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Discretizing L_p norms and frame theory ☆

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ABSTRACT

Given an N-dimensional subspace X^N of $L_p(\Omega)$, we consider the problem of choosing M-sampling points which may be used to discretely approximate the L_p norm on the subspace. We are particularly interested in knowing when the number of sampling points M can be chosen on the order of the dimension N. For the case $p = 2$ it is known that M may always be chosen on the order of N as long as the subspace X^N satisfies a natural L_∞ bound, and for the case $p = \infty$ there are examples where M may not be chosen on the order of N. We show for all $1 \leq p < 2$ that there exist classes of subspaces of $L_p([0, 1])$ which satisfy the L_∞ bound, but where the number of sampling points M cannot be chosen on the order of N. We show as well that the problem of discretizing the L_p norm of subspaces is directly connected with frame theory. In particular, we prove that discretizing a continuous frame to obtain a discrete frame which does stable phase retrieval requires discretizing both the L_2 norm and the L_1 norm on the range of the analysis operator of the continuous frame.

1. Introduction

Many problems in applied math, physics, and engineering are stated in terms of some continuous structure, but only become computationally feasible when appropriately discretized. Some examples of this are using numerical integration to estimate the integral of a function over a measure space, or using a fast Fourier transform as a discretization of the Fourier transform. We will be considering the problem of discretizing the L_p norm on finite dimensional subspaces of $L_p(\Omega)$ where $1 \leq p < \infty$ and Ω is a probability space. We let $X^N \subseteq L_p(\Omega)$ be an N-dimensional subspace and note that

$$\|f\|_p^p = \int_\Omega |f(t)|^p \, d\mu(t) \qquad \text{for all } f \in X^N. \tag{1.1}$$

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Given some $A < 1 < B$, we are interested in the problem of discretizing (1.1) by choosing sampling points $(t_j)_{j=1}^M \subseteq \Omega$ such that

$$A\|f\|_p^p \leq \frac{1}{M}\sum_{j=1}^{M}|f(t_j)|^p \leq B\|f\|_p^p \qquad \text{for all } f \in X^N. \tag{1.2}$$

As X^N is finite dimensional, the law of large numbers gives that if $M \in \mathbb{N}$ is large enough and $(t_j)_{j=1}^M \subseteq \Omega$ are chosen randomly and independently then (1.2) is satisfied with probability arbitrarily close to one. We provide a proof of this result in the appendix. However, the law of large numbers does not give any bounds on how large M must be for a given $X^N \subseteq L_p(\Omega)$, and as one of our motivations is to make a problem computationally feasible, it is of fundamental importance to obtain good bounds on the number of sampling points M. This problem was first considered by Marcinkiewicz and later by Zygmund for the case of discretizing the L_p norm on spaces of trigonometric polynomials [22][27]. Because of this, modern papers on the subject often refer to this class of problems as Marcinkiewicz-type discretization problems. It has been recently proven that for all $1 \leq p < \infty$ there are certain entropy conditions on $X^N \subseteq L_p(\Omega)$ which guarantee that the L_p-norm on X^N can be discretized to satisfy (1.2) using M on the order of $N(\log(N))^2$ sampling points for $1 \leq p < \infty$ [10],[11],[19],[26]. These entropy conditions can be fairly technical, but they imply in particular that the subspace satisfies a $(2,\infty)$-Nikol'skii-type inequality, which implies a (p,∞)-Nikol'skii-type inequality for $1 \leq p < 2$. Here, we say that an N-dimensional subspace $X^N \subseteq L_p(\Omega)$ satisfies a (p,∞)-Nikol'skii-type inequality for some $1 \leq p < \infty$ if there exists $\beta > 0$ such that

$$\|x\|_{L_\infty} \leq \beta N^{1/p}\|x\|_{L_p} \qquad \text{for all } x \in X^N. \tag{1.3}$$

Having $X^N \subseteq L_p(\Omega)$ satisfy inequality (1.3) is a very natural Banach space condition as it is equivalent to for all $t \in \Omega$ point evaluation at t is a bounded linear functional on X^N with norm at most $\beta N^{1/p}$. For the case $p = 2$, the celebrated solution to the Kadison-Singer problem [23] can be applied to show that if Ω is a probability space and $X^N \subseteq L_2(\Omega)$ satisfies the inequality (1.3) then the L_2-norm on X^N can be discretized using M on the order of N sampling points [21]. The main theorem in [23] was first used in discretization to prove that if $\Omega \subseteq \mathbb{R}$ is a subset with finite measure then $L_2(\Omega)$ has a frame of exponentials [24], and was later used to prove that if $(x_t)_{t\in\Omega}$ is a bounded continuous frame of a Hilbert space H then there is a sampling $(t_j)_{j\in J} \subseteq \Omega$ such that $(x_{t_j})_{j\in J}$ is a frame of H [17].

It was not previously known that if $X^N \subseteq L_p(\Omega)$ satisfied the Nikol'skii-type inequality (1.3) for some $1 \leq p < \infty$ with $p \neq 2$ then the L_p-norm on X^N could be discretized using M on the order of N sampling points. In Section 2 we give a construction of a class of subspaces $X^N \subseteq L_1(\Omega)$ which uniformly satisfy (1.3) but the L_1-norm on X^N cannot be discretized using M on the order of N sampling points, and in Section 3 we give for each $1 \leq p < 2$ a construction of a class of subspaces $X^N \subseteq L_p(\Omega)$ which uniformly satisfy (1.3) but the L_p-norm on X^N cannot be discretized using M on the order of N sampling points. These subspaces of $L_p(\Omega)$ which we construct in Section 3 are all uniformly isomorphic to Hilbert spaces. For the case $2 < p < \infty$, it is shown in section D20 of [18] that if $X^N \subseteq L_p(\Omega)$ are uniformly isomorphic to Hilbert spaces then the L_p-norm on X^N cannot be discretized using M on the order of N sampling points. Furthermore, for all $2 < p < \infty$, there exist constructions of subspaces of $X^N \subseteq L_p(\Omega)$ which are uniformly isomorphic to Hilbert spaces and which uniformly satisfy the $(2,\infty)$-Nikol'skii-type inequality. We complement these results by proving for all $2 < p < \infty$ that if $X^N \subseteq L_p(\Omega)$ are uniformly isomorphic to Hilbert spaces then they cannot uniformly satisfy the (p,∞)-Nikol'skii-type inequality.

In Section 4 we introduce the topics of frame theory and phase retrieval, and then show the important connection between them and norm discretization of subspaces of L_p. In particular, we prove that discretizing a continuous frame to obtain a discrete frame which does stable phase retrieval requires discretizing both the L_2 norm and the L_1 norm on the range of the analysis operator of the continuous frame.

We explain in Section 4 that the solution to the discretization problem for continuous frames [17] implies that there exist constants $A, B > 0$ such that if Ω is any σ-finite measure space and $X \subseteq L_2(\Omega)$ is a subspace such that there exists $\beta > 0$ with $\|x\|_{L_\infty} \leq \beta\|x\|_{L_2}$ for all $x \in X$ then there exists sampling points $(t_j)_{j\in J} \subseteq \Omega$ such that $\beta^2 A\|x\|_{L_2} \leq \sum_{j\in J} |x(t_j)|^2 \leq \beta^2 B\|x\|_{L_2}$ for all $x \in X$. In Section 5 we use the finite dimensional results in Sections 2 and 3 to prove that the corresponding result strongly fails for $1 \leq p < 2$. That is, for all $1 \leq p < 2$ there exists a subspace $X \subseteq L_p(\mathbb{R})$ with X isomorphic to ℓ_p and $\|x\|_{L_\infty} \leq \|x\|_{L_p}$ for all $x \in X$ such that if $(t_j)_{j\in J} \subseteq \mathbb{R}$ is such that $\sum_{j\in J} |x(t_j)|^p > 0$ for all $x \in X \setminus \{0\}$ then there exists $y \in X$ with $\sum_{j\in J} |y(t_j)|^p = \infty$.

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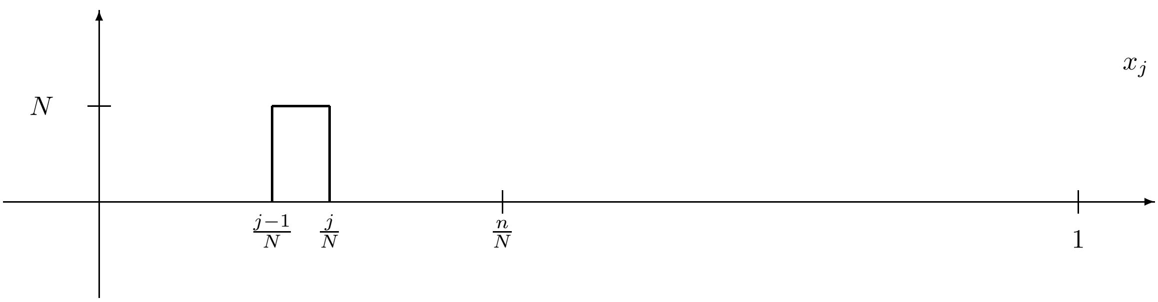
2. Constructing subspaces of L_1

In this section we will show how to construct subspaces of L_1 where the L_1-norm cannot be discretized using a number of sampling points on the order of the dimension of the subspace. In particular, for all $\varepsilon > 0$, we construct a class of subspaces $X^N \subseteq L_1[0,1]$ of the form $X^N = T_N(\text{span}(\mathbb{1}_{[(j-1)/N, j/N)})_{j=1}^N)$ where $T_N : L_1[0,1] \to L_1[0,1]$ is a linear operator such that $\|I_{L_1[0,1]} - T_N\| < \varepsilon$ (where $I_{L_1[0,1]}$ is the identity operator on $L_1[0,1]$). The subspace $X^N \subseteq L_1[0,1]$ that we construct satisfies $\|f\|_{L_\infty} \leq (1+\varepsilon)N\|f\|_{L_1}$ for all $f \in X^N$ and yet the L_1-norm on X^N cannot be discretized using $M = o(N\log(N)/(\log\log(N))$ sampling points. That is, we can consider the simplest N-dimensional subspace of $L_1[0,1]$ and perturb it an arbitrarily small amount to create a subspace which still satisfies the boundedness condition (1.3) and yet the subspace cannot be discretized using $M = o(N\log(N)/(\log\log(N))$ sampling points.

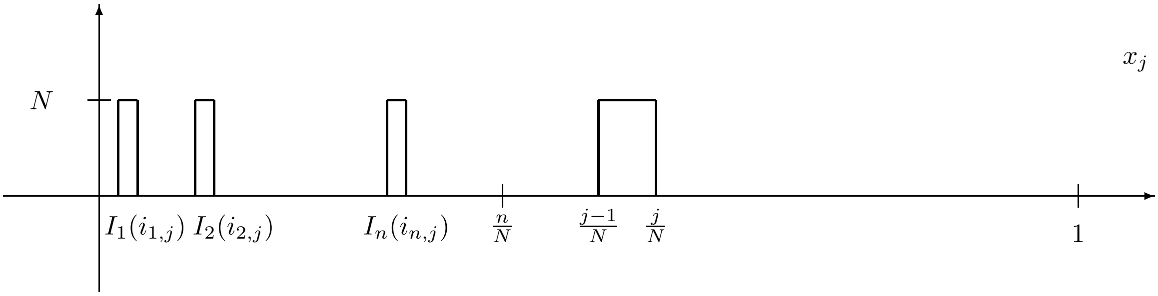
We now describe how to construct the subspace $X^N \subseteq L_1[0,1]$. Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Without loss of generality, we may assume that $1/\varepsilon$ is an integer and let $N = n + (\varepsilon^{-1}n)^n$. We will construct a basis $(x_j)_{j=1}^N$ of X^N which will be a perturbation of the sequence of indicator functions $(\mathbb{1}_{[(j-1)/N, j/N)})_{j=1}^N$. For $1 \leq j \leq n$ we let,

$$x_j = N\mathbb{1}_{[(j-1)/N, j/N)}.$$



For each $1 \leq k \leq n$ we partition the interval $[(k-1)/N, k/N)$ into $\varepsilon^{-1}n$ intervals $(I_k(i))_{i=1}^{\varepsilon^{-1}n}$ each of width $\varepsilon n^{-1}N^{-1}$. As $N = n + (\varepsilon^{-1}n)^n$, we can now enumerate $\{1, 2, ..., \varepsilon^{-1}n\}^n$ as $((i_{k,j})_{k=1}^n)_{j=n+1}^N$. For $n < j \leq N$ we let,

$$x_j = N\sum_{k=1}^{n} \mathbb{1}_{I_k(i_{k,j})} + N\mathbb{1}_{[\frac{j-1}{N}, \frac{j}{N})}.$$



We have created a sequence $(x_j)_{j=1}^N \subseteq L_1[0,1]$, and we now prove that it satisfies the following theorem.

Theorem 2.1. *Let $n \in \mathbb{N}$ and $1 > \varepsilon > 0$ with $\varepsilon^{-1} \in \mathbb{N}$. Consider $N = n + (\varepsilon^{-1}n)^n$ and $X^N \subseteq L_1[0,1]$ with basis $(x_j)_{j=1}^N$ as defined above. Then the following holds.*

(1) There is a linear map $T_N : L_1[0,1] \to L_1[0,1]$ with $T_N(N\mathbb{1}_{[(j-1)/N,j/N)}) = x_j$ for all $1 \le j \le N$ such that $\|I_{L_1[0,1]} - T_N\| \le \varepsilon$, where $I_{L_1[0,1]}$ is the identity operator.
(2) The basis $(x_j)_{j=1}^N$ of X^N satisfies that

$$(1-\varepsilon)\sum_{j=1}^N |a_j| \le \Big\|\sum_{j=1}^N a_j x_j\Big\|_{L_1} \le (1+\varepsilon)\sum_{j=1}^N |a_j| \quad \text{for all } (a_j)_{j=1}^N \in \ell_1^N,$$

(3) For all $x \in X^N$, $\|x\|_{L_\infty[0,1]} \le (1-\varepsilon)^{-1}N\|x\|_{L_1[0,1]}$.
(4) Let $M \in \mathbb{N}$ so that $(t_j)_{j=1}^M \subseteq [0,1]$ and $\sum |x(t_j)| \ne 0$ for all $x \in X^N \setminus \{0\}$. Then there exists $x \in X^N$ with

$$\frac{nN}{(1+\varepsilon)M}\|x\|_{L_1} \le \frac{1}{M}\sum_{j=1}^M |x(t_j)|$$

In particular, it follows from (4) that the L_1 norm on X^N cannot be discretized using M on the order of N sampling points. Furthermore, the L_1 norm on X^N cannot be discretized using $M = o(N\log(N)/(\log\log(N))$ sampling points. However, M can be chosen on the order of $N\log(N)/(\log\log(N))$ so that there exists sampling points $(t_j)_{j=1}^M \subseteq [0,1]$ such that $\|x\|_{L_1} = \frac{1}{M}\sum_{j=1}^M |x(t_j)|$ for all $x \in X^N$.

Proof. Consider the linear map $P_N : L_1[0,1] \to L_1[0,1]$ given by $P_N = \sum_{j=1}^N \mathbb{E}_{[(j-1)/N,j/N)}$ where $\mathbb{E}_{[(j-1)/N,j/N)}$ is conditional expectation on $[(j-1)/N, j/N)$. Note that P_N is a projection onto the subspace $\mathrm{span}(\mathbb{1}_{[(j-1)/N,j/N)})_{j=1}^N$ with $\|P_N\| = 1$. We define the operator $S_N : \mathrm{span}(\mathbb{1}_{[(j-1)/N,j/N)})_{j=1}^N \to L_1[0,1]$ by $S_N(\sum_{j=1}^N a_j N\mathbb{1}_{[(j-1)/N,j/N)}) = \sum_{j=1}^N a_j x_j$. We now let $T_N = (I_{L_1[0,1]} - P_N) + S_N P_N$. We have the following estimate.

$$\begin{aligned}
\|I_{L_1[0,1]} - T_N\| &= \|S_N P_N - P_N\| \\
&= \sup_{\sum a_j = 1} \Big\|\sum_{j=1}^N a_j x_j - \sum_{j=1}^N a_j N\mathbb{1}_{[(j-1)/N,j/N)}\Big\|_{L_1} \\
&\le \sup_{\sum a_j = 1} \sum_{j=n+1}^N \big\|a_j x_j - a_j N\mathbb{1}_{[(j-1)/N,j/N)}\big\|_{L_1} \\
&= \sup_{\sum a_j = 1} \sum_{j=n+1}^N a_j N\Big\|\sum_{k=1}^n \mathbb{1}_{I_k(i_{k,j})}\Big\|_{L_1}
\end{aligned}$$

$$= \varepsilon \qquad \text{(because } I_k(i_{k,j}) \text{ has length } \varepsilon n^{-1}N^{-1} \text{ for all } 1 \le k \le n).$$

This shows that $\|I_{L_1[0,1]} - T_N\| \le \varepsilon$ which proves (1). We have that $\|x_j\|_{L_1} \le 1+\varepsilon$ for all $1 \le j \le N$ and hence $\| \sum_{j=1}^N a_j x_j\|_{L_1} \le (1+\varepsilon)\sum_{j=1}^N |a_j|$ for all $(a_j)_{j=1}^N \in \ell_1^N$. We now consider a fixed $(a_j)_{j=1}^N \in \ell_1^N$ with $\sum_{j=1}^N |a_j| = 1$.

$$\begin{aligned}
\Big\| \sum_{j=1}^N a_j x_j \Big\|_{L_1} &= \Big\| \sum_{j=1}^N a_j N \mathbb{1}_{[(j-1)/N, j/N)} - a_j (N \mathbb{1}_{[(j-1)/N, j/N)} - x_j) \Big\|_{L_1} \\
&\ge \sum_{j=1}^N |a_j| - \sum_{j=1}^N |a_j| \big\| (I_{L_1[0,1]} - T_N) N \mathbb{1}_{[(j-1)/N, j/N)} \big\|_{L_1} \\
&\ge \sum_{j=1}^N |a_j| - \|I_{L_1[0,1]} - T_N\| \sum_{j=1}^N |a_j| \\
&= 1 - \varepsilon
\end{aligned}$$

Thus, we have proven (2) by showing that

$$(1-\varepsilon)\sum_{j=1}^N |a_j| \le \Big\| \sum_{j=1}^N a_j x_j \Big\|_{L_1} \le (1+\varepsilon)\sum_{j=1}^N |a_j| \text{ for all } (a_j)_{j=1}^N \in \ell_1^N.$$

Note that $\|x_j\|_{L_\infty} = N$ for all $1 \le j \le N$. Thus for all $(a_j)_{j=1}^N \in \ell_1^N$ we have by (2) that

$$\Big\| \sum_{j=1}^N a_j x_j \Big\|_{L_\infty} \le \sum_{j=1}^N |a_j| N \le (1-\varepsilon)^{-1} N \Big\| \sum_{j=1}^N a_j x_j \Big\|_{L_1}.$$

Thus we have proven (3).

We now let $M \in \mathbb{N}$ and $(t_k)_{k=1}^M \subseteq [0,1]$ with $\sum |x(t_k)| \neq 0$ for all $x \in X^N$. Hence, after reordering $(t_k)_{k=1}^M$ we may assume without loss of generality that $|x_k(t_k)| = N$ for all $1 \le k \le n$. There exists unique $n < j \le N$ such that $t_k \in I_k(i_{k,j})$ for all $1 \le k \le n$. Thus, we have that

$$\sum_{k=1}^M |x_j(t_k)| \ge \sum_{k=1}^n N \mathbb{1}_{I_k(i_{k,j})}(t_k) = nN.$$

As $\|x_j\|_{L_1} \le 1+\varepsilon$, this proves (4). To prove that the L_1 norm on X^N cannot be discretized using $M = o(N\log(N)/(\log\log(N))$ sampling points it follows from (4) that we need to prove that there exists $C > 0$ and $n_0 \in \mathbb{N}$ such that $N\log(N)/(\log\log(N) \le CNn$ for all $n \ge n_0$. We have that $N = n + (\varepsilon^{-1}n)^n$. Thus if $n \in \mathbb{N}$ is large enough then $N \le n^{2n}$ and hence $\log(N) \le 2n\log(n)$. On the other hand, $N \ge (\varepsilon^{-1}n)^n$ and hence $\log(N) \ge n$ which implies that $\log\log(N) \ge \log(n)$. Thus we have if $n \in \mathbb{N}$ is large enough then

$$2Nn \ge \frac{N\log(N)}{\log(n)} \ge \frac{N\log(N)}{\log\log(N)}.$$

We now prove that the L_1 norm on X^N can be perfectly discretized using M on the order of $N\log(N)/(\log\log(N)$ sampling points. For $n \in \mathbb{N}$ we let $M = \varepsilon^{-1}nN$ and $t_j = \frac{j-1}{M}$. As each $x \in X^N$ is constant on the interval $[\frac{j-1}{M}, \frac{j}{M})$ for all $1 \le j \le M$, we have that $\|x\|_{L_1} = \frac{1}{M}\sum_{j=1}^M |x(t_j)|$ for all $x \in X^N$. Thus we just need to prove that there exists $C > 0$ and $n_0 \in \mathbb{N}$ such that $CN\log(N)/\log\log(N) \ge Nn$ for all $n \ge n_0$.

As $N = n + (\varepsilon^{-1}n)^n$ we have that $N \geq n^n$ and hence $\log(N) \geq n\log(n)$. On the other hand, if $n \in \mathbb{N}$ is large enough then $\log(N) \leq n^2$ and hence $\log\log(N) \leq 2\log(n)$. Thus we have if $n \in \mathbb{N}$ is large enough then

$$Nn \leq \frac{N\log(N)}{\log(n)} \leq \frac{2N\log(N)}{\log\log(N)}. \quad \square$$

3. Constructing subspaces of L_p for $1 \leq p < 2$

In section 2 we constructed subspaces $X^N \subseteq L_1[0,1]$ which satisfied the Nikol'skii-type inequality $\|x\|_{L_\infty} \leq (1+\varepsilon)\|x\|_{L_1}$ for all $x \in X^N$ and yet the L_1 norm on X^N cannot be discretized using $M = o(N\log(N)/\log\log(N))$ sampling points. That construction only works for L_1, but we now introduce a different method which works for L_p when $1 \leq p < 2$.

For $n \in \mathbb{N}$, the nth Rademacher function R_n on $[0,1]$ is given by

$$R_n = \sum_{j=1}^{2^n} (-1)^j \mathbb{1}_{[(j-1)/2^n, j/2^n)}.$$

Note that $(R_n)_{n=1}^\infty$ is an independent sequence of mean-zero ± 1 random variables on $[0,1]$. This sequence is very useful when using probabilistic techniques in the geometry of Banach spaces, and we will rely on the following theorem.

Theorem 3.1 *(Khintchine's Inequality). For all $1 \leq p < \infty$ there exist constants $0 < A_p \leq B_p$ such that if $(R_j)_{j=1}^N$ is a sequence of Rademacher functions on $[0,1]$ then for all $N \in \mathbb{N}$ and all scalars $(a_j)_{j=1}^N$, we have that*

$$A_p\Big(\sum_{j=1}^N |a_j|^2\Big)^{1/2} \leq \Big(\int |\sum_{j=1}^N a_j R_j(s)|^p ds\Big)^{1/p} \leq B_p\Big(\sum_{j=1}^N |a_j|^2\Big)^{1/2}$$

We now use the Rademacher functions and Khintchine's inequality to build a class of subspaces of $L_p[0,1]$ for $1 \leq p < 2$ where the L_p norm cannot be discretized using M on the order of the dimension number of sampling points.

Proposition 3.2. *Let $1 \leq p < 2$ and $N \in \mathbb{N}$. For each $1 \leq j \leq N$ let $y_j(t) = N^{1/p-1/2}R_j(N^{1-p/2}t)$ and $X^N = \mathrm{span}(y_j)_{j=1}^N$. Then,*

(1) $(y_j)_{j=1}^N$ is 1-equivalent to the Rademacher sequence $(R_j)_{j=1}^N$ in L_p.
(2) $\|x\|_{L_\infty} \leq A_p^{-1}N^{1/p}\|x\|_{L_p}$ for all $x \in X^N$ where A_p is the constant in Khintchine's inequality.
(3) Suppose that $(t_j)_{j=1}^M \subseteq [0,1]$ are such that $\sum_{j=1}^M |x(t_j)|^p > 0$ for all $x \in X^N \setminus \{0\}$. Then

$$\frac{N^{2-p/2}}{M}\|y_1\|_{L_p}^p \leq \frac{1}{M}\sum_{j=1}^M |y_1(t_j)|^p.$$

Thus, the L_p-norm on X^N cannot be discretized using M on the order of N sampling points.

Proof. Let $(a_j)_{j=1}^N \in \ell_2^N$. We have that

$$\Big\|\sum_{j=1}^N a_j y_j\Big\|_{L_p}^p = \int |\sum_{j=1}^N a_j N^{1/p-1/2}R_j(N^{1-p/2}t)|^p dt$$

$$= \int \Big|\sum_{j=1}^{N} a_j R_j(s)\Big|^p ds \qquad \text{by substituting } s = N^{1-p/2}t.$$

Thus, $(y_j)_{j=1}^N$ is 1-equivalent to the Rademacher sequence $(R_j)_{j=1}^N$ in $L_p[0,1]$. By Khintchine's inequality we have that

$$A_p\Big(\sum_{j=1}^{N} |a_j|^2\Big)^{1/2} \leq \Big\|\sum_{j=1}^{N} a_j y_j\Big\|_{L_p} \leq B_p\Big(\sum_{j=1}^{N} |a_j|^2\Big)^{1/2}. \tag{3.1}$$

Let $x = \sum_{j=1}^N a_j y_j \in X^N$. We now give an upper bound for $\|x\|_{L_\infty}$.

$$\begin{aligned}
\|x\|_{L_\infty} &= \sup_{t\in[0,1]} \Big|\sum_{j=1}^{N} a_j N^{1/p-1/2} R_j(N^{1-p/2}t)\Big| \\
&= N^{1/p}\sum_{j=1}^{N} N^{-1/2}|a_j| \\
&\leq N^{1/p}\Big(\sum_{j=1}^{N} |a_j|^2\Big)^{1/2} \qquad \text{by Cauchy-Schwartz,} \\
&\leq A_p^{-1} N^{1/p}\|x\|_{L_p} \qquad \text{by (3.1).}
\end{aligned}$$

Thus we have proven (2). We now suppose that $(t_j)_{j=1}^M \subseteq [0,1]$ are such that $\sum_{j=1}^M |x(t_j)|^p > 0$ for all $x \in X^N$. As X^N is N-dimensional and supported on $[0, N^{p/2-1}]$ we have that there exists a subset $(t_j)_{j\in J} \subseteq [0, N^{p/2-1}]$ with $|J| \geq N$. Note that, $|y_1(t_j)| = N^{1/p-1/2}$ for all $j \in J$. We thus have that,

$$\sum_{j=1}^{M} |y_1(t_j)|^p \geq |J|N^{1-p/2} \geq NN^{1-p/2} = N^{2-p/2}$$

As $\|y_1\|_{L_p} = 1$ we have proven (3). $\square$

In Proposition 3.2 we constructed for all $1 \leq p < 2$ a collection of subspaces $(X^N)_{N=1}^\infty$ of $L_p[0,1]$ where X^N is (B_p/A_p)-isomorphic to ℓ_2^N and satisfies $\|x\|_{L_\infty} \leq A_p^{-1}\|x\|_{L_p}$ for all $x \in X^N$, and yet the L_p-norm on X^N could not be discretized using M on the order of N sampling points. Our proof only worked for $1 \leq p < 2$, but it is natural to ask if the result was still true for $2 < p < \infty$. We now show that this hypothesis fails for all $2 < p$. That is, there do not exist uniform constants $C_p, D_p > 0$ such that for all $N \in \mathbb{N}$, there is a probability space (M, Σ, μ) such that ℓ_2^N is C_p-isomorphic to a subspace $X^N \subseteq L_p(\mu)$ with $\|x\|_{L_\infty(\mu)} \leq D_p N^{1/p}\|x\|_{L_\infty(\mu)}$ for all $x \in X^N$. The proof follows the classical argument in [16] which proves that for all $2 < p$ there does not exist a constant $C > 0$ so that ℓ_2^n uniformly embeds into ℓ_p^{Cn}.

Proposition 3.3. *Let $2 < p < \infty$ and $N \in \mathbb{N}$. Let $X^N \subseteq L_p(\Omega)$ where Ω is a probability space and X^N has a basis $(x_j)_{j=1}^N$ such that $\|x_j\|_{L_p(\Omega)} = 1$ and the following holds for some $A, B, \beta > 0$:*

(1) For all $(a_j)_{j=1}^N \in \ell_2^N$,

$$A\Big(\sum_{j=1}^{N} |a_j|^2\Big)^{1/2} \leq \Big\|\sum_{j=1}^{N} a_j x_j\Big\|_{L_p(\Omega)} \leq B\Big(\sum_{j=1}^{N} |a_j|^2\Big)^{1/2}$$

(2) For all $x \in X^N$,

$$\|x\|_{L_\infty(\Omega)} \leq \beta N^{1/p} \|x\|_{L_p(\Omega)}$$

Then, $\frac{A}{B\,B_p} N^{1/2-1/p} \leq \beta$ where B_p is the constant from Khintchine's inequality. In particular, there do not exist constants $A, B, \beta > 0$ such that for all $N \in \mathbb{N}$ there exists a subspace $X^N \subseteq L_p(\Omega)$ such that X^N is $A^{-1}B$-isomorphic to ℓ_2^N and $X^N \subseteq L_p(\Omega)$ satisfies the boundedness condition $\|x\|_{L_\infty(\Omega)} \leq \beta N^{1/p} \|x\|_{L_p(\Omega)}$ for all $x \in X^N$.

Proof. Let $t \in \Omega$. By (2) we have without loss of generality that

$$|x(t)| \leq \beta N^{1/p} \|x\|_{L_p(\Omega)} \qquad \text{for all } x \in X^N. \tag{3.2}$$

By scaling (1), we have the following inequality

$$A \leq \left\| \sum_{j=1}^{N} \frac{x_j(t)}{\left(\sum_{j=1}^{N} |x_j(t)|^2\right)^{1/2}} x_j \right\|_{L_p(\Omega)} \leq B$$

Now using inequality (3.2) for $x = \sum_{j=1}^{N} \frac{x_j(t)}{\left(\sum_{j=1}^{N} |x_j(t)|^2\right)^{1/2}} x_j$ we get that

$$\sum_{j=1}^{N} \frac{|x_j(t)|^2}{\left(\sum_{j=1}^{N} |x_j(t)|^2\right)^{1/2}} \leq \beta N^{1/p} B$$

Therefore,

$$\left(\sum_{j=1}^{N} |x_j(t)|^2\right)^{1/2} \leq \beta N^{1/p} B \tag{3.3}$$

Let $(R_j)_{j=1}^{\infty}$ be the sequence of Rademacher functions on $[0,1]$. For $s \in [0,1]$ we have by (1) that

$$A^p N^{p/2} \leq \left\| \sum_{j=1}^{N} R_j(s) x_j \right\|_{L_p(\Omega)}^p = \int_\Omega \left| \sum_{j=1}^{N} R_j(s) x_j(t) \right|^p dt$$

By integrating with respect to s we get

$$\begin{aligned} A^p N^{p/2} &\leq \int_0^1 \int_\Omega \left| \sum_{j=1}^{N} R_j(s) x_j(t) \right|^p dt\, ds \\ &= \int_\Omega \int_0^1 \left| \sum_{j=1}^{N} R_j(s) x_j(t) \right|^p ds\, dt \\ &\leq \int_\Omega \left(B_p \left(\sum_{j=1}^{N} |x_j(t)|^2 \right)^{1/2} \right)^p dt \qquad \text{(by Khintchine's inequality)} \end{aligned}$$

$$\leq \int_\Omega \left(B_p \beta N^{1/p} B\right)^p dt \qquad \text{by (3.3)}$$
$$= \left(B_p \beta N^{1/p} B\right)^p$$

Thus we have that $AN^{1/2} \leq B_p \beta N^{1/p} B$ and hence $\frac{A}{B\,B_p} N^{1/2-1/p} \leq \beta$. As $2 < p$, we have that $N^{1/2-1/p}$ is unbounded and hence a uniform constant β cannot exist. $\square$

4. Discretization and frame theory

In previous sections we have considered the problem of discretizing the L_p norm on a subspace $X \subseteq L_p$. We now show how this problem is naturally connected with frame theory. A family of vectors $(x_j)_{j\in J}$ in a Hilbert space H is called a *frame* of H if there are constants $0 < A \leq B < \infty$ so that for all $x \in \mathbb{H}^N$,

$$A\|x\|^2 \leq \sum_{j\in J} |\langle x, x_j\rangle|^2 \leq B\|x\|^2. \tag{4.1}$$

A frame is called *tight* if the optimal frame bounds satisfy $A = B$, and a frame is called *Parseval* if the optimal frame bounds satisfy $A = B = 1$. The *analysis operator* of a frame $(x_j)_{j\in J}$ of H is the map $T : H \to \ell_2(J)$ given by $T(x) = (\langle x, x_j\rangle)_{j\in J}$. Note that $(x_j)_{j\in J}$ has upper frame bound B and lower frame bound A if and only if the analysis operator is an embedding and satisfies $A\|x\|^2 \leq \|Tx\|^2 \leq B\|x\|^2$ for all $x \in H$.

The notion of a frame can be generalized to a continuous frame by changing the summation in (4.1) to integration over a measure space. A collection of vectors $(x_t)_{t\in\Omega}$ in a Hilbert space H is called a *continuous frame* of H over a measure space (Ω, Σ, μ) if there are constants $0 < A \leq B < \infty$ so that for all $x \in H$,

$$A\|x\|^2 \leq \int_\Omega |\langle x, x_j\rangle|^2 d\mu \leq B\|x\|^2. \tag{4.2}$$

A continuous frame is called *tight* if the optimal frame bounds satisfy $A = B$, and a continuous frame is called *Parseval* if the optimal frame bounds satisfy $A = B = 1$. The *analysis operator* of a continuous frame $(x_t)_{t\in\Omega}$ of H is the map $T : H \to L_2(\Omega)$ given by $T(x) = (\langle x, x_t\rangle)_{t\in\Omega}$. Note that the analysis operator of a frame $(x_j)_{j\in J}$ is an embedding of H into $\ell_2(J)$ and that the analysis operator of a continuous frame $(x_t)_{t\in\Omega}$ is an embedding of H into $L_2(\Omega)$.

Continuous frames are widely used in mathematical physics and are particularly prominent in quantum mechanics and quantum optics. Though continuous frames such as the short time Fourier transform naturally characterize many different physical properties, discrete frames are much better suited for computations. Because of this, when working with continuous frames, researchers often create a discrete frame by sampling the continuous frame and then use the discrete frame for computations instead of the entire continuous frame. That is given, a continuous frame $(x_t)_{t\in\Omega}$ of H we are interested in choosing $(t_j)_{j\in J} \subseteq \Omega$ so that $(x_{t_j})_{j\in J}$ is a frame of H. The notion of creating a frame by sampling a continuous frame has its origins in the very start of modern frame theory. Indeed, Daubechies, Grossmann, and Meyer [12] popularized modern frame theory in their seminal paper "Painless nonorthogonal expansions", and their constructions of frames for Hilbert spaces were all done by sampling different continuous frames. Sampling continuous frames continues to be an important subject in applied harmonic analysis, and there are many modern research papers on the subject in various contexts [2][14][13]. The discretization problem, posed by Ali, Antoine, and Gazeau in their physics textbook *Coherent States, Wavelets, and Their Generalizations* [3], asks when a continuous frame of a Hilbert space can be sampled to obtain a frame. They state that a positive answer to the question is crucial for practical applications of coherent states, and chapter 16 of the book is devoted to the discretization problem. The first author with Darrin Speegle solved the discretization problem in its full generality by characterizing exactly when a continuous frame may be sampled to obtain a frame [17].

Theorem 4.1 *([17]). Let $(x_t)_{t\in\Omega}$ be a continuous frame of a separable Hilbert space H over a measure space (Ω, Σ, μ) such that singletons are measurable. Then there exists $(t_j)_{j\in J} \subseteq \Omega$ such that $(x_{t_j})_{j\in J}$ is a frame of a H if and only if there is a measure ν on (Ω, Σ) such that $(x_t)_{t\in\Omega}$ is a continuous frame of a H over the measure space (Ω, Σ, ν) and there is a constant $\beta > 0$ so that $\|x_t\| \leq \beta$ for almost every $t \in \Omega$.*

Furthermore, they prove the following quantized version.

Theorem 4.2 *([17]). There exist uniform constants $A, B > 0$ such that the following holds. Let $(x_t)_{t\in\Omega}$ be a continuous Parseval frame of a separable Hilbert space H over a measure space (Ω, Σ, μ) such that $\|x_t\| \leq 1$ for all $t \in \Omega$. Then there exists $(t_j)_{j\in J} \subseteq \Omega$ such that $(x_{t_j})_{j\in J}$ is a frame of H and*

$$A\|x\|^2 \leq \sum_{j\in J} |\langle x, x_{t_j}\rangle|^2 \leq B\|x\|^2 \qquad \text{for all } x \in H.$$

We now consider how these results are connected to the problem of discretizing L_p norms for subspaces. If $(x_t)_{t\in\Omega}$ is a continuous Parseval frame for a Hilbert space H then the analysis operator $T : H \to L_2(\Omega)$ is an isometric embedding of H into $L_2(\Omega)$. Being able to discretize the continuous frame $(x_t)_{t\in\Omega}$ to get a frame $(x_{t_j})_{j\in J}$ of H with lower frame bound A and upper frame bound B is then equivalent to discretizing the L_2 norm on the range of the analysis operator $T(H) \subseteq L_2(\Omega)$ so that

$$A\|y\|^2 \leq \sum_{j\in J} |y(t_j)|^2 \leq B\|y\|^2 \text{ for all } y \in T(H)$$

where if $y = Tx$ then $y(t) = \langle x, x_t\rangle$ for all $t \in \Omega$. Furthermore, suppose that $Y \subseteq L_2(\Omega)$ is a closed subspace and $\beta > 0$. Then, $Y \subseteq L_2(\Omega)$ satisfies that $\|y\|_{L_\infty} \leq \beta\|y\|_{L_2}$ for all $y \in Y$ if and only if Y is the range of the analysis operator of a continuous Parseval frame $(x_t)_{t\in\Omega}$ of a Hilbert space H such that $\|x_t\| \leq \beta$ for all $t \in \Omega$. This gives the following corollary.

Corollary 4.3. *There exist uniform constants $A, B > 0$ such that the following holds. Let (Ω, μ) be a σ-finite measure space and $\beta \geq 1$. Suppose that $Y \subseteq L_2(\Omega)$ is a closed subspace such that $\|y\|_{L_\infty} \leq \beta\|y\|_{L_2}$ for all $y \in Y$. Then there exists $(t_j)_{j\in J} \subseteq \Omega$ such that*

$$\beta^2 A\|y\|^2 \leq \sum_{j\in J} |y(t_j)|^2 \leq \beta^2 B\|y\|^2 \qquad \text{for all } y \in Y.$$

Note that Corollary 4.3 applies to Ω being either a finite or infinite measure space and to $Y \subseteq L_2(\Omega)$ being either finite or infinite dimensional. In the case that Ω is a probability space then the following theorem gives the relationship between the dimension of the subspace, the L_∞-bound on the subspace, and a bound on the number of sampling points required for discretization.

Theorem 4.4 *([21]). There exist uniform constants $A, B, C > 0$ such that the following holds. Let (Ω, μ) be a probability space and $\beta \geq 1$. Suppose that $Y \subseteq L_2(\Omega)$ is a closed subspace such that $\|y\|_{L_\infty} \leq \beta N^{1/2}\|y\|_{L_2}$ for all $y \in Y$. Then there exists $M \leq C\beta^2 N$ and sampling points $(t_j)_{j=1}^M \subseteq \Omega$ such that*

$$A\|y\|^2 \leq \frac{1}{M}\sum_{j=1}^{M} |y(t_j)|^2 \leq \beta^2 B\|y\|^2 \qquad \text{for all } y \in Y.$$

One of the many applications of frame theory is in the implementation of phase retrieval. A frame $(x_j)_{j\in J} \subseteq H$ for a Hilbert space H allows for any vector $x \in H$ to be linearly recovered from the collection of frame coefficients $(\langle x, x_j\rangle)_{j\in J}$. However, there are many instances in physics and engineering where one is able to obtain only the magnitude of linear measurements such as in speech recognition [5] and X-ray crystallography [25]. Let $T : H \to \ell_2(J)$ be the analysis operator of $(x_j)_{j\in J}$ given by $T(x) = (\langle x, x_j\rangle)_{j\in J}$ for all $x \in H$. For $x \in H$, the goal of phase retrieval is to recover x (up to a unimodular scalar) from the absolute value of the frame coefficients $|T(x)| = (|\langle x, x_j\rangle|)_{j\in J}$. We say that a frame $(x_j)_{j\in J}$ does phase retrieval if whenever $x, y \in H$ and $|Tx| = |Ty|$ we have that $x = \lambda y$ for some scalar λ with $|\lambda| = 1$. We say that $(x_j)_{j\in J}$ does *C-stable* phase retrieval if $\min_{|\lambda|=1} \|x - \lambda y\|_H \le C\||Tx| - |Ty|\|_{\ell_2(J)}$ for all $x, y \in H$. If we consider the equivalence relation $\sim$ on H to be $x \sim y$ if and only if $x = \lambda y$ for some $|\lambda| = 1$ then a frame $(x_j)_{j\in J}$ does C-stable phase retrieval is equivalent to the map $|Tx| \mapsto x/\!\sim$ is well defined and is C-Lipschitz. As any application will involve some error, having a good stability bound for phase retrieval is of fundamental importance in applications. Likewise, if $(x_t)_{t\in\Omega}$ is a continuous frame of H with frame operator $T : H \to L_2(\Omega)$ then we say that $(x_t)_{t\in\Omega}$ does *C-stable* phase retrieval if $\min_{|\lambda|=1} \|x - \lambda y\|_H \le C\||Tx| - |Ty|\|_{L_2(\Omega)}$ for all $x, y \in H$.

Every frame for a finite dimensional Hilbert space which does phase retrieval does C-stable phase retrieval for some constant $C > 0$ [4][7]. On the other hand, phase retrieval using a frame or continuous frame for an infinite dimensional Hilbert space is always unstable [6][1]. Given some $C > 0$ and dimension $N \in \mathbb{N}$, it is very difficult to explicitly construct a frame of ℓ_2^N which does C-stable phase retrieval. However, there are random constructions where it is possible to choose $C > 0$ such that a frame $(x_j)_{j=1}^m$ of random vectors does C-stable phase retrieval with high probability and the number of vectors m can be chosen on the order of the dimension N [9][15][20][8]. Each of these results can be thought of as sampling a continuous Parseval frame over a probability space which does stable phase retrieval to obtain a frame which does stable phase retrieval. This naturally leads to the following problem.

Problem 4.5. Let $\kappa, \beta > 0$. Do there exist constants $C, D > 0$ so that for all $N \in \mathbb{N}$ there exists $M \le DN$ such that the following holds? Suppose that H is an N-dimensional Hilbert space, (Ω, μ) is a probability space, and $(x_t)_{t\in\Omega}$ is a continuous Parseval frame of H which does κ-stable phase retrieval such that $\|x_t\| \le \beta\sqrt{N}$ for all $t \in \Omega$. Then there exists a sequence of sampling points $(t_j)_{j=1}^M \subseteq \Omega$ such that $(\frac{1}{\sqrt{M}} x_{t_j})_{j=1}^M$ is a frame of H which does C-stable phase retrieval.

This problem seems particularly difficult as Theorem 4.4 which relies on [23] can be thought of as a random sampling result which produces a good frame with low but positive probability. However, all known methods of producing frames which do stable phase retrieval using a number of vectors on the order of the dimension use sub-Gaussian random variables where a random sampling will produce a good frame with high probability. In the following theorem we connect the problem of constructions of frames which do stable phase retrieval to the problem of discretizing the L_1-norm on a subspace of $L_1(\Omega)$. In particular, we prove that in order to sample a continuous Parseval frame to obtain a frame which does stable phase retrieval, it is necessary to simultaneously discretize both the L_1-norm and the L_2-norm on the range of the analysis operator.

Theorem 4.6. *Let $(x_t)_{t\in\Omega}$ be a continuous Parseval frame for an N-dimensional real Hilbert space H over a probability space Ω which does κ-stable phase retrieval and $\|x_t\|_H \le \beta\sqrt{N}$ for all $t \in \Omega$. Let $T : H \to L_2(\Omega)$ be the analysis operator of $(x_t)_{t\in\Omega}$. Suppose that $(t_j)_{j=1}^M \subseteq \Omega$ is such that $(\frac{1}{\sqrt{M}} x_{t_j})_{j=1}^M$ is a frame of H with upper frame bound B and lower frame bound A which does C-stable phase retrieval. Then both the L_2 norm and the L_1 norm on the range of the analysis operator are discretized in the following way for all $y \in T(H)$,* *(1)* $A\|y\|^2_{L_2(\Omega)} \leq \frac{1}{M}\sum_{j=1}^{M}|y(t_j)|^2 \leq B\|y\|^2_{L_2(\Omega)}$,

(2) $A^{1/2}B^{-3/2}C^{-3}(1+A^{-1}\beta^2)^{-3/2}\|y\|_{L_1(\Omega)} \leq \frac{1}{M}\sum_{j=1}^{M}|y(t_j)| \leq B^{1/2}\kappa^3(1+\beta^2)^{3/2}\|y\|_{L_1(\Omega)}$.

Before proving Theorem 4.6 we will prove the following lemma.

Lemma 4.7. *Let Ω be a probability space and let X^N be an N-dimensional subspace of $L_2(\Omega)$. Suppose $\kappa, \beta > 0$ are such that $\|x\|_{L_\infty} \leq \beta\sqrt{N}\|x\|_{L_2}$ for all $x \in X^N$ and that*

$$\min(\|f-g\|_{L_2}, \|f+g\|_{L_2}) \leq \kappa\Big\||f|-|g|\Big\|_{L_2} \qquad \textit{for all } f,g \in X^N. \tag{4.3}$$

Then, $\|x\|_{L_1} \leq \|x\|_{L_2} \leq \kappa^3(1+\beta^2)^{3/2}\|x\|_{L_1}$ for all $x \in X^N$.

Proof. Let $x \in X^N$ with $\|x\|_{L_2} = 1$. Note that $\|x\|_{L_1} \leq \|x\|_{L_2}$ as Ω is a probability space. Let $\gamma = \|x\|^{1/3}_{L_1}$. We have by Markov's inequality that

$$\|x\|_{L_1} \geq \gamma Prob(|x| > \gamma).$$

Hence, $Prob(|x| > \gamma) \leq \|x\|^{2/3}_{L_1}$. Let $S = \{t \in \Omega : |x(t)| > \gamma\}$ and P_S be the restriction operator from $L_2(\Omega)$ to $L_2(S)$. Let $(e_j)_{j=1}^N$ be an orthonormal basis for X^N. For each $t \in S$ there exists $\psi_t \in X$ such that $\langle x, \psi_t\rangle = x(t)$ for all $x \in X^N$. Note that $\|\psi_t\|_{L_2} \leq \beta\sqrt{N}$ for all $t \in S$. We have that

$$\begin{aligned}
\sum_{j=1}^{N}\|P_S e_j\|^2_{L_2} &= \sum_{j=1}^{N}\int_S |e_j(t)|^2 dt \\
&= \sum_{j=1}^{N}\int_S |\langle e_j, \psi_t\rangle|^2 dt \\
&= \int_S \sum_{j=1}^{N}|\langle e_j, \psi_t\rangle|^2 dt \\
&= \int_S \|\psi_t\|^2 dt \\
&\leq Prob(S)\beta^2 N
\end{aligned}$$

Thus, there exists $1 \leq j \leq N$ such that $\|P_S e_j\|_{L_2} \leq (Prob(S))^{1/2}\beta$. In particular, there exists $y \in X^N$ with $\|y\|_{L_2} = 1$ and $\|P_S y\|_{L_2} \leq (Prob(S))^{1/2}\beta$.

Let $f = x+y$ and $g = x-y$. As $\|x\|_{L_2} = \|y\|_{L_2} = 1$ we have that $\|f-g\|_{L_2} = \|f+g\|_{L_2} = 2$. We now obtain an upper bound for $\Big\||f|-|g|\Big\|_{L_2}$.

$$\begin{aligned}
\Big\||f|-|g|\Big\|^2_{L_2} &= \Big\||x+y|-|x-y|\Big\|^2_{L_2} \\
&= \int (2\min(|x(t)||y(t)|)^2 dt \\
&= 4\int_S (\min(|x(t)||y(t)|)^2 dt + 4\int_{S^c} (\min(|x(t)||y(t)|)^2 dt
\end{aligned}$$

$$\begin{aligned}
&\le 4\int_S |y(t)|^2dt + 4\int_{S^c} |x(t)|^2dt \\
&\le 4\,Prob(S)\beta^2 + 4\gamma^2 \qquad \text{(as } \|P_S y\|_{L_2}^2 \le Prob(S)\beta^2 \text{ and } |x(t)| \le \gamma \text{ for all } t \in S^c\text{).} \\
&\le 4\|x\|_{L_1}^{2/3}\beta^2 + 4\|x\|_{L_1}^{2/3}
\end{aligned}$$

Thus, we have that

$$\frac{1}{8}\Big\||f| - |g|\Big\|_{L_2}^3 \le (1+\beta^2)^{3/2}\|x\|_{L_1} \tag{4.4}$$

As $\|x\|_{L_2} = 1$ and $\|f - g\|_{L_2} = \|f + g\|_{L_2} = 2$ we have by (4.3) and (4.4) that

$$\|x\|_{L_2} = \|x\|_{L_2}^3 = \frac{1}{8}\min(\|f-g\|_{L_2}^3, \|f+g\|_{L_2}^3) \le \frac{1}{8}\kappa^3\Big\||f| - |g|\Big\|_{L_2}^3 \le \kappa^3(1+\beta^2)^{3/2}\|x\|_{L_1}. \quad \square$$

We now show that Theorem 4.6 follows from Lemma 4.7

Proof. Let $y = T(x) \in T(H)$ with $\|y\|_{L_2(\Omega)} = 1$. As $(x_t)_{t\in\Omega}$ is a Parseval frame for H, we have that $\|y\|_{L_2(\Omega)} = \|x\|_H = 1$. Note that $y(t) = \langle x, x_t\rangle$ for all $t \in \Omega$. As $(\frac{1}{\sqrt{M}}x_{t_j})_{j=1}^M$ is a frame of H with lower frame bound A and upper frame bound B, we have that

$$A\|y\|_{L_2(\Omega)}^2 \le \frac{1}{M}\sum_{j=1}^M |y(t_j)|^2 \le B\|y\|_{L_2(\Omega)}^2.$$

We now have the following upper bound.

$$\begin{aligned}
\frac{1}{M}\sum_{j=1}^M |y(t_j)| &\le \Big(\frac{1}{M}\sum_{j=1}^M |y(t_j)|^2\Big)^{1/2} \\
&\le B^{1/2}\|y\|_{L_2(\Omega)} \\
&\le B^{1/2}\kappa^3(1+\beta^2)^{3/2}\|y\|_{L_1(\Omega)} \qquad \text{by Lemma 4.7.}
\end{aligned}$$

We have that $(\frac{1}{\sqrt{M}}x_{t_j})_{j=1}^M$ is a frame with lower frame bound A and upper frame bound B which does C-stable phase retrieval. Let the set $[M] = \{1, 2, .., M\}$ be given the uniform probability measure. Then, $(x_{t_j})_{j\in[M]}$ is a continuous frame with lower frame bound A and upper frame bound B which does C-stable phase retrieval. Let $T_{[M]} : H \to L_2([M])$ be the analysis operator of $(x_{t_j})_{j\in[M]}$. Thus, we have for all $f, g \in H$ that

$$\min_{|\lambda|=1} \|T_{[M]}f - \lambda T_{[M]}g\|_{L_2([M])} \le B^{1/2}\min_{|\lambda|=1}\|f - \lambda g\|_H \le B^{1/2}C\Big\||T_{[M]}f| - |T_{[M]}g|\Big\|_{L_2([M])}.$$

Furthermore, as $\|x_t\|_H \le \beta N^{1/2}$ for all $t \in \Omega$, we have for all $f \in H$ that

$$\|T_{[M]}f\|_{L_\infty([M])} = \sup_{j\in[M]} |\langle f, x_{t_j}\rangle| \le \|f\|_H\beta N^{1/2} \le A^{-1/2}\beta N^{1/2}\|T_{[M]}f\|_{L_2([M])}.$$

We can thus apply Lemma 4.7 to the subspace $T_{[M]}H \subseteq L_2(\Omega)$ with stability constant $B^{1/2}C$ and L_∞ bound $A^{-1/2}\beta$ to calculate the following.

$$\begin{aligned}\frac{1}{M}\sum_{j=1}^{M}|y(t_j)| &= \|y\|_{L_1([M])} \\ &\geq B^{-3/2}C^{-3}(1+A^{-1}\beta^2)^{-3/2}\|y\|_{L_2([M])} \qquad \text{by Lemma 4.7} \\ &\geq A^{1/2}B^{-3/2}C^{-3}(1+A^{-1}\beta^2)^{-3/2}\|y\|_{L_2(\Omega)} \\ &\geq A^{1/2}B^{-3/2}C^{-3}(1+A^{-1}\beta^2)^{-3/2}\|y\|_{L_1(\Omega)} \quad \square\end{aligned}$$

5. Discretizing infinite dimensional subspaces of L_p

In sections 2 and 3 we showed that Theorem 4.4 does not hold for finite dimensional subspaces of $L_p[0,1]$ for $1 \leq p < 2$. We now show that Corollary 4.3 fails in a strong way for infinite dimensional subspaces of $L_p(\mathbb{R})$ for $1 \leq p < 2$.

Proposition 5.1. *For all $1 \leq p < 2$ there exists a subspace $Y \subseteq L_p(\mathbb{R})$ such that Y is isomorphic to ℓ_p and $\|x\|_{L_\infty} \leq \|x\|_{L_p}$ for all $x \in Y$, and the following holds. If $J \subseteq \mathbb{R}$ is such that $\sup_{t\in J}|x(t)| \neq 0$ for all $x \in Y \setminus \{0\}$ then there exists $y \in Y$ such that $\sum_{t\in J}|y(t)|^p = \infty$.*

Proof. We first consider the case $1 < p < 2$. By Proposition 3.2 we have for all $N \in \mathbb{N}$ that there exists a subspace $X^N \subseteq L_p[0,1]$ such that X^N is $A_p^{-1}B_p$-isomorphic to ℓ_2^N and the following hold,

(1) $\|x\|_{L_\infty} \leq A_p^{-1}N^{1/p}\|x\|_{L_p}$ for all $x \in X^N$,
(2) If $(t_j)_{j=1}^M \subseteq [0,1]$ are such that $(\sum_{j=1}^M |x(t_j)|^p)^{1/p} > 0$ for all $x \in X^N \setminus \{0\}$ then

$$N^{2-p/2}\|x\|_{L_p}^p \leq \sum_{j=1}^{M}|x(t_j)|^p \qquad \text{for some } x \in X^N.$$

Let $(M_N)_{N=1}^\infty$ be an increasing sequence of real numbers such that $M_{N+1} > M_N + A_p^{-p}N$ for all $N \in \mathbb{N}$. For each $N \in \mathbb{N}$, we let $D_N : X^N \to L_2([M_N, M_N + A_p^{-p}N])$ be the operator defined by for $x \in X^N$,

$$(D_N x)(t) = A_p N^{-1/p} x\big(A_p^p N^{-1}(t - M_N)\big) \qquad \text{for all } t \in [M_N, M_N + A_p^{-p}N].$$

Note that D_N is an isometric embedding of X^N into $L_2([M_N, M_N + A_p^{-p}N])$. We let $Y^N = D_N(X^N)$. By (1) we have that $\|y\|_{L_\infty} \leq \|y\|_{L_p}$ for all $y \in Y^N$. Let $J_N \subseteq \mathbb{R}$ such that $(\sum_{t\in J_N}|y(t)|^p)^{1/p} > 0$ for all $y \in Y^N \setminus \{0\}$. Let $I_N = (J_N - M_N)A_p^pN^{-1}$. By (2), there exists $x_N \in X^N$ with $\|x_N\|_{L_p}^p = N^{p/2-2}$ and $\sum_{t\in I_N}|x_N(t)|^p \geq 1$. We let $y_N = D_N x_N$ and hence $\|y_N\|_{L_p}^p = N^{p/2-2}$ and $\sum_{t\in J_N}|y_N(t)|^p \geq A_p^pN^{-1}$.

We now let $Y = \overline{\mathrm{span}}(Y^N)_{N\in\mathbb{N}} \subseteq L_p(\mathbb{R})$. As each Y^N is $A_p^{-1}B_p$-isomorphic to ℓ_2^N and is composed of functions which are supported on the interval $[M_N, M_N + A_p^{-p}N]$, we have that Y is isomorphic to $(\oplus \ell_2^N)_{\ell_p}$ and hence Y is isomorphic to ℓ_p by Pełczyński's decomposition theorem. For all $y \in Y$, we have that $\|y\|_{L_\infty} \leq \|y\|_{L_p}$. As, $\|y_N\|_{L_p}^p = N^{p/2-2}$ and $1 < p < 2$, we have that $\sum_{N=1}^\infty \|y_N\|^p < \infty$. Thus, $\sum_{N=1}^\infty y_N \in Y$. We now suppose that $J \subseteq \mathbb{R}$ is such that $\sup_{t\in J}|y(t)| \neq 0$ for all $y \in Y \setminus \{0\}$. As $(y_N)_{N=1}^\infty$ have pairwise disjoint support, we have that

$$\sum_{t\in J}\Big|\sum_{N=1}^{\infty} y_N(t)\Big|^p = \sum_{N=1}^{\infty}\sum_{t\in J}|y_N(t)|^p \geq \sum_{N=1}^{\infty} A_p^p N^{-1} = \infty.$$

Thus, we have completed the proof for the case $1 < p < 2$. Note that the exact same proof does not work for $p = 1$ as $(\oplus \ell_2^n)_{\ell_1}$ is not isomorphic to ℓ_1. However, if we instead use Theorem 2.1 instead of Proposition 3.2 then the proof follows in the same way. $\square$

6. Appendix

In terms of integral norm discretization, the law of large numbers essentially states that if $f \in L_p(\Omega)$ for some probability space Ω and $1 \le p < \infty$, then for all $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that if $M \ge m$ and $(t_j)_{j=1}^M \subseteq \Omega$ are independent random samples then with probability at least $(1-\varepsilon)$ we have that

$$(1-\varepsilon)\|f\|_p^p \le \frac{1}{M}\sum_{j=1}^M |f(t_j)|^p \le (1+\varepsilon)\|f\|_p^p \tag{6.1}$$

Note that the Law of Large Numbers applies to only a fixed $f \in L_p(\Omega)$. The following proposition extends (6.1) to any finite dimensional subspace of $L_p(\Omega)$. We expect that this is well known, but we include a proof for completeness.

Proposition 6.1. *Let $1 \le p < \infty$ and let $X \subseteq L_p(\Omega)$ be a finite dimensional subspace, where (Ω, μ) is a probability space. Then for all $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that if $M \ge m$ and $(t_j)_{j=1}^M \subseteq \Omega$ are independent random samples then with probability at least $(1-\varepsilon)$ we have that*

$$(1-\varepsilon)\|f\|_p^p \le \frac{1}{M}\sum_{j=1}^M |f(t_j)|^p \le (1+\varepsilon)\|f\|_p^p \qquad \textit{for all } f \in X.$$

Proof. As X is a finite dimensional subspace of $L_p(\Omega)$ we have for almost every $t \in \Omega$ that point evaluation at t defines a linear functional on X. After discarding a set of measure zero, we may assume for each $t \in \Omega$ that there exists $\psi_t \in X^*$ so that $\psi_t(f) = f(t)$ for all $f \in X$. We now claim that $\int_{t\in\Omega} \|\psi_t\|^p d\mu < \infty$.

Indeed, let $(x_j)_{j=1}^n \subseteq S_X$ be a finite $(1/2)$-net in the unit sphere of X. Thus, for all $t \in \Omega$ there exists $1 \le j \le n$ such that $|\psi_t(x_j)| \ge (1/2)\|\psi_t\|$. Thus, we have that

$$\int\limits_{t\in\Omega} \|\psi_t\|^p d\mu \le 2^p \int\limits_{t\in\Omega} \sup_{1\le j\le n} |\psi_t(x_j)|^p d\mu \le 2^p \int\limits_{t\in\Omega} \sum_{1\le j\le n} |\psi_t(x_j)|^p d\mu = 2^p \sum_{1\le j\le n} \int\limits_{t\in\Omega} |x_j|^p d\mu = 2^p n$$

This proves our claim that $\int_{t\in\Omega} \|\psi_t\|^p d\mu < \infty$.

Let $\varepsilon > 0$ and choose $K > 0$ so that for $\Omega_K = \{t \in \Omega : \|\psi_t\| \le K\}$ we have that $\int_{t\in\Omega_K^c} \|\psi_t\|^p d\mu < \varepsilon$. In particular, we have for all $f \in X$ that,

$$\int\limits_{\Omega_K^c} |f|^p d\mu = \int\limits_{t\in\Omega_K^c} |\psi_t(f)|^p d\mu \le \int\limits_{t\in\Omega_K^c} \|\psi_t\|^p \|f\|_p^p \, d\mu \le \varepsilon \|f\|_p^p \tag{6.2}$$

Hence, we have for all $f \in X$ that

$$(1-\varepsilon)\|f\|_p^p \le \int\limits_{\Omega_K} |f|^p \, d\mu \le \|f\|_p^p \tag{6.3}$$

We now choose $0 < \delta < K^{-1}\varepsilon^p$. Let $(f_j)_{j\in J} \subseteq S_X$ be a finite δ-net. We apply the law of large numbers to the functions ψ and $(f_j)_{j\in J}$ to obtain $m \in \mathbb{N}$ such that if $M \ge m$ then with probability at least $1-\varepsilon$ we have that if $(t_j)_{j=1}^M \subseteq \Omega$ are independent random samples then for $T_K = (t_j)_{j=1}^M \cap \Omega_K$ we have that

$$\frac{1}{M}\sum_{t\in T_K^c} \|\psi_t\|^p < 2\varepsilon \quad \text{and} \quad (1-2\varepsilon) \le \frac{1}{M}\sum_{t\in T_K} |f_j(t)|^p \le (1+\varepsilon) \text{ for all } j \in J.$$

We now let $f \in S_X$. Choose $j \in J$ such that $\|f - f_j\| < \delta$. We have for all $t \in T_k$ that $|f(t) - f_j(t)| \leq \|\psi_t\|\|f - f_j\| < K\delta$. The sum over T_K satisfies that for all $j \in J$,

$$\Big|\Big(\frac{1}{M}\sum_{t\in T_K}|f(t)|^p\Big)^{1/p} - \Big(\frac{1}{M}\sum_{t\in T_K}|f_j(t)|^p\Big)^{1/p}\Big| \leq \Big(\frac{1}{M}\sum_{t\in T_K}|f(t) - f_j(t)|^p\Big)^{1/p} < K\delta < \varepsilon$$

Thus, we have that

$$((1-2\varepsilon)^{1/p} - \varepsilon)^p \leq \frac{1}{M}\sum_{t\in T_K}|f(t)|^p \leq ((1+\varepsilon)^{1/p} + \varepsilon)^p \tag{6.4}$$

The sum over T_K^c satisfies that

$$\frac{1}{M}\sum_{t\in T_K^c}|f(t)|^p \leq \frac{1}{M}\sum_{t\in T_K^c}\|\psi_t\|^p < 2\varepsilon \tag{6.5}$$

By summing (6.4) and (6.5) we have that

$$((1-2\varepsilon)^{1/p} - \varepsilon)^p \leq \frac{1}{M}\sum_{j=1}^{M}|f(t_j)|^p \leq ((1+\varepsilon)^{1/p} + \varepsilon)^p + 2\varepsilon \quad \square$$

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