}/2)^3 \times \mu_2^3$. In either case, G_v does not depend on the choice of generic element $v \in V_1$. Lemma 5.8 gives that G_v is the s.g.p. for the action of G on V . $\square$

We now treat the remaining case. The argument is similar to the preceding.

Proposition 6.2. *Suppose $\text{char } k = 3$ and $(G, V) = (\text{SL}_9 / \mu_3, \wedge^3 k^9)$. Then the s.g.p. exists for the action of G on V and is $(\mathbb{Z}/3)^2 \times \mu_3^2$.*

Proof. We follow the outline of the proof of the previous proposition, where $\tilde{G} = E_8$, replacing throughout the prime 2 with 3. We find a subsystem of type A_2^4 with simple roots $\gamma_1, \dots, \gamma_8$:

$$\begin{array}{cccccccc} 1000000 & 0100000 & 0001000 & 0000100 & 0000000 & 1232100 & 0000001 & 2465431 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 3 \end{array}$$

Note that γ_i, γ_{i+1} span a subsystem of type A_2 for $i = 1, 3, 5, 7$. The sublattice of T^* generated by the γ_i s defines a quotient $\bar{T}$ of T , and we set $H := \ker[T \rightarrow \bar{T}]$. We find $H \cong \mu_3^2$, generated by cocharacters $\alpha_1 + 2\alpha_3$ and $\alpha_5 + 2\alpha_6$.

As in the preceding proof, $C_G(H)$ and $C_G(\mathfrak{h})$ have identity component T . We set V_1 to be the 12-dimensional subspace of V spanned by $\tilde{\mathfrak{g}}_\beta$ for β a root in the A_2^4 subsystem of height congruent to 1 mod 3, i.e., $\beta = \gamma_i, \gamma_{i+1}$, or $-\gamma_i - \gamma_{i+1}$ for $i = 1, 3, 5, 7$. Lemma 5.8 applies.

As the 3-by-3 matrix of the cyclic permutation (1 2 3) has determinant 1, it follows that the element $s_{\gamma_i} s_{\gamma_{i+1}}$ of order 3 in the Weyl group of $\tilde{G}$ is the image of an element of order three in $N_{\tilde{G}}(T)$. An elementary argument as in the preceding proof shows that $N_{\tilde{G}}(T)/T$ is the subgroup of the Weyl group generated by these simple reflections, isomorphic to $(\mathbb{Z}/3)^4$. The subgroup of this stabilizing the roots of G is $(\mathbb{Z}/3)^2$. It follows, therefore, that G_v is isomorphic to $(\mathbb{Z}/3)^2 \times \mu_3^2$. $\square$

7. INFINITESIMAL STABILIZERS

Table 4 lists four representations that have infinitesimal generic stabilizers. That is, for generic $v \in V$ they have $G_v(k) = 1$ by [GuL] and $\mathfrak{g}_v \neq 0$ by [GaGu II]. In this section, we show that these representations have an s.g.p. and determine it as a group scheme.

Proposition 7.1. *If $(G, \text{char } k, V)$ is (i) $(\text{Spin}_5, 5, L(\omega_1 + \omega_2))$ or (ii) $(\text{Sp}_8, 3, L(\omega_3))$, then V has a s.g.p., which is isomorphic to (i) μ_5 or (ii) $\mu_3 \times \mu_3$ respectively.*

Proof. Put $p := \text{char } k$. Fix a pinning for G , which includes a maximal torus T and a Chevalley basis for $\mathfrak{g}$. As G is simply connected, the cocharacter lattice $\text{Hom}(\mathbb{G}_m, T)$ is identified with the coroot lattice for the root system of G . Put H for the subtorus of T generated by $\text{im } \beta^\vee$ for $\beta^\vee$ as follows:

$$(7.2) \quad (i) \quad \alpha_1^\vee + 2\alpha_2^\vee \qquad (ii) \quad \alpha_1^\vee + \alpha_4^\vee, \quad \alpha_2^\vee + \alpha_4^\vee.$$

Because T normalizes H , it also normalizes $V^{\mathfrak{h}}$, the subspace of V annihilated by $\mathfrak{h}$, and in particular $V^{\mathfrak{h}}$ is a sum of weight spaces. (Recall that all weights of V have multiplicity 1.) We find that $V^{\mathfrak{h}}$ has weights:

- (i) $2\omega_1 - \omega_3, -\omega_1 + 3\omega_2$
- (ii) $2\omega_1 - \omega_2 - \omega_3 + \omega_4, \omega_3, \omega_1 + \omega_2 - \omega_4, -\omega_1 + 2\omega_2 - 2\omega_3 + \omega_4.$

and their negatives.

Note that $\dim G - \text{rank } G = \dim V - \dim V^h$. As in the proof of Lemma 4.1 of [GaGu II], for generic $v \in V^h$, we have $\text{Trans}_G(v, V^h) \subseteq N_G(\mathfrak{h}) \subset N_G(T)$, so the transporter has dimension at most $\text{rank } G$ and Lemma 5.8 applies with $V_1 := V^h$. Moreover, there is a dense open subset U of V^h such that G_u is a closed subgroup of $N_G(T)$ for each $u \in U(k)$. On the other hand, for generic $v \in V$, $G_v(k) = 1$ by [GuL, §2.7], which shows that (after possibly shrinking U), G_u is a closed subgroup of T for each $u \in U(k)$.

That is, G_u is the intersection of $\ker \omega|_T$ as ω varies over weights displayed above. The quotient of the weight lattice T^* by the sublattice generated by those weights is (i) $\mathbb{Z}/5$ or (ii) $\mathbb{Z}/3 \times \mathbb{Z}/3$ respectively, proving the claim. $\square$

Proposition 7.3. *Let $(G, \text{char } k, V) = (\text{SL}_4, p, L(p^e \omega_1 + \omega_2))$. If*

- (i) *p is odd and $e \geq 1$ or*
- (ii) *$p = 2$ and $e \geq 2$,*

then V has an s.g.p., which is isomorphic to (i) μ_{p^e} or (ii) $\mu_{p^{e+1}}$ respectively.

Proof. Put $q := p^e > 1$. Identify the natural representation k^4 with $L(\omega_1)$. Put W_1 (resp. W_2) for the subspace of vectors with the last (resp. first) two coordinates zero, so $k^4 = W_1 \oplus W_2$. Put $\hat{J}$ for the subgroup of SL_4 normalizing or interchanging W_1 and W_2 , equivalently, the normalizer of $W_1 \otimes W_2$ in $L(\omega_2) = \wedge^2(W_1 \oplus W_2)$. Its identity component consists of block diagonal matrices with diagonal elements $g_1, g_2 \in \text{GL}_2(k)$ such that $\det g_1 \det g_2 = 1$. Put

$$U := (W_1 \oplus W_2)^{[q]} \otimes (W_1 \otimes W_2) \subseteq L(\omega_1)^{[q]} \otimes L(\omega_2) = V.$$

We verify the hypotheses of Lemma 5.8, with $V_1 := U$. A generic element of U is $u = \sum x_i \otimes y_i$ with the y_i a basis for $W_1 \otimes W_2$. If $gu' = u$ for some $u' \in U$, then g preserves $W_1 \otimes W_2$ and so the dimension of the transporter is at most

$$\dim \hat{J} = 7 = \dim G + \dim V_1 - \dim V,$$

as required. We know by [GuL, Prop. 2.8.3] that the stabilizer H of a generic $v \in U$ is infinitesimal, and in particular is connected and contained in the identity component J of $\hat{J}$.

Let T denote the copy of $\mathbb{G}_m$ in J such that $t \in k^\times$ acts on W_1 as multiplication by t and on W_2 as multiplication by t^{-1} . The group J is evidently generated by T and the subgroup $\text{SL}_2 \times \text{SL}_2$ of block diagonal matrices in SL_4 mentioned in the first paragraph of the proof. The two groups overlap in a copy of μ_2 (diagonally embedded in $\text{SL}_2 \times \text{SL}_2$) and we find that J is $(T \times \text{SL}_2 \times \text{SL}_2)/\mu_2$.

For R a commutative k -algebra write $[a, b, c]$ for the image of $(a, b, c) \in T(R) \times \text{SL}_2(R) \times \text{SL}_2(R) \in J(R)$. We remark that there are two “obvious” copies of μ_4 in J , namely a $\mu_4 < T$ and the center of SL_4 . Indeed, suppose $\zeta \in \mu_4(R)$ has order 4. The element $[\zeta, 1, 1] \in J$ belongs to $\mu_4(R) \subset T(R)$, whereas $[\zeta, \zeta^2, 1] = [\zeta^3, 1, \zeta^2]$ belongs to the center of SL_4 .

The kernel K of the action. Let us determine the group scheme $K := \ker[J \rightarrow \text{GL}(U)]$. If $[a, b, c] \in K(R)$, then b, c belong to $Z(\text{SL}_2)(R) = \mu_2(R)$ and in particular $c^2 = 1$. If $c \neq 1$, then we have $[a, b, c] = [a, b, c][c, c, c] = [ac, bc, 1]$ in J , so every element in $K(R)$ is of the form $[a, b, 1]$ for some $b \in \mu_2(R)$.

On $W_1^{[q]} \otimes W_1 \otimes W_2$, the element $[a, b, 1]$ acts by a scalar $a^q b^{q+1}$. For the other summand in U , the scalar is $a^{-q} b$. So $[a, b, 1]$ is in $K(R)$ if and only if

$$(7.4) \quad a^q b^{q+1} = 1 = a^{-q} b.$$

In this way, we find a copy of μ_q contained in K as elements $[a, 1, 1]$ for $a \in \mu_q(R)$. This subgroup contains all the elements of $K(R)$ of the form $[a, 1, 1]$.

If p is odd, then (7.4) reduces to $a^q = 1 = ba^{-q}$. That is, $b = 1$ and we find that $K = \mu_q \subset T$.

If $p = 2$, then (7.4) reduces to $a^q b = 1 = a^{-q} b$, so $a^{2q} = 1$ and $b = a^q$. That is, there is an isomorphism $\mu_{2q} \rightarrow K$ defined on R -points via $a \mapsto [a, a^q, 1] = [a^{q+1}, 1, a^q]$ and $K \cap T = \mu_q$.

Alternatively, consider a maximal torus $S := T \times \mathbb{G}_m \times \mathbb{G}_m \subset T \times \mathrm{SL}_2 \times \mathrm{SL}_2$ where $\mathbb{G}_m$ stands for the diagonal matrices in SL_2 . The image of S in J contains the kernel K . Writing $(1, 0, 0)$ for a fundamental weight on T and similarly for the other two components of S , the weights of U are

$$(7.5) \quad (q, \pm q \pm 1, \pm 1) \quad \text{and} \quad (-q, \pm 1, \pm q \pm 1)$$

where the signs may be chosen independently. The kernel of S on U is the intersection of the kernels of these weights; to say the same thing differently, the image $\bar{S}$ of S in J/K has character lattice $\bar{S}^*$ in S^* generated by the weights of U . When $p = 2$, the lattice has index $4q$ in S^* and basis $(q, 1, 1), (0, 2, 0), (0, 0, 2)$.

The center of $\mathrm{Lie}(J/K)$. Put $\bar{T}$ for the central torus in J/K , the image of T . If p is odd, then the center of $\mathrm{SL}_2 \times \mathrm{SL}_2$ injects into J/K , and we find that $Z(J/K) \cong \bar{T} \times \mu_2$. If $p = 2$, then the image of the center of $\mathrm{SL}_2 \times \mathrm{SL}_2$ in J/K is a copy of μ_2 , which is contained in $\bar{T}$, so $Z(J/K) = \bar{T}$. In either case, $\mathrm{Lie}(Z(J/K)) = \mathrm{Lie}(\bar{T})$.

Verification that $H = K$. Trivially the generic stabilizer $H = J_v$ contains K , and we claim they are equal. To see this, we verify that J/K acts generically freely on U . Indeed, $H/K = (J/K)_v$, and $(H/K)(k) = 1$ because $H(k) = 1$ [Milne, Prop. 5.47]. Consequently, we are reduced to showing that $\mathrm{Lie}(J/K)_v = 0$.

Suppose for the moment that

$$(7.6) \quad \dim U^x \leq 8 \quad \text{for nilpotent or semisimple } x \in \mathrm{Lie}(J/K) \setminus \mathrm{Lie}(\bar{T}).$$

As $\dim(\mathrm{Ad} J/K)x \leq \dim J/K - \mathrm{rank} J/K = 4$, we find that

$$\dim(\mathrm{Ad} J/K)x + \dim U^x < \dim U,$$

whence $\mathrm{Lie}(J/K)_v = \mathrm{Lie}(\bar{T})_v$ by [GaGu I, Lemma 1.6(2)]. As J/K is reductive and $\mathrm{Lie}(J/K)_v$ is contained in the center of $\mathrm{Lie}(J/K)$, it follows easily that $\mathrm{Lie}(J/K)_v = 0$, compare [GaGu I, Lemma 1.7].

Verification of (7.6). We now go back and verify claim (7.6). Suppose first that x is semisimple. There is a maximal torus of J/K whose Lie algebra contains x [Hu 67, Th. 13.3, Rem. 13.4]; since all maximal tori are conjugate we may assume that x is in $\mathrm{Lie}(\bar{S})$. If p is odd, then $\mathrm{Lie}(J/K) \cong \mathfrak{t} \oplus \mathfrak{so}_4$ (because $T/\mu_q \cong T$) where $t \in \mathfrak{t}$ acts on $W_i^{[q]} \otimes W_1 \otimes W_2$ via $(-1)^{i+1}t$ and $\mathfrak{so}_4$ acts on it as a sum of 2 copies of its natural representation. Write $x = t + x_0$ for $t \in \mathfrak{t}$ and $x_0 \in \mathfrak{so}_4$. As x is not central, $x_0 \neq 0$, so the largest eigenspace of x_0 on the natural representation has dimension at most 2. It follows that $\dim(W_i^{[q]} \otimes W_1 \otimes W_2)^x \leq 4$, whence (7.6).

Suppose now that x is semisimple and $p = 2$. For each weight of U on S as in (7.5), we express it in terms of the basis for the sublattice $\bar{S}^*$ of S^* and reduce mod 2 to find the weights of U on $\mathrm{Lie}(\bar{S})$; these are

$$(1, 0, 0), \quad (1, 0, 1), \quad (1, 1, 0), \quad (1, 1, 1),$$

each with multiplicity 4. If $\dim U^x > 8$, then at least three of these vanish on x . But any three of these are linearly independent, so $x = 0$, a contradiction, verifying (7.6).

Suppose now that x is nilpotent. Then x is the image of an element in

$$\mathfrak{so}_4 = \text{Lie}((\text{SL}_2 \times \text{SL}_2)/\mu_2) \subset J/\mu_q,$$

and x acts on U as on 4 copies of the natural representation k^4 . Since a nonzero nilpotent in $\mathfrak{so}_4$ has kernel of dimension at most 2 on the natural module, (7.6) follows. $\square$

In the previous proposition when $p = 2$, SL_4 does not act faithfully on V . To address this, we provide:

Corollary 7.7. *If $(G, \text{char } k, V) = (\text{SL}_4/\mu_2, 2, L(2^e\omega_1 + \omega_2))$ for some $e \geq 2$, then V is a faithful representation of G and has an s.g.p. that is isomorphic to μ_{2^e} .*

Proof. SL_4 acts on V with kernel μ_2 and generic stabilizer isomorphic to $\mu_{2^{e+1}}$ (Prop. 7.3), so Lemma 5.6 gives that SL_4/μ_2 has an s.g.p. and it is isomorphic to $\mu_{2^{e+1}}/\mu_2 \cong \mu_{2^e}$. $\square$

8. EXISTENCE OF AN S.G.P.: PROOF OF THEOREM 1.1

We will now prove Theorem 1.1, where V is an irreducible representation of a simple algebraic group G . Suppose first that G acts faithfully on V . The case $V = L(\tilde{\alpha})$ was treated in Lemma 3.1. If G acts generically freely on V , then there is nothing to prove, so assume $G_v \neq 1$ for generic v .

Consider the case $\dim V > \dim G$. By Theorem 4.1(1), we may assume that $(G, \text{char } k, V)$ is listed in Table 4.

In the cases where G_v is finite étale for generic v , [GuL] shows that the s.g.p. exists for the action by $G(k)$. Since $\mathfrak{g}_v = 0$ for generic v , it follows that there is an s.g.p. for the action of G (Lemma 5.3).

For $(\text{HSpin}_{16}, \text{any, half-spin})$ see [GaGu 17, Th. 1.2] or Prop. 6.1. For $(\text{SL}_8/\mu_4, 2, \wedge^4 k^8)$ and $(\text{SL}_9/\mu_3, 3, \wedge^3 k^9)$, see Propositions 6.1 and 6.2 respectively. In all three cases, $G_* = (\mathbb{Z}/p)^r \times \mu_p^r$ for $p := \text{char } k$ and some $r > 1$, so the identity component of G_* is μ_p^r . The proofs in §6 show that in each case μ_p^r is contained in a torus of G .

The four cases in Table 4 where $G_v(k) = 1$ were treated in §7. In each case, G_* is connected and contained in a maximal torus of G .

We note that in all of these cases with $\dim V > \dim G$, the s.g.p. G_* is a finite group scheme. For the third claim, since G_* is finite, its identity component G_*° is non-trivial exactly when G_* is not smooth; those cases are covered in the two preceding paragraphs. Since G_*° is contained in a maximal torus T of G , we have $\text{Lie}(G_*) = \text{Lie}(G_*^\circ) \subseteq \text{Lie}(T)$.

Consider now the cases in Table 3. If $\dim V/G \leq 1$, then there is an open G -orbit in $\mathbb{P}(V)$ [BeGuL, §6], hence an s.g.p. exists. For $(\text{PSp}_{2\ell}, \text{any, } L(\omega_2))$, the existence of an s.g.p. was established in Lemma 5.5. The s.g.p. for the representation $(\text{Spin}_{13}, \text{any, spin})$ was calculated in [GaGu 17, §8,9] and [GuL, Prop. 5.2.9], see Remark 4.2.

There is one final case from Table 3, which we treat in the following lemma.

Lemma 8.1 (natural representation of F_4). *Let G be a group of type F_4 , and let V be its “natural” Weyl module $V(\omega_4)$ of dimension 26 or the irreducible quotient $L(\omega_4)$ of the Weyl module. Then the s.g.p. exists for G acting on V , and it is isomorphic to Spin_8 .*

Proof. Suppose first that V is the Weyl module. The group G can be viewed as the (algebraic) group of automorphisms of an Albert algebra J , where V is the codimension-1 subspace of elements of trace zero. See [Pe 18] and [Jac 68] for background on Albert algebras.

As a vector space, J can be written as the set of hermitian 3-by-3 matrices with entries in an octonion algebra. The set E of diagonal matrices is a cubic étale algebra, and the sub-group-scheme of G fixing E elementwise is isomorphic to Spin_8 , by the arguments in [KnMRT, §38] and [Jac 71, Th. 6]. (Although stated under the hypothesis that $\mathrm{char} k \neq 2, 3$, the arguments go through without this hypothesis with only cosmetic changes. Alternatively, in case $\mathrm{char} = 2$ or 3 , one can check this on the level of k -points and use a computer to verify that the generic stabilizer in $\mathfrak{g}$ has dimension at most 28.)

A generic element $v \in V$ generates a cubic étale subalgebra E_v of J . (This is essentially when $\mathrm{char} k \neq 2$, because in that case every element generates a commutative associative subalgebra, and properties of the generic minimal polynomial show that a generic element generates a separable subalgebra of degree 3. When $\mathrm{char} k = 2$, the same reasoning works by invoking [McC, Prop. 1].) Therefore the stabilizer G_v of v is the subgroup of G fixing E_v elementwise. As k is algebraically closed, E_v is isomorphic to E as a k -algebra, and so is conjugate under G to E by Jacobson's Embedding Theorem (which, in this generality, is [Pe 15, §4.12]). This proves the claim for the Weyl module.

The Weyl module only fails to be irreducible when $\mathrm{char} k = 3$ [Lü]. In that case, the irreducible representation is the quotient of V by the span of the identity element of J . For v generic in the irreducible quotient, any inverse image of it in the Weyl module again generates a cubic étale subalgebra of J , and the Weyl module case again shows that the s.g.p. exists for the action of G . $\square$

We have now accounted for all of the representations in Table 3, proving Theorem 1.1 under the assumption that V is faithful. So drop the assumption that V is faithful and put $N := \ker[G \rightarrow \mathrm{GL}(V)]$. Then G/N is simple (Lemma 2.1) and acts faithfully on V , so it has an s.g.p. $(G/N)_*$. The inverse image of $(G/N)_*$ in G is an s.g.p. for the action of G (Lemma 5.6). If $\dim V > \dim G$, then since V is irreducible, G acts nontrivially on V and N is a finite group scheme. We conclude that $\dim G_* = \dim N + \dim(G/N)_* = 0$, i.e., G_* is also finite. If additionally N is central, then N is contained in every maximal torus T of G and there is a bijection between maximal tori of G and G/N given by $T \leftrightarrow T/N$ [BoTi, Th. 2.20(ii)]. Therefore, the inverse image in G of any maximal torus of G/N containing $(G/N)_*$ is a maximal torus of G containing G_* . This completes the proof of Theorem 1.1.

9. THE S.G.P. IS COMMUTATIVE FOR LARGE V

Building on what has gone before, we easily obtain the following result concerning when the s.g.p. is commutative.

Proposition 9.1. *Let V be a faithful and irreducible representation of a simple algebraic group G . If $\dim V > \dim G + 1$, then either*

- (1) *for generic $v \in V$ the stabilizer G_v is a commutative group scheme or*
- (2) *$(G, \mathrm{char} k, V) = (\mathrm{SL}_9 / \mu_3, 2, \wedge^3 k)$, up to graph automorphism.*

Proof. Assume $G_v \neq 1$, for otherwise there is nothing to prove. By Theorem 4.1(1), up to graph automorphism $(G, \mathrm{char} k, V)$ belongs to Table 4. In that table, only four rows have G_v non-commutative. (This claim relies on the results of §6 and §7.) Of those four, only the row $(\mathrm{SL}_9 / \mu_3, 2, \wedge^3 k^9)$ has dimension at least $\dim G + 1$. $\square$

10. SMOOTHNESS: PROOF OF THEOREM 1.2

We now address the question of whether the group scheme G_v stabilizing a generic $v \in V$, is smooth. Specifically, we prove Theorem 1.2.

We may assume that the stabilizer G_v of a generic $v \in V$ is not the trivial group scheme. If $V = L(\tilde{\alpha})$, then the generic stabilizer has identity component a maximal torus (Lemma 3.1). Therefore, by Theorem 4.1, we may assume that, up to graph automorphism, $(G, \text{char } k, V)$ belongs to Table 3 or 4.

We complete the proof of Theorem 1.2 by noting that the representations in Table 3 have G_v smooth except for G_2 in characteristic 2. For SO_n with $n \geq 5$, when n is odd or $\text{char } k \neq 2$, the stabilizer of an anisotropic vector in the tautological representation k^n is SO_{n-1} . The stabilizers in spin or half-spin representations of groups of type B and D are determined in [GaGu 17].

The natural representation of F_4 is treated in Lemma 8.1. (For $\text{char } k \neq 3$, the smoothness was established by different means in [Stw, Th. 3.2].) One can check easily the cases of the natural representations of SL_n and $\text{Sp}_{2\ell}$. For SL_n acting on $S^2 k^n$ with $\text{char } k \neq 2$ or $\wedge^2 k^n$ with n even, the stabilizer of a generic element is the definition of the special orthogonal or symplectic group. If n is odd for $\wedge^2 k^n$, it is the stabilizer of a degenerate form but it is straightforward to compute that it is smooth.

Here are two techniques that leverage our knowledge of the stabilizer $G(k)_v$ on the level of k -points. First, G_v is smooth if and only if $\dim \mathfrak{g}_v = \dim G(k)_v$. On the other hand, we may view $\mathfrak{g}$ as a representation of the abstract group $G(k)_v$, i.e., the k -points of the smooth subgroup $(G_v)_{\text{red}}$ of G_v . The group $G(k)_v$ normalizes the Lie algebra $\mathfrak{g}_v$, which contains $\mathfrak{h} := \text{Lie}((G_v)_{\text{red}})$ as a $G(k)_v$ -invariant subalgebra. Moreover, $\dim \mathfrak{h} = \dim G(k)_v$. By hypothesis, $\mathfrak{g}$ acts faithfully on V , so $\mathfrak{g}_v \neq \mathfrak{g}$ (otherwise $\mathfrak{g}$ would act trivially) and $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_v = 0$ (if a nonzero central element of $\mathfrak{g}$ annihilates a nonzero vector in V , it annihilates V). This provides many constraints on $\mathfrak{g}_v$.

Second, for any particular characteristic $p = \text{char } k$, we may construct $\mathfrak{g}$ and V over $\mathbb{F}_p$. For every field $k' \supseteq \mathbb{F}_p$ and every $v' \in V \otimes k'$, we have $\dim \mathfrak{g}_v \leq \dim \mathfrak{g}_{v'}$. In particular, it suffices to find such a v' , say with $k' = \mathbb{F}_{p^3}$, such that $\dim \mathfrak{g}_{v'} = \dim G(k)_v$; such a v' can be sought using a computer as described in [GaGu II, §3].

The next result concerns the “natural” representation of a group G of type G_2 , which has highest weight ω_2 . The Weyl module $V(\omega_2)$ is 7-dimensional and has a nonzero G -invariant quadratic form q , see for example [GaN, §4.4]. One can argue as in [FH, §23] or [SpV, 1.6.4, 1.7.3, 2.2.4] that the orbit of the highest weight vector is the set of nonzero v such that $q(v) = 0$. The stabilizer of v in G has codimension 1 in a parabolic subgroup, and we denote it by $A_1 U_5$, where U_5 stands for the unipotent radical of the parabolic subgroup. The representation $V(\omega_2)$ is irreducible if and only if $\text{char } k \neq 2$.

Lemma 10.1 (natural representation of G_2). *For G of type G_2 and $V = L(\omega_2)$, the stabilizer G_v is smooth (resp. reductive) for generic $v \in V$ if and only if $\text{char } k \neq 2$. If $\text{char } k = 2$, then $G(k)$ acts transitively on the nonzero vectors in V , G_v is not smooth, and $(G_v)_{\text{red}} = A_1 U_5$.*

Proof. Over $\mathbb{Z}$, we take an ordered basis of the 7-dimensional Weyl module $V(\omega_2)$ consisting of weight vectors v_μ for the weights $\mu = 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1, 0, -\alpha_1, -(\alpha_1 + \alpha_2), -(2\alpha_1 + \alpha_2)$ respectively and such that the root element e_{α_1} and corresponding 1-parameter subgroup $x_{\alpha_1} : \mathbb{G}_a \rightarrow G_2$ corresponding to the root α_1 are given by the matrices

$$(10.2) \quad e_{\alpha_1} = \begin{pmatrix} 0 & -1 & & & & & \\ & 0 & & & & & \\ & & 0 & 1 & & & \\ & & & 0 & 2 & & \\ & & & & 0 & & \\ & & & & & 0 & -1 \\ & & & & & & 0 \end{pmatrix} \quad \text{and} \quad x_{\alpha_1}(t) = \exp(te_{\alpha_1}) = \begin{pmatrix} 1 & -t & & & & & \\ & 1 & & & & & \\ & & 1 & t & t^2 & & \\ & & & 1 & 2t & & \\ & & & & 1 & & \\ & & & & & 1 & -t \\ & & & & & & 1 \end{pmatrix}$$

as in the proof of [GuL, Prop. 5.2.14].

If $\text{char } k \neq 2$, then V is the base change to k of the Weyl module, and the stabilizer G_v is SL_3 . This can be seen from the further explicit calculations in the proof of [GuL, Prop. 5.2.14] or as in [Stw] or by identifying V with the space of trace zero octonions as in [KnMRT, p. 507, Exercise 6c].

If $\text{char } k = 2$, then V is obtained from the Weyl module over k by modding out by the span of v_0 . The action of G on $L(\omega_2)$ preserves the alternating bilinear form obtained from q on $L(\omega_2)$ and so gives an inclusion $G \rightarrow \text{Sp}_6$. For any finite field K of characteristic 2, one has $\text{Sp}_6(K) = G(K) \text{Sp}_6(K)_v$ for any nonzero $v \in L(\omega_2)$ [LiPS]. The same factorization of Sp_6 therefore holds over the algebraic closure of $\mathbb{F}_2$ and so over the algebraically closed field k of characteristic 2. The transitivity of the action for $G(k)$ now follows from the same transitivity for $\text{Sp}_6(k)$.

Here is an alternate argument that $G(k)$ acts transitively. For each $y \in V$, pick $x \in V(\omega_2)$ such that $x \mapsto y$. One can argue (e.g., by interpreting the Weyl module as the trace zero subspace of the octonions) that the G -invariant quadratic from q on the Weyl module is not zero on v_0 , so by scaling q we may assume that $q(v_0) = 1$. For each $\lambda \in k$, we have $q(x + \lambda v_0) = q(x) + \lambda^2$, so there is a unique choice of x such that $x \mapsto y$ and $q(x) = 0$. Since $G(k)$ has two orbits on the hypersurface $q = 0$, it has two orbits on $L(\omega_2)$. Note that this argument shows that, on the level of k -points, the stabilizer agrees with the stabilizer of the highest weight vector, i.e., is $A_1 U_5$. (Or see [GuL, Prop. 5.2.14].)

It remains to verify that the stabilizer is not smooth. One can read off the action of elements of a Chevalley basis on V by writing them as matrices as we have done above and deleting the 4th row and column. By the transitivity of $G(k)$, we may pick the highest weight vector $v = v_{2\alpha_1 + \alpha_2}$ as a generic vector. The Lie algebra stabilizer $\mathfrak{g}_v$ is normalized by the maximal torus T in G underlying these calculations (because $Tv \subseteq kv$), so $\mathfrak{g}_v$ is a sum of $\mathfrak{g}_v \cap \mathfrak{t}$ and those root subalgebras $\mathfrak{g}_\alpha$ that belong to $\mathfrak{g}_v$. We note from (10.2) that e_{α_1} annihilates the image of $v_{-\alpha_1}$ in V . Since α_1 and $2\alpha_1 + \alpha_2$ are both short roots, they are in the same orbit under the Weyl group, and it follows that $e_{-(2\alpha_1 + \alpha_2)}$ annihilates v , so $\dim \mathfrak{g}_v \geq 9$. One can check that this is all of $\mathfrak{g}_v$ by verifying that each of the remaining four root subalgebras $\mathfrak{g}_\alpha$ for negative α do not annihilate v or by using a computer to find a vector $v' \in V$ such that $\dim \mathfrak{g}_{v'} = 9$. $\square$

We remark that in the proof above one can read off from (10.2) that the stabilizer of $v_{-\alpha_1}$ in $\text{im } x_{\alpha_1}$ has R -points $\{x_{\alpha_1}(t) \mid t \in R \text{ such that } 2t = 0\}$ and Lie algebra the root subspace $\mathfrak{g}_{\alpha_1}$. (The isomorphism class of this group scheme is generally denoted α_2 .) In this way, we can concretely see the source of the extra dimension in $\mathfrak{g}_v$.

Lemma 10.3 ($\wedge^3 k^7$). *For $G = \text{SL}_7$ and $V = \wedge^3 k^7$, the stabilizer G_v of a generic vector $v \in V$ is a simple algebraic group of type G_2 . In particular, G_v is smooth.*

Proof. In the case $k = \mathbb{C}$, this result goes back at least to [E], see [Ag] for context. For general k , [GuL, Prop. 5.2.17] shows that $G(k)_v$ are the k -points of a subgroup of type G_2 so that the tautological representation of SL_7 restricts to the Weyl module $V(\omega_2)$ of G_2 . In the notation established above, $\dim \mathfrak{h} = 14$. It remains to show that G_v is smooth, i.e., that $\mathfrak{g}_v = \mathfrak{h}$, equivalently that $\dim \mathfrak{g}_v = 14$.

If $\text{char } k \neq 2, 7$, then, as a representation of G_2 , $\mathfrak{sl}_7 = \mathfrak{so}_7 \oplus L(2\omega_2)$, where $\mathfrak{so}_7$ can be identified with skew-symmetric matrices and $L(2\omega_2)$ with the trace zero symmetric matrices, and $\mathfrak{so}_7/\mathfrak{h}$ is the natural representation of G_2 . The only Lie algebra lying between $\mathfrak{h}$ and $\mathfrak{sl}_7$ is $\mathfrak{so}_7$. However, the restriction of V to $\mathfrak{so}_7$ has head the spin representation with

generic stabilizer G_2 , so the stabilizer in $\mathfrak{so}_7$ of a generic vector in V can be no larger than $\mathfrak{h}$, whence $\mathfrak{g}_v \neq \mathfrak{so}_7$, completing the proof in this case.

If k has characteristic 2 or 7, one uses a computer to find a $v' \in V$ such that $\dim \mathfrak{g}_{v'} = 14$.

(Alternatively, [GuL, Prop. 5.2.17] shows that an s.g.p. exists on the level of k -points. Then, for any specific choice of char k — whether 2, 7, or something else — it suffices by Lemma 5.3 to use a computer to find a $v \in V$ such that $\dim \mathfrak{g}_v = 14$.) $\square$

Some unusual features of Lie algebras of groups of type G_2 when char k is 2, 3, or 7 are discussed in [CaRE].

For $(G, V) = (\mathrm{SL}_8, \wedge^3 k^8)$, one needs to show that $\dim \mathfrak{g}_v \leq 8$. This can be argued in a manner similar to Lemma 10.3. Alternatively, the stabilizer $\mathfrak{g}_v$ is computed explicitly in [SatoK, pp. 87–90]. The latter proof goes through if char $k \neq 2, 3$. In the remaining characteristics, $\dim \mathfrak{g}_v \leq 8$ can be verified by computer.

Consider now the case $(G, V) = (E_6, \text{minuscule})$. As a representation of F_4 , $\mathfrak{e}_6/\mathfrak{f}_4$ is the smallest nontrivial Weyl module of F_4 , $V(\omega_4)$, of dimension 26. (See [ChevS] for a view of this statement from the perspective of Jordan algebras.) The image of $\mathfrak{g}_v$ in $V(\omega_4)$ is contained in the radical. If char $k \neq 3$, then this radical is zero and $\mathfrak{g}_v = \mathfrak{h}$. If char $k = 3$, then the radical is $\mathfrak{z}(\mathfrak{e}_6)$, and again the image of $\mathfrak{g}_v$ is zero. (For this case and the case of $(E_7, \text{minuscule})$ in the following paragraph, smoothness of the generic stabilizer is also contained in [Stw].)

The representations $(G, V) = (\mathrm{SL}_6/\mu_3, \wedge^3 k^6)$, $(\mathrm{Sp}_6, \text{“spin”})$, and $(E_7, \text{minuscule})$, all belong to a family of representations considered in [Rö] and [Ga09a, §12]. In each case, there is a group $\tilde{G}$ and a simple root α of $\tilde{G}$ that is the only one not orthogonal to the highest root of $\tilde{G}$, G is a subgroup of $\tilde{G}$ generated by the roots of α -height zero, and the root subalgebras of α -height 1 in $\tilde{\mathfrak{g}}$ span a G -submodule of $\tilde{\mathfrak{g}}$ equivalent to V . Moreover, there is a unique simple root β of G not orthogonal to the highest weight of V , and β has coefficient 1 in the highest root of G so that the root subgroups of G corresponding to roots of β -height zero generate a group G_0 of the same description as $G(k)_v$. If char $k \neq 2$, the G -orbits in V are described in terms of the root system of $\tilde{G}$ in [Rö] and a representative generic vector v is provided such that $G_0(k) \subseteq G(k)_v$, whence equality. Grading $\mathfrak{g}$ by β -height, we find $\mathfrak{g}$ contains $\mathfrak{g}_0$, a rank 1 torus $\mathfrak{t}$, and subspaces $\mathfrak{g}_1, \mathfrak{g}_{-1}$ spanned by roots of β -height 1 and -1 respectively; these latter two subspaces are irreducible representations of G_0 [AzBS, Th. 2c]. That is, the composition series of $\mathfrak{g}_v/\mathfrak{h}$, as a representation of $G_0(k)$, has simple factors contained in $\mathfrak{t}, \mathfrak{g}_1, \mathfrak{g}_{-1}$. The explicit description of v from [Rö] shows that these cannot be contained in $\mathfrak{g}_v$ as in [Ga09a, 12.2], whence G_v is smooth. If char $k = 2$, we verify that G_v is smooth using Magma.

The case $(G, V) = (\mathrm{PSp}_{2\ell}, L(\omega_2))$ for $\ell \geq 3$ has a smooth s.g.p. by Lemma 5.5. This completes the proof of Theorem 1.2.

11. REPRESENTATIONS WITH “FEW” INVARIANTS: PROOF OF THEOREM 1.3

We now classify those irreducible representations V such that the $\dim V/G < \dim G$; there are relatively few.

Proof of Theorem 1.3. If $\dim V \leq \dim G$, then $V = L(\tilde{\alpha})$ or $(G, \text{char } k, V)$ belongs to Table 3 by Theorem 4.1(2) and (3), so assume $\dim V > \dim G$. Then, by Theorem 4.1(1), the stabilizer of a generic vector in V is a finite group scheme, whence

$$(11.1) \quad \dim k[V]^G = \dim V - \dim G.$$

That is, we are reduced to determining the faithful irreducible representations of G such that $\dim G < \dim V < 2 \dim G$. This is done in the next proposition, completing the proof of Theorem 1.3. $\square$

Proposition 11.2. *Let V be a faithful and irreducible representation of a simple algebraic group G over an algebraically closed field k . If $\dim G < \dim V < 2 \dim G$, then up to graph automorphism $(G, \text{char } k, V)$ appears in Table 4 or 5.*

Proof. Suppose first that the highest weight λ of V is restricted. Then the tables in [Lü] list all possibilities for $(G, \text{char } k, V)$, completing the proof in this case.

If $\text{char } k = 0$, then that is the only case, so suppose $p := \text{char } k \neq 0$.

We use the following observation. Put m for the smallest dimension of a nontrivial irreducible restricted representation of G . We note that $m^2 > 2 \dim G$ unless G has type A_ℓ or C_2 and that $m^3 > 2 \dim G$ regardless of the type of G .

Write $\lambda = \sum_{e \geq 0} p^e \lambda_e$ where λ_e is restricted for all e , so $\dim V = \prod_e \dim L(\lambda_e)$. Because V is faithful and not restricted, $\lambda_0 \neq 0$ and $\lambda_e \neq 0$ for some $e > 0$. If a third summand is not zero, then $\dim V \geq m^3 > 2 \dim G$, and we conclude that $\lambda = \lambda_0 + p^e \lambda_e$ for some $e > 0$ with $\lambda_0, \lambda_e \neq 0$.

For G of type A_ℓ , $m = \ell + 1$, and the representations of this dimension are $L(\omega_1)$ and $L(\omega_1)^* = L(\omega_\ell)$. The representations $L(\omega_1 + p^e \omega_\ell)$ and $L(\omega_1 + p^e \omega_1)$ appear in Table 4. Similarly, for G of type C_2 , $m = 4$, corresponding to $L(\omega_1)$ and the representation $L(\omega_1 + p^e \omega_1)$ appears in Table 5.

So suppose that $\dim L(\lambda_i) > m$ for $i = 0$ or e . Noting that $\dim L(\lambda_i) < 2(\dim G)/m$, there are few possibilities for λ_i . If G has type C_2 , then the upper and lower bounds on $\dim L(\lambda_i)$ in the preceding two sentences are 4 and 5, so there are no possibilities. If G has type A_ℓ , then we find only one possibility, that G has type A_3 and $\lambda_i = \omega_2$ with $\dim L(\omega_2) = 6$. The resulting representations appear in the third row of Table 5. $\square$

When $k[V]^G$ is a polynomial ring. Regarding cases where $\dim V/G$ is small, it is clear that $\dim V/G = 0$ if and only if there is a dense open G -orbit in V , if and only if $V/G = \text{Spec } k = \mathbb{A}^0$. Similarly, $\dim V/G \leq 1$ if and only if there is a dense open G -orbit in $\mathbb{P}(V)$, see, for example, [Po 80, Prop. 12] for $\text{char } k = 0$ or [BeGuL, Prop. 6.1] for $\text{char } k$ arbitrary. We have: If $\dim V/G = 1$, then $V/G \cong \mathbb{A}^1$.

If $\dim V/G = 2$, G simple, and $\text{char } k = 0$, then $V/G \cong \mathbb{A}^2$ by [Kempf, Th. 2.4]. Is the same conclusion true if k is allowed to have prime characteristic? We prove the following.

Proposition 11.3. *Let V be a faithful irreducible representation of a simple algebraic group G . If $\dim V/G = 2$, then $V/G \cong \mathbb{A}^2$ unless perhaps $(G, \text{char } k, V)$ is $(\text{Spin}_5, 5, L(\omega_1 + \omega_2))$ or $(\text{Spin}_{13}, \text{any}, \text{spin})$.*

Proof. We apply the classification of possibilities for $(G, \text{char } k, V)$ provided by Theorem 1.3. By Corollary 3.11, we may assume that $V \neq L(\tilde{\alpha})$. The minimum of $\dim k[V]^G$ for the representations in Table 5 is 5, so $(G, \text{char } k, V)$ belongs to Table 3 or 4.

For $(\text{PSp}_6, \neq 3, L(\omega_2))$ and $(\text{PSp}_8, 2, L(\omega_2))$, the ring $k[V]^G$ is described in [GaGu 15, Examples 8.3, 8.5] and it is polynomial.

The representations $(F_4, \neq 2, 3, \text{natural})$, $(\text{PGL}_2, \neq 2, 3, S^4 k^2)$, and $(\text{PGL}_3, \neq 2, 3, S^3 k^3)$ arise as θ -groups (a.k.a. Vinberg representations) where the overgroup is of type E_6 , A_2 , and D_4 and the automorphism θ of the overgroup has order 2, 2, and 3 respectively. (See, e.g., [PoV, p. 260–262] or [ReLYG, p. 1154].) By [Le, Th. 4.23], $k[V]^G$ is a polynomial ring.

The case $(F_4, 2, \text{natural})$ is not covered by the result from [Le]. Instead we refer to [GaGu 15, Example 11.4], which shows that $k[V]^G$ is a polynomial ring with generators of degree 2 and 3. $\square$

12. REGULAR ORBITS

In this section we consider when a simple algebraic group G acting on an irreducible module V has a regular orbit, i.e., when there is some $v \in V$ with $G_v = 1$. We will consider this first just for the points and then consider the same problem for group schemes, resulting in a proof Theorem 1.4.

The following is an immediate consequence of [GuL, Table 1]. We only state the result in characteristics other than 2, 3, and 5 as this is what we shall use.

Proposition 12.1. *Let G be a simple algebraic group over the algebraically closed field k of characteristic $p := \text{char } k \neq 2, 3, 5$ such that $G_v(k)$ acts faithfully and irreducibly on V . If $G_v(k)$ is finite for some v , then either $G_w(k) = 1$ for some w or the following holds (up to twists by Frobenius or graph automorphisms), $p \neq 0$, G is a quotient of $\text{SL}_{\ell+1}$, and $V = L(\omega_1 + p^e \omega_1)$ or $L(\omega_1 + p^e \omega_\ell)$ for some $e \geq 1$.*

We now consider the case when there is a regular orbit for the group scheme. Of course, if $\text{char } k = 0$, then the previous result implies that there exists a regular orbit if and only if the generic stabilizer is finite, if and only if $\dim V > \dim G$ (Theorem 4.1).

In the following, we write $P\Omega_n$ for $n = 3$ or $n \geq 5$ for the adjoint group of type B_ℓ (when $n = 2\ell + 1$ is odd) or D_ℓ (when $n = 2\ell$ is even).

Lemma 12.2. *Let $G = P\Omega_n$ for $n = 3$ or $n \geq 5$ with $\text{char } k \neq 2$ and V the nontrivial irreducible composition factor of the symmetric square of the orthogonal module. Then there exists $v \in V$ with G_v the trivial group scheme.*

Proof. Let W denote the space of symmetric $n \times n$ matrices of trace 0. View $\text{SO}_n(k)$ as the subgroup of $\text{SL}_n(k)$ of matrices A with $AA^\top = 1$. Then $\text{SO}_n(k)$ acts on W by conjugation with kernel the center. Note that $W = V$ unless p divides n , in which case $V = W/W_0$ where W_0 are the scalar matrices.

We recall that over an algebraically closed field, every matrix is similar to a symmetric matrix, see for example [BuGS, Lemma 3.1]. In particular, let A be a symmetric nilpotent matrix with minimal polynomial of degree n . The centralizer of A in $M_n(k)$ is the subalgebra $k[A]$, which consists of symmetric matrices. This shows that $\mathfrak{g}_A = 0$, since $\mathfrak{g}$ is the Lie algebra of skew symmetric matrices. If U is orthogonal and $UA = AU$, then $U = f(A)$ for some polynomial f . As U is symmetric and orthogonal, it is an involution. There are no non-scalar involutions in $k[A]$ and so $G_A(k) = 1$, whence G_A is the trivial group scheme.

This completes the proof if p does not divide n . If p does divide n , the result follows by observing that if any matrix commutes with A modulo scalars, it commutes with A , since the only nilpotent matrix in the set $A + \lambda I$ is A . $\square$

Proof of Theorem 1.4. Recall that V is a faithful and irreducible representation of a simple group G over a field k of characteristic $\neq 2, 3, 5$.

In case (b), i.e., $\dim V \leq \dim G$, then $\dim G_v > 0$ for generic $v \in V$ (Th. 4.1), so there cannot be a regular orbit. In case (c), $\dim \mathfrak{g}_v > 0$ for generic $v \in V$, so $\dim \mathfrak{g}_{v'} > 0$ for all $v' \in V$ and there cannot be a regular orbit. In case (d), there is no $v \in V$ with $G_v(k) = 1$ [GuL, Prop. 5.1.8], so there cannot be a regular orbit. In summary, if any of (b), (c), or (d) hold, then the other three conditions fail.

Therefore, we assume that $\dim V > \dim G$ and that we are not in case (c) nor (d), and we aim to prove the existence of a regular orbit. We may assume the s.g.p. is not the trivial group scheme, whence by Theorem 4.1, up to graph automorphism $(G, \text{char } k, V)$ belongs to Table 4. Lemma 12.2 handles four rows of the table, including $(\text{SL}_2 / \mu_2, \neq 2, 3, S^4 k^2)$ with $n = 3$ and $(\text{SL}_4 / \mu_4, \neq 2, L(2\omega_2))$ with $n = 6$. The row $(\text{SL}_2, \neq 2, 3, S^3 k^2)$ is from Example 2.2.

The remaining cases in Table 4 are examples of θ -groups, where G is the identity component of the subgroup of an overgroup H fixed by an automorphism θ of finite order. The cases for (G, V) are:

- (i) $(\text{PGL}_3, L(3\omega_1))$, with H of type D_4 and θ of order 3;
- (ii) $(\text{SL}_8 / \mu_4, L(\omega_4))$, with H of type E_7 and θ of order 2;
- (iii) $(\text{SL}_9 / \mu_3, L(\omega_3))$, with H of type E_8 and θ of order 3;
- (iv) $(\text{Spin}_{16} / \mu_2, \text{half-spin})$, with H of type E_8 and θ of order 2; or
- (v) $(\text{PSP}_8, L(\omega_4))$, with H of type E_6 and θ of order 2.

So Let $G < H$ be as in the five cases above. Since $p \neq 2, 3$ or 5 , $\text{char } k$ is good for H . Then $G = C_H(\theta)^\circ$ and $\mathfrak{h}$ is the direct sum of the eigenspaces of θ . Note that $\mathfrak{g}$ is the trivial eigenspace and V can be identified with the nontrivial eigenspace or one of the nontrivial eigenspaces if θ has order 3 and in that case V^* is the other eigenspace. It follows by [GuL, 5.1.4] that there exists a regular nilpotent element $v \in V$ and $G_v(k) = 1$. It also follows by the computation in [GuL] that the centralizer of v in H is contained in the sum of the nontrivial eigenspaces of θ whence its intersection with $\mathfrak{h}$ is 0. Let $J = C_H(n)$. So J is an abelian group of dimension equal to the rank of H and θ normalizes H with $C_J(\theta) = 1$. In good characteristic, the centralizer of v in $\mathfrak{h}$ is the Lie algebra of $C_J(\theta)$ and this is 0 since θ acts without fixed points on $C_H(v)$ and so also on its Lie algebra. $\square$

13. NOT-NECESSARILY-SEMISIMPLE REPRESENTATIONS

Let W be a section of a representation V of an algebraic group G . That is, there are G -invariant subspaces $V_1 \subseteq V_2 \subseteq V$ so that $W \cong V_2/V_1$ as representations of G . In this section, we discuss connections between the stabilizers G_w and G_v of generic $w \in W$ and $v \in V$ respectively.

If W is a summand of V , then one can take w to be a projection of v in W , in which case G_w evidently contains G_v , compare [Lö, Lemma 2.15]. Unfortunately, this statement does not easily extend to the case where W is not a summand of V , see [GaGu I, Example 2.6], which gives an example with $G = \mathbb{G}_a$ where W is a subspace of V and $G_w = 1$, yet $G_v \neq 1$. (See also Example 2.2 for a different but related phenomenon.)

We do know, by an easy argument using upper semicontinuity of dimension, that

$$(13.1) \quad \dim G_w \geq \dim G_v \quad \text{and} \quad \dim \mathfrak{g}_w \geq \dim \mathfrak{g}_v$$

for generic $w \in W$ and $v \in V$ when W is a section of V , see for example [GaGu I, Example 2.2].

Recall that a representation V of G is *generically free* if $G_v = 1$ for generic $v \in V$. We have: *If $\text{char } k = 0$ and a representation V of G has a section W that is generically free, then V is generically free because W is a summand of V .* Theorem 13.3 below provides a version of this in prime characteristic.

Separably free actions. Note that if V is generically free, the kernel N of the action is necessarily trivial. To accommodate the possibility that $N \neq 1$, we make the following definition.

Definition 13.2. A representation V of G is *separably free* if the kernel $N := \ker[G \rightarrow \mathrm{GL}(V)]$ of the action is étale and $(G/N)_v = 1$ for generic $v \in V$.

Theorem 13.3. *Let V be a representation of a simple algebraic group G . If V has an irreducible section that is separably (resp. generically) free as a representation of G , then V is separably (resp. generically) free.*

Before proving the theorem, we note some lemmas. The following is well known, and a proof is contained in [GaGu 15, §10], compare [GaGu 17, Lemma 2.6(i)].

Lemma 13.4. *Let V be a representation of a semisimple algebraic group G over an algebraically closed field k . If, for every $g \in G(k)$ that is (i) noncentral semisimple and whose image in $\mathrm{GL}(V)$ has prime order or (ii) unipotent, we have $\dim V^g + \dim g^G < \dim V$, then for generic $v \in V$, $G_v(k)$ is central in $G(k)$.*

As a consequence of previous work on irreducible representations of simple groups, we have the following converse:

Lemma 13.5 (Corollary 7 in [GuL]). *Let V be a faithful and irreducible representation of a simple algebraic group G over an algebraically closed field k . The stabilizer $G_v(k) = 1$ for generic $v \in V$ if and only if $\dim V^g + \dim g^G < \dim V$ for all $g \in G(k)$ of prime order (modulo the center) and, if $\mathrm{char} k = 0$, all unipotent g .*

The analogue of Lemma 13.5 for Lie algebras is false. For example, when $\mathrm{char} k = p \neq 0$, for any vector space W , the irreducible and faithful representation $V = W \otimes W^{[p]}$ of $G = \mathrm{SL}(W)$ has $\mathfrak{g}_v = 0$ for generic $v \in V$, yet $\dim x^G + \dim V^x = \dim V + \dim W - 2$ for x a root element, see [GaGu I, §10].

Proof of Theorem 13.3. Suppose first that the irreducible section W of V is separably free. Put $N := \ker[G \rightarrow \mathrm{GL}(W)]$ and $Z := \ker[G \rightarrow \mathrm{GL}(V)]$, so $Z \subseteq N$. By hypothesis, N is étale, so Z is also. In particular, both N and Z are central in G .

If any summand of V is separably free, then V is separably free. Writing V as a direct sum of the $Z(G)$ -homogeneous components we may assume that V is $Z(G)$ -homogeneous.

By Lemma 13.5, the inequality $\dim W^g + \dim g^G < \dim W$ holds for the relevant $g \in G(k)$. Then it is easy to see that $\dim V^g + \dim g^G < \dim V$ for those same g . The easier lemma, Lemma 13.4, now shows that $G_v(k)$ is a central subgroup for generic $v \in V$. As V is $Z(G)$ -homogeneous, $G_v(k) = Z(k)$.

Because $\mathrm{Lie}(N) = \mathrm{Lie}(Z) = 0$, the natural maps $\mathrm{Lie}(G) \rightarrow \mathrm{Lie}(G/N)$ and $\mathrm{Lie}(G) \rightarrow \mathrm{Lie}(G/Z)$ are isomorphisms, and we obtain isomorphisms of generic stabilizers $\mathrm{Lie}(G/N)_w \cong \mathfrak{g}_w$ and $\mathrm{Lie}(G/Z)_v \cong \mathfrak{g}_v$. As $\dim \mathfrak{g}_v \leq \dim \mathfrak{g}_w = 0$, we find that G_v is étale, so $G_v = Z$, i.e., V is separably free.

In case W is generically free, then (1) $N = 1$ so $Z = 1$ and (2) V is separably free by the above. So V is generically free. $\square$

Conjecture 13.6. *If G is reductive and W is a generically free section of V , then V is generically free.*

14. REPRESENTATIONS WITH THE SAME INVARIANTS

We give a new proof of one of the main results of [GaGu 15], Theorem 14.1 below, which characterizes inclusions where the subgroup and overgroup have the same invariants. The original proof relied on results from [Seitz], whereas the following, quite different proof avoids that reference and instead uses the information about the dimension of the

generic stabilizer. This approach leaves very few cases to examine. In this final section of the paper, we stop viewing algebraic groups as affine schemes and instead view them in a naive way as the group of k -points under the Zariski topology, as is done, for example in [Hu 81].

Theorem 14.1 (Theorem 13.1 in [GaGu 15]). *Suppose that $G < H \leq \mathrm{SL}(V)$ with G a simple algebraic group over an algebraically closed field k acting irreducibly on V , and H connected and closed in $\mathrm{SL}(V)$. If $\dim k[V]^G = \dim k[V]^H$, then $k[V]^G = k[V]^H$ and one of the following holds, up to a Frobenius twist and/or a twist by a graph automorphism:*

- (1) $H = \mathrm{SL}(V)$ and $k[V]^G = k$ (i.e., $\dim k[V]^G = 0$).
- (2) $H = \mathrm{Sp}(V)$, $\mathrm{char} k = 2$, $G = G_2$, $\dim V = 6$, and $k[V]^G = k$.
- (3) $H = \mathrm{SO}(V)$, $k[V]^G = k[q]$ for a homogeneous quadratic form q (in particular, $\dim k[V]^G = 1$).
- (4) $(G, H, V, \mathrm{char} k)$ is in Table 6.

The possibilities for G in (1) and (3) can be extracted from Tables 3 and 4; see Tables C and D in [GaGu 15] for an explicit list.

Remark 14.2. The statement of Theorem 14.1 is slightly different from the one in [GaGu 15]. The earlier version erroneously omitted the hypothesis that H is connected. Of course, the dimension of the ring of invariants only depends on the connected component of the identity but certainly the actual ring of invariants can change.

The earlier version also omitted the inclusion $G_2 < \mathrm{PSP}_6$ from Table 6. For this inclusion, G_2 embeds in PSP_6 when $\mathrm{char} k = 2$ via the natural irreducible representation of G_2 . The 14-dimensional representation $L(\omega_2)$ of PSP_6 restricts to the adjoint representation of G_2 . To see this, note that the representation $\wedge^2 k^6$ of PSP_6 is $k \oplus L(\omega_2)$, and compare the restrictions of this and the adjoint module to an A_2 subgroup of G_2 . The fact that the ring of invariants on this representation is polynomial with generators of degrees 2 and 3 as in Table 6 is [GaGu 15, Example 8.5] for PSP_6 and Example 3.8 for G_2 . The equality $k[V]^G = k[V]^H$ in the other cases was proved in [GaGu 15].

G	H	$\dim V$	$\mathrm{char} k$	degrees
PGL_3	G_2	7	3	2
G_2	PSP_6	14	2	2, 3
Spin_{11}	HSpin_{12}	32	all	$\begin{cases} 4 & \text{if } \mathrm{char} k \neq 2 \\ 2 & \text{if } \mathrm{char} k = 2 \end{cases}$
$\mathrm{SO}_{2n} (n \geq 3)$	PSp_{2n}	$\begin{cases} 2n^2 - n - 2 & \text{if } n \text{ even} \\ 2n^2 - n - 1 & \text{if } n \text{ odd} \end{cases}$	2	$\begin{cases} ? \\ 2, 3, \dots, n \end{cases}$
SO_8 or Sp_8	F_4	26	2	2, 3
SL_n	$\mathrm{SL}_n \otimes \mathrm{SL}_n$	n^2	$\neq 0$	n

TABLE 6. Representations referred to in Theorem 14.1, copied from Table E in [GaGu 15]. The representations in the last row are denoted $L(\omega_1 + p^e \omega_\ell)$ and $L(\omega_1 + p^e \omega_1)$ in Table 4. Note that since this section views algebraic groups in the naive sense, when $\mathrm{char} k = 2$ we have natural identifications $\mathrm{SO}_n = P\Omega_n$ and $\mathrm{Sp}_{2n} = \mathrm{PSP}_{2n}$.

We use the following.

Lemma 14.3. *Let $G < H \leq \mathrm{SL}(V)$ with G a simple algebraic group acting irreducibly on V and H a closed connected subgroup. Then $\dim k[V]^H = \dim k[V]^G$ if and only if GH_v is dense in H for generic $v \in V$.*

Proof. If GH_v is dense in H , then $Gv = GH_v v$ is dense in Hv for generic v and so the maximal dimension of G and H orbits are the same. Since the dimension of the ring of invariants is the codimension of a maximal dimensional orbit, the density assumption implies that the dimensions of the ring of invariants are the same.

Conversely, if the dimensions of the ring of invariants are the same, then there exists a nonempty subset $\mathcal{O}$ of V such that $\dim Gv = \dim Hv$ for all $v \in \mathcal{O}$ and since Hv is an irreducible variety Gv is dense in Hv . Thus, $\dim GH_v = \dim G + \dim H_v - \dim G_v = \dim H_v + \dim H - \dim H_v = \dim H$ and so GH_v is dense in H . $\square$

Lemma 14.4. *Let G be a simple algebraic irreducible subgroup of $H = \mathrm{Sp}_{2\ell}$ with $\ell \geq 3$. If $\dim G \geq \dim H - 3\ell$, then $\mathrm{char} k = 2$ and either (1) $G = \mathrm{SO}_{2\ell}$ or (2) $G = G_2$ and $\ell = 3$.*

Proof. The right side of the inequality in the statement is $2\ell^2 - 2\ell$, which is increasing for $\ell \geq 3$. Its minimum value is 12, so G cannot have type A_1 , A_2 , B_2 , or C_2 . Otherwise, the natural representation of H is an irreducible self-dual representation $L(\lambda)$ of G of dimension between 6 and $2\ell_G$, where ℓ_G is the real number ≥ 3 such that $2\ell_G^2 - 2\ell_G = \dim G$.

In case λ is restricted, the tables in [Lü] verify the claim. Indeed, for other group types, the smallest self-dual irreducible module with restricted highest weight is already too large.

In general, we may write $\lambda = \lambda_0 + p\lambda_1$ with λ_0 restricted and $p := \mathrm{char} k \neq 0$. Since G acts faithfully on $L(\lambda)$, $\lambda_0 \neq 0$. By hypothesis, $L(\lambda)$ is self-dual, i.e., λ is fixed by $-w_0$ for w_0 the longest element of the Weyl group, so the same is true for λ_0, λ_1 . Then $\dim L(\lambda) = \dim L(\lambda_0) \cdot \dim L(\lambda_1)$, so the examination of the dimensions of self-dual representations in the preceding paragraph shows that $\lambda_1 = 0$, i.e., λ is restricted. $\square$

Proof of Theorem 14.1. In view of the remarks just after the statement of the theorem, it suffices to show that equality of the dimension of the ring of invariants only occurs in the cases listed in the conclusion.

We first consider the case that H is not simple. If not, then V is tensor decomposable for H and so also G whence by [GuL], G_v is generically finite (and almost always generically trivial by [GaGu II]). In particular, $\dim k[V]^G = \dim V - \dim G$.

We may assume that $H = H_1 \times H_2$, G embeds in H_i by the projection π_i and that $V = V_1 \otimes V_2$ where V_i is an irreducible H_i -module. Let $J := \pi_1(H) \times \pi_2(H) \cong G \times G$. So $G \leq J \leq H$.

Let $d_i = \dim V_i$ and assume that $d = d_1 \leq d_2$. Let $v = \sum_{i=1}^d e_i \otimes f_i$ where the e_i constitute a basis for V_1 . Observe that π_2 restricted to H_v has trivial kernel. Indeed, if $h \in H_v$ and $\pi_2(h) = 1$, then h fixes each f_i and so fixes v if and only if $he_i = e_i$ for all i , whence h is trivial on V_1 and so on V .

If $d_1 < d_2$, then we see that $J_v \cong \pi_2(H_v)$ stabilizes the span of $f_1, \dots, f_d$ and so $\pi_2(H_v)$ is properly contained in H_2 . Thus $\dim J_v < \dim G$ and so $\dim k[V]^H \leq \dim k[V]^J < \dim V - \dim H + \dim G = \dim V - \dim G = \dim k[V]^G$.

So we may assume that $d_1 = d_2$ and identify $V_1 = V_2 = W$ (as vector spaces rather than G -modules). Note that in $L = \mathrm{SL}(W) \otimes \mathrm{SL}(W)$, the stabilizer in L of a generic vector is a diagonal subgroup D , i.e., it is isomorphic to $\mathrm{SL}(W)$ and the projection onto either factor is a bijection. If $\pi_2(G) \neq \mathrm{SL}(W)$, then clearly $\dim \pi_2(G \cap D) < \dim G$ for generic D . Thus, $\dim J_v < \dim G$ for generic v and so arguing as above, we see

that $\dim k[V]^H < \dim k[V]^G$. If $\pi_2(G) = \mathrm{SL}(W)$, then $G \cong \mathrm{SL}(W)$ and V is a tensor product of two G -modules each of which are Frobenius twists of the natural module or its dual. In this case, $\dim k[V]^G = 1$ and (G, H, V) are as in the last line of Table 6.

So now assume that H is simple. If $\dim H_v = 0$ for some $v \in V$, then $\dim k[V]^H = \dim V - \dim H < \dim V - \dim G = \dim k[V]^G$, a contradiction. So $\dim V \leq \dim H$, $(H, \mathrm{char} k, V)$ is in Table 3 or V is the irreducible part of the adjoint module for H .

First consider the case that V is the irreducible part of the adjoint module for H . As H acts faithfully, $\mathrm{char} k$ is not special and H is adjoint. Lemma 3.5 provides a contradiction.

Next consider the case that $H = \mathrm{Sp}(W) = \mathrm{Sp}_{2\ell}$ for some $\ell > 2$ and $V = L(\omega_2)$. Thus, $\dim V = \ell(2\ell - 1) - \delta$ with $\delta = 1$ or 2 . Suppose that G is not irreducible on W . Let X be a G -composition factor of W of maximal dimension e . So $e \leq 2\ell - 2$. If $e = 2$, then the largest composition factor of G on $\wedge^2 W$ is 4-dimensional, a contradiction. Otherwise, G has a nontrivial composition factor on V of dimension at most $\dim \wedge^2 X = (\ell - 2)(2\ell - 3) < \dim V$, whence G is not irreducible on V , a contradiction.

Also, GH_v is dense in H so $\dim G \geq \dim H - 3\ell$, since H_v is generically of dimension 3ℓ . Now apply Lemma 14.4 to deduce that $(G, H, V, \mathrm{char} k)$ are as in the 2nd or 4th lines of Table 6.

In the remaining cases from Table 3, we have $d = \dim k[V]^H \leq 2$. If $d = 1$, then inspection of Tables 3, 4, and 5 show (1), (2), or (3) of the theorem hold or we are in the case of line 1 of Table 6. So we may assume that $d = 2$.

First assume that $\dim V > \dim G$ and so $d := \dim k[V]^G = \dim V - \dim G > 0$. Then by Theorem 1.3 one of the following holds:

- $G = A_1, p \neq 2, 3, V = L(\omega_4)$ with $\dim V = 5$;
- $G = A_2, p \neq 2, 3, V = L(3\omega_1)$ with $\dim V = 10$; or
- $G = B_2, p = 5, V = L(\omega_1 + \omega_2)$ with $\dim V = 12$.

The possibilities for H with $\dim k[V]^H = 2$ are also given in Theorem 1.3 and we see that there are no examples.

Next consider the case that $\dim V \leq \dim G$. If V is the nontrivial composition factor of the adjoint module (and $\mathrm{char} k$ is not special for G) and $\dim k[V]^G \leq 2$, then (G, p) appears in Table 2. Again, the possibilities for H are all given in the tables and we see the only examples are captured in the 2nd and 4th rows of Table 6.

Finally assume that $\dim V < \dim G$ and we are not in the case of the adjoint module. Thus, G and H both occur in Table 3 and we see that there are no containments (when $d = 2$). $\square$

An immediate consequence of Theorem 14.1 is:

Corollary 14.5. *Under the hypotheses of Theorem 14.1: If G_v is finite for generic $v \in V$ or $\dim V > \dim G$, then $\mathrm{char} k \neq 0$ and $(G, H, V, \mathrm{char} k)$ are as in the last row of Table 6.*

Corollary 14.6. *Let V be a faithful and irreducible representation of a simple algebraic group G . Let e be the greatest common divisor of the degrees of the homogeneous elements of $k[V]^G$. Assume that (G, V) is not given (up to twist) in the statement of Theorem 14.1. If m is a sufficiently large multiple of e , then for almost all homogeneous $f \in k[V]^G$ of degree m , G is the identity component of the stabilizer of f .*

Proof. There are only finitely many closed subgroups H of $\mathrm{SL}(V)$ containing G (see [Li] or [GaGu 15, Prop. 9.2]) and by the previous result $\dim k[V]^H < \dim k[V]^G$. Thus, for m sufficiently large (and a multiple of e), the set of $f \in k[V]^G$ that are homogeneous

whose stabilizer has connected component strictly containing G is a finite union of proper subspaces of the degree m invariants of G , whence the result. $\square$

In fact using results of Seitz and Testerman and others (see [Seitz], [BuGMT, BuGT, BuMT]), for most (G, V) it is the case that G is maximal in the corresponding classical group ($\mathrm{SO}(V)$, $\mathrm{Sp}(V)$, or $\mathrm{SL}(V)$) and so any homogeneous G -invariant (other than a scalar times a power of the invariant quadratic form in the case $G < \mathrm{SO}(V)$) has stabilizer whose connected component is G . See [GaGu 15] for many examples of this, e.g., if $G = E_8$ and V is the adjoint module.

REFERENCES

- [Ag] I. Agricola, *Old and new on the exceptional group G_2* , Notices of the AMS **55** (2008), no. 8, 922–929. [26](#)
- [AnP] E.M. Andreev and V.L. Popov, *Stationary subgroups of points of general position in the representation space of a semisimple Lie group*, Functional Anal. Appl. **5** (1971), no. 4, 265–271. [4](#)
- [AzBS] H. Azad, M. Barry, and G. Seitz, *On the structure of parabolic subgroups*, Comm. Algebra **18** (1990), no. 2, 551–562. [18](#), [27](#)
- [BeGuL] H. Bermudez, S. Garibaldi, and V. Larsen, *Linear preservers and representations with a 1-dimensional ring of invariants*, Trans. Amer. Math. Soc. **366** (2014), no. 9, 4755–4780. [9](#), [23](#), [28](#)
- [BuGMT] T. Burness, S. Ghandour, C. Marion, and D. Testerman, *Irreducible almost simple subgroups of classical algebraic groups*, Mem. Amer. Math. Soc. **236** (2015), no. 1114. [35](#)
- [BuGS] T. Burness, R. Guralnick, and J. Saxl, *On base sizes for algebraic groups*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 8, 2269–2341. [29](#)
- [BuGT] T. Burness, S. Ghandour, and D. Testerman, *Irreducible geometric subgroup of classical algebraic groups*, Mem. Amer. Math. Soc. **239** (2016), no. 1130. [35](#)
- [BuMT] T. Burness, C. Marion, and D. Testerman, *On irreducible subgroups of simple algebraic groups*, Math. Ann. **367** (2017), no. 3–4, 1259–1309. [35](#)
- [Bour 02] N. Bourbaki, *Lie groups and Lie algebras: Chapters 4–6*, Springer-Verlag, Berlin, 2002. [5](#), [19](#)
- [Bour 05] ———, *Lie groups and Lie algebras: Chapters 7–9*, Springer-Verlag, Berlin, 2005. [9](#)
- [BoTi] A. Borel and J. Tits, *Compléments à l'article: Groupes réductifs*, Publ. Math. IHES **41** (1972), 253–276. [24](#)
- [CaRE] A. Castillo-Ramirez and A. Elduque, *Some special features of Cayley algebras, and G_2 , in low characteristics*, J. Pure Appl. Algebra **220** (2016), 1188–1205. [27](#)
- [ChaR] P.-E. Chaput and M. Romagny, *On the adjoint quotient of Chevalley groups over arbitrary base schemes*, J. Inst. Math. Jussieu **9** (2010), no. 4, 673–704. [9](#)
- [ChevS] C. Chevalley and R.D. Schafer, *The exceptional simple Lie algebras F_4 and E_6* , Proc. Nat. Acad. Sci. U. S. A. **36** (1950), 137–141. MR 0034378 (11,577b) [27](#)
- [CoGP] B. Conrad, O. Gabber, and G. Prasad, *Pseudo-reductive groups*, 2nd ed., Cambridge University Press, 2015. [6](#)
- [Dem] M. Demazure, *Invariants symétriques entiers des groupes de Weyl et torsion*, Invent. Math. **21** (1973), 287–301. [5](#), [9](#)
- [DemG] M. Demazure and A. Grothendieck, *Schémas en groupes II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*, Lecture Notes in Mathematics, vol. 152, Springer, 1970. [7](#)
- [DerK] H. Derksen and G. Kemper, *Computational invariant theory*, 2nd ed., Encyclopedia of Mathematical Sciences, no. 130, Springer, 2015. [10](#)
- [E] F. Engel, *Ein neues, dem linearen Komplexe analoges Gebilde*, Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Mathematisch-Physische Klasse **52** (1900), 63–76, 220–239. [26](#)
- [FH] W. Fulton and J. Harris, *Representation theory: a first course*, Graduate texts in mathematics, vol. 129, Springer, 1991. [25](#)
- [Ga 09a] S. Garibaldi, *Cohomological invariants: exceptional groups and spin groups*, Memoirs Amer. Math. Soc., no. 937, Amer. Math. Soc., Providence, RI, 2009, with an appendix by Detlev W. Hoffmann. [27](#)
- [Ga 09b] ———, *Vanishing of trace forms in low characteristic*, Algebra & Number Theory **3** (2009), no. 5, 543–566, with an appendix by A. Premet. [6](#)

- [GaGu 15] S. Garibaldi and R.M. Guralnick, *Simple groups stabilizing polynomials*, Forum of Mathematics: Pi **3** (2015), e3 (41 pages). [3](#), [9](#), [11](#), [17](#), [28](#), [29](#), [31](#), [32](#), [34](#), [35](#)
- [GaGu 16] ———, *Essential dimension of algebraic groups, including bad characteristic*, Arch. Math. **107** (2016), 101–119. [6](#), [7](#), [11](#)
- [GaGu 17] ———, *Spinors and essential dimension*, Compositio Math. **153** (2017), 535–556, with an appendix by A. Premet. [12](#), [18](#), [19](#), [23](#), [25](#), [31](#)
- [GaGu I] ———, *Generically free representations I: Large representations*, Algebra & Number Theory **14** (2020), no. 6, 1577–1611. [2](#), [5](#), [7](#), [16](#), [22](#), [30](#), [31](#)
- [GaGu II] ———, *Generically free representations II: Irreducible representations*, Transform. Groups **25** (2020), 793–817. [2](#), [5](#), [11](#), [18](#), [20](#), [21](#), [25](#), [33](#)
- [GaGu III] ———, *Generically free representations III: Extremely bad characteristic*, Transform. Groups **25** (2020), 819–841. [2](#), [5](#), [6](#), [11](#), [15](#), [17](#)
- [GaN] S. Garibaldi and D.K. Nakano, *Bilinear and quadratic forms on rational modules of split reductive groups*, Canad. J. Math. **68** (2016), no. 2, 395–421. [25](#)
- [GoGu] D. Goldstein and R. Guralnick, *Alternating forms and self-adjoint operators*, J. Algebra **308** (2007), no. 1, 330–349. [17](#)
- [GuL] R.M. Guralnick and R. Lawther, *Generic stabilizers in actions of simple algebraic groups*, October 2021, arxiv:1904.13375v3. [2](#), [3](#), [5](#), [11](#), [12](#), [15](#), [16](#), [19](#), [20](#), [21](#), [23](#), [26](#), [27](#), [29](#), [30](#), [31](#), [33](#)
- [Gurevich] G.B. Gurevich, *Foundations of the theory of algebraic invariants*, P. Noordhoff Ltd., Groningen, 1964, Translated by J.R.M. Radok and A.J.M. Spencer. [4](#)
- [Hiss] G. Hiss, *Die adjungierten Darstellungen der Chevalley-Gruppen*, Arch. Math. (Basel) **42** (1984), 408–416. [6](#), [7](#), [8](#)
- [Hog] G.M.D. Hogeweij, *Almost-classical Lie algebras. I, II*, Nederl. Akad. Wetensch. Indag. Math. **44** (1982), no. 4, 441–460. [8](#)
- [Howe] R. Howe, *Perspectives on invariant theory*, The Schur Lectures (1992), Bar-Ilan University, 1995. [15](#)
- [Hu 67] J.E. Humphreys, *Algebraic groups and modular Lie algebras*, Memoirs of the American Mathematical Society, No. 71, American Mathematical Society, Providence, R.I., 1967. [22](#)
- [Hu 81] ———, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 21, Springer, 1981. [32](#)
- [Jac 68] N. Jacobson, *Structure and representations of Jordan algebras*, Coll. Pub., vol. 39, Amer. Math. Soc., Providence, RI, 1968. [23](#)
- [Jac 71] ———, *Exceptional Lie algebras*, Lecture notes in pure and applied mathematics, vol. 1, Marcel-Dekker, New York, 1971. [24](#)
- [Jan 03] J.C. Jantzen, *Representations of algebraic groups*, second ed., Math. Surveys and Monographs, vol. 107, Amer. Math. Soc., 2003. [4](#), [5](#)
- [Jan 04] ———, *Nilpotent orbits in representation theory*, Lie theory, Progr. Math., vol. 228, Birkhäuser Boston, Boston, MA, 2004, pp. 1–211. [9](#)
- [KaPV] V.G. Kac, V.L. Popov, and E.B. Vinberg, *Sur les groupes linéaires algébriques dont l’algèbre des invariants est libre*, C. R. Acad. Sci. Paris Sér. A-B **283** (1976), no. 12, A1, A875–A878. [10](#)
- [KeM] G. Kemper and G. Malle, *The finite irreducible linear groups with polynomial ring of invariants*, Transform. Groups **2** (1997), no. 1, 57–89. [10](#)
- [Kempf] G. Kempf, *Some quotient surfaces are smooth*, Michigan Math. J. **27** (1980), no. 3, 295–299. [28](#)
- [KnMRT] M.-A. Knus, A.S. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, Colloquium Publications, vol. 44, Amer. Math. Soc., 1998. [24](#), [26](#)
- [Le] P. Levy, *Vinberg’s θ -groups in positive characteristic and Kostant-Weierstrass slices*, Transform. Groups **14** (2009), no. 2, 417–461. [18](#), [28](#), [29](#)
- [LiPS] M.W. Liebeck, C.E. Praeger, and J. Saxl, *The maximal factorizations of the finite simple groups and their automorphism groups*, Mem. Amer. Math. Soc. **86** (1990), no. 432, iv+151. [26](#)
- [LiT] M.W. Liebeck and D. Testerman, *Irreducible subgroups of algebraic groups*, Q. J. Math. **55** (2004), no. 1, 47–55. [34](#)
- [Lö] R. Lötscher, *Essential dimension of separable algebras embedding in a fixed central simple algebra*, Documenta Math. (2015), 443–459, Extra volume dedicated to A. Merkurjev. [30](#)
- [LöM] R. Lötscher and M. MacDonald, *The slice method for G -torsors*, Adv. Math. **320** (2017), 329–360. [2](#), [18](#)
- [Lü] F. Lübeck, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math. **4** (2001), 135–169. [2](#), [5](#), [11](#), [24](#), [28](#), [33](#)
- [LuR] D. Luna and R.W. Richardson, *A generalization of the Chevalley restriction theorem*, Duke Math. J. **46** (1979), no. 3, 487–496. [18](#)

- [McC] K. McCrimmon, *The Freudenthal-Springer-Tits constructions revisited*, Trans. Amer. Math. Soc. **148** (1970), 293–314. [24](#)
- [Milne] J.S. Milne, *Algebraic groups: the theory of group schemes of finite type over a field*, Cambridge studies in Advanced Math., vol. 170, Cambridge University Press, 2017. [2](#), [3](#), [4](#), [6](#), [9](#), [16](#), [17](#), [22](#)
- [N] H. Nakajima, *Invariants of finite groups generated by pseudo-reflections in positive characteristic*, Tsukuba J. Math. **3** (1979), no. 1, 109–122. [9](#), [10](#)
- [Pe 15] H.P. Petersson, *An embedding theorem for reduced Albert algebras over arbitrary fields*, Comm. Alg. **43** (2015), 2062–2088. [24](#)
- [Pe 18] ———, *A survey on Albert algebras*, Transf. Groups **24** (2018), no. 1, 219–278. [23](#)
- [Po 72] V.L. Popov, *On the stability of the action of an algebraic group on an algebraic variety*, Math. USSR Izv. **6** (1972), no. 2, 367–379, [Russian original: Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972)]. [15](#)
- [Po 80] ———, *Classification of spinors of dimension 14*, Trans. Moscow Math. Soc. (1980), no. 1, 181–232. [28](#)
- [Po 94] ———, *Sections in invariant theory*, The Sophus Lie Memorial Conference (Oslo, 1992), Scand. Univ. Press, Oslo, 1994, pp. 315–361. [18](#)
- [PoV] V.L. Popov and E.B. Vinberg, *Invariant theory*, Encyclopedia of Mathematical Sciences, vol. 55, pp. 123–284, Springer-Verlag, 1994. [5](#), [10](#), [15](#), [18](#), [28](#)
- [PrS] A.A. Premet and I.D. Suprunenko, *The Weyl modules and the irreducible representations of the symplectic group with the fundamental highest weights*, Comm. Algebra **11** (1983), no. 12, 1309–1342. [17](#)
- [ReLYG] M. Reeder, P. Levy, J.-K. Yu, and B.H. Gross, *Gradings of positive rank on simple Lie algebras*, Transform. Groups **17** (2012), no. 4, 1123–1190. [18](#), [28](#)
- [Ri 72] R.W. Richardson, *Principal orbit types for algebraic transformation spaces in characteristic zero*, Invent. Math. **16** (1972), 6–14. [2](#), [15](#)
- [Ri 77] ———, *Affine coset spaces of reductive algebraic groups*, Bull. London Math. Soc. **9** (1977), no. 1, 38–41. [15](#)
- [Rö] G. Röhrle, *On extraspecial parabolic subgroups*, Linear algebraic groups and their representations (Los Angeles, CA, 1992), Contemp. Math., vol. 153, Amer. Math. Soc., Providence, RI, 1993, pp. 143–155. [27](#)
- [SatoK] M. Sato and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155. [27](#)
- [Seitz] G.M. Seitz, *The maximal subgroups of classical algebraic groups*, Mem. Amer. Math. Soc. **67** (1987), no. 365. [15](#), [31](#), [35](#)
- [Sesh] C.S. Seshadri, *Geometric reductivity over arbitrary base*, Adv. Math. **26** (1977), 225–274. [2](#)
- [SpSt] T.A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Lecture Notes in Math., vol. 131, Springer, Berlin, 1970, [= *Robert Steinberg: collected papers*, pp. 293–394], pp. 167–266. [9](#)
- [SpV] T.A. Springer and F.D. Veldkamp, *Octonions, Jordan algebras and exceptional groups*, Springer-Verlag, Berlin, 2000. [25](#)
- [St 63] R. Steinberg, *Representations of algebraic groups*, Nagoya Math. J. **22** (1963), 33–56, [= *Collected Papers*, pp. 149–172]. [6](#)
- [St 75] ———, *Torsion in reductive groups*, Adv. Math. **15** (1975), no. 1, 63–92, [= *Collected Papers*, pp. 415–444]. [7](#)
- [Stw] D.I. Stewart, *On the minimal modules for exceptional Lie algebras: Jordan blocks and stabilizers*, LMS J. Comput. Math. **19** (2016), no. 1, 235–258. [25](#), [26](#), [27](#)
- [TZ] D.M. Testerman and A.E. Zalesski, *Subgroups of simple algebraic groups containing regular tori, and irreducible representations with multiplicity 1 non-zero weights*, Transform. Groups **20** (2015), no. 3, 831–861. [15](#)
- [V] E.B. Vinberg, *The Weyl group of a graded Lie algebra*, Math. USSR Izv. **10** (1976), 463–495. [18](#)
- [ZS] A.E. Zalesskiĭ and I.D. Suprunenko, *Representations of dimension $(p^n \pm 1)/2$ of the symplectic group of degree $2n$ over a field of characteristic p* , Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk (1987), no. 6, 9–15, 123. [15](#)

GARIBALDI: IDA CENTER FOR COMMUNICATIONS RESEARCH-LA JOLLA, 4320 WESTERRA CT, SAN DIEGO, CA 92121

Email address: skip@garibaldibros.com

GURALNICK: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90089

Email address: guralnic@usc.edu