Coprime automorphisms of finite groups

Cristina Acciarri, Robert M. Guralnick, and Pavel Shumyatsky

ABSTRACT. Let G be a finite group admitting a coprime automorphism α of order e. Denote by $I_G(\alpha)$ the set of commutators $g^{-1}g^{\alpha}$, where $g \in G$. Suppose that each subgroup generated by a subset of $I_G(\alpha)$ can be generated by at most r elements. We show that the rank of $[G, \alpha]$ is (e, r)-bounded. Along the way, we establish several results of independent interest. In particular, we prove that if every element of $I_G(\alpha)$ has odd order, then $[G, \alpha] \leq O(G)$. Further, if every pair of elements from $I_G(\alpha)$ generates a soluble, or nilpotent, subgroup, then $[G, \alpha]$ is soluble, or respectively nilpotent.

1. Introduction

An automorphism α of a finite group G is coprime if $(|G|, |\alpha|) = 1$. We denote by $C_G(\alpha)$ the fixed-point subgroup $\{x \in G;\ x^{\alpha} = x\}$ and by $I_G(\alpha)$ the set of all commutators $g^{-1}g^{\alpha}$, where $g \in G$. Then $[G, \alpha]$ stands for the subgroup generated by $I_G(\alpha)$.

It is well known that properties of the centralizer $C_G(\alpha)$ of a coprime automorphism have strong influence over the structure of G. There is a wealth of results illustrating this phenomenon, probably the most famous of which is Thompson's Theorem that if α has prime order and $C_G(\alpha) = 1$, then G is nilpotent [**22**]. Over the years, this was generalized in several directions. In particular, Khukhro proved that if G admits an automorphism α of prime order p with $C_G(\alpha)$ of order m, then G has a nilpotent subgroup of (m, p)-bounded index and p-bounded class [**13**]. Further, if G admits a coprime automorphism

2010 *Mathematics Subject Classification.* 20D45.

Key words and phrases. Finite groups, automorphisms.

The first and the third authors were supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), and Fundação de Apoio à Pesquisa do Distrito Federal (FAPDF), Brazil. The second author was partially supported by a Simons Foundation fellowship and NSF grant DMS-1901595.

α of prime order p with $C_G(\alpha)$ of rank r, then G has characteristic subgroups R and N such that N/R is nilpotent of p-bounded class, while R and G/N have (p,r)-bounded ranks [14]. Recall that a rank of a finite group G is the least number r such that each subgroup of G can be generated by at most r elements. Throughout, we use the term $(a,b,c\dots)$-bounded to mean "bounded from above by some function depending only on the parameters $a,b,c\dots$".

Given a coprime automorphism α of a finite group G, there is a kind of (rather vague) duality between $C_G(\alpha)$ and $I_G(\alpha)$. Our purpose in the present article is to show that also properties of $I_G(\alpha)$ may strongly impact the structure of G. It is easy to see that if $|I_G(\alpha)| \leq m$, then the order of $[G,\alpha]$ is m-bounded. We address the question whether a rank condition imposed on the set $I_G(\alpha)$ has an impact on the structure of G. We emphasize that $I_G(\alpha)$ in general is not a subgroup and therefore the usual concept of rank does not apply to $I_G(\alpha)$. Instead we consider the condition that each subgroup of G generated by a subset of $I_G(\alpha)$ can be generated by at most r elements. Our main result is as follows.

THEOREM 1.1. *Let G be a finite group admitting a coprime automorphism α of order e and suppose that any subgroup generated by a subset of $I_G(\alpha)$ can be generated by r elements. Then $[G,\alpha]$ has (e,r)-bounded rank.*

The much easier particular case of the theorem for $e=2$ was earlier dealt with in [1]. As might be expected, Theorem 1.1 depends on the classification of finite simple groups. Along the way, we establish several results of independent interest. In particular, we prove the following result.

THEOREM 1.2. *Let G be a finite group admitting a coprime automorphism α such that $g^{-1}g^{\alpha}$ has odd order for every $g \in G$. Then $[G,\alpha] \leq O(G)$.*

As usual, $O(G)$ stands for the maximal normal subgroup of odd order of G. Recall that an immediate corollary of Glauberman's celebrated Z^*-theorem is that if G contains an involution x such that $[g,x]$ has odd order for every $g \in G$, then $[G,x] \leq O(G)$ [4]. A theorem obtained in [10] states that if G contains an element x of prime order p such that $[g,x]$ has p-power order for every $g \in G$, then $[G,x] \leq O_p(G)$. Thus, one may wonder whether the assumption that α is coprime in Theorem 1.2 is really necessary. In Section 3 we give examples showing that the theorem is no longer true if the assumption is omitted.

It is well known that if any pair of elements of a finite group generates a soluble (respectively nilpotent) subgroup, then the whole group

is soluble (respectively nilpotent). Indeed, Guest [**7**] showed that if g is of prime order at least 5 and if every two conjugates of g generate a soluble group, then g is in the soluble radical of G. We will establish a similar result for groups with coprime automorphisms.

THEOREM 1.3. *Let G be a finite group admitting a coprime automorphism α. If any pair of elements from $I_G(\alpha)$ generates a soluble subgroup, then $[G,\alpha]$ is soluble. If any pair of elements from $I_G(\alpha)$ generates a nilpotent subgroup, then $[G,\alpha]$ is nilpotent.*

Guest [**7**] also gave examples of almost simple groups and elements of prime orders 2 or 3 so that any three conjugates generate a soluble group (possibly 4 for $p = 2$). This shows that the coprimeness assumption is needed in the previous theorem. It seems likely that the assumption can be removed in the case where α is of prime order at least 5.

2. Preliminary results

All groups considered in this paper are finite. The Feit-Thompson theorem that groups of odd order are soluble [**3**] will be used without explicit references. We start with a collection of well-known facts about coprime automorphisms of finite groups (see for example [**5**]).

LEMMA 2.1. *Let a group G admit a coprime automorphism α. The following conditions hold:*

(i) *$G = [G,\alpha]C_G(\alpha)$ and $|I_G(\alpha)| = [G : C_G(\alpha)]$;*
(ii) *If N is any α-invariant normal subgroup of G we have $C_{G/N}(\alpha) = C_G(\alpha)N/N$, and $I_{G/N}(\alpha) = \{gN \mid g \in I_G(\alpha)\}$;*
(iii) *If N is any α-invariant normal subgroup of G such that $N = C_N(\alpha)$, then $[G,\alpha]$ centralizes N;*
(iv) *The group G possesses an α-invariant Sylow p-subgroup for each prime $p \in \pi(G)$.*

Throughout, by a simple group we mean a nonabelian simple group. We will often use without special references the well-known corollary of the classification that if a simple group G admits a coprime automorphism α of order e, then $G = L(q)$ is a group of Lie type and α is a field automorphism. Furthermore, $C_G(\alpha) = L(q_0)$ is a group of the same Lie type defined over a smaller field such that $q = q_0^e$ (see [**6**]).

LEMMA 2.2. *Let r be a positive integer and G a simple group admitting a coprime automorphism α of order $e > 1$.*

(1) *If the order of $[P,\alpha]$ is at most r whenever P is an α-invariant Sylow subgroup of G, then the order of G is r-bounded.*

(2) *If the rank of $[P,\alpha]$ is at most r whenever P is an α-invariant Sylow subgroup of G, then the rank of G is r-bounded.*

PROOF. We know that $G = L(q_0^e)$ is a group of Lie type and $q_0 = p^s$ is a p-power for some prime p. Moreover $C_G(\alpha) = L(q_0)$ is a group of the same Lie type. Choose an α-invariant Sylow p-subgroup U in G such that $C_U(\alpha)$ is a Sylow p-subgroup of $C_G(\alpha)$.

Note that $|G| \leq |U|^3$ and so for (1), it suffices to show that U is bounded in terms of $|[U,\alpha]|$. Note that $|U| = q_0^{ed}$ for some d and $|C_U(\alpha)| = q_0^d$, it follows that $|[U,\alpha]| = q_0^{d(e-1)} > |U|^{1/2}$.

We now prove (2). To bound the rank of G, it suffices to bound the rank of each Sylow ℓ-subgroup of G [**9, 18**]. If $\ell \neq p$, it is well known that the rank of a Sylow ℓ-subgroup of G is at most the (untwisted) Lie rank of G plus the rank of the Weyl group [**6**, Sec. 4.10].

Now consider U. By [**6**, 3.3.1, Thm. 3.3.3], there exists an elementary abelian α-invariant subgroup A of U of order $p^{esf(m)}$ where m is the untwisted Lie rank of G and $f(m)$ is some function which grows quadratically in the Lie rank of m. Moreover, this subgroup A is a product of root subgroups if the group is untwisted and a product of abelian subgroups of root subgroups in the twisted case and each of these subgroups is α-invariant. It follows as in the proof of (1), that $[A,\alpha]$ has rank at least $(e-1)sf(m)$. Our hypothesis implies that e, s and m are bounded. Since $|U| \leq pesm^2$, this bounds the rank of U and the Lie rank of G. The result now follows. □

LEMMA 2.3. *Let G be a simple group admitting a coprime automorphism α. There is a prime $p \in \pi(G)$ such that G is generated by two p-subgroups P_1 and P_2 with the property that $P_1 = [P_1,\alpha]$ and $P_2 = [P_2,\alpha]$.*

PROOF. Again, G is a group of Lie type, say over the field of q elements and α is a field automorphism of odd order e (and so $e \geq 5$ or $e = 3$ and $G = Sz(q)$). Furthermore, $C_G(\alpha)$ is the group of the same Lie type over the field of q_0 elements where $q = q_0^e$. In particular, α normalizes a Borel subgroup $B = UT$, where T is a torus and U is a Sylow p-subgroup, where q is a p-power. Then $B^- = U^-T$ is the opposite Borel subgroup (with $U \cap U^- = 1$). This is obtained by conjugating B by the longest element in the Weyl group. We claim that $G = \langle [U,\alpha], [U^-,\alpha] \rangle$.

First consider the rank 1 groups: $PSL(2,q), PSU(3,q), {}^2G_2(q), Sz(q)$. Note that $|[U,\alpha]| \geq (q/q_0)^m$ where $|U| = q^m$ and by inspection of the maximal subgroups (cf [**23**]), deduce that the only maximal subgroup containing $[U,\alpha]$ is B and so the result holds.

In a similar way we treat the group ${}^2F_4(2^d)$ – all maximal subgroups are known ([**23**, 4.9.3]) and none of them contains both $[U, \alpha]$ and $[U^-, \alpha]$.

So assume that G has (twisted) Lie rank at least 2. The automorphism α leaves invariant each parabolic subgroup P containing B (and similarly for B^-). Note that $P = QL$ where Q is the unipotent radical and L is the standard Levi subgroup. By induction, we have $[L, L] \leq \langle [U \cap L, \alpha], [U^- \cap L, \alpha] \rangle$. Observe that $[L, L]$ is generated by the root subgroups corresponding to the system of a subset of positive roots. So every simple root is contained in some parabolic subgroup. We conclude that in particular the root subgroups $U_{\pm a}$ are contained in $\langle [U, \alpha], [U^-, \alpha] \rangle$ for each positive simple root a. Since the positive simple roots U_a generate U (and U_{-a} generate U^-), we see that U and U^- are contained in $\langle [U, \alpha], [U^-, \alpha] \rangle$ and it is well known (see Section 2.9 in [**6**]) that these generate G. □

Throughout, the term "semisimple group" means direct product of simple groups.

LEMMA 2.4. *Let C be a positive integer and G a finite group admitting a coprime automorphism α such that $G = [G, \alpha]$. Suppose that the order of $[P, \alpha]$ is at most C whenever P is an α-invariant Sylow subgroup of G. Then the order of G is C-bounded.*

PROOF. First, note that the result is obvious if G is abelian so we assume that G is nonabelian. If G is simple, then the result is immediate from Lemma 2.2(1). If G is semisimple and α transitively permutes the simple factors, then any α-invariant Sylow subgroup Q is a product $Q_1 \times \cdots \times Q_l$, where α transitively permutes the factors Q_i. Observe that $|[Q, \alpha]| \geq |Q_1|^{l-1}$ and the result follows. So suppose that G has proper α-invariant normal subgroups. Let $\pi(G) = \{p_1, \ldots, p_k\}$ and for each $i \leq k$ choose an α-invariant Sylow p_i-subgroup P_i in G. Let $s(G)$ denote the product $\prod_{1 \leq i \leq k} |[P_i, \alpha]|$. Obviously $s(G) \leq C^k$ and note that k is C-bounded. Thus $s(G)$ is C-bounded and so we will use induction on $s(G)$. Suppose first that α acts nontrivially on every α-invariant normal subgroup of G. Let M be a minimal α-invariant normal subgroup. By induction the order of G/M is C-bounded. The subgroup M is either an elementary abelian p-group for some prime $p \leq C$ or a semisimple group. In any case $[M, \alpha]$ has C-bounded order. Since $[M, \alpha]$ is normal in M, which has C-bounded index in G, we conclude that the normal closure $\langle [M, \alpha]^G \rangle$ has C-bounded order. Because of minimality of M we have $\langle [M, \alpha]^G \rangle = M$ and so the order

of G is C-bounded. This completes the proof in the particular case where α acts nontrivially on every α-invariant normal subgroup of G.

Next, suppose that G has nontrivial normal subgroups contained in $C_G(\alpha)$. Let N be the product of all such subgroups. In view of the above G/N has C-bounded order. Since $N \leq Z(G)$, we deduce from Schur's Theorem [**19**, Theorem 4.12] that G' has C-bounded order. Hence the result. □

LEMMA 2.5. *Let $G = H\langle a\rangle$ be a group with a normal subgroup H and an element a such that $(|H|, |a|) = 1$ and $H = [H, a]$. Suppose that G faithfully acts by permutations on a set Ω in such a way that the element a moves only m points. Then the order of G is m-bounded.*

PROOF. First, note that the order of a is obviously m-bounded. Another useful observation is that because of Lemma 2.4 without loss of generality we can assume that H is a p-group for some prime p. Also remark that without loss of generality the action of G on Ω can be assumed transitive and so it is sufficient to bound the cardinality of Ω. Consider the corresponding permutational representation of G over $\mathbb{C}$. So G naturally acts on the $|\Omega|$-dimensional linear space V. The dimension of $[V, a]$ is $m-1$. The space V is a direct sum of irreducible G-modules and there are at most m of these (the trivial module and at most $m-1$ nontrivial ones). Each of the irreducible G-modules has $(m, |a|)$-bounded dimension by the Hartley-Isaacs Theorem B [**11**]. It follows that the dimension of V is m-bounded, as required. □

The following lemma will be useful.

LEMMA 2.6. *Let G be a group admitting a coprime automorphism α such that $G = [G, \alpha]$. Suppose that $G = NH$ is a product of an α-invariant normal subgroup N and an α-invariant subgroup H. Assume that $[H, \alpha]$ is generated by $a_1, \ldots, a_s$ while $[N, \alpha]$ is generated by $b_1, \ldots, b_t$. Then $G = \langle a_1, \ldots, a_s, b_1, \ldots, b_t\rangle$.*

PROOF. Since $G = N[H, \alpha]$, without loss of generality we can assume that $H = [H, \alpha]$. Thus, $H = \langle a_1, \ldots, a_s\rangle$. Hence the subgroup $\langle [N, \alpha]^H\rangle$ is contained in $\langle a_1, \ldots, a_s, b_1, \ldots, b_t\rangle$. By Lemma 2.1(iii) the image of N in the quotient group $G/\langle [N, \alpha]^H\rangle$ becomes central and therefore the image of H becomes normal. Hence, $\langle [N, \alpha]^H\rangle H$ is normal in G. Obviously, α acts trivially on $G/\langle [N, \alpha]^H\rangle H$. Since $G = [G, \alpha]$, we conclude that $G = \langle [N, \alpha]^H\rangle H$ and the result follows. □

In the sequel we will require the following well-known fact (see [**20**, 618 at p. 271]).

LEMMA 2.7. *Let N be a normal subgroup of a finite group G. Let H be a minimal subgroup of G such that $G = NH$. Then $H \cap N \leq \Phi(H)$.*

3. Theorems 1.2 and 1.3

In this section Theorem 1.2 and Theorem 1.3 will be proved. Naturally, the proof relies on the classification of simple groups. We first prove results about $PSL(2, q)$.

LEMMA 3.1. *Let $e \geq 5$ be an odd positive integer, q_0 an odd prime power, and let $q = q_0^e$. Let α be a field automorphism of $H := PSL(2, q)$ of order e (necessarily with $C_H(\alpha) = PSL(2, q_0)$). Let $C = \alpha^H$. Then $CC^{-1} = H$.*

PROOF. For convenience it is easier to work with $J = PGL(2, q)$. Since $J = HC_J(\alpha)$, this suffices (i.e. $\alpha^J = \alpha^H$). Set $M = J\langle\alpha\rangle$.

Note that J has $q + 1$ conjugacy class of order prime to q and 1 conjugacy class of elements whose order is not relatively prime to q. So J has $q + 2$ nontrivial irreducible characters of dimension $q - 1, q$, and $q + 1$. Note that α leaves precisely $q_0 + 2$ conjugacy classes invariant and so by Brauer's permutation lemma, the same is true for irreducible characters.

By the class equation formula, the lemma is equivalent to saying that

$$\sum \phi(\alpha)\phi(\alpha^{-1})\phi(y)/\phi(1) \neq 0, \qquad (*)$$

for every $y \in H$. Here the sum is taken over the irreducible characters ϕ of M.

Note that if ϕ is an irreducible character of M and is not irreducible when restricted to J, then $\chi(\alpha) = 0$. So it suffices to consider the α-invariant characters of J. Each has precisely e distinct extensions to M, which are the same up to a twist by a linear character of M/J and so give the same value in the sum above – thus, to show that the class sum is nonzero, it suffices to pick one extension of each invariant character to $\langle J, \alpha\rangle$.

By inspection of the character table for $PGL(2, q)$, it follows that if χ is an irreducible character, $|\chi(y)| \leq 2$ for each $y \in H$. If χ is α-invariant, we claim that $|\chi(\alpha)| \leq q_0 + 1$.

To see this, note that if U is a Sylow p-subgroup of order q, then every nontrivial character of U occurs once and the trivial character with multiplicity at most 2. Thus, α permutes the nontrivial characters of U fixing exactly $q_0 - 1$ of them and so the claim holds.

If ϕ is not irreducible over J (equivalently not α-invariant), then $\phi(\alpha) = 0$ and so does not contribute to the sum in $(*)$. If $\phi(1) = 1$,

then $\phi(y) = 1$ as well (since $y \in H = [J, J]$) and so the contribution is 1 for each. There are two such characters.

There are q_0 irreducible characters not of degree 1 that are α-invariant. For any such character ϕ, we have

$$|\phi(\alpha)\phi(\alpha^{-1})\phi(y)/\phi(1)| \leq 2(q_0+1)^2/(q-1).$$

Since there are q_0 such characters, the absolute value of the sum of these is at most $2q_0(q_0+1)^2/(q-1) < 2$. The last inequality follows since $q \geq q_0^5$. This implies the claim. □

Essentially the same proof yields:

LEMMA 3.2. *Let $e \geq 4$ be a positive integer, q_0 a power of 2, and $q = q_0^e$. Let α be a field automorphism of $H := PSL(2, q)$ of order e (so with centralizer $PSL(2, q_0)$). Let $C = \alpha^H$. Then $CC^{-1} = H$.*

PROOF. Note that since q is even, $PSL_2(q) = PGL_2(q)$. In this case, there are q conjugacy classes of elements of odd order and 1 conjugacy class of involutions for a total of $q + 1$ conjugacy classes. As in the previous result, we see that there are $q_0 + 1$ α-invariant irreducible characters of H of possible dimensions $q - 1, q$ and $q + 1$. The trivial character of H is the only linear character. The estimates for the character values are the same as in the previous lemma and so each nontrivial character contributes at most $2(q_0 + 1)^2/(q - 1)$. Since $e \geq 4$ and there are q_0 characters to account for, the absolute value of this sum is at most $2q_0(q_0 + 1)^2/(q - 1) < 1$ (since $e \geq 4$) unless possibly $q_0 = 2$ and $e \leq 5$. In those two cases, one just computes the class sum directly to obtain the result. □

We are ready to prove Theorem 1.2. For the reader's convenience we restate it here.

THEOREM 3.3. *Let G be a finite group admitting a coprime automorphism α such that $g^{-1}g^\alpha$ has odd order for every $g \in G$. Then $[G, \alpha] \leq O(G)$.*

PROOF. Assume that this is false and let G be a counterexample of minimal order. Then $G = [G, \alpha]$ and $O(G) = 1$. Let M be a minimal α-invariant normal subgroup of G. By induction, G/M has odd order. The subgroup M is either elementary abelian or semisimple.

If M is abelian, then M is a 2-subgroup and so by hypotheses $M \leq C_G(\alpha)$. It follows that $M \leq Z(G)$, which leads to a contradiction since G/M has odd order and $O(G) = 1$.

Hence, we can assume that M is a direct product of isomorphic nonabelian simple groups and α transitively permutes the simple factors. Moreover because of minimality $G = M$. If M is a product of

more than one simple group, and if S is a simple factor in which g is an involution, observe that $g^{-1}g^{\alpha}$ has order two, a contradiction.

So we are reduced to the case that G is simple. It follows that G is a group of Lie type and α is a field automorphism, say of coprime order e.

If G is a group of Lie type in characteristic 2, then field automorphisms do not centralize Sylow 2-subgroups (but do normalize one) and so we have a contradiction again.

So G is a group of Lie type in odd characteristic p, defined over a field of size $q = q_0^e$ with all divisors of e at least 5 (the only case where there is a coprime automorphism of order 3 is for the Suzuki groups which are in characteristic 2).

First suppose that G has (twisted) Lie rank 1. If $G = PSL(2,q)$ we apply Lemma 3.1. If $G = PSU(3,q)$ or ${}^2G_2(q)$, we observe that there is an α-invariant subgroup isomorphic to $PSL(2,q)$ which is not centralized by α and so again the lemma applies.

So we may assume that G has rank at least 2. Note that α normalizes a Borel subgroup and, by the structure of field automorphisms, α normalizes each parabolic subgroup of G containing B. We see that α normalizes a Levi subgroup L (and so its derived subgroup). By choosing the parabolic subgroup to be minimal properly containing the Borel subgroup, we can assume that L is of rank 1. There may be a center but since the rank 1 case reduces to $PSL(2,q)$, the center will be a 2-group. The result follows. □

We will now show that the coprimeness assumption is really necessary in Theorem 1.2. First, quote a linear algebra result from [8].

THEOREM 3.4. *Let k be an algebraically closed field and let $A, B \in M_n(k)$ be matrices such that $AB - BA$ has rank 1. Then A and B can be simultaneously triangularized.*

This implies that if F is any field and $x, y \in GL(n,F)$ such that the group commutator $[x,y]$ is a transvection, then $\langle x,y\rangle$ has unipotent derived group, and in particular $\langle x,y\rangle$ is soluble.

To see this, we can assume that F is algebraically closed and write $x^{-1}y^{-1}xy = I + A$, where A is a nilpotent rank one matrix. Thus, $xy - yx = yxA$ has rank 1 and so x, y are simultaneously triangular, whence the commutator subgroup $\langle x,y\rangle'$ is unipotent.

COROLLARY 3.5. *Let $x \in GL(V)$ act on V irreducibly. Then for any $y \in GL(V)$ the commutator $[x,y]$ is not a transvection.*

PROOF. By way of contradiction suppose that there is $y \in GL(V)$ such that the commutator $[x,y]$ is a transvection. On the one hand, the

subgroup $\langle x, y\rangle$ is irreducible because so is x. On the other hand, $\langle x, y\rangle'$ is unipotent and therefore $C_V(\langle x, y\rangle') \neq 0$. This is a contradiction. $\square$

The following example shows that Theorem 1.2 is no longer true if the assumption that the automorphism α is coprime is omitted.

Let $q = 2^a > 2$ and let $G = SL(2, q)$. Let $x \in G$ be an element of order $q+1$. Then $[x, y]$ has odd order for all $y \in G$. This is because the only elements of even order in G are transvections and, by Corollary 3.5, these are not commutators $[x, y]$.

We will now prove Theorem 1.3.

Proof of Theorem 1.3. Recall that G is a group admitting a coprime automorphism α such that any pair of elements from $I_G(\alpha)$ generates a soluble subgroup. We wish to prove that $[G, \alpha]$ is soluble. Assume that this is false and let G be a counterexample of minimal order. Arguing precisely as in the proof of Theorem 3.3, we see that either G is simple or G is a product of $r > 1$ copies of a simple group L and α permutes the factors transitively.

Consider the second case. By conjugating (in $\mathrm{Aut}(G)$), we may assume that $\alpha = (x, 1, \ldots, 1)\rho$ where x is an automorphism of L and ρ permutes the coordinates of G. It is clear that an element of $I_G(\alpha)$ can have an arbitrary first coordinate in L and since L can be generated by 2 elements, the result holds in this case.

If G is simple, it follows by minimality that any proper nonsolvable α-invariant subgroup must be contained in the centralizer of α. As in the proof of Theorem 3.3, it follows that $G = PSL(2, q)$ or $Sz(q)$.

Suppose first that $G = PSL(2, q)$. Lemma 3.1 implies that every element of G is conjugate to some element in $I_G(\alpha)$. In particular, there are elements of order $(q \pm 1)$ (if q is even) or $(q \pm 1)/2$ (if q is odd). Since $q \geq 32$, there are no proper subgroups containing elements of both orders.

Therefore $G = Sz(q)$ and proper α-invariant subgroups of G are either soluble or contained in $C_G(\alpha)$. Choose two α-invariant cyclic Hall subgroups J and K of order $q + \sqrt{2q} + 1$ and $q - \sqrt{2q} + 1$, respectively. It is straightforward that $\langle [J, \alpha], [K, \alpha]\rangle$ is a nontrivial α-invariant subgroup generated by two elements from $I_G(\alpha)$. Hence, $\langle [J, \alpha], [K, \alpha]\rangle$ is soluble. The subgroup structure of the Suzuki groups is given in [**23**, p. 117]. We see that no proper subgroup of G contains $\langle [J, \alpha], [K, \alpha]\rangle$, a contradiction.

Thus, $[G, \alpha]$ is soluble, as claimed. We will now show that if any pair of elements from $I_G(\alpha)$ generates a nilpotent subgroup, then $[G, \alpha]$ is nilpotent.

Again, let G be a counterexample of minimal order. Then $G = [G, \alpha] = NH$, where N is an α-invariant elementary abelian normal p-subgroup and H is an α-invariant nilpotent p'-subgroup such that $H = [H, \alpha]$. By Lemma 2.6, $G = \langle I_N(\alpha), I_H(\alpha)\rangle$. Since any pair of elements from $I_G(\alpha)$ generates a nilpotent subgroup and since $(|N|, |H|) = 1$, we deduce that $I_N(\alpha)$ centralizes $I_H(\alpha)$. Taking into account that N is abelian deduce that $I_N(\alpha) \leq Z(G)$. It follows that $G/Z(G)$ is nilpotent and this completes the proof. □

4. Proof of Theorem 1.1

We are ready to embark on the proof of Theorem 1.1. It will be convenient to deal separately with the case where G is nilpotent.

4.1. The case of nilpotent groups. As usual, we write $Z_i(H)$ and $\gamma_i(H)$ for the ith term of the upper and lower central series of a group H, respectively.

LEMMA 4.1. *Let p be a prime and G a group admitting a coprime automorphism α such that $G = [G, \alpha]$. Let M be an α-invariant normal p-subgroup of G and assume that $|I_M(\alpha)| = p^m$ for some nonnegative integer m. Then $M \leq Z_{2m+1}(O_p(G))$.*

PROOF. If $m = 0$, then the result is immediate from Lemma 2.1(iii), so assume that $m \geq 1$ and use induction on m.

Let $K = O_p(G)$ and $N = M \cap Z_2(K)$. If $N \not\leq Z(K)$, then Lemma 2.1(iii) implies that $I_N(\alpha) \neq 1$, in which case we have $|I_{M/N}(\alpha)| < |I_M(\alpha)| = p^m$. By induction $M/N \leq Z_{2m-1}(K/N)$, whence $M \leq Z_{2m+1}(K)$. If $N \leq Z(K)$, then it turns out that $M \cap Z(K) = M \cap Z_i(K)$ for any $i \geq 2$ and so, obviously, $M \leq Z(K)$. This concludes the proof. □

The following result is well known (see for example [**21**, Lemma 2.2]). It will be useful later on.

LEMMA 4.2. *Let G be a group of prime exponent p and rank r_0. Then there exists a number $s = s(r_0)$, depending only on r_0, such that $|G| \leq p^s$.*

Throughout this subsection, unless stated otherwise, G is a p-group admitting a coprime automorphism α such that $G = [G, \alpha]$ and any subgroup generated by a subset of $I_G(\alpha)$ can be generated by at most r elements.

LEMMA 4.3. *Suppose that G is of prime exponent p. There exists a number $l = l(r)$, depending on r only, such that the rank $r(G)$ of G is at most l.*

PROOF. Let C be Thompson's critical subgroup of G (see [**5**, Theorem 5.3.11], and set $A = Z(C)$. Observe that $[A, \alpha]$ is an r-generated abelian subgroup of exponent p and so the order of $[A, \alpha]$ is at most p^r. By Lemma 4.1 A is contained in $Z_{2r+1}(G)$. Since $[G, C]$ is contained in A, we conclude that C is contained in $Z_{2r+2}(G)$. Recall that $\gamma_{2r+2}(G)$ commutes with $Z_{2r+2}(G)$ and so in particular $\gamma_{2r+2}(G)$ centralizes C. Again by Thompson's theorem, $C_G(C) = A$. Thus $\gamma_{2r+2}(G)$ is contained in A, that is, the quotient group G/A is nilpotent of class $2r + 1$. We deduce that G has r-bounded nilpotency class. Since $G = [G, \alpha]$ is r-generated by hypothesis, it follows that the rank $r(G)$ of G is r-bounded, as desired. □

We will require the concept of powerful p-groups. These were introduced by Lubotzky and Mann in [**17**]: a finite p-group H is powerful if and only if $H^p \leq [H, H]$ for $p \neq 2$ (or $H^4 \leq [H, H]$ for $p = 2$). The reader can consult books [**2**] or [**12**] for more information on these groups.

LEMMA 4.4. *There exists a number $\lambda = \lambda(r)$, depending only on r, such that $\gamma_{2\lambda+1}(G)$ is powerful.*

PROOF. Let $s(r_0)$ be as in Lemma 4.2 and let $l(r)$ be as in Lemma 4.3. Take $N = \gamma_{2\lambda+1}(G)$, where $\lambda = s(l(r))$. In order to show that $N' \leq N^p$, we assume that N is of exponent p and prove that N is abelian.

Note that the subgroup $[N, \alpha]$ is of exponent p. By Lemma 4.3 the rank of $[N, \alpha]$ is at most $l(r)$. It follows from Lemma 4.2 that $|[N, \alpha]| \leq p^{s(l(r))} = p^\lambda$. Now Lemma 4.1 yields $N \leq Z_{2\lambda+1}(G)$. Since $[\gamma_i(G), Z_i(G)] = 1$ for any positive integer i, we conclude that N is abelian, as required. □

LEMMA 4.5. *For any $i \geq 1$, there exists a number $m_i = m_i(i, r)$, depending only on i and r, such that $\gamma_i(G)$ is an m_i-generated group.*

PROOF. Let $N = \gamma_i(G)$. We can pass to the quotient $G/\Phi(N)$ and assume that N is elementary abelian. It follows that $|I_N(\alpha)| \leq p^r$. Thus, by Lemma 4.1, we have $N \leq Z_{2r+1}(G)$ and deduce that G has nilpotency class bounded only in terms of i and r. Since $G = [G, \alpha]$ is r-generated, we conclude that $r(G)$ is (i, r)-bounded as well. Therefore N is m_i-generated for some (i, r)-bounded number m_i. This concludes the proof. □

The next proposition shows that Theorem 1.1 is valid in the case where G is a p-group.

PROPOSITION 4.6. *The rank of G is r-bounded.*

PROOF. Let $s(r_0)$ be as in Lemma 4.2 and $l(r)$ as in Lemma 4.3. Take $N = \gamma_{2\lambda+1}(G)$, where $\lambda = \lambda(r) = s(l(r))$. Let d be the minimal number such that N is d-generated. Lemma 4.5 tells us that d is an r-bounded integer. Moreover, by Lemma 4.4 N is powerful. It follows from [**2**, Theorem 2.9] that $r(N) \leq d$, and so the rank of N is r-bounded. Since the nilpotency class of G/N is r-bounded (recall that λ depends only on r) and $G = [G, \alpha]$ is r-generated, we conclude that $r(G/N)$ is r-bounded as well. Note that $r(G) \leq r(G/N) + r(N)$ and the result follows. □

COROLLARY 4.7. *Assume the hypotheses of Theorem 1.1 and let G be nilpotent. Then the rank of $[G, \alpha]$ is r-bounded.*

PROOF. The rank of $[G, \alpha]$ is equal to the rank of $[P, \alpha]$, where P is some Sylow p-subgroup of G, and the result easily follows from Proposition 4.6. □

4.2. The case of soluble groups. As usual, we denote by $F(G)$ the Fitting subgroup of a group G. Write $F_0(G) = 1, F_1(G) = F(G)$ and let $F_{i+1}(G)$ be the inverse image of $F(G/F_i(G))$. If G is soluble, then the least number h such that $F_h(G) = G$ is called the Fitting height of G.

The purpose of this subsection is to show that if under the hypotheses of Theorem 1.1 the group G is soluble, then $h([G, \alpha])$ is (e, r)-bounded and moreover $[G, \alpha]$ can be generated by (e, r)-boundedly many elements from $I_G(\alpha)$. One key step consists in showing that there exists an (e, r)-bounded number f such that the fth term of the derived series of $[G, \alpha]$ is nilpotent. For this we will require the following result which is an immediate corollary of Hartley-Isaacs Theorem B in [**11**].

PROPOSITION 4.8. *Let H be a finite soluble group admitting a coprime automorphism α of order e such that $H = [H, \alpha]$. Let k be any field with characteristic prime to e, and V a simple $kH\langle\alpha\rangle$-module. Suppose that $\dim[V, \alpha] = r$. There exists an (e, r)-bounded number $\delta = \delta(e, r)$ such that $\dim V \leq \delta$.*

In the proof of the next proposition we will use the well-known theorem of Zassenhaus (see [**24**, Satz 7] or [**19**, Theorem 3.23]) stating that for any $n \geq 1$ there exists a number $j = j(n)$, depending only on n, such that, whenever k is a field, the derived length of any soluble subgroup of $GL(n, k)$ is at most j.

PROPOSITION 4.9. *Assume the hypotheses of Theorem 1.1. Suppose that G is soluble and $G = [G, \alpha]$. There exists a number $f = f(e, r)$,*

depending only on e and r, such that the fth term $G^{(f)}$ of the derived series of G is nilpotent.

PROOF. Let $\delta = \delta(e,r)$ be as in Proposition 4.8 and $f = j(\delta)$ be the number given by the Zassenhaus theorem.

Suppose that the proposition is false and let G be a group of minimal possible order such that the hypotheses hold while $G^{(f)}$ is not nilpotent. Then G has a unique minimal α-invariant normal subgroup M. Indeed, suppose that G has two minimal α-invariant normal subgroups, say M_1 and M_2. Then $M_1 \cap M_2 = 1$. Since $|G/M_1| < |G|$, the minimality of G implies that $(G/M_1)^{(f)}$ is nilpotent. By a symmetric argument $(G/M_2)^{(f)}$ is nilpotent too. This yields a contradiction since $G^{(f)}$ can be embedded into a nilpotent subgroup of $G/M_1 \times G/M_2$.

We claim that $M = C_G(M)$. Since M is a p-subgroup for some prime p and because of the uniqueness of M, the Fitting subgroup $F = F(G)$ is a p-subgroup too. If $\Phi(F)$ is nontrivial, then we immediately get a contradiction because $F(G/\Phi(F)) = F/\Phi(F)$ and, again by the minimality of G, we know that $(G/\Phi(F))^{(f)}$ is nilpotent, so in particular $G^{(f)} \leq F$.

So assume that $\Phi(F) = 1$ and thus F is elementary abelian. If $M = F$, then $M = C_G(M)$ since the Fitting subgroup of a soluble group contains its own centralizer (see, for example, [**5**, Theorem 1.3, Chap. 6]). Thus we can assume that $M < F$. By hypotheses, on one hand, we know that $G^{(f)} \leq F_2(G)$ and, on the other hand, $(G/M)^{(f)}$ is nilpotent (again by the minimality of G). Now let T be an α-invariant Hall p'-subgroup of $G^{(f)}$. It follows that both FT and MT are α-invariant normal subgroups of G. Indeed, FT/F is normal in G/F, since $(G/F)^{(f)}$ is nilpotent and, similarly, MT/M is normal in G/M since $(G/M)^{(f)}$ is nilpotent as well.

Suppose that $C_F(T) \neq 1$. Note that $C_F(T) = Z(FT)$ since F is abelian. Thus $C_F(T)$ is an α-invariant normal subgroup of G because FT is normal and α-invariant. Hence $M \leq C_F(T)$. This implies that T centralizes M and so $MT = T \times M$. Recall that $T \leq F_2(G)$ and $T \cap F = 1$. It follows that T is nilpotent. Then $T \times M$ is normal nilpotent and $T \leq F$, a contradiction.

Thus, $C_F(T) = 1$. On the other hand, we see that $[F,T] \leq M$, since the nilpotent p'-subgroup MT/M and the p-subgroup F/M are both contained in $F(G/M)$ and therefore commute. Now we have $M < F$ and $F = [F,T] \times C_F(T)$, so it should be $C_F(T) \neq 1$, a contradiction. Thus $M = C_G(M)$, as claimed above.

Therefore G/M acts faithfully and irreducibly on M. Moreover $[M,\alpha]$ is r-generated and elementary abelian, so $|[M,\alpha]| \leq p^r$. We view

M as a $G/M\langle\alpha\rangle$-module over the field with p elements. Observe that p does not divide e, since α is a coprime automorphism. By Proposition 4.8 we have $\dim(M) \leq \delta(e,r)$. Applying the theorem of Zassenhaus conclude that the derived length of G/M is at most $f = f(\delta(e,r))$. Then $G^{(f)} \leq F$, which concludes the proof. □

As a by-product of the previous result we deduce that the Fitting height of G is (e,r)-bounded.

COROLLARY 4.10. *Under the hypothesis of Proposition 4.9 the Fitting height $h(G)$ is (e,r)-bounded.*

PROOF. By Proposition 4.9 we know that $G^{(f)}$ is nilpotent for some (e,r)-bounded number f. The result follows since $h(G) \leq f+1$. □

PROPOSITION 4.11. *Under the hypothesis of Proposition 4.9 the group G is generated by (e,r)-boundedly many elements from $I_G(\alpha)$.*

PROOF. If G is a p-group, then the claim follows from the Burnside Basis Theorem since $G = [G,\alpha]$ is r-generated. In the case where G is nilpotent, we have $G = [P_1,\alpha] \times \cdots \times [P_s,\alpha]$, where $\{P_1,\dots,P_s\}$ are the Sylow subgroups of G. So it follows from the case of p-groups that G is generated by r elements from $I_G(\alpha)$.

Assume that G is not nilpotent. Let $h = h(G) \geq 2$. Since we know from Corollary 4.10 that h is (e,r)-bounded, we argue by induction on h. Let $F = F(G)$. By induction there are (e,r)-boundedly many elements $a_1,\dots,a_d \in I_G(\alpha)$ such that $G = F\langle a_1,\dots,a_d\rangle$. We can choose $a_1,\dots,a_d$ in such a way that the subgroup $H = \langle a_1,\dots,a_d\rangle$ is α-invariant. We have seen in the previous paragraph that $[F,\alpha]$ can be generated by at most r elements from $I_F(\alpha)$. Thus, Lemma 2.6 tells us that G can be generated by $d+r$ elements from $I_G(\alpha)$. □

4.3. The general case. Let G be a finite group admitting a coprime automorphism α of order e such that any subset of $I_G(\alpha)$ generates an r-generator subgroup. We want to prove that $[G,\alpha]$ has (e,r)-bounded rank. Thus, throughout the remaining part of the paper we assume that $G = [G,\alpha]$.

LEMMA 4.12. *If G is simple, then the rank of G is r-bounded.*

PROOF. This is immediate from Lemma 2.2 (2). □

LEMMA 4.13. *Suppose that G is semisimple and α transitively permutes the simple factors. Then the rank of G is (e,r)-bounded.*

PROOF. Write $G = S_1 \times \cdots \times S_k$. Since the case $k = 1$ was considered in Lemma 4.12, we assume that $k \geq 2$. Here k is a divisor of e

and so it is sufficient to show that the rank of S_1 is at most r. Suppose that this is not the case and choose a subgroup $H \leq S_1$ which cannot be generated with less than $r+1$ elements. Consider the subgroup $K \leq S_1 \times S_1^\alpha$ generated by all elements of the form $x^{-1}x^\alpha$, where $x \in H$. On the one hand, K is generated by a subset of $I_G(\alpha)$ and so it can be generated by r elements. On the other hand, H is a homomorphic image of K and so we have a contradiction with the fact that H cannot be generated with less than $r+1$ elements. □

LEMMA 4.14. *Suppose that G is semisimple. Then the rank of G is (e,r)-bounded.*

PROOF. Since $G = [G,\alpha]$, it follows that $G = G_1 \times \cdots \times G_m$, where each factor G_i is either simple such that $G_i = [G_i,\alpha]$ or a direct product of more than one simple groups which are transitively permuted by α. We already know from the two previous lemmas that the rank of G_i is (e,r)-bounded so it remains to show that the number m of such factors is (e,r)-bounded too. In view of Theorem 1.2 each subgroup G_i has an element g_i such that $x_i = g_i^{-1}g_i^\alpha$ has even order. The abelian subgroup $\langle x_1,\dots,x_m\rangle$ has Sylow 2-subgroup of rank m and so it cannot be generated with less than m elements. Hence, $m \leq r$. □

Write $G_0 = G\langle\alpha\rangle$.

LEMMA 4.15. *Let N be an α-invariant normal subgroup of G and assume that $N = S_1 \times \cdots \times S_l$ is a direct product of nonabelian simple factors S_i. Then both l and the rank of N are (e,r)-bounded.*

PROOF. In view of Lemma 4.14 the rank of $[N,\alpha]$ is (e,r)-bounded. Since all factors S_i have even order and since the rank of the Sylow 2-subgroup of $[N,\alpha]$ is (e,r)-bounded, it follows that only (e,r)-boundedly many, say m, of the subgroups $S_1,\dots,S_l$ are not contained in $C_G(\alpha)$. On the other hand, because of Lemma 2.1(iii) no nontrivial normal subgroup of G can be contained in $C_N(\alpha)$. Thus, every simple factor in the list $S_1,\dots,S_l$ is conjugate in G with a factor which is not centralized by α and so by Lemma 4.14 each S_i has (e,r)-bounded rank. Hence, we only need to show that l is (e,r)-bounded.

The group G_0 naturally acts on the set $\{S_1,\dots,S_l\}$ by conjugation. The above argument shows that there are at most m G_0-orbits in this action. It is sufficient to show that each G_0-orbit has (e,r)-bounded length. Let K be the kernel of the action, that is, the intersection of normalizers of S_i. It is straightforward from Lemma 2.5 that the index of K in G_0 is m-bounded. Since the length of each G_0-orbit is at most the index $[G_0 : K]$, the result follows. □

LEMMA 4.16. *Suppose that G has an α-invariant subgroup K of index i such that $[K,\alpha]$ is of rank s. Then the rank of G is (i,s)-bounded.*

PROOF. We can assume that K is normal in G. Since $[K,\alpha]$ is normal in K, it follows that the index of the normalizer of $[K,\alpha]$ in G is a divisor of i. Using the fact that the rank of $[K,\alpha]$ is s we conclude that the rank of the normal closure K_1 of $[K,\alpha]$ is (i,s)-bounded. In view of Lemma 2.1(iii) the quotient K/K_1 is central in G/K_1. Hence, by Schur's Theorem [**19**, Theorem 4.12], the image in G/K_1 of the commutator subgroup G' has i-bounded order. Therefore we can pass to the quotient G/K_1G' and assume that G is abelian. In this case the lemma is obvious. □

We will now establish several lemmas about generation of G by elements from $I_G(\alpha)$. Recall that Proposition 4.11 tells us that if G is soluble, then G can be generated by an (e,r)-bounded number of elements from $I_G(\alpha)$.

LEMMA 4.17. *If G is semisimple, then G can be generated by an (e,r)-bounded number of elements from $I_G(\alpha)$.*

PROOF. Let $G = S_1 \times \cdots \times S_l$ where the factors S_i are simple. The automorphism α permutes the simple factors and the proof of Lemma 4.14 shows that there are at most r orbits under this action. Therefore without loss of generality we assume that α transitively permutes the factors S_i and so l is a divisor of e. If G is simple, then by Lemma 2.3 G is generated by two nilpotent subgroups P_1 and P_2 such that $[P_1,\alpha] = P_1$ and $[P_2,\alpha] = P_2$. Each of the subgroups P_i is generated by at most r elements from $I_G(\alpha)$ and so G is generated by at most $2r$ such elements. We will therefore assume that $l \geq 2$.

We will use the fact each nonabelian simple group can be generated by two elements. Let a, b generate S_1.

Set

$$x_1 = a^{-1}a^{\alpha},\ x_2 = b^{-1}b^{\alpha} \text{ and } x_3 = ab((ab)^{-1})^{\alpha}.$$

Note that all x_i belong to $I_G(\alpha)$. Let K be the minimal α-invariant subgroup of G containing x_1, x_2, and x_3. Obviously K is generated by at most $3e$ elements from $I_G(\alpha)$. Observe that $1 \neq x_1x_2x_3 = [a,b] \in S_1 \cap K$. Evidently, the projection of K to S_1 is the whole group S_1, that is, K is a subdirect product of the factors S_i. We deduce that the conjugacy class $[a,b]^K$ generates S_1 and so S_1 is contained in K. Since K is α-invariant we are forced to conclude that $K = G$ and the result follows. □

LEMMA 4.18. *Suppose that G is semisimple-by-soluble. Then G is generated by an (e,r)-bounded number of elements from $I_G(\alpha)$.*

PROOF. Let N be an α-invariant normal semisimple subgroup of G such that G/N is soluble. Choose a minimal subgroup H_0 of G_0 such that $G_0 = NH_0$. Without loss of generality we can assume that $\alpha \in H_0$. Set $H = H_0 \cap G$ and note that $H = [H, \alpha]$. Note that H is soluble by Lemma 2.7. We have $G = NH$ and we know from Proposition 4.11 and Lemma 4.17 that both H and $[N, \alpha]$ can be generated by an (e,r)-bounded number of elements from $I_G(\alpha)$. The result follows from Lemma 2.6. □

LEMMA 4.19. *Assume that $G_0/\Phi(G_0)$ is semisimple-by-soluble. Then G is generated by an (e,r)-bounded number of elements from $I_G(\alpha)$.*

PROOF. Let $\overline{G_0} = G_0/\Phi(G_0)$ and denote by $\overline{G}$ the image of G in $\overline{G_0}$. By Lemma 4.18 $\overline{G}$ is generated by (e,r)-boundedly many elements from $I_{\overline{G}}(\alpha)$, say $\overline{x_1}, \dots, \overline{x_s}$ and so $\overline{G_0} = \langle \overline{\alpha}, \overline{x_1}, \dots, \overline{x_s} \rangle$. Hence $G_0 = \langle \alpha, x_1, \dots, x_s \rangle$, where $x_1, \dots, x_s \in I_G(\alpha)$. Thus G is generated by the α-orbits of $x_1, \dots, x_s$ and the result follows. □

In what follows $S(K)$ denotes the soluble radical of a group K.

LEMMA 4.20. *Suppose that G is soluble-by-semisimple-by-soluble. Then G is generated by an (e,r)-bounded number of elements from $I_G(\alpha)$.*

PROOF. Let $S = S(G)$. Let H_0 be a minimal subgroup of G_0 such that $G_0 = SH_0$. Again, without loss of generality we can assume that $\alpha \in H_0$. Since $H_0 \cap S \leq \Phi(H_0)$, Lemma 4.19 shows that $H = H_0 \cap G$ is generated by an (e,r)-bounded number of elements from $I_G(\alpha)$. Note that by Proposition 4.11 also $[S, \alpha]$ is generated by an (e,r)-bounded number of elements from $I_G(\alpha)$. The result follows from Lemma 2.6. □

LEMMA 4.21. *Assume that $S(G) = 1$. Then G has an α-invariant semisimple-by-soluble normal subgroup of (e,r)-bounded index.*

PROOF. Let $N = S_1 \times \cdots \times S_l$ be the socle of G. Here $S_1, \dots, S_l$ are the subnormal simple subgroups. In view of Lemma 4.15 l is (e,r)-bounded. The group G_0 naturally acts on the set $\{S_1, \dots, S_l\}$ by conjugation. Let K be the kernel of this action. By [**15**, Lemma 2.1], the quotient K/N is soluble. Therefore K is a semisimple-by-soluble normal subgroup of (e,r)-bounded index. □

LEMMA 4.22. *The group G_0 has a soluble-by-semisimple-by-soluble normal subgroup of (e,r)-bounded index.*

PROOF. This is immediate from Lemma 4.21. □

In view of Lemma 4.16 it is sufficient to prove Theorem 1.1 in the case where G is soluble-by-semisimple-by-soluble. We already know that in this case G is generated by (e,r)-boundedly many elements from $I_G(\alpha)$.

LEMMA 4.23. *Assume that G is soluble-by-semisimple-by-soluble. Let N be an α-invariant abelian normal subgroup of G. Then $[N,G]$ has (e,r)-bounded rank.*

PROOF. By Lemma 4.20 we know that G is generated by (e,r)-boundedly many elements from $I_G(\alpha)$, say $b_1,\dots,b_t$. Note that $[N,G] = [N,b_1]\dots[N,b_t]$. So it is sufficient to bound the rank of $[N,b_i]$, for each $b_i \in \{b_1,\dots,b_t\}$. By Lemma 2.1(i) we have $N = C_N(\alpha)\times[N,\alpha]$. Take any $b \in \{b_1,\dots,b_t\}$ and choose $a \in G$ such that $b = a^{-1}a^\alpha$. Set $N_0 = C_N(\alpha)\cap C_N(\alpha)^{a^{-1}}$. Since $[N,\alpha]$ has rank r by hypothesis, we have $r(N/N_0) \le 2r$. We claim that $N_0 \le C_G(b)$. Indeed, choose $x \in C_N(\alpha)$ such that $x^{a^{-1}} \in C_N(\alpha)$. Then, we have $x^{a^{-1}} = (x^{a^{-1}})^\alpha$ and so x commutes with $b = a^{-1}a^\alpha$ as claimed. Choose now elements $x_1,\dots,x_{2r}$ that generate N modulo N_0. By using linearity in N and the fact that N_0 centralizes b, we deduce that $[N,b]$ is generated by $[x_1,b],\dots,[x_{2r},b]$. Hence the result. □

PROOF OF THEOREM 1.1. Recall that G is a finite group admitting a coprime automorphism α of order e such that any subgroup generated by a subset of $I_G(\alpha)$ can be generated by r elements. We wish to prove that $[G,\alpha]$ has (e,r)-bounded rank. Without loss of generality we assume that $G = [G,\alpha]$.

As noted above, the combination of Lemma 4.16 and Lemma 4.22 ensures that it is sufficient to prove the result in the case where G is soluble-by-semisimple-by-soluble. Hence, we assume that G has a characteristic series

$$1 \le S \le T \le G$$

such that S and G/T are soluble while T/S is semisimple. Corollary 4.10 shows that the Fitting height of G/T and $[S,\alpha]$ is (e,r)-bounded. Note that $[S,\alpha]$ is subnormal in G. Therefore the Fitting height of the normal closure $\langle[S,\alpha]^G\rangle$ equals that of $[S,\alpha]$. In view of Lemma 2.1(iii) the quotient $S/\langle[S,\alpha]^G\rangle$ is central in $G/\langle[S,\alpha]^G\rangle$. Therefore G has a characteristic series of (e,r)-bounded length, say $l = l(e,r)$, all of whose factors are either semisimple or nilpotent. Moreover, there is at most one semisimple factor in the series and, by Lemma 4.14, it is of (e,r)-bounded rank. We will prove the theorem by induction on l.

If $l = 1$, then G is either semisimple or nilpotent. In the former case the result follows from Lemma 4.14 and in the latter one from Corollary 4.7. Therefore we assume that $l \geq 2$. Let N be the last term of the series. By induction G/N has (e,r)-bounded rank.

If $N \leq Z(G)$, then the rank of $G/Z(G)$ is bounded and a theorem of Lubotzky and Mann [**17**] guarantees that G' has (e,r)-bounded rank (see also [**16**]). Thus we can pass to G/G' and simply assume that G is abelian, whence the result is immediate. We therefore assume that N is not central in G. If N is semisimple, we have nothing to prove since the semisimple quotient of the series has (e,r)-bounded rank. Hence, we assume that N is nilpotent. In this case the rank of N is equal to the rank of some Sylow p-subgroup P of N. Thus, passing to $G/O_{p'}(N)$ without loss of generality, we can assume that $N = P$.

We note that P has an (e,r)-bounded number of generators. Really, pass to the quotient $G/\Phi(P)$ and assume that P is elementary abelian. By Lemma 4.23 $[P,G]$ has (e,r)-bounded rank and by the above this is also true for $G/[P,G]$. Hence, P has (e,r)-boundely many generators as well.

Next, we claim that for any $i \geq 2$ there exists a number $m_i = m_i(i,e,r)$, depending only on i, e and r, such that $V = \gamma_i(P)$ has m_i-bounded number of generators. We can pass to the quotient $G/\Phi(V)$ and assume that V is elementary abelian. Now $[V,\alpha]$ is an elementary abelian r-generated group, so $|[V,\alpha]| \leq p^r$. Thus, by Lemma 4.1, we have $V \leq Z_{2r+1}(O_p(G))$ so, in particular, $V \leq Z_{2r+1}(P)$ and deduce that the nilpotency class of $P/\Phi(V)$ is bounded in terms of i and r only. Since P has an (e,r)-bounded number of generators, we conclude that $r(P/\Phi(V))$ is (i,e,r)-bounded as well. Therefore V is m_i-generated for some (i,e,r)-bounded number m_i, as claimed.

Let $s(r_0)$ be as in Lemma 4.2 and let $l(r)$ be as in Lemma 4.3. Take $M = \gamma_{2\lambda+1}(P)$, where $\lambda = s(l(r))$. We want to prove that M is powerful. In order to show that $M' \leq M^p$, we assume that M is of exponent p and prove that M is abelian. Note that the subgroup $[M,\alpha]$ is of exponent p. By Lemma 4.3 the rank of $[M,\alpha]$ is at most $l(r)$. It follows from Lemma 4.2 that $|I_M(\alpha)| \leq p^{s(l(r))} = p^\lambda$. Now Lemma 4.1 yields that $M \leq Z_{2\lambda+1}(P)$. Since $[\gamma_i(P), Z_i(P)] = 1$, for any positive integer i, we conclude that M is abelian, as required.

Let now d_0 be the minimal number such that M is d_0-generated. It was shown above that d_0 is an (e,r)-bounded integer. Since M is powerful, it follows from [**2**, Theorem 2.9] that $r(M) \leq d_0$, and so the rank of M is (e,r)-bounded. Since the nilpotency class of P/M is (e,r)-bounded and P has (e,r)-boundely many generators, we deduce

that $r(P/M)$ is (e,r)-bounded as well. Now $r(P) \leq r(P/M) + r(M)$ and the result follows. This concludes the proof. □

References

[1] C. Acciarri and P. Shumyatsky, On the rank of a finite group of odd order with an involutory automorphism, preprint.

[2] J. D. Dixon, M. P. F. du Sautoy, A. Mann and D. Segal, Analytic pro-p groups. Cambridge 1991.

[3] W. Feit and J. Thompson, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 773–1029.

[4] G. Glauberman, Central elements in core-free groups, *J. Algebra* **4** (1966), 403–420.

[5] D. Gorenstein, Finite Groups, Chelsea Publishing Company, New York, 1980.

[6] D. Gorenstein, R. Lyons and R. Solomon, The classification of the finite simple groups. Number 3. Part I. Chapter A: Almost Simple K-groups, Mathematical Surveys and Monographs, vol. 40, AMS, Providence, RI, 1998.

[7] S. Guest, A solvable version of the Baer-Suzuki theorem, *Trans. Amer. Math. Soc.* **362** (2010), 5909–5946.

[8] R. M. Guralnick, A note on pairs of matrices with rank one commutator, *Linear and Multilinear Algebra* **8** (1979), 97–99.

[9] R. M. Guralnick, On the number of generators of a finite group, *Arch. Math. (Basel)* **53** (1989), 521–523.

[10] R. M. Guralnick and G. R. Robinson, On extensions of the Baer-Suzuki theorem, *Israel J. Math.* **82** (1993), 281–297.

[11] B. Hartley and I. M. Isaacs, On characters and fixed points of coprime operator groups. *J. Algebra* **131** (1990), 342–358.

[12] E. I. Khukhro, p-Automorphisms of finite p-groups, London Math. Soc. Lecture Note Ser., vol. 246, Cambridge Univ. Press, 1998.

[13] E. I. Khukhro, Groups and Lie rings admitting an almost regular automorphism of prime order, Mat. Sb., **181** (1990), 1197–1219; English translation Math. USSR Sb. **71** (1992), 51—63.

[14] E. I. Khukhro, Groups with an automorphism of prime order that is almost regular in the sense of rank, *J. London Math. Soc.*(2) **77** (2008), 130–148.

[15] E. I. Khukhro and P. Shumyatsky, Nonsoluble and non-p-soluble length of finite groups, *Israel J. Math.* **207** (2015), 507–525.

[16] L. A. Kurdachenko and P. Shumyatsky, The ranks of central factor and commutator groups. *Math. Proc. Camb. Phil. Soc.***154**, (2013) 63–69.

[17] A. Lubotzky and A. Mann, Powerful p-groups I, *J. Algebra* **105** (1987), 484–505.

[18] A. Lucchini, A bound on the number of generators of a finite group, *Arch.Math.* 53 (1989), 313–317.

[19] D. J. S. Robinson, Finiteness conditions and generalized soluble groups. part 1, Springer-Verlag, 1972.

[20] J. S. Rose, A Course on Group Theory, Cambridge Univ. Press, Cambridge, 1978.

[21] P. Shumyatsky, Involutory automorphisms of finite groups and their centralizers, *Arch. Math.* **71** (1998), 425–432.

[22] J. G. Thompson, Finite groups with fixed-point-free automorphisms of prime order. *Proc. Nat. Acad. Sci. U.S.A.***45** (1959), 578–581.
[23] R. A. Wilson, The Finite Simple Groups, Graduate Texts in Mathematics, vol. 251. Springer-Verlag London, Ltd., London, 2009.
[24] H. Zassenhaus, Beweis eines Satzes über diskrete Gruppen, *Abh. Math. Sem. Univ. Hamburg* **12** (1938), 289–312.

Cristina Acciarri: Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 Brazil
Email address: acciarri@unb.br

Robert M. Guralnick: Department of Mathematics, University of Southern California, Los Angeles, CA90089-2532, USA
Email address: guralnic@usc.edu

Pavel Shumyatsky: Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 Brazil
Email address: pavel@unb.br