PAIR CORRELATION OF ZEROS OF $\Gamma_1(q)$ L-FUNCTIONS

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Abstract. We study the analogue of the Montgomery's pair correlation function $F(\alpha)$ for a family of $\Gamma_1(q)$ L-functions. Assuming the Generalized Riemann Hypoth-esis, we evaluate this analogous pair correlation function in the range $|\alpha| < 2$; where the off-diagonal terms also contribute to the main term. As applications, we obtain that more than 93.5% of the low-lying zeros of $\Gamma_1(q)$ L-functions are simple, and we achieve upper and lower bounds for the number of pairs of zeros which are closer than μ times the average spacing apart.

1. Introduction

1.1. Background and main results. In this paper, we examine the pair correlation function of a family of L-functions attached to automorphic forms on GL(2). In the context of the Riemann zeta function $\zeta(s)$, Montgomery [19] introduced the function

$$F(\alpha) := \frac{2\pi}{T\log T}\sum_{0<\gamma,\gamma'\le T} T^{i\alpha(\gamma-\gamma')}w(\gamma-\gamma')$$

where $\beta+i\gamma$ and $\beta'+i\gamma'$ are nontrivial zeros of ζ, α is real, $T \ge 2$ and $w(u) = 4/(4+u^2)$.

Assuming RH, he showed that

$$F(\alpha) = \left(T^{-2|\alpha|}\log T + |\alpha|\right)\left(1 + O\left(\sqrt{\frac{\log\log T}{\log T}}\right)\right) \tag{1.1}$$

uniformly for $0 \le |\alpha| \le 1$. Understanding $F(\alpha)$ for wider ranges of α is a deep and difficult problem. In this direction, Montgomery conjectured that $F(\alpha) = 1 + o(1)$ as $T \to \infty$ uniformly in bounded intervals $1 \le |\alpha| \le b < \infty$ for any constant b.

The function $F(\alpha)$ is closely related to the distribution of the differences $|\gamma' - \gamma|$. For instance, understanding $F(\alpha)$ in the range $|\alpha| \le 1$ quickly leads to understanding of (1.2)

$$\sum\sum_{0<\gamma,\gamma'\le T} \omega(\gamma-\gamma')f\left(\frac{\log T}{2\pi}(\gamma-\gamma')\right);$$

for Schwartz class functions f with the Fourier transform $\hat{f}$ supported on $(-1; 1)$. Indeed, understanding the quantity in (1.2) quickly reduces to understanding $F(\alpha)$ by writing f as an inverse transform of its Fourier transform $\hat{f}$. Note that the restriction

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on the support of $\hat{f}$ is the same as on the range of in (1.1). Further, (1.2) is visibly related to the quantity

$$(1.3) \qquad N(T;) := \sum_{\substack{0<;^0 T \; 0<^0 \\ \frac{\log T}{2}}} 1;$$

which gives a count of those pairs $;^0$ which are closer than times the average spacing apart.

Montgomery's work suggested zeros of the Riemann zeta function are vertically distributed like the eigenvalues of a random unitary matrix, and this has led to signicant further work. Notably, the quantities introduced above are also intimately connected to other deep arithmetic questions.

For instance, understanding these pair correlation functions is related to the existence of Siegel zeros [7], the distribution of primes in short intervals ([10, 11, 12]), as well as to the Simple Zero conjecture. Indeed, Montgomery used (1.1) to show that at least 2/3 of the zeros of the Riemann zeta function are simple.

In all these applications, it is of great interest to extend our understanding of $F()$ to wider ranges of , or equivalently, understand (1.2) for test functions f with $\hat{f}$ having larger support. In the context of large families of Dirichlet L-functions, Ozluk [24, 25] and Chandee, Lee, Liu and Radziwill [4] extended the range of .

In particular, Ozluk's studied the pair correlation of low lying zeros of Dirichlet L-functioins $L(s;)$ on average over mod q for $Q \; q \; 2Q$. More recently, Chandee, Lee, Liu and Radziwill [4] were able to improve upon Ozluk's results by averaging over primitive characters. This yielded superior results since it avoided the over-counting inherent in Ozluk's work. In these two works, it is apparent that the larger size of the family, compared to the conductor, yields superior ranges of . As a consequence of their work, Chandee, Lee, Liu and Radziwill [4] improve upon Ozluk's work to get that 91% of the nontrivial zeros of all primitive Dirichlet L-functions are simple assuming GRH. This was later improved by Sono [26] to over 93.22% and by Chirre, Gonalves and de Laat [6] to over 93:5%.

In this paper, we consider a large family of GL(2) L-functions. To be precise, let k and q be positive integers, and $S_k(\ _0(q);)$ be the space of cusp forms of weight $k \; 3$ for the group $\ _0(q)$ and the nebentypus character (mod q), where as usual,

$$_0(q) = \quad \begin{matrix} a & b \\ c & d \end{matrix} \quad ad \quad bc = 1; \;\; c \; 0 \pmod q \quad :$$

Let $S_k(\ _1(q))$ be the space of holomorphic cusp forms for the group

$$_1(q) = \quad \begin{matrix} a & b \\ c & d \end{matrix} \quad ad \quad bc = 1; \;\; c \; 0 \pmod q; \qquad a \;\; d \;\; 1 \pmod q \quad :$$

Note that

$$S_k(\ _1(q)) = \bigoplus_{\pmod q} S_k(\ _0(q);):$$

Let $H \subset S_k(\Gamma_0(q);\chi)$ be an orthogonal basis of $S_k(\Gamma_0(q);\chi)$ consisting of Hecke cusp forms, normalized so that the first Fourier coefficient is 1. For each $f \in H$, we let $L(f;s)$ be the L-function associated to f, defined for $\mathrm{Re}(s) > 1$ as

$$L(s;f) = \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}; \tag{1.4}$$

where $\{\lambda_f(n)\}$ are the Hecke eigenvalues of f. When f is a newform, $L(s;f)$ can analytically continued to the entire complex plain and satisfies the functional equation

$$\Lambda\left(\tfrac{1}{2} + s; f\right) = i^k \epsilon_f \Lambda\left(\tfrac{1}{2} - s; \bar{f}\right) \tag{1.5}$$

where the completed L-function $\Lambda(s;f)$ is defined by

$$\Lambda\left(\tfrac{1}{2} + s; f\right) = \left(\frac{q}{4\pi^2}\right)^{\frac{s}{2}} \Gamma\left(s + \frac{k}{2}\right) L\left(\tfrac{1}{2} + s; f\right);$$

and the root number ϵ_f satisfies $|\epsilon_f| = 1$.

Suppose for each $f \in H$, we have an associated number α_f. Then we define the harmonic average of α_f over H to be

$$\sum_{f\in H}^{h} \alpha_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f \in H} \frac{\alpha_f}{\|f\|^2};$$

where $\|f\|^2 = \int_{\Gamma_0(q)\backslash \mathbb{H}} |f(z)|^2 y^{k-2}\, dx\, dy$.

Throughout this paper, we will assume the Generalized Riemann Hypothesis (GRH) for $L(s;f)$ and Dirichlet L-functions. We also assume q is a prime number. Let Φ be a smooth function which is real and compactly supported in $(a;b)$ with $0 < a < b$, and its Mellin transform is defined to be

$$\hat{\Phi}(s) = \int_0^\infty \Phi(x) x^{s-1} dx.$$

We consider the $\Gamma_1(q)$-analogue of the pair correlation function

$$F(\alpha) := \frac{2}{N(q)\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1) = (-1)^k}} \sum_{f\in H}^{h} \left| \sum_{\gamma_f} \hat{\Phi}(i\gamma_f) q^{i\alpha\gamma_f} \right|^2, \tag{1.6}$$

where

$$N(q) := \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1) = (-1)^k}} \sum_{f\in H}^{h} \sum_{\gamma_f} \left|\hat{\Phi}(i\gamma_f)\right|^2; \tag{1.7}$$

and $\frac{1}{2} + i\gamma_f$ are the non-trivial zeros of $L(s;f)$. Note that $F(\alpha)$ is an even function in α. Our main result is below.

Theorem 1.1. Assume GRH. Let q be a prime number. Then for any xed > 0 and k 3,

$$F(q) = (1 + o(1)) \quad f() + (q^{\ jj})^2 \log q \quad 2 \quad \frac{1}{\quad} Z_R\, j\hat{b}(it)j^2\, dt \quad {}_1 !$$
$$+\ O(q^{\ jj})\ \overset{p}{}\ f()\ \overline{\log q} + (q^{\ jj})q^{\ jj}$$

holds uniformly for jj 2 as q ! 1 where

$$f() := \begin{cases} jj & jj \ \ 1 \ jj > \\ 1 & 1: \end{cases}$$

The implied constant in the error terms depend on k and :

For the rest of the paper, we will x the > 0 appearing in Theorem 1.1. Our Theorem 1.1 yields results about the statistics of zeros of $L(s; f)$ that support a pair correlation conjecture for $_1(q)$ L-functions. In the theorem, our acceptable range for is essentially (2; 2). This should be compared with the range (1; 1) in Montgomery's work [19] in the case of the Riemann zeta function. A major dierence is that when the range is extended to (2; 2), the o diagonal terms contribute to the main term and need to be precisely understood.

Naively, it is natural to assume GRH for the L-functions $L(s; f)$ but the assumption of GRH for Dirichlet L-functions may appear odd at rst sight. Actually, the phenomenon that one needs to assume GRH for GL(1) L-functions when studying the vertical distribution of zeros of GL(2) L-functions was previously observed in the context of low lying zeros of certain GL(2) L-functions in the work of Iwaniec, Luo and Sarnak [16]. Note also that this phenomenon closes o an avenue of attack towards the Siegel zero problem (assuming GRH for GL(2) L-functions) by proceeding along similar lines to [7] for instance. From a technical point of view, the necessity of assuming GRH for Dirichlet L-functions arises because of the appearance of Kloosterman sums in the o-diagonal term. This o-diagonal term needs to be understood well when the support of ex-tends beyond (1; 1). In our work, this is done by inputting strong information on the distribution of primes in arithmetic progressions, which reduces to GRH for Dirichlet L-functions through standard lines. More structurally, in our family, this phenomenon is related to the observation of Iwaniec and Xiaoqing Li [17] that the $_1(q)$ harmonics are not perfectly orthogonal. To be more precise, their work showed that $_f(n)$ tend to point in the direction of Kloosterman-Bessel products. Practically speaking, this forces us to input information that implies certain coecients (which, in this case, morally look like (n)) do not correlate with Kloosterman-Bessel products.

As usual, an application of the explicit formula reduces the proof of Theorem 1.1 to studying a mean square of a sum over primes. Indeed, Theorem 1.1 follows quickly from the Proposition below.

Proposition 1.2. Assume GRH. Let q be a prime number, $\varepsilon > 0$, $k \geq 3$, and $X = q^{\alpha}$. Let

$$c_f(n) = \begin{cases} \alpha_f(p)^{\ell} + \beta_f(p)^{\ell} & \text{if } n = p^{\ell} \\ 0 & \text{otherwise:} \end{cases} \tag{1.8}$$

Then

$$\frac{2}{\varphi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} \sum_{f \in H_\chi}^{h} \Big| \sum_n \frac{\Lambda(n) c_f(n)}{\sqrt{n}} \Big(\frac{n}{X}\Big)^{-\frac{1}{2}+i\gamma_f} \Big|^2 \sim \log q \int_{\mathbb{R}} |\hat{\Phi}(it)|^2 \, dt \ \ b$$

uniformly for $0 \le \alpha \le 2 - \varepsilon$ as $q \to \infty$. Here $\Lambda(n)$ is the usual von Mangoldt function, which is $\log p$ when $n = p^{\ell}$ and 0 otherwise.

For completeness, we provide the standard proof of Theorem 1.1 assuming Proposition 1.2 in detail in Section 10. The bulk of our paper is devoted to proving Proposition 1.2 which is provided in Sections 4 through 9. Before diving into the proof of our main results we discuss two illustrative applications of Theorem 1.1.

1.2. Applications.

As with other primitive L-functions, it is hypothesized that all non-real zeros of $L(s; f)$ are simple. Roughly speaking, we can show that more than 93.5% of the zeros of $\Gamma_1(q)$ L-functions are simple on GRH. To be precise, we show the following.

Theorem 1.3. Assume GRH. Let q be a prime number. The proportion of simple zeros of all $\Gamma_1(q)$ L-functions is greater than or equal to L in the sense of the inequality

$$\frac{2}{\varphi(q) N(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} \sum_{f \in H_\chi}^{h} \sum_{\substack{\gamma_f \\ \text{simple}}} |\Phi(\gamma_f)|^2 \geq 0{:}9350 + o(1);$$

where Φ is chosen so that $\hat{\Phi}(i\tau) = \Big(\frac{\sin \pi\tau}{\pi\tau}\Big)^2$:

Chirre, Gonçalves and De Laat [6] showed that at least $93{:}5\%$ of zeros of a large family of primitive Dirichlet L-functions are simple, and our result is analogous. In particular, the proof of the theorem follows from Theorem 5 in [6], and we refer the reader there for details. In the approach for deriving such a proportion of simple zeros, one wants to minimize

$$\frac{g(0)}{\hat{g}(0)} + \frac{1}{\hat{g}(0)} \int_{-2}^{2} g(x)(1 - (1 - |x|)_+) \, dx; \tag{1.9}$$

where $a_+ = \max\{0; a\}$ for certain nice test functions g. In [4], the function $g(x)$ is chosen so that $\hat{g}(u) = \Big(\frac{\sin((2-\delta)\pi u)}{(2-\delta)\pi u}\Big)^2$, where δ is a small positive number. Later, Sono [26] considered the family of functions $g(x)$, where $\hat{g}(x)$ is a non-negative even-valued function in $L^1(\mathbb{R})$, $\hat{g}(0) = 1$, and g is compactly supported in $(-2; 2)$. He improves the simple zero proportion by choosing a g satisfying these conditions which minimizes (1.9). Chirre, Goncalves and de Laat [6] further improve this proportion by studying an even larger family of functions. In their case, their choice of g is eventually non-positive [1] and is not compactly supported.

As a further application of Theorem 1.1, we achieve bounds on the limsup and liminf of the pair correlation function. To state the result, we first define the $\mathcal{H}_1(q)$-analogue of Montgomery's pair correlation function $N^*(T;\beta)$ introduced in (1.3). To be precise, we let

$$N(q;\beta) := \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^k}} \sum^h_{f\in H} \left(\sum_{\substack{\gamma_f,\gamma'_f\\ 0<\gamma_f-\gamma'_f\le \frac{2\pi\beta}{\log q}}} \hat{h}(i\gamma_f)\hat{h}(i\gamma'_f) \right).$$

Similarly to the definition of $N(q)$ in (1.7), define

$$N^*(q) := \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^k}} \sum^h_{f\in H} \sum_{\gamma_f} \left(\hat{h}(i\gamma_f)\right)^2 m_{\gamma_f};$$

where m_{γ_f} denotes the multiplicity of the zero $1/2 + i\gamma_f$ of $L(s;f)$. If all the zeros of $\mathcal{H}_1(q)$ L-functions are simple then

$$N^*(q) = N(q) := \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^k}} \sum^h_{f\in H} \sum_{\gamma_f} \left(\hat{h}(i\gamma_f)\right)^2 .$$

The analogue of the Pair Correlation conjecture implies that

$$N(q;\beta) \sim N(q)\left(\frac{\beta}{2} + \frac{1}{2\pi^2\beta} + O\left(\frac{1}{\beta^2}\right)\right)$$

as $\beta \to \infty$ sufficiently slowly and q is large. In support of this we prove the following.

Theorem 1.4. *Assume GRH. Let q be a prime number and let $\varepsilon > 0$ be arbitrary. Then for $\beta > 0$, we have*

$$\limsup_{q\to\infty} \frac{N(q;\beta)}{N(q)} \le \frac{\beta}{4} + \frac{1}{4} + \varepsilon + \frac{1}{2\pi^2\beta} + O\left(\frac{1}{\beta^2}\right)$$

and, if $N^*(q) \sim N(q)$, as $q \to \infty$

$$\liminf_{q\to\infty} \frac{N(q;\beta)}{N(q)} \ge \frac{\beta}{4} - \frac{3}{4} - \varepsilon + \frac{1}{2\pi^2\beta} + O\left(\frac{1}{\beta^2}\right).$$

The bounds in Theorem 1.4 below are similar to those found by Carneiro et al. in [1]. The proof uses Beurling-Selberg majorants and minorants and follows similarly to that of Theorem 18 in Section 2.3 of [1] so we omit the proof for brevity.

[1] f is eventually non-positive if $f(x) \le 0$ for all sufficiently large $|x|$.

2. Preliminary Results

We collect necessary lemmas in this section. We start with the orthogonality relation for Dirichlet characters (e.g. see [8]) and Petersson's formula (e.g. see [14]). We conclude this section with an asymptotic for $N(q)$ and an explicit formula for $L(s,f)$.

Lemma 2.1. For $q \geq 3$, the orthogonality relation for Dirichlet characters is

$$\frac{2}{\varphi(q)} \sum_{\substack{\chi \ (\mathrm{mod}\ q) \\ \chi(-1)=(-1)^k}} \chi(m)\overline{\chi}(n) = \begin{cases} 1 & \text{if } m \equiv n \ (\mathrm{mod}\ q); \quad (mn,q)=1 \\ (-1)^k & \text{if } m \equiv -n \ (\mathrm{mod}\ q); \quad (mn,q)=1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Petersson's formula gives

$$\sum_{f \in H}^{h} \lambda_f(m)\overline{\lambda_f}(n) = \delta_{m=n} + \Delta(m,n), \tag{2.2}$$

where

$$\Delta(m,n) = 2\pi i^{-k} \sum_{c \equiv 0 \ (\mathrm{mod}\ q)} c^{-1} S(m,n;c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

and S is the Kloosterman sum defined by

$$S(m,n;cq) = \sum_{a \ (\mathrm{mod}\ cq)}^{*} \overline{\chi}(a) e\left(\frac{am+\overline{a}n}{cq}\right), \quad a\overline{a} \equiv 1. \tag{2.3}$$

The next lemma collects some well known properties and formulas for the J-Bessel function.

Lemma 2.2. Let J_{k-1} be the J-Bessel function of order $k-1$. We have

$$J_{k-1}(2\pi x) = \frac{1}{2\pi\sqrt{x}}\left(W_k(2\pi x) e\left(x - \frac{k}{4} + \frac{1}{8}\right) + \overline{W}_k(2\pi x) e\left(-x + \frac{k}{4} - \frac{1}{8}\right)\right)$$

where $W_k^{(j)}(x) \ll_{j,k} x^{-j}$. Moreover,

$$J_{k-1}(2x) = \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{x^{2\ell+k-1}}{\ell!(\ell+k-1)!} \tag{2.4}$$

and

$$J_{k-1}(x) \ll \min(x^{-1/2}, x^{k-1}).$$

The Mellin integration representation of J_{k-1} is

$$J_{k-1}(x) = \frac{1}{2\pi i}\int_{(\sigma)} 2^{-w-1} \frac{\Gamma\left(\frac{k-w-1}{2}\right)}{\Gamma\left(\frac{k+w+1}{2}\right)} x^{w}\, dw \tag{2.5}$$

where $0 < \sigma < k-1$.

The proof of the first three claims of Lemma 2.2 can be found in [28], and the statement of the last claim is modified from Equation 16 of Table 17.43 in [13].

The following standard lemma is a slight extension of Lemma 3.5 in [3], with a similar proof.

Lemma 2.3. Assume GRH for $L(s;\chi)$ with χ mod q. Let ϕ be a smooth function supported in $(a;b)$, where $0 < a < b$. Write $z = \sigma + it$ with $\sigma \in \left[\frac{1}{2} - \frac{10}{\log q}; \frac{1}{2} + \frac{10}{\log q}\right]$ and t real. If χ is a non-principal character, then

$$\sum_p \frac{\chi(p)\log(p)\,\phi(p/X)}{p^z} \ll \log^2(q + |t|) \max_{a\le x\le b} |\phi^{(3)}(x)|.$$

If χ is the principal character, then

$$\sum_p \frac{\chi(p)\log(p)\,\phi(p/X)}{p^z} \ll \log^2(q + |t|) \max_{a\le x\le b} |\phi^{(3)}(x)| + \frac{\sqrt{X}}{(1 + |t|)^A} \max_{a\le x\le b} |\phi^{(A)}(x)|;$$

for any $A > 0$.

Proof. When $\sigma \geq \frac{1}{2} + \frac{1}{\log q}$, under GRH,

$$\frac{L'}{L}(z;\chi) = O\left(\log^2(q + |t|)\right)$$

(see Chapter 19 of [8]). By Lemma 2 of [4],

$$\sum_p \frac{\chi(p)\log(p)\,\phi(p/X)}{p^z} = \sum_n \frac{\Lambda(n)\chi(n)\,\phi(n/X)}{n^z} + O(1).$$

Let δ_χ be 1 if χ is principal, and 0 otherwise.

Recall that

$$\hat{\phi}(s) = \int_0^\infty \phi(x)x^{s-1}\,dx. \tag{2.6}$$

By doing integration by parts B times, we have that

$$\hat{\phi}(s) = \int_0^\infty \phi^{(B)}(x)\,\frac{x^{s+B-1}}{s(s+1)\cdots(s+B-1)}\,dx.$$

Thus if $s = \sigma_1 + iv$, where $\sigma_1 > 0$, then

$$|\hat{\phi}(s)| \ll \frac{1}{|\sigma_1 + iv|(|v| + 1)^{B-1}} \max_{a\le x\le b} |\phi^{(B)}(x)| \tag{2.7}$$

By (2.6) and (2.7), we have

$$\begin{aligned}
\sum_n \frac{\Lambda(n)\chi(n)\,\phi(n/X)}{n^z} &= \frac{1}{2\pi i}\int_{(1)} \hat{\phi}(s)X^s \sum_n \frac{\Lambda(n)\chi(n)}{n^{z+s}}\,ds \\
&= -\frac{1}{2\pi i}\int_{(1)} \hat{\phi}(s)X^s \frac{L'}{L}(z + s;\chi)\,ds \\
&= -\frac{1}{2\pi i}\int_{\left(-\frac{20}{\log q}\right)} \hat{\phi}(s)X^s \frac{L'}{L}(z + s;\chi)\,ds + \delta_\chi\hat{\phi}(1 - z)X^{1-z}
\end{aligned}$$

$$\ll \int_{-\infty}^{\infty} \left|\hat\Phi\left(\frac{20}{\log q}+iv\right)\right| \log^2(q+|t|+|v|)\,dv + |\hat\Phi(1-it)|\sqrt{X}$$

By choosing $A = 2$ for the first term and $B = A$ for the second term, the above is bounded by

$$\ll \int_{-\infty}^{\infty} \frac{1}{\left|\frac{20}{\log q}+iv\right|(|v|+1)^2}\log^2(q+|t|+|v|)\,dv + \frac{\sqrt{X}}{(1+|t|)^A}\max_{a\le x\le b}|\Phi^{(A)}(x)|$$

$$\ll \log^2(q+|t|)\max_{a\le x\le b}|\Phi^{(3)}(x)| + \frac{\sqrt{X}}{(1+|t|)^A}\max_{a\le x\le b}|\Phi^{(A)}(x)|;$$

for any $A>0$ by the decay of $\hat\Phi$ on vertical lines.

The following lemma provides an asymptotic formula for $N(q)$ as defined in (1.7).

Lemma 2.4. Assume GRH. We have

$$N(q) \sim \frac{\log q}{2\pi}\int_{\mathbb{R}} |\hat\Phi(it)|^2\,dt$$

as $q\to\infty$.

Proof. Let $N(f;T)$ denote the number of zeros of $L(s,f)$ for $s=\sigma+it$ satisfying $0<\sigma<1$ and $-T\le t\le T$. Assuming GRH, the number of critical zeros of $L(s,f)$ below height T is

$$N(f;T) = \frac{T}{\pi}\log\frac{qT^2}{(2\pi e)^2} + O\left(\frac{\log(q(T+|k|)^2)}{\log\log(q(T+|k|)^2)}\right)$$

uniformly (e.g. see [2]). By integration by parts, we have

$$\begin{aligned}
\sum_{\gamma_f} |\hat\Phi(i\gamma_f)|^2 &= \int_0^\infty |\hat\Phi(it)|^2\,dN(f;t)\\
&= \frac{1}{\pi}\log q\int_0^\infty |\hat\Phi(it)|^2\,dt + O_k\left(\frac{\log q}{\log\log q}\right)\\
&= \frac{1}{2\pi}\log q\int_{-\infty}^{\infty} |\hat\Phi(it)|^2\,dt + O_k\left(\frac{\log q}{\log\log q}\right);
\end{aligned}$$

where an implied constant depends on k, and so

$$\begin{aligned}
N(q) := \frac{2}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^k}}\ \sideset{}{^h}\sum_{f\in H}\ \sum_{\gamma_f}|\hat\Phi(i\gamma_f)|^2
&= \frac{2}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi(-1)=(-1)^k}}\ \sideset{}{^h}\sum_{f\in H}\left[\frac{1}{2\pi}\log q\int_{-\infty}^{\infty}|\hat\Phi(it)|^2\,dt + O_k\left(\frac{\log q}{\log\log q}\right)\right]\\
&= \frac{\log q}{2\pi}\int_{-\infty}^{\infty}|\hat\Phi(it)|^2\,dt + O_k\left(\frac{\log q}{\log\log q}\right).
\end{aligned}$$

Finally, we have an explicit formula which relates zeros of $L(s;f)$ with a sum over its prime power coefficients.

Lemma 2.5. *Assume GRH for $L(s;f)$. Let $X \geq 1$. Then*

$$\sum_{\gamma_f} \hat{\phi}(i\gamma_f) X^{i\gamma_f} = -\sum_{n=1}^{\infty} \frac{\phi(n/X)\Lambda(n)c_f(n)}{\sqrt{n}} + \phi(X^{-1}) \log\frac{q}{4\pi^2} + O\left(X^{-1}\log|k|\right),$$

where $c_f(n)$ is defined in (1.8).

Proof. We will first examine the integral

$$\frac{1}{2\pi i}\int_{(3/4)} \frac{L'}{L}\left(s+\frac{1}{2};f\right)\hat{\phi}(s)X^{s}\,ds.$$

Note that we assume GRH for $L(s;f)$, and $L(s;f)$ is entire. Thus

(2.8)
$$\frac{1}{2\pi i}\int_{(3/4)} \frac{L'}{L}\left(s+\frac{1}{2};f\right)\hat{\phi}(s)X^{s}\,ds = \frac{1}{2\pi i}\int_{(-1)} \frac{L'}{L}\left(s+\frac{1}{2};f\right)\hat{\phi}(s)X^{s}\,ds + \sum_{\gamma_f}\hat{\phi}(i\gamma_f)X^{i\gamma_f}.$$

For the integral on the right hand side along the line $\mathrm{Re}(s) = -1$, we will examine the logarithmic derivative of the functional equation in (1.5). We then have

$$\frac{1}{2}\log\frac{q}{4\pi^2} + \frac{\Gamma'}{\Gamma}\left(s+\frac{k}{2}\right) + \frac{L'}{L}\left(s+\frac{1}{2};f\right) = -\frac{1}{2}\log\frac{q}{4\pi^2} - \frac{\Gamma'}{\Gamma}\left(-s+\frac{k}{2}\right) - \frac{L'}{L}\left(\frac{1}{2}-s;f\right).$$

For $s = -1+it$, $\frac{L'}{L}\left(\frac{1}{2}-s;f\right) \ll 1$ and the Gamma terms contribute $\ll \log(|t|+|k|)$. Thus the integral on the right hand side of (2.8) is

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(-1)} \frac{L'}{L}\left(s+\frac{1}{2};f\right)\hat{\phi}(s)X^{s}\,ds \\
&= -\frac{1}{2\pi i}\int_{(-1)} \log\frac{q}{4\pi^2}\,\hat{\phi}(s)X^{s}\,ds + O\left(X^{-1}\log|k|\right) \\
&= -\log\frac{q}{4\pi^2}\,\phi(X^{-1}) + O\left(X^{-1}\log|k|\right)
\end{aligned}$$

We conclude the proof by expressing the original integral in terms of the series representation for $\frac{L'}{L}(s;f)$. To be precise, since

$$\frac{L'}{L}(s;f) = -\sum_{n=1}^{\infty}\frac{\Lambda(n)c_f(n)}{n^{s}}$$

for $\mathrm{Re}(s) > 1$, hence

$$\frac{1}{2\pi i}\int_{(3/4)} \frac{L'}{L}\left(s+\frac{1}{2};f\right)\hat{\phi}(s)X^{s}\,ds = -\sum_{n=1}^{\infty}\frac{\Lambda(n)c_f(n)}{\sqrt{n}}\,\frac{1}{2\pi i}\int_{(3/4)}\hat{\phi}(s)\left(\frac{X}{n}\right)^{s}ds$$

$$= \sum_{n=1}^{\infty} \frac{\Lambda(n) c_f(n)}{\sqrt{n}} \hat{\Phi}\left(\frac{\log n}{\log X}\right).$$

3. Outline of the proof of Proposition 1.2

Since the proof of Proposition 1.2 is somewhat lengthy and has some delicate points, we provide an outline here to help orient the reader.

The main object to be studied is

$$\sum_{p,r} a_p a_r \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} \sum_{f \in H} {}^{h}\, \psi_f(p)\overline{\psi_f(r)}$$

where

$$a_n = \frac{\Lambda(n)\hat{\Phi}(\log n/\log X)}{\sqrt{n}}. \tag{3.1}$$

Applying Petersson's formula, we obtain the diagonal term arising from $p = r$, which is easy to evaluate (see Lemma 4.1), and the off-diagonal terms which are

$$\sum_{p \neq r} a_p a_r \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} 2\pi i^{-k} \sum_{\substack{c>0 \\ c \equiv 0 \bmod q}} c^{-1} S(p; r; c) J_{k-1}\left(\frac{4\pi\sqrt{pr}}{c}\right).$$

Recall that $X = q^{\Delta}$. When $1 < |\Delta| < 2$, there will be the additional main term from the off-diagonal terms.

By applying orthogonality of Dirichlet characters and rearranging the resulting exponential sum it suffices to understand

$$\sum_{s,t} \frac{2\pi}{qst} \sum_{\substack{x \pmod t \\ (x(x+qs);t)=1}} \sum_{m \bmod t}{}^{*} \sum_{n \bmod t}{}^{*} e\left(\frac{m\bar{x} + n\overline{x+qs}}{t}\right) \sum_{\substack{p,r \\ p \equiv m \bmod t \\ r \equiv n \bmod t}} a_p a_r e\left(\frac{p+r}{qst}\right) J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qst}\right).$$

We express the congruence conditions $p \equiv m \pmod t$ and $r \equiv n \pmod t$ using Dirichlet characters modulo t. The off-diagonal main contribution comes from the principal characters, and the rest give error terms. In contrast to the diagonal term, the evaluation of the off-diagonal main terms requires careful analysis which is performed in §6.

To illustrate what happens with the error terms, let us consider only the transition region for the Bessel function where $st \asymp \frac{X}{q}$. In this region, the factor $e\left(\frac{p+r}{qst}\right)$ has small derivatives and can be absorbed as a smooth function. Applying Cauchy-Schwarz's inequality and orthogonality relations, we would like to bound the sum of the form

$$\frac{1}{qS} \sum_{\substack{s \sim S \\ t \sim T \\ ST \asymp \frac{X}{q}}} \sum_{;t} \frac{1}{\phi(t)} \sum_{\substack{\psi \pmod t \\ \psi \neq \psi_0}} \left| \sum_{p \sim X} \psi(p) \frac{\log p}{\sqrt{p}} \right|^2$$

This is where we require GRH for Dirichlet L-functions to conclude that the sum over p is very small and in particular bounded by q when χ is a non-principal character. Executing the rest of the sum trivially gives that the error terms are bounded by $\frac{X}{q^2 S}$ q when X q^2. This also indicates the barrier of the method, i.e. we cannot evaluate $F(\phi)$ for $|\sigma| > 2$: The full details of this error term bound can be found in §7 and §8.

4. Initial Setup for the Proof of Proposition 1.2

First, we note that the inner sum in Proposition 1.2 can be expressed as a sum over primes p:

$$\sum_n \frac{\Lambda(n) c_f(n) \Phi(n/X)}{\sqrt{n}} = \sum_p \frac{\Lambda(p) c_f(p) \Phi(p/X)}{\sqrt{p}} + \sum_p \frac{\Lambda(p^2) c_f(p^2) \Phi(p^2/X)}{p}$$
$$+ \sum_p \sum_{k \geq 3} \frac{\Lambda(p^k) c_f(p^k) \Phi(p^k/X)}{p^{k/2}}$$
$$= \sum_p \frac{\Lambda(p) \lambda_f(p) \Phi(p/X)}{\sqrt{p}} + \sum_p \frac{\Lambda(p^2)(\lambda_f(p^2) \qquad \chi(p)) \Phi(p^2/X)}{p} + O(1):$$

Since Φ has compact support in (a,b) for some $0 < a < b$, we see

$$\sum_p \frac{\log p (\lambda_f(p^2) \qquad \chi(p)) \Phi(p^2/X)}{p} \qquad \sum_{\sqrt{aX} < p < \sqrt{bX}} \frac{\log p}{p}$$
$$= \log \sqrt{bX} \quad \log \sqrt{aX} + O(1)$$
$$= O(1):$$

Thus the asymptotic in Proposition 1.2 is equivalent to

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} \sum_{f \in H}^{h} \sum_p \frac{\log(p) \lambda_f(p) \Phi(p/X)}{\sqrt{p}} \quad f(\phi)^2 \log q \quad \frac{\qquad}{2} \int_{\mathbb{R}} |t| |\hat{\phi}(t)|^2 \, dt:$$

Let a_n be dened as in (3.1). By Petersson's formula (Lemma 2.1), we have

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} \sum_{f \in H}^{h} \Big| \sum_p a_p \lambda_f(p) \Big|^2 = \sum_{p;r} a_p a_r \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} \sum_{f \in H}^{h} \lambda_f(p) \lambda_f(r)$$
$$= \sum_p a_p^2 + \sum_{p;r} a_p a_r \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} \quad (p; r)$$
$$=: S_D + S_N:$$

For the diagonal term, Lemma 3 in [4] shows:

Lemma 4.1. As $X \; ! \;\; 1$,

$$S_D = \sum_p \frac{\log^2 p \; ^2(p=X)}{p} = \frac{1}{2}\int_{\mathbb{R}} |(it)|^2 dt \; \log X + O(1):$$

The non-diagonal term S_N will contribute to the main term when $1 < \;\; 2$. We evaluate S_N in the following lemma and the proof of Proposition 1.2 follows from this computation.

Lemma 4.2. Assume GRH. Let q be a prime, be dened as in Proposition 1.2, $k \;\; 3$, and $X = q$. Then

$$S_N = \begin{cases} \frac{1}{2}\log \frac{q}{X}\int_{\mathbb{R}} |(it)|^2 \, dt + O(1) & \text{if } 1 < \;\; 2 \\ O(1) & \text{if } 0 \;\; 1: \end{cases}$$

uniformly for $0 \;\; 2$ as $q \; ! \;\; 1$. The implied constant depends on and k.

The bulk of the work of this paper is in proving Lemma 4.2. This will be done in Sections 5 - 9.

5. The non-diagonal term S_N

First we will extract the main term from S_N. To do this we aim to rewrite the congruence in terms of Dirichlet characters but rst we must manipulate the sum. Recall that

$$S_N = \sum_{p;r} \sum a_p a_r \frac{2}{(q)} \sum_{\substack{\text{mod } q \\ (1)=(1)^k}} (p;r)$$

$$= \sum_{p;r} \sum a_p a_r \frac{2}{(q)} \sum_{\substack{\text{mod } q \\ (1)=(1)^k}} 2i^k \sum_{\substack{c>0 \\ c \equiv 0 \text{ mod } q}} c^{1} S(p;r;c) J_{k-1}\left(\frac{4\sqrt{pr}}{c}\right);$$

where from (2.3)

$$S(p;r;c) = \sum_{a\bar{a} \equiv 1 \; (\text{mod } c)} \overline{(a)} e\left(\frac{ap + \bar{a}r}{c}\right):$$

Let $Kf := i^{-k} f + i^k \bar{f}$. Then by orthogonality of Dirichlet characters in (2.1),

$$\frac{2}{(q)} \sum_{\substack{\text{mod } q \\ (1)=(1)^k}} S(p;r;c) = \frac{2}{(q)} \sum_{\substack{\text{mod } q \\ (1)=(1)^k}} \sum_{a\bar{a} \equiv 1 \; (\text{mod } c)} \overline{(a)} e\left(\frac{ap + \bar{a}r}{c}\right)$$

$$= i^k K \sum_{\substack{a \text{ mod } c \\ a \equiv 1 \text{ mod } q}}{}^{*} e\left(\frac{ap + \bar{a}r}{c}\right):$$

Thus, after a change of variable in c, we obtain that

$$S_N = \sum_{p,r} a_p a_r \, 2\pi \sum_{c>0} \frac{1}{cq} J_{k-1}\left(\frac{4\pi\sqrt{pr}}{cq}\right) K \sideset{}{^*}\sum_{\substack{a \bmod cq\\ a\equiv 1 \bmod q}} e\left(\frac{ap+\bar a r}{cq}\right).$$

Now, we follow Iwaniec and Li [17] and set $a = 1+qy$ with y running modulo c. Let $(y,c) = s$ so that $c = st$ and $a = 1+qsx$ with x running modulo t. Furthermore, $(x(1+qsx),t) = 1$. We have

$$\sideset{}{^*}\sum_{\substack{a \bmod cq\\ a\equiv 1 \bmod q}} e\left(\frac{ap+\bar a r}{cq}\right) = \sum_{st=c} \sum_{\substack{x \pmod t\\ (x(1+qsx),t)=1}} e\left(\frac{p(1+qsx)+r\overline{(1+qsx)}}{qst}\right).$$

Then we write $\overline{1+u} \equiv 1 - u\overline{(1+u)} \pmod t$ for $u = qsx$. Next, we change x to $\bar x$ and $\overline{(1+qs\bar x)}\,\bar x \equiv \overline{x+qs} \pmod t$. Thus we have

$$\sideset{}{^*}\sum_{\substack{a \bmod cq\\ a\equiv 1 \bmod q}} e\left(\frac{ap+\bar a r}{cq}\right) = e\left(\frac{p+r}{cq}\right) \sum_{\substack{s,t\\ st=c}} V_{qs}(p,r;t)$$

where

$$V_d(p,r;t) := \sum_{\substack{x \bmod t\\ (x(x+d),t)=1}} e\left(\frac{px - r\overline{x+d}}{t}\right).$$

Therefore

$$S_N = \sum_{p,r} a_p a_r K \sum_{s,t} \frac{2\pi}{qst} e\left(\frac{p+r}{qst}\right) V_{qs}(p,r;t) J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qst}\right).$$

We now write

$$S_N = M_{off} + E_{off},$$

where M_{off} is the contribution from terms $(t,pr) = 1$, and E_{off} is the rest. The purpose is to extract the terms with $(t,pr) = 1$ and later rewrite the congruence condition $\pmod t$ in terms of Dirichlet characters.

Lemma 5.1. For $k \geq 3$, we have

$$E_{off} := \sum_{p,r} a_p a_r K \sum_{\substack{s,t\\ (t,pr)\neq 1}} \frac{2\pi}{qst} e\left(\frac{p+r}{qst}\right) V_{qs}(p,r;t) J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qst}\right) \ll q^{-1}.$$

Proof. We have

$$E_{off} = \sum_{\substack{p,r\\ p=r}} a_p a_r K \sum_{\substack{s,t\\ p\mid t}} \frac{2\pi}{qst} e\left(\frac{p+r}{qst}\right) V_{qs}(p,r;t) J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qst}\right)$$

$$+\sum_{\substack{p,r\\ p\neq r}}\sum a_p a_r\, K \sum_{\substack{s,t\\ r\mid t;\, p\nmid t}}\sum \frac{2\pi}{qst}\, e\left(\frac{p+r}{qst}\right) V_{qs}(p;r;t) J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qst}\right)$$

$$+\sum_{p} a_p^2\, K \sum_{\substack{s,t\\ p\mid t}}\sum \frac{2\pi}{qst}\, e\left(\frac{2p}{qst}\right) V_{qs}(p;p;t) J_{k-1}\left(\frac{4\pi p}{qst}\right)$$

$$=: E_1+E_2+E_3.$$

We will focus on bounding E_1. The bound for E_2 and E_3 follows similarly.

By the change of variable $t \to tp$,

$$E_1=\sum_{\substack{p,r\\ p\neq r}}\sum \frac{a_p a_r}{\sqrt{p}}\, K \sum_{s,t}\sum \frac{2\pi}{qst}\, e\left(\frac{p+r}{qstp}\right) V_{qs}(p;r;tp) J_{k-1}\left(\frac{4\pi\sqrt{p}\sqrt{r}}{qst\sqrt{p}\sqrt{p}}\right)$$

Using trivial bound for V_{qs}, the bound in Lemma 2.2, which is

$$J_{k-1}\left(\frac{4\pi\sqrt{r}}{qst\sqrt{p}}\right) \ll \left(\frac{4\pi\sqrt{r}}{qst\sqrt{p}}\right)^{k-1}$$

$X \ll q^{2-\delta}$ and $k \geq 3$, we obtain that

$$\begin{aligned}
E &\ll \sum_{aX<p,r<bX}\sum \frac{\log p\log r}{\sqrt{p}\sqrt{r}}\sum_s \frac{1}{qs}\sum_t \left(\frac{4\pi\sqrt{r}}{qst\sqrt{p}}\right)^{k-1}\\
&\ll \sum_{aX<p,r<bX}\sum \frac{\log p}{p^{k/2}}(\log r) r^{k/2-1}\sum_s \frac{1}{q^k s^k}\\
&\ll \frac{1}{q^k X}\sum_{p,r\asymp X}\sum \log p\log r\\
&\ll \frac{X}{q^k} \ll q^{-1}
\end{aligned}$$

upon choosing small enough δ. By the same arguments, we can show that $E_2 \ll q^{-1}$, and $E_3 \ll q^{-4}$.

Therefore

$$S_N = M_{off} + O(q^{-1}),$$

where

$$M_{off} = K\sum_{s,t}\sum \frac{2\pi}{qst}\sum_{m \bmod t}^{*}\sum_{n \bmod t}^{*} V_{qs}(m;n;t)\sum_{\substack{p,r\\ p\equiv m \bmod t\\ r\equiv n \bmod t}}\sum a_p a_r\, e\left(\frac{p+r}{qst}\right) J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qst}\right).$$

Now we express the condition $p \equiv m \bmod t$ and $r \equiv n \bmod t$ using Dirichlet charac-ters. In particular,

$$
\begin{aligned}
M_{off} = K & \sum_{s;t} \sum \frac{2}{qst} \sum_{\substack{m \bmod t \\ n \bmod t}}^{*} \sum^{*} V_{qs}(m;n;t) \frac{1}{(t)^2} \sum_{\bmod t} \sum {}^{0}\overline{(m)(n)}{}^{0} \bmod t \\
& \sum_{p;r} \sum {}^{0}(p)(r) a_p a_r \, e\left(\frac{p r}{qst}\right) J_{k-1}\left(\frac{4\sqrt{pr}}{qst}\right) \\
= K & \sum_{s;t} \sum \frac{2}{qst(t)^2} \sum_{\substack{m \bmod t \\ (pr;t)=1}}^{*} \sum^{*} V_{qs}(m;n;\ ) \Bigg[t \sum_{p;r} \sum a_p a_r \, e\left(\frac{p+r}{qst}\right) J_{k-1}\left(\frac{4\sqrt{pr}}{qst}\right) \overline{\ }_{n \bmod t} \\
& + \sum_{\substack{\bmod t \\ {}^{0} \bmod t \\ =0;\,{}^{0}=0}} \sum {}^{0}\overline{(m)(n)} \sum_{;\ p\,r} \sum {}^{0}(p)(r) a_p a_r \, e\left(\frac{p\ r}{qst}\right) J_{k-1}\left(\frac{4\sqrt{pr}}{qst}\right) \\
& + 2 \sum_{(p;t)=1} \sum_{\substack{\bmod t \\ =0}} \overline{(n)} \sum_{r} (r) a_p a_r \, e\left(\frac{p+r}{qst}\right) J_{k-1}\left(\frac{4\sqrt{pr}}{qst}\right) \Bigg] \\
=: & \ S_1 + S_2 + S_3.
\end{aligned}
$$

When $1 < \ \ 2$, S_1 contributes to the main term while S_2 and S_3 are always subsumed in the error term. We will treat S_1, S_2 and S_3 in Section 6, 7 and 8, respectively.

6. Main Term S_1

In this section, our goal is to prove the following.

Lemma 6.1. Assume RH. Let q be a prime, be dened as in Proposition 1.2, $k\ \ 3$, and $X = q$. Then

$$
S_1 = \begin{cases} \frac{1}{2}\log \frac{q}{X} \int_{\mathbb{R}} |(it)|^2 \, dt + O(1) & \text{if } 1 < \ \ 2 \\ O(1) & \text{if } 0 \ \ 1 \end{cases}
$$

uniformly for $0 \ \ 2$ as $q \to \infty$. The implied constant depends on and k.

6.1. Preliminary lemmas for evaluating S_1. We will collect necessary lemmas to calculate S_1 in this section. The rst lemma taken from [5], and its proof can be found there.

Lemma 6.2. Let $(a;\ell) = 1$. We have

$$
f(c;\ell) := \sum_{\substack{x \ (\mathrm{mod}\ c\ell) \\ x \equiv a \ (\mathrm{mod}\ \ell)}}^{*} 1 = c \prod_{\substack{p|c \\ p\nmid \ell}} \left(1 - \frac{1}{p}\right) = (c) \prod_{p|(\ell;c)} \left(1 - \frac{1}{p}\right)^{-1}.
$$

The next lemma is a slight modication of Lemma 7.2 in [5]. The proof follows by the same arguments.

Lemma 6.3. Let $\ell; m; n$ be positive integers satisfying $aX \le m; n \le bX$ and $q \asymp X$. We recall that $X = q^{\theta}$, where $0 < \theta < 2$. Define

$$T = T(q;\ell;m;n) := \sum_{\ell=1}^{\infty} J_{k-1}\left(\frac{4\pi\sqrt{n\ell}}{q}\right)\frac{K(\ell, m; q)}{q} \quad m \quad n$$

Further, let $L = \frac{Xq}{q}$ and w be a smooth function on $\mathbb{R}^+$ with $w(x) = 1$ if $0 \le x \le 1$, and $w(x) = 0$ if $x > 2$. Then for any $A > 0$, we have

$$T = 2\sum_{\ell=1}^{\infty} w\left(\frac{\ell}{L}\right) J_{k-1}\left(4\pi\sqrt{\frac{m\ell}{q}}\right) J_{k-1}\left(4\pi\sqrt{\frac{n\ell}{q}}\right) - 2\int_0^{\infty} w\left(\frac{\ell}{L}\right) J_{k-1}\left(4\pi\sqrt{\frac{m\ell}{q}}\right) J_{k-1}\left(4\pi\sqrt{\frac{n\ell}{q}}\right) d\ell + O_A(q^{-A}).$$

Our next lemma is similar to Lemma 7.3 in [5], and we will include the proof here for completeness.

Lemma 6.4. Assume RH. Let w be a smooth function on $\mathbb{R}^+$ with $w(x) = 1$ if $0 \le x \le 1$; and $w(x) = 0$ if $x > 2$. Also we let $L > 0$ be any real parameter. Then for $\mathrm{Re}(s) = \varepsilon_0$, where ε_0 is a small positive real number, and $0 \le a \le \frac{1}{2}$,

$$\sum_{\ell=1}^{\infty} w\left(\frac{\ell}{L}\right)\ell^{-s-a} - \int_0^{\infty} w\left(\frac{\ell}{L}\right)\ell^{-s-a}\, d\ell = O\left(L^{\frac{1}{2}-a+\varepsilon_0}(1+|\mathrm{Im}(s)|)\right). \tag{6.1}$$

Moreover,

$$\sum_{\ell=1}^{\infty} w\left(\frac{\ell}{L}\right)\frac{1}{\ell^{1-s}} - \int_0^{\infty} w\left(\frac{\ell}{L}\right)\frac{d\ell}{\ell^{1-s}} = \zeta(1-s) + O\left(L^{-\frac{1}{2}}(1+|\mathrm{Im}(s)|)\right). \tag{6.2}$$

Proof. Let $\hat{w}(z)$ be the Mellin transform of w, defined by

$$\hat{w}(z) = \int_0^{\infty} w(t)\frac{t^z}{t}\, dt.$$

By integration by parts as in Lemma 7.3 of [5], $\hat{w}(z)$ can be analytically continued to $\mathrm{Re}\, z > -1$ except a simple pole at $z = 0$ with residue $w(0) = 1$. For (6.1), let $\sigma > 1 - a + \varepsilon_0$ so we have

$$\sum_{\ell=1}^{\infty} w\left(\frac{\ell}{L}\right)\ell^{-s-a} = \sum_{\ell=1}^{\infty}\frac{1}{2\pi i}\int_{(\sigma)}\hat{w}(z)\left(\frac{L}{\ell}\right)^z \ell^{-s-a}\, dz = \frac{1}{2\pi i}\int_{(\sigma)}\hat{w}(z)L^z\zeta(z+s+a)\, dz.$$

Then we shift the contour to $\mathrm{Re}(z) = \frac{1}{2} - a + \varepsilon_0$, picking up the simple pole of $\zeta(z+s+a)$ at $z = 1 - s - a$. Thus

$$\sum_{\ell=1}^{\infty} w\left(\frac{\ell}{L}\right)\ell^{-s-a} = \hat{w}(1-s-a)L^{1-s-a} + \frac{1}{2\pi i}\int_{(\frac{1}{2}-a+\varepsilon_0)}\hat{w}(z)L^z\zeta(z+s+a)\, dz.$$

From the denition of $\hat{w}$,

$$\hat{w}(1+s-a)L^{1+s-a} = \int_0^1 w\left(\frac{\ell}{L}\right)\ell^{s-a}\,d\ell.$$

Since $\mathrm{Re}(z-s+a) = \frac{1}{2}$ when $\mathrm{Re}(z) = \frac{1}{2} - a + \sigma_0$, and for any $\epsilon > 0$, under RH,

$$\zeta(\sigma + it) \ll (|t|+1)^{\epsilon} \tag{6.3}$$

uniformly for $\frac{1}{2} \le \sigma \le 1 - \sigma_1$ (e.g. see [27]), we obtain (6.1).

For (6.2), let $c > \sigma_0$; we have

$$\sum_{\ell=1}^{\infty} w\left(\frac{\ell}{L}\right)\frac{1}{\ell^{1-s}} = \frac{1}{2\pi i}\int_{(c)} \hat{w}(z)L^{z}\zeta(z+1-s)\,dz. \tag{6.4}$$

Shifting the contour to $\mathrm{Re}(z) = -\frac{1}{4}$, we pass through poles at $z = s$ and $z = 0$. Thus (6.4) is

$$\zeta(1-s) + \int_0^1 w\left(\frac{\ell}{L}\right)\frac{1}{\ell^{1-s}}\,d\ell + \frac{1}{2\pi i}\int_{(-1/4)} \hat{w}(z)L^{z}\zeta(z+1-s)\,dz.$$

Equation (6.2) follows from the bound in (6.3).

6.2. Proof of Lemma 6.1. We recall that

$$S_1 = K\sum_{s}\sum_{t}\frac{2}{qst\phi(t)^2}\sideset{}{^*}\sum_{m \bmod t}\sideset{}{^*}\sum_{n \bmod t} V_{qs}(m;n;t)\sum_{\substack{p,r\\(pr;t)=1}} a_p\bar{a}_r\, e\left(\frac{p+r}{qst}\right)J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qst}\right).$$

We would like to apply Lemma 6.3 to the sum over s in S_1 but some manipulation must be done rst to t the assumptions. After applying Lemma 6.3, we shift the resulting contours and use Lemma 6.4 to extract the main term and bound the error (see the proof of Lemma 6.6).

First we see that

$$\sideset{}{^*}\sum_{\substack{m \bmod t\\ n \bmod t}}\sideset{}{^*}\sum V_{qs}(m;n;t) = \sideset{}{^*}\sum_{\substack{x \bmod t\\(x+qs;t)=1}}\ \sideset{}{^*}\sum_{m \bmod t} e\left(\frac{mx}{t}\right)\sideset{}{^*}\sum_{n \bmod t} e\left(\frac{n(\bar{x}+\overline{qs})}{t}\right) = \mu^2(t)\sideset{}{^*}\sum_{\substack{x \bmod t\\(x+qs;t)=1}} 1$$

since the sums over m and n are Ramanujan sums, each equal to $\mu(t)$. Next we can remove the coprimality condition $(pr;t) = 1$ by the same arguments as bounding E_{off} in Lemma 5.1. Moreover, we can express the coprimality condition $(x+qs;t) = 1$ as $\sum_{d|(x+qs;t)}\mu(d)$. Thus up to $O(q^{-1})$,

$$S_1 = \sum_{s}\sum_{t}\frac{2\mu^2(t)}{qst\phi(t)^2}\sideset{}{^*}\sum_{\substack{x \bmod t\\(x+qs;t)=1}} 1\sum_{p,r} a_p\bar{a}_r\, K\, e\left(\frac{p+r}{qst}\right)J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qst}\right)$$

$$= \sum_{d} \frac{\mu(d)}{d} \sum_{s,t} \frac{\mu^2(td)}{qst\phi(td)^2}\left(\sum^{*}_{\substack{x \bmod td\\ x\equiv qs \bmod d}} 1\right) \sum_{p,r} a_p a_r K e\left(\frac{p+r}{qstd}\right) J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qstd}\right)$$

$$= \sum_{\substack{d\\(d,q)=1}} \frac{\mu(d)}{d} \sum_{\substack{s,t\\(s,d)=1}} \frac{\mu^2(td)}{qst\phi(td)^2}\phi(t) \prod_{p|(t,d)}\left(1-\frac{1}{p}\right)^{-1} \sum_{p,r} a_p a_r K e\left(\frac{p+r}{qstd}\right) J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qstd}\right)$$

by Lemma 6.2. In the above, we have noted that the sum over $x \bmod td$ is zero if $(qs,d)\neq 1$. Further, the presence of the factor $\mu^2(td)$ forces $(t,d)=1$. Therefore

$$S = \sum_{(d,q)=1} \frac{\mu(d)}{d\phi^2(d)} \sum_{\substack{s,t\\(st,d)=1}} \frac{\mu^2(t)}{qst\phi(t)} \sum_{p,r} a_p a_r K e\left(\frac{p+r}{qstd}\right) J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qstd}\right) + O\left(q^{-1}\right).$$

Now we will include terms of the form

$$a_{p^{j_1}} a_{r^{j_2}} K e\left(\frac{p^{j_1}+r^{j_2}}{qstd}\right) J_{k-1}\left(\frac{4\pi\sqrt{p^{j_1}r^{j_2}}}{qstd}\right)$$

where $\max\{j_1,j_2\}\geq 2$. The contribution of these terms is very small, and we include them now for technical convenience.

Lemma 6.5. Let

$$E_{\text{high power}} := \sum_{\substack{p^{j_1},r^{j_2}\\ \max\{j_1,j_2\}\geq 2}} a_{p^{j_1}} a_{r^{j_2}} \sum_{(d,q)=1} \frac{\mu(d)}{d\phi^2(d)} \sum_{\substack{s,t\\(st,d)=1}} \frac{\mu^2(t)}{qst\phi(t)} K e\left(\frac{p^{j_1}+r^{j_2}}{qstd}\right) J_{k-1}\left(\frac{4\pi\sqrt{p^{j_1}r^{j_2}}}{qstd}\right).$$

Then

$$E_{\text{high power}} \ll q^{-\epsilon/4}.$$

Proof. From the bound of J_{k-1} in Lemma 2.2 and the definition of a_p in (3.1), we have that

$$E_{\text{high power}} \ll \frac{1}{q} \sum_{\substack{j_1,j_2\leq \log X\\ \max\{j_1,j_2\}\geq 2}} \sum_{p\leq X^{1/j_1};\, r\leq X^{1/j_2}} \frac{\log p}{p^{j_1/2}}\frac{\log r}{r^{j_2/2}} \sum_d \frac{1}{d\phi^2(d)} \sum_t \frac{1}{t\phi(t)} \sum_s \frac{1}{s} \min\left\{\left(\frac{\sqrt{p^{j_1}r^{j_2}}}{qstd}\right)^{1/2}, \left(\frac{\sqrt{p^{j_1}r^{j_2}}}{qstd}\right)^{k-1}\right\}$$

$$\ll \frac{1}{q} \sum_{\substack{j_1,j_2\leq \log X\\ \max\{j_1,j_2\}\geq 2}} \sum_{p\leq X^{1/j_1};\, r\leq X^{1/j_2}} \frac{\log p}{p^{j_1/2}}\frac{\log r}{r^{j_2/2}} \sum_d \frac{1}{d\phi^2(d)} \sum_t \frac{1}{t\phi(t)} \left(\sum_{s<\frac{\sqrt{p^{j_1}r^{j_2}}}{qtd}} \frac{1}{s}\left(\frac{\sqrt{p^{j_1}r^{j_2}}}{qstd}\right)^{1/2} + \sum_{s\geq\frac{\sqrt{p^{j_1}r^{j_2}}}{qtd}} \frac{1}{s}\left(\frac{\sqrt{p^{j_1}r^{j_2}}}{qstd}\right)^{k-1}\right)$$

$$\begin{aligned}
&\ll \frac{1}{q}\sum_{\substack{j_1,j_2\le \log X\\ \max\{j_1,j_2\}\ge 2}}\ \sum_{p\le X^{1/j_1}}\ \sum_{r\le X^{1/j_2}} \frac{\log p}{p^{j_1/2}}\frac{\log r}{r^{j_2/2}}\sum_d \frac{1}{d^2\phi(d)}\sum_t \frac{1}{t\phi(t)}\\
&\ll \frac{1}{q}\sum_{\substack{j_1,j_2\le \log X\\ \max\{j_1,j_2\}\ge 2}}\ \sum_{p\le X^{1/j_1}}\ \sum_{r\le X^{1/j_2}} \frac{\log p}{p^{j_1/2}}\frac{\log r}{r^{j_2/2}}\\
&\ll \frac{\sqrt{X}(\log X)^3}{q} \ll q^{-\varepsilon/2} \ll q^{-\varepsilon/4}
\end{aligned}$$

upon choosing ε small enough.

From the lemma above, we have

$$S_1 = \sum_{\substack{d\\(d,q)=1}} \frac{\mu(d)}{d\,\phi(d)} \sum_{\substack{s,t\\(st,d)=1}} \frac{2\mu^2(t)}{qst\,\phi(t)} \sum_{m,n} a_m a_n\, K\, e\!\left(\frac{m+n}{qstd}\right) J_{k-1}\!\left(\frac{4\pi\sqrt{mn}}{qstd}\right) + O\!\left(q^{-\varepsilon/4}\right).$$

Next we remove the condition $(s,d)=1$ by replacing it with $\sum_{h|(s,d)}\mu(h)$. Thus,

$$\begin{aligned}
S_1 = \frac{2}{q}\sum_{\substack{d\\(d,q)=1}} \frac{\mu(d)}{d^2\phi(d)} \sum_{h|d}\frac{\mu(h)}{h} \sum_{\substack{t\\(t,d)=1}} \frac{\mu^2(t)}{t\,\phi(t)} \sum_{m,n} a_m a_n \\
\sum_s \frac{1}{s} K\, e\!\left(\frac{m+n}{qtdhs}\right) J_{k-1}\!\left(\frac{4\pi\sqrt{mn}}{qtdhs}\right) + O(q^{-\varepsilon/4})\\
=: S_{t;big} + S_{t;small} + O(q^{-\varepsilon/4});
\end{aligned}$$

where $S_{t;big}$ is the contribution from $t \ge \frac{X}{qdh}$, and the rest is from $t < \frac{X}{qdh}$. Note that Lemma 6.3 may only be applied to when $t < \frac{X}{qdh}$.

Case 1: $t \ge \frac{X}{qdh}$. For this case, we apply Lemma 2.2 to bound J_{k-1}. Thus

$$\begin{aligned}
S_{t;big} &\ll \frac{1}{q}\sum_d \frac{1}{d^2\phi(d)}\sum_{h|d}\frac{1}{h}\sum_{t\ge \frac{X}{qdh}} \frac{\mu^2(t)}{t\,\phi(t)} \sum_{aX\le m,n\le bX} \frac{\tau(m)\tau(n)}{\sqrt{mn}} \sum_s \frac{1}{s}\left(\frac{\sqrt{mn}}{qtdhs}\right)^{k-1}\\
&\ll \frac{X^k}{q^k}\sum_d \frac{1}{d^{k+2}\phi(d)}\sum_{h|d}\frac{1}{h^k}\sum_{t\ge \frac{X}{qdh}} \frac{\mu^2(t)}{t^k\phi(t)} \qquad (6.5)\\
&\ll \frac{X^k}{q^k}\sum_d \frac{1}{d^{k+2}\phi(d)}\sum_{h|d}\frac{1}{h^k}\left(\frac{qdh}{X}\right)^k = O(1).
\end{aligned}$$

When $0 < \varepsilon \le 1$, $S_{t;small}$ is empty as $\frac{X}{qdh} \le 1$. Thus $S_1 = S_{t;big} + O\left(q^{-\varepsilon/4}\right)$, and so the above suffices for Lemma 6.1.

Case 2: $X > qtdh$. For this case, we apply Lemma 6.3 to the sum over s with $L = \frac{Xq^{0}}{qtdh}$ for a small $^0 > 0$ to be specied later. Therefore, we have

$$
(6.6)\quad S_{d^2(d)} = \frac{4^2}{h} \sum_{\substack{d \\ (d;q)=1}} \frac{(d)}{\frac{X}{t(t)}} \sum_{hjd} \frac{(h)}{L} \sum_{\substack{t< \frac{X}{qdh} \\ (t;d)=1}} \frac{^2(t)}{L} \left(\sum_{\ell=1}^{1} w\left(\frac{\ell}{}\right) - \int_0^1 w\left(\frac{\ell}{}\right) d\ell \right)^{t;small} q \sum_{m;n} \sum a_m a_n J_{k-1}\left(4\sqrt{\frac{m\ell}{qtdh}}\right) J_{k-1}\left(4\sqrt{\frac{n\ell}{qtdh}}\right):
$$

Recall that $a_n = \frac{(n)(n)}{\sqrt{n}}\,(n/X)$. Next, we separate variables by introducing the Mellin transforms for (x) and J-Bessel functions in (2.5). Then we write for some small $_1 > 0$

$$
S_{t;small} = \frac{4^2}{q}\frac{1}{(2i)^4} \int_{(\frac12+2_1)} \int_{(\frac12+2_1)} \int_{(_1)} \int_{(_1)} (s_1)(s_2)\,2^{w_1 w_2 2}\, \frac{\left(\frac{k-w_1-1}{2}\right)}{\left(\frac{k+w_1+1}{2}\right)} \frac{\left(\frac{k-w_2-1}{2}\right)}{\left(\frac{k+w_2+1}{2}\right)}
$$
$$
\frac{(4)^{w_1+w_2} X^{s_1+s_2}}{q^{\frac{w_1}{2}+\frac{w_2}{2}}} \sum_{\substack{d \\ (d;q)=1}} \frac{(d)}{d^{1+\frac{w_1}{2}+\frac{w_2}{2}}\,^2(d)} \sum_{hjd} \frac{(h)}{h^{1+\frac{w_1}{2}+\frac{w_2}{2}}} \sum_{\substack{t< \frac{X}{qdh} \\ (t;d)=1}} \frac{^2(t)}{t^{1+\frac{w_1}{2}+\frac{w_2}{2}}(t)}
$$
$$
\left(\sum_{\ell=1}^{1} w\left(\frac{\ell}{L}\right) \ell^{\frac{w_1}{2}+\frac{w_2}{2}} - \int_0^1 w\left(\frac{\ell}{L}\right) \ell^{\frac{w_1}{2}+\frac{w_2}{2}} d\ell \right) \sum_{m;n} \sum \frac{(m)}{m^{\frac12+s_1-\frac{w_1}{2}}} \frac{(n)}{n^{\frac12+s_2-\frac{w_2}{2}}}
$$
$$
dw_1\, dw_2\, ds_1\, ds_2
$$

$$
= \frac{4^2}{q}\frac{1}{(2i)^4} \int_{(\frac12+2_1)} \int_{(\frac12+2_1)} \int_{(_1)} \int_{(_1)} (s_1)(s_2)\,2^{w_1 w_2 2}\, \frac{\left(\frac{k-w_1-1}{2}\right)}{\left(\frac{k+w_1+1}{2}\right)} \frac{\left(\frac{k-w_2-1}{2}\right)}{\left(\frac{k+w_2+1}{2}\right)}
$$
$$
\frac{(4)^{w_1+w_2} X^{s_1+s_2}}{q^{\frac{w_1}{2}+\frac{w_2}{2}}} \sum_{\substack{d \\ (d;q)=1}} \frac{(d)}{d^{1+\frac{w_1}{2}+\frac{w_2}{2}}\,^2(d)} \sum_{hjd} \frac{(h)}{h^{1+\frac{w_1}{2}+\frac{w_2}{2}}} \sum_{\substack{t< \frac{X}{qdh} \\ (t;d)=1}} \frac{^2(t)}{t^{1+\frac{w_1}{2}+\frac{w_2}{2}}(t)}
$$
$$
D\left(L; -\frac{w_1}{2}-\frac{w_2}{2}\right) {}^{0}\left(\frac12+s_1-\frac{w_1}{2}\right) \left(\frac12+s_2-\frac{w_2}{2}\right) dw_1\, dw_2\, ds_1\, ds_2
$$

where $\mathrm{Re}(s_1) = \mathrm{Re}(s_2) = \frac12 + 2_1$, $\mathrm{Re}(w_1) = \mathrm{Re}(w_2) = {}_1$, and $_1$

$$
D(L; s) = \sum_{\ell=1}^{1} w\left(\frac{\ell}{L}\right) \ell^{s} - \int_0^1 w\left(\frac{\ell}{L}\right) \ell^{s}\, d\ell:
$$

We move the contour in w_1 to the right to $\mathrm{Re}(w_1) = 6_1$ and pick up a simple pole at $w_1 = 2s_1 - 1$. The residue of the pole is -2 because ${}^{0}\left(\frac12 + s_1 - \frac{w_1}{2}\right) = \frac{1}{w_1-(2s_1-1)}\,2 + $

$O(1)$ when w_1 is close to $2s_1 \quad 1$. Thus we obtain that

$$S_{t;\text{small}} = C_0 + C_1;$$

where

$$C_0 := \frac{8^{\ 2}}{q^{\frac{1}{2}}}\frac{1}{(2i)^3}\int_{(\frac{1}{2}+2_1)}\int_{(\frac{1}{2}+2_1)}\int_{(1)} (s_1)b(s_2)2b^{\ 2s_1 \quad w_2 \quad 1}\frac{\frac{k \quad 2s_1}{2}}{\frac{k+2s_1}{2}}\frac{\frac{k \quad w_2 \quad 1}{2}}{\frac{k+w_2+1}{2}}$$

$$\frac{(4)^{2s_1+w_2 \quad 1}}{q^{s_1+\frac{w_2}{2}}}\sum^{s_1+s_2}\sum_{\substack{d\\ d;q)=1\\0}}\frac{(d)}{d^{\frac{1}{2}+s_1+\frac{w_2}{2}\,2}(d)}\sum_{hjd}\frac{(h)}{h^{\frac{1}{2}+s_1+\frac{w_2}{2}}}\sum^{2}_{\substack{t<\frac{X}{qdh}\\(t;d)=1}}\frac{^2(t)}{t^{\frac{1}{2}+s_1+\frac{w_2}{2}}(t)}$$

$$D \quad L; s_1 \quad \frac{1}{2}+\frac{w_2}{2} \quad (\quad \frac{1}{2}+s_2 \quad \frac{w_2}{2} \qquad dw_2\, ds_1\, ds_2;$$

and

(6.7)

$$C_1 := \frac{4^{\ 2}}{q}\frac{1}{(2i)^{\ 4}}\int_{(\frac{1}{2}+2_1)}\int_{(\frac{1}{2}+2_1)}\int_{(1)}\int_{(6_1)} (s_1)(s_2)b\,2\,{}_2^{\ w_1 \quad w_2 \quad 2}\frac{\frac{k \quad w_1 \quad 1}{2}}{\frac{k+w_1+1}{2}}\frac{\frac{k \quad w_2 \quad 1}{2}}{\frac{k+w_2+1}{2}}$$

$$\frac{(4)^{w_1+w_2}}{q^{\frac{w_1}{2}+\frac{w_2}{2}}}\sum^{s_1+s_2}\sum_{\substack{d\\(d \quad ;q)= \quad 1\\0}}\frac{(d)}{d^{1+\frac{w_1}{2}+\frac{w_2}{2}\,2}(d)}\sum_{hjd}\frac{(h)}{h^{1+\frac{w_1}{2}+\frac{w_2}{2}}}\sum_{\substack{t<\frac{X}{qdh}\\(t;d)=1}}\frac{^2(t)}{t^{1+\frac{w_1}{2}+\frac{w_2}{2}}(t)}$$

$$D \quad L; \ \frac{w_1}{2} \quad \frac{w_2}{2} \quad (\quad \frac{1}{2}+s_1 \quad \frac{w_1}{2} \quad \frac{1}{2}+s_2 \quad \frac{w_2}{2} \qquad dw_1\, dw_2\, ds_1\, ds_2:$$

For C_0, we move the contour integral in w_2 to the right to $\mathrm{Re}(w_2) = 6_1$ and encounter the pole at $w_2 = 2s_2 \quad 1$ with residue equal to $\quad 2$. Thus

$$C_0 = MS_1 + C_2;$$

where

(6.8)

$$MS_1 := \frac{1}{(2i)^2}\int_{(\frac{1}{2}+2_1)}\int_{(\frac{1}{2}+2_1)} \frac{4 \quad X}{q}^{s_1+s_2} (s_1)(s_2)b\,b\,\frac{\frac{k \quad 2s_1}{2}}{\frac{k+2s_1}{2}}\frac{\frac{k \quad 2s_2}{2}}{\frac{k+2s_2}{2}}$$

$$\sum_{\substack{d\\(d;q)=1}}\frac{(d)}{d^{s_1+s_2\,2}(d)}\sum_{hjd}\frac{(h)}{h^{s_1+s_2}}\sum_{\substack{t<\frac{X}{qdh}\\(t;d)=1}}\frac{(^2 t)}{t^{s_1+s_2}(t)}D(L; s_1+s_2 \quad 1)\, ds_1\, ds_2;$$

and

(6.9)

$$C_2 := \frac{8\ ^2}{q^{\frac{1}{2}}} \frac{1}{(2i)\ ^3} \int_{(\frac{1}{2}+2_1)} \int_{(\frac{1}{2}+2_1)} \int_{(6_1)} (s_1)(s_2) b\ 2^{\ 2s_1\ \ w_2\ \ 1} \frac{\frac{k\ \ 2s_1}{2}\ \frac{k\ \ w_2\ \ 1}{2}}{\frac{k+2s_1}{2}\ \frac{k+w_2+1}{2}}$$

$$\frac{(4)^{2s_1+w_2\ \ 1}}{q^{s_1+\frac{w_2}{2}}} \sum_{d;q\ \ 1}^{s_1+s_2} \sum_{d} \frac{(d)}{d^{\frac{1}{2}+s_1+\frac{w_2}{2}}\,{}^2(d)} \sum_{hjd} \frac{(h)}{h^{\frac{1}{2}+s_1+\frac{w_2}{2}}} \sum_{\substack{t<\frac{X}{qdh} \\ (t;d)=1}} \frac{{}^2(t)}{t^{\frac{1}{2}+s_1+\frac{w_2}{2}}(t)}$$

$$D\ \ L;\ s_1\ \ (\ \ )=\ ^0 \ \frac{1}{2}+\frac{w_2}{2} \quad \frac{1}{2}+s_2 \quad \frac{w_2}{2} \quad dw_2\,ds_1\,ds_2:$$

Now we consider C_1 in (6.7). We move contour integration in w_2 to the right to Re $(w_2) = 6_1$, and we encounter the pole at $w_2 = 2s_2 \quad 1$ with residue equal to $\quad 2$. Thus we obtain that

$$C_1 = C_3 + C_4;$$

where

$$C_3 := \frac{8^2}{q^{\frac{1}{2}}} \frac{1}{(2i)\ ^3} \int_{(\frac{1}{2}+2_1)} \int_{(\frac{1}{2}+2_1)} \int_{(6_1)} (s_1)(s_2) b\ 2^{\ w_1\ \ 2s_2\ \ 1} \frac{\frac{k\ \ w_1\ \ 1}{2}\ \frac{k\ \ 2s_2}{2}}{\frac{k+w_1+1}{2}\ \frac{k+2s_2}{2}}$$

$$\frac{(4)^{w_1+2s_2\ \ 1}}{q^{\frac{w_1}{2}+s_2}} \sum_{(d;q)=1}^{s_1+s_2} \sum_{d} \frac{(d)}{d^{\frac{1}{2}+\frac{w_1}{2}+s_2}\,{}^2(d)} \sum_{hjd} \frac{(h)}{h^{\frac{1}{2}+\frac{w_1}{2}+s_2}} \sum_{\substack{t<\frac{X}{qdh} \\ (t;d)=1}} \frac{{}^2(t)}{t^{\frac{1}{2}+\frac{w_1}{2}+s_2}(t)}$$

$$D\ \ L;\ ^0 \ \frac{1}{2} \quad \frac{w_1}{2} \quad s_2 \quad \frac{1}{2}+s_1 \quad \frac{w_1}{2} \quad dw_1\,ds_1\,ds_2;$$

and

(6.10)

$$C_4 := \frac{4\ ^2}{q} \frac{1}{(2i)^4} \int_{(\frac{1}{2}+2_1)} \int_{(\frac{1}{2}+2_1)} \int_{(6_1)} \int_{(6_1)} (s_1)(s_2) b\ 2^{\ w_1\ \ w_2\ \ 2} \frac{\frac{k\ \ w_1\ \ 1}{2}\ \frac{k\ \ w_2\ \ 1}{2}}{\frac{k+w_1+1}{2}\ \frac{k+w_2+1}{2}}$$

$$\frac{(4)^{w_1+w_2}}{q^{\frac{w_1}{2}+\frac{w_2}{2}}} \sum_{(d;q)=1}^{s_1+s_2} \sum_{d} \frac{(d)}{d^{1+\frac{w_1}{2}+\frac{w_2}{2}}\,{}^2(d)} \sum_{hjd} \frac{(h)}{h^{1+\frac{w_1}{2}+\frac{w_2}{2}}} \sum_{\substack{t<\frac{X}{qdh} \\ (t;d)=1}} \frac{{}^2(t)}{t^{1+\frac{w_1}{2}+\frac{w_2}{2}}(t)}$$

$$D\ \ L;\ ^0 \ \frac{w_1}{2} \quad \frac{w_2}{2} \quad \frac{1}{2}+s_1 \quad \frac{w_1}{2} \quad \frac{1}{2}+s_2 \quad \frac{w_2}{2} \quad dw_1\,dw_2\,ds_1\,ds_2:$$

Therefore

(6.11) $$S_{t;small} = MS_1 + C_2 + C_3 + C_4:$$

To prove Lemma 6.1 for $X = q$, where $1 < \quad 2$; it suces to show the following lemma.

Lemma 6.6. Let $X = q$, where $1 < \quad 2$: With notation as above, we have

$$(6.12) \qquad MS_1 = \frac{1}{2}\log \frac{q}{X} \int_{\mathbb{R}} j\hat{h}(t)j^2 \, dt + O(1);$$

$$(6.13) \qquad C_2; \; C_3 \quad q^{1=4};$$

and

$$(6.14) \qquad C_4 \quad q^{1=4};$$

where the implied constants depend on and k.

The lemma above along with (6.5) and (6.11) completes the proof of Lemma 6.1.

Proof of (6.12) { the main term MS_1. Now we focus on calculating MS_1 in (6.8). We move the line integration over s_i to $\mathrm{Re}(s_i) = {}_1$ so

$$MS_1 = \frac{1}{(2i)^2} \int_{(1)} \int_{(1)} \frac{4X^2}{q}^{s_1+s_2} b(s_1) b(s_2) \frac{\frac{k-2s_1}{2}}{\frac{k+2s_1}{2}} \frac{\frac{k-2s_2}{2}}{\frac{k+2s_2}{2}}$$
$$\sum_{\substack{d \\ (d;q)=1}} \frac{s(d)}{d^{s_1+s_2} \, 2(d)} \sum_{hjd} \frac{(h)}{h^{s_1+s_2}} \sum_{\substack{t < \frac{X}{qdh} \\ (t;d)=1}} \frac{2(t)}{t^{s_1+s_2}(t)} D(L; s_1+s_2 \quad 1) \, ds_1 \, ds_2$$

From (6.2) in Lemma 6.4,

$$D(L; s_1+s_2 \quad 1) = (1 \quad s_1 \quad s_2) + O\left(\frac{Xq}{qtdh}^{0 \; \frac{1}{} \; 4} (1 + j\mathrm{Im}(s_1+s_2)j) \right)!:$$

so

$$MS_1 = M_1 + ES_1;$$

where M_1 is the contribution from $(1 \quad s_1 \quad s_2)$ and ES_1 is the contribution of the rest.

First we consider ES_1. By Stirling's formula and writing $s_i = {}_1 + it_i$, we have

$$\frac{\frac{k-2s_i}{2}}{\frac{k+2s_i}{2}} \quad (jt_ij + 1)^{2_1}:_2$$

Thus

(6.15)
$$ES_1 \quad \frac{X}{q}^{2_1} \int_1^1 \int_1^1 (\hat{b}_1 + it_1)(_1 + \hat{b}it_2)(1 + jt_1j)^{2_1}(1 + jt_2j)^{2_1}$$
$$\sum_d \frac{1}{d^{2_1}{}^2(d)} \sum_{hjd} \frac{1}{h^{2_1}} \sum_{t<\frac{X}{qdh}} \frac{{}^2(t)}{t^{2_1}(t)} \frac{Xq}{qtdh}^{0} \frac{1}{4} (1 + jt_1 + t_2j)\, dt_1\, dt_2$$
$$q \quad \frac{X}{}^{2_1} q^{\frac{1}{}=4^4} \sum_d{}^{c} \frac{1}{d^{2_1}{}^2(d)} \sum_{hjd} \frac{1}{h^{2_1}} (dh)^{4} \frac{1}{qdh} \quad \frac{1}{} \quad \frac{X}{}^{\frac{1}{4}2_1}$$
$$q^{0=4}:$$

Next we consider M_1: First we add back terms corresponding to large t and note that their contribution is small
$$EM_1 := \frac{1}{(2i)^2} \int_{(_1)} \int_{(_1)} \frac{4^2 X}{q}^{s_1+s_2} \hat{b}(s_1)\hat{b}(s_2) \frac{\frac{k-2s_1}{2}}{\frac{k+2s_1}{2}} \frac{\frac{k-2s_2}{2}}{\frac{k+2s_2}{2}}$$
$$\sum_{\substack{d \\ (d;q)=1}} \frac{(d)}{d^{s_1+s_2}{}^2(d)} \sum_{hjd} \frac{(h)}{h^{s_1+s_2}} \sum_{\substack{t \;(\frac{X}{qdh}) \\ t;d\;1}} \frac{{}^2(t)}{t^{s_1+s_2}(t)} (1 - s_1 - s_2)\, ds_1\, ds_2$$
$$\sum_d \frac{X}{}^{2_1} \frac{1}{d^{2_1}{}^2(d)} \sum_{hjd} \frac{1}{h^{2_1}} \sum_{t>\frac{X}{qdh}} \frac{1}{t^{2_1}(t)} \frac{{}^2(t)\; q}{X}$$
$$1$$

since $\hat{b}$ has a rapid decay on vertical lines and by the bound (6.3) for . Therefore
$$M_1 = \frac{1}{(2i)^2} \int_{(_1)} \int_{(_1)} \frac{4^2 X}{q}^{s_1+s_2} \hat{b}(s_1)\hat{b}(s_2) \frac{\frac{k-2s_1}{2}}{\frac{k+2s_1}{2}} \frac{\frac{k-2s_2}{2}}{\frac{k+2s_2}{2}}$$
$$\sum_{\substack{d \\ (d;q)=1}} \frac{(d)}{d^{s_1+s_2}{}^2(d)} \sum_{hjd} \frac{(h)}{h^{s_1+s_2}} \sum_{\substack{t \\ (t;d)=1}} \frac{{}^2(t)}{t^{s_1+s_2}(t)} (1 - s_1 - s_2)\, ds_1\, ds_2 + O(1):$$

On the other hand, by multiplicativity the sum over t is
$$\sum_{\substack{t \\ (t;d)=1}} \frac{{}^2(t)}{t^{s_1+s_2}(t)} = (1 + s_1 + s_2)G(s_1 + s_2)B_d(s_1 + s_2);$$

where
$$G(z) := \prod_p 1 + \frac{1}{p(p-1)}p^z - \frac{1}{p(p-1)p^{2z}}$$
$$B_d(z) := \prod_{pjd} 1 + \frac{1}{(p-1)p^z}^{-1}:$$

Note that $G(z)$ is absolutely convergent when $\mathrm{Re}(z) > 1/2$, and $G(0) = 1$. Therefore $M_1 + O(1)$ is

$$\frac{1}{(2\pi i)^2}\int_{(1)}\int_{(1)} \left(\frac{4\pi^2 X}{q}\right)^{s_1+s_2} (s_1)(s_2)\, b\,(1 - s_1 - s_2)(1+s_1+s_2)G(s_1+s_2) \sum_{\substack{d \\ (d;q)=1}} \frac{(d)}{d^{s_1+s_2}\,{}^2(d)} \sum_{h|d} \frac{(h)}{h^{s_1+s_2}} B_d(s_1+s_2) \frac{s_1 + \frac{k}{2}\; s_2 + \frac{k}{2}}{s_1 + \frac{k}{2}\; s_2 + \frac{k}{2}}\, ds_2\, ds_1$$

Recall that $X = q$, where $1 < 2$. We shift the contour of s_2 to the left to $\mathrm{Re}(s_2) = -\frac{1}{4}$, picking up the contribution of the double poles at $s_2 = -s_1$ from $(1 - s_1 - s_2)(1 + s_1 + s_2)$. Thus we write that $M_1 + O(1)$ is

$$R_M + \frac{1}{(2\pi i)^2}\int_{(1)}\int_{(-\frac{1}{4})} \left(\frac{4\pi^2 X}{q}\right)^{s_1+s_2} (s_1)(s_2)\, b\,(1 - s_1 - s_2)(1+s_1+s_2)G(s_1+s_2) \sum_{\substack{d \\ (d;q)=1}} \frac{(d)}{d^{s_1+s_2}\,{}^2(d)} \sum_{h|d} \frac{(h)}{h^{s_1+s_2}} B_d(s_1+s_2) \frac{s_1 + \frac{k}{2}\; s_2 + \frac{k}{2}}{s_1 + \frac{k}{2}\; s_2 + \frac{k}{2}}{}^2\, ds_2\, ds_1$$

where R_M denotes the residue at $s_2 = -s_1$. We note that the contour integral above is $\left(\frac{X}{q}\right)^{-1/4+\epsilon_1} \ll 1$. Further, write

$$R_M = \log\frac{X}{q}\,\frac{1}{2\pi i}\int_{(1)} (s_1)\, b\,(-s_1)\, b\, G(0) \sum_{\substack{d \\ (d;q)=1}} \frac{(d)}{{}^2(d)} \sum_{h|d} (h) B_d(0)\, ds_1 + E_R$$

where

$$E_R = \frac{1}{2\pi i}\int_{(1)} (s_1)\, b \frac{s_1 + \frac{k}{2}}{s_1 + \frac{k}{2}}\, d s_2 \,(4\pi^2)^{s_1+s_2}(s_2)(s_1+s_2)\, b\,{}^2(1 - s_1 - s_2)(1+s_1+s_2)\, G(s_1+s_2) \sum_{\substack{d \\ (d;q)=1}} \frac{(d)}{d^{s_1+s_2}\,{}^2(d)} \sum_{h|d} \frac{(h)}{h^{s_1+s_2}} B_d(s_1+s_2) \left(\frac{s_2 + \frac{k}{2}}{s_2 + \frac{k}{2}}\right)^{1}_{s_2=-s_1} ds_1$$

$= O(1)$.

Since $G(0) = 1$ and $\sum_{h|d} (h) = 0$ unless $d = 1$, we obtain that

$$R_M = \log\frac{X}{q}\,\frac{1}{2\pi i}\int_{(1)} (s_1)\, b\,(-s_1)\, b\, ds_1 + O(1) = \log\frac{X}{q}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} (it)\, b\,(-it)\, b\, dt + O(1)$$

$$= \log\frac{X}{q}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} |(it)\, b|^2\, dt + O(1).$$

Thus

$$M_1 = \frac{1}{2}\log\frac{q}{X}\int_{\mathbb{R}} j\hat{h}(t)j^2\, dt + O(1): \tag{6.16}$$

Combining (6.15) and (6.16) gives (6.12) in Lemma 6.6.

Proof of (6.13) { error terms $C_2; C_3$. Due to symmetry of C_2 and C_3, it is sucient to prove the bound for C_2. Recall that C_2 is dened in (6.9).

First, we move the lines of integration in s_1 and s_2 to $\mathrm{Re}(s_1) = {}_1$ and $\mathrm{Re}(s_2) = 4_1$, respectively. Under RH, no poles are encountered when moving the lines of integration. We write $\mathrm{Im}\,(s_i) = t_i$ and $\mathrm{Im}\,(w_2) = y_2$. By (6.1) in Lemma 6.4 for $a = \frac{1}{2}$ and $s = s_1 + \frac{w_2}{2}$ we have that

$$D \quad L; s_1 \quad \frac{1}{2} + \frac{w_2}{2} \qquad L^{4_1}(1 + jt_1j + jy_2j) \tag{6.17}$$

Under RH, we also have the bound

$$\frac{(}{(}\ + it) \ \log^2(jtj + 2); \tag{6.18}$$

uniformly when $\frac{1}{2} + {}_1 \quad {}_4$ (e.g. $\frac{}{3}$ see [8]), and so we derive that

$$C_2 \quad \frac{1}{q^{\frac{1}{1}}}\int_{1}^{1}\int_{1}^{1}\int_{1}^{1} j\hat{h}(1 + it_1)(4_1 + it_2)j(1 + jt_1j)^{\ 2_1}(1 + jy_2j)^{\ 1\ 6_{1\,2}} \quad {}_1$$

$$\frac{X^{5_1}}{q^{4_1}} \sum_d \frac{1}{d^{\frac{1}{2}+4_1\,2}(d)} \sum_{hjd} \frac{1}{h^{\frac{1}{2}+4_1}} \sum_{t<\frac{X}{qdh}} \frac{{}^2(t)}{t^{\frac{1}{2}+4_1}(t)} \quad \frac{Xq^0}{qtdh}^{\ 4_1}$$

$$(1 + jt_1j + jy_2j)\ \log^2(jt_2j + jy_2j + 2)\ dy_2\, dt_1\, dt_2$$

$$\frac{X^{\ 9\ _1}q^{4^0\ _1}}{q^{\frac{1}{2}+8_1}} \quad q^{\ _4;\ \frac{}{1}}$$

upon choosing small enough ${}_1;{}_0$:

Proof of (6.14) { error term C_4. For C_4 as in (6.10), we move the integral in s_i to $\mathrm{Re}(s_i) = 4_1$. We encounter no poles under RH. We use the bounds in (6.1) in Lemma 6.4 for $a = 0$ and (6.18). Thus we obtain that

$$C_4 \quad \frac{1}{q}\int_{1}^{1}\int_{1}^{1}\int_{1}^{1}\int_{1}^{1} j\hat{h}(4_1 + it_1)(4_1 + it_2)j(1 + jy_1j)^{\ 1\ 6_1}(1 + jy_2j)^{\ 1\ 6_1}$$

$$\frac{X^{8_1}}{q^{6_1}} \sum_d \frac{1}{d^{1+6_1\,2}(d)} \sum_{hjd} \frac{(h)}{h^{1+6_1}} \sum_{t<\frac{X}{qdh}} \frac{{}^2(t)}{t^{1+6_1}(t)} \quad \frac{Xq^0}{qtdh}^{\frac{1}{2}+6_1}$$

$$(1 + jy_1j + jy_2j)\ \log^2(jt_1j + jy_1j + 2)\ \log^2(jt_2j + jy_2j + 2)\ dy_1\, dy_2\, dt_1\, dt_2$$

$$\frac{1}{q}\frac{X^{\frac{1}{2}+14\ _1}q^{0(1=2+6\ )_1}}{q^{\frac{1}{2}+12_1}} \quad q^{\ _4\ \frac{}{1}}$$

upon choosing small enough ${}_1$ and ^{0}.

7. Error Term S_2

Let us recall the denition of S_2:

$$S_2 = K \sum_{s;t} \frac{2}{qst(t)^2} \sideset{}{^*}\sum_{\substack{m \bmod t \\ n \bmod t}} V_{qs}(m;n;t) \sum_{\substack{\bmod t \\ {}^0 \bmod t \\ =_0;{}^0=_0}} \sum \overline{{}^0(m)(n)}$$
$$\sum_{p;r} \sum {}^0(p)(r)a_p a_r \, e\left(\frac{p \quad r+}{qst}\right) J_{k-1}\left(\frac{4}{qst}\sqrt{pr}\right):$$

In what follows we will bound S_2 by $q^{=4}$. Note that this is the same that was xed in Section 1.1 so that jj 2 .

Lemma 7.1. With notation as above, we have

$$S_2 \quad q^{=4}:$$

We will consider sums over s and t in dyadic intervals of the form $T \quad t \quad 2T$ and S $s \quad 2S$ denoted by $t \quad T$ and $s \quad S$ respectively. Thus in order to prove Lemma 7.1, it is sucient to show that

$$D(S;T) \quad q^{=2}; \tag{7.1}$$

where

$$D(S;T) := \sum_{s S;\, t T} \sum \frac{1}{qst(t)^2} \sideset{}{^*}\sum_{\substack{m \bmod t \\ n \bmod t}} \sideset{}{^*}\sum V_{qs}(m;n;t) \sum_{\substack{\bmod t \\ {}^0 \bmod t \\ =_0;{}^0=_0}} \sum \overline{{}^0(m)(n)}$$
$$\sum_{p;r} \sum {}^0(p)(r)a_p a_r \, e\left(\frac{p \quad r+}{qst}\right) J_{k-1}\left(\frac{4}{qst}\sqrt{pr}\right):$$

We will apply the power series expansion for J-Bessel function when T is large, but when T is small, we will use an argument of Deshouillers and Iwaniec [9] to separate variables in the J-Bessel function. Thus the claimed bound for $D(S;T)$ follows from the following lemmas.

Lemma 7.2. Let $T \quad \frac{30bX}{qS}$. Then

$$D(S;T) \quad q^{=2}:$$

Lemma 7.3. Let $T \quad \frac{30bX}{qS}$. Then

$$D(S;T) \quad q^{=2}:$$

7.1. Proof of Lemma 7.2 — small T. First note that we have $aX < \sqrt{pr} \le bX$ so the sum over p and r remains unchanged if a smooth weight $\Phi(\sqrt{pr}/X)$ is attached to each term provided that

$$\Phi(x) = \begin{cases} 1 & \text{if } x \in (a;b] \\ 0 & \text{if } x \notin (\frac{a}{2};2b) \end{cases}:$$

From Lemma 2.2,

$$J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qst}\right) = \frac{\sqrt{qst}}{\pi\sqrt{2}\sqrt{\sqrt{pr}}}\mathrm{Re}\left(W\left(\frac{4\pi\sqrt{pr}}{qst}\right)e\left(\frac{2\sqrt{pr}}{qst}-\frac{k}{4}+\frac{1}{8}\right)\right):$$

Now we write

$$\Phi(y)W\left(\frac{4\pi yX}{qst}\right)e\left(\frac{2yX}{qst}\right) = \frac{1}{2\pi i}\int_{(1)} M_1(s)\,y^{-s}\,ds \tag{7.2}$$

and

$$\Phi(y)e\left(\frac{yX}{qst}\right) = \frac{1}{2\pi i}\int_{(1)} M_2(s)\,y^{-s}\,ds \tag{7.3}$$

where

$$M_1(1+iv) = \int_0^\infty \Phi(y)W\left(\frac{4\pi yX}{qst}\right)e\left(\frac{2yX}{qst}\right)y^{iv}\,dy;\quad M_2(1+iv) = \int_0^\infty \Phi(y)e\left(\frac{yX}{qst}\right)y^{iv}\,dy:$$

Moreover,

$$M_i(1+iv) \ll \begin{cases} (1+|v|)^{-1/2} & \text{if } |v| \le \frac{c_iX}{qst} \\ v^{-2} & \text{if } |v| > \frac{c_iX}{qst} \end{cases} \tag{7.4}$$

for some absolute constant c_i. The bound in (7.4) follows by the same arguments as in [9]. From (7.4), for $s \sim S$ and $t \sim T$, we have

$$\int_{(1)} |M_i(z)|\,|dz| \ll \left(\frac{X}{qST}\right)^{1/2} \tag{7.5}$$

since $\frac{XT}{qS} \gg 1$ for this case.

Now Lemma 2.2 and (7.2) give

$$D(S;T) = \frac{1}{\sqrt{2}\pi}\sum_{t\sim T}\sum \frac{1}{qst\,\phi(t)^2}\frac{\sqrt{qst}}{\sqrt{\pi X}}\sum^{*}_{m \bmod t}\sum^{*}_{n \bmod t} V_{qs}(m;n;t)\sum_{\substack{\chi \bmod t\\ \chi' \bmod t\\ \chi\chi_0;\ \chi'\chi_0}}\sum \overline{\chi'(m)\chi(n)}\,\epsilon_{sS}$$

$$\sum_p\sum_r \chi(p)\chi'(r)a_pa_r\,e\left(\frac{(p+r)}{qst}\right)\left(\frac{\sqrt{r}}{\sqrt{p}}\right)\left(\frac{X^2}{}\right)^{1/4}\Phi_{p;r}$$

$$\mathrm{Re}\left[\left\{\frac{1}{2\pi i}\int_{(1)} M_1(z)\left(\frac{\sqrt{pr}}{X}\right)^{z} dz\right\} e\left(\frac{}{4}+\frac{k}{8}\right)\right] \ll \frac{1}{}.$$

To prove Lemma 7.2, it suces to bound

$$D_1(S;T) := \sum_{s\sim S}\sum_{t\sim T} \frac{1}{qst\,\phi(t)^2}\frac{1}{2\pi i}\int_{(1)} \left(\frac{\sqrt{qst}}{X}\right)^{z} M_1(z) \sum_{m \bmod t}{}^{*}\sum_{n \bmod t}{}^{*} V_{qs}(m;n;t) \sum_{\substack{\psi \bmod t\\ \psi'\bmod t\\ \psi=\psi_0;\,\psi'=\psi'_0}} \overline{\psi'(m)\psi(n)}\,\ldots$$

$$\sum\sum_{p;r} \lambda(p)\lambda(r)a_p a_r\, e\left(\frac{\overline{p}+\overline{r}}{qst}\right)\left(\frac{X^2}{pr}\right)^{1/4+z/2} dz$$

$$= \sum_{s\sim S}\sum_{t\sim T}\frac{1}{qst\,\phi(t)^2}\frac{1}{2\pi i}\int_{(1)}\left(\frac{\sqrt{qst}}{X}\right)^{z} M_1(z)\sum_{\substack{x \bmod t\\ (x(x+qs);t)=1}} T_1(x;z)T_2(x;z)\, dz$$

where

$$T_1(x;z) := \sum_{m \bmod t}{}^{*} e\left(\frac{mx}{t}\right)s_1(m); \qquad T_2(x;z) := \sum_{n \bmod t}{}^{*} e\left(\frac{n\overline{x}\pm\overline{qs}}{t}\right)s_1(n);$$

and

$$s_1(m) = \sum_{\substack{\psi \bmod t\\ \psi=\psi_0}} \overline{\psi(m)} \sum_{p} \lambda(p)\, a_p\, \psi(p)\left(\frac{X}{p}\right)^{1/4+z/2} e\left(\frac{\overline{p}}{qst}\right). \tag{7.6}$$

By setting $z = 1 + iv$ and Cauchy-Schwarz's inequality, we obtain that

$$D_1(S;T) \ll \sqrt{\frac{1}{q}}\,\frac{\sqrt{ST}}{X}\sum_{s\sim S}\sum_{t\sim T}\frac{1}{\phi(t)^2}\int_{-\infty}^{\infty}|M_1(1+iv)|\,(S_1S_2)^{1/2}\,dv; \tag{7.7}$$

where

$$S_1 := \sum_{\substack{x \bmod t\\ (x(x+qs);t)=1}} |T_1(x;1+iv)|^2; \quad\text{and}\quad S_2 := \sum_{\substack{x \bmod t\\ (x(x+qs);t)=1}} |T_2(x;1+iv)|^2.$$

Now we bound S_i. By a change of variables and completing the sum over x,

$$S_1 = \sum_{\substack{x \bmod t\\ (x(x+qs);t)=1}}\left|\sum_{m \bmod t}{}^{*} e\left(\frac{mx}{t}\right)s_1(m)\right|^2 \le \sum_{x \bmod t}\left|\sum_{m \bmod t}{}^{*} e\left(\frac{xm}{t}\right)s_1(m)\right|^2;$$

and

$$S_2 = \sum_{\substack{x \bmod t\\ (x(x+qs);t)=1}}\left|\sum_{n \bmod t}{}^{*} e\left(\frac{n\overline{x+qs}}{t}\right)s_1(n)\right|^2 \le \sum_{x \bmod t}\left|\sum_{n \bmod t}{}^{*} e\left(\frac{xn}{t}\right)s_1(n)\right|^2.$$

Opening up the square and applying orthogonality, we obtain that

$$\sum_{t}\ \sum_{m \bmod t}^{*} \left| e\left(\frac{xm}{t}\right) s_1(m)\right|^2 = t \sum_{m \bmod t}^{*} |s_1(m)|^2 : \quad x \bmod$$

Now from the denition of $s_1(m)$, we open up the square and write $z = \frac{1}{2} + iv$ to get

$$(7.8)\qquad \begin{aligned} S_i \quad & t \sum_{m \bmod t}^{*}\ \sum_{\substack{\bmod t\\ =0}} \sum_{p} (m)\,\overline{(p)}\, a_p\, p^{1=4}\ \frac{p}{X}^{1=2\ iv=2}\ e\,\frac{p}{qst} \\ & = t(t) \sum_{\substack{\bmod t\\ =0}} \sum_{p} \frac{(p)\log(p)}{p^{1=2+iv=2}}\, e\,\frac{p}{qst}\ \ _1\,\frac{p}{X}^{2} \ ; \end{aligned}$$

by the denition of a_p.

In the above, we have written $_1(y) = (y)y^{3=4}$ so that $_1$ is a bounded real smooth function supported in $(a; b)$. Since $\frac{p}{qst}$ 1, we do not absorb the exponential factor into the smooth function. Instead, we will use Mellin transform in (7.3). Let $_2(y) = y^{1}{}_1(y)$. Again $_2$ is a smooth function with compact support in $(a; b)$. Also note that the sum over p does not change by introducing the smooth weight $(p=X)$ attached to each term. Therefore, using (7.3) the sum over p in (7.8) is

$$\begin{aligned} \sum \frac{(p)\log(p)}{p^{1=2+iv=2}}\, e\,\frac{p}{qst}\ \ _1\,\frac{p}{X} & = \frac{1}{2i}\int_{(1)} M_2(z) \sum_{p} \frac{(p)\log(p)}{p^{1=2+iv=2}}\, p^{z}\, \frac{p}{X}\ \ _1\,\frac{p}{X}^{p}\ dz \\ & \int |M_2(1+iw)| \sum_{p} \frac{(p)\log(p)}{p^{1=2+iv=2+iw}}\, {}_2\,X^{iw}\ dw \quad \int_{R} |M_2(1 \\ & + iw)|\log^2(q + |v=2 + w|)\, dw \\ (7.9)\qquad & \sqrt{\frac{X}{qST}}^{R}\ \log^2(q + |v|) \end{aligned}$$

by applying Lemma 2.3 and the bound (7.5). Hence from (7.8), we have

$$S_i \quad t^2(t)\, q\sqrt{\frac{X}{T}}\ \log^4(q + |v|):$$

Then from the above bound, (7.7) and applying (7.5),

$$\begin{aligned} D_1(S;T) & \frac{1}{qS}\ \sqrt{\frac{qST}{X}}^{r}\ \sum_{s}\ \sum_{tT} \int_1^{1} |M_1(1+iv)|\ qS\frac{X}{T}\ \log^4(q+|v|)\,dv \\ & \frac{1}{qS}\ \frac{qX}{(qST)}\ \sqrt{\frac{qST}{X}}^{r}\ \frac{s}{\ }\ \frac{X}{qST}\ \sum_{sS}\ \sum_{tT} 1 \end{aligned}$$

$$\frac{q}{S}\sum_{q^2} \frac{X}{q} \quad q^{=2}$$

upon choosing small enough , which gives us the desired bound for $D(S;T)$.

7.2. Proof of Lemma 7.3 { big T. Note that $\frac{qst}{4}\frac{\sqrt{pr}}{p}\ \frac{2}{1}$. We now use the power series expansion for the J-Bessel function (2.4) to separate variables. To be precise, we have

$$D(S;T) = \sum_{tT}\sum \frac{1}{qst(t)^2}\sum_{m \bmod t}{}^{*}\sum_{n \bmod t}{}^{*} V_{qs}(m;n;t) \sum_{\bmod t}\sum_{\substack{\bmod t \\ =0;\,^0=0}} \overline{{}^0(m)(n)\ sS}$$

$$\sum_{;}\sum_{pr} {}^0(p)(r)a_p a_r\, e\ \frac{r+}{qst}\ \sum_{`=0}^{1} \frac{(\ 1)^{`}}{`!(`+k\ \ 1)!}\ \frac{2\sqrt{p r}}{qst}^{2`+k\ \ 1}$$

$$p\quad \frac{1}{qS}\sum_{`=0}^{1} \frac{1}{`!(`+k\ \ 1)!}\ \frac{2bX}{qST}^{2`+k\ \ 1}\sum_{sS}\sum_{tT} \frac{1}{t(t)^2}(W_1W_2)^{1=2};$$

where

$$W_{1;`} := \sum_{\substack{x \bmod t \\ (x(x+qs);t)=1}} jT_3(x;z)j^2; \quad \text{and} \quad W_{2;`} := \sum_{\substack{x \bmod t \\ (x(x+qs);t)=1}} jT_4(x;z)j^2:$$

Moreover,

$$T_3(x;z;`) := \sum_{m \bmod t}{}^{*} e\ \frac{mx}{t}\ s_{2;`}(m); \qquad T_4(x;z;`) := \sum_{n \bmod t}{}^{*} e\ \frac{n\ \bar{x} + \bar{q}s}{t}\ s_{2;`}(n);$$

and

$$s_{2;`}(m) = \sum_{\substack{\bmod t \\ =0}} \overline{(m)}\sum_{p}(p)a_p\, e\ \frac{p}{qst}\ \frac{r}{\frac{p}{bX}}^{2`+k\ \ 1}: \tag{7.10}$$

By the same arguments as in the proof of Lemma 7.2, we derive that

$$W_{i;`}\ \ t(t)\sum_{\substack{(\bmod\ t) \\ =0}}\sum_{p} \frac{(p)\log p}{p}\ \frac{p}{\overline{p}}_{`}\ ;\ \overline{X}^{2}$$

where

$$= (y)e\ \frac{y X_b}{qst}\ \frac{r}{\overline{y}}^{2`+k\ \ 1}{}_{`}(y) :$$

Note that $_`$ is also a smooth function which is compactly supported in $(a;b)$ with $0 < a < b$.

By Lemma 2.3, the sum over p is $\log^2 q \max_{axb} j\ ^{(3)}(x)j$, so we have $W_{i;`\ k}$

$$t(t)^2\ \ (\log q)^{2`3\ \ 2}:$$

Hence

$$D(S;T) \quad \frac{q}{qS} \sum_{\ell=0}^{1} \ell \frac{\ell 6}{!(\ell+k \quad 1)!} \frac{2bX}{qST}^{2\ell+k \quad 1} ST$$

$$\frac{Xq}{q^2 S} \sum_{\ell=0}^{1} \frac{\ell 6}{\ell!(\ell+k \quad 1)!} \frac{2bX}{qST}^{2 \ \ell + k \quad 2}$$

$$S\frac{Xq}{q^2 q} \quad {}^{=2}$$

since $\frac{2bX}{qST} < \frac{1}{2}$ so the sum over ℓ is bounded by an absolute constant.

8. Error term S_3

We recall that

$$S_3 = K \sum_{s;t} \sum \frac{4}{qst(t)^2} \sum_{\substack{m \bmod t \\ n \bmod t}}{}^{*} \sum{}^{*} V_{qs}(m;n;t) \sum_{(p;t)=1} \sum_{\substack{\bmod t \\ 0 =}} \overline{(n)}$$

$$\sum_{r} (r) a_p a_r e \quad \frac{+}{pqstr} J_{k \quad 1} \frac{}{qst\, p} pr: \quad \frac{4}{} \quad —$$

We will exploit a combination of arguments from the treatment of S_1 and S_2 to bound S_3 and prove the following.

Lemma 8.1. With notations as above,

$$S_3 \quad q^{\ =8}:$$

Again note that this is the same that was xed in Section 1.1 so that jj 2 .
Similar to bounding S_2; it is enough to show

$$D_2(S;T) \quad q^{\ =4} \tag{8.1}$$

where $D_2(S;T)$ is

$$\sum_{sS;tT} \sum \frac{1}{qst(t)^2} \sum_{\substack{m \bmod t \\ n \bmod t}}{}^{*} \sum{}^{*} V_{qs}(m;n;t) \sum_{(p;t)=1} \sum_{\bmod t} \overline{(n)} \sum_{r} (r) a_p a_r e \frac{}{pqstr} J_k \frac{1}{qst}{}^{p} pr: {}_{=0} \frac{4}{} \quad —$$

Moreover, after opening $V_{qs}(m;n;t)$ the sum over m is a Ramanujan sum, which is equal to (t). Therefore,

$$D_2(S;T) = \sum_{sS;tT} \sum \frac{(t)}{qst(t)^2} \sum_{x \ (\bmod\ t)}{}^{*} \sum_{n \bmod t}{}^{*} e \quad n\left(\frac{\overline{x} + qs)}{t}\right)! \sum_{\substack{\bmod t \\ =0}} (n) \sum_{r} \overline{(r)} a_r e \frac{}{qst}{}^{(x+qs;t)=1} \frac{}{}$$

$$\sum_{(p;t)=1} a_p e \frac{p}{qst} J_{k \quad 1} \quad \frac{4}{qst} \frac{p}{\sqrt{p}\, r} :$$

We will then separate p and r in the J-Bessel function $J_{k-1}\left(\frac{4\pi\sqrt{pr}}{qst}\right)$. Similar to Section 7, we do it in two ways depending on the size of T. The bound of $D_2(S;T)$ will follow from the followings lemmas.

Lemma 8.2. Let $T \le \frac{30bX}{qS}$. Then

$$D_2(S;T) \ll q^{-\varepsilon/4}.$$

Lemma 8.3. Let $T \ge \frac{30bX}{qS}$. Then

$$D_2(S;T) \ll q^{-\varepsilon/4}.$$

8.1. Proof of Lemma 8.2 { small T. We will apply the Mellin transform in equation (7.2), and by similar reasoning as in Section 7.1, it is sucient to bound

$$D_3(S;T) = \sum_{s\sim S}\sum_{t\sim T}\frac{1}{qst\phi(t)^2}\frac{\sqrt{qst}}{\sqrt{X}}\int_{(1)} \mathcal{M}_1(z) \sum^*_{\chi \,(\mathrm{mod}\ t)}\sum^*_{\substack{n \bmod t\\ (x+qs;t)=1}} e\left(\frac{\overline{n(x+qs)}}{t}\right)$$
$$\sum_{\substack{\chi \bmod t\\ \chi\neq\chi_0}}\chi(n)\sum_r \overline{\chi}(r) a_r\, e\left(\frac{r}{qst}\right)\left(\frac{X}{r}\right)^{1/4+z/2}\sum_{(p;t)=1} a_p\, e\left(\frac{p}{qst}\right)\left(\frac{X}{p}\right)^{1/4+z/2} dz$$

By Cauchy-Schwarz's inequality, we have

$$D_3(S;T) \ll \frac{1}{qS}\sqrt{\frac{qST}{X}}\sum_{t\sim T}\frac{1}{\phi(t)^2}\int_{-\infty}^{\infty}|\mathcal{M}_1(1+iv)|\,(U_1U_2)^{1/2}\,dv; \tag{8.2}$$

where

$$U_1 := \sum_{\substack{x \bmod t\\ (x(x+qs);t)=1}}\left|\sum^*_{n \bmod t} e\left(\frac{\overline{n(x+qs)}}{t}\right)s_1(n)\right|^2$$

for $s_1(n)$ dened in (7.6), and

$$U_2 := \sum_{\substack{x \bmod t\\ (x(x+qs);t)=1}}\left|\sum_{(p;t)=1} a_p\, e\left(\frac{p}{qst}\right)\left(\frac{p}{X}\right)^{-3/4-iv/2}\right|^2. \tag{8.3}$$

By the same argument as bounding S_2 in Section 7.1,

$$U_1 \ll t^2\phi(t)\sqrt{\frac{qX}{ST}}. \tag{8.4}$$

For U_2, we need a somewhat dierent treatment. For clarity, we rst prove the following technical lemma.

Lemma 8.4.

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|\mathcal{M}_1(1+iv)\mathcal{M}_2(1+iw)|}{(1+|v/2+w|)^B}\,dw\,dv \ll q^{\varepsilon}; \tag{8.5}$$

for M_i satisfying (7.4) and any $B \geq 2$.

Proof. By (7.4), we note that the contribution of the ranges $|v| > c_1 \frac{X}{qst}$ or $|w| > c_2 \frac{X}{qst}$ is $O(1)$. Hence we will assume that $|v|, |w| \ll \frac{X}{qst}$. Further, we write $u = \frac{v}{2} + w$, so that the left side of (8.5) is

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|M_1(1+iv)M_2(1+iw)|}{(1+|v/2+w|)^B}\,dw\,dv &\ll \int_{|w| \ll \frac{X}{qst}}\int_{|v| \ll \frac{X}{qst}} \frac{1}{(1+|v|)^{1/2}(1+|w|)^{1/2}(1+|v/2+w|)^B}\,dv\,dw \\
&\ll \int_{|w| \ll \frac{X}{qst}}\int_{|u| \ll \frac{X}{qst}} \frac{1}{(1+2|u-w|)^{1/2}(1+|w|)^{1/2}(1+|u|)^B}\,du\,dw \\
&\ll \log \frac{X}{qst}, \qquad (8.6)
\end{aligned}$$

where the last line follows by considering the cases $|u-w| > |w|/2$ and $|u-w| \leq |w|/2$ separately.

Similarly to (7.9), we write the inner sum of U_2 as

$$\begin{aligned}
\sum_{(p,t)=1} a_p\, e\left(\frac{p}{qst}\right) p^{-3/4 - iv/2}\,\Phi\left(\frac{p}{X}\right) &\ll \int |M_2(1+iw)| \left|\sum \frac{\log p}{p^{1/2+iv/2+iw}}\Phi\left(\frac{p}{X}\right)\right|^2 dw \\
&\ll \int_{\mathbb{R}} |M_2(1+iw)| \left(\log^2(q+|v/2+w|) + \frac{\sqrt{X}}{(1+|v/2+w|)^{20}}\right) dw
\end{aligned}$$

by Lemma 2.3. The contribution from the term $\log^2(q+|v/2+w|)$ is $\ll \sqrt{\frac{TX}{qS}}\log^2(q+|v|)$ as in (7.9). The bound for this portion then proceeds similarly as when bounding D_1. It follows that

$$\begin{aligned}
U_2 &\ll \sum_{\substack{x \bmod t \\ (x(x+qs),t)=1}} \left(\int_{\mathbb{R}} |M_2(1+iw)| \frac{\sqrt{X}}{(1+|v/2+w|)^{20}}\,dw\right)^2 \\
&\ll t \left(\int_{\mathbb{R}} |M_2(1+iw)| \frac{\sqrt{X}}{(1+|v/2+w|)^{20}}\,dw\right)^2
\end{aligned}$$

and it remains to bound the contribution of

$$\int_{\mathbb{R}} |M_2(1+iw)| \frac{\sqrt{X}}{(1+|v/2+w|)^{20}}\,dw$$

to $D_3(S;T)$.

Putting this into (8.3) and combining with (8.4) and (8.2) gives

$$D_3(S;T) \ll \frac{\sqrt{X}q}{qS}\sum_{s\sim S}\sum_{t\sim T}\frac{\tau(t)}{\phi(t)}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|M_1(1+iv)M_2(1+iw)|}{(1+|v/2+w|)^{20}}\,dw\,dv \ll \frac{\sqrt{X}q}{q} \ll q^{-1/2+\varepsilon}$$

by Lemma 8.4.

8.2. **Proof of Lemma 8.3 – big T.** Similar to Section 7.2, we will use the power series expansion for the J-Bessel function (2.4) to separate variables p, r. This gives us

$$D_2(S;T) = \sum_{s\sim S}\sum_{t\sim T}\frac{1}{qst\phi(t)^2}\sum^{*}_{\substack{x\ (\mathrm{mod}\ t)\\(x+qs;t)=1}}\sum^{*}_{n\ \mathrm{mod}\ t} e\left(\frac{n(\bar{x}+\overline{qs})}{xt}\right)\sum_{\chi\ \mathrm{mod}\ t}\chi(n)\sum_{r}\bar{\chi}(r)a_r e\left(\frac{r}{qst}\right)$$
$$\sum_{(p;t)=1} a_p e\left(\frac{p}{qst}\right)\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!(\ell+k-1)!}\left(\frac{2\pi\sqrt{pr}}{qst}\right)^{2\ell+k-1}$$
$$\ll \frac{1}{qS}\sum_{\ell=0}^{\infty}\frac{1}{\ell!(\ell+k-1)!}\left(\frac{2bX}{qST}\right)^{2\ell+k-1}\sum_{s\sim S}\sum_{t\sim T}\frac{1}{t\phi(t)^2}\sum_{\chi}(V_{1;\ell}V_{2;\ell})^{1/2};$$

where

$$V_{1;\ell} := \sum_{\substack{x\ \mathrm{mod}\ t\\(x(x+qs);t)=1}}\left|\sum^{*}_{n\ \mathrm{mod}\ t} e\left(\frac{n\bar{x}+\overline{qs}}{t}\right)s_{2;\ell}(n)\right|^2$$

for $s_{2;\ell}(n)$ defined in (7.10), and

$$V_{2;\ell} := \sum_{\substack{x\ \mathrm{mod}\ t\\(x(x+qs);t)=1}}\left|\sum_{p}\frac{\log p}{\sqrt{p}}\,e\left(\frac{p}{qst}\right)\left(\frac{\sqrt{p}}{\sqrt{bX}}\right)^{2\ell+k-1}\chi(p)\right|^2 :$$

By the same arguments as in Section 7.2, we have

$$V_{1;\ell} \ll t^2\phi(t)((\log q)^{2\ell^3})^2:$$

For $V_{2;\ell}$, we bound the sum over p by

$$\sum_{p\le bX}\frac{\log p}{\sqrt{p}}\sqrt{p} \ll X$$

upon using the prime number theorem. Therefore

$$V_{2;\ell} \ll \phi(t)X:$$

Hence

$$D_2(S;T) \ll \frac{q}{qS}\sum_{\ell=0}^{\infty}\frac{\ell^3}{\ell!(\ell+k-1)!}\left(\frac{2bX}{qST}\right)^{2\ell+k-1}\sum_{s\sim S}\sum_{t\sim T}\frac{1}{t\phi(t)^2}\sqrt{X}\sqrt{t\phi(t)}$$

$$\frac{\sqrt{X}}{q} X^{1} \sum_{\ell=0} \frac{\ell 3}{\ell!(\ell+k-1)!} \frac{2bX^{2\ell+k-1}}{qST} \quad q^{=4}$$

as desired.

9. Conclusion for S_N

From Section 5, we have that

$$S_N = S_1 + S_2 + S_3 + O(q^{=2})$$

where $S_1 = MS_1 + O(q^{=4})$ from Section 6, $S_2 \; q^{=4}$ from Section 7, and $S_3 \; q^{=8}$ from Section 8. Thus

$$S_N = MS_1 + O(1) = \begin{cases} \frac{1}{2}\log\frac{q}{X} \int_{\mathbb{R}} |\hat{h}(it)|^2\,dt + O(1) & \text{if } 1 < \;\; 2 \\ O(1) & \text{if } 0 \;\; 1: \end{cases}$$

which concludes the proof of Proposition 1.2.

10. Proof of Theorem 1.1

The proof follows similarly to that of Theorem 2 in Section 4 of [4].

As mentioned earlier, $F(q)$ is an even function in . So we prove it for 0: By the Cauchy-Schwarz inequality and Lemma 2.5,

$$F(q) = M_1 + M_2 + M_3 + O(\sqrt{M_1M_2}) + O(\sqrt{M_1M_3}) + O(\sqrt{M_2M_3});$$

where

$$M_1 := \frac{2}{N(q)(q)} \sum_{\substack{\text{mod } q \\ (1)=(1)^k}} \sum_{f \in H} \Big| \sum_{n} h(n)c_f(n)(n)\hat{h}(\ldots X) \Big|^2 ;$$

$$M_2 := \frac{2}{N(q)(q)} \sum_{\substack{\text{mod } q \\ (1)=(1)^k}} \sum_{f \in H} h(X^{1}) \log \frac{q}{4^2} \ldots^2$$

and

$$M_3 = O\left(X^{2} \log^2 |k| \frac{\log q}{\log 2} \int_{\mathbb{R}} |\hat{h}(it)|^2\,dt \right)^{1}!$$

By Proposition 1.2,

$$M_1 \;\; f()$$

for $0 \;\; 2$: Also, by Lemma 2.4,

$$M_2 \;\; \frac{1}{N(q)} (X^{1}) \log^2 \frac{q}{4^2} \; (q\, |q|^2 (\log q)^2 \;\; \frac{\log q}{2} \int_{\mathbb{R}} |\hat{h}(it)|^2\,dt \;\; 1 :$$

Finally

$$M_3 + O(\sqrt{M_1M_2}) + O(\sqrt{M_1M_3}) + O(\sqrt{M_2M_3})$$

$$= O\left(q^{|\,|}\right)\sqrt{f()\log q} + O\left(q^{|\,|}\right)q^{|\,|}\log|k| \; :$$

Therefore we have

$$F(q) = (1+o(1))\left(f() + (q^{|\,|})^2 \log q \; \frac{1}{2}\int_{\mathbb{R}} |\hat{}(it)|^2\, dt \right)$$
$$+ O\left(q^{|\,|}\right)\left(\sqrt{f()\log q} + (q^{|\,|})q^{|\,|}\log|k|\right) \; :$$

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