Comments on "Stability Regions of Nonlinear Autonomous Dynamical Systems"

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***Abstract*— The proofs of the groundbreaking theorems of [1] rely on a lemma which states that if the stable manifold of a first hyperbolic closed orbit intersects transversely the unstable manifold of a second (possibly the same) hyperbolic closed orbit, then the dimension of the unstable manifold of the first is strictly less than the dimension of the unstable manifold of the second. However, we provide an example meeting the conditions of the lemma where the dimensions of the unstable manifolds are equal, thereby disproving the lemma. In particular, we present a hyperbolic closed orbit of a C^∞ vector field over $\mathbb{R}^3$ whose stable and unstable manifolds have nonempty, transverse intersection.**

I. Introduction

Consider the system:

$$\dot{x} = f(x) \tag{1}$$

where f is a C^1 vector field over Euclidean space $\mathbb{R}^n$ for some $n > 0$. In [1, Theorem 4.2], the authors claim that if a stable equilibrium point of (1) satisfies their assumptions, then the boundary of its region of attraction (RoA) is equal to the union of the stable manifolds of the equilibrium points and periodic orbits it contains. The main technical result behind their proof of this impressive theorem is [1, Theorem 3-8]. In turn, [1, Lemma 3-5] is crucial in their proof of [1, Theorem 3-8]. This note is devoted to the construction of a counterexample to [1, Lemma 3-5].

The exact statement of [1, Lemma 3-5] is as follows:

Lemma 3-5 of [1]. Let x_i and x_j be hyperbolic critical elements of (1). Suppose that the intersection of stable and unstable manifolds of x_i, x_j satisfy the transversality condition and $\{W^u(x_i) - x_i\} \bigcap \{W^s(x_j) - x_j\} \neq \emptyset$. Then dim $W^u(x_i) \geq$ dim $W^u(x_j)$, where the equality sign is true only when x_i is an equilibrium point and x_j is a closed orbit.

II. Context

Before proceeding we provide some preliminary definitions. Let a critical element refer to either an equilibrium point or a closed orbit of (1). For any $x \in \mathbb{R}^n$, $\omega(x)$ is the set of limit points of its forward orbit under (1) as $t \to \infty$, and $\alpha(x)$ is the set of limit points of its backwards orbit under (1) as $t \to -\infty$. An equilibrium point x_e of (1) is hyperbolic if df_{x_e} has no purely imaginary eigenvalues. A periodic orbit X of (1) is hyperbolic if there exists $x \in X$, a smooth cross section S containing x, and a C^1 first return map $\tau : S \to S$ such that $d\tau_x$ has no eigenvalues of norm one. A hyperbolic critical element X possesses a local stable manifold $W^s_{\text{loc}}(X)$ that is forward invariant under the flow, and a local unstable manifold $W^u_{\text{loc}}(X)$ that is backward invariant under the flow. Then its stable manifold $W^s(X)$ is constructed by flowing $W^s_{\text{loc}}(X)$ backward in time for all negative times, and its unstable manifold is constructed by flowing $W^u_{\text{loc}}(X)$ forward in time for all positive times. Hence, $W^s(X)$ and $W^u(X)$ are both invariant under the flow. If A is a C^1 manifold and $x \in A$, let T_xA denote the tangent space of A at x. A pair of C^1 submanifolds N and S of the smooth manifold M satisfy the transversality condition, or have transversal intersection, if either they are disjoint or for every $x \in N \bigcap S$, $T_xN + T_xS = T_xM$. If N and M are smooth manifolds, V is a vector field on N, and $F : N \to M$ is an embedding, then the pushforward of V under F is the unique vector field $\hat{V}$ on the image of F defined by $\hat{V}(p) = dF_{F^{-1}(p)}(V(F^{-1}(p)))$.

The authors of [1] attribute their Lemma 3-5 to a survey paper by Smale [2] and do not provide a proof of the lemma. However, we have not found the lemma in [2]. Furthermore, in another paper by Smale [3] - which deals with the particular setting of flows on compact boundaryless (closed) manifolds, whereas [2] does not necessarily - he considers a set of assumptions in which his conditions 1–4 include the requirements that all equilibrium points and periodic orbits are hyperbolic, and their stable and unstable manifolds have transverse intersection. Yet Smale also includes a further condition 5, which states that for any closed orbit X there does not exist y such that $\omega(y) = \alpha(y) = X$. In other words, condition 5 states for every closed orbit X that its stable and unstable manifolds have empty intersection outside of X. In addition, Smale writes: "It is true that conditions 1–5 are independent." If [1, Lemma 3-5] were correct, then conditions 1–4 would imply condition 5[1]. Therefore Smale's statement that conditions 1–5 are independent would necessarily be incorrect. Consequently, attributing [1, Lemma 3-5] to Smale seems inconsistent with his own claims in [3].

It should be noted that the example presented here does not contradict [1, Theorem 4.2], and that the theorem may be correct as stated. However, we have not seen a proof that avoids the use of [1, Lemma 3-5]. We have recently shown that the result of Theorem 4.2 can be proved analogously to the original proof under a slightly stronger assumption than the original theorem: namely, that the nonwandering set on the RoA boundary consists of a finite union of

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[1] As a consequence of [1, Lemma 3-5], hyperbolic closed orbits could not have nonempty, transverse intersections between their stable and unstable manifolds.

critical elements [4]. The assumption of a finite number of closed orbits makes it possible to complete the proof of [1, Theorem 4.2] without the use of [1, Lemma 3-5].

Furthermore, it is possible that under the additional assumptions of Theorem 4.2, which were not assumed in the statement of Lemma 3-5, the conclusion of Lemma 3-5 may in fact be true. However, such a result is not immediate, and the validity of Lemma 3-5 even under these additional assumptions is unclear.

III. Example

To disprove [1, Lemma 3-5], we will provide an example in which $\dim W^u(x_i) = \dim W^u(x_j)$ but both x_i and x_j are closed (periodic) orbits. In particular, we will give an example where $x_i = x_j$ is a hyperbolic periodic orbit whose stable and unstable manifolds are transverse and $\{W^u(x_i) - x_i\} \bigcap \{W^s(x_j) - x_j\} \neq \emptyset$. Then $\dim W^u(x_i) = \dim W^u(x_j)$ because $x_i = x_j$.

Consider the classic example of the Duffing equation with negative linear stiffness, weak damping, and weak periodic forcing, which is given by the following C^∞ nonautonomous vector field on $\mathbb{R}^2$ [5, p. 191]:

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= u - u^3 + \epsilon(\gamma \cos t - \delta v), \end{aligned}$$

where $\epsilon, \delta, \gamma \geq 0$ are parameters and $u, v \in \mathbb{R}$. At $\epsilon = 0$ this system possesses a hyperbolic saddle equilibrium point at $(0, 0)$ whose stable manifold and unstable manifold are equal (they consist of homoclinic orbits). We can rewrite this system as an autonomous vector field on $\mathbb{R}^2 \times \mathbb{S}^1$ by introducing a time coordinate τ:

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= u - u^3 + \epsilon(\gamma \cos \tau - \delta v) \\ \dot{\tau} &= 1. \end{aligned}$$

Letting $w = [u\ v\ \tau]^T \in \mathbb{R}^2 \times \mathbb{S}^1$, we can write the above vector field as:

$$\dot{w} = V(w). \tag{2}$$

Note that V is a C^∞ vector field. For $\epsilon = 0$, this system has a hyperbolic periodic orbit given by $\Gamma_0 = \{(0, 0, \tau) : \tau \in \mathbb{S}^1\}$ whose stable and unstable manifolds are each two-dimensional. Since Γ_0 is hyperbolic, for $\epsilon > 0$ sufficiently small there exists a unique hyperbolic periodic orbit Γ_ϵ which is C^1-close to Γ_0 [6, Chapter 16].

For any $\tau \in \mathbb{S}^1$, let $S_\tau = \mathbb{R}^2 \times \{\tau\}$. Let $g : S_\tau \to S_\tau$ be the first return map, which is well-defined and C^∞. Let $p_\epsilon^\tau = \Gamma_\epsilon \bigcap S_\tau$, which is a single point. Then it is straightforward to see that p_ϵ^τ is a hyperbolic fixed point of g. Let $W^s(p_\epsilon^\tau)$ and $W^u(p_\epsilon^\tau)$ denote its stable and unstable manifolds, respectively, in S_τ. Using Melnikov's method [5, Theorem 4.5.3], it can be shown [5, p. 193] that for any $\tau \in \mathbb{S}^1$, $\epsilon > 0$ sufficiently small, and $\frac{\gamma}{\delta}$ sufficiently large, $W^s(p_\epsilon^\tau)$ and $W^u(p_\epsilon^\tau)$ are transverse and $\{W^s(p_\epsilon^\tau) - p_\epsilon^\tau\} \bigcap \{W^u(p_\epsilon^\tau) - p_\epsilon^\tau\} \neq \emptyset$. Fig. 1 (originally appearing in [5, p. 208]) shows $W^s(p_\epsilon^\tau)$ and $W^u(p_\epsilon^\tau)$ for a particular

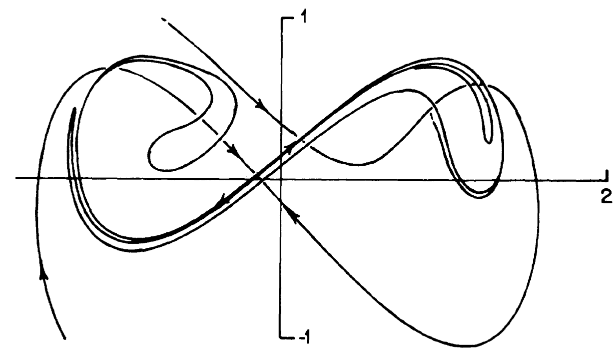

Fig. 1. This figure originally appeared in [5, p. 208] and was computed numerically by Y. Ueda. It shows the stable and unstable manifolds of the first return map of a cross section of the Duffing equation for parameter values $\epsilon\delta = 0.25$ and $\epsilon\gamma = 0.30$. Note that the stable and unstable manifolds have nonempty, transverse intersection.

choice of τ, $\epsilon > 0$ sufficiently small, and $\frac{\gamma}{\delta}$ sufficiently large such that the intersection of $\{W^s(p_\epsilon^\tau) - p_\epsilon^\tau\}$ and $\{W^u(p_\epsilon^\tau) - p_\epsilon^\tau\}$ is nonempty and transverse. As $\tau \in \mathbb{S}^1$ was arbitrary, it follows that $W^s(\Gamma_\epsilon)$ and $W^u(\Gamma_\epsilon)$ are transverse and $\{W^s(\Gamma_\epsilon) - \Gamma_\epsilon\} \bigcap \{W^u(\Gamma_\epsilon) - \Gamma_\epsilon\} \neq \emptyset$. We fix $\epsilon, \gamma, \delta > 0$ to preserve those properties.

However, the vector field (2) is over $\mathbb{R}^2 \times \mathbb{S}^1$, whereas [1, Lemma 3-5] is stated for a vector field over Euclidean space. So, we will modify the example to obtain a related vector field over Euclidean space. To do so, it is natural to embed $\mathbb{R}^2 \times \mathbb{S}^1$ into $\mathbb{R}^3$ as an open full torus (ie. a full torus that does not contain its boundary), push the vector field forward along this embedding, and then extend it to a smooth vector field over all of $\mathbb{R}^3$. To ensure that the desired vector field extension exists, and that the critical elements of V, along with their stable and unstable manifolds, remain unchanged after the extension, we will multiply V by a scalar function h which ensures that the product hV rapidly converges to zero as the boundary of the open full torus is approached. Then the vector field on the open full torus will be trivially extended to $\mathbb{R}^3$ by defining it to be identically zero on the complement of the open full torus. We will see that multiplication by the scalar function h does not affect the critical elements or their stable and unstable manifolds, so the nonempty, transversal intersection of $W^s(\Gamma_\epsilon)$ with $W^u(\Gamma_\epsilon)$ will be preserved.

We begin by defining the embedding from $\mathbb{R}^2 \times \mathbb{S}^1$ into $\mathbb{R}^3$ whose image is an open full torus. Towards that end, we will use standard cylindrical coordinates on $\mathbb{R}^3$, so let $x := [r\ \theta\ z]^T \in \mathbb{R}^3$. In this setting, the θ coordinate can be related to $\mathbb{S}^1$, and for each θ the image of $\mathbb{R}^2 \times \{\theta\}$ will be an open ball of radius one, which is the cross section of the open full torus at angle θ. In particular, the second coordinate of $(u, v) \in \mathbb{R}^2$ transforms to the length coordinate z of the cylinder, and the first coordinate of $(u, v) \in \mathbb{R}^2$ is shifted by the value 2 (so it is always positive) to become the r coordinate. This defines the interior of an open full torus centered at the origin in $\mathbb{R}^3$ using cylindrical coordinates, and allows us to embed $\mathbb{R}^2 \times \mathbb{S}^1$ into $\mathbb{R}^3$ as desired. More concretely, this embedding is accomplished by the following smooth change of coordinates:

$$r = \frac{u}{\sqrt{1+u^2+v^2}} + 2 \quad (3)$$

$$\theta = \tau \quad (4)$$

$$z = \frac{v}{\sqrt{1+u^2+v^2}}. \quad (5)$$

Note that $(u,v) \to \left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}\right)$ is a diffeomorphism that maps $\mathbb{R}^2$ onto the open ball of radius one [7, Example 2.14]. Thus, the image of the above embedding of $\mathbb{R}^2 \times \mathbb{S}^1$ is in fact the open full torus in $\mathbb{R}^3$, and the boundary of the open full torus in $\mathbb{R}^3$ is approached as $(u,v) \to \infty$ in $\mathbb{R}^2$. Note that proximity to the boundary of the open full torus is independent of the coordinate $\tau \in \mathbb{S}^1$.

Next we construct the scalar function h such that $hV \to 0$ as the boundary of the open full torus is approached. For the embedding of (3)-(5), the boundary of the open full torus is approached as $(u,v) \to \infty$ in $\mathbb{R}^2$. Therefore, we select a function $h : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}$ which is positive and such that for any sequence $\{w_k\}_{k=1}^\infty = \{(u_k, v_k, \tau_k)\}_{k=1}^\infty \subset \mathbb{R}^2 \times \mathbb{S}^1$, $\lim_{k\to\infty} h(w_k)V(w_k) = 0$ and $\lim_{k\to\infty} \frac{d^m h(w)V(w)}{dw^m}|_{w=w_k} = 0$ for all positive integers m. One such function is given by

$$h(w) = h(u,v,\tau) = \frac{1}{1+(u^2+v^2)^2} = \frac{1}{1+||(u,v)||_2^4}.$$

Multiplying (2) by h yields the following vector field on $\mathbb{R}^2 \times \mathbb{S}^1$:

$$\tilde{V}(w) = h(w)V(w). \quad (6)$$

Since h is a positive scalar function, by [8, Proposition 1.2.2] and its proof the flow of $\tilde{V}$ is a time change of the flow of V. This means that the orbits of $\tilde{V}$ are precisely the orbits of V. The intuition is that h represents a time rescaling of V and affects the rate at which the orbits of V are traversed, but does not change the orbits themselves. In particular, the critical elements and their stable and unstable manifolds under $\tilde{V}$ are precisely the same as the critical elements and their stable and unstable manifolds under V. Thus, Γ_ϵ is a hyperbolic periodic orbit under $\tilde{V}$, $W^s(\Gamma_\epsilon)$ and $W^u(\Gamma_\epsilon)$ are transverse, and $\{W^s(\Gamma_\epsilon) - \Gamma_\epsilon\} \bigcap \{W^u(\Gamma_\epsilon) - \Gamma_\epsilon\} \neq \emptyset$.

Finally, we push $\tilde{V}$, given by (6), forward along the embedding of (3)-(5), recalling that the image of this embedding is the open full torus, and define it to be identically zero on the complement of the open full torus. More explicitly, let $H : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^3$ send (u,v,τ) to (r,θ,z) by (3)-(5), and let $\mathbb{T}^2$ be the image of H. Then we define the vector field $\hat{V}$ for $x \in \mathbb{R}^3$ by

$$\hat{V}(x) = \begin{cases} dH_{H^{-1}(x)}(\tilde{V}(H^{-1}(x))) & x \in \mathbb{T}^2 \\ 0 & x \notin \mathbb{T}^2 \end{cases}.$$

For any sequence $\{x_k\}_{k=1}^\infty \subset \mathbb{T}^2$ with $x_k \to \partial\mathbb{T}^2$, by definition of H the sequence $\{H^{-1}(x_k)\}_{k=1}^\infty =: \{(u_k, v_k, \tau_k)\}_{k=1}^\infty$ satisfies $(u_k, v_k) \to \infty$. By the choice of h above, $\lim_{k\to\infty} \tilde{V}(H^{-1}(x_k)) = 0$ and $\lim_{k\to\infty} \frac{d^m \tilde{V}(w)}{dw^m}|_{w=H^{-1}(x_k)} = 0$ for all positive integers m. As dH is smooth and $dH_{(u,v,\tau)}$ (along with all higher order derivatives) is bounded as $(u,v) \to \infty$, this implies that $\lim_{k\to\infty} \hat{V}(x_k) = 0$ and $\lim_{k\to\infty} \frac{d^m \hat{V}(x)}{dx^m}|_{x=x_k} = 0$ for all positive integers m. Thus, as $\hat{V}$ is defined piecewise by two smooth vector fields which agree (including all derivatives) on their boundary of definition $\partial\mathbb{T}^2$, it is a smooth vector field over all of $\mathbb{R}^3$. As $\hat{V}|_{\mathbb{T}^2}$ is the pushforward of $\tilde{V}$ by the smooth embedding H, and H is a diffeomorphism onto its image, the flow of $\tilde{V}$ on $\mathbb{R}^2 \times \mathbb{S}^1$ is conjugate by a C^∞ diffeomorphism (namely, H with its codomain restricted to $\mathbb{T}^2$) to the flow of $\hat{V}$ on $\mathbb{T}^2$ [7, Corollary 9.14]. Now the flows of $\hat{V}|_{\mathbb{T}^2}$ and $\tilde{V}$ are smoothly conjugate, $\hat{V}$ is identically zero on the complement of $\mathbb{T}^2$, and smooth conjugacies preserve critical elements, hyperbolicity, and transversal intersection of stable and unstable manifolds. Therefore, $\Gamma := H(\Gamma_\epsilon)$ is a hyperbolic periodic orbit of $\hat{V}$ such that $W^s(\Gamma)$ and $W^u(\Gamma)$ are transverse and $\{W^s(\Gamma) - \Gamma\} \bigcap \{W^u(\Gamma) - \Gamma\} \neq \emptyset$. As $\hat{V}$ is a vector field over $\mathbb{R}^3$, the setting of Lemma 3-5, it is therefore a counterexample to [1, Lemma 3-5].

IV. Conclusion

A counterexample to a key lemma [1, Lemma 3-5] used in the proof of a highly regarded classical theorem [1, Theorem 4.2] is presented. A consequence of the lemma is that, for any C^1 vector field over Euclidean space, it is not possible for a hyperbolic periodic orbit to have nonempty, transverse intersection of its stable and unstable manifolds. However, a C^∞ vector field over Euclidean space is constructed which possesses a hyperbolic periodic orbit whose stable and unstable manifolds do have nonempty, transverse intersection, thereby disproving the lemma. Under the setting of [1, Theorem 4.2], in which additional assumptions are made, it may be possible that the conclusion of [1, Lemma 3-5] is correct when restricted to critical elements in the RoA boundary. However, this is not immediate, and may not hold true even in this setting. The proof of [1, Theorem 4.2] has been completed under the slightly stronger assumption that the intersection of the nonwandering set with the RoA boundary consists of a finite union of critical elements [4].

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