

REAL CONSTITUENTS OF PERMUTATION CHARACTERS

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Dedicated to the memory of our good friend Jan Saxl

ABSTRACT. We prove a broad generalization of a theorem of W. Burnside about the existence of real characters of finite groups to permutation characters. If G is a finite group, under the necessary hypothesis of $\mathbf{O}^{2'}(G) = G$, we can also give some control on the parity of multiplicities of the constituents of permutation characters (a result that needs the Classification of Finite Simple Groups). Along the way, we give a new characterization of the 2-closed finite groups using odd-order real elements of the group. All this can be seen as a contribution to Brauer's Problem 11 which asks how much information about subgroups of a finite group can be determined by the character table.

1. INTRODUCTION

Recall that W. Burnside proved that a finite group of odd order has no nontrivial irreducible real valued characters. This paper began with an elementary generalization of this theorem of Burnside, which we have not found in the literature.

Before stating the result, we recall the trichotomy for irreducible complex characters of a finite group G . If χ is an irreducible complex character of G , then the Frobenius Schur indicator $\nu_2(\chi) = |G|^{-1} \sum_{g \in G} \chi(g^2)$ has possible values $0, \pm 1$ [Is, Theorem 4.5]. We say that χ is of $+$ type if $\nu_2(\chi) = 1$. This happens if there is a representation affording χ which is defined over $\mathbb{R}$, or equivalently, if there is a G -invariant nonzero quadratic form on the corresponding module. We have that $\nu_2(\chi) = -1$ if χ is real valued but the representation cannot be defined over $\mathbb{R}$. If χ is not real valued, then $\nu_2(\chi) = 0$, or equivalently, the representation is not self dual. (See [Is] for a more detailed discussion.)

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Theorem A. *Let G be a finite group, and let H be a subgroup of G . Then the permutation character $(1_H)^G$ has a unique real-valued + type irreducible constituent if and only if $[G : \text{core}_G(H)]$ is odd.*

Proof. Write $K = \text{core}_G(H)$. If G/K has odd order, then the trivial character 1_G is the only real-valued irreducible constituent of $(1_H)^G$, by Burnside's Theorem. Conversely, if $|G/K|$ is not odd, then let xK be an involution of G/K . Then $|G : H| = (1_H)^G(x^2) > (1_H)^G(x)$, since $\ker((1_H)^G) = K$. Therefore, using this and the fact that $(1_H)^G$ is a permutation character, we have that

$$\frac{1}{|G|} \sum_{g \in G} (1_H)^G(g^2) > \frac{1}{|G|} \sum_{g \in G} (1_H)^G(g) = [(1_H)^G, 1_G] = 1.$$

Then

$$1 < \frac{1}{|G|} \sum_{g \in G} (1_H)^G(g^2) = \sum_{\chi \in \text{Irr}(G)} [(1_H)^G, \chi] \nu_2(\chi).$$

Since $\nu_2(\chi)$ is 1, -1 or 0, the proof easily follows. $\square$

In most cases that we have observed, it turns out that there is a non-trivial real-valued *odd multiplicity* irreducible constituent χ in $(1_H)^G$ as long as $[G : \text{core}_G(H)]$ is not odd. (As we shall point out in Lemma 4.4 below, this implies that χ has + type.) Multiplicities of constituents of irreducible characters in permutation characters correspond to the dimensions of the irreducible representations of the Hecke algebra $\text{End}((1_H)^G)$, so they are of interest.

We now give an example to show that, unfortunately, it is not always the case that there is a nontrivial real valued irreducible constituent of odd multiplicity in $(1_H)^G$ when $[G : \text{core}_G(H)]$ is even. Suppose that $p \equiv 3 \pmod{4}$ is a prime, and let $q = p^a$, where $a > 1$ is odd. Let $G = \text{AGL}(1, q)$, the semidirect product of the Galois field $F = \text{GF}(q)$ with the multiplicative group $\text{GF}(q)^\times$. If $H \leq G$ has order $2p$, then G/FH is a cyclic group of odd order $\frac{q-1}{2}$, and

$$(1_H)^G = \frac{q^{a-1} - 1}{2} \theta + \sum_{\lambda \in \text{Irr}(G/FH)} \lambda,$$

where $\theta \in \text{Irr}(G)$ is the unique irreducible character of order $q - 1$. Also, notice that $\frac{q^{a-1}-1}{2}$ is even and θ is rational-valued.

Our main result in this paper is that this does not happen under the additional hypothesis that $\mathbf{O}^{2'}(G) = G$. (Recall that $\mathbf{O}^{2'}(G)$ is the smallest normal subgroup N of G such that G/N has odd order.) For the proof, we require the Classification of Finite Simple Groups.

Theorem B. *Let G be a finite group and let H be a proper subgroup of G . If $\mathbf{O}^{2'}(G) = G$, then $(1_H)^G$ has a non-trivial real-valued irreducible constituent of odd multiplicity.*

There is a great deal of literature on permutation characters, much inspired by Richard Brauer's Problem 11 of his celebrated list: *Given the character table of a group G , how much information about the existence of subgroups can be obtained?* Notice that Theorem B adds, in principle, a new condition to check for a given character of a finite group if it is a permutation character or not. (See [Is, Theorem 5.18]).

As we shall see, a key idea in the proof of Theorem B is that if 1_G is the only real-valued irreducible constituent of $(1_H)^G$ with odd multiplicity, then $[G : H]$ is odd, and every real conjugacy class of G meets H . (See Lemma 4.1.) If H is, say, a non-normal Sylow 2-subgroup of G , then an elementary argument (using Baer's theorem and involutions, see Proposition 6.4 of [DNT]) produces a non-trivial odd order real element, hence proving Theorem B in this case. In general, however, there are examples of proper subgroups H meeting all the real classes of groups G . And in order to prove Theorem B, we shall need to classify, essentially, all the cases whenever H is a maximal subgroup of odd index in a simple group G and H intersects every real conjugacy class. We believe that this result might have some interest on its own. Indeed, we classify all primitive permutation groups in which every real element has a fixed point. We only state the result for n odd since that is what we use. See Theorem 2.8 for the corresponding result for n even.

Theorem C. *Let G be a primitive permutation group acting on a set X of odd cardinality. Assume that every real element of the socle of G fixes a point of X . Let M be a point stabilizer. Then G has a unique minimal normal subgroup A and either G has odd order or $A = L_1 \times \dots \times L_t$ with $L_i \cong L$ a nonabelian simple group. Also, $M \cap A = D_1 \times \dots \times D_t$, D_i is maximal in L_i and intersects every real conjugacy class of G which intersects L_i . Furthermore $(1_M)^G$ has a nontrivial orthogonal constituent of multiplicity 1. Moreover either G/A has odd order or $L \cong \text{PSL}_3(4)$ and G/A_1 has odd order where A_1/A is an elementary abelian 2-group and A_1 normalizes each L_i . Indeed, $L_i \cong M_{22}, M_{23}$ or $\text{PSL}_n(q)$ with n odd and D_i is as given in Theorem 2.3.*

In the special case where H is a proper 2-Sylow normalizer of G , a case with particular interest, we shall prove (again using the Classification) that not only H does not meet some real class of G , but something stronger.

Theorem D. *Let G be a finite group, and let $P \in \text{Syl}_2(G)$. Then the following are equivalent.*

- (i) P is normal in G .
- (ii) There are no nontrivial real elements in G of odd order.

- (iii) *The 2-Brauer character $(1_{\mathbf{N}_G(P)}^0)^G$ has no nontrivial real-valued 2-Brauer irreducible constituent with odd multiplicity.*
- (iv) *Every real element of odd order of G normalizes a G -conjugate of P .*

We remark that the equivalence of the first three conditions is elementary, while the fourth is proved using the Classification.

Theorem D(iii) naturally suggests a question: is Theorem B true for Brauer characters? And the answer is “no.” For instance $G = \mathrm{PSL}_3(2)$ has $H = \mathbf{S}_4$ as a subgroup and the 2-Brauer character $(1_H^0)^G$ decomposes as the trivial 2-Brauer character of G plus two non-real-valued irreducible 2-Brauer characters of degree 3. Indeed, Theorem B fails for p -Brauer characters for any p . Let $G = \mathrm{PSL}_n(p)$ with n odd. By [ZS, 1.8], the composition factors in characteristic p of $(1_H)^G$, with H the stabilizer of a 1-space, are the trivial module and $(p-1)\omega_i$ where the ω_i are the fundamental dominant weights for $\mathrm{SL}_n(p)$. In particular, since n is odd, none of the nontrivial composition factors are self dual and so none of the nontrivial irreducible p -Brauer characters are real.

On the other hand, in Section 5 below, we shall prove that analogs of Theorems A and B do hold for subgroups of odd index and for 2-Brauer characters if certain composition factors are not involved.

It is often the case that theorems on real-valued characters can be improved to rational-valued characters. This is clearly not the case with the results in this paper. For instance, if $G = \mathbf{D}_{10}$ is the dihedral group and $P \in \mathrm{Syl}_2(G)$, then $(1_P)^G$ has a unique rational-valued irreducible constituent with odd multiplicity.

2. PRIMITIVE GROUPS AND REAL ELEMENTS

In this section, we classify the primitive permutation groups such that all real elements have fixed points. We apply the result in the case that the degree is odd but the methods to deal with the general case are only a bit more onerous than the odd case and there are very few examples in the even case.

The critical case is when the group is almost simple. The existence of the Mathieu group examples shows that it is unlikely that there is a proof without the classification of finite simple groups.

Before we start the proofs of the the simple case, we note the following elementary results.

Lemma 2.1. *Let N be a normal subgroup of G and let $x \in N$.*

- (i) *If x is real in N , then so is x^g for any $g \in G$.*
- (ii) *If $[G : \mathrm{NC}_G(x)] = m$ is odd and x is real in G , then x is real in N .*

Proof. The first statement is obvious. Consider (ii). Then x^G is the union of m distinct conjugacy classes of N . Note the inversion permutes these m conjugacy

classes and since m is odd, it must fix at least one of them. Thus, some element of x^G is real in N and so by (i), x is real in N . $\square$

In particular, this gives the following result:

Corollary 2.2. *Let $G = \mathrm{SL}_n(q)$ and let $g \in G$ be unipotent. Assume that either q is even or that g has a Jordan block of odd size (which is always the case when n is odd). Then g is real in G .*

Proof. By Jordan canonical forms, g is real in $H = \mathrm{GL}_n(q)$. Our assumptions imply that $[H : GC_H(g)]$ is odd and so the result follows by the previous lemma. $\square$

Note that if $q \equiv 3 \pmod{4}$, then transvections in $\mathrm{SL}_2(q)$ are not real.

We shall use in several parts of this paper an elementary result (for instance, Lemma 3.1(d) of [DMN]): if a 2-group S acts non-trivially on a group of odd order T , then there exists $t \in T$ and $s \in S$ such that $t^s = t^{-1}$.

We state the results for odd and even degree separately.

Theorem 2.3. *Let L be a finite nonabelian simple group. Let M be a maximal subgroup of L such that $[L : M]$ is odd and M intersects every real conjugacy class of L . Then one of the following holds:*

- (i) $L = M_{22}$, $M = 2^4 : \mathrm{Alt}_6$, $[L : M] = 77$ and $(1_M)^L = 1a + 21a + 55a$;
- (ii) $L = M_{22}$, $M = 2^4 : \mathrm{Sym}_5$, $[L : M] = 231$ and $(1_M)^L = 1a + 21a + 55a + 154a$;
- (iii) $L = M_{23}$, $M = M_{22}$, $[L : M] = 23$ and $(1_M)^L = 1a + 22a$;
- (iv) $L = M_{23}$, $M = \mathrm{PSL}_3(4).2_2$, $[L : M] = 253$ and $(1_M)^L = 1a + 22a + 230a$;
- (v) $L = M_{23}$, $M = 2^4 : \mathrm{Alt}_7$, $[L : M] = 253$ and $(1_M)^L = 1a + 22a + 230a$;
- (vi) $L = M_{23}$, $M = 2^4 : (3 \times \mathrm{Alt}_5).2$, $[L : M] = 1771$ and $(1_M)^L = 1a + 22a + 230aa + 253a + 1035a$.
- (vii) $L = \mathrm{PSL}_n(q)$ with $n \geq 3$ and odd, M is the stabilizer of 1-space or hyperplane, $[L : M] = \frac{q^n - 1}{q - 1}$ and L acts 2-transitively on the cosets of M .

Theorem 2.4. *Let L be a finite nonabelian simple group. Let M be a maximal subgroup of L such that $[L : M]$ is even and M intersects every real conjugacy class of L . Then one of the following holds:*

- (i) $L = M_{11}$, $M = \mathrm{Sym}_5$, $[L : M] = 66$ and $(1_M)^L = 1a + 10a + 11a + 44a$;
- (ii) $L = M_{23}$, $M = M_{11}$, $[L : M] = 1288$ and $(1_M)^L = 1a + 22a + 230a + 1035a$;
- (iii) $L = M_{24}$, $M = M_{12}.2$, $[L : M] = 1288$ and $(1_M)^L = 1a + 252a + 1035a$.

We prove the two results together.

Proof. If L is a sporadic simple group, then this is quickly computed in GAP. (We thank Thomas Breuer for doing the computation [Br].) In the statement of Theorem 2.3, $230aa$, for instance, means that the unique irreducible character of degree 230 of M_{23} appears with multiplicity 2.

Suppose that $L = \text{Alt}_n$ with $n \geq 5$. First suppose that n is odd and $n \geq 9$. Note that the following Sym_n -conjugacy classes consist of real elements in Alt_n : $n - 2$ cycles, elements with exactly three orbits of size 3, 3 and $n - 6$ and 3-cycles. Since the first two classes intersect M , it follows that M is primitive and since it contains a 3-cycle, $M = L$, a contradiction.

Next suppose that n is even and $n \geq 10$. Then consider the following Sym_n real classes in Alt_n : $n - 3$ cycles, elements with two cycles of size 4 and $n - 4$, elements with orbits of size 1, 3, 3 and $n - 7$ and 3-cycles. Again it follows that D is primitive and contains 3-cycles, a contradiction.

If $n < 9$, one can compute using GAP (or check by hand) that there are no possibilities.

So L is a finite simple group of Lie type in characteristic p . Let X be the corresponding algebraic group with X simply connected. We will assume that $L = X^F/Z(X^F)$ where F is the corresponding Lang-Steinberg endomorphism. The very few cases where this fails can be checked in GAP (and many arise in different contexts – e.g. $\text{Sp}_4(2) \cong \text{Sym}_6$).

First consider the case that $L = \text{PSL}_2(q)$ with $q \geq 11$ and q odd (if $q = 5, 7$ or 9 , these groups arise in other forms). Then elements of order $(q \pm 1)/2$ are real and no proper subgroup contains elements of both orders. If $L = \text{PSL}_2(q)$ with $q \geq 4$ and even, then all elements are real.

Next suppose that -1 is in the Weyl group of $L \neq \text{PSL}_2(q)$. Then all semisimple elements are real. We claim that there exist nontrivial real unipotent elements. If the characteristic of L is even, then any element of order 2 is real and unipotent. Suppose that the characteristic is not 2. It suffices to prove this for groups of rank at most 2. In $\text{PSL}_3(q)$ or $\text{PSU}_3(q)$, transvections are real. Since $\text{Sp}_4(q)$ contains $\text{GL}_2(q)$, the claim holds for Sp_4 . Since $G_2(q)$ contains $\text{SL}_3(q)$, the claim holds there as well. The only remaining case is ${}^2G_2(q)'$ with q an odd power of 3. Then $\text{PSL}_2(8)$ is a subgroup and contains real elements of order 3.

Thus every prime dividing the order of L also divides the order of M . By [LPS, 10.7], this can only occur in the following cases:

- (i) $L \cong \Omega_{2n+1}(q)$, $n \geq 2$, n even and M is the stabilizer of a nondegenerate hyperplane of $-$ type (when q is even, $L = \text{Sp}_{2n}(q)$);
- (ii) $L \cong \Omega_{2n}^+(q)$, $n \geq 4$, even and M is the stabilizer of a nondegenerate hyperplane;
- (iii) $L \cong \text{PSp}_4(7)$ and $M = \text{Alt}_7$;
- (iv) $L \cong \text{Sp}_6(2)$ and $M = \text{Sym}_8$;
- (v) $L \cong \Omega_8^+(2)$ and M a maximal end node parabolic or $M = \text{Alt}_9$;
- (vi) $L \cong G_2(3)$ and $M = \text{PSL}_2(13)$; or
- (vii) $L \cong {}^2F_4(2)'$ and $M = \text{PSL}_2(25)$.

In all but the first two cases, the result follows easily by inspection of the character tables. Suppose that $L \cong \Omega_{2n+1}(q)$, $n \geq 2$ with n even and M the stabilizer of a nondegenerate hyperplane of $-$ type. Let x be a real element stabilizing two complementary totally singular subspaces each of dimension n which generate a hyperplane of $+$ type. Then x is real and does not fix a hyperplane of $-$ type and so is not conjugate to an element of M .

Suppose that $L \cong \Omega_{2n}^+(q)$, $n \geq 4$, even and M is the stabilizer of a nondegenerate hyperplane. Let x be a real element stabilizing two complementary totally singular subspaces each of dimension n . Then x does not preserve any hyperplane and in particular is not conjugate to an element of M .

So we may assume that -1 is not in the Weyl group of L .

First consider the case that $L = \Omega_{2n}^\epsilon$, $n \geq 5$ and odd. Consider the following real conjugacy classes. The first is an element of order $(q^{n-1} + 1)$ acting irreducibly on a nondegenerate subspace of codimension 2 (of $-$ type) and trivial on its perpendicular complement. The second is an element of order $q^{n-1} - 1$ acting irreducibly on each of a pair of totally singular spaces of dimension $n - 1$ which generate a nondegenerate subspace of dimension $2n - 2$ of $+$ type and again acting trivially on the 2 dimensional orthogonal complement.

Then M must act either absolutely irreducibly or is irreducible on a nondegenerate hyperplane. Moreover, M is not a subfield group either.

Consider a unipotent element x with two Jordan blocks of size n . Note that x is real in the algebraic group (since it is contained in GL_n). Since n is odd, this class is invariant under the graph automorphism and any Frobenius automorphism. By [LS2, Theorem 3.1], the centralizer of x is connected (in SO_{2n}) and so we can choose x real in L (if q is even, this is clear – if q is odd, we can also see this element in $\mathrm{SO}_n(q) \times \mathrm{SO}_n(q)$ and the inverting element can be chosen of spinor norm 1). Since x is conjugate to an element of M , M cannot preserve a nondegenerate hyperplane and so must act irreducibly.

Note that root elements are real and so by either [Ka, Theorem 1] or [GS, Theorem 7.1] there are no possibilities for irreducible M intersecting all the conjugacy classes above.

Next consider $L = E_6^\epsilon(q)$. Then long root elements are real as well as elements in the unipotent class $E_6(a_1)$ [LS1, Chapter 22]. The latter class rules out $M = F_4$. Also every semisimple conjugacy class which intersects $F_4(q)$ is real. In particular, M contains elements of order $q^4 \pm 1$ and $q^4 - q^2 + 1$. In particular, this implies that M is not parabolic.

We claim that there exists a conjugacy class of real regular unipotent elements. If $p \geq 3$, this follows from the fact that a regular unipotent element is real in the algebraic group and the number of connected components of the centralizer is odd. If $p = 2$, it is still true that there is a class of real regular unipotent elements by inspection of the character table of $F_4(2)$ (and noting that regular unipotent elements

in F_4 are also regular unipotent in E_6). A list of all possible maximal subgroups containing a regular unipotent element in an exceptional group is given in [GM] and none of them contain all the real elements described above.

Next suppose that $L = \mathrm{PSL}_n(q)$ with $n \geq 3$. We first show that if M is reducible, then n is odd and M must be the stabilizer of a 1-space or hyperplane.

Suppose that n is odd. Then there exists a real element $x \in L$ acting irreducibly on a hyperplane (take an irreducible element in $\mathrm{Sp}_{n-1}(q) \leq \mathrm{SL}_{n-1}(q)$). Thus, if M is reducible, the only possibility is that M is the stabilizer of a hyperplane or 1-space. Note that since $\mathrm{SL}_n(q)$ has odd center, real elements lift to real elements in $\mathrm{SL}_n(q)$. Any real element of $\mathrm{SL}_n(q)$ must have 1 as eigenvalue and so fixes a 1-space and a hyperplane and so this example is allowed in the conclusion. Suppose that $n \geq 4$ is even. Then there is a real element x that acts irreducibly and so is not in any parabolic.

So we may assume that M is irreducible. Note that long root elements are real. If q is odd, it follows by [Ka] that M would have to be a symplectic group, a linear group or a unitary group. Since M is irreducible, M is either a symplectic, unitary or linear group over some subfield [Ka]. If n is even, there are irreducible real elements not defined over any subfield (and clearly not in any unitary group) and then the only possibility for M is $\mathrm{PSp}_n(q)$. If we take x to be unipotent with single nontrivial Jordan block which is of size 3, then x is real but is not contained in a symplectic group. If n is odd and M is irreducible, then there are real elements of order $q^{(n-1)/2} + 1$ acting irreducibly on a hyperplane and again this rules out all the possibilities for M .

Essentially the same proof works when q is even aside from a very few small cases. Note that all unipotent elements are real by Corollary 2.2. Now argue as above using either [Ka] or for $d \geq 6$ [GS, Theorem 7.1] to conclude there are no such groups. If $d < 6$, a straightforward inspection of the maximal subgroups yields the result [BHR].

Finally suppose that $L = \mathrm{PSU}_n(q)$ with $n \geq 3$. If $n = 3$, then an element of order $q + 1$ with eigenvalues $1, a, a^{-1} = a^q$ is real and is contained in no parabolic. If $n = 4$, then similarly as long as $q > 2$, we can find a real element of order $q + 1$ with distinct eigenvalues not contained in any parabolic subgroup. Note that $L = \mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$, a case already handled. Since transvections are real, by inspection there are no proper subgroups intersecting each real class of L .

So assume that $n \geq 5$. First suppose that n is odd. Then there exists a conjugacy of real regular unipotent elements (since the centralizer has an odd number of components in the algebraic group and the element is real in the algebraic group). Thus, M cannot preserve a nondegenerate space. There are real elements in L that have irreducible submodules of dimensions $(n-1)/2, n(n-1)/2$ and 1 (all nonisomorphic) with the first two totally singular. Thus, the only possible parabolic that M could be is the stabilizer of totally singular $(n-1)/2$ -dimensional space. We can choose $y \in L$ real that has irreducible non-isomorphic modules of dimensions $(n-3)/2, (n-3)/2$

and 3 one-dimensional modules (all nonsingular). Thus element does not stabilize a totally singular $(n-1)/2$ space and so M must be irreducible. Since transvections are real, M is absolutely irreducible. Applying [Ka] or [GS] as in the linear case, we see that there are no possibilities for M .

Next suppose that $n \geq 6$ is even. Note that unipotent elements with two Jordan blocks of size $n/2$ are real (since they are in $\mathrm{GL}_{n/2}(q^2)$). A unipotent element with a single Jordan block of size $n-1$ is also real. Thus, M cannot preserve a nondegenerate space. There exists $x \in L$ real with irreducible submodules of dimensions $n/2$ and $n/2$ and also one with irreducible submodules of dimensions $n/2-1, n/2-1, 1$ and 1 where the 1-dimensional submodules are nondegenerate. This rules out the possibility that M is parabolic. As in the odd case, we see that since M contains transvections, there are no examples with M irreducible. $\square$

Next we consider subgroups H which are not necessarily maximal but still satisfy the hypotheses of Theorem 2.3. Since we only use the result when n is odd, we assume that and leave the case n even to the interested reader.

Corollary 2.5. *Let L be a finite nonabelian simple group and H a proper subgroup of L such that H intersects every conjugacy class of real elements in L and $[L : H]$ is odd. Then either H is maximal in L or one of the following holds:*

- (i) $L = M_{23}$ and there are three possible conjugacy classes of subgroups H each self normalizing. In each case, $(1_H)^L$ contains a real-valued nontrivial constituent of multiplicity 1; or
- (ii) $L = \mathrm{PSL}_n(q)$ with n odd, H is contained in a unique maximal subgroup M which is the stabilizer of a 1-space or hyperplane and $(1_H)^L$ contains a real-valued constituent of degree $(q^{n-1} - 1)/(q - 1) - 1$ with multiplicity 1.

Proof. If L is a sporadic group, then by the previous result $L \cong M_{22}$ or M_{23} and one computes using GAP (again, we thank Thomas Breuer).

Otherwise, $L \cong \mathrm{PSL}_n(q)$ with n odd. Let M be a maximal subgroup containing H and we assume that $H \neq M$. Then M is the stabilizer of a 1-space or hyperplane. As we noted in the proof of the previous result, H contains a real element x acting irreducibly on a hyperplane. Note that x fixes a unique 1-space and a unique hyperplane which do not intersect. Thus, if H is contained in two maximal subgroups, H would be contained in their intersection. This intersection is the Levi subgroup preserving the 1-space and hyperplane preserved by x . Since this subgroup contains no regular unipotent elements, this is not possible.

By applying an outer automorphism if necessary, we may assume that M is the stabilizer of 1-space. Let U be the unipotent radical of M . By Corollary 2.2, H intersects every conjugacy class of unipotent elements.

If q is odd, then considering real elements that have -1 as eigenvalue odd multiplicity and arguing as in the case of $\mathrm{SL}_{n-1}(q)$, we see that HU contains $[M, M]$. If

$H \cap U = 1$, then as $H^1(\mathrm{SL}_{n-1}(q), U) = 0$ [JP], H would be a Levi complement but this is a contradiction as H would not contain a regular unipotent element. Thus, $H \geq [M, M]$ and H has exactly 2 orbits on 1-spaces, whence the statement about the real constituent follows.

Suppose that q is even. Since H contains unipotent elements of every possible Jordan form and H intersects every conjugacy of real semisimple elements, then arguing as in the proof of Theorem 2.3, it follows that $HU/U \geq \mathrm{Sp}_{n-1}(q)$. If $n \geq 9$, then $\mathrm{Sp}_{n-1}(q)U$ does not contain any unipotent elements with exactly 3 Jordan blocks with the blocks of odd distinct sizes and so for $n \geq 9$, $HU \geq [M, M]$ and we argue as in the case q is odd.

In all cases we see that either $H \geq U$ or H is isomorphic to a subgroup of $\mathrm{GL}_{n-1}(q)$. The number of unipotent classes in that group is less than in G and since H intersects each unipotent class, that is not possible. Thus, $H \geq U\mathrm{Sp}_{n-1}(q)$. Since $\mathrm{Sp}_{n-1}(q)$ is transitive on nonzero vectors, it follows that H has two orbits on lines and so the statement about the real constituent follows. $\square$

The almost simple case also follows easily:

Theorem 2.6. *Let G be a finite almost simple group with socle L (a nonabelian simple group). Let M be a maximal subgroup of G such that $[G : M]$ is odd and $D := L \cap M \neq L$ intersects all real G -classes contained in L . Then one of the following holds:*

- (i) $G = L = M_{22}$, $M = 2^4 : \mathrm{Alt}_6$, $[G : M] = 77$ and $(1_M)^L = 1a + 21a + 55a$;
- (ii) $G = L = M_{22}$, $M = 2^4 : \mathrm{Sym}_5$, $[G : M] = 231$ and $(1_M)^L = 1a + 21a + 55a + 154a$;
- (iii) $G = L = M_{23}$, $M = M_{22}$, $[G : M] = 23$ and $(1_M)^L = 1a + 22a$;
- (iv) $G = L = M_{23}$, $M = \mathrm{PSL}_3(4).2_2$, $[G : M] = 253$ and $(1_M)^L = 1a + 22a + 230a$;
- (v) $G = L = M_{23}$, $M = 2^4 : \mathrm{Alt}_7$, $[G : M] = 253$ and $(1_M)^L = 1a + 22a + 230a$;
- (vi) $G = L = M_{23}$, $M = 2^4 : (3 \times \mathrm{Alt}_5).2$, $[G : M] = 1771$ and $(1_M)^L = 1a + 22a + 230aa + 253a + 1035a$.
- (vii) $L = \mathrm{PSL}_n(q)$ with $n \geq 3$ and odd, M is the stabilizer of 1-space or hyperplane, $[G : M] = \frac{q^n - 1}{q - 1}$ and L acts 2-transitively on the cosets of M . Moreover, $[G : L]$ is odd unless possibly $L = \mathrm{PSL}_3(4)$ and $G/\mathbf{O}^2(G)$ is generated by a field automorphism of order 2.

Proof. If $G = L$, the previous result applies.

Suppose that $G > L$. If L is sporadic, then $L = M_{22}$ and if an outer automorphism is present, then elements of order 11 in L are real and we see there are no proper maximal subgroups of L containing all real elements (the only possibilities are the D already mentioned and they do not contain elements of order 11).

The only other possibility is that $L = \mathrm{PSL}_n(q)$ with n odd and by the case $G = L$ we see that D is contained in a unique maximal subgroup P of L , a parabolic subgroup. Since $M = N_G(D)$, it follows that M normalizes P whence $D = P$. Since $[PGL_n(q) : \mathrm{PSL}_n(q)] = \gcd(n, q-1)$ is odd, if G/L has even order, G contains an outer involution that is either a graph, field or graph-field automorphism. Since $G = LN_G(D)$, G cannot contain a graph or graph-field automorphism.

Suppose that G contains a field automorphism of order 2. Then $q = q_0^2$. Aside from the case that $n = 3$ and $q = 4$, there exists a primitive prime divisor p of $q^n - 1$ and so p divides $q_0^n + 1$. Thus, there is $x \in \mathrm{PSU}_n(q_0) < L$ acting irreducibly. Since a field automorphism induces the transpose inverse automorphism on $\mathrm{PSU}_n(q_0)$, x is real in G , a contradiction.

In the remaining case, $L = \mathrm{PSL}_3(4)$ and $G \leq \mathrm{P}\Gamma\mathrm{L}(3, 4)$. Any real element of G contained in L does have a fixed point. $\square$

We note that if $L = \mathrm{PSL}_3(4)$ and every real element of G has a fixed point, then either $[G : L] = 3$ or $[G : L] = 2$.

We next consider general primitive groups of odd degree. This is Theorem C of the introduction. Recall that real-valued irreducible characters are either of $+$ type (or orthogonal) if they can be afforded by a representation with real entries, or of $-$ type (or symplectic), if they cannot.

Theorem 2.7. *Let G be a primitive permutation group acting on a set X of odd cardinality. Assume that every real element of the socle of G fixes a point of X . Let M be a point stabilizer. Then G has a unique minimal normal subgroup A and either G has odd order or $A = L_1 \times \dots \times L_t$ with $L_i \cong L$ a nonabelian simple group. Also, $M \cap A = D_1 \times \dots \times D_t$, D_i is maximal in L_i (and is described in Theorem 2.3) and intersects every real conjugacy class of G which intersects L_i . Furthermore $(1_M)^G$ has a nontrivial orthogonal constituent of multiplicity 1. Moreover either G/A has odd order or $L \cong \mathrm{PSL}_3(4)$ and G/A_1 has odd order where A_1/A is an elementary abelian 2-group and A_1 normalizes each L_i . Indeed, $L_i \cong M_{22}, M_{23}$ or $\mathrm{PSL}_n(q)$ with n odd.*

Proof. We have that M is maximal in G with $\mathrm{core}_G(M) = 1$. Since M contains a Sylow 2-subgroup S of G , we have that $\mathbf{O}_2(G) = 1$. Suppose that $\mathbf{O}_p(G) \neq 1$ with p odd. Let A be a minimal normal subgroup contained in $\mathbf{O}_p(G)$. Then $G = AM$ and $A \cap M = 1$. If G does not have odd order, then S cannot centralize A , and therefore there is $1 \neq a \in A$ which is inverted by some element of S (see for instance Lemma 3.1(d) of [DMN]). By hypothesis, some G -conjugate of a is in M , but this is impossible since $A \cap M = 1$.

Let A be a minimal normal subgroup of G . Thus, A is a direct product $L_1 \times \dots \times L_t$ with $L_i \cong L$ a nonabelian simple group. Since L_i contains real elements, $D_i = M \cap L_i \neq 1$ for some i . If B is another minimal normal subgroup of G , then $[A, B] = 1$ and thus $M \cap B$ is normal in $AM = G$. Hence $M \cap B = 1$ and therefore $|B| = [G : M]$

is odd. But this cannot happen. Hence A is the unique minimal normal subgroup of G .

We can view X as $L_1/D_1 \times \dots \times L_t/D_t$ and so we see that $x \in L_1$ fixes a point on X if and only if it fixes a point on L_1/D_1 . It follows that D_i intersects any real class of G that intersects some L_j . Let $G_i = \mathbf{N}_G(L_i)/\mathbf{C}_G(L_i)$. This is an almost simple group with socle L_i and $\mathbf{N}_{G_i}(D_i)$ is a maximal subgroup of G_i . Our assumptions imply that D_i intersects every real class of L_i (in G_i) whence $(G_i, L_i, \mathbf{N}_{G_i}(D_i), D_i)$ satisfies the conclusions of Theorem 2.3. In particular, D_i is maximal in L_i and G_i/L_i has odd order unless $L_i \cong \text{PSL}_3(4)$.

We next show that G/A has odd order if $L_i \neq \text{PSL}_3(4)$. Suppose that there exists $g \in G \setminus A$ an outer automorphism with $g^2 \in A$. We may assume that g has order a power of 2. Then g cannot normalize all components L_i (as the automizer in G of any component has odd order). Thus, we may assume that g interchanges L_1 and L_2 and g^2 induces inner automorphisms on L_1 and L_2 . Replacing g by gu with $u \in L_1 \times L_2$ allows us to assume that g interchanges the coordinates of L_1 and L_2 . Choose $x \in L_1$ that has no fixed points. Then g inverts $c := (x, x^{-1}, 1, \dots, 1)$ with $c \in A$. Then c has no fixed points but is real, a contradiction. A similar analysis when $L_i \cong \text{PSL}_3(4)$ completes the proof. $\square$

Here is the analog for n even.

Theorem 2.8. *Let G be a primitive permutation group acting on a set X of even cardinality n . Assume that every real element of the socle of G fixes a point of X . Let M be a point stabilizer. Then G has a unique minimal normal subgroup A and $A = L_1 \times \dots \times L_t$ with $L_i \cong L$ a nonabelian simple group. Also, $M \cap A = M_1 \times \dots \times M_t$, M_i is maximal in L_i and intersects every real conjugacy class of G which intersects L_i . Furthermore G/A has odd order and (L_i, M_i) are described in Theorem 2.4.*

Proof. Let A be a minimal normal subgroup of G . Then A acts transitively whence A has even order. If A acts regularly, then some involution of A has no fixed points, a contradiction. Thus, G has no regular normal subgroups and so $A = L_1 \times \dots \times L_t$ with $L_i \cong L$, a nonabelian simple group. Since $M \cap L_i \neq 1$ for some i (since L_j contains real elements), it follows that $M \cap A = M_1 \times \dots \times M_t$ where M_i are maximal in the almost simple group $\mathbf{N}_G(M_i)/\mathbf{C}_G(M_i)$. It follows by Theorem 2.4 that L_i is a Mathieu group and has trivial outer automorphism group, whence $\mathbf{N}_G(M_i)/\mathbf{C}_G(M_i) = M_i$ and M_i is maximal in L_i . Arguing as in the case that n is odd shows that G/A has odd order. $\square$

We close this section by proving a result about transitive groups which we require for 2-Brauer characters.

Corollary 2.9. *Let G be a (faithful) transitive permutation group of odd degree n . Let $A = L_1 \times \dots \times L_t$ be a normal subgroup of G such that each $L_i \cong L$, a nonabelian*

simple group not isomorphic to any of M_{22}, M_{23} or $\mathrm{PSL}_n(q)$ with n odd. Then some real element (in G) of A is fixed point free.

Proof. Let H be a point stabilizer and set $B = H \cap A \neq A$. Since n is odd, $B \geq S_1 \times \dots \times S_t$ where $S_i \in \mathrm{Syl}_2(L_i)$. Let $Q_i = \pi_i(B)$ where π_i is the projection from A onto L_i . If $Q_i = L_i$, then $L_i = [Q_i, S_i] = [B, S_i] \leq B$. Thus (reordering if necessary), $B \leq M_1 \times L_2 \times \dots \times L_t$ where M_1 is a maximal subgroup of L_1 containing Q_1 . Let $a \in L$ be a real element. Then $y := (a, \dots, a)$ has a fixed point, whence $y^G \cap H$ is nonempty. This implies that $M_1 \geq Q_1$ contains an L_1 conjugate of a of every real element in L_1 . By Theorem 2.3 and the fact that we are excluding the simple groups occurring in the conclusion of that theorem, we obtain a contradiction. $\square$

3. SYLOW 2-SUBGROUPS AND THEOREM D

In this section we prove Theorem D. For Brauer characters we use the notation in [N]. If Ψ is a Brauer character of G , and we write $\Psi = \sum_{\varphi \in \mathrm{IBr}(G)} a_\varphi \varphi$, we call a_φ the multiplicity of φ in Ψ .

Recall that the equivalence of (i) and (ii) in Theorem D is elementary and well-known. (See for instance Proposition 6.4 in [DNT].) We start with condition (iii).

Theorem 3.1. *Let G be a finite group and assume that $P \in \mathrm{Syl}_2(G)$ is not normal in G . Then the 2-Brauer character $(1_{\mathbf{N}_G(P)}^0)^G$ contains a non-trivial irreducible real 2-Brauer character with odd multiplicity.*

Proof. Arguing by contradiction, suppose that

$$(1_{\mathbf{N}_G(P)}^0)^G = a 1_G^0 + \sum_i b_i (\varphi_i + \bar{\varphi}_i) + \sum_j c_j \mu_j,$$

where $\varphi_i \in \mathrm{IBr}(G)$ are non-real, $\mu_j \in \mathrm{IBr}(G)$ are real but non-trivial, a is a positive integer, and c_j is even for every j . Now, by Theorem 2.30 of [N], we have that $\mu_j(1)$ is even for every j . Since $[G : \mathbf{N}_G(P)]$ is odd, we conclude that a is odd.

Let $1 \neq x \in G$ be a real element of odd order (using that P is not normal and Proposition 6.4 in [DNT]). Suppose that x normalizes some Sylow 2-subgroup Q . Let $y \in G$ invert x with y a 2-element. Let Ω be the set of Sylow 2-subgroups of G normalized by x . Then notice that $\langle y \rangle$ acts on Ω and has no fixed point on it. Indeed, if $Q \in \Omega$ and $Q^y = Q$, then $y \in Q$. Then $y^x \in Q^x = Q$. However, $y^x y^{-1} = x^{-2} \in Q$, a contradiction. We conclude that $|\Omega|$ is even. Therefore $b = (1_{\mathbf{N}_G(P)}^0)^G(x)$ is even. Thus

$$b = (1_{\mathbf{N}_G(P)}^0)^G(x) = a + 2\alpha$$

for some algebraic integer α . We deduce that $-a/2$ is an algebraic integer, but this is not possible because a is odd. $\square$

We briefly digress to recall that, unlike ordinary characters, the trivial 2-Brauer character does not necessarily appear with multiplicity 1 in the 2-Brauer character $((1_P)^0)^G$, even if P is self-normalizing. In the following, Φ_φ is the projective indecomposable character associated with $\varphi \in \text{IBr}(G)$.

Lemma 3.2. *If p is a prime, G is a finite group and $P \in \text{Syl}_p(G)$, then*

$$(1_P^0)^G = \sum_{\varphi \in \text{IBr}(G)} \frac{\Phi_\varphi(1)}{|P|} \varphi.$$

Proof. Use the induction formula and compare the character $(1_P^0)^G$ with the restriction of the regular character of G to the p -regular elements of G . $\square$

(Notice that Lemma 3.2 reproves the well-known fact that $|G|_p$ divides the degrees of the projective indecomposable characters.) Now, if $G = \text{Alt}(6)$, then $P = \mathbf{N}_G(P)$ and the 2-Brauer trivial character of G enters with multiplicity 5 in the Brauer character $(1_P^0)^G$.

Next we show that condition (iv) of Theorem D does not happen in simple groups.

Theorem 3.3. *Let G be a finite nonabelian simple group and let P be a Sylow 2-subgroup of G . Then there exists $g \in G$ with g real and of odd order with g not normalizing any G -conjugate of P .*

Proof. Let us remark that if $P = \mathbf{N}_G(P)$, it follows by the Baer-Suzuki theorem that there exists a real element $g \in G$ of odd order, whence the result follows. This handles 21 of the 26 sporadic groups. In the remaining 5 cases, there is a real element of order 5 and 5 does not divide the order of $\mathbf{N}_G(P)$ (these are the cases $G = J_1, J_2, J_3, Sz$ and HN).

If $G = \text{Alt}_n, n \geq 6$, then $P = \mathbf{N}_G(P)$ and the result follows. If $n = 5$, then 5-cycles are real and do not normalize a Sylow 2-subgroup.

Now suppose that G is a simple group of Lie type. Since the proof is quite similar to that of Theorem 2.3, we just indicate the modifications required.

Suppose first that G is a group in characteristic 2. In this case, the proof given in Theorem 2.3 shows that there exist odd order real elements not contained in any parabolic subgroup aside from the cases with $G = \text{PSL}_n(q)$ with n odd. The proof shows that the only parabolics which intersect every real class are the two maximal end node parabolics and in particular not the Borel subgroup, which is the normalizer of a Sylow 2-subgroup.

Finally, suppose that G is a simple group of Lie type in characteristic not 2. If $G = \text{PSL}(2, q), q \geq 11$, then either P is self normalizing and the result follows as above or $|P| = 4$ and $|\mathbf{N}_G(P)| = 12$. Note that there is a real element of odd order either $(q+1)/2$ or $(q-1)/2$ and this element does not normalize a Sylow 2-subgroup.

If $G = \text{PSL}_n(q)$ or $\text{PSU}_n(q)$ with $n \geq 3$, then transvections are real. Let V be the natural module for $\text{SL}_n(q)$ or $\text{SU}_n(q)$. Note that a transvection normalizing a Sylow

2-subgroup would have to preserve each irreducible P -submodule of V of dimension at least 2. Thus, if unless P is diagonalizable, we can reduce to the case that P is irreducible. However, by [Ka], the irreducible subgroups containing transvections are classified and $\mathbf{N}_G(P)$ is not a possibility. If P is diagonalizable, then by induction, we can reduce to the case $n = 2$ which was handled above.

Next assume that -1 is in the Weyl group of G . Then every semisimple element is real. Note that there exist unipotent real elements (since a Borel subgroup B has even order and $\mathbf{O}_2(B) = 1$, this follows by the Baer-Suzuki theorem). It follows that any prime dividing $|G|$ also divides $|\mathbf{N}_G(P)|$. By [LPS, 10.7], the only possibilities for (G, M) with M containing $\mathbf{N}_G(P)$ are $G = \Omega_{2m+1}(q)$ and M the stabilizer of a hyperplane of $-$ type. Let x be an element whose order is the odd part of $q^m - 1$. Then the only irreducible subspaces of x are two totally singular hyperplanes of dimension m and a nondegenerate 1-space. Note that x is real and not conjugate to an element of M , whence the result.

The only remaining possibility for G a classical group is $\Omega_{2m}^\pm(q)$ with $m \geq 5$ odd. Note that in this case P has a unique two dimensional irreducible submodule and so we can work in $\Omega_{2m-2}^\pm(q)$ and use the previous result.

This leaves only the case that $G = E_6^\pm(q)$. It is easy to see that $\mathbf{N}_G(P)/P$ embeds in the Weyl group and so in particular contains no elements of prime order greater than 5. On the other hand, if $g \in F_4(q) < G$ is an element whose order is a primitive prime divisor of $q^{12} - 1$, then g is real and has order at least 13, whence g normalizes no conjugate of P . $\square$

In order to prove the next result, which easily closes the proof of Theorem D, we use the fact that odd order real elements of factor groups can be lifted to real elements of the group. This is key in an inductive hypothesis.

Theorem 3.4. *Suppose that G is a finite group, $P \in \text{Syl}_2(G)$. If every odd order real element of G lies in some G -conjugate of $\mathbf{N}_G(P)$, then P is normal in G .*

Proof. Argue by induction on $|G|$. Suppose that $1 < N$ is normal in G , and let $Nx \in G/N$ of odd order. By Lemma 3.2 of [NST], there is $y \in G$ real such that $Nx = Ny$. Since Nx has odd order, then by taking $y_{2'}$, we may assume that y has odd order. Then $y^g \in \mathbf{N}_G(P)$ for some $g \in G$, so $Nx^g \in \mathbf{N}_{G/N}(PN/N)$, and by induction, we may assume that PN is normal in G .

In particular, assume that N is a minimal normal subgroup, and let $K = PN$. If N is a 2-group, then $N \leq P$ and so P is normal in G . If N is an elementary abelian subgroup of odd order, then either P centralizes N whence P is characteristic in K and so normal in G or $\mathbf{C}_P(N) = 1$ and $P = \mathbf{N}_K(P)$. Then Baer-Suzuki implies that there exist a real element g of odd order in N . Note that g does not normalize a Sylow 2-subgroup of K . Since K is normal in G , any Sylow 2-subgroup of G is contained in K and we have a contradiction.

So N is a direct product of nonabelian simple groups. Let L be a component of N . By Theorem 3.3, there is an element $g \in L$ that is real and of odd order which does not normalize any Sylow 2-subgroup of L . Then g normalizes no Sylow 2-subgroup of N (since if g normalizes some Sylow 2-subgroup T of G , then g also normalizes $T \cap L$, a Sylow 2-subgroup of L). $\square$

4. PROOF OF THEOREM B

Our notation for characters follows [Is]. If $N \triangleleft G$ and $\theta \in \text{Irr}(N)$, then $\text{Irr}(G|\theta)$ is the set of irreducible constituents of the induced character θ^G . By the Frobenius reciprocity, these are the characters χ such that θ is a constituent of the restriction χ_N . Also, $\nu_2(\chi)$ denotes the Frobenius-Schur indicator of $\chi \in \text{Irr}(G)$. If χ is real valued, then $\nu_2(\chi) = \pm 1$, according to its type.

Lemma 4.1. *Let G be a finite group, and let H be a subgroup of G . If $(1_H)^G$ has a unique real-valued irreducible constituent with odd multiplicity, then $x^G \cap H \neq \emptyset$ for every real element $x \in G$. Furthermore, $[G : H]$ is odd.*

Proof. Write

$$(1_H)^G = 1_G + 2 \sum_i a_i \epsilon_i + \sum_j b_j (\tau_j + \bar{\tau}_j),$$

where $\epsilon_i \in \text{Irr}(G)$ are real-valued, and $\tau_j \in \text{Irr}(G)$ are not. If $x \in G$ is real and $x^G \cap H = \emptyset$, then by evaluating in x the expression above, we conclude that

$$0 = 1 + 2\alpha,$$

where α is an algebraic integer. This is impossible. For the second part, evaluate the character $(1_H)^G$ in the trivial element. $\square$

Indeed, in Lemma 4.1, the conclusion $x^G \cap H \neq \emptyset$ can be replaced by the stronger condition $(1_H)^G(x)$ is odd for every real element $x \in G$. Unfortunately, this does not seem to shorten the proofs much – in all the cases with G primitive of odd degree, the condition that every real element has a fixed point turns out to be equivalent to the condition that every real element has an odd number of fixed points.

The following is well-known.

Lemma 4.2. *Suppose that G/N has odd order.*

- (i) *If $\chi \in \text{Irr}(G)$ is real valued, then every irreducible constituent of χ_N is real-valued.*
- (ii) *If $\theta \in \text{Irr}(N)$ is real-valued, then there exists a unique real-valued $\chi \in \text{Irr}(G|\theta)$.*

Proof. Let $\theta \in \text{Irr}(N)$ be under χ . By Clifford's theorem, we have that $\bar{\theta}$ and θ are G -conjugate by some $g \in G$. Then g^2 fixes θ , and therefore g fixes θ , using that G/N has odd order. Hence, $\theta = \theta^g = \bar{\theta}$. The second part is Corollary 2.2 of [NT]. $\square$

We should warn the reader that if $[G : H]$ is odd and $\theta \in \text{Irr}(H)$ is real valued, even of $+$ type, it is not necessarily true that θ^G contains a real-valued irreducible constituent. The $\text{SmallGroup}(48, 30) = C_3 : Q_8$ is an example. Also, it is often the case, that odd-degree real-valued irreducible constituents of $(1_H)^G$ appear with odd multiplicity whenever $[G : H]$ is odd. But not always, as shown by $G = \text{PSL}_2(25)$ and $H = D_{24}$.

The following includes Theorem B of the introduction.

Theorem 4.3. *Let G be a finite group and let H be a proper subgroup of G . Assume that $G = KH$ where K is the smallest normal subgroup of G with G/K of odd order. Then $(1_H)^G$ contains a non-trivial real-valued irreducible constituent with odd multiplicity.*

Proof. Let G be a counterexample minimizing $|G|$. By Lemma 4.1, we have that $[G : H]$ is odd and that H intersects every real conjugacy class of G . Let $1 < V$ be a normal subgroup of G . By working in G/V , we may assume that $HV = G$ (since if ϕ is an irreducible constituent of $(1_{HV})^G$, then its multiplicity in $(1_H)^G$ is the same as in $(1_H)^G$). In particular, $\text{core}_G(H) = 1$.

Note that $\mathbf{O}_2(G) \leq H$ and so $\mathbf{O}_2(G) = 1$. Let A be a minimal normal subgroup of G , so that $G = HA$. Suppose that A is an elementary abelian p -group for some odd prime p . Then $H \cap A \triangleleft G$ and thus $H \cap A = 1$ and $\mathbf{C}_H(A) = 1$. In particular, if P is a Sylow 2-subgroup of H , then P does not centralize A . Hence, A contains a non-trivial real element x (see for instance Lemma 3.1(d) of [DMN]). However, $x^G \cap H \subseteq A \cap H = 1$, a contradiction.

We claim that A is the unique minimal normal subgroup of G . Let B be another such. Then $G = HB$ and B centralizes $H \cap A$, whence $H \cap A$ is normal, a contradiction.

It follows that $A = L_1 \times \dots \times L_t$ with $L_i \cong L$ a nonabelian simple group. Let $C_i = H \cap L_i$. If $x \in L_i$ is real, then $x^g \in H$. Using that $g = ah$ for some $a \in A$ and $h \in H$, and that $[L_i, L_j] = 1$ for $i \neq j$, we have that $x^{l_i} \in C_i$ for some $l_i \in L_i$. Therefore L_i satisfies the hypotheses of Corollary 2.5 with respect to the subgroup C_i .

Let M be a maximal subgroup of G containing $A \cap H$. Notice that $\text{core}_G(M) = 1$, since otherwise, $A \leq M$ but $G = AH$. By Theorem 2.7, we have that $K = A$ unless possibly $L \cong \text{PSL}_3(4)$ and K/A is an elementary abelian 2-group normalizing each L_i . In particular, H acts transitively on the L_i .

Let $C = C_1 \times \dots \times C_t$ and note that $H \leq \mathbf{N}_G(C)$. Also, $\mathbf{N}_A(C) = B_1 \times \dots \times B_t$, where $B_i = \mathbf{N}_{L_i}(C_i)$. Let $\pi_i : H \cap A \rightarrow L_i$ be the projection map. So $C_i \leq \pi_i(H \cap C) \leq B_i$. We now apply Corollary 2.5. If $L = M_{22}$, then $B_i = C_i$ is maximal in L_i . If $L = M_{23}$, then $B_i = C_i$ since C_i is self-normalizing. Thus, in these cases $A \cap H = C$.

In the case that $L = \text{PSL}_n(q)$ with n odd, B_i is the maximal parabolic containing C_i .

By Corollary 2.5, we have that the characters

$$(1_{C_1})^{L_1} \times \dots \times (1_{C_t})^{L_t}$$

and

$$(1_{B_1})^{L_1} \times \dots \times (1_{B_t})^{L_t}$$

have a non-trivial real-valued constituent $\eta \in \text{Irr}(K)$ of multiplicity 1. Since $C \leq H \cap K \leq B$, in all cases we conclude that

$$((1_H)^G)_K = (1_{H \cap K})^K$$

contains a non-trivial irreducible constituent η with multiplicity 1. (Here we are using that $(1_{C_i})^{L_i}$ and $(1_{B_i})^{L_i}$ contain the same non-trivial irreducible constituent τ_i with multiplicity 1, by Corollary 2.5.)

Since G/K is odd, let $\nu \in \text{Irr}(G)$ be the unique real-valued irreducible constituent lying over η , by Lemma 4.2. Now, if we write

$$(1_H)^G = a\nu + \sum_i a_i(\nu_i + \bar{\nu}_i) + \Delta$$

where Δ is a character such that $[\Delta_K, \eta] = 0$ and $\nu_i \in \text{Irr}(G)$ are non-real and lie over η , we have that

$$1 = [((1_H)^G)_K, \eta] = a + 2 \sum_i a_i [\nu_i, \eta],$$

and we deduce that $a = 1$. □

Notice that Lemma 4.4 below implies that the odd-multiplicity irreducible character in Theorem 4.3 is of $+$ type. The following is well-known.

Lemma 4.4. *Suppose that G is a finite group, and H is a subgroup of G . Let ρ be a character of G which admits a G -invariant non degenerate quadratic form. If $\chi \in \text{Irr}(G)$ is real-valued and $[\chi_H, \rho]$ is odd, then χ has $+$ type.*

Proof. Let V be a $\mathbb{C}G$ -module affording ρ . Let W be the homogeneous component of V corresponding to χ . Note that W is orthogonal to every other homogeneous component (since it is self dual), whence W is non degenerate.

So we may assume that $\rho = m\chi$ with m odd. We induct on m . If $m = 1$, the result is clear. Otherwise, let U be an irreducible submodule of W . If $\nu_2(\chi) = -1$, then U must be totally singular with respect to the quadratic form. Then $U^\perp/U$ is non degenerate with respect to the quadratic form with character $(m-2)\chi$ and the result follows. □

In particular, Lemma 4.4 applies to the case where $\rho = (1_H)^G$ or more generally if $\rho = \psi^G$ with $\psi \in \text{Irr}(G)$ irreducible of $+$ type under the assumption that $[\rho_H, \psi]$ is odd. One can also give a proof of the previous result using Schur indices. The proof above shows that the Schur index over $\mathbb{R}$ is odd and divides 2 whence the result.

To summarize the results in this section, we have shown the following.

Theorem 4.5. *Let H be a subgroup of G . Then $(1_H)^G$ has a non-trivial real-valued irreducible constituent of odd multiplicity under any of the following hypotheses.*

- (i) $[G : H]$ is even.
- (ii) $H \cap x^G = \emptyset$ for some real element x of G .
- (iii) $(1_H)^G(x)$ is even for some real element x of G .
- (iv) H is maximal in G and $G/\text{core}_G(H)$ has even order.
- (v) $G = \mathbf{O}^{2'}(G)H$ and H does not contain $\mathbf{O}^{2'}(G)$.

Proof. The first three results follow from Lemma 4.1 and its proof, while (iv) and (v) follow from Theorem 4.3. $\square$

5. MORE ON 2-BRAUER CHARACTERS

In this final section, we prove that there are versions of Theorems A and B for 2-Brauer characters. First we need some results for p -Brauer characters, where p is any prime, which are essentially well-known. If $\Psi = \sum_{\varphi \in \text{IBr}(G)} a_\varphi \varphi$ is a Brauer character of G , with certain abuse of notation, let us write $a_\varphi = [\Psi, \varphi]$.

Lemma 5.1. *Suppose that G/N has odd order.*

- (i) *If $\chi \in \text{IBr}(G)$ is real valued, then every irreducible constituent of χ_N is real-valued.*
- (ii) *If $\theta \in \text{IBr}(N)$ is real-valued, then there exists a unique real-valued $\chi \in \text{IBr}(G|\theta)$. Furthermore, $[\chi_N, \theta] = 1$.*

Proof. Part (i) follows precisely as in the proof of Lemma 4.2(i). The first part of (ii) follows from Lemma 5 and Corollary 1 of [NT06]. Now, notice, in that proof, that $\chi = \psi^G$, where $\psi \in \text{Irr}(T)$ is the unique real extension of θ to the stabilizer T of θ in G . By the Clifford correspondence for Brauer characters (Theorem 8.9 of [N]), we have that $[\chi_N, \theta] = [\psi_N, \theta] = 1$. $\square$

We also need the following result, which is well-known for ordinary characters. We have been unable to find a proof without using projective representations. (The analogs of this result for induction of Brauer characters and Isaacs π -partial characters can be found in [NRS].)

Lemma 5.2. *Suppose that $G = NH$, where $N \triangleleft G$ and $H \leq G$. Let $D = N \cap H$. Let $\theta \in \text{IBr}(N)$ be G -invariant such that $\theta_D = \varphi \in \text{IBr}(D)$. Then restriction defines a bijection $\text{IBr}(G|\theta) \rightarrow \text{IBr}(H|\varphi)$.*

Proof. Let $\mathcal{P}$ be a projective representation of G associated with θ with factor set α . (That is, $\mathcal{P}_N$ affords θ , $\mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n)$ and $\mathcal{P}(ng) = \mathcal{P}(n)\mathcal{P}(g)$ for $g \in G, n \in N$, see Theorem 8.14 of [N].) We know that $\alpha(xn, ym) = \alpha(x, y)$ for $x, y \in G$ and $n, m \in$

N . View $\alpha \in \mathbf{Z}^2(G/N, \mathbb{C}^\times)$. Then notice that $\mathcal{P}_H$ is a projective representation of H associated with φ .

Now, let $\mu \in \text{IBr}(G|\theta)$. By Theorem 8.16 in [N], there is a projective representation $\mathcal{Q}$ of G/N with factor set α^{-1} such that $\mathcal{Q} \otimes \mathcal{P}$ affords μ .

By Theorem 8.18 of [N], we have that $\mathcal{Q}$ is irreducible. Now, notice that $\mathcal{Q}_H$ is irreducible since $Nh \mapsto Dh$ is an isomorphism $G/N \rightarrow H/D$. By Theorem 8.18 of [N], we have that the representation $\mathcal{Q}_H \otimes \mathcal{P}_H = (\mathcal{Q} \otimes \mathcal{P})_H$ is irreducible. Therefore μ_H is irreducible.

Suppose now that $\tau \in \text{IBr}(G|\theta)$, and let $\mathcal{Q}'$ as before such that $\mathcal{Q}' \otimes \mathcal{P}$ affords τ . If $\tau_H = \mu_H$ then $\mathcal{Q}'_H \otimes \mathcal{P}_H$ and $\mathcal{Q}_H \otimes \mathcal{P}_H$ are similar. By Theorem 8.16 of [N], $\mathcal{Q}'_H$ and $\mathcal{Q}_H$ are similar. By the isomorphism, $\mathcal{Q}'$ and $\mathcal{Q}$ are similar, and therefore so are $\mathcal{Q} \otimes \mathcal{P}$ and $\mathcal{Q}' \otimes \mathcal{P}$. Hence $\tau = \mu$. Surjectivity is proved similarly. $\square$

Lemma 5.3. *Suppose that $N \leq H \leq G$, where N is normal in G , G/N has odd order, and $\theta \in \text{IBr}(H)$ is real-valued. Then there is a unique real-valued $\chi \in \text{IBr}(G)$ such that θ is an irreducible constituent of χ_H . Furthermore, $[\chi_H, \theta] = 1$. If $p = 2$, then $[\theta^G, \chi] = 1$.*

Proof. By Lemma 5.1(i), let $\nu \in \text{Irr}(N)$ be real-valued under θ . First of all notice that if χ exists necessarily it is unique. Indeed, if $\chi_i \in \text{Irr}(G)$ are real-valued over θ , then χ_i lies over ν , and therefore $\chi_1 = \chi_2$ by Lemma 5.1(ii). Also, the fact that if χ exists then $[\chi_H, \theta] = 1$, follows from the second part of Lemma 5.1.

In this paragraph, we prove that if $p = 2$, then χ is necessarily an irreducible constituent of θ^G . By the second part of Corollary 8.7 of [N], we have that $[\chi_N, \nu] = [\nu^G, \chi]$, that is, Frobenius reciprocity holds for $N \triangleleft G$ if the characteristic does not divide $|G/N|$. Hence, by Lemma 5.1, we can write $\nu^G = \theta + \sum_i a_i(\eta_i + \bar{\eta}_i)$, where $\eta_i \in \text{IBr}(H)$ are not real-valued. Now $\nu^G = \theta^G + \sum_i a_i(\eta_i + \bar{\eta}_i)$. Since χ is real-valued and appears with multiplicity one in ν^G , necessarily χ is a constituent of θ^G , and appears with multiplicity 1.

Hence, we only need to prove that there exists some real-valued $\chi \in \text{Irr}(G)$ lying over θ . Suppose that (G, H, N) is a counterexample minimizing first $|G|$, then $[G : N]$, and finally $[G : H]$. Using our inductive hypotheses, it is straightforward to check that we may assume that H is maximal and that N is the core of H in G . If $H = N$, then the claim is Lemma 5.1. Now, let K/N be a chief factor of G . Then $KH = G$ and $K \cap H = N$, using that G/N is solvable. Let I be the stabilizer of ν in H and let $\tau \in \text{Irr}(I)$ be the Clifford correspondent of θ over ν . By the uniqueness in the Clifford correspondence, we have that τ is real-valued. If $IK < G$, then by induction, there is a real-valued $\mu \in \text{Irr}(IK)$ over τ . Since $[G : K] < [G : N]$, by induction, there is $\chi \in \text{Irr}(G)$ real-valued over μ . Then χ lies over τ . Thus there is $\epsilon \in \text{IBr}(H)$ under χ such that ϵ_I contains τ . By the Clifford correspondence for Brauer characters, necessarily $\epsilon = \tau^H = \theta$, and we are done in this case. Hence, we may assume that $IK = G$ and therefore $I = H$. If H is the stabilizer of ν in

G , then θ^G is irreducible and lies over θ by the Clifford correspondence, and we are done. Hence, we may assume that ν is G -invariant. By Lemma 5.1(ii), there is a unique real-valued $\hat{\nu} \in \text{Irr}(K)$ over ν , that also extends ν . By uniqueness, notice that $\hat{\nu}$ is G -invariant. By Lemma 5.2, we have that restriction defines a bijection $\text{Irr}(G|\hat{\nu}) \rightarrow \text{Irr}(H|\nu)$. Now, if $\gamma \in \text{Irr}(G|\hat{\nu})$ extends θ , then γ is real-valued by the uniqueness of the restriction map. $\square$

We have mentioned in the introduction that the conclusion of Theorem B does not hold for Brauer characters, and we have given some counterexamples. However, using similar arguments as those used before, we can prove that essentially these are all the counterexamples. We only sketch a proof.

Theorem 5.4. *Let G be a group with a proper subgroup H of odd index. Set $K = \mathbf{O}^2(G)$ and assume that H does not contain K . If K does not surject onto M_{23} or $\text{PSL}_n(q)$ with n odd and q even, then the 2-Brauer character $((1_H)^0)^G$ has a non-trivial real-valued irreducible constituent ϕ .*

Proof. First assume that $G = KH$.

Arguing by induction on $|G|$ and then on the index $[G : H]$, it suffices to assume that H is maximal in G (note that this maximal subgroup does not contain K). In this case, we will prove that there is a real constituent of odd multiplicity.

We may assume that H is corefree. Since H has odd index and every nontrivial self dual irreducible Brauer character has even degree, it follows that the trivial character occurs an odd number of times in $((1_H)^0)^G$. Arguing as in the proof of Theorem 3.1 shows that the result follows unless every odd order real element has an odd number of fixed points on $\{Hx \mid x \in G\}$. Since $[G : H]$ is odd, this implies the same for any element whose odd part is real (because if $g \in G$, then $\langle g_2 \rangle$ acts on the set of points fixed by g_2). Hence H must intersect every conjugacy class of real elements. Now by Theorem C, K is a direct product of copies of a simple nonabelian group L and L is isomorphic to M_{22} , M_{23} or $\text{PSL}_n(q)$ with n odd.

Since we are excluding the possibility that $L = M_{23}$ or $\text{PSL}_n(q)$ with q even, it follows by Theorem C that $L \cong M_{22}$ or $\text{PSL}_n(q)$ with n odd and q odd and moreover the structure of $M \cap K$ is given in Theorem C. Arguing as in the case of ordinary characters, the result follows once we know that $((1_{M \cap L})^0)^L$ has a nontrivial real constituent. If $L = M_{22}$, then the result follows by inspection. If $G = \text{PSL}_n(q)$ with q odd, then the permutation module in characteristic 2 is $k \oplus W$ with W irreducible (and so self dual). This follows from the fact that the smallest representation of $\text{PSL}_n(q)$ with nq odd in characteristic 2 has dimension $\frac{q^n-1}{q-1} - 1$ [GPPS].

The result follows in this case. Now consider the general case. Set $J = HK$. Then we know that $((1_H)^0)^J$ has a nontrivial real constituent θ . By Lemma 5.3, then there exists a nontrivial real irreducible Brauer character ϕ such that $[\theta^G, \phi] = 1$ whence ϕ is a real nontrivial constituent of $((1_H)^0)^G$. $\square$

If we strengthen the hypotheses a bit, we can prove the analog of Theorem B for 2-Brauer characters for subgroups of odd index. We first state an elementary lemma.

Lemma 5.5. *Let G be a finite group, H a subgroup of G , and A a normal subgroup of G of order not divisible by p . Let ϕ be an irreducible p -Brauer character with $A \leq \ker(\phi)$. Then $[((1_H)^0)^G, \phi] = [((1_{HA})^0)^G, \phi]$.*

Proof. Let k be an algebraically closed field of characteristic p and let $M = k[G/H]$ be the permutation module. Since $|A|$ has order prime to p , $M = [A, M] \oplus C_M(A)$. Also, note that there is a surjection f of kG -modules from $k[G/H]$ onto $k[G/J]$. Since A acts trivially on $k[G/J]$ and since the dimension of the fixed points of A on $k[G/H]$ is $[G : J]$, we see that if $A \leq \ker(\phi)$, then ϕ is not a composition factor of the kernel of f whence the claim. $\square$

Theorem 5.6. *Let G be a finite group with a proper subgroup H of odd index. Set $K = \mathbf{O}_2'(G)$ and assume that $G = HK$. Assume that K has no composition factor isomorphic to M_{22} , M_{23} or $\mathrm{PSL}_n(q)$ with n odd. Then the 2-Brauer character $((1_H)^0)^G$ has a nontrivial real-valued irreducible constituent ϕ of odd multiplicity.*

Proof. We induct on $|G|$ and then on $[G : H]$. Consider a minimal counterexample. So H is corefree and $\mathbf{O}_2(G) = 1$. We use induction on the index. If there exists $x \in G$ real of odd order with $(1_H)^G(x)$ even, then as we have seen before the result follows. So we may assume that $(1_H)^G(x)$ is odd for such x , whence the same is true for all real x . In particular, we may assume that $H \cap x^G$ is nonempty for all real x .

Suppose there exists a minimal normal abelian subgroup A of G contained in K .

First suppose that $J := HA \neq G$. By minimality, it follows that there exists a nontrivial real constituent ϕ of $((1_J)^0)^G$ of odd multiplicity. By Lemma 5.5, $[((1_H)^0)^G, \phi] = [((1_J)^0)^G, \phi]$ is odd, a contradiction.

Suppose that $G = J$. Then $H \cap A$ is normal in G , whence $H \cap A = 1$. If a Sylow 2-subgroup centralizes A , then $[K, A] = 1$, whence $A \leq \mathbf{Z}(G)$ and so $G = H \times A$ and $K \leq H$, a contradiction. If not, then A contains real elements but if $1 \neq x \in A$ is real, then $x^G \subseteq A$ is disjoint from H , a contradiction.

So we may assume that $\mathbf{F}(K) = 1$. Let A be a minimal normal subgroup of G contained in K . So $A = L_1 \times \dots \times L_t$ where $L_i \cong L$ is a nonabelian simple group (and not isomorphic to one of the forbidden composition factors). Since every real element of A is conjugate to an element of H , Corollary 2.9 implies the result. $\square$

We close by noting that if $[G : H]$ is even, it is easy to see that the trivial 2-Brauer character occurs with even multiplicity in $((1_H)^0)^G$ and so in particular there are at least two real valued irreducible constituents.

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