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A Finsler type Lipschitz optimal transport metric for a quasilinear wave equation

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Abstract

We consider the global well-posedness of weak energy conservative solution to a general quasilinear wave equation through variational principle, where the solution may form finite time cusp singularity, when energy concentrates. As a main result in this paper, we construct a Finsler type optimal transport metric, then prove that the solution flow is Lipschitz under this metric. We also prove a generic regularity result by applying Thom's transversality theorem, then find piecewise smooth transportation paths among a dense set of solutions. The results in this paper are for large data solutions, without restriction on the size of solutions.

MSC: 35L70; 35B30; 35G25

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1. Introduction

Consider a class of quasilinear wave equations derived from a variational principle whose action is a quadratic function of derivatives of the field with coefficients depending on both the field and independent variables

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$$\delta \int A^{ij}_{\mu\nu}(\mathbf{x}, u)\frac{\partial u^\mu}{\partial x_i}\frac{\partial u^\nu}{\partial x_j}d\mathbf{x} = 0, \tag{1.1}$$

where we use the summation convention, see [1]. Here $\mathbf{x} \in \mathbb{R}^{d+1}$ are the space-time variables and $u : \mathbb{R}^{d+1} \to R^n$ are the dependent variables. We assume the coefficients $A^{ij}_{\mu\nu} : \mathbb{R}^{d+1} \times \mathbb{R}^n \to \mathbb{R}$ are smooth and satisfy $A^{ij}_{\mu\nu} = A^{ij}_{\nu\mu} = A^{ji}_{\mu\nu}$. The Euler-Lagrange equations associated with (1.1) are

$$\frac{\partial}{\partial x_i}\Big(A^{ij}_{k\mu}\frac{\partial u^\mu}{\partial x_j}\Big) = \frac{1}{2}\frac{\partial A^{ij}_{\mu\nu}}{\partial u^k}\frac{\partial u^\mu}{\partial x_i}\frac{\partial u^\nu}{\partial x_j}. \tag{1.2}$$

In this paper, we consider the special case of (1.1) when $n = 1$ and $d = 1$, where the Euler-Lagrange equation (1.2) reads that

$$(A^{11}u_t + A^{12}u_x)_t + (A^{12}u_t + A^{22}u_x)_x = \frac{1}{2}\big(\frac{\partial A^{11}}{\partial u}u_t^2 + 2\frac{\partial A^{12}}{\partial u}u_t u_x + \frac{\partial A^{22}}{\partial u}u_x^2\big). \tag{1.3}$$

Moreover, assume the coefficients satisfy

$$(A^{ij})_{2\times 2} = \begin{pmatrix} \alpha^2 & \beta \\ \beta & -\gamma^2 \end{pmatrix}(x, u),$$

then equation (1.3) exactly gives the following nonlinear variational wave equation

$$(\alpha^2 u_t + \beta u_x)_t + (\beta u_t - \gamma^2 u_x)_x = \alpha\alpha_u u_t^2 + \beta_u u_t u_x - \gamma\gamma_u u_x^2, \tag{1.4}$$

with initial data

$$u(x, 0) = u_0(x) \in H^1, \quad u_t(x, 0) = u_1(x) \in L^2. \tag{1.5}$$

Here the variable $t \geq 0$ is time, and x is the spatial coordinate. The coefficients $\alpha = \alpha(x, u)$, $\beta = \beta(x, u)$, $\gamma = \gamma(x, u)$ are smooth functions on x and u, satisfying that, there exist positive constants $\alpha_1, \alpha_2, \beta_2, \gamma_1$ and γ_2, such that for any $z = (x, u)$,

$$\begin{cases} 0 < \alpha_1 \leq \alpha(z) \leq \alpha_2, \ \ |\beta(x, u)| \leq \beta_2, \ \ 0 < \gamma_1 \leq \gamma(z) \leq \gamma_2, \\ \sup\limits_z\{|\nabla\alpha(z)|, |\nabla\beta(z)|, |\nabla\gamma(z)|\} < \infty, \ \ \forall z \in \mathbb{R}^2. \end{cases} \tag{1.6}$$

Then system (1.4) is strictly hyperbolic with two eigenvalues

$$\lambda_- := \frac{\beta - \sqrt{\beta^2 + \alpha^2\gamma^2}}{\alpha^2} < 0, \qquad \lambda_+ := \frac{\beta + \sqrt{\beta^2 + \alpha^2\gamma^2}}{\alpha^2} > 0. \tag{1.7}$$

Moreover, in this paper we always assume that the following generic condition is satisfied

$$\partial_u\lambda_\pm(x, u) = 0 \Rightarrow \partial_{uu}\lambda_\pm(x, u) \neq 0 \quad \text{or} \quad \partial_{ux}\lambda_\pm(x, u) \neq 0. \tag{1.8}$$

In this paper, we will always call waves in the families of λ_- and λ_+ as backward and forward waves, respectively. By (1.6), $-\lambda_-(x,u)$ and $\lambda_+(x,u)$ are both smooth on x and u, bounded and uniformly positive.

1.1. Physical background and earlier results

There are various physical models related to equations (1.1) and (1.4). For example, see [1] for the background in general relativity. There are many classical results, such as in [11].

A particular physical example leading to (1.1) and (1.4) is the motion of a massive director field in a nematic liquid crystal. A nematic liquid crystal can be described by a director field of unit vectors $\mathbf{n} \in \mathbb{S}^2$. In the regime in which inertia effects dominate viscosity, the propagation of orientation waves in the director field is modeled by the least action principle (see [2])

$$\delta \int \Big(\partial_t \mathbf{n} \cdot \partial_t \mathbf{n} - W(\mathbf{n}, \nabla \mathbf{n}) \Big) \, d\mathbf{x} dt = 0, \quad \mathbf{n} \cdot \mathbf{n} = 1, \tag{1.9}$$

where $W(\mathbf{n}, \nabla \mathbf{n}) = K_1 |\mathbf{n} \times (\nabla \times \mathbf{n})|^2 + K_2 (\nabla \cdot \mathbf{n})^2 + K_3 (\mathbf{n} \cdot \nabla \times \mathbf{n})^2$ is the well-known Oseen-Franck potential energy density. Here K_1, K_2 and K_3 are positive elastic constants. This variational principle is in the form of (1.1).

When $\mathbf{n} = (\cos u(x,t), \sin u(x,t), 0)$, with $x \in \mathbb{R}$, the dynamics are described by the variational principle

$$\delta \int (u_t^2 - c^2(u) u_x^2) \, dx \, dt = 0, \tag{1.10}$$

with wave speed c given by $c^2(u) = K_1 \cos^2 u + K_2 \sin^2 u$. This gives the variational wave equation

$$u_{tt} - c(u)(c(u) u_x)_x = 0, \tag{1.11}$$

which is a special example of (1.4), so does the inhomogeneous case: (1.10) with $c = c(x,u)$.

It is known that solutions for the initial value problem of (1.11) generically have finite time cusp singularity [3,6,10]. The global existence and uniqueness of Hölder continuous energy conservative solution was established by Bressan-Zheng and Bressan-Chen-Zhang in [7,5], respectively. See other existence result for (1.11) in [14].

The breakthrough on the Lipschitz continuous dependence happened later in [4] by Bressan and Chen, where the solution flow was proved to be Lipschitz continuous on a new Finsler type optimal transport metric.

The main target of this paper is to extend this Lipschitz continuous dependence result to the general equation (1.4), where the existence and uniqueness of energy conservative Hölder continuous solution for (1.4)–(1.5) has been established by [15,9], respectively.

1.2. Main results of this paper

Due to finite time singularity formation [3,6,10], one needs to consider weak solutions for (1.4)–(1.5).

Definition 1.1 *(Weak solution).* The function $u = u(x,t)$, defined for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, is a **weak solution** to the Cauchy problem (1.4)–(1.5) if it satisfies following conditions.

(i) In the x-t plane, the function $u(x,t)$ is locally Hölder continuous with exponent $1/2$. The function $t \mapsto u(\cdot,t)$ is continuously differentiable as a map with values in L^p_{loc}, for all $1 \le p < 2$. Moreover, it is Lipschitz continuous with respect to (w.r.t.) the L^2 distance, that is, there exists a constant L such that

$$\left\| u(\cdot,t) - u(\cdot,s) \right\|_{L^2} \le L\,|t-s|,$$

for all $t, s \in \mathbb{R}^+$.

(ii) The function $u(x,t)$ takes on the initial conditions in (1.5) pointwise, while their temporal derivatives hold in L^p_{loc} for $p \in [1,2)$.

(iii) The equations (1.4) hold in distributional sense, that is

$$\int\int \left[\varphi_t(\alpha^2 u_t + \beta u_x) + \varphi_x(\beta u_t - \gamma^2 u_x) + \varphi(\alpha\alpha_u u_t^2 + \beta_u u_t u_x - \gamma\gamma_u u_x^2)\right] dx\, dt = 0$$

for any test function $\varphi \in C^1_c(\mathbb{R} \times \mathbb{R}^+)$.

When a finite time gradient blowup forms, the solution flow fails to be Lipschitz in the energy space, i.e. H^1 space. We will construct a Finsler-type distance that renders the conservative solution flows of (1.4)–(1.5) Lipschitz continuous. The new distance is determined by the minimum cost to transport from one solution to another. We consider a double optimal transportation problem which equips the metric with information on the quasilinear structure of the wave equation. To control the energy transfer between two characteristic directions, we add a wave potential capturing future wave interactions in the metric.

The main result of this paper is:

Theorem 1.1 *(Lipschitz continuous dependence). We consider the unique conservative solution given in Theorems 2.1 and 2.2 for (1.4)–(1.5). Let the conditions (1.6)–(1.8) be satisfied, then the geodesic distance $d(\cdot,\cdot)$, defined in Definition 7.2, provides solution flow the following Lipschitz continuous property. Consider two initial data $(u_0,u_1)(x)$ and $(\hat{u}_0,\hat{u}_1)(x)$ in (1.5), then for any $T > 0$, the corresponding solutions $u(x,t)$ and $\hat{u}(x,t)$ satisfy*

$$d\Big((u,u_t)(t), (\hat{u},\hat{u}_t)(t)\Big) \le C d\Big((u_0,u_1), (\hat{u}_0,\hat{u}_1)\Big),$$

when $t \in [0,T]$, where the constant C depends only on T and the total energy.

There are many variations in the construction of new Lipschitz metric and the proof of Lipschitz property for (1.4), comparing to the theory for variational wave equation in [4], especially on the subtle relative shift terms. A slight change in the metric may ruin the Lipschitz property.

Another crucial obstruction in establishing the new distance is how to prove the existence of regular enough transportation planes between two solutions. Here we prove the following generic regularity result showing that on a dense set of initial data the corresponding solutions are piecewise smooth and only including generic singularities. Then we prove the existence of piecewise smooth solution path between any two generic solutions in Theorem A.1.

Theorem 1.2 *(Generic regularity). Let the condition* (1.6)–(1.8) *be satisfied and let $T>0$ be given, then there exists an open dense set of initial data*

$$\mathcal{M}\subset\Big(\mathcal{C}^3(\mathbb{R})\cap H^1(\mathbb{R})\Big)\times\Big(\mathcal{C}^2(\mathbb{R})\cap L^2(\mathbb{R})\Big),$$

such that, for $(u_0,u_1)\in\mathcal{M}$, the conservative solution $u=u(x,t)$ of (1.4)–(1.5) *is twice continuously differentiable in the complement of finitely many characteristic curves, within the domain $\mathbb{R}\times[0,T]$.*

This paper will be divided into seven sections. Section 2 is a short review on the existence and uniqueness of conservative solution to (1.4)–(1.5). In Section 3, we will introduce the main idea and steps used in Sections 4 to 7, where we construct the metric and prove main theorems step by step.

2. Previous existence and uniqueness results

We begin, in this paper, by reviewing the existence and uniqueness of conservative weak solution to the Cauchy problem (1.4)–(1.5) in [9,15].

Theorem 2.1 *(Existence [15]). Let the condition* (1.6) *be satisfied, then the Cauchy problem* (1.4)–(1.5) *admits a global weak solution $u=u(x,t)$ defined for all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$.*

To introduce the uniqueness result, let's first introduce some notations. Denote wave speeds as

$$c_1:=\alpha\lambda_-=\frac{\beta-\sqrt{\beta^2+\alpha^2\gamma^2}}{\alpha}<0,\qquad c_2:=\alpha\lambda_+=\frac{\beta+\sqrt{\beta^2+\alpha^2\gamma^2}}{\alpha}>0,$$

and Riemann variables as

$$R:=\alpha u_t+c_2u_x,\quad S:=\alpha u_t+c_1u_x. \tag{2.1}$$

By (1.6), the wave speeds $-c_1$ and c_2 are smooth, bounded and uniformly positive.

For a smooth solution of (1.4), the variables R and S satisfy

$$\begin{cases}\alpha(x,u)R_t+c_1(x,u)R_x=a_1R^2-(a_1+a_2)RS+a_2S^2+c_2bS-d_1R,\\ \alpha(x,u)S_t+c_2(x,u)S_x=-a_1R^2+(a_1+a_2)RS-a_2S^2+c_1bR-d_2S,\\ u_t=\dfrac{c_2S-c_1R}{\alpha(c_2-c_1)}\quad\text{or}\quad u_x=\dfrac{R-S}{c_2-c_1},\end{cases} \tag{2.2}$$

where

$$a_i=\frac{c_i\partial_u\alpha-\alpha\partial_uc_i}{2\alpha(c_2-c_1)},\qquad b=\frac{\alpha\partial_x(c_1-c_2)+(c_1-c_2)\partial_x\alpha}{2\alpha(c_2-c_1)},$$
$$d_i=\frac{c_2\partial_xc_1-c_1\partial_xc_2}{2(c_2-c_1)}+\frac{\alpha\partial_xc_i-c_i\partial_x\alpha}{2\alpha},\quad (i=1,2).$$

Here ∂_x and ∂_u denote partial derivatives with respect to x and u, respectively.

Multiplying the first equation in (2.2) by $2R$ and the second one by $2S$, one has the balance laws for energy densities in two directions, namely

$$\begin{cases}(R^2)_t+(\frac{c_1}{\alpha}R^2)_x=\frac{2a_2}{\alpha}(RS^2-R^2S)+\frac{2c_2b}{\alpha}RS-\frac{c_2\partial_xc_1-c_1\partial_xc_2}{\alpha(c_2-c_1)}R^2,\\ (S^2)_t+(\frac{c_2}{\alpha}S^2)_x=\frac{2a_1}{\alpha}(RS^2-R^2S)-\frac{2c_1b}{\alpha}RS-\frac{c_2\partial_xc_1-c_1\partial_xc_2}{\alpha(c_2-c_1)}S^2.\end{cases} \tag{2.3}$$

Moreover, we have

$$\begin{cases}(\tilde{R}^2)_t+(\frac{c_1}{\alpha}\tilde{R}^2)_x=G,\\ (\tilde{S}^2)_t+(\frac{c_2}{\alpha}\tilde{S}^2)_x=-G,\end{cases} \tag{2.4}$$

where

$$\tilde{R}^2=\frac{-c_1}{c_2-c_1}R^2,\quad \tilde{S}^2=\frac{c_2}{c_2-c_1}S^2,\quad \text{and}$$
$$G=\frac{2c_2a_1}{\alpha(c_2-c_1)}R^2S-\frac{2c_1a_2}{\alpha(c_2-c_1)}RS^2-\frac{2c_1c_2b}{\alpha(c_2-c_1)}RS,$$

which indicates the following conserved quantities

$$\alpha^2u_t^2+\gamma^2u_x^2=\tilde{R}^2+\tilde{S}^2,$$

and the corresponding energy conservation law

$$(\tilde{R}^2+\tilde{S}^2)_t+(\frac{c_1}{\alpha}\tilde{R}^2+\frac{c_2}{\alpha}\tilde{S}^2)_x=0.$$

Now, we state the uniqueness result in [9], which together with the energy conservation proved in [15] show that the problem (1.4)–(1.5) has a unique conservative solution under Definition 1.1.

Theorem 2.2 *(Uniqueness [9] and energy conservation [15]). Let the condition* (1.6) *be satisfied, then there exists a unique conservative weak solution $u(x,t)$ for* (1.4)–(1.5).

Here a weak solution $u(x,t)$ defined in Definition 1.1 is said to be (energy) conservative *if one can find two families of positive Radon measures on the real line: $\{\mu_-^t\}$ and $\{\mu_+^t\}$, depending continuously on t in the weak topology of measures, with the following properties.*

(i) *At every time t one has*

$$\mu_-^t(\mathbb{R})+\mu_+^t(\mathbb{R}) \;=\; \mathcal{E}_0 \;:=\; \int_{-\infty}^{\infty}\Big[\alpha^2\big(x,u_0(x)\big)u_1^2(x)+\gamma^2\big(x,u_0(x)\big)u_{0,x}^2(x)\big)\Big]\,dx\,.$$

(ii) *For each t, the absolutely continuous parts of μ_-^t and μ_+^t with respect to the Lebesgue measure have densities respectively given by*

$$\tilde{R}^2 = \frac{-c_1}{c_2 - c_1}(\alpha u_t + c_2 u_x)^2, \qquad \tilde{S}^2 = \frac{c_2}{c_2 - c_1}(\alpha u_t + c_1 u_x)^2.$$

(iii) *For almost every $t \in \mathbb{R}^+$, the singular parts of μ_-^t and μ_+^t are concentrated on the set where $\partial_u \lambda_- = 0$ or $\partial_u \lambda_+ = 0$.*

(iv) *The measures μ_-^t and μ_+^t provide measure-valued solutions respectively to the balance laws*

$$\begin{cases} \xi_t + (\frac{c_1}{\alpha}\xi)_x = \frac{2c_2 a_1}{\alpha(c_2 - c_1)} R^2 S - \frac{2c_1 a_2}{\alpha(c_2 - c_1)} R S^2 - \frac{2c_1 c_2 b}{\alpha(c_2 - c_1)} R S, \\ \eta_t + (\frac{c_2}{\alpha}\eta)_x = -\frac{2c_2 a_1}{\alpha(c_2 - c_1)} R^2 S + \frac{2c_1 a_2}{\alpha(c_2 - c_1)} R S^2 + \frac{2c_1 c_2 b}{\alpha(c_2 - c_1)} R S. \end{cases}$$

Furthermore, for above conservative weak solution, the total energy represented by the sum $\mu_- + \mu_+$ is showed to be conserved in time. This energy may only be concentrated on a set of zero measure or at points where $\partial_u \lambda_-$ or $\partial_u \lambda +$ vanishes. In particular, if $\partial_u \lambda_\pm \neq 0$ for any (x, u), then the set

$$\left\{\tau;\ \mathcal{E}(\tau) := \int_{-\infty}^{\infty} \Big[|\alpha^2\big(x, u(x,\tau)\big) u_t^2(x,\tau) + \gamma^2\big(x, u(x,\tau)\big) u_x^2(x,\tau)\Big]\, dx \ < \ \mathcal{E}_0\right\}$$

has measure zero.

3. Main idea and structure of the proof for main theorems

Due to the finite time energy concentration at the gradient blowup, solution flow of (1.4) is not Lipschitz in the energy space, i.e. H^1 space. One can find examples showing this instability in [8] for some unitary direction models, or in [10] for the variational wave equation.

It is natural to use an optimal transport metric. To capture the quasilinear structure of solutions, we consider a double transportation problem, which means that we study the propagation of waves in backward and forward directions, respectively.

More precisely, to keep track of the cost in the transportation, we are led to construct the geodesic distance. That is, for two given solution profiles $u(t)$ and $u^\epsilon(t)$, we consider all possible smooth deformations/paths $\gamma^t : \theta \mapsto \big(u^\theta(t), u_t^\theta(t)\big)$ for $\theta \in [0, 1]$ with $\gamma^t(0) = \big(u(t), u_t(t)\big)$ and $\gamma^t(1) = \big(u^\epsilon(t), u_t^\epsilon(t)\big)$, and then measure the length of these paths through integrating the norm of the tangent vector $d\gamma^t/d\theta$; see Fig. 1 (a).

Roughly speaking, the distance between u and u^ϵ will be calculated by the optimal path length

$$d\left(u(t), u^\epsilon(t)\right) = \inf_{\gamma^t} \|\gamma^t\| := \inf_{\gamma^t} \int_0^1 \|v^\theta(t)\|_{u^\theta(t)}\, d\theta, \quad \text{where } v^\theta(t) = \frac{d\gamma^t}{d\theta}.$$

The subscript $u^\theta(t)$ emphasizes the dependence of the norm on the flow u. The most important element is how to define the Finsler norm $\|v^\theta(t)\|_{u^\theta(t)}$ by capturing behaviors of the quasilinear wave equation, such that, for regular solutions,

$$\|\gamma^t\| \le C\|\gamma^0\|, \qquad \forall\, t \in [0, T]. \tag{3.1}$$

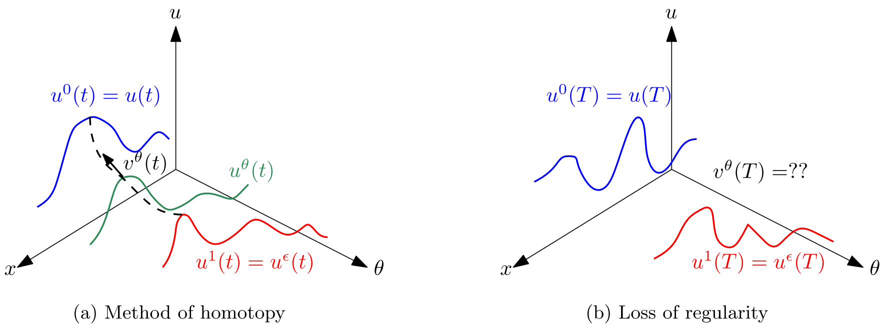

Fig. 1. Compare two solutions $u(x)$ and $u^\epsilon(x)$ at a given time t.

Here C only depends on the total initial energy and T, and is uniformly bounded when the solution approaches a singularity.

The norm $\|v^\theta(t)\|_{u^\theta(t)}$ measures the cost in shifting from one solution to the other one. We will measure the cost in forward and backward directions, with energy densities μ^+, μ^- defined in Theorem 2.2, respectively. As a payback of using the double transportation problem, now we can use both balance laws on forward and backward energy densities along characteristics, such as (2.3) and (2.4). We need to control the growth of norm caused by the energy transfer between two families during nonlinear wave interactions. The idea is to add interaction potentials in the norm. More detail will be given in Section 4.

We will construct the metric d and prove that the solution map is Lipschitz continuous under this metric in several steps.

1. We construct a Lipschitz metric for smooth solution (Section 4).
2. By an application of Thom's transversality theorem, we prove that the piecewise smooth solutions with only generic singularities are dense in $H^1 \times L^2$ space. This proves Theorem 1.2 (Section 5).
3. We extend the Lipschitz metric to piecewise smooth solutions with generic singularities. (Section 6).
4. We finally define the metric d for weak solutions with general $H^1 \times L^2$ initial data, and complete the proof of Theorems 1.1. We also compare the metric d with some Sobolev metrics and Kantorovich-Rubinstein metric (Section 7).

4. The norm of tangent vectors for smooth solutions

The first goal is to define a Finsler norm on tangent vectors measuring the cost of transport, and show this norm satisfies the desired Lipschitz property (3.1) for any smooth solutions.

Let (u, R, S) be any smooth solution to (1.4), (2.2), and then take a family of perturbed solutions $(u^\epsilon, R^\epsilon, S^\epsilon)(x)$ to (1.4), (2.2), which can be written as

$$u^\epsilon(x) = u(x) + \epsilon v(x) + o(\epsilon), \quad \text{and} \quad \begin{cases} R^\epsilon(x) = R(x) + \epsilon r(x) + o(\epsilon), \\ S^\epsilon(x) = S(x) + \epsilon s(x) + o(\epsilon). \end{cases} \tag{4.1}$$

Here both u and u^ϵ satisfy the Definition 1.1 of weak solution. Because of finite speed of propagation, for any time $T>0$, there exists a compact subset on the x-t plane with $t\in[0,T]$, out of which the solution is smooth.

Here and in the sequel, we will omit the variables t, x when we use any functions if it does not cause any confusion.

Let the tangent vectors r, s be given, in terms of (2.2) (equation on u_x) and (4.1), the perturbation v can be uniquely determined by

$$v_x=\frac{r-s}{c_2-c_1}-\frac{\partial_u c_2-\partial_u c_1}{(c_2-c_1)^2}(R-S)v, \qquad v(0,t)=0. \tag{4.2}$$

Moreover, it holds that

$$v_t=\frac{c_2 s-c_1 r}{\alpha(c_2-c_1)}-\frac{c_2 S-c_1 R}{\alpha^2(c_2-c_1)}v\partial_u\alpha+\frac{c_2\partial_u c_1-c_1\partial_u c_2}{\alpha(c_2-c_1)^2}(S-R)v. \tag{4.3}$$

Furthermore, by a straightforward calculation, the first order perturbations v, s, r must satisfy the equations

$$\begin{aligned}\alpha^2 v_{tt}-\gamma^2 v_{xx}+2\beta v_{xt}&=[2\gamma\gamma_u u_x+2\gamma\gamma_x-\beta_u u_t]v_x-[2\alpha\alpha_u u_t+\beta_u u_x+\beta_x]v_t\\&\quad-[\alpha_u^2u_t^2+\alpha_{uu}\alpha u_t^2+2\alpha\alpha_u u_{tt}+2\beta_u u_{xt}+\beta_{xu}u_t+\beta_{uu}u_t u_x]v\\&\quad+[\gamma_u^2u_x^2+\gamma\gamma_{uu}u_x^2+2\gamma\gamma_u u_{xx}+2\gamma_u\gamma_x u_x+2\gamma\gamma_{xu}u_x]v,\end{aligned} \tag{4.4}$$

and

$$\begin{cases}\alpha r_t+c_1 r_x=2a_1Rr-(a_1+a_1)(Rs+Sr)+2a_2Ss+c_2bs-d_1r+2a_1(c_2-c_1)R_xv\\ \qquad\qquad +\alpha B_1R^2v-\alpha(B_1+B_2)RSv+\alpha B_2S^2v+\alpha B_4Sv-\alpha B_5Rv,\\ \alpha s_t+c_2s_x=-2a_1Rr+(a_1+a_1)(Rs+Sr)-2a_2Ss+c_1br-d_2s+2a_2(c_2-c_1)S_xv\\ \qquad\qquad -\alpha B_1R^2v+\alpha(B_1+B_2)RSv-\alpha B_2S^2v-\alpha B_6Sv+\alpha B_3Rv,\end{cases} \tag{4.5}$$

where $B_i=\frac{\alpha\partial_u a_i-a_i\partial_u\alpha}{\alpha^2}, i=1,2$, $B_3=\frac{\alpha\partial_u(c_1b)-c_1b\partial_u\alpha}{\alpha^2}$, $B_4=\frac{\alpha\partial_u(c_2b)-c_2b\partial_u\alpha}{\alpha^2}$, $B_5=\frac{\alpha\partial_u d_1-d_1\partial_u\alpha}{\alpha^2}$, $B_6=\frac{\alpha\partial_u d_2-d_2\partial_u\alpha}{\alpha^2}$.

To measure the cost in shifting from one solution to the other one, it is nature to consider both vertical and horizontal shifts in the energy space. With this in mind, since the tangent flows v, r, s only measure the vertical shifts between two solutions, we also need to add quantities $w(x,t), z(x,t)$ measuring the horizontal shifts, corresponding to backward and forward directions, respectively. And it is very tentative to embed some important information of waves into $w(x,t), z(x,t)$ in order to focus only on reasonable transports between two solutions. Here we require $w(x,t)$ to satisfy

$$\epsilon w(x,t)+o(\epsilon)=x^\epsilon(t)-x(t),$$

where $x^\epsilon(t)$ and $x(t)$ are two backward characteristics starting from initial points $x^\epsilon(0)$ and $x(0)$. Symmetrically, the function $\epsilon z(x,t)$ measures the difference of two forward characteristics. Then it is easy to see that w, z satisfy the following system

$$\begin{cases} \alpha w_t + c_1 w_x = \dfrac{\alpha \partial_x c_1 - c_1 \partial_x \alpha}{\alpha} w - 2a_1(c_2 - c_1)(v + u_x w), \\ \alpha z_t + c_2 z_x = \dfrac{\alpha \partial_x c_2 - c_2 \partial_x \alpha}{\alpha} z - 2a_2(c_2 - c_1)(v + u_x z), \\ w(x,0) = w_0(x), \qquad z(x,0) = z_0(x). \end{cases} \tag{4.6}$$

Next, we define *interaction potentials* $\mathcal{W}^+/\mathcal{W}^-$ for forward/backward directions as follows.

$$\mathcal{W}^- := 1 + \int_{-\infty}^{x} S^2(y)\,dy, \quad \mathcal{W}^+ := 1 + \int_{x}^{+\infty} R^2(y)\,dy.$$

Essentially, when tracking a backward wave, $\mathcal{W}^-$ measures the total forward energy, that this backward wave will meet in the future.

Now, let's show the decay of interaction potentials. In view of (2.3), it holds that

$$\begin{cases} \mathcal{W}_t^- + \dfrac{c_1}{\alpha}\mathcal{W}_x^- = -\dfrac{c_2 - c_1}{\alpha} S^2 + \displaystyle\int_{-\infty}^{x} \Big[\dfrac{2a_1}{\alpha}(RS^2 - R^2 S) - \dfrac{2c_1 b}{\alpha} RS - \dfrac{c_2 \partial_x c_1 - c_1 \partial_x c_2}{\alpha(c_2 - c_1)} S^2\Big]\,dy, \\ \mathcal{W}_t^+ + \dfrac{c_2}{\alpha}\mathcal{W}_x^+ = -\dfrac{c_2 - c_1}{\alpha} R^2 + \displaystyle\int_{x}^{+\infty} \Big[\dfrac{2a_2}{\alpha}(RS^2 - R^2 S) + \dfrac{2c_2 b}{\alpha} RS - \dfrac{c_2 \partial_x c_1 - c_1 \partial_x c_2}{\alpha(c_2 - c_1)} R^2\Big]\,dy. \end{cases}$$

This together with condition (1.6) implies that

$$\begin{cases} \mathcal{W}_t^- + \dfrac{c_1}{\alpha}\mathcal{W}_x^- \le -\dfrac{2\gamma_1}{\alpha_2} S^2 + G_1(t), \\ \mathcal{W}_t^+ + \dfrac{c_2}{\alpha}\mathcal{W}_x^+ \le -\dfrac{2\gamma_1}{\alpha_2} R^2 + G_2(t), \end{cases} \tag{4.7}$$

where the functions

$$G_1(t) := \int_{-\infty}^{+\infty} \Big|\frac{2a_1}{\alpha}(RS^2 - R^2 S) + \frac{2c_1 b}{\alpha} RS - \frac{c_2 \partial_x c_1 - c_1 \partial_x c_2}{\alpha(c_2 - c_1)} S^2\Big|\,dy,$$

$$G_2(t) := \int_{-\infty}^{+\infty} \Big|\frac{2a_2}{\alpha}(RS^2 - R^2 S) + \frac{2c_2 b}{\alpha} RS - \frac{c_2 \partial_x c_1 - c_1 \partial_x c_2}{\alpha(c_2 - c_1)} R^2\Big|\,dy.$$

As proved in [9] (see equation (3.16) in [9]), we obtain

$$\int_0^T G_i(t) \le C_T, \quad i = 1, 2, \tag{4.8}$$

for some constant C_T depending only on T and the total energy.

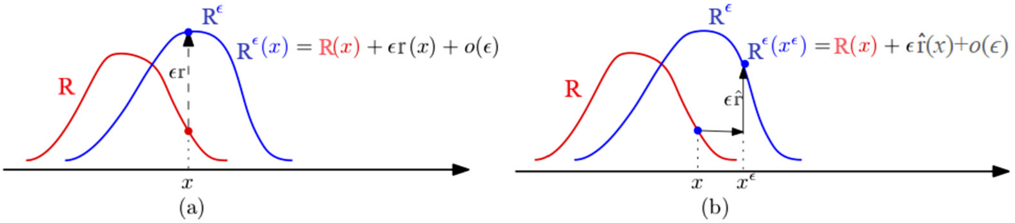

Fig. 2. A sketch of how to deform from R to R^ϵ: (a) a vertical shift ϵr; (b) a horizontal shift $\epsilon R_x w$ followed by a vertical displacement ϵr. Here $x^\epsilon := x + \epsilon w(x)$. We denote the total shift as $\epsilon \hat{r} = \epsilon(r + R_x w)$.

Up to now, we are ready to define a Finsler norm for the tangent vectors v, r, s as

$$\|(v, r, s)\|_{(u,R,S)} := \inf_{v,w,\hat{r},z,\hat{s}} \|(v, w, \hat{r}, z, \hat{s})\|_{(u,R,S)}, \tag{4.9}$$

where the infimum is taken over the set of vertical displacements $v, \hat{r}, \hat{s}$ and horizontal shifts w, z which satisfy equations (4.2), (4.3), (4.6) and relations

$$\begin{cases} \hat{r} = r + wR_x + \dfrac{a_2(w-z)}{c_2 - c_1} S^2 - \dfrac{a_1 + a_2}{c_2 - c_1}(w - z)RS + \dfrac{c_2 b(w-z)}{c_2 - c_1} S, \\ \hat{s} = s + zS_x - \dfrac{a_1(w-z)}{c_2 - c_1} R^2 + \dfrac{a_1 + a_2}{c_2 - c_1}(w - z)RS + \dfrac{c_1 b(w-z)}{c_2 - c_1} R. \end{cases} \tag{4.10}$$

Here, to motivate the explicit construction of $\|(v, r, s)\|_{(u,R,S)}$, we consider a reference solution R together with a perturbation R^ϵ. As shown in Fig. 2, the tangent vector r can be expressed as a horizontal part ϵw and a vertical part $\epsilon \hat{r}$, that is $r = \hat{r} - R_x w$. The other terms in (4.10) take account of relative shift. We will give more details on the relative shift terms later. Now, we define the following norm:

$$\begin{aligned} &\|(v, w, \hat{r}, z, \hat{s})\|_{(u,R,S)} \\ := {}& \kappa_0 \int_{\mathbb{R}} \left[|w| \mathcal{W}^- + |z| \mathcal{W}^+ \right] dx + \kappa_1 \int_{\mathbb{R}} \left[|w|(1 + R^2) \mathcal{W}^- + |z|(1 + S^2) \mathcal{W}^+ \right] dx \\ &+ \kappa_2 \int_{\mathbb{R}} \left| v + \frac{Rw - Sz}{c_2 - c_1} \right| \left[(1 + R^2) \mathcal{W}^- + (1 + S^2) \mathcal{W}^+ \right] dx + \kappa_3 \int_{\mathbb{R}} \left[|\hat{r}| \mathcal{W}^- + |\hat{s}| \mathcal{W}^+ \right] dx \\ &+ \kappa_4 \int_{\mathbb{R}} \left[\left| w_x + \frac{2a_1(w-z)}{c_2 - c_1} S \right| \mathcal{W}^- + \left| z_x - \frac{2a_2(w-z)}{c_2 - c_1} R \right| \mathcal{W}^+ \right] dx \\ &+ \kappa_5 \int_{\mathbb{R}} \left[\left| Rw_x + \frac{2a_1(w-z)}{c_2 - c_1} RS \right| \mathcal{W}^- + \left| Sz_x - \frac{2a_2(w-z)}{c_2 - c_1} RS \right| \mathcal{W}^+ \right] dx \\ &+ \kappa_6 \int_{\mathbb{R}} \left[\left| 2R\hat{r} + R^2 w_x + \frac{2a_1(w-z)}{c_2 - c_1} R^2 S \right| \mathcal{W}^- + \left| 2S\hat{s} + S^2 z_x - \frac{2a_2(w-z)}{c_2 - c_1} RS^2 \right| \mathcal{W}^+ \right] dx \end{aligned}$$

$$=: \sum_{i=0}^{6} \kappa_i \Big(\int_{\mathbb{R}} J_i^- \mathcal{W}^- \, dx + \int_{\mathbb{R}} J_i^+ \mathcal{W}^+ \, dx \Big) =: \sum_{i=0}^{6} \kappa_i I_i, \tag{4.11}$$

where κ_i with $i = 0, 1, 2 \cdots, 6$ are the constants to be determined later, and I_i, J_i^-, J_i^+ are the corresponding terms in the above equation.

Next we give more details on how to obtain (4.11).

[I]. For I_1, the integrand $|w|(1+R^2)$ accounts for the cost of transporting the base measure with density $1+R^2$ from the point x to the point $x+\epsilon w(x)$.

The integrand $|z|(1+S^2)$ accounts for the cost of transporting the base measure with density $1+S^2$ from the point x to the point $x+\epsilon z(x)$. There are no relative shift terms.

The terms in I_0 are corresponding to the variation of $|x|$ with base measure with density 1, which are added for a technical purpose.

[II]. I_2 can be interpreted as: [change in u] in the energy space. Indeed, the change in u can be estimated as

$$\begin{aligned}
\frac{u^\epsilon\big(x+\epsilon w(x)\big) - u(x)}{\epsilon} &= v(x) + u_x(x) w(x) + o(\epsilon) \\
&= v(x) + \frac{R(x) - S(x)}{c_2 - c_1} w(x) + o(\epsilon) \\
&= v(x) + \frac{Rw - Sz}{c_2 - c_1} + \frac{z - w}{c_2 - c_1} S + o(\epsilon).
\end{aligned}$$

Here the last term $\frac{z-w}{c_2-c_1}S$ on the right hand side of the above equality is just balanced by the relative shift term.

Here we use this term to introduce how to calculate the relative shift term. Recall that

$$\alpha u_t + c_2 u_x = R, \quad \alpha u_t + c_1 u_x = S.$$

So the difference of these two equations give

$$u_x = \frac{1}{c_2 - c_1}(R - S).$$

Roughly speaking,

$$\Delta u \approx \frac{\Delta x}{c_2 - c_1}(R - S) = \frac{z - w}{c_2 - c_1}(R - S). \tag{4.12}$$

Here the S term balances $\frac{z-w}{c_2-c_1}S$. We omit the R term since it is a lower order term.

As we can see from (4.12) that there is only a general philosophy on how to choose terms taking account of the relative shift. One needs to adjust them very carefully to meet the demand. Comparing to the variational wave equation (1.11) in which two wave speeds have same magnitude but different signs, it is much harder to find the relative shift terms for the general equation (1.4).

[III]. I_3 accounts for the vertical displacements in the graphs of R and S. More precisely, the integrand $|\hat{r}|$ as the change in arctan R times the density $1+R^2$ of the base measure. Notice that, for $x^\epsilon = x + \epsilon w(x) + o(\epsilon)$,

$$\begin{aligned}\arctan R^\epsilon(x^\epsilon) &= \arctan\Big(R(x^\epsilon) + \epsilon r(x^\epsilon) + o(\epsilon)\Big)\\ &= \arctan\Big(R(x) + \epsilon w(x) R_x(x) + \epsilon r(x) + o(\epsilon)\Big)\\ &= \arctan R(x) + \epsilon \frac{r(x) + w(x) R_x(x)}{1 + R^2(x)} + o(\epsilon),\end{aligned}$$

which together with the relative shift term gives J_3^-. Here we add some subtle adjustments in the relative shift terms to take account of interactions between forward and backward waves using (2.2).

The change in arctan S times the density $1+S^2$ of the base measure is explained similarly.

[IV]. I_6 can be interpreted as the change in the base measure with densities R^2 and S^2, produced by the shifts w, z. Indeed,

$$\begin{aligned}\big(R^\epsilon(x^\epsilon)\big)^2 &= R^2(x^\epsilon) + 2\epsilon R(x^\epsilon) r(x^\epsilon) + o(\epsilon)\\ &= R^2(x) + 2\epsilon w(x) R(x) R_x(x) + 2\epsilon R(x) r(x) + o(\epsilon),\end{aligned}$$

we obtain that

$$\big(R_x^\epsilon(x^\epsilon)\big)^2 dx^\epsilon - R^2(x) dx = \Big(2\epsilon R(x) R_x(x) w(x) + 2\epsilon R(x) r(x) + \epsilon R^2(x) w_x(x) + o(\epsilon)\Big)\, dx. \tag{4.13}$$

Moreover, as in (4.12), in view of (2.3), if the mass with density S^2 is transported from x to $x + \epsilon z(x)$, the relative shift between forward and backward waves will contribute

$$[2a_2(RS^2 - R^2 S) + 2c_2 bRS] \cdot \epsilon \frac{z - w}{c_2 - c_1}. \tag{4.14}$$

Hence subtracting (4.14) from (4.13) yields the term J_6^-. Symmetrically, we have J_6^+.

[V]. In order to close the time derivatives of I_2 and I_6, we have to add two additional terms I_4 and I_5. Here I_4 accounts for the change in the Lebesgue measure produced by the shifts w, z, while I_5 account for the change in the base measure with densities R and S, produced by the shifts w, z. These two terms are in some sense lower order terms of I_6.

The main goal of this section is to prove the following lemma by showing that the norm of tangent vectors defined in (4.9) satisfies a Gröwnwall type inequality.

Lemma 4.1. *Let $T > 0$ be given, and $(u, R, S)(x, t)$ be a smooth solution to (1.4) and (2.2) when $t \in [0, T]$. Assume that the first order perturbations (v, r, s) satisfy the corresponding equations (4.4)–(4.5). Then it follows that*

$$\|(v, r, s)(t)\|_{(u,R,S)(t)} \le \mathcal{C}(T) \|(v, r, s)(0)\|_{(u,R,S)(0)}, \tag{4.15}$$

for some constant $\mathcal{C}(T)$ depending only on the initial total energy and T.

Proof. To achieve (4.15), it suffices to show that

$$\frac{d}{dt}\|(v,w,\hat{r},z,\hat{s})(t)\|_{(u,R,S)(t)} \le a(t)\|(v,w,\hat{r},z,\hat{s})(t)\|_{(u,R,S)(t)}, \tag{4.16}$$

for any w, z and $\hat{r}$, $\hat{s}$ satisfying (4.6) and (4.10), with a local integrable function $a(t)$. Here and in the sequel, unless specified, we will use $C>0$ to denote a constant depending on the initial total energy and T, where C may vary in different estimates. Now we prove (4.16) by seven steps.

0). We first treat the time derivative of I_0. By (2.2) and (4.6), a straightforward calculation yields that

$$\begin{aligned}
&w_t+(\frac{c_1}{\alpha}w)_x\\
&=2\frac{\alpha\partial_x c_1-c_1\partial_x\alpha}{\alpha^2}w-\frac{2a_1}{\alpha}(c_2-c_1)(v+\frac{Rw-Sz}{c_2-c_1})+\frac{2a_1S}{\alpha}(w-z)-\frac{2a_1w}{\alpha}(R-S).
\end{aligned}$$

This together with the uniform bounds (4.7) on the weights implies that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}J_0^-\mathcal{W}^-\,dx=\frac{d}{dt}\int_{\mathbb{R}}|w|\mathcal{W}^-\,dx\\
&\le C\int_{\mathbb{R}}|w|(1+|R|+|S|)\mathcal{W}^-\,dx+C\int_{\mathbb{R}}|z||S|\mathcal{W}^+\,dx+C\int_{\mathbb{R}}\Big|v+\frac{Rw-Sz}{c_2-c_1}\Big|\mathcal{W}^-\,dx\\
&\quad+G_1(t)\int_{\mathbb{R}}|w|\mathcal{W}^-\,dx-\frac{2\gamma_1}{\alpha_2}\int_{\mathbb{R}}|w|S^2\mathcal{W}^-\,dx.
\end{aligned}$$

Repeating the above process for the time derivative of $\int_{\mathbb{R}}J_0^+\mathcal{W}^+\,dx$ yields

$$\begin{aligned}
&\frac{d}{dt}I_0=\frac{d}{dt}\int_{\mathbb{R}}\Big(J_0^-\mathcal{W}^-+J_0^+\mathcal{W}^+\Big)\,dx\\
&\le C\sum_{k=1,2}\int_{\mathbb{R}}\Big((1+|S|)J_k^-\mathcal{W}^-+(1+|R|)J_k^+\mathcal{W}^+\Big)\,dx\\
&\quad+\int_{\mathbb{R}}\Big(G_1(t)J_0^-\mathcal{W}^-+G_2(t)J_0^+\mathcal{W}^+\Big)\,dx-\frac{2\gamma_1}{\alpha_2}\int_{\mathbb{R}}\Big(S^2J_0^-\mathcal{W}^-+R^2J_0^+\mathcal{W}^+\Big)\,dx.
\end{aligned} \tag{4.17}$$

1). For I_1, using (2.2) and (2.3), by a direct computation, one has

$$\begin{aligned}
(1+R^2)_t+[\frac{c_1}{\alpha}(1+R^2)]_x=&\frac{2a_2}{\alpha}(RS^2-R^2S)+\frac{2c_2b}{\alpha}RS-\frac{c_2\partial_x c_1-c_1\partial_x c_2}{\alpha(c_2-c_1)}R^2\\
&-\frac{2a_1}{\alpha}(R-S)+\frac{\alpha\partial_x c_1-c_1\partial_x\alpha}{\alpha^2}.
\end{aligned} \tag{4.18}$$

With this help and by (4.6), we obtain

$$
\begin{aligned}
&\big[w(1+R^2)\big]_t+\big[\frac{c_1}{\alpha}w(1+R^2)\big]_x\\
&=(w_t+\frac{c_1}{\alpha}w_x)(1+R^2)+w[(1+R^2)_t+\big(\frac{c_1}{\alpha}(1+R^2)\big)_x]\\
&\quad+\frac{2c_2b}{\alpha}RS-\frac{c_2\partial_x c_1-c_1\partial_x c_2}{\alpha(c_2-c_1)}R^2-\frac{2a_1}{\alpha}(R-S)+\frac{\alpha\partial_x c_1-c_1\partial_x\alpha}{\alpha^2}]\\
&=-\frac{2a_1}{\alpha}(v+\frac{Rw-Sz}{c_2-c_1})(1+R^2)(c_2-c_1)-\frac{2a_1Sz}{\alpha}(1+R^2)+\frac{w}{\alpha}\big[2(a_1-a_2)R^2S\\
&\quad+2a_2RS^2+2c_2bRS+\frac{\partial_x(c_2-c_1)}{c_2-c_1}c_1R^2-\frac{\partial_x\alpha}{\alpha}c_1R^2-2a_1R+4a_1S+2\frac{\alpha\partial_x c_1-c_1\partial_x\alpha}{\alpha}\big].
\end{aligned}
$$

This together with (4.7) and the similar estimate for the other terms of I_1 gives that

$$
\begin{aligned}
\frac{d}{dt}I_1\le{}& C\sum_{k=1,2}\int_{\mathbb{R}}\Big((1+|S|)J_k^-\mathcal{W}^-+(1+|R|)J_k^+\mathcal{W}^+\Big)\,dx\\
&+C\int_{\mathbb{R}}\Big(S^2J_0^-\mathcal{W}^-+R^2J_0^+\mathcal{W}^+\Big)\,dx\\
&+\int_{\mathbb{R}}\Big(G_1(t)J_1^-\mathcal{W}^-+G_2(t)J_1^+\mathcal{W}^+\Big)\,dx-\frac{\gamma_1}{\alpha_2}\int_{\mathbb{R}}\Big(S^2J_1^-\mathcal{W}^-+R^2J_1^+\mathcal{W}^+\Big)\,dx.
\end{aligned}
\tag{4.19}
$$

Here we have used the fact that $|RS^2|\le\frac{1}{2}(\frac{\gamma_1}{\alpha_2}R^2S^2+\frac{\alpha_2}{\gamma_1}S^2)$ and $|R^2S|\le\frac{1}{2}(\frac{\gamma_1}{\alpha_2}R^2S^2+\frac{\alpha_2}{\gamma_1}R^2)$.

2). To estimate the time derivative of I_2, recalling (4.2) and (4.3), we get the equation for the first order perturbation v:

$$
v_t+\frac{c_1}{\alpha}v_x=\frac{s}{\alpha}+\frac{2a_1R-2a_2S}{\alpha}v+\frac{\partial_u c_1-\partial_u c_2}{\alpha(c_2-c_1)}Sv. \tag{4.20}
$$

Next, by (2.2) and (4.6), it holds that

$$
\begin{aligned}
&\Big[\frac{Rw-Sz}{c_2-c_1}\Big]_t+\frac{c_1}{\alpha}\Big[\frac{Rw-Sz}{c_2-c_1}\Big]_x\\
&=\frac{w}{c_2-c_1}(R_t+\frac{c_1}{\alpha}R_x)-\frac{z}{c_2-c_1}(S_t+\frac{c_2}{\alpha}S_x)+\frac{z}{\alpha}S_x+\frac{R}{c_2-c_1}(w_t+\frac{c_1}{\alpha}w_x)\\
&\quad-\frac{S}{c_2-c_1}(z_t+\frac{c_2}{\alpha}z_x)+\frac{S}{\alpha}z_x+(Rw-Sz)\big[(\frac{1}{c_2-c_1})_t+\frac{c_1}{\alpha}(\frac{1}{c_2-c_1})_x\big]\\
&=\frac{w}{\alpha(c_2-c_1)}\Big[a_1R^2-\big(a_1+a_2+\frac{\partial_u c_2-\partial_u c_1}{c_2-c_1}\big)RS+a_2S^2+c_2bS+(d_1-\partial_x c_1)R\Big]\\
&\quad-\frac{z}{\alpha(c_2-c_1)}\Big[-a_1R^2+(a_1+a_2)RS-(a_2+\frac{\partial_u c_2-\partial_u c_1}{c_2-c_1})S^2+c_1bR+(d_2-\partial_x c_1)S\Big]
\end{aligned}
$$

$$+\frac{z}{\alpha}S_x+\frac{S}{\alpha}z_x-\frac{2a_1R}{\alpha}\big(v+\frac{R-S}{c_2-c_1}w\big)+\frac{2a_2S}{\alpha}\big(v+\frac{R-S}{c_2-c_1}z\big). \tag{4.21}$$

Consequently, in accordance with (4.18), (4.20) and (4.21), we have

$$\begin{aligned}
&\Big[\big(v+\frac{Rw-Sz}{c_2-c_1}\big)(1+R^2)\Big]_t+\Big[\frac{c_1}{\alpha}\big(v+\frac{Rw-Sz}{c_2-c_1}\big)(1+R^2)\Big]_x\\
=&\Big[(v_t+\frac{c_1}{\alpha}v_x)+\big(\frac{Rw-Sz}{c_2-c_1}\big)_t+\frac{c_1}{\alpha}\big(\frac{Rw-Sz}{c_2-c_1}\big)_x\Big](1+R^2)\\
&+\big(v+\frac{Rw-Sz}{c_2-c_1}\big)\big[(1+R^2)_t+\big(\frac{c_1}{\alpha}(1+R^2)\big)_x\big]\\
=&\frac{1+R^2}{\alpha}\Big[s+zS_x-\frac{a_1(w-z)}{c_2-c_1}R^2+\frac{a_1+a_2}{c_2-c_1}(w-z)RS+\frac{c_1b(w-z)}{c_2-c_1}R\Big]\\
&+\frac{w(1+R^2)}{\alpha(c_2-c_1)}\big[a_2S^2+c_2bS\big]-\frac{z(1+R^2)}{\alpha(c_2-c_1)}\big[a_2S^2+(d_2-\partial_xc_1)S\big]\\
&+\frac{1+R^2}{\alpha}(Sz_x-\frac{2a_2(w-z)}{c_2-c_1}RS)+\big(v+\frac{Rw-Sz}{c_2-c_1}\big)\Big[\frac{2a_2}{\alpha}(RS^2-R^2S)+\frac{2c_2b}{\alpha}RS\\
&-\frac{c_2\partial_xc_1-c_1\partial_xc_2}{\alpha(c_2-c_1)}R^2+\frac{\partial_uc_1-\partial_uc_2}{\alpha(c_2-c_1)}(1+R^2)S+\frac{\alpha\partial_xc_1-c_1\partial_x\alpha}{\alpha^2}-\frac{2a_1}{\alpha}(R-S)\Big].
\end{aligned}$$

This together with (4.7) and the similar estimate for the other terms of I_2 gives that

$$\begin{aligned}
\frac{d}{dt}I_2\le\ & C\int_{\mathbb{R}}\Big((1+|S|)J_2^-\mathcal{W}^-+(1+|R|)J_2^+\mathcal{W}^+\Big)\,dx\\
&+C\sum_{i=1,3,5}\int_{\mathbb{R}}\Big((1+S^2)J_i^-\mathcal{W}^-+(1+R^2)J_i^+\mathcal{W}^+\Big)\,dx\\
&+\int_{\mathbb{R}}\Big(G_1(t)J_2^-\mathcal{W}^-+G_2(t)J_2^+\mathcal{W}^+\Big)\,dx-\frac{2\gamma_1}{\alpha_2}\int_{\mathbb{R}}\Big(S^2J_2^-\mathcal{W}^-+R^2J_2^+\mathcal{W}^+\Big)\,dx.
\end{aligned} \tag{4.22}$$

3). We now turn to the time derivative of I_3, which is much more delicate than the other terms. Differentiating $(2.2)_1$ with respect to x, we have

$$\begin{aligned}
R_{tx}+\big(\frac{c_1}{\alpha}R_x\big)_x=&\frac{2a_1}{\alpha}RR_x-\frac{a_1+a_2}{\alpha}(RS_x+SR_x)+\frac{2a_2}{\alpha}SS_x+\frac{c_2b}{\alpha}S_x-\frac{d_1}{\alpha}R_x\\
&+\frac{B_1R^3-B_2S^3}{c_2-c_1}+\frac{B_1+2B_2}{c_2-c_1}RS^2-\frac{2B_1+B_2}{c_2-c_1}R^2S+(A_1-\frac{B_5}{c_2-c_1})R^2\\
&+(A_2-\frac{B_4}{c_2-c_1})S^2-(A_1+A_2-\frac{B_4+B_5}{c_2-c_1})RS-A_4R+A_3S,
\end{aligned} \tag{4.23}$$

where we have used the notations in (4.5) and

$$A_i = \frac{\alpha\partial_x a_i - a_i\partial_x\alpha}{\alpha^2}, i = 1, 2, \quad A_3 = \frac{\alpha\partial_x(c_2 b) - c_2 b\partial_x\alpha}{\alpha^2},$$

and

$$A_4 = \frac{\alpha\partial_x d_1 - d_1\partial_x\alpha}{\alpha^2}, \quad A_5 = \frac{\alpha\partial_x d_2 - d_2\partial_x\alpha}{\alpha^2}.$$

Then it follows from (4.5), (4.6) and (4.23) that

$$\begin{aligned}
&\big[r + wR_x\big]_t + \big[\frac{c_1}{\alpha}(r + wR_x)\big]_x \\
&= (r_t + \frac{c_1}{\alpha}r_x) + \big(\frac{c_1}{\alpha}\big)_x r + R_x(w_t + \frac{c_1}{\alpha}w_x) + w[R_{tx} + \big(\frac{c_1}{\alpha}R_x\big)_x] \\
&= (v + \frac{R-S}{c_2 - c_1}w)\big(B_1R^2 - (B_1 + B_2)RS + B_2S^2 + B_3S - B_4R\big) - \frac{a_1 + a_2}{\alpha}R(s + S_x w) \\
&\quad + \frac{2a_2}{\alpha}S(s + S_x w) + \frac{c_2 b}{\alpha}(s + S_x w) + \frac{a_1 - a_2}{\alpha}S(r + R_x w) - \frac{d_1}{\alpha}(r + R_x w) \\
&\quad + \frac{\alpha\partial_x c_1 - c_1\partial_x\alpha}{\alpha^2}(r + R_x w) + w\big[A_1R^2 - (A_1 + A_2)RS + A_2S^2 + A_3S - A_4R\big].
\end{aligned} \tag{4.24}$$

Furthermore, for the third term of J_3^-, we derive from (2.2) (2.3) and (4.6) that

$$\begin{aligned}
&\Big[\frac{a_2(w-z)}{c_2 - c_1}S^2\Big]_t + \Big[\frac{c_1}{\alpha}\frac{a_2(w-z)}{c_2 - c_1}S^2\Big]_x \\
&= \frac{a_2S^2}{c_2 - c_1}(w_t + \frac{c_1}{\alpha}w_x) - \frac{a_2S^2}{c_2 - c_1}(z_t + \frac{c_2}{\alpha}z_x) + \frac{a_2S^2}{\alpha}z_x + \frac{a_2(w-z)}{c_2 - c_1}\big[(S^2)_t + \big(\frac{c_2}{\alpha}S^2\big)_x\big] \\
&\quad - \frac{a_2(w-z)}{c_2 - c_1}\big[\big(\frac{c_2}{\alpha}S^2\big)_x - \big(\frac{c_1}{\alpha}S^2\big)_x\big] + (w-z)S^2\big[(\frac{a_2}{c_2 - c_1})_t + \frac{c_1}{\alpha}(\frac{a_2}{c_2 - c_1})_x\big] \\
&= \frac{2a_2(a_2 - a_1)S^2}{\alpha}\big(v + \frac{Rw - Sz}{c_2 - c_1}\big) + \frac{w-z}{\alpha(c_2 - c_1)}(2a_1a_2 + \alpha B_2)S^3 - \frac{2a_1a_2(w-z)}{\alpha(c_2 - c_1)}R^2S \\
&\quad + \frac{a_2(\alpha\partial_x c_1 - c_1\partial_x\alpha)}{\alpha^2(c_2 - c_1)}(w-z)S^2 + [\frac{\partial_x c_1 - \partial_x c_2}{\alpha(c_2 - c_1)} + \frac{\partial_x\alpha}{\alpha^2}]a_2S^2w + \frac{a_2}{\alpha}S^2z_x \\
&\quad + \frac{c_1\partial_x a_2 - a_2\partial_x c_1}{\alpha(c_2 - c_1)}(w-z)S^2 + \frac{2a_2c_1b(w-z)}{\alpha(c_2 - c_1)}RS - \frac{2a_2}{\alpha}SS_x(w-z).
\end{aligned} \tag{4.25}$$

The other terms of J_3^- can be estimated similarly. On the one hand, using (2.2) and (4.6) we have

$$\begin{aligned}
&\Big[-\frac{a_1 + a_2}{c_2 - c_1}(w-z)RS\Big]_t + \Big[-\frac{c_1}{\alpha}\frac{a_1 + a_2}{c_2 - c_1}(w-z)RS\Big]_x \\
&= -\frac{a_1 + a_2}{c_2 - c_1}RS(w_t + \frac{c_1}{\alpha}w_x) + \frac{a_1 + a_2}{c_2 - c_1}RS(z_t + \frac{c_2}{\alpha}z_x) - \frac{a_1 + a_2}{\alpha}RSz_x \\
&\quad - \frac{a_1 + a_2}{c_2 - c_1}(w-z)S\big[R_t + \frac{c_1}{\alpha}R_x\big] - \frac{a_1 + a_2}{c_2 - c_1}(w-z)R\big[S_t + \frac{c_2}{\alpha}S_x\big]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{a_1+a_2}{\alpha}(w-z)RS_x-(w-z)RS\Big[\big(\frac{a_1+a_2}{c_2-c_1}\big)_t+\big(\frac{c_1}{\alpha}\frac{a_1+a_2}{c_2-c_1}\big)_x\Big]\\
=&\frac{2(a_1^2-a_2^2)}{\alpha}RS\big(v+\frac{Rw-Sz}{c_2-c_1}\big)+\frac{w-z}{\alpha(c_2-c_1)}(2a_2^2-3a_1^2-a_1a_2)RS^2-\frac{a_1+a_2}{\alpha}RSz_x\\
&+\frac{w-z}{\alpha(c_2-c_1)}(2a_2^2+a_1^2+3a_1a_2)R^2S+\frac{a_1(a_1+a_2)}{\alpha(c_2-c_1)}(w-z)R^3-\frac{a_2(a_1+a_2)}{\alpha(c_2-c_1)}(w-z)S^3\\
&+\frac{(a_1+a_2)RS}{\alpha^2(c_2-c_1)}\Big[2(c_1\partial_x\alpha-\alpha\partial_x c_1)w+\big(\alpha(\partial_x c_1+\partial_x c_2)-(c_2+c_1)\partial_x\alpha\big)z\Big]\\
&-\frac{a_1+a_2}{\alpha(c_2-c_1)}(w-z)\big[(a_1+a_2)R^2S+c_1bR^2+c_2bS^2-(d_1+d_2)RS\big]\\
&-\frac{c_1(w-z)}{\alpha(c_2-c_1)^2}\big[(c_2-c_1)\partial_x(a_1+a_2)-(a_1+a_2)\partial_x(c_2-c_1)\big]RS+\frac{a_1+a_2}{\alpha}RS_x(w-z)\\
&-\frac{w-z}{\alpha(c_2-c_1)^2}\big[(c_2-c_1)\partial_u(a_1+a_2)-(a_1+a_2)\partial_u(c_2-c_1)\big]RS^2. \qquad (4.26)
\end{aligned}
$$

On the other hand, by (2.2) and (4.6) we obtain

$$
\begin{aligned}
&\Big[\frac{c_2b(w-z)}{c_2-c_1}S\Big]_t+\Big[\frac{c_1}{\alpha}\frac{c_2b(w-z)}{c_2-c_1}S\Big]_x\\
=&\frac{c_2bS}{c_2-c_1}(w_t+\frac{c_1}{\alpha}w_x)-\frac{c_2bS}{c_2-c_1}(z_t+\frac{c_2}{\alpha}z_x)+\frac{c_2bS}{\alpha}z_x+\frac{c_2b(w-z)}{c_2-c_1}\big(S_t+\frac{c_2}{\alpha}S_x\big)\\
&-\frac{c_2bS_x(w-z)}{\alpha}+(w-z)S\Big[\big(\frac{c_2b}{c_2-c_1}\big)_t+\big(\frac{c_1}{\alpha}\frac{c_2b}{c_2-c_1}\big)_x\Big]\\
=&\frac{2c_2b(a_2-a_1)S}{\alpha}\big(v+\frac{Rw-Sz}{c_2-c_1}\big)+\frac{c_2b(w-z)}{\alpha(c_2-c_1)}(4a_1-a_2)S^2+\frac{c_2b}{\alpha}Sz_x\\
&+\frac{(c_2-c_1)\partial_u(c_2b)-c_2b\partial_u(c_2-c_1)}{\alpha(c_2-c_1)^2}(w-z)S^2-\frac{2c_2b(a_1+a_2)(w-z)}{\alpha(c_2-c_1)}RS\\
&-\frac{a_1c_2b(w-z)}{\alpha(c_2-c_1)}R^2+\frac{c_2b(w-z)}{\alpha(c_2-c_1)}[(a_1+a_2)RS+c_1bR-d_2S]\\
&+\frac{c_2bS}{\alpha^2(c_2-c_1)}\Big[2(\alpha\partial_x c_1-c_1\partial_x\alpha)w-\big(\alpha\partial_x(c_1+c_2)-(c_1+c_2)\partial_x\alpha\big)z\Big]\\
&+\frac{c_1(w-z)S}{\alpha(c_2-c_1)^2}\big[(c_2-c_1)\partial_x(c_2b)-c_2b\partial_x(c_2-c_1)\big]-\frac{c_2b}{\alpha}S_x(w-z). \qquad (4.27)
\end{aligned}
$$

In view of (4.10), combining (4.24)–(4.27) yields

$$
\begin{aligned}
&\hat{r}_t+\big(\frac{c_1}{\alpha}\hat{r}\big)_x\\
=&(v+\frac{Rw-SZ}{c_2-c_1})\Big(B_1R^2-(B_1+B_2)RS+B_2S^2+B_3S-B_4R-\frac{2a_2(a_1-a_2)}{\alpha}S^2\Big)\\
&+(v+\frac{Rw-Sz}{c_2-c_1})\Big(\frac{2(a_1^2-a_2^2)}{\alpha}RS-\frac{2(a_1-a_2)c_2b}{\alpha}S\Big)+\frac{c_2b-(a_1+a_2)R}{\alpha}\hat{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(\frac{(a_1-a_2)S-d_1}{\alpha}+\frac{\alpha\partial_x c_1-c_1\partial_x\alpha}{\alpha^2}\Big)\hat{r}+\frac{a_2}{\alpha}(2S\hat{s}+S^2z_x-\frac{2a_2(w-z)}{c_2-c_1}RS^2)\\
&+\frac{c_2b-(a_1+a_2)R}{\alpha}\big(Sz_x-\frac{2a_2(w-z)}{c_2-c_1}RS\big)+\frac{R^2S(w-z)}{\alpha(c_2-c_1)}[a_1^2+a_1a_2-\alpha B_1]\\
&+A_1wR^2-\frac{RS^2(w-z)}{\alpha(c_2-c_1)}[a_2^2+3a_1a_2]+\frac{(a_1+a_2)RS}{\alpha(c_2-c_1)}[w\partial_x c_2-z\partial_x c_1-d_2(w-z)]\\
&+\frac{RS}{\alpha(c_2-c_1)}\big[(c_1z-c_2w)\partial_x(a_1+a_2)+(\alpha B_4-2a_1c_2b)(w-z)\big]-A_4wR\\
&+\frac{S}{\alpha(c_2-c_1)}\big[(c_2w-c_1z)\partial_x(c_2b)+(d_1+d_2)c_2b(w-z)+c_2b(z\partial_x c_1-w\partial_x c_2)\big]\\
&+\frac{S^2}{\alpha(c_2-c_1)}\big[(c_2w-c_1z)\partial_x a_2+a_2(d_1+c_2b)(w-z)+a_2(z\partial_x c_1-w\partial_x c_2)\big].
\end{aligned}\tag{4.28}
$$

This together with (4.7) and the similar estimate for the other terms of I_3 gives that

$$
\begin{aligned}
\frac{d}{dt}I_3\le\ & C\sum_{k=2,3,5,6}\int_{\mathbb{R}}\Big((1+|S|)J_k^-\mathcal{W}^-+(1+|R|)J_k^+\mathcal{W}^+\Big)\,dx\\
&+C\int_{\mathbb{R}}\Big((1+S^2)J_1^-\mathcal{W}^-+(1+R^2)J_1^+\mathcal{W}^+\Big)\,dx\\
&+\int_{\mathbb{R}}\Big(G_1(t)J_3^-\mathcal{W}^-+G_2(t)J_3^+\mathcal{W}^+\Big)\,dx-\frac{2\gamma_1}{\alpha_2}\int_{\mathbb{R}}\Big(S^2J_3^-\mathcal{W}^-+R^2J_3^+\mathcal{W}^+\Big)\,dx.
\end{aligned}\tag{4.29}
$$

4). Consider I_4, we first differentiate $(4.6)_1$ with respect to x to get

$$
\begin{aligned}
w_{tx}+\big(\frac{c_1}{\alpha}w_x\big)_x=&-\frac{2a_1}{\alpha}(r+R_xw)+\frac{2a_1}{\alpha}(s+S_xw)-\frac{2a_1}{\alpha}(R-S)w_x\\
&-\frac{2a_1}{\alpha}(\partial_x c_2-\partial_x c_1)v-2(c_2-c_1)(A_1+B_1\frac{R-S}{c_2-c_1})(v+\frac{R-S}{c_2-c_1}w)\\
&+\frac{\alpha\partial_x c_1-c_1\partial_x\alpha}{\alpha^2}w_x+\big(\frac{\alpha\partial_x c_1-c_1\partial_x\alpha}{\alpha^2}\big)_xw.
\end{aligned}\tag{4.30}
$$

With this help, utilizing the estimates (2.2), (4.6) and (4.30), we can derive

$$
\begin{aligned}
&\Big[w_x+\frac{2a_1(w-z)}{c_2-c_1}S\Big]_t+\Big[\frac{c_1}{\alpha}\big(w_x+\frac{2a_1(w-z)}{c_2-c_1}S\big)\Big]_x\\
=\ &w_{xt}+\big(\frac{c_1}{\alpha}w_x\big)_x+\frac{2a_1S}{c_2-c_1}(w_t+\frac{c_1}{\alpha}w_x)-\frac{2a_1S}{c_2-c_1}(z_t+\frac{c_2}{\alpha}z_x)+\frac{2a_1S}{\alpha}z_x\\
&+\frac{2a_1(w-z)}{c_2-c_1}\big(S_t+\frac{c_2}{\alpha}S_x\big)-\frac{2a_1(w-z)}{\alpha}S_x+2(w-z)S\big[(\frac{a_1}{c_2-c_1})_t+(\frac{c_1}{\alpha}\frac{a_1}{c_2-c_1})_x\big]\\
=&-\frac{2a_1}{\alpha}\hat{r}+\frac{2a_1}{\alpha}\hat{s}-\frac{2a_1}{\alpha}\big(Rw_x+\frac{2a_1(w-z)}{c_2-c_1}RS\big)+\frac{2a_1}{\alpha}\big(Sz_x-\frac{2a_2(w-z)}{c_2-c_1}RS\big)
\end{aligned}
$$

$$
\begin{aligned}
&-2\Big[A_1(c_2-c_1)+B_1(R-S)+\frac{2a_1S}{\alpha}(a_1-a_2)+\frac{a_1(\partial_x c_2-\partial_x c_1)}{\alpha}\Big]\big(v+\frac{Rw-Sz}{c_2-c_1}\big)\\
&+\big(\frac{2a_1S}{\alpha}+\frac{\alpha\partial_x c_1-c_1\partial_x\alpha}{\alpha^2}\big)\big(w_x+\frac{2a_1(w-z)}{c_2-c_1}S\big)+\big(\frac{\alpha\partial_x c_1-c_1\partial_x\alpha}{\alpha^2}\big)_x w\\
&+\frac{2a_1(\partial_x c_2-\partial_x c_1)}{\alpha(c_2-c_1)}Rw+\frac{2a_1Sz}{\alpha(c_2-c_1)}[2\partial_x(c_1-c_2)-\frac{c_1-c_2}{\alpha}\partial_x\alpha]\\
&+\frac{2(w-z)S}{\alpha(c_2-c_1)}\big[2a_1a_2S-a_1(a_1+a_2)R+a_1(d_2+c_2b-\partial_x c_2)+\alpha B_1R+c_2\partial_x a_1\big].
\end{aligned}
\tag{4.31}
$$

This together with (4.7) and the similar estimate for the other terms of I_4 gives that

$$
\begin{aligned}
\frac{d}{dt}I_4\le& C\sum_{k=2,3,4,5}\int_{\mathbb{R}}\Big((1+|S|)J_k^-\mathcal{W}^-+(1+|R|)J_k^+\mathcal{W}^+\Big)\,dx\\
&+C\int_{\mathbb{R}}\Big((1+S^2)J_1^-\mathcal{W}^-+(1+R^2)J_1^+\mathcal{W}^+\Big)\,dx\\
&+\int_{\mathbb{R}}\Big(G_1(t)J_4^-\mathcal{W}^-+G_2(t)J_4^+\mathcal{W}^+\Big)\,dx-\frac{2\gamma_1}{\alpha_2}\int_{\mathbb{R}}\Big(S^2J_4^-\mathcal{W}^-+R^2J_4^+\mathcal{W}^+\Big)\,dx.
\end{aligned}
\tag{4.32}
$$

5). Next, we deal with the time derivative of I_5. By (2.2) and (4.31), it holds that

$$
\begin{aligned}
&\Big[R\big(w_x+\frac{2a_1(w-z)}{c_2-c_1}S\big)\Big]_t+\Big[\frac{c_1}{\alpha}R\big(w_x+\frac{2a_1(w-z)}{c_2-c_1}S\big)\Big]_x\\
=&(R_t+\frac{c_1}{\alpha}R_x)\big(w_x+\frac{2a_1(w-z)}{c_2-c_1}S\big)\\
&+R\Big[\big(w_x+\frac{2a_1(w-z)}{c_2-c_1}S\big)_t+\big(\frac{c_1}{\alpha}(w_x+\frac{2a_1(w-z)}{c_2-c_1}S)\big)_x\Big]\\
=&-\frac{a_1}{\alpha}[2R\hat{r}+R^2w_x+\frac{2a_1(w-z)}{c_2-c_1}R^2S]+\frac{2a_1}{\alpha}R\hat{s}+\frac{2a_1R}{\alpha}\big(Sz_x-\frac{2a_2(w-z)}{c_2-c_1}RS\big)\\
&+\big(\frac{a_1-a_2}{\alpha}S-\frac{d_1}{\alpha}+\frac{\alpha\partial_x c_1-c_1\partial_x\alpha}{\alpha^2}\big)\big(Rw_x+\frac{2a_1(w-z)}{c_2-c_1}RS\big)\\
&-2R\Big[A_1(c_2-c_1)+B_1(R-S)+\frac{2a_1S}{\alpha}(a_1-a_2)+\frac{a_1(\partial_x c_2-\partial_x c_1)}{\alpha}\Big]\big(v+\frac{Rw-Sz}{c_2-c_1}\big)\\
&+\frac{S}{\alpha}(a_2S+c_2b)\big(w_x+\frac{2a_1(w-z)}{c_2-c_1}S\big)+\big(\frac{\alpha\partial_x c_1-c_1\partial_x\alpha}{\alpha^2}\big)_x Rw\\
&+\frac{2a_1(\partial_x c_2-\partial_x c_1)}{\alpha(c_2-c_1)}R^2w+\frac{2a_1RSz}{\alpha(c_2-c_1)}[2\partial_x(c_1-c_2)-\frac{c_1-c_2}{\alpha}\partial_x\alpha]\\
&+\frac{2(w-z)RS}{\alpha(c_2-c_1)}\big[2a_1a_2S-a_1(a_1+a_2)R+a_1(d_2+c_2b-\partial_x c_2)+\alpha B_1R+c_2\partial_x a_1\big].
\end{aligned}
$$

This together with (4.7) and the similar estimate for the other terms of I_5 gives that

$$\begin{aligned}
\frac{d}{dt}I_5 \leq & C\sum_{k=2,3,5,6}\int_{\mathbb{R}}\Big((1+|S|)J_k^-\mathcal{W}^- + (1+|R|)J_k^+\mathcal{W}^+\Big)\,dx \\
& + C\sum_{i=1,4}\int_{\mathbb{R}}\Big((1+S^2)J_i^-\mathcal{W}^- + (1+R^2)J_i^+\mathcal{W}^+\Big)\,dx \\
& + \int_{\mathbb{R}}\Big(G_1(t)J_5^-\mathcal{W}^- + G_2(t)J_5^+\mathcal{W}^+\Big)\,dx - \frac{2\gamma_1}{\alpha_2}\int_{\mathbb{R}}\Big(S^2J_5^-\mathcal{W}^- + R^2J_5^+\mathcal{W}^+\Big)\,dx.
\end{aligned} \tag{4.33}$$

6). Finally, we repeat the same procedure on I_6. Using (2.2), (2.3), (4.28) and (4.30), we achieve

$$\begin{aligned}
& \big[2R\hat{r} + R^2w_x\big]_t + \big[\frac{c_1}{\alpha}(2R\hat{r} + R^2w_x)\big]_x \\
= & 2R\Big(\hat{r}_t + (\frac{c_1}{\alpha}\hat{r})_x\Big) + 2\hat{r}(R_t + \frac{c_1}{\alpha}R_x) + R^2[w_{xt} + (\frac{c_1}{\alpha}w_x)_x] + w_x[(R^2)_t + \frac{c_1}{\alpha}\big(R^2\big)_x] \\
= & 2(v + \frac{Rw - Sz}{c_2 - c_1})\Big[B_2RS^2 - B_2R^2S + B_3RS - B_4R^2 - A_1(c_2 - c_1)R^2 + \frac{2(a_1^2 - a_2^2)}{\alpha}R^2S \\
& - \frac{2a_2(a_1 - a_2)}{\alpha}RS^2 - \frac{a_1(\partial_x c_2 - \partial_x c_1)}{\alpha}R^2 + \frac{2c_2b(a_2 - a_1)}{\alpha}RS\Big] + \frac{2\hat{r}}{\alpha}(a_2S^2 + c_2bS) \\
& - \frac{2\hat{s}}{\alpha}(a_2R^2 - c_2bR) + (\frac{c_1\partial_x c_2 - c_2\partial_x c_1}{\alpha(c_2 - c_1)} - \frac{2a_2S}{\alpha})\big(2R\hat{r} + R^2w_x + \frac{2a_1(w - z)}{c_2 - c_1}R^2S\big) \\
& + \frac{2a_2R}{\alpha}\big(2S\hat{s} + S^2z_x - \frac{2a_2(w - z)}{c_2 - c_1}RS^2\big) + \frac{2c_2bS + 2a_2S^2}{\alpha}\big(Rw_x + \frac{2a_1(w - z)}{c_2 - c_1}RS\big) \\
& + \frac{2c_2bR - 2(a_1 + a_2)R^2}{\alpha}\big(Sz_x - \frac{2a_2(w - z)}{c_2 - c_1}RS\big) + \frac{2a_1(w - z)}{\alpha}R^2S_x + \frac{2a_1^2(w - z)}{\alpha(c_2 - c_1)}R^4 \\
& + \frac{2a_1(\partial_x c_2 - \partial_x c_1)}{\alpha(c_2 - c_1)}R^3w - \frac{2a_1c_1b(w - z)}{\alpha(c_2 - c_1)}R^3 + 2A_1wR^3 - 2A_4wR^2 - \frac{4a_1a_2(w - z)}{\alpha(c_2 - c_1)}RS^3 \\
& + \frac{2RS}{\alpha(c_2 - c_1)}\big[(c_2w - c_1z)\partial_x(c_2b) + c_2b(d_1 + d_2)(w - z) + c_2b(z\partial_x c_1 - w\partial_x c_2)\big] \\
& + \frac{2RS^2}{\alpha(c_2 - c_1)}\big[(c_2w - c_1z)\partial_x a_2 + (a_2(d_1 + c_2b) - 2a_1c_2b)(w - z) + a_2(z\partial_x c_1 - w\partial_x c_2)\big] \\
& + \frac{2R^2S}{\alpha(c_2 - c_1)}\Big[(c_1z - c_2w)\partial_x(a_1 + a_2) - a_1(\partial_x c_2 - \partial_x c_1)z + \alpha(B_4 + A_1(c_2 - c_1))(w - z) \\
& - a_1(c_2b + \frac{c_1\partial_x c_2 - c_2\partial_x c_1}{c_2 - c_1})(w - z)\Big] + \frac{2(a_1 + a_2)}{\alpha(c_2 - c_1)}R^2S[w\partial_x c_2 - z\partial_x c_1 - d_2(w - z)] \\
& - \frac{2(w - z)}{\alpha(c_2 - c_1)}R^2S^2[\alpha B_1 + a_2^2] - \frac{2(w - z)}{\alpha(c_2 - c_1)}R^3S(a_1^2 + a_1a_2) + (\frac{\alpha\partial_x c_1 - c_1\partial_x\alpha}{\alpha^2})_xR^2w.
\end{aligned} \tag{4.34}$$

Moreover, it follows from (2.2), (2.3) and (4.6) that

$$
\begin{aligned}
&\Big[\frac{2a_1(w-z)}{c_2-c_1}R^2S\Big]_t+\Big[\frac{c_1}{\alpha}\frac{2a_1(w-z)}{c_2-c_1}R^2S\Big]_x\\
=&\frac{2a_1R^2S}{c_2-c_1}(w_t+\frac{c_1}{\alpha}w_x)-\frac{2a_1R^2S}{c_2-c_1}(z_t+\frac{c_2}{\alpha}z_x)+\frac{2a_1R^2S}{\alpha}z_x-\frac{2a_1(w-z)}{\alpha}R^2S_x\\
&+\frac{2a_1R^2(w-z)}{c_2-c_1}\big(S_t+\frac{c_2}{\alpha}S_x\big)+\frac{2a_1S(w-z)}{c_2-c_1}\big[(R^2)_t+(\frac{c_1}{\alpha}R^2)_x\big]\\
&+2(w-z)R^2S\big[(\frac{a_1}{c_2-c_1})_t+\frac{c_1}{\alpha}(\frac{a_1}{c_2-c_1})_x\big]\\
=&\frac{4a_1(a_2-a_1)R^2S}{\alpha}\Big(v+\frac{Rw-Sz}{c_2-c_1}\Big)+\frac{2a_1R^2}{\alpha}\Big(Sz_x-\frac{2a_2(w-z)}{c_2-c_1}RS\Big)-\frac{2a_1(w-z)}{\alpha}R^2S_x\\
&+\frac{2R^2S^2(w-z)}{\alpha(c_2-c_1)}(\alpha B_1-a_1a_2)+\frac{2a_1R^2S}{\alpha(c_2-c_1)}\Big[\frac{\alpha\partial_xc_1-c_1\partial_x\alpha}{\alpha}w-\frac{\alpha\partial_xc_2-c_2\partial_x\alpha}{\alpha}z\Big]\\
&-\frac{2R^2S(w-z)}{\alpha(c_2-c_1)}[a_1d_2+a_1\partial_xc_1-c_1\partial_xa_1]+\frac{4a_1a_2(w-z)}{\alpha(c_2-c_1)}RS^3+\frac{4a_1c_2b(w-z)}{\alpha(c_2-c_1)}RS^2\\
&+\frac{2a_1(a_1+a_2)(w-z)}{\alpha(c_2-c_1)}R^3S+\frac{2a_1c_1b(w-z)}{\alpha(c_2-c_1)}R^3-\frac{2a_1^2(w-z)}{\alpha(c_2-c_1)}R^4.
\end{aligned}
\tag{4.35}
$$

Combining (4.34) and (4.35), we obtain

$$
\begin{aligned}
&\big[2R\hat r+R^2w_x+\frac{2a_1(w-z)}{c_2-c_1}R^2S\big]_t+\big[\frac{c_1}{\alpha}\big(2R\hat r+R^2w_x+\frac{2a_1(w-z)}{c_2-c_1}R^2S\big)\big]_x\\
=&2(v+\frac{Rw-Sz}{c_2-c_1})\Big[B_2(RS^2-R^2S)+B_3RS-B_4R^2-A_1(c_2-c_1)R^2-\frac{2(a_1-a_2)^2}{\alpha}R^2S\\
&-\frac{2a_2(a_1-a_2)}{\alpha}RS^2-\frac{a_1(\partial_xc_2-\partial_xc_1)}{\alpha}R^2+\frac{2c_2b(a_2-a_1)}{\alpha}RS\Big]+\frac{2\hat r}{\alpha}(a_2S^2+c_2bS)\\
&-\frac{2\hat s}{\alpha}(a_2R^2-c_2bR)+\Big(\frac{c_1\partial_xc_2-c_2\partial_xc_1}{\alpha(c_2-c_1)}-\frac{2a_2S}{\alpha}\Big)\Big(2R\hat r+R^2w_x+\frac{2a_1(w-z)}{c_2-c_1}R^2S\Big)\\
&+\frac{2a_2R}{\alpha}\Big(2S\hat s+S^2z_x-\frac{2a_2(w-z)}{c_2-c_1}RS^2\Big)+\frac{2c_2bS+2a_2S^2}{\alpha}\Big(Rw_x+\frac{2a_1(w-z)}{c_2-c_1}RS\Big)\\
&+\frac{2c_2bR-2a_2R^2}{\alpha}\Big(Sz_x-\frac{2a_2(w-z)}{c_2-c_1}RS\Big)-\frac{2(w-z)}{\alpha(c_2-c_1)}R^2S^2[a_1a_2+a_2^2]\\
&+\frac{2R^2S}{\alpha(c_2-c_1)}\Big[(c_1-c_2)z\partial_xa_1+(c_1z-c_2w)\partial_xa_2+(\alpha B_4-a_1c_2b-a_2d_2)(w-z)\Big]\\
&+\frac{2a_1R^2S}{\alpha(c_2-c_1)}\Big[(\partial_xc_1-\partial_xc_2)z+\frac{(c_2-c_1)\partial_x\alpha}{\alpha}z\Big]+\frac{2a_2R^2S}{\alpha(c_2-c_1)}[w\partial_xc_2-z\partial_xc_1]\\
&+\frac{2RS^2}{\alpha(c_2-c_1)}\big[(c_2w-c_1z)\partial_xa_2+a_2(d_1+c_2b)(w-z)+a_2(z\partial_xc_1-w\partial_xc_2)\big]\\
&+\frac{2RS}{\alpha(c_2-c_1)}\big[(c_2w-c_1z)\partial_x(c_2b)+c_2b(d_1+d_2)(w-z)+c_2b(z\partial_xc_1-w\partial_xc_2)\big]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\alpha\partial_u(\alpha\partial_x c_1 - c_1\partial_x\alpha) - 2(\alpha\partial_x c_1 - c_1\partial_x\alpha)\partial_u\alpha}{\alpha^3}R^2 Sw - 2A_4R^2 w\\
&+\frac{\alpha\partial_x(\alpha\partial_x c_1 - c_1\partial_x\alpha) - 2(\alpha\partial_x c_1 - c_1\partial_x\alpha)\partial_x\alpha}{\alpha^3}R^2 w.
\end{aligned}
$$

Together with (4.7) and the similar estimate for the other terms of I_6, we have

$$
\begin{aligned}
\frac{d}{dt}I_6 \le & C\sum_{k=2,6}\int_{\mathbb{R}}\Big((1+|S|)J_k^-\mathcal{W}^- + (1+|R|)J_k^+\mathcal{W}^+\Big)\,dx\\
&+C\sum_{i=1,3,5}\int_{\mathbb{R}}\Big((1+S^2)J_i^-\mathcal{W}^- + (1+R^2)J_i^+\mathcal{W}^+\Big)\,dx\\
&+\int_{\mathbb{R}}\Big(G_1(t)J_6^-\mathcal{W}^- + G_2(t)J_6^+\mathcal{W}^+\Big)\,dx - \frac{2\gamma_1}{\alpha_2}\int_{\mathbb{R}}\Big(S^2J_6^-\mathcal{W}^- + R^2J_6^+\mathcal{W}^+\Big)\,dx.
\end{aligned}
\tag{4.36}
$$

Combining the estimates in (4.17), (4.19), (4.22), (4.29), (4.32), (4.33) and (4.36), and using (4.8), we have

$$
\begin{aligned}
\frac{dI_k}{dt} \le\ & C\sum_{\ell\in\mathcal{F}_k^l}\left(\int_{\mathbb{R}}(1+|S|)\,J_\ell^-\,\mathcal{W}^-\,dx + \int_{\mathbb{R}}(1+|R|)J_\ell^+\,\mathcal{W}^+\,dx\right)\\
&+C\sum_{\ell\in\mathcal{F}_k^h}\left(\int_{\mathbb{R}}(1+S^2)\,J_\ell^-\,\mathcal{W}^-\,dx + \int_{\mathbb{R}}(1+R^2)\,J_\ell^+\,\mathcal{W}^+\,dx\right)\\
&+\int_{\mathbb{R}}\Big(G_1(t)J_k^-\mathcal{W}^- + G_2(t)J_k^+\mathcal{W}^+\Big)\,dx - \frac{\gamma_1}{\alpha_2}\left(\int_{\mathbb{R}}S^2\,J_k^-\,\mathcal{W}^-\,dx + \int_{\mathbb{R}}R^2\,J_k^+\,\mathcal{W}^+\,dx\right).
\end{aligned}
$$

Here $\mathcal{F}_k^l, \mathcal{F}_k^h \subset \{0,1,2,\cdots,6\}$ are suitable sets of indices from the estimates (4.17), (4.19), (4.22), (4.29), (4.32), (4.33) and (4.36), where a graphical summary of $\mathcal{F}_k^h$ is illustrated in Fig. 3. For example, by (4.36), $\mathcal{F}_6^l = \{2,6\}$ and $\mathcal{F}_6^h = \{1,3,5\}$.

Since there is no cycle for the relation tree $\mathcal{F}_k^h$, we can choose a suitable small constant $\delta > 0$, with the weighted norm defined by

$$
\|(w,\hat{r},z,\hat{s})\|_{(u,R,S)} := I_0 + \delta I_1 + \delta^4 I_2 + \delta^2 I_3 + \delta^2 I_4 + \delta^3 I_5 + \delta^4 I_6,
$$

such that the desired estimate (4.16) holds. This completes the proof of Lemma 4.1. □

5. Generic regularity of conservative solutions

Aim of this section is to study generic singularities to (1.4)–(1.5) and thus prove Theorem 1.2 by an application of the Thom's Transversality theorem. Furthermore, for any two generic solutions, we show that there exist a family of regular solutions connecting them.

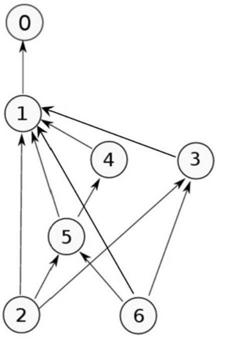

Fig. 3. $k \to \mathcal{F}_k^h \subset \{0, 1, \cdots 6\}$ has **no cycle!** Choose κ_k in a certain order ($\kappa_0 \gg \kappa_1 \gg \kappa_3, \kappa_4 \gg \kappa_5 \gg \kappa_2, \kappa_6$) to prove (4.16).

5.1. The semi-linear system on new coordinates

As a start, we briefly review the semi-linear system introduced in [15], which will be used in both this and next sections. Please find detail calculations and derivations in [15].

Consider the equations for the forward and backward characteristics as follows

$$\begin{cases} \frac{d}{ds}x^{\pm}(s;x,t) = \lambda_{\pm}(x^{\pm}(s;x,t), u(s;x^{\pm}(s;x,t))), \\ x^{\pm}|_{s=t} = x, \end{cases}$$

where $\lambda_{\pm}$ are defined in (1.7). Then introduce a new coordinate transformation $(x,t) \to (X,Y)$ as

$$X := \int_0^{x^-(0;x,t)} [1 + R^2(y,0)]\,dy, \quad \text{and} \quad Y := \int_{x^+(0;x,t)}^{0} [1 + S^2(y,0)]\,dy,$$

which implies that

$$\alpha(x,u)X_t + c_1(x,u)X_x = 0, \quad \alpha(x,u)Y_t + c_2(x,u)Y_x = 0. \tag{5.1}$$

Thus, for any smooth function f, we have

$$\begin{cases} \alpha(x,u)f_t + c_2(x,u)f_x = (\alpha X_t + c_2 X_x)f_X = (c_2 - c_1)X_x f_X, \\ \alpha(x,u)f_t + c_1(x,u)f_x = (\alpha Y_t + c_1 Y_x)f_Y = (c_1 - c_2)Y_x f_Y. \end{cases} \tag{5.2}$$

For convenience to deal with possibly unbounded values of R and S, we introduce a new set of dependent variables

$$\ell := \frac{R}{1+R^2}, \quad h := \frac{1}{1+R^2}, \quad p := \frac{1+R^2}{X_x},$$
$$m := \frac{S}{1+S^2}, \quad g := \frac{1}{1+S^2}, \quad q := \frac{1+S^2}{-Y_x}.$$

Making use of (2.2), (5.1) and the above definitions, one obtains a semi-linear hyperbolic system with smooth coefficients for the variables $\ell, m, h, g, p, q, u, x$ in (X, Y) coordinates, cf. [15].

$$\begin{cases} \ell_Y = \dfrac{q(2h-1)}{c_2-c_1}[a_1 g + a_2 h - (a_1+a_2)(gh+m\ell) + c_2 bhm - d_1 g\ell], \\ m_X = \dfrac{p(2g-1)}{c_2-c_1}[-a_1 g - a_2 h + (a_1+a_2)(gh+m\ell) + c_1 bg\ell - d_2 hm], \end{cases} \tag{5.3}$$

$$\begin{cases} h_Y = -\dfrac{2q\ell}{c_2-c_1}[a_1 g + a_2 h - (a_1+a_2)(gh+m\ell) + c_2 bhm - d_1 g\ell], \\ g_X = -\dfrac{2pm}{c_2-c_1}[-a_1 g - a_2 h + (a_1+a_2)(gh+m\ell) + c_1 bg\ell - d_2 hm], \end{cases} \tag{5.4}$$

$$\begin{cases} p_Y = \dfrac{2pq}{c_2-c_1}[a_2(\ell-m) + (a_1+a_2)(hm-g\ell) + c_2 bm\ell + d_1 gh + \dfrac{c_1\partial_x c_2 - c_2\partial_x c_1}{2(c_2-c_1)} g], \\ q_X = \dfrac{2pq}{c_2-c_1}[a_1(\ell-m) + (a_1+a_2)(hm-g\ell) + c_1 bm\ell + d_2 gh + \dfrac{c_1\partial_x c_2 - c_2\partial_x c_1}{2(c_2-c_1)} h], \end{cases} \tag{5.5}$$

$$\begin{cases} u_X = \dfrac{p\ell}{c_2-c_1}, \quad (\text{or} \quad u_Y = \dfrac{qm}{c_2-c_1}), \\ x_X = \dfrac{c_2}{c_2-c_1} ph, \quad (\text{or} \quad x_Y = \dfrac{c_1}{c_2-c_1} qg). \end{cases} \tag{5.6}$$

Setting $f = t$ in the (5.2), we obtain the equations for t,

$$t_X = \frac{\alpha ph}{c_2-c_1}, \quad t_Y = \frac{\alpha qg}{c_2-c_1}. \tag{5.7}$$

The system (5.3)–(5.7) must now be supplemented by non-characteristic boundary conditions, corresponding to the initial data (1.5). Toward this goal, along the curve

$$\gamma_0 := \{(X, Y);\ X + Y = 0\} \subset \mathbb{R}^2$$

parameterized by $x \mapsto (\bar{X}(x), \bar{Y}(x)) := (x, -x)$, we assign the boundary data $(\bar{u}, \bar{\ell}, \bar{m}, \bar{h}, \bar{g}, \bar{p}, \bar{q})$ by setting

$$\begin{aligned} &\bar{u} = u_0(x), \quad \bar{h} = \frac{1}{1+R^2(x,0)}, \quad \bar{g} = \frac{1}{1+S^2(x,0)}, \\ &\bar{\ell} = R(x,0)\bar{h}, \quad \bar{m} = S(x,0)\bar{g}, \quad \bar{p} = 1 + R^2(x,0), \quad \bar{q} = 1 + S^2(x,0), \end{aligned} \tag{5.8}$$

with

$$\begin{aligned} R(x,0) &= \alpha\big(x, u_0(x)\big)u_1(x) + c_2\big(x, u_0(x)\big)u_{x,0}(x), \\ S(x,0) &= \alpha\big(x, u_0(x)\big)u_1(x) + c_1\big(x, u_0(x)\big)u_{x,0}(x). \end{aligned}$$

Obviously, the coordinate transformation $\mathcal{F} : (X, Y) \mapsto (x, t)$ maps the point $(x, -x) \in \gamma_0$ to the point $(x, 0)$, for every $x \in \mathbb{R}$.

For future reference, we state the following result of the above construction in [9,15].

Lemma 5.1. *Let $(u, \ell, m, h, g, p, q, x, t)$ be a smooth solution to the system (5.3)–(5.8) with $p, q > 0$. Then the function $u = u(x, t)$ whose graph is*

$$\{(x(X, Y), t(X, Y), u(X, Y));\ (X, Y) \in \mathbb{R}^2\} \tag{5.9}$$

provide the unique conservative solution to the variational wave equation (1.4)–(1.6).

As a preliminary, we examine the boundary data should satisfy some compatibility conditions. Instead of (5.8), we can assign a more general boundary data for (5.3)–(5.7), along a line $\gamma_\kappa = \{(X, Y);\ X + Y = \kappa\}$, say

$$u(s, \kappa - s) = \bar{u}(s), \quad \begin{cases} \ell(s, \kappa - s) = \bar{\ell}(s), \\ m(s, \kappa - s) = \bar{m}(s), \end{cases} \quad \begin{cases} h(s, \kappa - s) = \bar{h}(s), \\ g(s, \kappa - s) = \bar{g}(s), \end{cases} \quad \begin{cases} p(s, \kappa - s) = \bar{p}(s), \\ q(s, \kappa - s) = \bar{q}(s), \end{cases} \tag{5.10}$$

and

$$x(s, \kappa - s) = \bar{x}(s), \quad t(s, \kappa - s) = \bar{t}(s). \tag{5.11}$$

If both equations in $(5.6)_1$ hold, then the boundary data should satisfy the compatibility condition

$$\begin{aligned} \frac{d}{ds}\bar{u}(s) &= \frac{d}{ds}u(s, \kappa - s) = (u_X - u_Y)(s, \kappa - s) \\ &= \frac{\bar{p}(s)\bar{\ell}(s)}{\bar{c}_2 - \bar{c}_1} - \frac{\bar{q}(s)\bar{m}(s)}{\bar{c}_2 - \bar{c}_1}. \end{aligned} \tag{5.12}$$

Moreover, according to $(5.6)_2$ and (5.7), the following compatibility conditions is also be required

$$\frac{d}{ds}\bar{x}(s) = \frac{d}{ds}x(s, \kappa - s) = \frac{\bar{c}_2\bar{p}(s)\bar{h}(s) - \bar{c}_1\bar{q}(s)\bar{g}(s)}{\bar{c}_2 - \bar{c}_1}, \tag{5.13}$$

$$\frac{d}{ds}\bar{t}(s) = \frac{d}{ds}t(s, \kappa - s) = \frac{\bar{p}(s)\bar{h}(s) - \bar{q}(s)\bar{g}(s)}{\bar{c}_2 - \bar{c}_1}\alpha\big(\bar{x}(s), \bar{u}(s)\big), \tag{5.14}$$

here, we have denoted

$$\bar{c}_1 := c_1\big(\bar{x}(s), \bar{u}(s)\big) \quad \text{and} \quad \bar{c}_2 := c_2\big(\bar{x}(s), \bar{u}(s)\big). \tag{5.15}$$

We take the following lemma as the starting point for our analysis.

Lemma 5.2. (i) *Let $(u, \ell, m, h, g, p, q)(X, Y)$ be smooth solutions of the system (5.3)–(5.6) with the boundary conditions (5.10) along the line $\gamma = \{(X, Y);\ X + Y = \kappa\}$. Assume that the compatibility condition (5.12) is satisfied. Then, for any $(X, Y) \in \mathbb{R}^2$, it holds that*

$$u_Y = \frac{qm}{c_2 - c_1}, \tag{5.16}$$

if and only if

$$u_X = \frac{p\ell}{c_2 - c_1}, \tag{5.17}$$

(ii) *Let* $(u, \ell, m, h, g, p, q)(X, Y)$ *be smooth solutions of the system* (5.3)–(5.6). *Then there exists a solution* $(t, x)(X, Y)$ *of* $(5.6)_2$–(5.7) *with the boundary data* (5.11) *if and only if the compatibility conditions* (5.13)–(5.14) *are satisfied.*

Proof. (i). By a direct calculation, we observe that

$$\begin{aligned}
\big(\frac{qm}{c_2-c_1}\big)_X &= \frac{pq}{(c_2-c_1)^2}\Big\{a_1(2m\ell - g) - (a_1+a_2)(m\ell - gh) + a_2(h-2gh) \\
&\quad - \frac{\partial_u(c_2-c_1)}{c_2-c_1}m\ell + c_1 bg\ell + (d_2 - \partial_x c_2)mh\Big\} \\
&= \frac{pq}{(c_2-c_1)^2}\Big\{a_1 g(h-1) + a_2 h(1-g) - \frac{\partial_u[\alpha(c_2-c_1)]}{2\alpha(c_2-c_1)}m\ell + b(c_1 g\ell + c_2 mh)\Big\},
\end{aligned}$$

and

$$\begin{aligned}
\big(\frac{p\ell}{c_2-c_1}\big)_Y &= \frac{pq}{(c_2-c_1)^2}\Big\{a_1(2gh - g) + (a_1+a_2)(m\ell - gh) + a_2(h-2m\ell) \\
&\quad - \frac{\partial_u(c_2-c_1)}{c_2-c_1}m\ell + c_2 bmh + (d_1 - \partial_x c_1)g\ell\Big\} \\
&= \frac{pq}{(c_2-c_1)^2}\Big\{a_1 g(h-1) + a_2 h(1-g) - \frac{\partial_u[\alpha(c_2-c_1)]}{2\alpha(c_2-c_1)}m\ell + b(c_1 g\ell + c_2 mh)\Big\},
\end{aligned}$$

which leads to

$$\big(\frac{qm}{c_2-c_1}\big)_X = \big(\frac{p\ell}{c_2-c_1}\big)_Y. \tag{5.18}$$

Assume that (5.16) holds, it follows from (5.18) that

$$u_{YX} = \big(\frac{qm}{c_2-c_1}\big)_X = \big(\frac{p\ell}{c_2-c_1}\big)_Y.$$

This together with the boundary condition (5.10), compatibility condition (5.12) and the assumption (5.16) gives that

$$\begin{aligned}
u_X(X,Y) &= u_X(X, \kappa - X) + \int_{\kappa - X}^{Y} \big(\frac{p\ell}{c_2-c_1}\big)_Y(X,s)\,ds \\
&= [u_X - u_Y](X, \kappa - X) + u_Y(X, \kappa - X) + \frac{p\ell}{c_2-c_1}(X,Y) - \frac{p\ell}{c_2-c_1}(X, \kappa - X) \\
&= \frac{p\ell}{c_2-c_1}(X,Y),
\end{aligned}$$

which is indeed the desired identity (5.17). Similar arguments yield the converse implication.

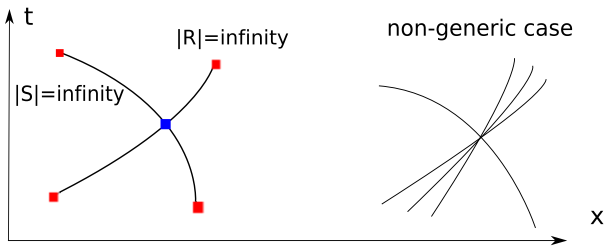

Fig. 4. The singular point in a solution $u(t, x)$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

(ii). We omit the proof here for brevity, since a similar approach of this result can be found in [3]. □

5.2. Three types of generic singularities

We observe that, for smooth initial data, the solution of the semilinear system (5.3)–(5.7) remains smooth on the X-Y plane. However, the solution $u(x, t)$ of system (1.4)–(1.6) can have singularities. This happens precisely at points where the Jacobian matrix $D\mathcal{F}$ is not invertible. In fact, the determinant of its Jacobian matrix is calculated as

$$det \begin{pmatrix} x_X & x_Y \\ t_X & t_Y \end{pmatrix} = \frac{\alpha pqgh}{c_2 - c_1}.$$

We recall that p, q remain uniformly positive and uniformly bounded on compact subsets of the X-Y plane. To analyze the set of points (x, t) where u is singular, we thus need to study in more details of the points where $g = 0$ or $h = 0$. It is natural to distinguish three generic types of singularities:

i. Points where $h = 0$ but $\ell_X \neq 0$ and $g \neq 0$ (or else, where $g = 0$ but $m_Y \neq 0$ and $h \neq 0$), their images under the map $\mathcal{F} : (X, Y) \mapsto \big(x(X, Y), t(X, Y)\big)$ yield a family of characteristic curves in the x-t plane where solution $u(x, t)$ is singular (Fig. 4, black curves, inner points of singular curves).
ii. Points where $h = 0$ and $\ell_X = 0$ but $\ell_{XX} \neq 0$ (or else, $g = 0$ and $m_Y = 0$ but $m_{YY} \neq 0$), their images in the x-t plane are point where singular curves start or end: (Fig. 4, red dots, initial and terminal points of singular curves).
iii. Points where $h = 0$ and $g = 0$, their images in the x-t plane are points where two singular curves cross: (Fig. 4, blue dot, intersection of singular curves in two directions).

Correspondingly, we give the following definition.

Definition 5.1. We say that a solution $u = u(x, t)$ of (1.4) has only **generic singularities** for $t \in [0, T]$ if it admits a representation of the form (5.9), where

(i) the functions $(u, \ell, m, h, g, p, q, x, t)(X, Y)$ are $\mathcal{C}^\infty$,

(ii) the following generic conditions

$$\begin{cases} h=0, \ell_X=0 \Longrightarrow \ell_Y \neq 0, \ell_{XX} \neq 0, \\ g=0, m_Y=0 \Longrightarrow m_X \neq 0, m_{YY} \neq 0, \\ h=0, g=0 \Longrightarrow \ell_X \neq 0, m_Y \neq 0, \end{cases} \tag{5.19}$$

hold for $t(X,Y) \in [0,T]$.

5.3. Families of perturbed solutions

Now we construct families of smooth solutions to the semi-linear system of (5.3)–(5.6), depending on parameters. Let a point (X_0, Y_0) be given, and consider the line

$$\gamma_\kappa = \{(X,Y);\ X+Y=\kappa\}, \quad \kappa \ = \ X_0+Y_0.$$

The following lemma is crucial in proving the generic regularity result.

Lemma 5.3. *Assume the generic condition* (1.8) *holds. Let a point* $(X_0,Y_0) \in \mathbb{R}^2$ *be given, and* (u,ℓ,m,h,g,p,q,x) *be a smooth solution of the semi-linear system* (5.3)–(5.6).

(1) *If* $(h,\ell_X,\ell_{XX})(X_0,Y_0)=(0,0,0)$*, then there exists a 3-parameter family of smooth solutions* $(u^\vartheta,\ell^\vartheta,m^\vartheta,h^\vartheta,g^\vartheta,p^\vartheta,q^\vartheta,x^\vartheta)$ *of* (5.3)–(5.6)*, depending smoothly on* $\vartheta \in \mathbb{R}^3$*, such that the following holds.*

(i) *When* $\vartheta=0 \in \mathbb{R}^3$*, one recovers the original solution, namely* $(u^0,\ell^0,m^0,h^0,g^0,p^0,q^0,x^0)=(u,\ell,m,h,g,p,q,x)$.

(ii) *At a point* (X_0,Y_0)*, when* $\vartheta=0$ *one has*

$$rank\ D_\vartheta(h^\vartheta,\ell^\vartheta_X,\ell^\vartheta_{XX}) \ = \ 3. \tag{5.20}$$

(2) *If* $(h,g,\ell_X)(X_0,Y_0)=(0,0,0)$*, then there exists a 3-parameter family of smooth solutions* $(u^\vartheta,\ell^\vartheta,m^\vartheta,h^\vartheta,g^\vartheta,p^\vartheta,q^\vartheta,x^\vartheta)$ *satisfying* (i)–(ii) *as above, with* (5.20) *replaced by*

$$rank\ D_\vartheta(h^\vartheta,g^\vartheta,\ell^\vartheta_X) \ = \ 3. \tag{5.21}$$

(3) *If* $\big(h,\ell_X,\partial_u\lambda_-(x,u)\big)(X_0,Y_0)=(0,0,0)$*, then there exists a 3-parameter family of smooth solutions* $(u^\vartheta,\ell^\vartheta,m^\vartheta,h^\vartheta,g^\vartheta,p^\vartheta,q^\vartheta,x^\vartheta)$ *satisfying* (i)–(ii) *as above, with* (5.20) *replaced by*

$$rank\ D_\vartheta\big(h^\vartheta,\ell^\vartheta_X,\partial_u\lambda_-(x^\vartheta,u^\vartheta)\big) \ = \ 3. \tag{5.22}$$

Proof. Let (u,ℓ,m,h,g,p,q,x) be a smooth solution to the semi-linear system (5.3)–(5.6), and $(\bar u,\bar\ell,\bar m,\bar h,\bar g,\bar p,\bar q,\bar x)(s)$ be the values along a line γ_κ as in (5.10). The main goal of this lemma is to consider families solution $(\bar u^\vartheta,\bar\ell^\vartheta,\bar m^\vartheta,\bar h^\vartheta,\bar g^\vartheta,\bar p^\vartheta,\bar q^\vartheta,\bar x^\vartheta)$ of (5.3)–(5.6) with perturbations on the data (5.10) along the curve γ_κ, so that the matrices in (5.20)–(5.22), computed at $\vartheta = 0$, have full rank at the given point (X_0, Y_0). These perturbations will have the form

$$\begin{cases} \bar{\ell}^{\vartheta}(s) = \bar{\ell}(s) + \displaystyle\sum_{j=1}^{3} \vartheta_j L_j(s), \\ \bar{m}^{\vartheta}(s) = \bar{m}(s) + \displaystyle\sum_{j=1}^{3} \vartheta_j M_j(s), \\ \bar{h}^{\vartheta}(s) = \bar{h}(s) + \displaystyle\sum_{j=1}^{3} \vartheta_j H_j(s), \end{cases} \qquad \begin{cases} \bar{g}^{\vartheta}(s) = \bar{g}(s) + \displaystyle\sum_{j=1}^{3} \vartheta_j G_{\mathrm{J}}(s), \\ \bar{p}^{\vartheta}(s) = \bar{p}(s) + \displaystyle\sum_{j=1}^{3} \vartheta_j P_j(s), \\ \bar{q}^{\vartheta}(s) = \bar{q}(s) + \displaystyle\sum_{j=1}^{3} \vartheta_j Q_j(s), \end{cases}$$

for some suitable functions $L_j, M_j, H_j, G_j, P_j, Q_j \in \mathcal{C}_c^\infty(\mathbb{R})$. Moreover, at point $s = X_0$, we set

$$\bar{u}^{\vartheta}(X_0) = \bar{u}(X_0) + \sum_{j=1}^{3} \vartheta_j U_j(X_0), \quad \bar{x}^{\vartheta}(X_0) = \bar{x}(X_0) + \sum_{j=1}^{3} \vartheta_j \mathcal{X}_j(X_0).$$

Notice that, with the above definitions and the compatibility conditions (5.12) and (5.13), we can obtain the values $\bar{u}^{\vartheta}(s)$ and $\bar{x}^{\vartheta}(s)$ for all $s \in \mathbb{R}$. In addition, we can derive a unique solution of the semi-linear system (5.3)–(5.6) for each $\vartheta \in \mathbb{R}^3$.

To prove our results, we proceed with the values of ℓ_X and ℓ_{XX} at the point (X_0, Y_0). To this end, we first observe that

$$\bar{z}'(s) = \frac{d}{ds} z(s, \kappa - s) = (z_X - z_Y)(s, \kappa - s),$$

at any point $(s, \kappa - s) \in \gamma_\kappa$, for $z = \ell, m, h, g, q$. Here and in the rest of this manuscript, unless specified, we will use a prime to denote the derivative with respect to the parameter s along the line γ_κ. Hence, it follows from (5.3)–(5.5) that,

$$\ell_X(X_0, Y_0) = \bar{\ell}' + \frac{\bar{q}(2\bar{h} - 1)}{\bar{c}_2 - \bar{c}_1} f_1, \quad m_Y(X_0, Y_0) = -\bar{m}' + \frac{\bar{p}(2\bar{g} - 1)}{\bar{c}_2 - \bar{c}_1} f_2, \tag{5.23}$$

$$h_X(X_0, Y_0) = \bar{h}' - \frac{2\bar{q}\bar{\ell}}{\bar{c}_2 - \bar{c}_1} f_1, \qquad g_Y(X_0, Y_0) = -\bar{g}' - \frac{2\bar{p}\bar{m}}{\bar{c}_2 - \bar{c}_1} f_2, \tag{5.24}$$

$$q_Y(X_0, Y_0) = -\bar{q}' + \frac{2\bar{p}\bar{q}}{\bar{c}_2 - \bar{c}_1} f_3, \tag{5.25}$$

where the right hand sides of (5.23)–(5.25) are evaluated at $s = X_0$ and we have denoted

$$\begin{aligned} f_1 &:= \bar{a}_1\bar{g} + \bar{a}_2\bar{h} - (\bar{a}_1 + \bar{a}_2)(\bar{g}\bar{h} + \bar{m}\bar{\ell}) + \bar{c}_2\bar{b}\bar{h}\bar{m} - \bar{d}_1\bar{g}\bar{\ell}, \\ f_2 &:= -\bar{a}_1\bar{g} - \bar{a}_2\bar{h} + (\bar{a}_1 + \bar{a}_2)(\bar{g}\bar{h} + \bar{m}\bar{\ell}) + \bar{c}_1\bar{b}\bar{g}\bar{\ell} - \bar{d}_2\bar{h}\bar{m}, \\ f_3 &:= \bar{a}_1(\bar{\ell} - \bar{m}) + (\bar{a}_1 + \bar{a}_2)(\bar{h}\bar{m} - \bar{g}\bar{\ell}) + \bar{c}_1\bar{b}\bar{m}\bar{\ell} + \bar{d}_2\bar{g}\bar{h} + \frac{\bar{c}_1\partial_x\bar{c}_2 - \bar{c}_2\partial_x\bar{c}_2}{2(\bar{c}_2 - \bar{c}_1)}\bar{h}, \end{aligned}$$

with $\bar{a}_i = a_i\big(\bar{x}(s), \bar{u}(s)\big)$, $\bar{b} = b\big(\bar{x}(s), \bar{u}(s)\big)$, $\bar{d}_i = d_i\big(\bar{x}(s), \bar{u}(s)\big)$ and $\bar{c}_i$ denoted in (5.15), for $i = 1, 2$.

On the other hand, a straightforward computation now shows

$$\frac{d^2}{ds^2}\bar{\ell}(s) = \frac{d}{ds}[\ell_X(s, \kappa - s) - \ell_Y(s, \kappa - s)] = (\ell_{XX} + \ell_{YY} - 2\ell_{XY})(s, \kappa - s). \tag{5.26}$$

By (5.3)–(5.6) and (5.24)–(5.25), further manipulation leads to the following estimates for ℓ_{YY} and ℓ_{YX}:

$$\begin{aligned}
\ell_{YY}(X_0, Y_0) = &- \frac{\bar{q}(2\bar{h}-1)}{(\bar{c}_2 - \bar{c}_1)^2}\Big[\partial_u(\bar{c}_2 - \bar{c}_1)\bar{u}_Y + \partial_x(\bar{c}_2 - \bar{c}_1)\bar{x}_Y\Big] f_1 \\
&+ \frac{(2\bar{h}-1) f_1}{\bar{c}_2 - \bar{c}_1}\bar{q}_Y + \frac{2\bar{q} f_1}{\bar{c}_2 - \bar{c}_1}\bar{h}_Y + \frac{\bar{q}(2\bar{h}-1)}{\bar{c}_2 - \bar{c}_1}\partial_Y f_1 \\
=&\frac{\bar{q}(2\bar{h}-1)}{(\bar{c}_2 - \bar{c}_1)^2}\Big[\frac{\partial_u \bar{c}_1 - \partial_u \bar{c}_2}{\bar{c}_2 - \bar{c}_1}\bar{q}\bar{m} + \frac{\partial_x \bar{c}_1 - \partial_x \bar{c}_2}{\bar{c}_2 - \bar{c}_1}\bar{c}_1\bar{q}\bar{m}\Big] f_1 \\
&+ \frac{(2\bar{h}-1) f_1}{\bar{c}_2 - \bar{c}_1}[-\bar{q}' + \frac{2\bar{p}\bar{q}}{\bar{c}_2 - \bar{c}_1} f_3] - \frac{4\bar{q}^2\bar{\ell} f_1^2}{(\bar{c}_2 - \bar{c}_1)^2} \\
&+ \frac{\bar{q}(2\bar{h}-1)}{\bar{c}_2 - \bar{c}_1}\partial_Y f_1 =: F_1,
\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}
\ell_{YX}(X_0, Y_0) = &- \frac{\bar{q}(2\bar{h}-1)}{(\bar{c}_2 - \bar{c}_1)^2}\Big[\partial_u(\bar{c}_2 - \bar{c}_1)\bar{u}_X + \partial_x(\bar{c}_2 - \bar{c}_1)\bar{x}_X\Big] f_1 \\
&+ \frac{(2\bar{h}-1) f_1}{\bar{c}_2 - \bar{c}_1}\bar{q}_X + \frac{2\bar{q} f_1}{\bar{c}_2 - \bar{c}_1}\bar{h}_X + \frac{\bar{q}(2\bar{h}-1)}{\bar{c}_2 - \bar{c}_1}\partial_X f_1 \\
=&\frac{\bar{q}(2\bar{h}-1)}{(\bar{c}_2 - \bar{c}_1)^2}\Big[\frac{\partial_u \bar{c}_1 - \partial_u \bar{c}_2}{\bar{c}_2 - \bar{c}_1}\bar{p}\bar{\ell} + \frac{\partial_x \bar{c}_1 - \partial_x \bar{c}_2}{\bar{c}_2 - \bar{c}_1}\bar{c}_2\bar{p}\bar{h}\Big] f_1 \\
&+ \frac{2(2\bar{h}-1)\bar{p}\bar{q} f_1 f_3}{(\bar{c}_2 - \bar{c}_1)^2} + \frac{2\bar{q} f_1}{\bar{c}_2 - \bar{c}_1}[\bar{h}' - \frac{2\bar{q}\bar{\ell}}{\bar{c}_2 - \bar{c}_1} f_1] \\
&+ \frac{\bar{q}(2\bar{h}-1)}{\bar{c}_2 - \bar{c}_1}\partial_X f_1 =: F_2,
\end{aligned} \tag{5.28}$$

with $\partial_r f_1 = \partial_r\big[\bar{a}_1\bar{g} + \bar{a}_2\bar{h} - (\bar{a}_1 + \bar{a}_2)(\bar{g}\bar{h} + \bar{m}\bar{\ell}) + \bar{c}_2\bar{b}\bar{h}\bar{m} - \bar{d}_1\bar{g}\bar{\ell}\big]$, for $r = X$ or Y. The equations (5.26)–(5.28) in turn yield

$$\ell_{XX} = \bar{\ell}'' - F_1 + 2F_2. \tag{5.29}$$

Now, we are ready to construct families of perturbed solutions satisfying (5.20)–(5.22).

(1). We choose suitable perturbations $(\bar{u}^\vartheta, \bar{\ell}^\vartheta, \bar{m}^\vartheta, \bar{h}^\vartheta, \bar{g}^\vartheta, \bar{p}^\vartheta, \bar{q}^\vartheta, \bar{x}^\vartheta)$, such that, at the point $s = X_0$ and $\vartheta = 0$, it holds that

$$D_\vartheta \begin{pmatrix} \bar{u} \\ \bar{h} \\ \bar{g} \\ \bar{\ell} \\ \bar{m} \\ \bar{p} \\ \bar{q} \\ \bar{x} \\ \bar{g}' \\ \bar{h}' \\ \bar{m}' \\ \bar{\ell}' \\ \bar{\ell}'' \\ \bar{q}' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, by using (5.23) and (5.29), we obtain the desired Jacobian matrix at the point (X_0, Y_0),

$$D_\vartheta \begin{pmatrix} h \\ \ell_X \\ \ell_{XX} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}.$$

This in turn yields (5.20).

(2). We choose suitable perturbations $(\bar{u}^\vartheta, \bar{\ell}^\vartheta, \bar{m}^\vartheta, \bar{h}^\vartheta, \bar{g}^\vartheta, \bar{p}^\vartheta, \bar{q}^\vartheta, \bar{x}^\vartheta)$, such that, at the point $s = X_0$ and $\vartheta = 0$, the Jacobian matrix of first order derivatives with respect to ϑ is given by

$$D_\vartheta \begin{pmatrix} \bar{u} \\ \bar{h} \\ \bar{g} \\ \bar{\ell} \\ \bar{m} \\ \bar{p} \\ \bar{q} \\ \bar{x} \\ \bar{\ell}' \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This together with (5.23) implies that at the point (X_0, Y_0),

$$D_\vartheta \begin{pmatrix} h \\ g \\ \ell_X \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{pmatrix}.$$

We thus conclude this matrix has full rank, that is, (5.21) holds.

(3). If $(h, \partial_u \lambda_-, \ell_X)(X_0, Y_0) = (0, 0, 0)$ is satisfied, the generic condition (1.8) gives that

$$\partial_{uu}\lambda_-(X_0, Y_0) \neq 0 \quad \text{or} \quad \partial_{ux}\lambda_-(X_0, Y_0) \neq 0. \tag{5.30}$$

By choosing suitable perturbations $(\bar{u}^\vartheta, \bar{\ell}^\vartheta, \bar{m}^\vartheta, \bar{h}^\vartheta, \bar{g}^\vartheta, \bar{p}^\vartheta, \bar{q}^\vartheta, \bar{x}^\vartheta)$, such that, at the point $s = X_0$ and $\vartheta = 0$, it holds that

$$D_\vartheta\begin{pmatrix}\bar u\\ \bar h\\ \bar g\\ \bar\ell\\ \bar m\\ \bar p\\ \bar q\\ \bar x\\ \bar\ell'\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&1\end{pmatrix}\quad\text{or}\quad D_\vartheta\begin{pmatrix}\bar u\\ \bar h\\ \bar g\\ \bar\ell\\ \bar m\\ \bar p\\ \bar q\\ \bar x\\ \bar\ell'\end{pmatrix}=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Here the first matrix corresponds to the assumption that $\partial_{uu}\lambda_-(X_0,Y_0)\neq 0$, while the second one corresponds to $\partial_{ux}\lambda_-(X_0,Y_0)\neq 0$. In terms of this construction and (5.23), one has

$$D_\vartheta\begin{pmatrix}h\\ \partial_u\lambda_-\\ \ell_X\end{pmatrix}=\begin{pmatrix}1&0&0\\ *&\partial_{uu}\lambda_-&0\\ *&*&1\end{pmatrix}\quad\text{or}\quad D_\vartheta\begin{pmatrix}h\\ \partial_u\lambda_-\\ \ell_X\end{pmatrix}=\begin{pmatrix}1&0&0\\ *&\partial_{ux}\lambda_-&0\\ *&*&1\end{pmatrix},$$

at the point (X_0,Y_0), which, in combination with (5.30) achieves (5.22). Here the first matrix corresponds to the assumption that $\partial_{uu}\lambda_-(X_0,Y_0)\neq 0$, while the second one corresponds to $\partial_{ux}\lambda_-(X_0,Y_0)\neq 0$. This completes the proof of Lemma 5.3. □

Once we proved Lemma 5.3, Theorem 1.2 can be proved by using a very similar method as in [3]. We leave this proof and the existence of generic regular path in the Appendix to make this paper self-contained.

6. Metric for piecewise smooth solutions

In this section, we extend the Lipschitz metric for smooth solutions in Section 4 to piecewise smooth solutions with only generic singularities.

6.1. Tangent vectors in transformed coordinates

To begin with, we express the norm of tangent vectors (4.11) in transformed coordinates X-Y.

Let $u(x,t)$ be a reference solution of (1.4) and $u^\varepsilon(x,t)$ be a family of perturbed solutions. In the (X,Y) plane, denote (u,ℓ,m,h,g,p,q,x,t) and $(u^\varepsilon,\ell^\varepsilon,m^\varepsilon,h^\varepsilon,g^\varepsilon,p^\varepsilon,q^\varepsilon,x^\varepsilon,t^\varepsilon)$ be the corresponding smooth solutions of (5.3)–(5.7), and moreover assume the perturbed solutions take the form

$$\begin{aligned}(u^\varepsilon,\ell^\varepsilon,m^\varepsilon,h^\varepsilon,g^\varepsilon,p^\varepsilon,q^\varepsilon,x^\varepsilon,t^\varepsilon)&=(u,\ell,m,h,g,p,q,x,t)\\&\quad+\varepsilon(U,L,M,H,G,P,Q,\mathcal{X},\mathcal{T})+o(\varepsilon).\end{aligned}$$

Here we denote the curve in (X,Y) plane by

$$\Gamma_\tau=\{(X,Y)\,|\,t(X,Y)=\tau\}=\{(X,Y(\tau,X));\,X\in\mathbb{R}\}=\{(X(\tau,Y),Y);\,Y\in\mathbb{R}\}\tag{6.1}$$

and the perturbed curve as

$$\Gamma_\tau^\varepsilon = \{(X,Y)\,|\,t^\varepsilon(X,Y)=\tau\} = \{(X, Y^\varepsilon(\tau, X));\, X\in\mathbb{R}\} = \{(X^\varepsilon(\tau,Y), Y);\, Y\in\mathbb{R}\}.$$

Notice that the coefficients of system (5.3)–(5.7) are smooth, it thus follows that the first order perturbations satisfy a linearized system and are well defined for $(X,Y)\in\mathbb{R}^2$.

Now, we are ready to derive an expression for I_0–I_6 of (4.11) in terms of $(U, L, M, H, G, P, Q, \mathcal{X}, \mathcal{T})$. First, we observe that

$$t^\varepsilon\big(X, Y^\varepsilon(\tau, X)\big) = t^\varepsilon\big(X^\varepsilon(\tau,Y), Y\big) = \tau.$$

By the implicit function theorem, at $\varepsilon=0$, it holds that

$$\frac{\partial X^\varepsilon}{\partial\varepsilon}\Big|_{\varepsilon=0} = -\mathcal{T}\frac{c_2-c_1}{\alpha hp}, \quad \text{and} \quad \frac{\partial Y^\varepsilon}{\partial\varepsilon}\Big|_{\varepsilon=0} = -\mathcal{T}\frac{c_2-c_1}{\alpha gq}. \tag{6.2}$$

(1). The change in x is

$$\begin{aligned} w &= \lim_{\varepsilon\to 0}\frac{x^\varepsilon\big(X, Y^\varepsilon(\tau,X)\big) - x\big(X, Y(\tau,X)\big)}{\varepsilon} \\ &= \mathcal{X}\big(X, Y(\tau,X)\big) + x_Y\cdot\frac{\partial Y^\varepsilon}{\partial\varepsilon}\Big|_{\varepsilon=0} = \big(\mathcal{X} - \frac{c_1}{\alpha}\mathcal{T}\big)(X, Y(\tau,X)). \end{aligned} \tag{6.3}$$

Similarly, we obtain

$$\begin{aligned} z &= \lim_{\varepsilon\to 0}\frac{x^\varepsilon\big(X^\varepsilon(\tau,Y), Y\big) - x\big(X(\tau,Y), Y\big)}{\varepsilon} \\ &= \mathcal{X}\big(X(\tau,Y), Y\big) + x_X\cdot\frac{\partial X^\varepsilon}{\partial\varepsilon}\Big|_{\varepsilon=0} = \big(\mathcal{X} - \frac{c_2}{\alpha}\mathcal{T}\big)(X(\tau,Y), Y). \end{aligned} \tag{6.4}$$

(2). For the change in u, we first observe from (5.6) and (6.2) that,

$$\begin{aligned} v + u_x w &= \lim_{\varepsilon\to 0}\frac{u^\varepsilon\big(X, Y^\varepsilon(\tau,X)\big) - u\big(X, Y(\tau,X)\big)}{\varepsilon} \\ &= U\big(X, Y(\tau,X)\big) + u_Y\cdot\frac{\partial Y^\varepsilon}{\partial\varepsilon}\Big|_{\varepsilon=0} = \big(U - \frac{\mathcal{T}m}{\alpha g}\big)(X, Y(\tau,X)). \end{aligned}$$

This together with (2.2) gives

$$v + \frac{Rw - Sz}{c_2-c_1} = v + u_x w + \frac{w-z}{c_2-c_1}S = U(X, Y(\tau,X)). \tag{6.5}$$

(3). In addition, we derive an expression for the terms $J_3^\pm$ in (4.11). Using (5.3), (5.4) and (6.2), a direct computation gives rise to

$$
\begin{aligned}
r + wR_x &= \frac{d}{d\varepsilon}\frac{\ell^\varepsilon}{h^\varepsilon}\big(X, Y^\varepsilon(\tau, X)\big)\Big|_{\varepsilon=0} = \lim_{\varepsilon\to 0}\frac{\frac{\ell^\varepsilon}{h^\varepsilon}\big(X, Y^\varepsilon(\tau, X)\big) - \frac{\ell}{h}\big(X, Y(\tau, X)\big)}{\varepsilon} \\
&= \frac{1}{h}\big(L + \ell_Y \cdot \frac{\partial Y^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}\big) - \frac{\ell}{h^2}\big(H + h_Y \cdot \frac{\partial Y^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}\big) \\
&= \frac{L}{h} - \frac{\ell H}{h^2} - \frac{\mathcal{T}}{\alpha h g}[a_1 g + a_2 h - (a_1 + a_2)(gh + m\ell) + c_2 bhm - d_1 g\ell],
\end{aligned}
$$

and

$$
\begin{aligned}
s + zS_x &= \frac{d}{d\varepsilon}\frac{m^\varepsilon}{g^\varepsilon}\big(X^\varepsilon(\tau, Y), Y\big)\Big|_{\varepsilon=0} \\
&= \frac{1}{g}\big(M + m_X \cdot \frac{\partial X^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}\big) - \frac{m}{g^2}\big(G + g_X \cdot \frac{\partial X^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}\big) \\
&= \frac{M}{g} - \frac{mG}{g^2} - \frac{\mathcal{T}}{\alpha h g}[-a_1 g - a_2 h + (a_1 + a_2)(gh + m\ell) + c_1 bg\ell - d_2 hm].
\end{aligned}
$$

Thus, we obtain from (4.10) that

$$
\hat{r} = \frac{L}{h} - \frac{\ell H}{h^2} - \frac{\mathcal{T}}{\alpha h}(a_1 - a_1 h - d_1 \ell), \tag{6.6}
$$

and

$$
\hat{s} = \frac{M}{g} - \frac{mG}{g^2} - \frac{\mathcal{T}}{\alpha g}(-a_2 + a_2 g - d_2 m). \tag{6.7}
$$

(4). To continue, we first use the same procedure to derive the change in the base measure with density $1 + R^2$ as

$$
\frac{d}{d\varepsilon}p^\varepsilon\big(X, Y^\varepsilon(\tau, X)\big)\Big|_{\varepsilon=0} = P + p_Y \cdot \frac{\partial Y^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} = P - \frac{c_2 - c_1}{\alpha g q}\mathcal{T} p_Y. \tag{6.8}
$$

Moreover, the change in base measure with density R^2 can be calculated as

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\Big(\big(p^\varepsilon(1 - h^\varepsilon)\big)\big(X, Y^\varepsilon(\tau, X)\big)\Big)\Big|_{\varepsilon=0} \\
&= \big(P + p_Y \cdot \frac{\partial Y^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}\big)(1 - h) - p\big(H + h_Y \cdot \frac{\partial Y^\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0}\big) \\
&= \big(P - \frac{c_2 - c_1}{\alpha g q}\mathcal{T} p_Y\big)(1 - h) - p\big(H - \frac{c_2 - c_1}{\alpha g q}\mathcal{T} h_Y\big).
\end{aligned} \tag{6.9}
$$

Finally, we can achieve the change in base measure with density 1 by subtracting (6.8) from (6.9)

$$
h\big(P - \frac{c_2 - c_1}{\alpha g q}\mathcal{T} p_Y\big) + p\big(H - \frac{c_2 - c_1}{\alpha g q}\mathcal{T} h_Y\big). \tag{6.10}
$$

Notice that

$$(1+R^2)dx = p\,dX, \quad (1+S^2)dx = -q\,dY.$$

Hence, according to (6.3)–(6.10), the weighted norm (4.11) can be rewritten as a line integral over the line Γ_τ defined in (6.1). More specifically, we have

$$\|(v, w, \hat{r}, z, \hat{s})\|_{(u,R,S)} = \sum_{k=0}^{6} \kappa_k \int_{\Gamma_\tau} \Big(|J_k| \mathcal{W}^- dX + |H_k| \mathcal{W}^+ dY \Big),$$

where

$$\begin{cases} J_0 = \big(\mathcal{X} - \frac{c_1}{\alpha}\mathcal{T}\big) ph, \\ J_1 = \big(\mathcal{X} - \frac{c_1}{\alpha}\mathcal{T}\big) p, \\ J_2 = Up, \\ J_3 = Lp - \frac{\ell H}{h} p - \frac{\mathcal{T}p}{\alpha}(a_1 - a_1 h - d_1 \ell), \\ J_4 = hP + pH + \frac{2p\mathcal{T}}{\alpha}\big(a_1 \ell + \frac{c_1 \partial_x \alpha - \alpha \partial_x c_1}{2\alpha} h\big), \\ J_5 = \ell P + \frac{\ell H}{h} p + \frac{2p\mathcal{T}\ell}{\alpha h}\big(a_1 \ell + \frac{c_1 \partial_x \alpha - \alpha \partial_x c_1}{2\alpha} h\big), \\ J_6 = (1-h)P - pH - \frac{c_1 \partial_x c_2 - c_2 \partial_x c_1}{\alpha(c_2 - c_1)} \mathcal{T} p (1-h), \end{cases}$$

and

$$\begin{cases} H_0 = \big(\mathcal{X} - \frac{c_2}{\alpha}\mathcal{T}\big) qg, \\ H_1 = \big(\mathcal{X} - \frac{c_2}{\alpha}\mathcal{T}\big) q, \\ H_2 = Uq, \\ H_3 = Mq - \frac{mG}{g} q - \frac{\mathcal{T}q}{\alpha}(-a_2 + a_2 g - d_2 m), \\ H_4 = gQ + qG + \frac{2q\mathcal{T}}{\alpha}\big(-a_2 m + \frac{c_2 \partial_x \alpha - \alpha \partial_x c_2}{2\alpha} g\big), \\ H_5 = mQ + \frac{mG}{g} q + \frac{2q\mathcal{T}m}{\alpha g}\big(-a_2 m + \frac{c_2 \partial_x \alpha - \alpha \partial_x c_2}{2\alpha} g\big), \\ H_6 = (1-g)Q - qG - \frac{c_1 \partial_x c_2 - c_2 \partial_x c_1}{\alpha(c_2 - c_1)} \mathcal{T} q (1-g). \end{cases}$$

It is easy to verify that each integrands J_k, H_k are smooth, for $k = 0, 1, 2, 4, 6$. On the other hand, for the term $\frac{\ell H}{h} p$ in J_3 and J_5, we first observe that,

$$(\ell^\varepsilon)^2 + (h^\varepsilon)^2 = h^\varepsilon.$$

Differentiating this equation with respect to ε, at $\varepsilon = 0$, it holds that

$$2\ell L + 2hH = H.$$

With this help, we achieve

$$\frac{\ell H}{h}p = \frac{2\ell^2 L + 2\ell h H}{h}p = \frac{2(h-h^2)L + 2\ell h H}{h}p = 2p[(1-h)L + \ell H],$$

here we have used the fact that $\ell^2 + h^2 = h$. Therefore, J_3 and J_5 are also smooth. In a similar way, we can get the smoothness of H_3 and H_5.

6.2. Length of piecewise regular paths

In this part, we define the length of a piecewise regular path $\gamma^t : \theta \mapsto \big(u^\theta(t), u_t^\theta(t)\big)$, and examine the appearance of the generic singularity will not impact the Lipschitz property of this metric.

Definition 6.1. The length $\|\gamma^t\|$ of the piecewise regular path $\gamma^t : \theta \mapsto \big(u^\theta(t), u_t^\theta(t)\big)$ is defined as

$$\|\gamma^t\| = \inf_{\gamma^t} \int_0^1 \Big\{ \sum_{k=0}^{6} \kappa_k \int_{\Gamma_t^\theta} \Big(|J_k^\theta| \mathcal{W}^- dX + |H_k^\theta| \mathcal{W}^+ dY \Big) \Big\} d\theta, \tag{6.11}$$

where the infimum is taken over all piecewise smooth relabellings of the X-Y coordinates and $\Gamma_\tau^\theta := \{(X,Y); t^\theta(X,Y) = \tau\}$

Remark 6.1. In general, there are many distinct solutions to the system (5.3)–(5.7) which yields the same solution $u = u(x,t)$ of (1.4). In fact, suppose $\varphi, \psi : \mathbb{R} \mapsto \mathbb{R}$ be two $\mathcal{C}^2$ bijections, with $\varphi', \psi' > 0$. Consider a particular solution $(u, \ell, m, h, g, p, q, x, t)$ to the system (5.3)–(5.7), and let the new independent and dependent variables $(\widetilde{X}, \widetilde{Y})$ and $(\tilde{u}, \tilde{\ell}, \tilde{m}, \tilde{h}, \tilde{g}, \tilde{p}, \tilde{q}, \tilde{x}, \tilde{t})$ be defined by

$$X = \varphi(\widetilde{X}), \quad Y = \psi(\widetilde{Y}), \tag{6.12}$$

$$\begin{cases} (\tilde{u}, \tilde{\ell}, \tilde{m}, \tilde{h}, \tilde{g}, \tilde{x}, \tilde{t})(\widetilde{X}, \widetilde{Y}) = (u, \ell, m, h, g, x, t)(X, Y), \\ \tilde{p}(\widetilde{X}, \widetilde{Y}) = p(X, Y) \cdot \varphi'(\widetilde{X}), \\ \tilde{q}(\widetilde{X}, \widetilde{Y}) = q(X, Y) \cdot \psi'(\widetilde{Y}). \end{cases}$$

It is easy to see that $(\tilde{u}, \tilde{\ell}, \tilde{m}, \tilde{h}, \tilde{g}, \tilde{p}, \tilde{q}, \tilde{x}, \tilde{t})(\widetilde{X}, \widetilde{Y})$ is also the solution to the same system (5.3)–(5.7), and the set

$$\{\big(\tilde{x}(\widetilde{X}, \widetilde{Y}), \tilde{t}(\widetilde{X}, \widetilde{Y}), \tilde{u}(\widetilde{X}, \widetilde{Y})\big); \quad (\widetilde{X}, \widetilde{Y}) \in \mathbb{R}^2\} \tag{6.13}$$

coincides with the set (5.9). We thus derive that the set (6.13) is another graph of the same solution $u(x,t)$ of (1.4). One can regard the variable transformation (6.12) simply as a relabeling of forward and backward characteristics, in the solution $u(x,t)$. We refer the readers to [13] for more details on the relabeling symmetries, in connection with the Camassa-Holm equation.

Our main result in this section is stated as follows, which extends the Lipschitz property in Lemma 4.1 to piecewise smooth solutions with generic singularities.

Theorem 6.1. *Let $T > 0$ be given, consider a path of solutions $\theta \mapsto \big(u^\theta(t), u_t^\theta(t)\big)$ of (1.4), which is piecewise regular for $t \in [0, T]$. Moreover, the total energy is less than some constants $E > 0$. Then there exists constants $\kappa_0, \kappa_1, \cdots, \kappa_6$ in (6.11) and $C > 0$, such that the length satisfies*

$$\|\gamma^t\| \le C\|\gamma^0\|, \tag{6.14}$$

where the constant C depends only on T and E.

Proof. Let the piecewise regular path $\theta \mapsto \big(u^\theta(t), u_t^\theta(t)\big)$ be given. From Definition A.1, for every $\theta \in [0, 1]\backslash\{\theta_1, \cdots, \theta_N\}$, the solution $u^\theta(t)$ has generic regularities for $t \in [0, T]$. More specifically, u^θ is smooth in the X-Y coordinates and piecewise smooth in the x-t coordinates, hence the tangent vector is well-defined for all $\theta \in [0, 1]$, $t \in [0, T]$.

To prove (6.14), it suffices to show that

$$\|\big(v^\theta, r^\theta, s^\theta\big)(t)\|_{(u^\theta, R^\theta, S^\theta)(t)} \le C_1\|\big(v^\theta, r^\theta, s^\theta\big)(0)\|_{(u^\theta, R^\theta, S^\theta)(0)}, \tag{6.15}$$

for $\theta \in [0, 1]\backslash\{\theta_1, \cdots, \theta_N\}$, here $C_1 >$ is a constant depending only on T and the upper bound of the total energy. Indeed, according to Definition 6.1, fix $\epsilon > 0$ and choose a relabeling of the variables X, Y, such that, at time $t = 0$, it holds that

$$\int_0^1 \left\{\sum_{k=0}^{6} \kappa_k \int_{\Gamma_0^\theta} \Big(|J_k^\theta|\mathcal{W}^- dX + |H_k^\theta|\mathcal{W}^+ dY\Big)\right\} d\theta \le \|\gamma^0\| + \epsilon.$$

Integrating (6.15) over $\theta \in [0, 1]$, we have

$$\|\gamma^t\| \le C(\|\gamma^0\| + \epsilon),$$

which yields the desired estimate (6.14) immediately, since $\epsilon > 0$ is arbitrary.

To complete the proof we need to achieve the estimate (6.15). Two cases can occur.

CASE 1: If u^θ is smooth in the x-t coordinates, (6.15) follows directly from (4.15).

CASE 2: If u^θ is piecewise smooth with generic singularities. In this case, we claim that the appearance of the generic singularities will not affect the estimate (4.15). Toward this goal, we first observe that there exist at most finitely many points $W_j = (X_j, Y_j)$, $j = 1, \cdots, N$ such that the generic conditions (5.19) hold when $t \in [0, T]$. Moreover, for each time $t_j = t(X_j, Y_j)$ corresponding to the point W_j, the map

$$t \mapsto \int_0^1 \left\{\sum_{k=0}^{6} \kappa_k \int_{\Gamma_t^\theta} \Big(|J_k^\theta|\mathcal{W}^- dX + |H_k^\theta|\mathcal{W}^+ dY\Big)\right\} d\theta$$

is continuous. Hence the metric will not be impacted at (at most finitely) time $t = t_j$ when there exist singularities such that the generic conditions (5.19) hold.

On the other hand, at time $t \neq t_j$, to obtain the estimate (4.15) it suffices to show that the time derivative

$$\frac{d}{dt}\sum_{k=0}^{6}\kappa_k \int\limits_{\Gamma_t^\theta} \left(|J_k^\theta|\mathcal{W}^- dX + |H_k^\theta|\mathcal{W}^+ dY\right)$$

will not be affected by the presence of singularity. Indeed, assume that the solution has the generic singularities along a backward characteristic. For a fixed time τ and denote $\Gamma_\tau := \{(X,Y);\ t^\theta(X,Y) = \tau\}$. Let the point $(X_\varepsilon, Y_\varepsilon)$ be the intersection of the curve $\Gamma_{\tau-\varepsilon} = \{(X,Y);\ t^\theta(X,Y) = \tau - \varepsilon\}$ and the singular curve $\{(X,Y);\ h^\theta(X,Y) = 0\}$, and the point $(X'_\varepsilon, Y'_\varepsilon)$ be the intersection of the curve $\Gamma_{\tau+\varepsilon} = \{(X,Y);\ t^\theta(X,Y) = \tau + \varepsilon\}$ and the singular curve $\{(X,Y);\ h^\theta(X,Y) = 0\}$. In addition, define the curves

$$\begin{cases}\Lambda_\varepsilon^+ := \Gamma_{\tau+\varepsilon} \cap \{(X,Y); X \in [X'_\varepsilon, X_\varepsilon]\},\\ \Lambda_\varepsilon^- := \Gamma_{\tau-\varepsilon} \cap \{(X,Y); X \in [X'_\varepsilon, X_\varepsilon]\},\end{cases} \qquad \begin{cases}\chi_\varepsilon^+ := \Gamma_{\tau+\varepsilon} \cap \{(X,Y); Y \in [Y_\varepsilon, Y'_\varepsilon]\},\\ \chi_\varepsilon^- := \Gamma_{\tau-\varepsilon} \cap \{(X,Y); Y \in [Y_\varepsilon, Y'_\varepsilon]\}.\end{cases}$$

Then, it follows that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\int\limits_{\Lambda_\varepsilon^+} - \int\limits_{\Lambda_\varepsilon^-}\Big)\sum_{k=0}^{6}|J_k^\theta|\mathcal{W}^- \, dX = 0,$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\int\limits_{\chi_\varepsilon^+} - \int\limits_{\chi_\varepsilon^-}\Big)\sum_{k=0}^{6}|H_k^\theta|\mathcal{W}^+ \, dY = 0.$$

The first limit holds since each integrand is continuous and $|X_\varepsilon - X'_\varepsilon| = O(\varepsilon)$. The second limit holds since each integrand is continuous and $|Y'_\varepsilon - Y_\varepsilon| = O(\varepsilon)$. Consequently, (4.15) follows even in the presence of singular curve where $h = 0$. Similarity, we can obtain the same result in the presence of singular curve where $g = 0$. This completes the proof of Theorem 6.1. □

7. Metric for general weak solutions

Finally, we prove Theorem 1.1, by extending the Lipschitz metric to general weak solutions. Then we compare our metric with some other distances.

7.1. Construction of geodesic distance

In this part, we construct a geodesic distance $d(\cdot,\cdot)$ on the space $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and prove the Lipschitz property. For the sake of convenience, fix any constant $E > 0$, we denote a set

$$\Omega_E := \{(u,u_t) \in H^1(\mathbb{R}) \times L^2(\mathbb{R});\ \mathcal{E}(u,u_t) := \int\limits_{\mathbb{R}} [\alpha^2 u_t^2 + \gamma^2 u_x^2]\, dx \le E\}.$$

Recall the generic regularity Theorem 1.2, that is, there exists an open dense set of initial data $\mathcal{M} \subset \left(\mathcal{C}^3(\mathbb{R}) \cap H^1(\mathbb{R})\right) \times \left(\mathcal{C}^2(\mathbb{R}) \cap L^2(\mathbb{R})\right)$, such that, for $(u_0, u_1) \in \mathcal{M}$, the conservative solution of (1.4) has only generic singularities. For future reference, we denote a set

$$\mathcal{M}^\infty := \mathcal{C}_0^\infty \cap \mathcal{M},$$

on which we define a geodesic distance by optimizing over all piecewise regular paths connecting two solutions of (1.4). Then by the semilinear system (5.3)–(5.7) and Theorem 6.1, we can extend this distance from space $\mathcal{M}^\infty$ to a larger space.

Definition 7.1. For solutions with initial data in $\mathcal{M}^\infty \cap \Omega_E$, we define the geodesic distance $d\big((u, u_t), (\hat{u}, \hat{u}_t)\big)$ as the infimum among the weighted lengths of all piecewise regular paths $\theta \mapsto (u^\theta, u_t^\theta)$, which connect (u, u_t) with $(\hat{u}, \hat{u}_t)$, that is, for any time t,

$$\begin{aligned} d\big((u, u_t), (\hat{u}, \hat{u}_t)\big) := \inf\{\|\gamma^t\| : &\gamma^t \text{ is a piecewise regular path}, \gamma^t(0) = (u, u_t), \\ &\gamma^t(1) = (\hat{u}, \hat{u}_t), \mathcal{E}(u^\theta, u_t^\theta) \le E, \text{ for all } \theta \in [0, 1]\}. \end{aligned}$$

The definition $d(\cdot, \cdot)$ is indeed a distance because after a suitable re-parameterization, the concatenation of two piecewise regular paths is still a piecewise regular path. Now, we can define the metric for the general weak solutions.

Definition 7.2. Let (u_0, u_1) and $(\hat{u}_0, \hat{u}_1)$ in $H^1(R) \times L^2(R)$ be two initial data as required in the existence and uniqueness Theorem 2.2. Denote u and $\hat{u}$ to be the corresponding global weak solutions, then for any time t, we define,

$$d\big((u, u_t), (\hat{u}, \hat{u}_t)\big) := \lim_{n\to\infty} d\big((u^n, u_t^n), (\hat{u}^n, \hat{u}_t^n)\big),$$

for any two sequences of solutions (u^n, u_t^n) and $(\hat{u}^n, \hat{u}_t^n)$ with the corresponding initial data in $\mathcal{M}^\infty \cap \Omega_E$, moreover

$$\|(u_0^n - u_0, \hat{u}_0^n - \hat{u}_0)\|_{H^1} \to 0, \quad \text{and} \quad \|(u_1^n - u_1, \hat{u}_1^n - \hat{u}_1)\|_{L^2} \to 0.$$

We claim that the definition of this metric is well-defined. Indeed, the limit in the definition is independent on the selection of sequences because the solution with initial data in $\mathcal{M}^\infty \cap \Omega_E$ are Lipschitz continuous.

On the other hand, when

$$\|u_0^n - u_0\|_{H^1} \to 0, \quad \|u_1^n - u_1\|_{L^2} \to 0,$$

by the semi-linear equations (5.3)–(5.7), we can get the corresponding solutions satisfy, for any $t > 0$,

$$\|u^n - u\|_{H^1} \to 0, \quad \|u_t^n - u_t\|_{L^2} \to 0.$$

Thus the Lipschitz property in Theorem 6.1 can be extended to the general solutions, this in turn yields the main theorem: Theorem 1.1.

7.2. Comparison with other metrics

Finally, by some calculations, we study the relations among our distance $d(\cdot,\cdot)$ and other types of metrics.

Proposition 7.1 *(Comparison with the Sobolev metric). For any two finite energy initial data (u_0,u_1) and $(\hat{u}_0,\hat{u}_1)\in\mathcal{M}^\infty$, there exists some constant C depends only on the initial energy, such that,*

$$d\big((u_0,u_1),(\hat{u}_0,\hat{u}_1)\big)\le C\Big(\|u_0-\hat{u}_0\|_{H^1}+\|u_0-\hat{u}_0\|_{W^{1,1}}+\|u_1-\hat{u}_1\|_{L^1}+\|u_1-\hat{u}_1\|_{L^2}\Big).$$

Proof. To find an upper bound of this optimal transport metric, we only have to consider one path (u_0^θ,u_1^θ) connecting (u_0,u_1) and $(\hat{u}_0,\hat{u}_1)$, satisfying the following conditions

$$R^\theta=\theta R+(1-\theta)\hat{R},\qquad S^\theta=\theta S+(1-\theta)\hat{S}.$$

In fact, it is easy to use above equations to recover a unique path (u^θ,u_t^θ); see (7.3). It is easy to check that the energy $\int(\tilde{R}^\theta)^2+(\tilde{S}^\theta)^2\,dx$ is bounded by the energies of (u_0,u_1) and $(\hat{u}_0,\hat{u}_1)$.

Then we choose $w=z=0$, so the norm becomes

$$\begin{aligned}\|(v^\theta,w^\theta,\hat{r}^\theta,z^\theta,\hat{s}^\theta)\|_{(u^\theta,R^\theta,S^\theta)}&=\kappa_2\int_{\mathbb{R}}\left|v^\theta\right|\left[(1+(R^\theta)^2)\,(\mathcal{W}^-)^\theta+(1+(S^\theta)^2)\,(\mathcal{W}^+)^\theta\right]dx\\&\quad+\kappa_3\int_{\mathbb{R}}\left[|r^\theta|\,(\mathcal{W}^-)^\theta+|s^\theta|\,(\mathcal{W}^+)^\theta\right]dx\\&\quad+\kappa_6\int_{\mathbb{R}}\left[\left|2R^\theta r^\theta\right|(\mathcal{W}^-)^\theta+\left|2S^\theta s^\theta\right|(\mathcal{W}^+)^\theta\right]dx.\end{aligned}\tag{7.1}$$

Now we come to estimate terms in the above equation. It is easy to see that

$$r^\theta=\frac{d}{d\theta}R^\theta=R-\hat{R},\qquad s^\theta=\frac{d}{d\theta}S^\theta=S-\hat{S}.\tag{7.2}$$

Finally, we estimate v^θ. First, by (2.1),

$$u_x^\theta=\frac{R^\theta-S^\theta}{(c_2-c_1)(x,u^\theta)}.\tag{7.3}$$

Since the right hand side is Lipschitz on u^θ and u^θ has compact support, one can easily prove the existence and uniqueness of $u^\theta(x)$. So $v^\theta=\frac{d}{d\theta}u^\theta$ satisfies

$$v_x^\theta = \frac{r^\theta - s^\theta}{(c_2 - c_1)(x, u^\theta)} - v^\theta \frac{R^\theta - S^\theta}{(c_2 - c_1)^2(x, u^\theta)}(\partial_u c_2 - \partial_u c_1)(x, u^\theta),$$

then, using (7.2), it is easy to see that

$$|v^\theta| \leq K\,(\|S - \hat{S}\|_{L^1} + \|R - \hat{R}\|_{L^1}) \tag{7.4}$$

for some constant K. Using (7.1) and (7.2)–(7.4), it is easy to prove this Proposition. □

Using the Lipschitz continuous dependence under Finsler norm, i.e. Theorem 1.1, this proposition tells that

$$d\big((u, u_t)(t), (\hat{u}, \hat{u}_t)(t)\big) \leq C\Big(\|u_0 - \hat{u}_0\|_{H^1} + \|u_0 - \hat{u}_0\|_{W^{1,1}} + \|u_1 - \hat{u}_1\|_{L^1} + \|u_1 - \hat{u}_1\|_{L^2}\Big)$$

for any $t \geq 0$. The path used in the proof of Proposition 7.1 is totally different from the one used before in [4], because in the general case we lose the special structure that variational wave equation holds. The following proposition can be proved in a similar way as in [4], we omit it here for brevity.

Proposition 7.2. *For any solutions $u(t), \hat{u}(t)$ of system (1.4) with initial data $u_0, \hat{u}_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$ and $u_1, \hat{u}_1 \in L^2(\mathbb{R})$, there exists some constant C depends only on the upper bound for the total energy, such that,*

- (Comparison with L^1 metric)

$$\|u - \hat{u}\|_{L^1} \leq C \cdot d\Big((u, u_t)(t), (\hat{u}, \hat{u}_t)(t)\Big).$$

- (Comparison with the Kantorovich-Rubinstein metric)

$$\sup_{\|f\|_{C^1} \leq 1} \left| \int f\, d\mu - \int f\, d\hat{\mu} \right| \leq C \cdot d\Big((u, u_t)(t), (\hat{u}, \hat{u}_t)(t)\Big), \tag{7.5}$$

where $\mu, \hat{\mu}$ are the measures with densities $\alpha^2(x, u)u_t^2 + \gamma^2(x, u)u_x^2$ and $\alpha^2(\hat{x}, \hat{u})\hat{u}_t^2 + \gamma^2(\hat{x}, \hat{u})\hat{u}_x^2$ with respect to the Lebesgue measure. The metric (7.5) is usually called a Kantorovich-Rubinstein distance, which is equivalent to a Wasserstein distance by a duality theorem [16].

Data availability

No data was used for the research described in the article.

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Appendix A

Now, we give the proof of Theorem 1.2 and the existence of generic regular path to make this paper self-contained. The proof in the Appendix is very similar to the proof in [3].

A.1. Proof of Theorem 1.2

To recover the singularities of the solution $u = u(x,t)$ of (1.4) in the original (x,t) plane, we will use Lemma 5.3 together with transversality argument (cf. [3,12]) to study the smooth solutions to the semi-linear (5.3)–(5.7), and hence determine the generic structure of the level sets $\{(X,Y);\ h(X,Y) = 0\}$ and $\{(X,Y);\ g(X,Y) = 0\}$. One can prove the following lemma in a very similar method as in [3], we omit it here for brevity.

Lemma A.1. *Assume the generic condition (1.8) holds. Consider a compact domain of the form*

$$\Omega := \{(X,Y);\ |X| \le M, \quad |Y| \le M\},$$

and denote $\mathcal{S}$ be the family of all $\mathcal{C}^2$ solutions (u,ℓ,m,h,g,p,q,x) to the semi-linear system (5.3)–(5.6), with $p,q>0$ for all $(X,Y)\in\mathbb{R}^2$. Moreover, denote $\mathcal{S}'\subset\mathcal{S}$ be the subfamily of all solutions (u,ℓ,m,h,g,p,q,x), such that for $(X,Y)\in\Omega$, none of the following values is attained:

$$\begin{cases}(h,\ell_X,\ell_{XX})=(0,0,0),\\(g,m_Y,m_{YY})=(0,0,0),\end{cases}\quad \begin{cases}(h,g,\ell_X)=(0,0,0),\\(h,g,m_Y)=(0,0,0),\end{cases}\quad \begin{cases}(h,\partial_u\lambda_-,\ell_X)=(0,0,0),\\(g,\partial_u\lambda_+,m_Y)=(0,0,0).\end{cases}\tag{A.1}$$

Then $\mathcal{S}'$ is a relatively open and dense subset of $\mathcal{S}$, in the topology induced by $\mathcal{C}^2(\Omega)$.

Now we introduce a new space

$$\mathcal{N} := \Big(\mathcal{C}^3(\mathbb{R})\cap H^1(\mathbb{R})\Big)\times\Big(\mathcal{C}^2(\mathbb{R})\cap L^2(\mathbb{R})\Big),$$

equipped with the norm

$$\|(u_0,u_1)\|_{\mathcal{N}} := \|u_0\|_{\mathcal{C}^3}+\|u_0\|_{H^1}+\|u_1\|_{\mathcal{C}^2}+\|u_1\|_{L^2}.$$

Applying a standard comparison argument, we deduce that, if the initial data $(u_0,u_1)\in\mathcal{N}$, then the corresponding solution remains smooth for all $|x|$ sufficiently large. The proof of this lemma is similar to [3], and we omit it here for brevity.

Lemma A.2. *Assume $(u_0,u_1)\in\mathcal{N}$ and let $T>0$ be given. Then there exists $r>0$ sufficiently large so that the solution $u=u(x,t)$ of (1.4)–(1.5) remains $\mathcal{C}^2$ on the domain $\{(x,t);\ t\in[0,T],\ |x|\ge r\}$.*

With the help of Lemma A.1, we can now prove the generic regularity of conservative solutions to (1.4)–(1.6) of the Theorem 1.2.

Proof of Theorem 1.2. Let the initial data $(\hat{u}_0, \hat{u}_1) \in \mathcal{N}$ be given and set the open ball

$$B_\delta := \{(u_0, u_1) \in \mathcal{N};\ \|(u_0, u_1) - (\hat{u}_0, \hat{u}_1)\|_{\mathcal{N}} < \delta\}.$$

To prove our main theorem, it suffices to prove that, for any $(\hat{u}_0, \hat{u}_1) \in \mathcal{N}$, there exists an open dense subset $\widehat{\mathcal{M}} \subset B_\delta$, such that, for every initial data $(u_0, u_1) \in \widehat{\mathcal{M}}$, the conservative solution $u = u(x,t)$ of (1.4)–(1.6) is twice continuously differentiable in the complement of finitely many characteristic curves, within the domain $\mathbb{R} \times [0, T]$. We prove this result by two steps.

(1). (Construction of an open dense set $\widehat{\mathcal{M}}$) Since $(\hat{u}_0, \hat{u}_1) \in \mathcal{N}$, in view of Lemma A.2, we can choose $r > 0$ large enough so that the corresponding functions $\hat{R}$, $\hat{S}$ in (2.1) being uniformly bounded on the domain of the form $\{(x,t); t \in [0,T], |x| \geq r\}$. In particular, we can choose $\delta > 0$, such that, for initial data $(u_0, u_1) \in B_\delta$, the corresponding solution $u = u(x,t)$ of (1.4) being twice continuously differentiable on the outer domain $\{(x,t);\ t \in [0,T], |x| \geq \varrho\}$, for some $\varrho > 0$ sufficiently large. This means the singularities of $u(x,t)$ in the set $\mathbb{R} \times [0, T]$ only appear on the compact set

$$\mathcal{U} := [-\varrho, \varrho] \times [0, T].$$

Denote $\mathcal{F}$ be the map of $(X,Y) \mapsto \mathcal{F}(X,Y) := (x(X,Y), t(X,Y))$. Then, we can easily obtain the inclusion $\mathcal{U} \subset \mathcal{F}(\Omega)$ by choosing M large enough and by possibly shrinking the radius δ, where Ω is a domain defined in Lemma A.1.

Now, we defined the subset $\widehat{\mathcal{M}} \subset B_\delta$ as follows: $(u_0, u_1) \in \widehat{\mathcal{M}}$ if the following items are satisfied

(I). $(u_0, u_1) \in B_\delta$;

(II). for any (X,Y) such that $(x(X,Y), t(X,Y)) \in \mathcal{U}$, the values (A.1) are never attained, here (u, ℓ, m, h, g, p, q) is the corresponding solution of (5.3)–(5.6) with boundary data (5.8).

We claim the set $\widehat{\mathcal{M}}$ is open and dense in B_δ, we omit the detailed proof here for brevity, since a similar procedure of this result can be found in [3].

(2). (u is piecewise smooth) Now, it remains to verify that for every initial data $(u_0, u_1) \in \widehat{\mathcal{M}}$, the corresponding solution $u(x,t)$ of (1.4) is piecewise $\mathcal{C}^2$ on the domain $[0,T] \times \mathbb{R}$. Toward this goal, we recall that $u(x,t)$ is $\mathcal{C}^2$ on the outer domain $\{(x,t);\ t \in [0,T], |x| \geq \varrho\}$, so in the following we just need to consider the singularities of solution $u(x,t)$ on the inner domain $\mathcal{U}$. Recall the inclusion $\mathcal{U} \subset \mathcal{F}(\Omega)$, we know that, for every point $(X_0, Y_0) \in \Omega$, there are two cases:

CASE 1. If $h(X_0, Y_0) \neq 0$ and $g(X_0, Y_0) \neq 0$, we can obtain the determinant of the Jacobian matrix

$$\det \begin{pmatrix} x_X & x_Y \\ t_X & t_Y \end{pmatrix} = \frac{\alpha p q h g}{c_2 - c_1} > 0,$$

by $(5.6)_2$ and (5.7). This implies that the map $(X,Y) \mapsto (x,t)$ is locally invertible in a neighborhood of (X_0, Y_0). The solution $u(x,t)$ is $\mathcal{C}^2$ in a neighborhood of $(x(X_0,Y_0), t(X_0,Y_0))$.

CASE 2. If $h(X_0, Y_0) = 0$, we can obtain $\ell = 0$, immediately. In this case, we claim that either $\ell_X \neq 0$ or $\ell_Y \neq 0$. In fact, by the equation (5.3) and the definition of a_1 in (2.2), at the point (X_0, Y_0), we have

$$\ell_Y(X_0, Y_0) = -\frac{q}{c_2 - c_1} a_1 g = \frac{\alpha q g}{2(c_2 - c_1)^2} \partial_u \lambda_-.$$

This together with the construction of $\widehat{\mathcal{M}}$ that the values $(h, g, \ell_X) = (0, 0, 0)$ and $(h, \partial_u\lambda_-, \ell_X) = (0, 0, 0)$ are never attained in Ω, it is easy to see that $\ell_X \neq 0$ or $\ell_Y \neq 0$.

By continuity, we can choose $\eta > 0$, so that in the open neighborhood

$$\Omega' := \{(X, Y);\ |X| < M + \eta, |Y| < M + \eta\},$$

the values listed in (A.1) are never attained. Applying the implicit function theorem, we derive that the sets

$$\chi^h := \{(X, Y) \in \Omega';\ h(X, Y) = 0\}, \quad \chi^g := \{(X, Y) \in \Omega';\ g(X, Y) = 0\}$$

are 1–dimensional embedded manifold of class $\mathcal{C}^2$. In particular, the set $\chi^h \cap \Omega$ has finite connected components. Indeed, assume on the contrary that there exists a sequence of points $P_1, P_2, \cdots \in \chi^h \cap \Omega$ belonging to distinct components. Then we can choose a subsequence, denote still by P_i, such that $P_i \to \bar{P}$ for some $\bar{P} \in \chi^h \cap \Omega$. Since $h(\bar{P}) = 0$, then $(\ell_X, \ell_Y)(\bar{P}) \neq (0, 0)$, which together with the implicit function theorem implies that there is a neighborhood Γ of $\bar{P}$ such that $\chi^h \cap \Gamma$ is a connected $\mathcal{C}^2$ curve. Thus, $P_i \in \chi^h \cap \Gamma$ for all i large enough, providing a contradiction on the assumption that P_i belongs to distinct components.

To complete the proof, we need to study more details on the image of the singular sets χ^h and χ^g, since the set for the singular points (t, x) of u coincides with the image of the two sets χ^h, χ^g under the C^2 map $(X, Y) \mapsto \mathcal{F}(X, Y) = (x(X, Y), t(X, Y))$.

By the previous argument, there are only finite many points $P_i = (X_i, Y_i), i = 1, \cdots, m$, inside set Ω', where $h = 0, \ell = 0$, and $\ell_X = 0$. Moreover, by (A.1), at a point $(X_0, Y_0) \in \chi^h \cap \chi^g$, we have $\ell_X \neq 0, \ell_Y = 0, m_X = 0, m_Y \neq 0$. Thus, the two curves $h = 0$ and $g = 0$ intersect perpendicularly. Therefore, there are only finitely many such intersection points $Q_J = (X'_J, Y'_J), J = 1, \cdots, n$, inside the compact set Ω.

Moreover, the set $\chi^h \backslash \{P_1, \cdots, P_m, Q_1, \cdots, Q_n\}$ has finitely many connected components which intersect Ω. Consider any one of these components, which is a connected curve, say γ_j, such that $h = 0, \ell = 0$ and $\ell_X \neq 0$ for any $(X, Y) \in \gamma_j$. Thus, for a suitable function φ_j, this curve can be expressed as

$$\gamma_j = \{(X, Y) :\ X = \varphi_j(Y), a_j < Y < b_j\}.$$

We claim that the image $\Lambda(\gamma_j)$ is a $\mathcal{C}^2$ curve in the x-t plane. Indeed, on the open interval (a_j, b_j), the differential of the map $Y \mapsto \big(x(\varphi_j(Y), Y), t(\varphi_j(Y), Y)\big)$ does not vanish. This is true, because by (5.7), we have

$$\frac{d}{dY} t(\varphi_j(Y), Y)) = t_X \varphi'_j + t_Y = 0 \cdot \varphi'_j + \frac{\alpha q g}{c_2 - c_1} > 0,$$

since $g, c_2 - c_1, q > 0$. As a consequence, the singular set $\mathcal{F}(\chi^h)$ is the union of the finitely points $p_i = \mathcal{F}(P_i), i = 1, \cdots, m$, $q_J = \mathcal{F}(Q_J), J = 1, \cdots, n$, together with finitely many $\mathcal{C}^2$-curve $\mathcal{F}(\gamma_j)$. Obviously, the same representation is valid for the image $\mathcal{F}(\chi^g)$. This completes the proof of Theorem 1.2. $\square$

Finally, we prove the existence of regular enough path of generic solutions between any two generic solutions, in order to define the tangent vectors in the norm (4.9).

Definition A.1. We say that a path of initial data $\gamma^0 : \theta \mapsto (u_0^\theta, u_1^\theta)$, $\theta \in [0, 1]$ is a **piecewise regular path** if the following conditions hold.

(i) There exists a continuous map $(X, Y, \theta) \mapsto (u, \ell, m, h, g, p, q, x, t)$ such that the semilinear system (5.3)–(5.7) holds for $\theta \in [0, 1]$, and the function $u^\theta(x, t)$ whose graph is

$$\text{Graph } (u^\theta) = \{(x, t, u)(X, Y, \theta);\ (X, Y) \in \mathbb{R}^2\}$$

provides the conservation solution of (1.4) with initial data $u^\theta(x, 0) = u_0^\theta(x)$, $u_t^\theta(x, 0) = u_1^\theta(x)$.

(ii) There exist finitely many values $0 = \theta_0 < \theta_1 < \cdots < \theta_N = 1$ such that the map $(X, Y, \theta) \mapsto (u, \ell, m, h, g, p, q, x, t)$ is $\mathcal{C}^\infty$ for $\theta \in (\theta_{i-1}, \theta_i)$, $i = 1, \cdots, N$, and the solution $u^\theta = u^\theta(x, t)$ has only generic singularities at time $t = 0$.

In addition, if for all $\theta \in [0, 1]\backslash\{\theta_1, \cdots, \theta_N\}$, the solution u^θ has only generic singularities for $t \in [0, T]$, then we say that the path of solution $\gamma^t : \theta \mapsto (u^\theta, u_t^\theta)$ is **piecewise regular** for $t \in [0, T]$.

Towards our goal, we state the following result, which is an application of Theorem 1.2. The proof of this result is similar to [3], and we omit it here for brevity.

Theorem A.1. *Assume the generic condition* (1.8) *holds. For any fixed* $T > 0$*, let* $\theta \mapsto (u^\theta, \ell^\theta, m^\theta, h^\theta, g^\theta, p^\theta, q^\theta, x^\theta, t^\theta), \theta \in [0, 1]$*, be a smooth path of solutions to the system* (5.3)–(5.7)*. Then there exists a sequence of paths of solutions* $\theta \mapsto (u_n^\theta, \ell_n^\theta, m_n^\theta, h_n^\theta, g_n^\theta, p_n^\theta, q_n^\theta, x_n^\theta, t_n^\theta)$*, such that*

(i) *For each* $n \geq 1$*, the path of the corresponding solution of* (1.4) $\theta \mapsto u_n^\theta$ *is regular for* $t \in [0, T]$ *in the sense of Definition A.1.*

(ii) *For any bounded domain* Σ *in the* (X, Y) *space, functions* $(u_n^\theta, \ell_n^\theta, m_n^\theta, h_n^\theta, g_n^\theta, p_n^\theta, q_n^\theta, x_n^\theta, t_n^\theta)$ *converge to* $(u^\theta, \ell^\theta, m^\theta, h^\theta, g^\theta, p^\theta, q^\theta, x^\theta, t^\theta)$ *uniformly in* $\mathcal{C}^k([0, 1] \times \Sigma)$*, for every* $k \geq 1$*, as* $n \to \infty$.

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