Emily Clader · Chiara Damiolini · Daoji Huang · Shiyue Li [iD] · Rohini Ramadas

Permutohedral complexes and rational curves with cyclic action

Abstract. We define a moduli space of rational curves with finite-order automorphism and weighted orbits, and we prove that the combinatorics of its boundary strata are encoded by a particular polytopal complex that also captures the algebraic structure of a complex reflection group acting on the moduli space. This generalizes the situation for Losev–Manin's moduli space of curves (whose boundary strata are encoded by the permutohedron and related to the symmetric group) as well as the situation for Batyrev–Blume's moduli space of curves with involution, and it extends that work beyond the toric context.

1. Introduction

The moduli space $\overline{\mathcal{M}}_{0,n}$ of genus-zero stable curves with n distinct marked points is a fundamental object in algebraic geometry, in part due to its applicability—to such fields as enumerative geometry, representation theory, and mathematical physics, to name a few—but also because it is an interesting variety in its own right. In particular, while $\overline{\mathcal{M}}_{0,n}$ is not toric when $n \geq 5$, it shares some of the combinatorial structure that a toric variety would enjoy. The Chow ring of a toric variety, for example, is generated by the toric boundary (the positive-codimension torus-invariant subvarieties) with relations described combinatorially in terms of

E. Clader: Department of Mathematics, San Francisco State University, San Francisco, CA, USA
e-mail: eclader@sfsu.edu

C. Damiolini: Department of Mathematics, Rutgers University, New Brunswick, NJ, USA
e-mail: chiarad@sas.upenn.edu

D. Huang: ICERM, Brown University, Providence, RI, USA
e-mail: huan0664@umn.edu

S. Li (✉): Department of Mathematics, Brown University, Providence, RI, USA
e-mail: shiyue_li@brown.edu

R. Ramadas: Warwick Mathematics Institute, University of Warwick, Coventry, UK
e-mail: Rohini.ramadas@warwick.ac.uk

C. Damiolini: Department of Mathematics, University of Pennsylvania, Philadelphia, PA, USA

D. Huang: School of Mathematics, University of Minnesota, Minneapolis, MN, USA

fan data; analogously, Keel proved in [14] that the Chow ring of $\overline{\mathcal{M}}_{0,n}$ is generated by the modular boundary (the positive-codimension boundary strata) with relations described combinatorially in terms of dual graphs.

One perspective on the close connection between $\overline{\mathcal{M}}_{0,n}$ and toric varieties is that the moduli problem can be tweaked to produce a space that is, in fact, toric. Losev and Manin studied a particularly significant such modification in [16], constructing a moduli space $\overline{\mathcal{L}}_n$ that parameterizes genus-zero curves with marked points $(y_1, y_2, z_1, \ldots, z_n)$, where the marked points y_1 and y_2 are "heavy"—that is, they cannot coincide with any other marked points—whereas the marked points $z_1, \ldots, z_n$ are "light" in the sense that they are allowed to coincide with one another. The space $\overline{\mathcal{L}}_n$ (which is birational to $\overline{\mathcal{M}}_{0,n+2}$) is a toric variety, and its associated polytope is the permutohedron Π_n: the convex hull in $\mathbb{R}^n$ of the $n!$ points obtained by permuting the coordinates of $(1, 2, \ldots, n)$. Moreover, Losev and Manin proved that the torus-invariant strata of $\overline{\mathcal{L}}_n$ are precisely the boundary strata, which implies that there is a dimension-preserving, inclusion-preserving bijection

$$\left\{ \begin{matrix} \text{boundary} \\ \text{strata in } \overline{\mathcal{L}}_n \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{faces} \\ \text{of } \Pi_n \end{matrix} \right\}.$$

From a combinatorial perspective, on the other hand, the faces of Π_n have another interpretation: they encode the generation of the symmetric group S_n by adjacent transpositions. Namely, the d-dimensional faces of Π_n are in inclusion-preserving bijection with the right cosets in S_n of subgroups of the form

$$\langle \tau_1, \ldots, \tau_d \rangle \subseteq S_n,$$

where $\tau_1, \ldots, \tau_d$ are adjacent transpositions.

Batyrev and Blume extended the work of Losev and Manin in [2,3], constructing a moduli space $\overline{\mathcal{L}}_n^2$ that parameterizes genus-zero curves with an involution σ, two light fixed points of σ, one heavy marked orbit of σ, and n light marked orbits. Again, this moduli space is toric, and its torus-invariant strata are precisely the boundary strata, so one obtains a dimension-preserving, inclusion-preserving bijection

$$\left\{ \begin{matrix} \text{boundary} \\ \text{strata in } \overline{\mathcal{L}}_n^2 \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{faces} \\ \text{of } \Delta_n^2 \end{matrix} \right\}.$$

Here, Δ_n^2 is the polytope known as the type-B permutohedron, which is the convex hull in $\mathbb{R}^n$ of the $2^n n!$ points obtained by permuting the coordinates of $(\pm 1, \pm 2, \ldots, \pm n)$. Also analogously to the Losev–Manin case, this polytope has a group-theoretic interpretation, this time in terms of the complex reflection group $S(2, n)$ of $n \times n$ matrices all of whose nonzero entries are ± 1, and with exactly one nonzero entry in each row and column.

The motivation for Batyrev and Blume's work comes from the theory of root systems. Specifically, they constructed a toric variety associated to any root system and proved that Losev–Manin space $\overline{\mathcal{L}}_n$ is the toric variety associated to the classical root system A_{n-1}, while working instead with the root system B_n yields their moduli space $\overline{\mathcal{L}}_n^2$. From the perspective of root systems, however, this seems to be the end

of the line: Batyrev and Blume proved that the toric varieties in types C and D do not have equally well-behaved modular interpretations.

In this paper, we propose a generalization of Losev–Manin and Batyrev–Blume's story in a different direction. Namely, rather than preserve the connection to root systems, we preserve from Batyrev–Blume's work the existence of an automorphism σ but allow it to have any finite order r. The result is a moduli space $\overline{\mathcal{L}}_n^r$ that parameterizes genus-zero curves with an order-r automorphism, two light fixed points, one heavy marked orbit, and n light marked orbits.

In one sense, the moduli spaces $\overline{\mathcal{L}}_n^r$ break the story, because when $r \geq 3$, they are not toric. In particular, then, their boundary strata are not encoded by the faces of a polytope. However, something perhaps more intriguing is true: we prove that there exists a polytopal complex Δ_n^r whose "Δ-faces" encode the boundary strata of $\overline{\mathcal{L}}_n^r$. More specifically, Δ_n^r is defined as the subset of

$$\left(\mathbb{R}^{\geq 0} \cdot \mu_r \right)^n \subseteq \mathbb{C}^n$$

bounded by certain hyperplanes (where μ_r denotes the group of rth roots of unity), and its Δ-faces are defined as the intersections of Δ_n^r with collections of the bounding hyperplanes. When $r = 2$, the complex Δ_n^r specializes to the type-B permutohedron Δ_n^2 and Δ-faces are ordinary faces, so Batyrev–Blume's result is recovered.

Furthermore, we generalize the group-theoretic interpretation of both Losev–Manin and Batyrev–Blume's moduli spaces. Namely, let $S(r, n)$ be the group of $n \times n$ matrices all of whose nonzero entries are rth roots of unity, and with exactly one nonzero entry in each row and column. Then $S(r, n)$ is generated by the set

$$\mathcal{T} := \left\{ \begin{matrix} \text{adjacent transpositions} \\ s_i \text{ in } S_n \subseteq S(r, n) \end{matrix} \right\} \cup \left\{ s_0 := \begin{pmatrix} \zeta & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \right\},$$

where ζ is a primitive rth root of unity, and we define a d-**dimensional** $\mathcal{T}$-**coset** to be a right coset of the form

$$\langle t_1, \ldots, t_d \rangle \cdot A \subseteq S(r, n)$$

with $t_1, \ldots, t_d \in \mathcal{T}$ and $A \in S(r, n)$.

Our main theorem is the following:

Theorem 1.1. *For any integers $r \geq 2$ and $n \geq 0$, there are dimension-preserving, inclusion-preserving bijections*

$$\left\{ \begin{matrix} \text{boundary} \\ \text{strata in } \overline{\mathcal{L}}_n^r \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \mathcal{T}\text{-cosets} \\ \text{in } S(r, n) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \Delta\text{-faces} \\ \text{of } \Delta_n^r \end{matrix} \right\}.$$

Remark 1.2. More precisely, all three of the sets appearing in Theorem 1.1 depend on the choice of a primitive rth root of unity ζ: the moduli space $\overline{\mathcal{L}}_n^r$ is a union of connected components $\overline{\mathcal{L}}_n^r(\zeta)$, the definition of $\mathcal{T}$ given above involves a choice of ζ, and ζ also appears in the definition of the bounding hyperplanes of Δ_n^r and hence of its Δ-faces. While we suppress this choice from the notation to make it less cumbersome, Theorem 1.1 should be interpreted as a statement for each choice of primitive rth root of unity; see Convention 4.3.

Remark 1.3. In fact, the bijections of Theorem 1.1 preserve quite a bit more of the structure of the above three types of objects. In particular, we show that there are product decompositions of boundary strata, $\mathcal{T}$-cosets, and Δ-faces, as well as $S(r, n)$-actions on each, and the bijections of Theorem 1.1 respect these. See Sect. 8 for the details.

When $r = 2$, Theorem 1.1 specializes to the results of Batyrev and Blume. For larger values of r, the theorem indicates that the spaces $\overline{\mathcal{L}}_n^r$ are unions of toric varieties with independent torus actions, and in this way they occupy a middle ground between toric varieties and the classical moduli spaces of genus-zero curves—a setting in which the applicability of polyhedral methods is an intriguing new avenue for study.

Let us illustrate the statement of Theorem 1.1 with two special cases, to give the reader a flavor of the combinatorics involved.

Example 1.4. Let $r = 2$ and $n = 2$, which is part of Batyrev–Blume's work. Then Δ_2^2 is the octagon in $\mathbb{R}^2$ with vertices $(\pm 2, \pm 1)$ and $(\pm 1, \pm 2)$, and the combinatorial content of Theorem 1.1 is that we can label the faces of Δ_2^2 in two different ways, both of which are dimension-preserving and inclusion-preserving and are illustrated in Fig. 1(A), (B).

First, in Fig. 1(A), we label each face of Δ_2^2 by a boundary stratum in $\overline{\mathcal{L}}_2^2$. Such a stratum is described by an odd-length chain of rational curves—depicted by its dual graph in the figure—where the involution reflects across the central component, together with two marked fixed points of the involution (on the central component) as well as light orbits (z_1^0, z_1^1) and (z_2^0, z_2^1) and a heavy orbit (y^0, y^1).

Second, as illustrated in Fig. 1(B), each face of Δ_2^2 can be labeled by a $\mathcal{T}$-coset in $S(2, 2)$. In this case, we have $\mathcal{T} = \{s_0, s_1\}$, where

$$s_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The 0-dimensional $\mathcal{T}$-cosets (which label the vertices) are the elements of $S(2, 2)$, while the 1-dimensional $\mathcal{T}$-cosets are the right cosets in $S(2, 2)$ of subgroups generated by a single element of $\mathcal{T}$. Since the two elements of $\mathcal{T}$ together generate $S(2, 2)$, there is only a single 2-dimensional $\mathcal{T}$-coset, which is the entire group and labels the unique 2-dimensional face.

Example 1.5. Now, let $r = 3$ and $n = 2$, which is a new case in the current work. In the previous example, the intersection of $\Delta_2^2 \subseteq \mathbb{R}^2$ with each quadrant is a pentagon,
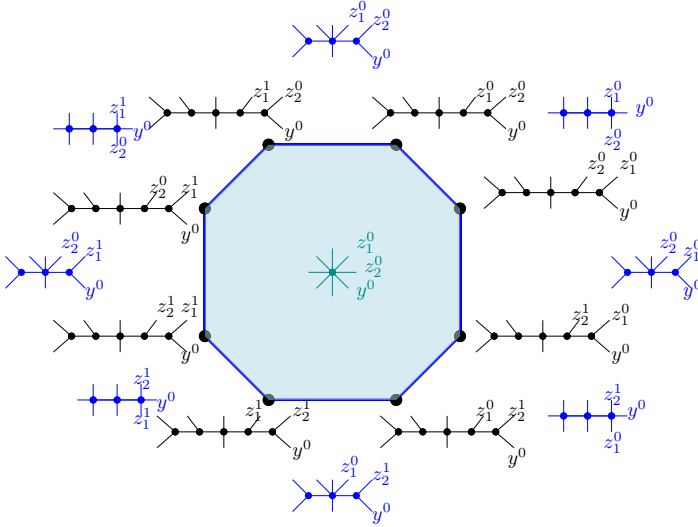
(A) The polytope Δ_2^2 with faces labeled by boundary strata in $\overline{\mathcal{L}}_2^2$. The markings on the left side of each stratum are omitted for clarity but are uniquely determined by the involution.
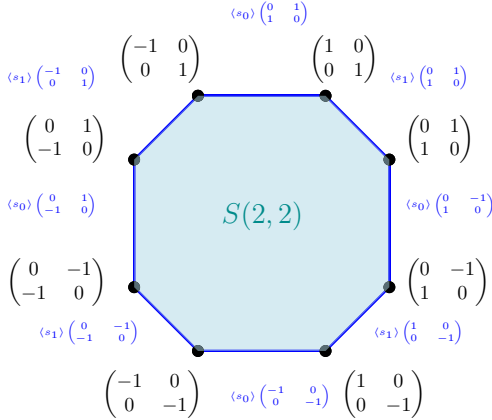
(B) The polytope Δ_2^2 with faces labeled by $\mathcal{T}$-cosets in $S(2,2)$.

Fig. 1. The example of $r = 2$ and $n = 2$

and these pentagons meet in pairs when a coordinate changes sign. When $r = 3$, by contrast, the polytopal complex Δ_2^3 is a subset of

$$\left(\mathbb{R}^{\geq 0} \cdot \{1, \zeta, \zeta^2\} \right) \times \left(\mathbb{R}^{\geq 0} \cdot \{1, \zeta, \zeta^2\} \right) \subseteq \mathbb{C}^2$$

where ζ is a primitive third root of unity. Its intersection with each of the subsets

$$\left(\mathbb{R}^{\geq 0} \cdot \zeta^a \right) \times \left(\mathbb{R}^{\geq 0} \cdot \zeta^b \right) \subseteq \mathbb{C}^2$$

for $a, b \in \{0, 1, 2\}$ is a pentagon, and these nine pentagons meet in triples when a or b changes. This complex is depicted in Fig. 2, where we illustrate the statement
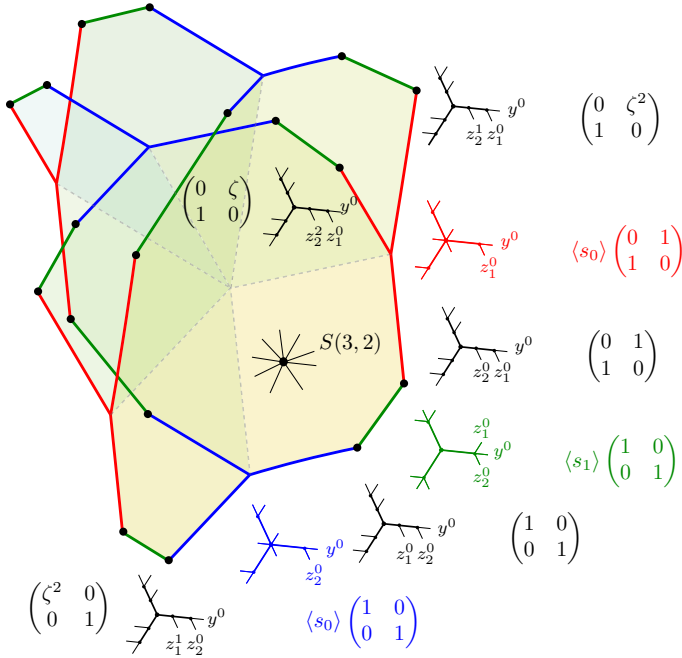
Fig. 2. A projection of the polytope Δ_2^3, with some Δ-faces labeled by both the boundary strata in $\overline{\mathcal{L}}_2^3$ and by the corresponding $\mathcal{T}$-cosets in $S(3, 2)$

of Theorem 1.1 again by labeling some of the Δ-faces in two ways: first by the boundary strata in $\overline{\mathcal{L}}_2^3$, which are described by marked trees of rational curves with μ_3-symmetry, and second by $\mathcal{T}$-cosets in $S(3, 2)$.

The precise definition of Δ-faces is given in Definition 7.2, but, for now, we simply remark that they are themselves polytopal complexes; in particular, the 0-dimensional Δ-faces are the vertices labeled in black, the 1-dimensional Δ-faces are the line segments labeled in green as well as the Y-shapes labeled in red and blue, and there is only a single 2-dimensional Δ-face, which is the entire complex Δ_2^3. The key observation is that a 1-dimensional Δ-face may be adjacent to either two or three 0-dimensional Δ-faces. This corresponds precisely to a fact about boundary strata and about $\mathcal{T}$-cosets:

- in $\overline{\mathcal{L}}_2^3$, a 1-dimensional boundary stratum contains either two or three different 0-dimensional boundary strata, depending on whether there is a light orbit on the central component;
- in $S(3, 2)$, the 1-dimensional $\mathcal{T}$-cosets $\langle t \rangle \cdot A$ have either two or three elements, depending on whether t is an adjacent transposition or $t = s_0$.

These are special cases of the combinatorial phenomena that arise more generally in what follows.

Future and related work

In the sequel [6] to this paper, we prove that $\overline{\mathcal{L}}_n^r$ can be realized as a wonderful compactification of a particular hyperplane arrangement in a product of projective spaces, from which we deduce a presentation of its Chow ring, analogously to Keel's computation of $A^*(\overline{\mathcal{M}}_{0,n})$ in [14] or the results of [4,5,15] for Hassett spaces. In future work, we hope to probe further geometric and combinatorial properties of $\overline{\mathcal{L}}_n^r$ to more fully exploit its proximity to being a toric variety. For example, one could study the tropical manifestation of $\overline{\mathcal{L}}_r^n$ (along the lines of the work carried out in [7,8,21] for $\overline{\mathcal{M}}_{g,n}$ and Hassett spaces) or its symmetries (along the lines of [18,19]). The birational geometry of $\overline{\mathcal{L}}_n^r$ would also be very interesting to study. In the case of $\overline{\mathcal{M}}_{0,n}$, the parallelism with toric varieties motivated Fulton's famous F-conjecture as well as the (now disproven) conjecture that the Cox ring of $\overline{\mathcal{M}}_{0,n}$ is finitely-generated [9,11]. Perhaps the fact that $\overline{\mathcal{L}}_n^r$ is "more toric" than $\overline{\mathcal{M}}_{0,n}$— in that it is combinatorially encoded by a polyhedral object—would make these birational-geometric questions more amenable to study in this setting.

Plan of the paper

In Sect. 2, we precisely define the objects parameterized by the moduli space $\overline{\mathcal{L}}_n^r$. The fact that there indeed exists a fine moduli space parameterizing these objects is the content of Sect. 3; readers wishing to accept the existence of $\overline{\mathcal{L}}_n^r$ are encouraged to skip that section and proceed directly to the combinatorial material that follows. Section 4 defines the combinatorial data of decorated chains of subsets of $\{1, \dots, n\}$, and Sects. 5, 6, and 7 show that this data can be used to describe the boundary strata, $\mathcal{T}$-cosets, and Δ-faces, respectively. Finally, in Sect. 8, we combine the results of the previous three sections to deduce Theorem 1.1, and we observe that the boundary strata, $\mathcal{T}$-cosets, and Δ-faces also have product decompositions and $S(r, n)$-actions that are respected by the bijections between them.

2. The moduli space

Fix integers $r \geq 2$ and $n \geq 0$. Denote by $\mu_r \subseteq \mathbb{C}^*$ the cyclic group of rth roots of unity, and denote $\mathbb{Z}_r = \{0, 1, \dots, r-1\}$.

2.1. Objects and families

We begin by specifying the underlying curves of the objects we are interested in parameterizing. Throughout, varieties are considered over the field $\mathbb{C}$.

Definition 2.1. An r-**pinwheel curve** is a tree of projective lines meeting at nodes, consisting of a central projective line from which r equal-length chains of projective lines ("spokes") emanate. If each of these spokes has k components, we say that the pinwheel curve has **length** k; in the case where $k = 0$, the curve is simply $\mathbb{P}^1$.
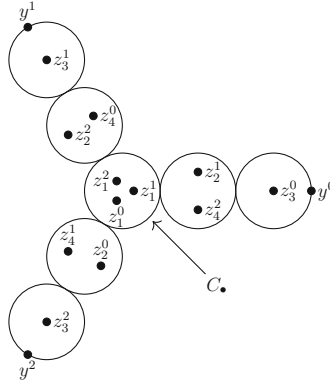
Fig. 3. A stable length-two $(3, 4)$-curve, where each circle represents a $\mathbb{P}^1$ component and σ is the rotational automorphism. Not pictured are the marked points x^+ and x^-, which are the two fixed points of σ and must both lie on the central component $C_\bullet$.

The objects of our moduli space are built from r-pinwheel curves as follows.

Definition 2.2. An (r, n)**-curve** consists of the following data:

- an r-pinwheel curve C;
- an order-r automorphism $\sigma : C \to C$;
- two distinct fixed points x^+ and x^- of σ;
- n labeled r-tuples $(z_1^0, \ldots, z_1^{r-1}), \ldots, (z_n^0, \ldots, z_n^{r-1})$ of points $z_i^j \in C$ satisfying

$$\sigma(z_i^j) = z_i^{j+1 \bmod r}$$

for each i and j, where we allow that $z_i^j = z_{i'}^{j'}$ and that $z_i^j = x^\pm$;
- an additional labeled r-tuple $(y^0, \ldots, y^{r-1})$ satisfying

$$\sigma(y^\ell) = y^{\ell+1 \bmod r}$$

for each ℓ, whose elements are distinct from one another as well as from $x^\pm$ and z_i^j.

We refer to an (r, n)-curve as **stable** if each irreducible component of C contains at least two "heavy" points—where the "heavy" points are the half-nodes and the points y^ℓ—and any irreducible component with exactly two heavy points contains at least one of the "light" points $x^\pm$ or z_i^j. (This is a special case of the stability condition for Hassett spaces, which will play a major role in the construction of the moduli space $\overline{\mathcal{L}}_n^r$ in Sect. 3 below.)

It is straightforward to see that the stability condition forces $y^0, \ldots, y^{r-1}$ to lie on the r leaves of the pinwheel curve, in which case σ must consist of a rotation taking each spoke of the pinwheel to another, and $x^\pm$ must both lie on the central component, which we denote $C_\bullet$; see Fig. 3.

Up to an automorphism of C, one can assume that x^+ and x^- are the points ∞ and 0, respectively, in the central component $C_\bullet \cong \mathbb{P}^1$, and that the node at which the y^0-spoke meets $C_\bullet$ is the point $1 \in C_\bullet \cong \mathbb{P}^1$. Under this identification, the fact that σ has order r ensures that the y^1-spoke meets $C_\bullet$ at $\zeta \in \mathbb{P}^1$, where ζ is a primitive rth root of unity. Once ζ is chosen, the points at which the remaining spokes meet $C_\bullet$ are determined, but ζ itself can be freely chosen to be any primitive rth root of unity. We encode this choice in the following terminology.

Definition 2.3. Let ζ be a primitive rth root of unity. Given a stable (r, n)-curve, let $p^\ell \in C_\bullet$ be the point at which the y^ℓ-spoke meets the central component, for each $\ell \in \mathbb{Z}_r$. We say that the curve has **type** ζ if, under the unique automorphism of the central component $C_\bullet$ that sends

$$x^+ \mapsto \infty, \quad x^- \mapsto 0, \quad p^0 \mapsto 1,$$

we have

$$p^\ell \mapsto \zeta^\ell$$

for all $\ell \in \mathbb{Z}_r$.

Remark 2.4. A stable (r, n)-curve of type ζ can be viewed as a curve with an action of the cyclic group μ_r, in which the generator $\zeta \in \mu_r$ acts by the automorphism σ.

Having defined the objects of interest, we now specify the notions of family and morphism of families.

Definition 2.5. A **family of stable (r, n)-curves over a base scheme B** is a flat, proper morphism $\pi : \mathcal{C} \to B$ equipped with an order-r automorphism σ of $\mathcal{C}$ such that $\pi \circ \sigma = \pi$, and sections $x^\pm$, $\{z_i^j\}$, and $\{y^\ell\}$ of π such that for any geometric point $b \in B$, the fiber

$$\left(\pi^{-1}(b); \sigma\big|_{\pi^{-1}(b)}; x^\pm(b), \{y^\ell(b)\}, \{z_i^j(b)\} \right)$$

is a stable (r, n)-curve. If, furthermore, each fiber has type ζ, we say that the family has type ζ.

Remark 2.6. If the base B is connected, then every fiber has the same type, so the type of the family can be deduced from considering any single fiber.

Definition 2.7. Given families

$$(\pi : \mathcal{C} \to B; \sigma; x^\pm, \{y^\ell\}, \{z_i^j\}) \text{ and } (\pi' : \mathcal{C}' \to B; \sigma'; X^\pm, \{Y^\ell\}, \{Z_i^j\})$$

over the same base B, a **morphism** of families is a morphism $s : \mathcal{C} \to \mathcal{C}'$ satisfying

- $\pi' \circ s = \pi$;
- $\sigma' \circ s = s \circ \sigma$;
- $X^\pm = s \circ x^\pm$, $Z_i^j = s \circ z_i^j$, and $Y^\ell = s \circ y^\ell$.

The goal of Sect. 3 is to identify a fine moduli space representing the moduli problem specified above, which we denote as follows.

Definition 2.8. For any integers $r \geq 2$ and $n \geq 0$, we denote by $\overline{\mathcal{L}}_n^r(\zeta)$ the moduli space of isomorphism classes of stable (r, n)-curves of type ζ, and we denote by $\overline{\mathcal{L}}_n^r = \bigsqcup_\zeta \overline{\mathcal{L}}_n^r(\zeta)$ the moduli space of all isomorphism classes of stable (r, n)-curves.

The reader willing to accept the existence of such a fine moduli space may wish to skip Sect. 3 entirely and proceed directly to the combinatorics in Sect. 4, and they are encouraged to do so. First, however, we must describe the boundary strata in $\overline{\mathcal{L}}_n^r$, which are critical to the combinatorics that follow.

2.2. Boundary strata

In any moduli space of curves, a boundary stratum is defined as the closure of the locus of curves of a fixed topological type. More precisely, we have the following definition of boundary strata in our case.

Definition 2.9. Any $(C; \sigma; x^\pm, \{y^\ell\}, \{z_i^j\}) \in \overline{\mathcal{L}}_n^r(\zeta)$ has an associated **dual graph** consisting of

- a vertex v for each irreducible component C_v of C;
- an edge between vertices v and w if the corresponding irreducible components C_v and C_w meet at a node;
- a half-edge attached to the vertex v for each marked point on C_v, labeled by the name $x^\pm$, y^ℓ, or z_i^j of the marked point.

Given such a dual graph G, the corresponding **boundary stratum** $S_G \subseteq \overline{\mathcal{L}}_n^r(\zeta)$ is defined as the closure of the set of curves with dual graph G.

In particular, passing to the closure means that S_G includes also degenerations of curves with dual graph G. It follows that one can detect in terms of dual graphs when there is an inclusion of boundary strata: we have $S_G \subseteq S_H$ if and only if H can be obtained from G by edge-contraction of some subset of the edges, a procedure that corresponds geometrically to degeneration of a curve in S_H to a curve in S_G. For example, if G is the top dual graph in Fig. 4 and G_1 and G_2 are the two dual graphs depicted below it, then we have $S_G \subseteq S_{G_1}$ and $S_G \subseteq S_{G_2}$, corresponding to the fact that both G_1 and G_2 can be obtained by edge-contraction from G.

3. Construction of the moduli space

We construct the moduli space of stable (r, n)-curves as a closed subscheme of a more well-known moduli space constructed by Hassett in [12], so we begin by recalling the necessary definitions from the theory of Hassett spaces. Throughout, we denote

$$[n] = \{1, 2, \ldots, n\}$$

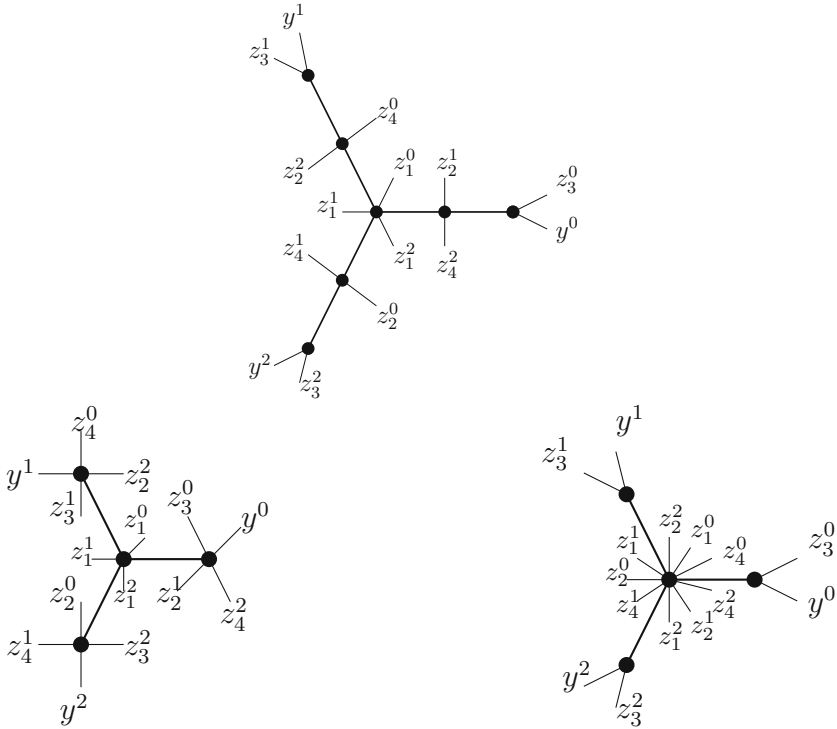and

$$[n]_0 = \{0, 1, \ldots, n\}.$$

Fig. 4. The dual graph G of the curve in Fig. 3, and below it, two graphs obtained by edge-contraction from G

3.1. Hassett spaces and maps between them

For any $g \geq 0$, $n \geq 0$ and any weight vector $\mathbf{w} = (w_1, \ldots, w_n) \in (\mathbb{Q} \cap (0, 1])^n$ such that $2g + \sum_{i=1}^{n} w_i > 2$, the associated Hassett space is a smooth Deligne–Mumford stack $\overline{\mathcal{M}}_{g,\mathbf{w}}$ that is an alternate modular compactification of the moduli space $\mathcal{M}_{g,n}$ of smooth projective curves of genus g with n distinct marked points.

Specifically, elements of $\overline{\mathcal{M}}_{g,\mathbf{w}}$ are tuples $(C; q_1, \ldots, q_n)$, where C is a projective curve of arithmetic genus g and only nodes as singularities, and $q_1, \ldots, q_n \in C$ are marked points with weights $w_1, \ldots, w_n$, satisfying the following two conditions:

- the sum of the weights of any collection of coinciding marked points is at most 1;
- for each irreducible component D of C, if g_D is the genus of D, n_D is the number of half-nodes of D, and $\{q_i\}_{i \in I_D \subseteq [n]}$ are the marked points of D, then

$$2g_D - 2 + n_D + \sum_{i \in I_D} w_i > 0.$$

We refer to elements of $\overline{\mathcal{M}}_{g,\mathbf{w}}$ as **w-stable curves**. Note that the usual moduli space of curves $\overline{\mathcal{M}}_{g,n}$ is recovered by taking $w_i = 1$ for all $i \in [n]$.
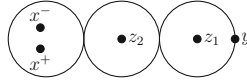
Fig. 5. A typical curve in $\overline{\mathcal{M}}_2^1$

The special case of this construction that is relevant for the current work is when $g = 0$ and the weight vector is

$$
\mathbf{w} = \left(\frac{1}{2} + \varepsilon, \frac{1}{2} + \varepsilon, \underbrace{1, \ldots, 1}_{r}, \underbrace{\varepsilon, \ldots, \varepsilon}_{nr} \right) \tag{1}
$$

for any fixed $0 < \varepsilon < 1/(nr + 2)$. Let

$$
\overline{\mathcal{M}}_n^r := \overline{\mathcal{M}}_{0, \mathbf{w}}
$$

denote the Hassett space with this weight vector, which is a smooth projective scheme. Let $\mathcal{C}_n^r$ denote the universal curve over $\overline{\mathcal{M}}_n^r$. Suggestively, we denote the marked points of $\overline{\mathcal{M}}_n^r$ with weights $1/2 + \varepsilon$ by x^+ and x^-, the r marked points with weight 1 by $y^0, \ldots, y^{r-1}$, and the nr marked points with weight ε by z_i^j with $i \in [n]$ and $j \in \mathbb{Z}_r$.

Remark 3.1. It is possible to let $r = 1$ in this construction, in which case $\overline{\mathcal{M}}_n^1$ is the Hassett space with weight vector $\mathbf{w} = (1/2 + \varepsilon, 1/2 + \varepsilon, 1, \varepsilon, \ldots, \varepsilon)$. We omit the $\mathbb{Z}_r$-superscripts on the marked points in this case, so an element of $\overline{\mathcal{M}}_n^1$ is denoted $(C; x^+, x^-, y, z_1, \ldots, z_n)$. Here, the dual graph of C is a chain with y on one leaf and (if C is reducible) x^+ and x^- on the unique other leaf; see an example of an element in $\overline{\mathcal{M}}_2^1$ in Fig. 5. While some readers may recognize such chains of projective lines as the underlying curves of the elements in Losev–Manin space, we stress that $\overline{\mathcal{M}}_n^1$ is *not* Losev–Manin space, which can instead be described as $\overline{\mathcal{M}}_{0, \mathbf{w}'}$ with $\mathbf{w}' = (1, 1, \varepsilon, \ldots, \varepsilon)$.

When $r \geq 2$, the choice of the weight vector $\mathbf{w}$ ensures that the r-pinwheel curves C that underlie stable (r, n)-curves are elements of $\overline{\mathcal{M}}_n^r$. However, not every element of $\overline{\mathcal{M}}_n^r$ is such a curve. Thus, the goal of the next subsection is to identify a closed subscheme

$$
\overline{\mathcal{L}}_n^r \subseteq \overline{\mathcal{M}}_n^r
$$

that is a fine moduli space for stable (r, n)-curves. The definition of this subscheme involves two families of morphisms between Hassett spaces, which we now describe.

First, for every map $\alpha \colon [n]_0 \to \mathbb{Z}_r$, there is a morphism

$$
\pi_\alpha \colon \overline{\mathcal{M}}_n^r \to \overline{\mathcal{M}}_n^1
$$

given on $\mathbb{C}$-points by

$$
\pi_\alpha(C; x^\pm, \{y^\ell\}, \{z_i^j\}) = (C; x^\pm, y^{\alpha(0)}, z_1^{\alpha(1)}, \ldots, z_n^{\alpha(n)}).
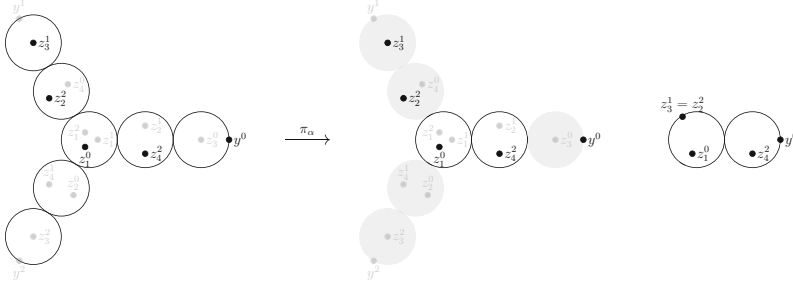$$

Fig. 6. A visualization of the map π_α in the case where $\alpha(0) = 0$, $\alpha(1) = 0$, $\alpha(2) = 2$, $\alpha(3) = 1$, $\alpha(4) = 2$. The marked points $x^\pm$ are on the central component and are not pictured. The faded marked points are forgotten under π_α and the faded solid circles indicate contracted components as a result of forgetting some of the marked points

That is, π_α forgets the marked points not in $\{x^\pm, y^{\alpha(0)}, z_1^{\alpha(1)}, \ldots, z_n^{\alpha(n)}\}$ and contracts any resulting unstable components of C; see Fig. 6.

In fact, one can upgrade π_α from a morphism on $\mathbb{C}$-points to a morphism on families. Namely, for any base scheme B and any map $f: B \to \overline{\mathcal{M}}_n^r$, there are associated maps

$$f_\alpha: B \to \overline{\mathcal{M}}_n^1 \qquad \text{and} \qquad \psi_{\alpha,f}: f^*\mathcal{C}_n^r \to f_\alpha^*\mathcal{C}_n^1$$

satisfying the following conditions:

- If $f^*\mathcal{C}_n^r$ is marked by the sections $x^\pm$, $\{y^\ell\}$, $\{z_i^j\}$, then $f_\alpha^*\mathcal{C}_n^1$ is the stabilization of

$$\left(f^*\mathcal{C}_n^r; x^\pm, y^{\alpha(0)}, z_1^{\alpha(1)}, \ldots, z_n^{\alpha(n)}\right).$$

- The map $\psi_{\alpha,f}$ is the contraction map, which contracts unstable components in each fiber.

The key special case for what follows is when $B = \overline{\mathcal{M}}_n^r$ and $f = \text{id}$, in which case $f_\alpha = \pi_\alpha$ and the map $\psi_{\alpha,\text{id}}$ is a birational morphism that we denote by ψ_α for simplicity. These maps then fit into the following diagram:

$$
\begin{array}{ccc}
\mathcal{C}_n^r & \xrightarrow{\psi_\alpha} \pi_\alpha^*\mathcal{C}_n^1 \longrightarrow & \mathcal{C}_n^1 \qquad\qquad (2) \\
\downarrow & \swarrow & \downarrow \\
\overline{\mathcal{M}}_n^r & \xrightarrow{\quad \pi_\alpha \quad} & \overline{\mathcal{M}}_n^1.
\end{array}
$$

To define the other family of morphisms between Hassett spaces, we first note that there is a natural isomorphism between $\overline{\mathcal{M}}_0^2$ and $\mathbb{P}^1$ given by associating $q \in \mathbb{P}^1$ to

$$(\mathbb{P}^1; \infty, 0, 1, q) \in \overline{\mathcal{M}}_0^2.$$

Given this, for every $i \in \mathbb{Z}_r$, we can define a morphism

$$\lambda_i : \overline{\mathcal{M}}_n^r \to \overline{\mathcal{M}}_0^2 \cong \mathbb{P}^1$$

by sending the point $(C; x^\pm, \{y^\ell\}, \{z_i^j\}) \in \overline{\mathcal{M}}_n^r$ to the element of $\mathbb{P}^1$ corresponding to the point $(C; x^+, x^-, y^i, y^{i+1}) \in \overline{\mathcal{M}}_0^2$.

As above, the morphism λ_i can be described functorially: to any scheme B and any map $f : B \to \overline{\mathcal{M}}_n^r$, we associate maps

$$f_i : B \to \overline{\mathcal{M}}_0^2 \cong \mathbb{P}^1 \qquad \text{and} \qquad \varphi_{i,f} : f^*\mathcal{C}_n^r \to f_i^*\mathcal{C}_0^2$$

satisfying the following conditions:

- If $f^*\mathcal{C}_n^r$ is marked by the sections $x^\pm$, $\{y^\ell\}$, $\{z_i^j\}$, then $f_i^*\mathcal{C}_n^1$ is the stabilization of

$$\left(f^*\mathcal{C}_n^r, x^+, x^-, y^i, y^{i+1} \right).$$

- The map $\varphi_{i,f}$ is the standard contraction map.

Again, we will be particularly interested in the case where $B = \overline{\mathcal{M}}_n^r$ and $f = \mathrm{id}$, in which case $f_i = \lambda_i$ and $\varphi_{i,\mathrm{id}}$ is a birational morphism that we denote simply by φ_i. These maps then fit into a diagram as follows:

$$
\begin{array}{ccccc}
\mathcal{C}_n^r & \xrightarrow{\varphi_i} & \lambda_i^*\mathcal{C}_0^2 & \longrightarrow & \mathcal{C}_0^2 \\
\downarrow & \swarrow & & & \downarrow \\
\overline{\mathcal{M}}_n^r & \xrightarrow{\hspace{1cm} \lambda_i \hspace{1cm}} & \overline{\mathcal{M}}_0^2 = \mathbb{P}^1.
\end{array}
\qquad (3)
$$

Equipped with these morphisms between Hassett spaces, we are prepared to describe our moduli space. In fact, we define separate moduli spaces

$$\overline{\mathcal{L}}_n^r(\zeta) \subseteq \overline{\mathcal{M}}_n^r$$

for each primitive rth root of unity ζ, parameterizing stable (r, n)-curves of type ζ. The moduli spaces for different rth roots of unity are all isomorphic to one another, and the full moduli space

$$\overline{\mathcal{L}}_n^r = \bigsqcup_\zeta \overline{\mathcal{L}}_n^r(\zeta)$$

parameterizing all stable (r, n)-curves is a disjoint union of these isomorphic components.

3.2. Construction of the moduli space

Let $\zeta \in \mathbb{C}$ be a primitive rth root of unity, and for any $\alpha : [n]_0 \to \mathbb{Z}_r$ and any $b \in \mathbb{Z}_r$, denote by $\alpha + b$ the function $[n]_0 \to \mathbb{Z}_r$ defined by

$$(\alpha + b)(x) = \big(\alpha(x) + b\big) \mod r.$$

We define $\overline{\mathcal{L}}_n^r(\zeta)$ as the subscheme of $\overline{\mathcal{M}}_n^r$ obtained from the following fiber diagram:

$$
\begin{array}{ccc}
\overline{\mathcal{L}}_n^r(\zeta) & \lhook\joinrel\longrightarrow & \overline{\mathcal{M}}_n^r \\
\downarrow & & \downarrow{\scriptstyle \prod_\alpha \pi_\alpha \times \pi_{\alpha+1} \times \prod_{i \in \mathbb{Z}_r} \lambda_i} \\
\prod\limits_{\alpha:[n]_0 \to \mathbb{Z}_r} \overline{\mathcal{M}}_n^1 \times \{\zeta\} & \overset{\Delta}{\longrightarrow} & \left(\prod\limits_{\alpha:[n]_0 \to \mathbb{Z}_r} \overline{\mathcal{M}}_n^1 \times \overline{\mathcal{M}}_n^1 \right) \times \prod\limits_{i \in \mathbb{Z}_r} \mathbb{P}^1.
\end{array}
$$

Here, the map Δ is the product of the diagonal embeddings of $\overline{\mathcal{M}}_n^1$ into $\prod_\alpha \left(\overline{\mathcal{M}}_n^1 \times \overline{\mathcal{M}}_n^1 \right)$ and of $\{\zeta\}$ into each $\mathbb{P}^1$, and the upper map is the inclusion of $\overline{\mathcal{L}}_n^r(\zeta)$ into $\overline{\mathcal{M}}_n^r$.

To interpret this fiber diagram more explicitly, note that elements of $\overline{\mathcal{L}}_n^r(\zeta) \subseteq \overline{\mathcal{M}}_n^r$ are defined by the condition that there are isomorphisms of pointed curves

$$\left(C; x^{\pm}, y^{\alpha(0)}, z_1^{\alpha(1)}, \ldots, z_n^{\alpha(n)} \right) \cong \left(C; x^{\pm}, y^{\alpha(0)+1}, z_1^{\alpha(1)+1}, \ldots, z_n^{\alpha(n)+1} \right) \quad (4)$$

for all $\alpha : [n]_0 \to \mathbb{Z}_r$, and

$$(C; x^+, x^-, y^i, y^{i+1}) \cong (C; \infty, 0, 1, \zeta) \quad (5)$$

for all $i \in \mathbb{Z}_r$. The first step in proving that $\overline{\mathcal{L}}_n^r(\zeta)$ is indeed a fine moduli space for stable (r, n)-curves of type ζ is to verify that its points are in bijection with isomorphism classes of such curves.

Lemma 3.2. *If $(C; x^{\pm}, \{y^\ell\}, \{z_i^j\}) \in \overline{\mathcal{L}}_n^r(\zeta)$, then there exists a unique automorphism σ of C making $(C; \sigma; x^{\pm}, \{y^\ell\}, \{z_i^j\})$ into a stable (r, n)-curve of type ζ, and the elements of $\overline{\mathcal{L}}_n^r(\zeta)$ are precisely the elements of $\overline{\mathcal{M}}_n^r$ for which such an automorphism exists.*

Proof. Fix an element $(C; x^{\pm}, \{y^\ell\}, \{z_i^j\}) \in \overline{\mathcal{L}}_n^r(\zeta)$, and let $C_\bullet$ be the component of C containing x^+. Then each y^ℓ lies on some (possibly empty) tree of projective lines attached to $C_\bullet$ at a point p^ℓ, and x^- lies on some (possibly empty) tree of projective lines attached to $C_\bullet$ at a point p^-.

A priori, some of these trees could coincide with one another. However, they cannot all be identical—that is, $x^-, y^0, \ldots, y^{r-1}$ cannot all lie on a single tree emanating from $C_\bullet$—since this would force the only special points on $C_\bullet$ to be a

single node, the marked point x^+, and possibly some of the z_i^j. With the weights (1), such a curve cannot be stable.

Thus, there must be at least one y^ℓ on a different tree than x^-. In this case, we can choose an automorphism s of $C_\bullet$ such that

$$s(x^+) = \infty, \quad s(p^-) = 0, \quad s(p^\ell) = 1,$$

and the condition (5) ensures that

$$s(p^{\ell+1}) = \zeta.$$

(In particular, note that this means that the $y^{\ell+1}$-tree is distinct from both the y^ℓ-tree and the x^--tree.) Applying (5) again with ℓ replaced by $\ell + 1$ shows that

$$s(p^{\ell+2}) = \zeta^2.$$

Continuing in this way proves that x^- and the y^ℓ's lie on $r + 1$ distinct trees emanating from $C_\bullet$, attached (under the automorphism s) at 0 and roots of unity.

Each such tree must end in one or more leaves. However, the weights (1) ensure that a leaf must either contain at least one y^ℓ or both of x^+ and x^-. Therefore, the x^--tree must be empty (that is, $x^- \in C_\bullet$) and each of the y^ℓ-trees must be a chain of projective lines ending in a single leaf with y^ℓ. What remains to be shown, then, is that each of these r chains has the same length—which, in particular, implies that C admits an automorphism σ taking y^ℓ to $y^{\ell+1}$ for each ℓ—and that this automorphism takes z_i^j to z_i^{j+1} for each $i \in [n]$ and $j \in \mathbb{Z}_r$.

To see this, first notice that a repeated application of (4) shows

$$\left(C; x^\pm, y^{\alpha(0)}, z_1^{\alpha(1)}, \ldots, z_n^{\alpha(n)} \right) \cong \left(C; x^\pm, y^{\alpha(0)+i}, z_1^{\alpha(1)+i}, \ldots, z_n^{\alpha(n)+i} \right)$$

for all $i \in \mathbb{Z}_r$. This implies that if z_i^0 lies on the $y^{\alpha(0)}$-spoke of C, then z_i^j lies on the $y^{j+\alpha(0)}$-spoke. In particular, for each fixed $i \in [n]$, no two of the marked points $z_i^0, \ldots, z_i^{r-1}$ can lie on the same spoke, and furthermore, if one of these lies on the central component, then they all do.

In light of this, we can define a function $\alpha : [n]_0 \to \mathbb{Z}_r$ as follows. First, set $\alpha(0) = 0$. Then, for each $i \in [n]$ such that the marked points $z_i^0, \ldots, z_i^{r-1}$ do not lie on the central component, set $\alpha(i) \in \mathbb{Z}_r$ so that $z_i^{\alpha(i)}$ lies on the y^0-spoke of C. Finally, for each $i \in [n]$ such that the marked points $z_i^0, \ldots, z_i^{r-1}$ do lie on the central component, set $\alpha(i) \in \mathbb{Z}_r$ to be any value. For example, for the curve in Fig. 3, we have

$$\alpha(0) = 0, \quad \alpha(1) = \text{anything in } \mathbb{Z}_3, \quad \alpha(2) = 1, \quad \alpha(3) = 0, \quad \alpha(4) = 2.$$

For this α, the morphism π_α does no contraction on the y^0-spoke of C but it contracts all of the other spokes; see Fig. 7.

By the same token, the morphism $\pi_{\alpha+1}$ contracts all spokes except for the y^1-spoke, which remains intact. The condition (4) thus implies that the y^0- and y^1-spokes are isomorphic pointed curves, meaning they are chains of the same length and the isomorphism between them takes any z_i^j in the y^0-spoke to z_i^{j+1}.
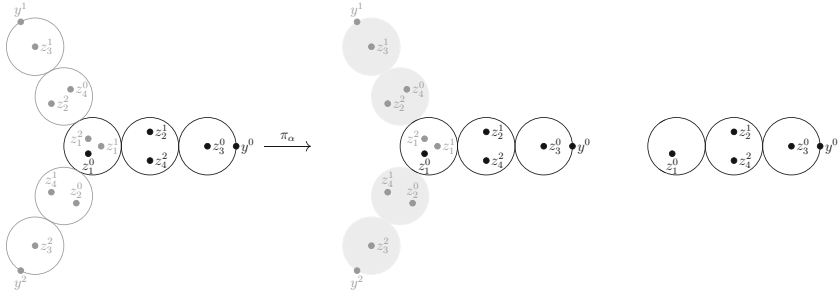
Fig. 7. The image of the curve in Fig. 3 under π_α, where $\alpha(0) = 0$, $\alpha(1) \in \mathbb{Z}_3$, $\alpha(2) = 1$, $\alpha(3) = 0$ and $\alpha(4) = 2$

Repeating this reasoning for each successive spoke shows that C has the appropriate radial symmetry and therefore there exists an automorphism σ that makes C into an (r, n)-curve of type ζ. Since the behavior of this automorphism is specified on at least three points of the central component, it is unique.

Conversely, it is clear that (4) and (5) hold for any (r, n)-curve of type ζ, so such curves are precisely the elements of $\overline{\mathcal{L}}_n^r(\zeta)$. $\qquad\square$

The recipe in the proof of Lemma 3.2 for producing a function $\alpha : [n]_0 \to \mathbb{Z}_r$ from a stable (r, n)-curve will be useful in what follows, so before proceeding, we take a moment to record it in the following definition.

Definition 3.3. Let $(C; \sigma; x^\pm, \{y^\ell\}, \{z_i^j\})$ be a stable (r, n)-curve, which we abbreviate by C for conciseness. We say that a function $\alpha : [n]_0 \to \mathbb{Z}_r$ is **compatible** with C if for each $i \in [n]$ such that the ith light orbit $z_i^0, \ldots, z_i^{r-1}$ does not lie on the central component, the marked point $z_i^{\alpha(i)}$ lies on the $y^{\alpha(0)}$-spoke of C.

Remark 3.4. The notion of compatibility satisfies the following properties:

(1) If $(C; x^\pm, \{y^\ell\}, \{z_i^j\})$ is compatible with α, then $(C; x^\pm, \{y^\ell\}, \{z_i^j\})$ is compatible with $\alpha + b$ for all $b \in \mathbb{Z}_r$.
(2) If $(C; x^\pm, \{y^\ell\}, \{z_i^j\})$ is compatible with α, then so are all other (r, n)-curves with the same dual graph.
(3) Let G and H be dual graphs of (r, n)-curves such that H is obtained via edge-contraction from G. If α is compatible with (r, n)-curves with dual graph G, then it is also compatible with (r, n)-curves with dual graph H. In particular, a smooth (r, n)-curve is compatible with every $\alpha : [n]_0 \to \mathbb{Z}_r$.

Remark 3.4 implies that, for fixed $\alpha : [n]_0 \to \mathbb{Z}_r$, the locus of curves not compatible with α is a union of boundary strata. Conversely, the locus of curves that are compatible with α forms an open set, which we denote by

$$U_\alpha \subseteq \overline{\mathcal{L}}_r^n(\zeta) \tag{6}$$

in what follows.

3.3. Proof of fine moduli space

At this point, we have shown via Lemma 3.2 that the points of $\overline{\mathcal{L}}_n^r(\zeta)$ are in bijection with stable (r, n)-curves of type ζ. In order to know that $\overline{\mathcal{L}}_n^r(\zeta)$ is a fine moduli space for these objects, though, we must also construct a universal family. This can nearly be bootstrapped from $\overline{\mathcal{M}}_n^r$: if

$$\iota : \overline{\mathcal{L}}_n^r(\zeta) \to \overline{\mathcal{M}}_n^r$$

denotes the inclusion, then we can define a universal curve $\mathcal{C}_n^r(\zeta)$ over $\overline{\mathcal{L}}_n^r(\zeta)$ by

$$\mathcal{C}_n^r(\zeta) := \iota^*\mathcal{C}_n^r, \tag{7}$$

where $\mathcal{C}_n^r$ is the universal curve over $\overline{\mathcal{M}}_n^r$. Furthermore, we can define sections $x^\pm$, y^ℓ, and z_j^i of $\mathcal{C}_n^r(\zeta)$ by pullback of the corresponding sections of $\mathcal{C}_n^r$. What remains, however, is to construct a universal automorphism σ of $\mathcal{C}_n^r(\zeta)$. This is the key content of the following theorem.

Theorem 3.5. *Let $r \geq 2$ and $n \geq 0$ be integers, and let ζ be a primitive rth root of unity. Then $\overline{\mathcal{L}}_n^r(\zeta)$ is a fine moduli space for stable (r, n)-curves of type ζ.*

Proof. Let $\mathcal{C}_n^r(\zeta)$ and its sections $x^\pm$, y^ℓ, and z_j^i be defined as in (7) and the subsequent paragraph. To prove the theorem, it suffices to construct an automorphism σ of $\mathcal{C}_n^r(\zeta)$ such that

$$\sigma \circ x^+ = x^+, \quad \sigma \circ x^- = x^-, \quad \sigma \circ y^\ell = y^{\ell+1}, \quad \text{and } \sigma \circ z_i^j = z_i^{j+1} \tag{8}$$

for all $i \in [n]$ and all $\ell, j \in \mathbb{Z}_r$. Indeed, if such an automorphism σ exists, then it makes $\mathcal{C}_n^r(\zeta)$ into a family of stable (r, n)-curves, and this family is of type ζ by Lemma 3.2. It follows that, for any base scheme B, one can restrict the bijection

$$\{\text{morphisms } B \to \overline{\mathcal{M}}_n^r\} \leftrightarrow \{\text{families of } \mathbf{w}\text{-stable curves over } B\}/\cong, \tag{9}$$

which exists by virtue of $\overline{\mathcal{M}}_n^r$ being a fine moduli space for $\mathbf{w}$-stable curves, to yield

$$\{\text{morphisms } B \to \overline{\mathcal{L}}_n^r(\zeta)\} \to \{\text{families of stable } (r, n)\text{-curves of type } \zeta \text{ over } B\}/\cong$$
$$f \mapsto f^*\Big(\mathcal{C}_n^r(\zeta)\Big). \tag{10}$$

Because it is the restriction of a bijection, the map in (10) is certainly injective. It is also surjective, because a family of stable, type-ζ (r, n)-curves over B yields a family of $\mathbf{w}$-stable curves by forgetting σ, and the latter is the pullback of the universal family on $\overline{\mathcal{M}}_n^r$ under some $f : B \to \overline{\mathcal{M}}_n^r$ by the surjectivity of (9). The fact that each fiber of the family is in fact a stable (r, n)-curve of type ζ implies, by Lemma 3.2, that f lands in $\overline{\mathcal{L}}_n^r(\zeta)$. Thus, the map in (10) is a bijection, which says precisely that $\overline{\mathcal{L}}_n^r(\zeta)$ is the requisite fine moduli space.

To construct the automorphism σ of $\mathcal{C}_n^r(\zeta)$, we patch together automorphisms defined locally on open subsets of $\mathcal{C}_n^r(\zeta)$. Toward defining these open sets, let

$\alpha : [n]_0 \to \mathbb{Z}_r$ be any function, and let $U_\alpha \subseteq \overline{\mathcal{L}}_n^r(\zeta)$ be as in (6). For any $b \in \mathbb{Z}_r$, consider the morphism

$$\pi_{\alpha+b}\big|_{U_\alpha} : U_\alpha \to \overline{\mathcal{M}}_n^1.$$

Geometrically, this morphism leaves the $y^{\alpha(0)+b}$-spoke unchanged but contracts all other spokes. In particular, since each individual spoke is isomorphic to each other spoke, we see that

$$\pi_{\alpha+b}\big|_{U_\alpha} = \pi_{\alpha+b'}\big|_{U_\alpha}$$

for any $b, b' \in \mathbb{Z}_r$.

Now, let $\mathcal{C}_\alpha \subseteq \mathcal{C}_n^r(\zeta)$ be the restriction of $\mathcal{C}_n^r(\zeta)$ to U_α. Each of the morphisms $\pi_{\alpha+b}\big|_{U_\alpha}$ can be lifted to a birational morphism

$$\psi_{\alpha+b} : \mathcal{C}_\alpha \to \pi_{\alpha+b}^*(\mathcal{C}_n^1)$$

as in (2), which performs the above contraction of all but the $y^{\alpha(0)+b}$ spoke on each fiber of $\mathcal{C}_\alpha$. Note that since the particular spokes being contracted depend on b, we have

$$\psi_{\alpha+b} \neq \psi_{\alpha+b'}$$

if $b \neq b'$.

For any $\alpha : [n]_0 \to \mathbb{Z}_r$ and any $b \in \mathbb{Z}_r$, denote by

$$V_{\alpha,b} \subseteq \mathcal{C}_\alpha$$

the largest open subset of $\mathcal{C}_\alpha$ on which the birational morphism $\psi_{\alpha+b}$ is an isomorphism. Geometrically, $V_{\alpha,b}$ is obtained from $\mathcal{C}_\alpha$ by removing all spokes in each fiber except for the $y^{\alpha(0)+b}$-spoke, including removing the points in the central component of the fiber at which these spokes are attached. In particular, from this geometric description we see that $\{V_{\alpha,b}\}_{b \in \mathbb{Z}_r}$ covers $\mathcal{C}_\alpha$, and therefore varying over all α, we obtain an open cover of $\mathcal{C}_n^r(\zeta)$.

Our goal, now, is to construct an isomorphism

$$\sigma_{\alpha,b} : V_{\alpha,b} \to V_{\alpha,b+1}$$

relative to U_α, which will serve as the local definition of the universal automorphism σ. The key observation is that

$$\psi_{\alpha+b}(V_{\alpha,b}) = \psi_{\alpha+b+1}(V_{\alpha,b+1}) \tag{11}$$

for every $b \in \mathbb{Z}_r$, from which it follows that $\sigma_{\alpha,b}$ can be defined as the composition of the two isomorphisms

$$V_{\alpha,b} \xrightarrow{\psi_{\alpha+b}|_{V_{\alpha,b}}} \psi_{\alpha+b}(V_{\alpha,b}) = \psi_{\alpha+b+1}(V_{\alpha,b+1}) \xrightarrow{(\psi_{\alpha+b+1}|_{V_{\alpha,b+1}})^{-1}} V_{\alpha,b+1}. \tag{12}$$

To prove (11), first observe that, by the definition of the maps $\psi_{\alpha+b}$ and π_α, we have

$$\psi_{\alpha+b}(V_{\alpha,b}) = \psi_{\alpha+b}(\mathcal{C}_\alpha^{\mathrm{sm}}) \cup \left(\pi_\alpha^*(\mathcal{C}_n^1) \setminus \bigcup_{\ell \neq 0} \mathrm{im}\left(\psi_{\alpha+b} \circ y^{\alpha(0)+b+\ell} \right) \right), \quad (13)$$

where $\mathcal{C}_\alpha^{\mathrm{sm}} \subseteq \mathcal{C}_\alpha$ is the open set whose fibers are smooth curves. For any $\ell \neq 0$, let us describe the morphism

$$\psi_{\alpha+b} \circ y^{\alpha(0)+b+\ell} : U_\alpha \to \pi_\alpha^*(\mathcal{C}_n^1)$$

in geometric terms, by describing its behavior on closed points. Let C be a closed point of U_α, which corresponds to a marked curve $(C; x^\pm, \{y^\ell\}, \{z_i^j\})$. The image of C under π_α is a point that corresponds to a marked curve C', and the image of C under $\psi_{\alpha+b} \circ y^{\alpha(0)+b+\ell}$ is a closed point of C' that we will denote $q^{b,\ell}$. By definition, C' is a chain of projective lines with $x^\pm$ together on an end component that we denote by $C_\bullet$. In this notation, we have $q^{b,\ell} \in C_\bullet$, and if we denote the node of $C_\bullet$ by p, then (5) implies

$$(C_\bullet; x^+, x^-, p, q^{b,\ell}) \cong (\mathbb{P}^1; \infty, 0, 1, \zeta^\ell).$$

Since this result does not depend on b, we see that the morphism $\psi_{\alpha+b} \circ y^{\alpha(0)+b+\ell}$ is independent of b for any $\ell \neq 0$. Given that $\psi_{\alpha+b}(\mathcal{C}_\alpha^{\mathrm{sm}})$ is also manifestly independent of b, it follows from (13) that $\psi_{\alpha+b}(V_{\alpha,b})$ is independent of b, which proves (11).

Lastly, we glue the local morphisms $\sigma_{\alpha,b}$ defined by (12) to give a global morphism

$$\sigma : \mathcal{C}_n^r(\zeta) \to \mathcal{C}_n^r(\zeta)$$

over the base $\overline{\mathcal{L}}_n^r(\zeta)$. To see that the morphisms $\sigma_{\alpha,b}$ indeed agree on the overlaps in their domain, we restrict to fibers of $\mathcal{C}_n^r(\zeta)$ and describe the maps $\sigma_{\alpha,b}$ on closed points. Given any $(C; x^\pm, \{y^\ell\}, \{z_i^j\}) \in \overline{\mathcal{L}}_n^r(\zeta)$, let us also denote by $C \subseteq \mathcal{C}_n^r(\zeta)$ the corresponding fiber of the universal curve. By Lemma 3.2, C is an (r, n)-curve; we denote its central component by $C_\bullet$ as usual, and we denote by $C(b)$ the subcurve obtained by taking the union of $C_\bullet$ and the spoke containing y^b. Suppose $\alpha : [n]_0 \to \mathbb{Z}_r$ is compatible with C. Setting

$$C^\circ(b) := C(b) \cap V_{\alpha,b} = C(b) \setminus \bigcup_{b' \neq b} C(b'), \quad (14)$$

we observe that $\sigma_{\alpha,b}|_{C^\circ(b)}$ defines an isomorphism onto $C^\circ(b+1)$ that fixes $x^\pm$, takes $y^{\alpha(0)}$ to $y^{\alpha(0)+1}$, and takes $z_i^{\alpha(i)}$ to $z_i^{\alpha(i)+1}$. As in the proof of Lemma 3.2, this forces $\sigma_{\alpha,b}|_{C_\bullet \cap C^\circ(b)}$ to be rotation by ζ, once coordinates are chosen on $C_\bullet$ such that $x_+ = \infty$, $x_- = 0$, and the node connecting $C_\bullet$ to $y^{\alpha(0)}$ is 1. This shows that $\sigma_{\alpha,b}|_{C^\circ(b)}$ agrees with $\tau|_{C^\circ(b)}$, where τ is the unique automorphism of C making it into an (r, n)-curve of type ζ. In particular, because this description of $\sigma_{\alpha,b}$ depends

only on the fiber of $\mathcal{C}_n^r(\zeta)$ and not on α or b, the local morphisms $\sigma_{\alpha,b}$ indeed glue to give a global automorphism σ of $\mathcal{C}_n^r(\zeta)$.

Having equipped the universal curve with a universal automorphism σ, which (by the argument of the previous paragraph) makes it into a family of (r, n)-curves of type ζ, the proof of the theorem is complete. $\qquad\qquad\square$

3.4. Geometric observations

We conclude this section with some geometric observations about the moduli space $\overline{\mathcal{L}}_n^r(\zeta)$. Since these are not needed for the current work, we give only brief indications of the proofs.

Observation 3.6. The moduli space $\overline{\mathcal{L}}_n^r(\zeta)$ is smooth, and its boundary (the union of the positive-codimension boundary strata) is a simple normal crossings divisor.

To prove Observation 3.6, one can leverage the analogous result for the Hassett space $\overline{\mathcal{M}}_n^1$. In particular, one can show that the morphisms

$$\pi_\alpha|_{U_\alpha} : U_\alpha \to \overline{\mathcal{M}}_n^1$$

are isomorphisms of U_α onto an open set $U \subseteq \overline{\mathcal{M}}_n^1$. (Specifically, expressing elements of $\overline{\mathcal{M}}_n^1$ as $(C; x^\pm, y, \{z_i\})$ as in Remark 3.1, let $C_\bullet$ be the component containing $x^\pm$. Then, after choosing coordinates on $C_\bullet$ in which $x^+ = \infty$, $x^- = 0$, and the half-node of $C_\bullet$ is 1, the open set $U \subseteq \overline{\mathcal{M}}_n^1$ consists of curves for which none of the light points z_i lies at an rth root of unity in $C_\bullet$.) The local isomorphisms π_α take the boundary stratification of $\overline{\mathcal{L}}_n^r(\zeta)$ to the boundary stratification of $\overline{\mathcal{M}}_n^1$, so Observation 3.6 follows from the analogous statement for Hassett spaces, which is shown in [12].

In addition to $\pi_\alpha|_{U_\alpha}$, there is another natural map from $\overline{\mathcal{L}}_n^r(\zeta)$ to $\overline{\mathcal{M}}_n^1$: rather than remembering a single spoke of an (r, n)-curve as $\pi_\alpha|_{U_\alpha}$ does, one can identify all r spokes with each other. This leads to the following observation.

Observation 3.7. There exists a surjective morphism

$$p : \overline{\mathcal{L}}_n^r(\zeta) \to \overline{\mathcal{M}}_n^1$$

that sends an (r, n)-curve C with automorphism σ to the quotient C/σ, and p realizes $\overline{\mathcal{M}}_n^1$ as the quotient $\overline{\mathcal{L}}_n^r(\zeta)/(\mathbb{Z}_r)^n$.

This relates $\overline{\mathcal{L}}_n^r(\zeta)$ with moduli spaces parameterizing coverings of rational curves with marked points: it provides a compactification of the moduli space of coverings with marked orbits that is related to the Harris–Mumford admissible covers spaces [13] in the same way in which Hassett spaces are related to $\overline{\mathcal{M}}_{g,n}$. Other works in this direction can be found in [10], where a different choice of weights is made that allows ramification points to collide.

Both $\pi_\alpha|_{U_\alpha}$ and p can be interpreted from a polytopal perspective, using that $\overline{\mathcal{M}}_n^1$ is a toric variety (and hence has an associated polytope) whereas $\overline{\mathcal{L}}_n^r(\zeta)$, as we will see below, has an associated polytopal complex. We return to this in Remark 7.14.

4. Decorated chains

Having constructed the moduli space $\overline{\mathcal{L}}_n^r$ of stable (r, n)-curves, we begin the combinatorial heart of the paper: proving that the same combinatorics encodes the boundary strata in $\overline{\mathcal{L}}_n^r$, the $\mathcal{T}$-cosets in the complex reflection group $S(r, n)$, and the Δ-faces of the polytopal complex Δ_n^r. The key idea that yields the correspondence between these three types of objects is that all three can be indexed by the discrete data of decorated chains, which we now define.

Definition 4.1. Let $r \geq 2$ and $n \geq 0$. A $\mathbb{Z}_r$-**decorated chain of subsets of** $[n]$ (or simply **chain**, for short) is a tuple

$$\widetilde{\mathbf{I}} = (I_1, \ldots, I_k, \mathfrak{a}),$$

where

$$\emptyset = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_k \subseteq [n]$$

and

$$\mathfrak{a} : I_k \to \mathbb{Z}_r.$$

We refer to k as the **length** of the chain. The possibility that $k = 0$ is allowed, in which case we make the convention that there is a unique length-0 chain given by $\widetilde{\mathbf{I}} = (\emptyset, \mathfrak{a})$ for the unique function $\mathfrak{a} : \emptyset \to \mathbb{Z}_r$. If $n = 0$, then the length-0 chain is the only chain.

In the following three sections, we describe a bijective procedure for producing, from a chain $\widetilde{\mathbf{I}}$, either a boundary stratum $S_{\widetilde{\mathbf{I}}}$ (Proposition 5.4), a $\mathcal{T}$-coset $C_{\widetilde{\mathbf{I}}}$ (Proposition 6.6), or a Δ-face $F_{\widetilde{\mathbf{I}}}$ of Δ_n^r (Proposition 7.15). Furthermore, we interpret both the dimension of a stratum (or $\mathcal{T}$-coset, or Δ-face) and the inclusion relation between strata (or $\mathcal{T}$-cosets, or Δ-faces) in terms of corresponding features of the chain $\widetilde{\mathbf{I}}$. In particular, the inclusion relation is described in terms of the following relation on chains.

Definition 4.2. Let $\widetilde{\mathbf{I}} = (I_1, \ldots, I_k, \mathfrak{a})$ and $\widetilde{\mathbf{J}} = (J_1, \ldots, J_\ell, \mathfrak{b})$ be chains of length k and ℓ respectively. We say that $\widetilde{\mathbf{I}}$ **refines** $\widetilde{\mathbf{J}}$ if

$$\{J_1, \ldots, J_\ell\} \subseteq \{I_1, \ldots, I_k\}$$

and

$$\mathfrak{b} = \mathfrak{a}\big|_{J_\ell}.$$

(Note that if a chain $\widetilde{\mathbf{I}}$ of length k refines a chain $\widetilde{\mathbf{J}}$ of length ℓ, then $J_\ell \subseteq I_k$ and so the restriction of $\mathfrak{a}$ on J_ℓ is well-defined.)

We will find in what follows that

- the boundary stratum $S_{\widetilde{\mathbf{I}}}$ has codimension k, where $\widetilde{\mathbf{I}}$ is a chain of length k,
- for boundary strata $S_{\widetilde{\mathbf{I}}}$ and $S_{\widetilde{\mathbf{J}}}$, we have $S_{\widetilde{\mathbf{I}}} \subseteq S_{\widetilde{\mathbf{J}}}$ if and only if $\widetilde{\mathbf{I}}$ refines $\widetilde{\mathbf{J}}$,

and the exact same statements hold with boundary strata replaced by $\mathcal{T}$-cosets or Δ-faces. Thus, passing through chains provides the dimension-preserving, inclusion-preserving bijection of Theorem 1.1.

More precisely, we should note that everything that follows depends on the choice of a primitive rth root of unity ζ: the boundary stratum $S_{\widetilde{\mathbf{I}}}$ lies inside a particular component $\overline{\mathcal{L}}_n^r(\zeta)$, the definition of the generating set $\mathcal{T}$ of $S(r, n)$ depends on a choice of ζ, and the Δ-faces of Δ_n^r are described by intersecting with certain hyperplanes whose definition depends on ζ. Thus, we make the following convention once and for all:

Convention 4.3. *Throughout what follows, ζ is a fixed choice of primitive rth root of unity.*

With this set-up in place, we are ready to flesh out the association of chains to each of the requisite objects.

5. Combinatorics of the boundary strata

Recall from Sect. 2.2 that the boundary strata in $\overline{\mathcal{L}}_n^r(\zeta)$ are the closures of the loci of curves of a fixed topological type. We make use of the following labeling scheme for the components of the underlying curves in a boundary stratum, illustrated in Fig. 8.

Notation 5.1. Let $(C; \sigma; x^{\pm}, \{y^{\ell}\}, \{z_i^j\})$ be a stable (r, n)-curve, where C is an r-pinwheel curve of length k. (Recall from Definition 2.1 that this means that each of the r spokes of C has k components.) For each $\ell \in \mathbb{Z}_r$, denote by

$$C_1^{\ell}, \ldots, C_k^{\ell}$$

the components of the spoke containing y^{ℓ}, where $y^{\ell} \in C_1^{\ell}$ and the other components are labeled in order from outermost to innermost. Denote the central component by C_{k+1}.

From here, the construction of a boundary stratum in $\overline{\mathcal{L}}_n^r(\zeta)$ from a chain is as follows.

Definition 5.2. Let $\widetilde{\mathbf{I}} = (I_1, \ldots, I_k, \mathfrak{a})$ be a chain. The associated boundary stratum $S_{\widetilde{\mathbf{I}}} \subseteq \overline{\mathcal{L}}_n^r(\zeta)$ is the closure of the locus of curves

$$(C; x^{\pm}, y^0, \ldots, y^{r-1}, z_1^1, \ldots, z_n^{r-1}) \in \overline{\mathcal{L}}_n^r(\zeta),$$

where C is a length-k r-pinwheel curve and, using Notation 5.1, we have

(1) for each $j \in \{1, \ldots, k\}$, the light marked points on C_j^0 are precisely

$$\{z_i^{\mathfrak{a}(i)} \mid i \in I_j \setminus I_{j-1}\},$$
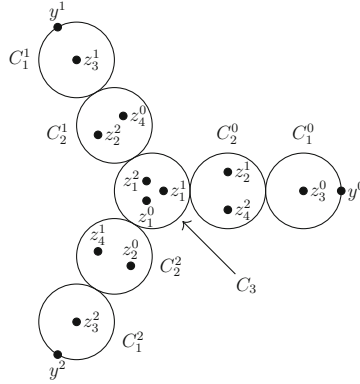
where $I_0 = \emptyset$;

Fig. 8. An element of $\overline{\mathcal{L}}_3^4$ with components labeled via Notation 5.1

(2) the light marked points on the central component C_{k+1} are

$$\{z_i^\ell \mid i \in [n] \setminus I_k, \ \ell \in \mathbb{Z}_r\} \cup \{x^\pm\}.$$

Example 5.3. Let $r = 3$ and $n = 4$, and consider the chain $\widetilde{\mathbf{I}} = (I_1, I_2, \mathfrak{a})$ of length 2, where

$$I_1 = \{3\}, \quad I_2 = \{2, 3, 4\}$$

and $\mathfrak{a} : I_2 \to \mathbb{Z}_3$ is given by

$$\mathfrak{a}(2) = 1, \quad \mathfrak{a}(3) = 0, \quad \mathfrak{a}(4) = 2.$$

The associated boundary stratum $S_{\widetilde{\mathbf{I}}}$ is the closure of the locus of elements of the topological type illustrated in Fig. 8 above. In particular, notice that I_1 indexes the orbits on the outermost components, I_2 indexes the orbits on the two outermost components, and elements of $[n] \setminus I_2$ correspond to orbits in the central component. The decoration $\mathfrak{a}$ indicates the member of each orbit that lies on the y^0-spoke of the pinwheel.

The key combinatorial proposition about boundary strata is the following.

Proposition 5.4. *Let $r \geq 2$ and $n \geq 0$. The association*

$$\widetilde{\mathbf{I}} \mapsto S_{\widetilde{\mathbf{I}}}$$

is a bijection from the set of $\mathbb{Z}_r$-decorated chains of subsets of $[n]$ to the set of boundary strata of $\overline{\mathcal{L}}_n^r(\zeta)$. Furthermore, this bijection satisfies

(i) *$length(\widetilde{\mathbf{I}}) = codim(S_{\widetilde{\mathbf{I}}})$,*
(ii) *$S_{\widetilde{\mathbf{I}}} \subseteq S_{\widetilde{\mathbf{J}}}$ if and only if $\widetilde{\mathbf{I}}$ refines $\widetilde{\mathbf{J}}$.*

Proof. The surjectivity of $\widetilde{\mathbf{I}} \mapsto S_{\widetilde{\mathbf{I}}}$ is clear, since if $S \subseteq \overline{\mathcal{L}}_n^r(\zeta)$ is a boundary stratum with associated dual graph G, then G has a pinwheel shape and thus one can define $I_j \subseteq [n]$ to index the marked points on the outermost j vertices of the y^0-spoke of the pinwheel. The requirement that $z_i^{\mathfrak{a}(i)}$ lies on the y^0-spoke of the pinwheel for all $i \in I_k$ thus defines a function $\mathfrak{a} : I_k \to \mathbb{Z}_r$ for which $S_{\widetilde{\mathbf{I}}} = S$.

Injectivity of the association $\widetilde{\mathbf{I}} \mapsto S_{\widetilde{\mathbf{I}}}$ follows from item (ii) of the proposition, since it is only possible that $\widetilde{\mathbf{I}}$ and $\widetilde{\mathbf{J}}$ refine one another if $\widetilde{\mathbf{I}} = \widetilde{\mathbf{J}}$. Item (ii), on the other hand, follows directly from the containment of boundary strata described in Sect. 2.2. In particular, if $S_{\widetilde{\mathbf{I}}}$ and $S_{\widetilde{\mathbf{J}}}$ are boundary strata with associated dual graphs $G_{\widetilde{\mathbf{I}}}$ and $G_{\widetilde{\mathbf{J}}}$, then $S_{\widetilde{\mathbf{I}}} \subseteq S_{\widetilde{\mathbf{J}}}$ if and only if $G_{\widetilde{\mathbf{J}}}$ can be obtained from $G_{\widetilde{\mathbf{I}}}$ by edge-contraction of some subset of the edges of $G_{\widetilde{\mathbf{I}}}$. Since contracting edges combines the marked points on adjacent vertices, this is the case if and only if $\widetilde{\mathbf{I}}$ refines $\widetilde{\mathbf{J}}$.

Finally, for item (i), let $S = S_{\widetilde{\mathbf{I}}}$ be a boundary stratum with associated dual graph G, where $\mathrm{length}(\widetilde{\mathbf{I}}) = k$ and therefore G is a pinwheel graph in which each spoke has k vertices. Choose any

$$\alpha : [n]_0 \to \mathbb{Z}_r$$

that is compatible with a generic element of S (that is, with any curve with dual graph exactly G), where compatibility is defined as in Definition 3.3. In the notation of Sect. 3.1, there is a morphism

$$\pi_\alpha : \overline{\mathcal{L}}_n^r(\zeta) \to \overline{\mathcal{M}}_n^1$$

that maps $S(\zeta)$ birationally onto the boundary stratum $S_0 \subseteq \overline{\mathcal{M}}_n^1$ whose dual graph G_0 consists of only the y^0-spoke of G together with the central vertex. This dual graph G_0 has k edges, so the well-known results on Hassett spaces imply that $\mathrm{codim}_{\overline{\mathcal{M}}_n^1}(S_0) = k$. Given that $\overline{\mathcal{L}}_n^r(\zeta)$ is birational to $\overline{\mathcal{M}}_n^1$, it follows that

$$\mathrm{codim}_{\overline{\mathcal{L}}_n^r(\zeta)}\big(S(\zeta)\big) = \mathrm{codim}_{\overline{\mathcal{M}}_n^1}(S_0) = k,$$

as claimed. $\qquad\square$

In addition to encoding the dimension and inclusion of boundary strata, we note that the chain $\widetilde{\mathbf{I}}$ also encodes one further piece of geometric information: the decomposition of a boundary stratum into a product of smaller-dimensional moduli spaces. More precisely, let $\overline{\mathcal{L}}_n$ denote the Losev–Manin space mentioned in the introduction; in the language of Hassett spaces, this can be described as

$$\overline{\mathcal{L}}_n = \overline{\mathcal{M}}_{0,(1,1,\varepsilon,\ldots,\varepsilon)},$$

where there are n marked points of weight ϵ and $\epsilon \leq 1/n$. In particular, elements of $\overline{\mathcal{L}}_n$ are chains of projective lines with two "heavy" marked points (one on each end component of the chain) and n "light" marked points. It is worth stressing that, while we used the Losev–Manin space in the introduction to motivate the present work, the spaces $\overline{\mathcal{L}}_n$ are not actually the $r = 1$ case of the spaces $\overline{\mathcal{L}}_n^r$. See Remark 8.1.

With this notation, the following proposition gives a product decomposition of the boundary stratum $S_{\widetilde{\mathbf{I}}}$, in which the factors can be read off directly from the chain $\widetilde{\mathbf{I}}$.

Proposition 5.5. *For any chain* $\widetilde{\mathbf{I}} = (I_1, \ldots, I_k, \mathfrak{a})$, *there is a natural isomorphism between the boundary stratum* $S_{\widetilde{\mathbf{I}}}$ *and a product of smaller-dimensional moduli spaces, as follows:*

$$S_{\widetilde{\mathbf{I}}} \cong \overline{\mathcal{L}}^r_{|[n]\setminus I_k|} \times \prod_{j=1}^{k} \overline{\mathcal{L}}_{|I_j \setminus I_{j-1}|}.$$

Proof. This is essentially immediate from the definition of $S_{\widetilde{\mathbf{I}}}$: the factor of $\overline{\mathcal{L}}^r_{|[n]\setminus I_k|}$ parameterizes the central component, and the factors $\overline{\mathcal{L}}_{|I_j \setminus I_{j-1}|}$ each parameterize a component of one spoke of the pinwheel (which determines all of the other spokes). □

6. Combinatorics of the complex reflection group

We next introduce the group whose structure is combinatorially related to the boundary stratification of $\overline{\mathcal{L}}^r_n(\zeta)$. This group, denoted $S(r, n)$, consists of all $n \times n$ matrices whose only nonzero entries are in the group μ_r of rth roots of unity, and with exactly one nonzero entry in each row and column. (By convention, $S(r, 0)$ is a trivial group.) Note that for any $n \geq 0$, there is a natural group isomorphism

$$S_n = S(1, n),$$

where the permutation $\sigma \in S_n$ is identified with the matrix whose ith column is the $\sigma(i)$th standard basis vector.

Remark 6.1. The group $S(r, n)$ is an example of a complex reflection group (a finite group acting on $\mathbb{C}^n$ generated by elements whose action fixes a complex hyperplane), and in the classification of complex reflection groups, it is denoted $G(r, 1, n)$. It is sometimes also referred to as the "generalized symmetric group" and can be equivalently described as the wreath product $\mu_r \wr S_n$. For more on complex reflection groups, see [17].

It is well-known that the symmetric group S_n is generated by the set of adjacent transpositions. The complex reflection groups $S(r, n)$ have an analogous generating set: define

$$\mathcal{T} = \{s_0, s_1, \ldots, s_{n-1}\} \subseteq S(r, n), \tag{15}$$

where

$$s_0 := \begin{pmatrix} \zeta & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and for $1 \leq i \leq n - 1$, the element $s_i \in S_n \subseteq S(r, n)$ is the adjacent transposition swapping i and $i + 1$, or in other words, the matrix obtained by swapping the ith and $(i + 1)$st columns of the $n \times n$ identity matrix. (Note that the definition of s_0 makes use of the choice of primitive rth root of unity ζ in Convention 4.3.)

It is straightforward to see that $\mathcal{T}$ generates $S(r, n)$: multiplying the identity matrix I by adjacent transpositions can bring any column to the first column, and multiplying by powers of s_0 can change the entry in the first column to any power of ζ. More generally, the following remark describes the structure of subgroups of $S(r, n)$ generated by elements of $\mathcal{T}$.

Remark 6.2. Fix a subset $\mathcal{S} \subseteq \mathcal{T}$ and let $H_{\mathcal{S}}$ be the subgroup of $S(r, n)$ generated by the subset $\mathcal{S}$. Define $\{s_{j_1}, \ldots, s_{j_k}\} := \mathcal{T} \setminus \mathcal{S}$, where $0 \leq j_1 < \cdots < j_k \leq n - 1$. Then $H_{\mathcal{S}}$ is equal to the group of block-diagonal matrices

$$
\begin{pmatrix}
S(r, j_1) & & & & & \\
& S_{j_2 - j_1} & & & & \\
& & S_{j_3 - j_2} & & & \\
& & & \ddots & & \\
& & & & S_{j_k - j_{k-1}} & \\
& & & & & S_{n - j_k}
\end{pmatrix} \subseteq S(r, n).
$$

That is, the upper-left block is an element of the complex reflection group $S(r, j_1)$, while the remaining blocks are elements of the indicated symmetric groups.

The key objects of interest for this paper are the right cosets in $S(r, n)$ of the subgroups described by Remark 6.2. The following definition establishes the terminology.

Definition 6.3. A $\mathcal{T}$-**coset** in $S(r, n)$ is a right coset of the form

$$
\langle t_1, \ldots, t_d \rangle \cdot A \subseteq S(r, n)
$$

for some $d \geq 0$, where $t_1, \ldots, t_d \in \mathcal{T}$ and $A \in S(r, n)$. We say that a $\mathcal{T}$-coset as above, where $t_1, \ldots, t_d$ are distinct, has **dimension** d or **codimension** $n - d$.

In particular, a singleton $\{A\} \subseteq S(r, n)$ is a 0-dimensional $\mathcal{T}$-coset. The 1-dimensional $\mathcal{T}$-cosets are of the form

$$
\langle s_i \rangle \cdot A
$$

for $s_i \in \mathcal{T}$, and they thus have either two elements or r elements, depending on whether $i \geq 1$ or $i = 0$. Since $\mathcal{T}$ generates $S(r, n)$, the only n-dimensional $\mathcal{T}$-coset is the entire group.

Analogously to Definition 5.2, we now describe a procedure for producing a $\mathcal{T}$-coset from a chain $\widetilde{\mathbf{I}}$.

Definition 6.4. Let $\widetilde{\mathbf{I}} = (I_1, \ldots, I_k, \mathfrak{a})$ be a chain. Define the subgroup $H_{\widetilde{\mathbf{I}}} \subseteq S(r, n)$ by

$$
H_{\widetilde{\mathbf{I}}} := \langle \{s_i \mid n - i \notin \{|I_1|, \ldots, |I_k|\}\} \rangle,
$$

and let $A \in S(r, n)$ be any matrix satisfying the following two conditions:

(i) for each $j \in \{1, \ldots, k\}$, the last $|I_j|$ rows of A have nonzero entries in the columns indexed by I_j—that is,

$$I_j = \{b \in [n] \mid A_{ab} \neq 0 \text{ for some } a > n - |I_j|\}, \tag{16}$$

where A_{ab} denotes the entry in the ath row and bth column of A;

(ii) for each $i \in I_k$, the unique nonzero entry in column i of A is $\zeta^{-\mathfrak{a}(i)}$.

We define the $\mathcal{T}$-coset associated to $\widetilde{\mathbf{I}}$ as

$$C_{\widetilde{\mathbf{I}}} = H_{\widetilde{\mathbf{I}}} \cdot A.$$

To make the elements of $C_{\widetilde{\mathbf{I}}}$ more explicit, note that in the notation of Remark 6.2, we have $H_{\widetilde{\mathbf{I}}} = H_{\mathcal{S}(\widetilde{\mathbf{I}})}$ for the set

$$\mathcal{S}(\widetilde{\mathbf{I}}) := \{s_i \mid n - i \notin \{|I_1|, \ldots, |I_k|\} \subseteq \mathcal{T}. \tag{17}$$

Thus, the set $\{s_{j_1}, \ldots, s_{j_k}\} := \mathcal{T} \setminus \mathcal{S}(\widetilde{\mathbf{I}})$ in Remark 6.2 is given by

$$j_1 = n - |I_k|$$
$$j_2 = n - |I_{k-1}|$$
$$\vdots$$
$$j_k = n - |I_1|,$$

and therefore $H_{\widetilde{\mathbf{I}}}$ is the group of block-diagonal matrices

$$H_{\widetilde{\mathbf{I}}} = \begin{pmatrix} S(r, |[n] \setminus I_k|) & & & & & \\ & S_{|I_k \setminus I_{k-1}|} & & & & \\ & & S_{|I_{k-1} \setminus I_{k-2}|} & & & \\ & & & \ddots & & \\ & & & & S_{|I_2 \setminus I_1|} & \\ & & & & & S_{|I_1|} \end{pmatrix} \subseteq S(r, n).$$

From here, one sees from Remark 6.2 and the definition of A in Definition 6.4 that $C_{\widetilde{\mathbf{I}}}$ is equal to the set of matrices illustrated in Fig. 9. The conditions defining the matrix A are equivalent to requiring that A belong to this set, which in particular implies that the definition of $C_{\widetilde{\mathbf{I}}}$ does not depend on the choice of A satisfying those conditions.

To illustrate the construction of Definition 6.4, we compute the associated $\mathcal{T}$-coset for the same chain that we considered in Sect. 5.

Example 6.5. As in Example 5.3, let $r = 3$ and $n = 4$, and let $\widetilde{\mathbf{I}}$ be the chain

$$\widetilde{\mathbf{I}} = (\{3\}, \{2, 3, 4\}, \mathfrak{a}),$$

where $\mathfrak{a} : I_2 \to \mathbb{Z}_3$ is given by

$$\mathfrak{a}(2) = 1, \quad \mathfrak{a}(3) = 0, \quad \mathfrak{a}(4) = 2.$$

Then

$$H_{\widetilde{\mathbf{I}}} = \langle s_i \mid 4 - i \notin \{1, 3\}\rangle = \langle s_0, s_2\rangle,$$

and $C_{\widetilde{\mathbf{I}}} = H_{\widetilde{\mathbf{I}}} \cdot A$ for any matrix A such that
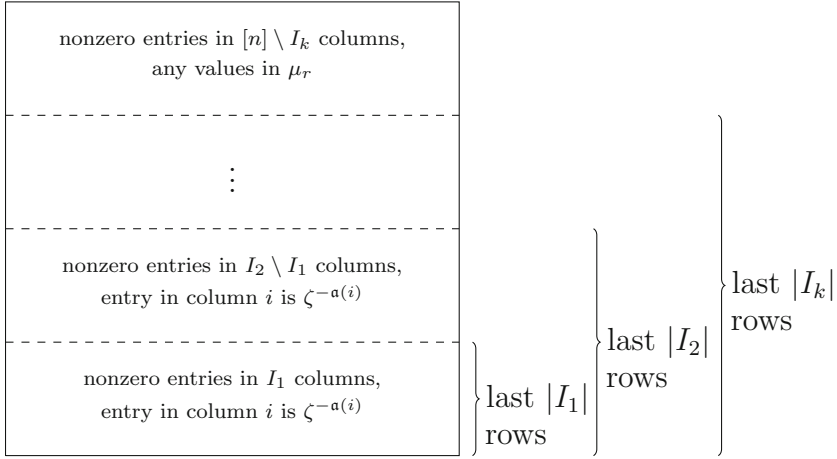
Fig. 9. A typical element of the $\mathcal{T}$-coset $C_{\widetilde{\mathbf{I}}}$. The action of $H_{\widetilde{\mathbf{I}}}$ permutes rows within each of the blocks separated by dotted lines, and multiplies elements of the top-most block by rth roots of unity

(i) the last row has its nonzero entry in column 3, and that entry is ζ^0; and
(ii) the last three rows have their nonzero entries in columns 2, 3, and 4, and those entries are $\zeta^{-1} = \zeta^2$, ζ^0, and $\zeta^{-2} = \zeta^1$, respectively.

The action of $H_{\widetilde{\mathbf{I}}}$ on matrices of this form multiplies the first row by roots of unity and swaps the second and third rows. Thus, we have

$$C_{\widetilde{\mathbf{I}}} = \left\{ \begin{pmatrix} \zeta^i & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^1 \\ 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & \zeta^0 & 0 \end{pmatrix} \,\Bigg|\, i \in \mathbb{Z}_3 \right\} \cup \left\{ \begin{pmatrix} \zeta^i & 0 & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & 0 & \zeta^1 \\ 0 & 0 & \zeta^0 & 0 \end{pmatrix} \,\Bigg|\, i \in \mathbb{Z}_3 \right\}.$$

With the association $\widetilde{\mathbf{I}} \mapsto C_{\widetilde{\mathbf{I}}}$ established, we are prepared to state an analogue for $\mathcal{T}$-cosets of Proposition 5.4.

Proposition 6.6. *Let $r \geq 2$ and $n \geq 0$. The association*

$$\widetilde{\mathbf{I}} \mapsto C_{\widetilde{\mathbf{I}}}$$

is a bijection from the set of $\mathbb{Z}_r$-decorated chains of subsets of $[n]$ to the set of $\mathcal{T}$-cosets in $S(r, n)$. Furthermore, this bijection satisfies

(i) *$length(\widetilde{\mathbf{I}}) = codim(C_{\widetilde{\mathbf{I}}})$,*
(ii) *$C_{\widetilde{\mathbf{I}}} \subseteq C_{\widetilde{\mathbf{J}}}$ if and only if $\widetilde{\mathbf{I}}$ refines $\widetilde{\mathbf{J}}$.*

Proof. The surjectivity of $\widetilde{\mathbf{I}} \mapsto C_{\widetilde{\mathbf{I}}}$ is clear, since given an arbitrary $\mathcal{T}$-coset

$$C = \langle s_{\ell_1}, \ldots, s_{\ell_{n-k}} \rangle \cdot A,$$

one can define sets $I_1 \subseteq \cdots \subseteq I_k$ by (16). Condition (ii) of Definition 6.4 then uniquely defines a function $\mathfrak{a} : I_k \to \mathbb{Z}_r$, and by construction, setting $\widetilde{\mathbf{I}} = (I_1, \ldots, I_k, \mathfrak{a})$ gives $C = C_{\widetilde{\mathbf{I}}}$.

Injectivity of this association will follow from item (ii), while item (i) is immediate from the definition of $\mathrm{codim}(C_{\widetilde{\mathbf{I}}})$. Thus, what remains is to prove item (ii).

Let

$$\widetilde{\mathbf{I}} = (I_1, \ldots, I_k, \mathfrak{a})$$

and

$$\widetilde{\mathbf{J}} = (J_1, \ldots, J_\ell, \mathfrak{b}),$$

and suppose that $\widetilde{\mathbf{I}}$ refines $\widetilde{\mathbf{J}}$. Then

$$\{I_1, \ldots, I_k\} \supseteq \{J_1, \ldots, J_\ell\}$$

and hence

$$\{|I_1|, \ldots, |I_k|\} \supseteq \{|J_1|, \ldots, |J_\ell|\}.$$

From here, it is straightforward to unpack that

$$\mathcal{S}(\widetilde{\mathbf{I}}) \subseteq \mathcal{S}(\widetilde{\mathbf{J}}),$$

where $\mathcal{S}(\widetilde{\mathbf{I}})$ is defined by (17) and $\mathcal{S}(\widetilde{\mathbf{J}})$ is defined analogously. Furthermore, a matrix A satisfying the conditions of Definition 6.4 for $\widetilde{\mathbf{I}}$ will satisfy the same conditions for $\widetilde{\mathbf{J}}$, so the same matrix can be chosen to represent both cosets. It follows that $C_{\widetilde{\mathbf{I}}} \subseteq C_{\widetilde{\mathbf{J}}}$.

Conversely, suppose that $C_{\widetilde{\mathbf{I}}} \subseteq C_{\widetilde{\mathbf{J}}}$. This means that one can choose the same representative for both cosets, so we have

$$H_{\widetilde{\mathbf{I}}} \cdot A \subseteq H_{\widetilde{\mathbf{J}}} \cdot A \tag{18}$$

for a matrix $A \in C_{\widetilde{\mathbf{I}}}$. The elements of $C_{\widetilde{\mathbf{I}}}$ differ from A by permuting the rows within blocks as in Fig. 9, so in order for the containment (18) to hold, the corresponding blocks for $C_{\widetilde{\mathbf{J}}}$ must be unions of the blocks for $C_{\widetilde{\mathbf{I}}}$. That is, we must have

$$\{|I_1|, \ldots, |I_k|\} \supseteq \{|J_1|, \ldots, |J_\ell|\}.$$

In particular, for any $i \in \{1, \ldots, \ell\}$, we must have $|J_i| = |I_j|$ for some $j \in \{1, \ldots, k\}$, so the set J_i indexing the columns with nonzero entries in the last $|J_i|$ rows of A is equal to the set I_j indexing the columns with nonzero entries in the last $|I_j|$ rows of A. That is, we have

$$\{I_1, \ldots, I_k\} \supseteq \{J_1, \ldots, J_\ell\}.$$

Since $\mathfrak{a}$ and $\mathfrak{b}$ are determined by the same matrix A, we also have $\mathfrak{b} = \mathfrak{a}|_{I_k}$, concluding the proof that $\widetilde{\mathbf{I}}$ refines $\widetilde{\mathbf{J}}$. $\qquad\square$

We end this section by noting that, analogously to the way in which Proposition 5.5 gives a product decomposition of a boundary stratum in terms of the combinatorial data of a chain, the $\mathcal{T}$-cosets in $S(r, n)$ have decompositions as products of groups dictated by their associated chains. To state the decomposition, for any chain $\widetilde{\mathbf{I}}$, let

$$\Phi_{\widetilde{\mathbf{I}}} : S(r, |[n] \setminus I_k|) \times S(r, |I_k \setminus I_{k-1}|) \times \cdots \times S(r, |I_2 \setminus I_1|) \times S(r, |I_1|) \hookrightarrow S(r, n) \tag{19}$$

be the embedding of the left-hand side as block-diagonal matrices in $S(r, n)$, where the first factor is embedded as the first $|[n] \setminus I_k|$ rows and the columns indexed by $[n] \setminus I_k$, and similarly for the remaining factors.

Proposition 6.7. *For any chain $\widetilde{\mathbf{I}} = (I_1, \cdots, I_k, \mathfrak{a})$, let $C_{\widetilde{\mathbf{I}}} = H_{\widetilde{\mathbf{I}}} \cdot A$ be the associated $\mathcal{T}$-coset. Then there is a natural isomorphism between $H_{\widetilde{\mathbf{I}}}$ and a product of complex reflection groups, as follows:*

$$H_{\widetilde{\mathbf{I}}} \cong S(r, |[n] \setminus I_k|) \times \prod_{j=1}^{k} S_{|I_j \setminus I_{j-1}|}. \tag{20}$$

More specifically, let $\Phi_{\widetilde{\mathbf{I}}}$ be as in (19) and let $A_j \in S(r, |I_j \setminus I_{j-1}|)$ be any matrix whose entry in the ℓth column is $\zeta^{-\mathfrak{a}(\ell)}$ for each $\ell \in I_j \setminus I_{j-1}$. Then

$$C_{\widetilde{\mathbf{I}}} = \Phi_{\widetilde{\mathbf{I}}} \left(S(r, |[n] \setminus I_k|) \times \prod_{j=1}^{k} S_{|I_j \setminus I_{j-1}|} \cdot A_j \right). \tag{21}$$

Proof. Recall from Remark 6.2 that, if $\mathcal{S} \subseteq \mathcal{T}$ and $H_{\mathcal{S}}$ denotes the subgroup of $S(r, n)$ generated by $\mathcal{S}$, then the block-diagonal decomposition gives an isomorphism

$$H_{\mathcal{S}} \cong S(r, j_1) \times S_{j_2 - j_1} \times S_{j_3 - j_2} \times \cdots \times S_{j_k - j_{k-1}} \times S_{n - j_k},$$

in which $0 \leq j_1 < \cdots < j_k \leq n - 1$ are defined by

$$j \in \{j_1, \ldots, j_k\} \iff s_j \notin \mathcal{S}. \tag{22}$$

The group $H_{\widetilde{\mathbf{I}}}$ is equal to $H_{\mathcal{S}(\widetilde{\mathbf{I}})}$ for the set $\mathcal{S}(\widetilde{\mathbf{I}}) \subseteq \mathcal{T}$ defined by

$$s_i \in \mathcal{S}(\widetilde{\mathbf{I}}) \iff n - i \notin \{|I_1|, \ldots, |I_k|\}. \tag{23}$$

Combining (22) and (23) shows that the isomorphism (20) is given by the block-diagonal decomposition. The elements of $C_{\widetilde{\mathbf{I}}}$ are precisely the matrices with this block-diagonal decomposition and entry $\zeta^{-\mathfrak{a}(i)}$ in column i for all $i \in I_k$, which proves (21). $\qquad\square$

7. Combinatorics of the permutohedral complex

The third setting that is combinatorially related to the boundary stratification of $\overline{\mathcal{L}}_n^r(\zeta)$ and to the $\mathcal{T}$-coset structure of $S(r, n)$ is the permutohedral complex. To define it, we set

$$\mathsf{Y} := \mathbb{R}^{\geq 0} \cdot \mu_r \subseteq \mathbb{C}$$

for any $r \geq 2$, and we set

$$\delta_k^n := n + (n - 1) + \cdots + (n - k + 1)$$

for any $n \geq 1$ and $k \in [n]$.

Definition 7.1. Let $r \geq 2$ and $n \geq 0$. The n-**dimensional** r-**permutohedral complex** is defined as

$$\Delta_n^r := \left\{ (x_1, \ldots, x_n) \in \mathsf{Y}^n \ \middle| \ \sum_{i \in I} |x_i| \leq \delta_{|I|}^n \text{ for all } I \subseteq [n] \right\}$$

if $n \geq 1$, or as a single point if $n = 0$.

Because in general Δ_n^r is not a polytope (rather, as we will prove in Corollary 7.12 below, it is a polytopal complex), defining the appropriate notion of "face" requires some care. We carry this out in the following definition.

Definition 7.2. A **decorated subset** of $[n]$ is $\widetilde{I} = (I, \mathfrak{a})$, where $I \subseteq [n]$ and $\mathfrak{a} : I \to \mathbb{Z}_r$ is any function; in other words, it is a length-1 decorated chain. Any decorated subset has an associated hyperplane

$$H_{\widetilde{I}} := \left\{ (x_1, \ldots, x_n) \in \mathbb{C}^n \ \middle| \ \sum_{i \in I} \zeta^{\mathfrak{a}(i)} \cdot x_i = \delta_{|I|}^n \right\} \subseteq \mathbb{C}^n.$$

A Δ-**face** of Δ_n^r is defined as any nonempty intersection

$$\Delta_n^r \cap H_{\widetilde{I}_1} \cap \cdots \cap H_{\widetilde{I}_k},$$

where $\widetilde{I}_1, \ldots, \widetilde{I}_k$ are a choice of distinct decorated subsets of $[n]$.

When $r = 2$, the complex Δ_n^2 is in fact a polytope in $\mathbb{R}^n$ (in particular, Δ_2^2 is the octagon illustrated in Fig. 1, and more generally, Δ_n^2 is known as the type-B permutohedron) and its Δ-faces are precisely its faces in the usual sense. When $r \geq 3$, on the other hand, the Δ-faces of Δ_n^r are themselves polytopal complexes. We illustrate this in an example before proving it in general.

Example 7.3. Let $r = 3$ and $n = 2$. Given $\widetilde{I} = (I, \mathfrak{a})$ with $I = \{1, 2\}$ and $\mathfrak{a} : I \to \mathbb{Z}_3$ defined by

$$\mathfrak{a}(1) = 2, \quad \mathfrak{a}(2) = 0,$$

the associated Δ-face is

$$\Delta_2^3 \cap H_{\widetilde{I}} = \{(x_1, x_2) \in \Delta_2^3 \mid \zeta^2 x_1 + \zeta^0 x_2 = 3\}.$$

Geometrically, this is a line segment, and there are nine such Δ-faces of Δ_2^3 given by changing the powers of ζ in the coefficients of the defining equations; see the nine green line segments labeled in Fig. 2. On the other hand, given $\widetilde{I}' = (I', a')$ where $I' = \{1\}$ and $a'(1) = 2$, the Δ-face associated to $\widetilde{I}'$ is

$$\Delta_2^3 \cap H_{\widetilde{I}'} = \{(x_1, x_2) \in \Delta_2^3 \mid \zeta^2 x_1 = 2\}.$$

One can check that this is equivalent to

$$\{2\zeta^1\} \times \{x_2 \in \mathsf{Y} \mid |x_2| \leq 1\},$$

which is the union of three line segments in a "Y" shape. There are three such Δ-faces given by changing the value of $a'(1)$, labeled in red in Fig. 2, and there are three similar Δ-faces given by the equations $\zeta^i x_2 = 2$ for $0 \leq i \leq 2$, labeled in blue in Fig. 2.

In addition, Δ_2^3 has 0-dimensional Δ-faces given by intersecting two 1-dimensional Δ-faces; it is straightforward to check that each such nonempty intersection is a single point. Finally, although Fig. 2 shows Δ_2^3 as the union of nine pentagons, these together constitute just a single 2-dimensional face.

This example illustrates that Δ_n^r is a polytopal complex glued from n-dimensional polytopes, and that a Δ-face given by intersecting Δ_n^r with k distinct hyperplanes is a polytopal complex glued from $(n - k)$-dimensional polytopes. Before proving these observations in general, it is useful to draw on another key observation: a condition on the decorated subsets $\widetilde{I}_1, \ldots, \widetilde{I}_k$ must be satisfied in order to ensure that the corresponding Δ-face is nonempty.

Lemma 7.4. *Let $\widetilde{I}_1, \ldots, \widetilde{I}_k$ be decorated subsets of $[n]$, where $\widetilde{I}_j = (I_j, \mathfrak{a}_j)$ for each j. Then*

$$\Delta_n^r \cap H_{\widetilde{I}_1} \cap \cdots \cap H_{\widetilde{I}_k} \neq \emptyset$$

if and only if, after possibly reordering $\widetilde{I}_1, \ldots, \widetilde{I}_k$, the tuple $(I_1, \ldots, I_k, \mathfrak{a}_k)$ is a $\mathbb{Z}_r$-decorated chain of subsets of $[n]$.

Proof. Since n is fixed throughout this proof, we write δ_k^n as simply δ_k to avoid cluttering the notation.

Without loss of generality, we assume that the decorated sets $\widetilde{I}_1, \cdots, \widetilde{I}_k$ are distinct. To prove the forward direction, it suffices to show that

$$\Delta_n^r \cap H_{\widetilde{I}} \cap H_{\widetilde{J}} \neq \emptyset \implies \widetilde{I} \subseteq \widetilde{J}.$$

Suppose, then, that $x \in \Delta_n^r \cap H_{\widetilde{I}} \cap H_{\widetilde{J}}$, and let $\widetilde{I} = (I, \mathfrak{a})$ and $\widetilde{J} = (J, \mathfrak{b})$. The fact that $x \in \Delta_n^r$ means in particular that $x \in Y^n$, so

$$x = (\lambda_1 c_1, \ldots, \lambda_n c_n)$$

for some $\lambda_i \in \mathbb{R}^{\geq 0}$ and $c_i \in \mu_r$. We claim that

$$c_i = \zeta^{-\mathfrak{a}(i)}$$

for all $i \in I$.

To see this, note that since $x \in H_{\widetilde{I}}$ we have

$$\sum_{i \in I} \lambda_i \cdot \zeta^{\mathfrak{a}(i)} c_i = \delta_{|I|}, \tag{24}$$

and $x \in \Delta_n^r$ we obtain

$$\sum_{i \in I} \lambda_i \leq \delta_{|I|}.$$

From the triangle inequality we obtain

$$\delta_{|I|} = \left| \sum_{i \in I} \lambda_i \cdot \zeta^{\mathfrak{a}(i)} c_i \right| \leq \sum_{i \in I} \left| \lambda_i \cdot \zeta^{\mathfrak{a}(i)} c_i \right| = \sum_{i \in I} \lambda_i \leq \delta_{|I|}.$$

Thus, the triangle inequality is in fact an equality, which is only possible if the complex numbers $\lambda_i \cdot \zeta^{\mathfrak{a}(i)} c_i$ are all non-negative real scalar multiples of one another. Since their sum is a positive real number by (24), they must each individually be non-negative real numbers; that is,

$$\lambda_i \cdot \zeta^{\mathfrak{a}(i)} c_i \in \mathbb{R}^{\geq 0} \tag{25}$$

for each i. Furthermore, $\lambda_i \neq 0$ for all $i \in I$, since if $\lambda_j = 0$ for some $j \in I$, then

$$\sum_{i \in I \setminus \{j\}} \lambda_i = \delta_{|I|} > \delta_{|I|-1},$$

violating one of the inequalities in the definition of Δ_n^r. Thus, we have $\lambda_i > 0$ for all i, so (25) implies $\zeta^{\mathfrak{a}(i)} c_i \in \mathbb{R}^{\geq 0} \cap \mu_r$. We conclude that $c_i = \zeta^{-\mathfrak{a}(i)}$, as claimed.

This proves that, once that ζ is fixed, the decorations on I are determined by the coefficients c_i on $x \in \Delta_n^r \cap H_{\widetilde{I}} \cap H_{\widetilde{J}}$, and the exact same argument shows that the decorations on J are given by the same formula. Thus, what remains to be proved is that either $I \subseteq J$ or $J \subseteq I$. To see this, notice that we can now express $x \in H_{\widetilde{I}} \cap H_{\widetilde{J}}$ by the equations

$$\sum_{i \in I} \lambda_i = \delta_{|I|} \quad \text{and} \quad \sum_{i \in J} \lambda_i = \delta_{|J|}.$$

From these equations, we deduce that

$$\sum_{i \in I \cup J} \lambda_i = \sum_{i \in I} \lambda_i + \sum_{i \in J} \lambda_i - \sum_{i \in I \cap J} \lambda_i = \delta_{|I|} + \delta_{|J|} - \sum_{i \in I \cap J} \lambda_i.$$

By the defining inequalities of Δ_n^r, we have

$$\sum_{i \in I \cap J} \lambda_i \leq \delta_{|I \cap J|},$$

so

$$\sum_{i \in I} \lambda_i + \sum_{i \in J} \lambda_i - \sum_{i \in I \cap J} \lambda_i \geq \delta_{|I|} + \delta_{|J|} - \delta_{|I \cap J|},$$

or in other words,

$$\sum_{i \in I \cup J} \lambda_i \geq \delta_{|I|} + \delta_{|J|} - \delta_{|I \cap J|}. \qquad (26)$$

But a straightforward calculation shows that if $|I \cap J|$ is strictly less than both $|I|$ and $|J|$, then

$$\delta_{|I|} + \delta_{|J|} - \delta_{|I \cap J|} > \delta_{|I|+|J|-|I \cap J|}.$$

Given that $\delta_{|I|+|J|-|I \cap J|} = \delta_{|I \cup J|}$, we would then obtain from (26) that

$$\sum_{i \in I \cup J} \lambda_i > \delta_{|I \cup J|},$$

contradicting one of the defining inequalities of Δ_n^r. Thus, we must have either $|I \cap J| = |I|$ or $|I \cap J| = |J|$, meaning that either $I \subseteq J$ or $J \subseteq I$. This concludes the proof of the forward direction of the lemma.

For the reverse direction, it suffices to show the statement when $k = n$ and $\widetilde{I}_1, \ldots, \widetilde{I}_n$ are distinct, because any chain can be extended to a maximal one. In light of this, let

$$\widetilde{\mathbf{I}} = (I_1, \ldots, I_n, \mathfrak{a})$$

be a maximal chain, which can equivalently be expressed as

$$I_1 = \{i_1\}$$
$$I_2 = \{i_1, i_2\}$$
$$\vdots$$
$$I_n = \{i_1, i_2, \ldots, i_n\} = [n]$$

for some $i_1, \ldots, i_n$. Setting

$$x_{i_j} = \zeta^{-\mathfrak{a}(i_j)} \cdot (n + 1 - j) \qquad (27)$$

for each j, it is straightforward to check that $(x_1, \ldots, x_n) \in \Delta_n^r \cap H_{\widetilde{I}_1} \cap \cdots \cap H_{\widetilde{I}_n}$, where $\widetilde{I}_j := (I_j, \mathfrak{a}|_{I_j})$. Therefore, $\Delta_n^r \cap H_{\widetilde{I}_1} \cap \cdots \cap H_{\widetilde{I}_n} \neq \emptyset$, so the reverse direction of the lemma is proved. $\qquad \square$

The key upshot of Lemma 7.4 is that the Δ-faces of Δ_n^r, like the boundary strata and $\mathcal{T}$-cosets, can be indexed by chains. The notation, analogously to the previous two sections, is as follows.

Definition 7.5. Let $\widetilde{\mathbf{I}} = (I_1, \ldots, I_k, \mathfrak{a})$ be a chain. Then the Δ-face of Δ_n^r associated to $\widetilde{\mathbf{I}}$ is

$$F_{\widetilde{\mathbf{I}}} := \Delta_n^r \cap H_{\widetilde{I}_1} \cap \cdots \cap H_{\widetilde{I}_k},$$

where $\widetilde{I}_j = \left(I_j, \mathfrak{a}|_{I_j} \right)$ for each j.

In the special case where the chain $\widetilde{\mathbf{I}}$ is maximal, the Δ-face $F_{\widetilde{\mathbf{I}}}$ is a single point of Δ_n^r, which we refer to as a **vertex**. Note that Eq. (27) makes the coordinates of the vertex associated to a maximal chain $\widetilde{\mathbf{I}}$ explicit. More generally, the proof of Lemma 7.4 gives an explicit description of the elements of $F_{\widetilde{\mathbf{I}}}$ for any chain $\widetilde{\mathbf{I}}$, which we collect in the following remark for future reference.

Remark 7.6. Suppose $\widetilde{\mathbf{I}} = (I_1, \ldots, I_k, \mathfrak{a})$ is a chain. Then $(x_1, \ldots, x_n) \in \mathsf{Y}^n$ lies in $F_{\widetilde{\mathbf{I}}}$ if and only if the following conditions are satisfied:

(C1) $(x_1, \ldots, x_n) \in \Delta_n^r$, or in other words,

$$\sum_{i \in I} |x_i| \leq \delta_{|I|}^n$$

for all $I \subseteq [n]$;
(C2) $x_i \in \mathbb{R}^{\geq 0} \cdot \zeta^{-\mathfrak{a}(i)}$ for all $i \in I_k$;
(C3) for all $j \in \{1, \ldots, k\}$,

$$\sum_{i \in I_j} |x_i| = \delta_{|I_j|}^n.$$

Let us illustrate the passage from a chain to its associated Δ-face for the same chain considered in Examples 5.3 and 6.5 above.

Example 7.7. Let $r = 3$ and $n = 4$, and consider again the chain $\widetilde{\mathbf{I}} = (I_1, I_2, \mathfrak{a})$, where

$$I_1 = \{3\}, \quad I_2 = \{2, 3, 4\}$$

and $\mathfrak{a} : I_2 \to \mathbb{Z}_3$ is given by

$$\mathfrak{a}(2) = 1, \quad \mathfrak{a}(3) = 0, \quad \mathfrak{a}(4) = 2.$$

The associated Δ-face $F_{\widetilde{\mathbf{I}}} \subseteq \Delta_4^3$ is, by definition,

$$F_{\widetilde{\mathbf{I}}} = \left\{ (x_1, \ldots, x_4) \in \mathsf{Y}^4 \;\middle|\; \begin{array}{c} \sum_{i \in I} |x_i| \leq \delta_{|I|}^4 \text{ for all } I \subseteq [4] \\ \zeta^0 x_3 = 4 \\ \zeta^1 x_2 + \zeta^0 x_3 + \zeta^2 x_4 = 9 \end{array} \right\}.$$

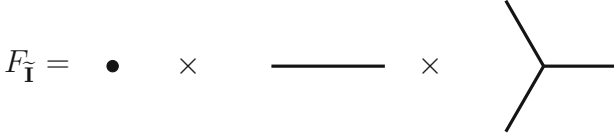**Fig. 10.** The $\triangle$-face $F_{\tilde{\mathbf{I}}} \subseteq \Delta_4^3$ of Example 7.7, decomposed as a product according to its x_3-coordinate, its (x_2, x_4)-coordinates, and its x_1-coordinate

It is illuminating to divide the four coordinates according to the decomposition

$$[4] = I_1 \cup \left(I_2 \setminus I_1 \right) \cup \left([4] \setminus I_2 \right)$$
$$= \{3\} \cup \{2, 4\} \cup \{1\}.$$

For the coordinates in each of these sets, we have the following conditions:

- The coordinate x_3 must satisfy

$$x_3 \in \mathbb{R}^{\geq 0} \cdot \zeta^0,$$

  and by the second equality in the above expression for $F_{\tilde{\mathbf{I}}}$, we have

$$|x_3| = 4. \tag{28}$$

  Thus, the value of x_3 is completely determined.
- The coordinates x_2 and x_4 must satisfy

$$x_2 \in \mathbb{R}^{\geq 0} \cdot \zeta^2 \text{ and } x_4 \in \mathbb{R}^{\geq 0} \cdot \zeta^1,$$

  and by the second and third equalities in the above expression for $F_{\tilde{\mathbf{I}}}$, we have

$$|x_2| + |x_4| = 5. \tag{29}$$

  Thus, (x_2, x_4) lie in a line segment.
- The coordinate x_1 can lie in any of the three branches of Y. Combining Eqs. (28) and (29) with the first inequality of $F_{\tilde{\mathbf{I}}}$, we see that $|x_1| \leq 1$. Thus,

$$x_1 \in \Delta_1^3,$$

  which is the Y-shaped region depicted in Fig. 10.

Combining these conditions, we find that $F_{\tilde{\mathbf{I}}}$ is the product shown in Fig. 10.

The product decomposition of $F_{\tilde{\mathbf{I}}}$ illustrated in Fig. 10 can be interpreted as a product of the standard permutohedra of dimensions zero and one, and the permutohedral complex Δ_1^3. (We recall the definition of the standard permutohedra in Definition 7.9 below.) To see the permutohedron arising more clearly, it is illuminating to consider one additional example.
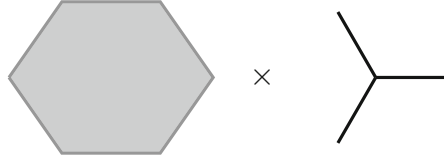
Fig. 11. The Δ-face $F_{\widetilde{\mathbf{I}}'} \subseteq \Delta_4^3$ of Example 7.8, decomposed as a product according to its (x_2, x_3, x_4)-coordinates and its x_1-coordinate

Example 7.8. Let $\widetilde{\mathbf{I}}'$ be the chain obtained by removing I_1 from the chain $\widetilde{\mathbf{I}}$ of Example 7.7. Then

$$F_{\widetilde{\mathbf{I}}'} = \left\{ (x_1, \ldots, x_4) \in \mathsf{Y}^4 \; \middle| \; \begin{array}{l} \sum_{i \in I} |x_i| \leq \delta_{|I|}^4 \;\; \text{for all } I \subseteq [4] \\ \zeta^1 x_2 + \zeta^0 x_3 + \zeta^2 x_4 = 9 \end{array} \right\}.$$

As before, $x_1 \in \Delta_1^3$ and is independent from x_2, x_3, x_4. The coordinates x_2, x_3, x_4, on the other hand, must satisfy

$$x_2 \in \mathbb{R}^{\geq 0} \cdot \zeta^2, \quad x_3 \in \mathbb{R}^{\geq 0} \cdot 1, \quad x_4 \in \mathbb{R}^{\geq 0} \cdot \zeta$$

as well as

$$|x_2| + |x_3| + |x_4| = 9,$$

One can check that these conditions, together with the inequalities in $F_{\widetilde{\mathbf{I}}'}$, shows that x_2, x_3, x_4 lie in a hexagon with vertices

$$(4\zeta^2, 3, 2\zeta), \; (4\zeta^2, 2, 3\zeta), \; (3\zeta^2, 4, 2\zeta), \; (3\zeta^2, 2, 4\zeta), \; (2\zeta^2, 4, 3\zeta), \; (2\zeta^2, 3, 4\zeta).$$

See Fig. 11.

Generalizing the above examples, we now prove that each Δ-face of Δ_n^r is equal to a product of smaller permutohedral complexes and permutohedra; this will provide the analogue of Propositions 5.5 and 6.7 (the product decompositions of boundary strata and $\mathcal{T}$-cosets, respectively), and furthermore, the product decomposition will be used to verify the dimension of each Δ-face.

First, we must recall the definition of the standard permutohedron.

Definition 7.9. The **permutohedron** Π_n is the set of points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying the inequalities

$$\sum_{i \in I} x_i \leq \delta_{|I|}^n$$

for all $I \subsetneq [n]$ and the equality

$$\sum_{i \in [n]} x_i = \delta_n^n.$$

(Analogously to the situation for Losev–Manin space, we point out again that while the analogy to the permutohedron motivates the definition of Δ_n^r, it is not literally the case that setting $r = 1$ in the definition of Δ_n^r recovers Π_n. See Remark 8.1.)

It is well-known that Π_n is a polytope of dimension $n - 1$. More generally, one obtains other polytopes of the same dimension by shifting Π_n, as follows.

Definition 7.10. Let $\gamma \in \mathbb{R}$. The γ-**shifted permutohedron** $\Pi_n + \gamma$ is the set of points $(x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying the inequalities

$$\sum_{i \in I} x_i \leq \delta^n_{|I|} + |I| \cdot \gamma$$

for all $I \subsetneq [n]$ and the equality

$$\sum_{i \in [n]} x_i = \delta^n_n + n\gamma.$$

Given these definitions, we have the following alternative description of a face $F_{\widetilde{\mathbf{I}}}$ of Δ^r_n.

Proposition 7.11. *Let* $\widetilde{\mathbf{I}} = (I_1, \ldots, I_k, \mathfrak{a})$ *be a chain. Then* $F_{\widetilde{\mathbf{I}}}$ *is equal to the set of points* $(x_1, \ldots, x_n) \in \mathbb{C}^n$ *that satisfy the following conditions:*

(C1') $(x_i)_{i \in [n] \setminus I_k} \in \Delta^r_{|[n] \setminus I_k|}$, *or in other words,*

$$\sum_{i \in I} |x_i| \leq \delta^{|[n] \setminus I_k|}_{|I|}$$

    for all $I \subseteq [n] \setminus I_k$;
(C2') $x_i \in \mathbb{R}^{\geq 0} \cdot \zeta^{-\mathfrak{a}(i)}$ *for all* $i \in I_k$;
(C3') *for all* $j \in \{1, \ldots, k\}$, *the point* $(|x_i|)_{i \in I_j \setminus I_{j-1}}$ *lies in the shifted permutohedron*

$$\Pi_{|I_j \setminus I_{j-1}|} + \gamma_j$$

    for $\gamma_j := |[n] \setminus I_j|$.

Proof. By Remark 7.6, it suffices to show that conditions **(C1)** – **(C3)** are equivalent to conditions **(C1')** – **(C3')**.

First, suppose that $(x_1, \ldots, x_n) \in \mathbb{C}^n$ satisfies **(C1)** – **(C3)**. Then, given $I \subseteq [n] \setminus I_k$, we have

$$\begin{aligned}
\sum_{i \in I} |x_i| &= \sum_{i \in I \cup I_k} |x_i| - \sum_{i \in I_k} |x_i| \\
&\leq \delta^n_{|I| + |I_k|} - \sum_{i \in I_k} |x_i| \\
&= \delta^n_{|I| + |I_k|} - \delta^n_{|I_k|} \\
&= \delta^{|[n] \setminus I_k|}_{|I|},
\end{aligned}$$

where the first equality follows from the fact that I and I_k are disjoint, the inequality from **(C1)**, and the second equality from **(C3)**. Thus, $(x_1, \ldots, x_n)$ satisfies **(C1')**.

Condition (**C2′**) is identical to condition (**C2**), so there is nothing to check. Finally, for condition (**C3′**), let $j \in \{1, \ldots, k\}$. Then

$$
\begin{aligned}
\sum_{i \in I_j \setminus I_{j-1}} |x_i| &= \sum_{i \in I_j} |x_i| - \sum_{i \in I_{j-1}} |x_i| \\
&= \delta^n_{|I_j|} - \delta^n_{|I_{j-1}|} \\
&= \delta^{|[n] \setminus I_{j-1}|}_{|I_j \setminus I_{j-1}|} \\
&= \delta^{|I_j \setminus I_{j-1}|}_{|I_j \setminus I_{j-1}|} + \big|[n] \setminus I_j\big| \cdot \big|I_j \setminus I_{j-1}\big|,
\end{aligned}
$$

which is the equality in the definition of the shifted permutohedron $\Pi_{|I_j \setminus I_{j-1}|} + \gamma_j$. Furthermore, for any $I \subsetneq I_j \setminus I_{j-1}$, we have

$$
\begin{aligned}
\sum_{i \in I} |x_i| &= \sum_{i \in I_{j-1} \cup I} |x_i| - \sum_{i \in I_{j-1}} |x_i| \\
&\leq \delta^n_{|I| + |I_{j-1}|} - \delta^n_{|I_{j-1}|} \\
&= \delta^{|[n] \setminus I_{j-1}|}_{|I|} \\
&= \delta^{|I_j \setminus I_{j-1}|}_{|I|} + \big|[n] \setminus I_j\big| \cdot |I|,
\end{aligned}
$$

which are the inequalities in the definition of the shifted permutohedron. Thus, $(x_1, \ldots, x_n)$ satisfies (**C3′**).

For the reverse direction, suppose that $(x_1, \ldots, x_n) \in \mathbb{C}^n$ satisfies (**C1′**)–(**C3′**). It is automatic that $(x_1, \ldots, x_n) \in Y^n$ satisfies (**C2**). For (**C1**), note that for any $I \subseteq [n]$, we have

$$
\begin{aligned}
\sum_{i \in I} |x_i| &= \sum_{i \in I \setminus I_k} |x_i| + \sum_{j=1}^{k} \sum_{i \in I \cap (I_j \setminus I_{j-1})} |x_i| \\
&\leq \delta^{|[n] \setminus I_k|}_{|I \setminus I_k|} + \sum_{j=1}^{k} \left(\delta^{|I_j \setminus I_{j-1}|}_{|I \cap (I_j \setminus I_{j-1})|} + \big|[n] \setminus I_j\big| \cdot \big|I \cap (I_j \setminus I_{j-1})\big| \right) \\
&= \delta^{|[n] \setminus I_k|}_{|I \setminus I_k|} + \sum_{j=1}^{k} \left(\delta^{|[n] \setminus I_{j-1}|}_{|I \cap (I_j \setminus I_{j-1})|} \right),
\end{aligned}
$$

For notational convenience, we define

$$
n_j := |[n] \setminus I_{j-1}| \quad \text{and} \quad a_j := |I \cap (I_j \setminus I_{j-1})|
$$

for each $j \in \{1, \ldots, k+1\}$, where $I_0 = \emptyset$ and $I_{k+1} = [n]$. In this notation, the above is expressed as

$$
\sum_{i \in I} |x_i| \leq \sum_{j=1}^{k+1} \delta^{n_j}_{a_j}.
$$

From here, using the fact that $n_1 \geq n_2 \geq \cdots \geq n_{k+1}$ and $a_j \leq n_{j+1} - n_j$ for each j, one can check that

$$\sum_{j=1}^{k+1} \delta_{a_j}^{n_j} \leq \delta_{a_1 + \cdots + a_{k+1}}^{n_1}$$

and hence

$$\sum_{i \in I} |x_i| \leq \delta_{a_1 + \cdots + a_{k+1}}^{n_1} = \delta_{|I|}^n,$$

proving that $(x_1, \ldots, x_n) \in \mathsf{Y}^n$ satisfies (**C1**). We also have, for $\ell \in \{1, \ldots, k\}$, that

$$\sum_{i \in I_\ell} |x_i| = \sum_{j=1}^{\ell} \sum_{i \in I_j \setminus I_{j-1}} |x_i|$$

$$= \sum_{j=1}^{\ell} \left(\delta_{|I_j \setminus I_{j-1}|}^{|I_j \setminus I_{j-1}|} + |[n] \setminus I_j| \cdot |I_j \setminus I_{j-1}| \right)$$

$$= \sum_{j=1}^{\ell} \left(\delta_{|I_j|}^n - \delta_{|I_{j-1}|}^n \right)$$

$$= \delta_{|I_\ell|}^n,$$

proving that $(x_1, \ldots, x_n)$ satisfies (**C3**) and completing the proof of the proposition. $\square$

As a corollary, we find that the Δ-faces of Δ_n^r have a product decomposition, which shows that they are polytopal complexes of the expected dimension.

Corollary 7.12. *For any chain* $\widetilde{\mathbf{I}}$, *we have*

$$F_{\widetilde{\mathbf{I}}} = \Delta_{|[n] \setminus I_k|}^r \times \varphi_{\mathfrak{a}} \left(\prod_{j=1}^{k} (\Pi_{|I_j \setminus I_{j-1}|} + \gamma_j) \right), \tag{30}$$

where $\varphi_{\mathfrak{a}} : \mathbb{R}^{|I_k|} \to \mathbb{C}^{|I_k|}$ *multiplies the ith coordinate by* $\zeta^{-\mathfrak{a}(i)}$, *and* $\gamma_j :=$ $|[n] \setminus I_j|$. *In particular,* $F_{\widetilde{\mathbf{I}}}$ *is a polytopal complex of dimension* $n - k$.

Proof. The product decomposition (30) follows immediately from Proposition 7.11. For the "in particular," it suffices to prove that Δ_n^r itself is a polytopal complex of dimension n. If this is the case, then (30) implies that $F_{\widetilde{\mathbf{I}}}$ is a product of polytopal complexes and polytopes, so it is a polytopal complex. Furthermore, using the fact that $\dim(\Pi_m) = m - 1$, we find that the dimension of $F_{\widetilde{\mathbf{I}}}$ is

$$|[n] \setminus I_k| + \sum_{j=1}^{k} \left(|I_j \setminus I_{j-1}| - 1 \right) = n - k,$$

as claimed.

To prove that Δ_n^r is a polytopal complex of dimension n, for every $\mathfrak{c} : [n] \to \mathbb{Z}_r$ define

$$Y_{\mathfrak{c}}^n = (\mathbb{R}^{\geq 0} \cdot \zeta^{\mathfrak{c}(1)}) \times \cdots \times (\mathbb{R}^{\geq 0} \cdot \zeta^{\mathfrak{c}(n)}) \subseteq Y^n.$$

Then $Y_{\mathfrak{c}}^n$ is naturally identified with $\mathbb{R}^n$, and under this identification, we have

$$\Delta_n^r \cap Y_{\mathfrak{c}}^n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \ \middle| \ \begin{array}{l} \sum_{i \in I} x_i \leq \delta_{|I|}^n \ \text{ for all } \ I \subseteq [n], \\ x_i \geq 0 \ \text{ for all } \ i \in [n] \end{array} \right\}.$$

In particular, $\Delta_n^r \cap Y_{\mathfrak{c}}^n$ is a polytope in $\mathbb{R}^n$, and it contains the origin as well as the n standard basis vectors $e_1, \ldots, e_n$. Since these are $n + 1$ affinely independent points in $\mathbb{R}^n$, it follows that $\Delta_n^r \cap Y_{\mathfrak{c}}^n$ has dimension n. Thus, we have an expression

$$\Delta_n^r = \bigcup_{\mathfrak{c}:[n] \to \mathbb{Z}_r} (\Delta_n^r \cap Y_{\mathfrak{c}}^n)$$

as a union of n-dimensional polytopes intersecting only along the faces where some subset of the coordinates is equal to zero. That is, Δ_n^r is a polytopal complex of dimension n. $\qquad\square$

Remark 7.13. The proof of Corollary 7.12 shows that the individual n-dimensional polytopes $\Delta_n^r \cap Y_{\mathfrak{c}}^n$ that are glued to form Δ_n^r are independent of r. Setting $r = 2$, we find that Δ_n^2 is, in fact, a polytope (the type-B permutohedron) and the polytopes that comprise any Δ_n^r are the intersection of Δ_n^2 with an octant. When $n = 2$, for example, Δ_2^2 is an octagon whose intersection with each quadrant is a pentagon, and these pentagons are visible in Fig. 2 as the building blocks of Δ_2^3.

Remark 7.14. The polytopes $\Delta_n^r \cap Y_{\mathfrak{c}}^n$ for different choices of $\mathfrak{c}$ are all isomorphic to one another, and they are precisely the polytope of the toric variety $\overline{\mathcal{M}}_n^1$ that arose in Sect. 3. In particular, one can interpret the maps

$$\pi_\alpha|_{U_\alpha} : U_\alpha \to \overline{\mathcal{M}}_n^1$$

and

$$p : \overline{\mathcal{L}}_n^r(\zeta) \to \overline{\mathcal{M}}_n^1$$

of Sect. 3.4 in polytopal terms: the map $\pi_\alpha|_{U_\alpha}$ amounts to looking locally at a single octant $\Delta_n^r \cap Y_{\mathfrak{c}}^n$ of Δ_n^r, while the map p amounts to identifying all of the octants of Δ_n^r with one another.

We are now ready to prove that—just as in the settings of boundary strata and $\mathcal{T}$-cosets—the Δ-face structure of Δ_n^r is precisely captured by chains.

Proposition 7.15. *Let $r \geq 2$ and $n \geq 0$. The association*

$$\widetilde{\mathbf{I}} \mapsto F_{\widetilde{\mathbf{I}}}$$

is a bijection from the set of $\mathbb{Z}_r$-decorated chains of subsets of $[n]$ to the set of Δ-faces of Δ_n^r. Furthermore, this bijection satisfies

(i) $length(\widetilde{\mathbf{I}}) = codim(F_{\widetilde{\mathbf{I}}})$,

(ii) $F_{\widetilde{\mathbf{I}}} \subseteq F_{\widetilde{\mathbf{J}}}$ if and only if $\widetilde{\mathbf{I}}$ refines $\widetilde{\mathbf{J}}$.

Proof. The surjectivity of $\widetilde{\mathbf{I}} \mapsto F_{\widetilde{\mathbf{I}}}$ is the content of Lemma 7.4. The injectivity will follow from item (ii) of the proposition, since the only way that $\widetilde{\mathbf{I}}$ is a refinement of $\widetilde{\mathbf{J}}$ and vice versa is if $\widetilde{\mathbf{I}} = \widetilde{\mathbf{J}}$. Item (i) is immediate from Corollary 7.12.

Thus, all that remains is to prove item (ii). One direction is clear: if $\widetilde{\mathbf{I}}$ is a refinement of $\widetilde{\mathbf{J}}$, then the hyperplanes intersected to form $F_{\widetilde{\mathbf{J}}}$ are a subset of the hyperplanes intersected to form $F_{\widetilde{\mathbf{I}}}$, so $F_{\widetilde{\mathbf{I}}} \subseteq F_{\widetilde{\mathbf{J}}}$.

Conversely, suppose that $F_{\widetilde{\mathbf{I}}} \subseteq F_{\widetilde{\mathbf{J}}}$, where

$$\widetilde{\mathbf{I}} = (I_1, \ldots, I_k, \mathfrak{a}),$$
$$\widetilde{\mathbf{J}} = (J_1, \ldots, J_\ell, \mathfrak{b}).$$

By completing $\widetilde{\mathbf{I}}$ to a maximal chain (which, in particular, involves extending $\mathfrak{a}$ to a function $\mathfrak{a} : [n] \to \mathbb{Z}_r$), one can find a vertex

$$v := (x_1, \ldots, x_n) \in F_{\widetilde{\mathbf{I}}}.$$

Equation (27) implies that, for each $i \in \{1, \ldots, n\}$, we have

$$x_i = \zeta^{-\mathfrak{a}(i)}\lambda_i$$

for some $\lambda_i \in [n]$. The fact that $v \in F_{\widetilde{\mathbf{I}}} \subseteq F_{\widetilde{\mathbf{J}}}$ then implies that

$$\sum_{i \in J_j} \zeta^{\mathfrak{b}(i)} \cdot \zeta^{-\mathfrak{a}(i)}\lambda_i = \delta_{|J_j|}$$

for each j, and from here, the same triangle inequality argument from Lemma 7.4 shows that $\mathfrak{a}(i) = \mathfrak{b}(i)$ for all $i \in J_j$. Thus, the decorations on $\widetilde{\mathbf{J}}$ agree with the decorations on $\widetilde{\mathbf{I}}$ where both are defined, and what remains is to prove that

$$\{J_1, \ldots, J_\ell\} \subseteq \{I_1, \ldots, I_k\}. \tag{31}$$

If not, then one possibility is that $J_\ell = [n]$ whereas $I_k \neq [n]$. In this case, however, if $i \in [n] \setminus I_k$, then for any $s \in \mathbb{Z}_r$ one can construct a vertex $(x_1, \ldots, x_n) \in F_{\widetilde{\mathbf{I}}}$ with $x_i = \zeta^s \lambda_i$. By contrast, any vertex of $F_{\widetilde{\mathbf{J}}}$ has $x_i = \zeta^{-\mathfrak{b}(i)}\lambda_i$, so it cannot be the case that $F_{\widetilde{\mathbf{I}}} \subseteq F_{\widetilde{\mathbf{J}}}$.

Having ruled out this possibility, the failure of (31) implies that

$$\{J_1, \ldots, J_\ell, [n]\} \not\subseteq \{I_1, \ldots, I_k, [n]\}, \tag{32}$$

and from here, we can cite the known face structure of the permutohedron Π_n (see, for example, [20, Proposition 2.6] or [1, Section 4.1]). In particular, faces of Π_n are indexed by subsets of $[n]$ in which the largest is $[n]$ itself, and if we set

$$F_{J_1, \ldots, J_\ell, [n]} := \left\{ (x_1, \ldots, x_n) \in \Pi_n \ \middle| \ \sum_{i \in J_j} x_i = \delta_{|J_j|} \text{ for all } j \right\}$$

and similarly

$$F_{I_1,\ldots,I_k,[n]} := \left\{ (x_1,\ldots,x_n) \in \Pi_n \;\middle|\; \sum_{i \in I_j} x_i = \delta_{|I_j|} \text{ for all } j \right\},$$

then it is known that (32) implies

$$F_{J_1,\ldots,J_\ell,[n]} \not\subseteq F_{I_1,\ldots,I_k,[n]}.$$

That is, there exists $(x_1,\ldots,x_n) \in F_{J_1,\ldots,J_\ell,[n]}$ with $(x_1,\ldots,x_n) \notin F_{I_1,\ldots,I_k,[n]}$. It is straightforward to see that, for any extension of $\mathfrak{a}$ to a function $[n] \to \mathbb{Z}_r$, we have

$$(\zeta^{-\mathfrak{a}(1)}x_1,\ldots,\zeta^{-\mathfrak{a}(n)}x_n) \in F_{\widetilde{\mathbf{J}}}$$

but

$$(\zeta^{-\mathfrak{a}(1)}x_1,\ldots,\zeta^{-\mathfrak{a}(n)}x_n) \notin F_{\widetilde{\mathbf{I}}}.$$

This contradicts our assumption that $F_{\widetilde{\mathbf{I}}} \subseteq F_{\widetilde{\mathbf{J}}}$ and thus completes the proof. $\qquad\square$

The previous two sections concluded with a product decomposition of the relevant objects, and we close this section by briefly noting that the analogous product decomposition also holds for Δ-faces.

Remark 7.16. For any chain $\widetilde{\mathbf{I}} = (I_1,\ldots,I_k,\mathfrak{a})$, there is an isomorphism

$$F_{\widetilde{\mathbf{I}}} \cong \Delta^r_{|[n]\setminus I_k|} \times \prod_{j=1}^k \Pi_{|I_j \setminus I_{j-1}|}.$$

(The word "isomorphism" here can be taken to mean "combinatorial equivalence," or more strongly, "isometry" under the standard inner products on $\mathbb{R}^n$ and $\mathbb{C}^n$.) This follows directly from Corollary 7.12.

8. Proof of Theorem 1.1 and Enhancements

We are now positioned to complete the proof of the main theorem:

Theorem 1.1 *For any integers $r \geq 2$ and $n \geq 0$, there are dimension-preserving, inclusion-preserving bijections*

$$\left\{ \begin{array}{c} \text{boundary} \\ \text{strata in } \overline{\mathcal{L}}^r_n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathcal{T}\text{-cosets} \\ \text{in } S(r,n) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \Delta\text{-faces} \\ \text{of } \Delta^r_n \end{array} \right\}.$$

Proof. Propositions 5.4, 6.6, and 7.15 give bijections between each of these three sets and the set of $\mathbb{Z}_r$-decorated chains of subsets of $[n]$, and in each case, the dimension is encoded by the co-length of a chain and the inclusion relation is encoded by refinement of chains. $\qquad\square$

In fact, the statement of Theorem 1.1 can be enhanced to incorporate two pieces of additional structure: product decompositions of the three types of objects and an action of $S(r,n)$ on each. The remainder of this last section of the paper is devoted to carrying out these enhancements.

8.1. Product decompositions

Theorem 1.1 is an analogue of results pertaining to the Losev–Manin moduli spaces $\overline{\mathcal{L}}_n$ studied in [16]. There, the relevant group is the symmetric group S_n (in which there is a precisely analogous definition of $\mathcal{T}$-cosets with $\mathcal{T}$ the set of adjacent transpositions), and the relevant polytopal complex is the permutohedron Π_n (which, in this case, is actually a polytope). The theorem, then, is that there are dimension-preserving and inclusion-preserving bijections

$$\left\{ \begin{matrix} \text{boundary} \\ \text{strata in } \overline{\mathcal{L}}_n \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \mathcal{T}\text{-cosets} \\ \text{in } S_n \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{faces} \\ \text{of } \Pi_n \end{matrix} \right\}. \tag{33}$$

Remark 8.1. The analogy between (33) and Theorem 1.1, and the fact that S_n is the $r = 1$ case of $S(r, n)$, suggests that $\overline{\mathcal{L}}_n$ and Π_n should be viewed "morally" as the $r = 1$ cases of $\overline{\mathcal{L}}_n^r$ and Δ_n^r, respectively, despite the fact that these objects are not literally recovered by setting $r = 1$ in the higher-r construction. Setting $r = 1$ in the definition of $\overline{\mathcal{L}}_n^r$, in particular, yields the space $\overline{\mathcal{M}}_n^1$, which we observed in Remark 3.1 is not equal to Losev–Manin space.

With this analogy established, we note that the bijections of Theorem 1.1 preserve rich geometric structures of $\overline{\mathcal{L}}_n^r$ in a way that incorporates the corresponding structures in Losev–Manin spaces encoded by (33). More precisely, we have seen in Proposition 5.5, Proposition 6.7, and Remark 7.16 that

- a boundary stratum in $\overline{\mathcal{L}}_n^r$ is isomorphic to a product with one factor $\overline{\mathcal{L}}_{n'}^r$ for some n' and all other factors Losev–Manin spaces;
- a $\mathcal{T}$-coset $C_{\widetilde{\mathbf{I}}}$ in $S(r, n)$ is a coset of a subgroup isomorphic to a product with one factor $S(r, n')$ for some n' and the other factors symmetric groups;
- a Δ-face $F_{\widetilde{\mathbf{I}}}$ in Δ_n^r is isomorphic to a product with one factor $\Delta_{n'}^r$ for some n' and all other factors permutohedra.

The following theorem says that the bijections of Theorem 1.1 are compatible with these product decompositions.

Theorem 8.1. *Under the bijections of Theorem 1.1, a boundary stratum $S_{\widetilde{\mathbf{I}}}$ naturally isomorphic (via Proposition 5.5) with*

$$\overline{\mathcal{L}}_{n_{k+1}}^r \times \prod_{j=1}^{k} \overline{\mathcal{L}}_{n_j}$$

corresponds to a $\mathcal{T}$-coset $C_{\widetilde{\mathbf{I}}}$ naturally identified (via Proposition 6.7) with

$$S(r, n_{k+1}) \times \prod_{j=1}^{k} S_{n_j} \cdot A_j,$$

and to a Δ-face $F_{\widetilde{\mathbf{I}}}$ naturally isomorphic (via Remark 7.16) with

$$\Delta_{n_{k+1}}^r \times \prod_{j=1}^{k} \Pi_{n_j}.$$

In particular, if $\widetilde{\mathbf{I}} = (I_1, \ldots, I_k, \mathfrak{a})$, *then*

$$n_{k+1} = |[n] \setminus I_k| \text{ and } n_j = |I_j \setminus I_{j-1}|$$

for all $j \in \{1, \ldots, k\}$.

Proof. This follows from Proposition 5.5, Proposition 6.7, and Remark 7.16, which describe the product decomposition of a boundary stratum, $\mathcal{T}$-coset, or Δ-face corresponding to a given chain $\widetilde{\mathbf{I}}$. $\qquad\square$

8.2. Actions and equivariance

Another key feature of each of the three settings of interest is the existence of a right action by $S(r, n)$. In particular:

- $S(r, n)$ acts on $\overline{\mathcal{L}}_n^r(\zeta)$, because an element of $\overline{\mathcal{L}}_n^r(\zeta)$ is determined by the choice of the curve C and the first element $z_1^0, \ldots, z_n^0$ in each light orbit. Thus, for any $A \in S(r, n)$ we can view the matrix-vector product

$$(z_1^0, \ldots, z_n^0) \cdot A$$

  as a new tuple of elements of C by identifying $\zeta^k \cdot z_j^i$ with $\sigma^k(z_j^i)$, so setting

$$(C; z_1^0, \ldots, z_n^0) \cdot A := (C; (z_1^0, \ldots, z_n^0) \cdot A)$$

  gives an action of $S(r, n)$ on $\overline{\mathcal{L}}_n^r(\zeta)$.
- $S(r, n)$ acts on itself by multiplication on the right.
- $S(r, n)$ acts on Δ_n^r by matrix-vector multiplication

$$(x_1, \ldots, x_n) \cdot A$$

for $(x_1, \ldots, x_n) \in \Delta_n^r$.

We will see in Lemma 8.4 that the action on $\overline{\mathcal{L}}_n^r(\zeta)$ takes 0-dimensional boundary strata to 0-dimensional boundary strata, and the action on Δ_n^r takes 0-dimensional Δ-faces to 0-dimensional Δ-faces. Moreover, the bijections of Theorem 1.1 are equivariant under these actions. Before proving this in general, let us illustrate it in two examples.

Example 8.2. In the case of $(r, n) = (2, 2)$, let

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then the action of A on Δ_2^2 rotates by $\pi/4$ counterclockwise. On $S(2, 2)$, the action is simply right-multiplication, whereas on $\overline{\mathcal{L}}_2^2$, the action is described by

$$(z_1^0, \ z_2^0) \cdot A = (z_2^1, \ z_1^0);$$

in other words, if $(C; z_1^0, z_2^0)$ specifies an element of $\overline{\mathcal{L}}_2^2$, then

$$(C; z_1^0, z_2^0) \cdot A = (C; \tilde{z}_1^0, \tilde{z}_2^0)$$

in which $\tilde{z}_1^0 = z_2^1$ and $\tilde{z}_2^0 = z_1^0$. From here, consulting Fig. 1(A) shows that the bijection between 0-dimensional boundary strata in $\overline{\mathcal{L}}_2^2$ and vertices of Δ_2^2 is $S(2, 2)$-equivariant: if $(C; z_1^0, z_2^0)$ is a 0-dimensional boundary stratum corresponding to a vertex $v \in \Delta_2^2$, then $(C; z_0^1, z_0^2) \cdot A$ is the 0-dimensional boundary stratum corresponding to the vertex $v \cdot A$. Similarly, from Fig. 1(B), one sees that if $B \in S(2, 2)$ is a group element (that is, a 0-dimensional $\mathcal{T}$-coset) corresponding to a vertex $v \in \Delta_2^2$, then $B \cdot A$ is the group element corresponding to $v \cdot A$.

Example 8.3. In the case of $(r, n) = (3, 4)$, let

$$A = \begin{pmatrix} 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & 0 & \zeta \\ 0 & 0 & \zeta^2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{34}$$

Then we have

$$(z_1^0, z_2^0, z_3^0, z_4^0) \begin{pmatrix} 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & 0 & \zeta \\ 0 & 0 & \zeta^2 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = (z_4^0, z_1^2, z_3^2, z_2^1),$$

which means that after the action of A, the first elements of the light orbits are located at the points where $z_4^0, z_1^2, z_3^2, z_2^1$ were located before the action of A. For instance, the action of A sends the element of $\overline{\mathcal{L}}_4^3(\zeta)$ in Fig. 12(A) to the element in Fig. 12(B).

Unpacking the bijections of Theorem 1.1, one sees that the curve in Fig. 12(A) is the 0-dimensional boundary stratum corresponding to the $\mathcal{T}$-coset $\{I\}$ containing only the identity matrix, and corresponding to the vertex $(1, 2, 3, 4) \in \Delta_2^3$. On the other hand, the curve in Fig. 12(B) is the 0-dimensional boundary stratum corresponding to the chain

$$\widetilde{\mathbf{I}} = (\{1\}, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 3, 4\}, \mathfrak{a}),$$

where

$$\mathfrak{a}(1) = 0, \quad \mathfrak{a}(2) = 1, \quad \mathfrak{a}(3) = 1, \quad \mathfrak{a}(4) = 2,$$

from which it is straightforward to check that it corresponds to the $\mathcal{T}$-coset $\{A\} = \{A \cdot I\}$ and to the vertex $(4, \zeta^2, 3\zeta^2, 2\zeta) = (1, 2, 3, 4) \cdot A$ of Δ_2^3. Thus, in this case, we again see that the bijections of Theorem 1.1 are $S(2, 3)$-equivariant.

(A) An element C in $\overline{\mathcal{L}}_4^3(\zeta)$.

(B) The element $C \cdot A$ in $\overline{\mathcal{L}}_4^3(\zeta)$.
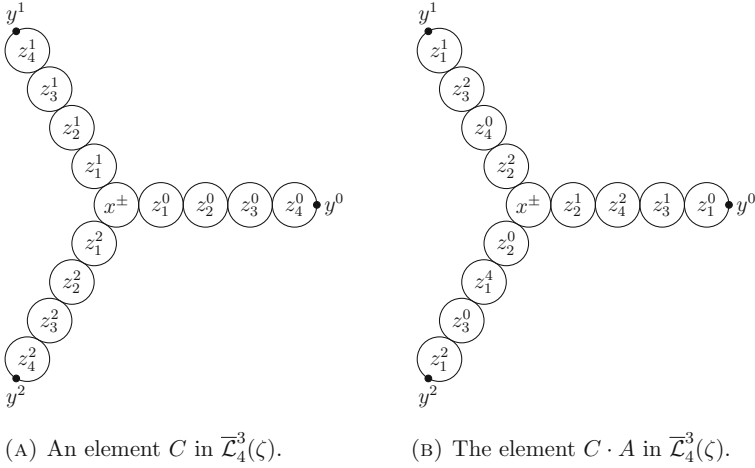
Fig. 12. The action of $A \in S(3,4)$ defined by (34) on an element of $\overline{\mathcal{L}}_4^3(\zeta)$

To confirm that the bijections between 0-dimensional objects are $S(r,n)$-equivariant in general, we denote by $S_0 \in \overline{\mathcal{L}}_n^r(\zeta)$ the zero-dimensional boundary stratum corresponding under Theorem 1.1 to the 0-dimensional $\mathcal{T}$-coset $\{I\} \subseteq S(r,n)$. Specifically, this means that the y^0-spoke of S_0 contains the light marked points $z_1^0, \ldots, z_n^0$, with one on each component in order from innermost to outermost, or in other words that

$$z_j^i \in C_{n+1-j}^i.$$

For instance, Fig. 12(A) illustrates $S_0 \subseteq \overline{\mathcal{L}}_4^3(\zeta)$.

Lemma 8.4. *The bijections*

$$\left\{ \begin{matrix} \textit{zero-dimensional} \\ \textit{boundary} \\ \textit{strata in } \overline{\mathcal{L}}_n^r(\zeta) \end{matrix} \right\} \longleftrightarrow S(r,n) \longleftrightarrow \left\{ \begin{matrix} \textit{vertices} \\ \textit{of } \Delta_n^r \end{matrix} \right\}$$

of Theorem 1.1 *identify* $A \in S(r,n)$ *with the boundary stratum* $S_0 \cdot A$ *and with the vertex* $(1, \ldots, n) \cdot A \in \Delta_n^r$.

Proof. All of these objects correspond to maximal chains

$$\widetilde{\mathbf{I}} = (I_1, \ldots, I_n, \mathfrak{a})$$

with $\mathfrak{a} : [n] \to \mathbb{Z}_r$, for which the sets $I_1, \ldots, I_n$ can be expressed as

$$I_1 = \{i_1\}$$
$$I_2 = \{i_1, i_2\}$$
$$\vdots$$
$$I_n = \{i_1, i_2, \ldots, i_n\} = [n]$$

for some $i_1, \ldots, i_n \in [n]$. In this notation:

(i) the associated boundary stratum $S_{\widetilde{\mathbf{I}}}$ is defined by the condition that

the unique light marked point on the component C_j^0 is $z_{i_j}^{\mathfrak{a}(i_j)}$

for each $j \in [n]$;

(ii) the associated $\mathcal{T}$-coset is the singleton $C_{\widetilde{\mathbf{I}}} = \{A\}$, where A is the matrix defined by the condition that

row $n + 1 - j$ of A has nonzero entry $\zeta^{-\mathfrak{a}(i_j)}$ in column i_j

for each $j \in [n]$;

(iii) the associated Δ-face is the vertex $F_{\widetilde{\mathbf{I}}} = \{(x_1, \ldots, x_n)\}$ with coordinates defined by

$$x_{i_j} = \zeta^{-\mathfrak{a}(i_j)} \cdot (n + 1 - j)$$

for each $j \in [n]$. (See Eq. (27).)

In particular, it is straightforward to see that the vertex in (iii) is equal to

$$(x_1, \ldots, x_n) = (1, \ldots, n) \cdot A$$

for the matrix A in (ii), which verifies one half of the lemma.

For the other half of the lemma, we must show that $S_{\widetilde{\mathbf{I}}} = S_0 \cdot A$. To see this, let $\{z_j^i\}$ denote the elements of the light orbits in S_0, so that, by the definition of S_0, we have

$$z_j^i \in C_{n+1-j}^i.$$

If $\{\tilde{z}_j^i\}$ denote the elements of the light orbits in $S_0 \cdot A$, then the definition of the action and of A implies that

$$\tilde{z}_{i_j}^0 = z_{n+1-j}^{-\mathfrak{a}(i_j)} \in C_j^{-\mathfrak{a}(i_j)},$$

In particular, $S_0 \cdot A$ has just one light marked point on each of the components C_j^0 for $j \in [n]$, and that marked point is $\tilde{z}_{i_j}^{\mathfrak{a}(i_j)}$. This exactly agrees with the above description of $S_{\widetilde{\mathbf{I}}}$, so $S_{\widetilde{\mathbf{I}}} = S_0 \cdot A$. $\qquad\square$

From here, the fact that the bijections of Theorem 1.1 are inclusion-preserving gives a concise reinterpretation of the bijections in general.

Proposition 8.5. *The bijections*

$$\left\{ \begin{matrix} boundary \\ strata\ in\ \overline{\mathcal{L}}_n^r(\zeta) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \mathcal{T}\text{-}cosets \\ in\ S(r, n) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \Delta\text{-}faces \\ of\ \Delta_n^r \end{matrix} \right\}$$

of Theorem 1.1 identify a boundary stratum S with

$$\{A \in S(r, n) \mid S_0 \cdot A \in S\} \subseteq S(r, n)$$

and identify a Δ-face F with

$$\{A \in S(r, n) \mid (1, \ldots, n) \cdot A \in F\} \subseteq S(r, n),$$

both of which are $\mathcal{T}$-cosets.

Proof. Let S be a boundary stratum, and let $C_S \subseteq S(r, n)$ be the $\mathcal{T}$-coset associated to it via the bijection of Theorem 1.1. By Lemma 8.4, the bijection associates each $A \in S(r, n)$ to the zero-dimensional boundary stratum $S_0 \cdot A$. And since it is inclusion-preserving, we have

$$A \in C_S \iff S_0 \cdot A \in S.$$

This proves that

$$C_S = \{A \in S(r, n) \mid S_0 \cdot A \in S\},$$

so in particular, the latter is indeed a $\mathcal{T}$-coset. The argument for the case of a Δ-face F is identical. $\qquad\square$

One reason to like this interpretation—in addition to the fact that it is much simpler to state than how we initially constructed the bijections of Theorem 1.1, and in particular does not require the auxiliary machinery of chains—is that it immediately shows that the bijections are $S(r, n)$-equivariant. To see this, we first should note that there is a right action of $S(r, n)$ on the sets of boundary strata, $\mathcal{T}$-cosets, and Δ-faces, in each case by setting

$$X \cdot A := \{x \cdot A \mid x \in X\}$$

for a boundary stratum, $\mathcal{T}$-coset, or Δ-face X. Here, the fact that $S(r, n)$ indeed acts on each of these sets is a result of the following observations:

- The action of $S(r, n)$ on $\overline{\mathcal{L}}_n^r$ preserves the topological type of C (in fact, it preserves C itself) while permuting marked points, so it takes boundary strata to boundary strata.
- The action of $S(r, n)$ on $\mathcal{T}$-cosets is equivalently described by

$$\left(\langle s_{\ell_1}, \dots, s_{\ell_d} \rangle B \right) \cdot A = \langle s_{\ell_1}, \dots, s_{\ell_d} \rangle \cdot (BA),$$

  so it takes $\mathcal{T}$-cosets to $\mathcal{T}$-cosets.
- The action of $S(r, n)$ on $\mathbb{C}^n$ on Δ_n^r takes points satisfying the conditions of Remark 7.6 to points satisfying an analogous set of conditions, so it takes Δ-faces to Δ-faces.

From here, it is essentially immediate from Proposition 8.5 that the bijections of Theorem 1.1 respect these actions.

Remark 8.6. One way to confirm that the bijections of Theorem 1.1 are equivariant is to verify that the above three $S(r, n)$-actions all correspond, under Theorem 1.1, to an action on chains. Indeed, this is the case: the image of a chain $(I_1, \dots, I_k, \mathfrak{a})$ under the action of the element $A \in S(r, n)$ is the chain $(I_1', \dots, I_k', \mathfrak{a}')$ characterized by

$$I_j' = \bigcup_{i \in I_j} \{\ell \in [n] \mid A_{i\ell} \neq 0\}$$

and

$$\mathfrak{a}'(\ell) = \mathfrak{a}(i) - m_{i\ell} \quad \text{for all } \ell \in I_k',$$

where $i \in [n]$ and $m_{i\ell} \in \mathbb{Z}_r$ are uniquely determined by the condition that $A_{i\ell} = \zeta^{m_{i\ell}}$. However, it requires some care to check that this action on chains indeed matches the three actions above, so we will instead prove the equivariance of Theorem 1.1 directly as a corollary of Proposition 8.5.

Corollary 8.7. *The bijections of Theorem* 1.1 *are* $S(r,n)$-*equivariant.*

Proof. Let S be a boundary stratum, and let C_S be the associated $\mathcal{T}$-coset. Under the action of $B \in S(r,n)$, we have

$$C_S \cdot B = \{AB \in S(r,n) \mid A \in C_S\} = \{A \in S(r,n) \mid AB^{-1} \in C_S\}.$$

The element $AB^{-1} \in S(r,n)$ corresponds to the boundary stratum $S_0 \cdot AB^{-1} \in \overline{\mathcal{L}}_n^r(\zeta)$, by Lemma 8.4. Together with the fact that the bijection from $\mathcal{T}$-cosets to boundary strata is inclusion-preserving, this implies that

$$AB^{-1} \in C_S \text{ if and only if } S_0 \cdot AB^{-1} \in S.$$

Thus, we have

$$C_S \cdot B = \{A \in S(r,n) \mid S_0 \cdot AB^{-1} \in S\} = \{A \in S(r,n) \mid S_0 \cdot A \in S \cdot B\},$$

which, by Proposition 8.5, is precisely equal to $C_{S \cdot B}$.

This proves that the bijection between the sets of boundary strata and $\mathcal{T}$-cosets is $S(r,n)$-equivariant, and an identical proof shows the same statement for the bijection between Δ-faces of Δ_n^r and $\mathcal{T}$-cosets. $\qquad\square$

Declarations

Conflict of interest The authors declare no conflict of interests.

References

[1] Aguiar, M., Ardila, F.: Hopf monoids and generalized permutahedra. (2017). arXiv preprint arXiv:1709.07504

[2] Batyrev, V., Blume, M.: The functor of toric varieties associated with Weyl chambers and Losev–Manin moduli spaces. Tohoku Math. J. (2) **63**(4), 581–604 (2011)

[3] Batyrev, V., Blume, M.: On generalisations of Losev–Manin moduli spaces for classical root systems. Pure Appl. Math. Q. **7**(4, Special Issue: In memory of Eckart Viewweg), 1053–1084 (2011)

[4] Bergström, J., Minabe, S.: On the cohomology of moduli spaces of (weighted) stable rational curves. Math. Z. **275**(3–4), 1095–1108 (2013)

[5] Bergström, J., Minabe, S.: On the cohomology of the Losev–Manin moduli space. Manuscr. Math. **144**(1), 241–252 (2014)

[6] Clader, E., Damiolini, C., Li, S., Ramadas, R.: Wonderful compactifications and rational curves with cyclic action. (2022). arXiv preprint arXiv:2208.05463

[7] Chan, M., Galatius, S., Payne, S.: Tropical curves, graph complexes, and top weight cohomology of $\mathcal{M}_g$. J. Am. Math. Soc., **1** (2021)

[8] Cavalieri, R., Hampe, S., Markwig, H., Ranganathan, D.: Moduli spaces of rational weighted stable curves and tropical geometry. Forum Math. Sigma **4**, e9, 35 (2016)

[9] Castravet, A.-M., Tevelev, J.: $\overline{M}_{0,n}$ is not a Mori dream space. Duke Math. J. **164**(8), 1641–1667 (2015)

[10] Deopurkar, A.: Compactifications of Hurwitz spaces. Int. Math. Res. Not. IMRN **14**, 3863–3911 (2014)

[11] González, J.L., Karu, K.: Some non-finitely generated Cox rings. Compos. Math. **152**(5), 984–996 (2016)

[12] Hassett, B.: Moduli spaces of weighted pointed stable curves. Adv. Math. **173**(2), 316–352 (2003)

[13] Harris, J., Mumford, D.: On the Kodaira dimension of the moduli space of curves. Invent. Math. **67**, 23–86 (1982)

[14] Keel, S.: Intersection theory of moduli space of stable n-pointed curves of genus zero. Trans. Am. Math. Soc. **330**(2), 545–574 (1992)

[15] Kannan, S., Karp, D., Li, S.: Chow rings of heavy/light Hassett spaces via tropical geometry. J. Combin. Theory Ser. A **178**, 105348 (2021)

[16] Losev, A., Manin, Y.: New moduli spaces of pointed curves and pencils of flat connections. Michigan Math. J. **48**(1), 443–472 (2000)

[17] Lehrer, G.I., Taylor, D.E.: Unitary Reflection Groups Australian. Mathematical Society Lecture Series, vol. 20. Cambridge University Press, Cambridge (2009)

[18] Massarenti, A.: The automorphism group of $\overline{M}_{g,n}$. J. Lond. Math. Soc. (2) **89**(1), 131–150 (2014)

[19] Massarenti, A., Mella, M.: On the automorphisms of Hassett's moduli spaces. Trans. Am. Math. Soc. **369**(12), 8879–8902 (2017)

[20] Postnikov, A.: Permutohedra, associahedra, and beyond. Int. Math. Res. Not. IMRN **6**, 1026–1106 (2009)

[21] Ulirsch, M.: Tropical geometry of moduli spaces of weighted stable curves. J. Lond. Math. Soc. (2) **92**(2), 427–450 (2015)