Single-Sample Direction-of-Arrival Estimation by Hankel-matrix Decompositions

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***Abstract*—Modern networked robotic platforms operating autonomously on the ground, in the air, or in space over high-frequency bands (e.g., mm-wave or future THz) require rapid and effective estimation of the direction of arrival (DoA) of signals of interest to maintain high data rate connectivity with each other and avoid interference from external in-band sources. High robotic platform mobility limits -or completely negates- our ability to wait and collect the necessary statistically stationary sequence of antenna-array-front measurements. As a result, conventional statistical DoA estimation optimization methods may not be applicable. In this paper, we present for the first time in the literature a single-sample DoA estimation algorithm based on Hankel-matrix-representation and singular-value decomposition (SVD) of the individual antenna-array snapshot. We compare the newly proposed estimator against the Maximum Likelihood (ML) single-sample estimator of the DoA of a signal observed in white Gaussian noise and -arguably surprisingly- demonstrate significant superiority in each metric of interest, such as mean-square estimation error, bias, and variance.**

***Index Terms*—Cramér–Rao bound (CRB), direction of arrival (DoA) estimation, Hankel matrices, maximum likelihood estimation, sensor arrays, small sample support.**

I. Introduction

The advent of networked autonomous platforms communicating over high-frequency bands in diverse environments (ground, air, and space) brings renewed interest and a new twist to the problem of real-time accurate localization and tracking by effective Direction of Arrival (DoA) estimation of signals of interest in the presence of noise [1], [2]. Historically, DoA estimation of signals impinging on antenna arrays is carried out by statistical optimization methods that can be broadly categorized into Maximum-Likelihood (ML) techniques (for example, [3], [4]) and vector subspace analysis techniques with subspaces drawn from the estimated space-domain signal autocorrelation matrix (for example, [5]–[8].) Achievable estimation accuracy has been established only asymptotically in the number of antenna array samples, the number of antenna elements, and the signal-to-noise ratio (SNR) [4]. Non-asymptotic (finite number of data samples, finite number of elements, finite SNR value) performance is yet generally unknown.

Modern high-mobility robotic platform applications may operate in environments of severely limited statistical coherence time. In this paper, we consider specifically the extreme case of attempting to carry out signal direction-of-arrival estimation in white Gaussian noise from one -just one- antenna array sample. For this purpose, we propose to harness the power of leading advancements in linear algebra. In particular, represent our single antenna-array sample in the form of a Hankel matrix and execute standard Singular-Value Decomposition (SVD) as pursued in conventional Singular-Spectrum-Analysis (SSA) literature [9] and other similar approaches [10] aiming to separate signal components from present noise. The calculated principal singular-vector component(s) constitutes our "filtered" single data point representation. Then, array-response-vector matched-filtering (MF) of the SVD-filtered data point and energy scanning over the angle-of-arrival horizon returns the estimate of the angle of arrival.

We compared numerically the proposed single-sample DoA estimator against the well-researched ML single-sample estimator under conditions of varying number of antenna elements and SNR values and we benchmarked against the known Cramér–Rao lower bound (CRB) on the variance of all estimators [4] (the ML estimator is asymptotically unbiased and asymptotically efficient.) Extensive studies demonstrated that the newly developed SVD-filtering single-sample DoA estimator has at all times lower bias, lower MS error and lower variance than the ML estimator. In the case of small and moderate antenna-array sizes and SNR values, the performance gains are -arguably unexpectedly- very significant.

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II. Signal Model and Background

For clarity and simplicity in our presentation, we consider a uniform linear antenna array (ULA) of M elements with array response vector

$$\mathbf{s}(\theta) \triangleq \left[1, e^{-j2\pi \frac{d}{\lambda}\sin\theta}, \cdots, e^{-j2\pi (M-1)\frac{d}{\lambda}\sin\theta}\right]^T, \quad \theta \in [-90^\circ, 90^\circ) \tag{1}$$

where d is the inter-element spacing, λ is the received-signal wavelength, θ represents the incidence angle with respect to broadside, and T denotes the transpose operator. In this work, we consider only the fundamental case where we observe one narrowband signal transmission from the far field in the presence of additive white Gaussian noise (AWGN) that is independent and identically distributed (i.i.d.) across the antenna elements. If we denote the baseband received antenna-array data vector by $\mathbf{x} \in \mathbb{C}^M$, then

$$\mathbf{x} = A\mathbf{s}(\theta) + \mathbf{n} \tag{2}$$

where $A > 0$ is the received signal amplitude that we treat as deterministic and $\mathbf{n} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2 \mathbf{I}_{\mathrm{M}}\right)$ is a complex white Gaussian vector with i.i.d. real and imaginary parts.

In this paper, we are interested in the practical case when we have available just *one* sample $\mathbf{x} \in \mathbb{C}^M$ from which we want to estimate the signal angle of arrival θ. In the single-sample case under the model in (2), it is straightforward to calculate [11] that the ML estimator of θ is independent of A and of the form

$$\hat{\theta}_{ML} = \arg \max_{\phi \in [-90^0, 90^\circ)} \left|\mathbf{s}(\phi)^H \mathbf{x}\right|^2. \tag{3}$$

Using the general Cramér–Rao lower bound (CRB) expression of Stoica and Nehorai [4] for the variance of estimators of $\omega = 2\pi \frac{d}{\lambda} sin\theta$, we can calculate that for single-sample estimation and Nyquist inter-element spacing $d = \frac{\lambda}{2}$ the CRB of estimators of $sin\theta$, say $\widehat{sin\theta}$, is

$$CRB(\widehat{sin\theta}) = \frac{6}{\mathrm{SNR} M(M^2-1)\pi^2} \tag{4}$$

where $\mathrm{SNR} = \frac{|A|^2}{\sigma^2}$. Evidently, as SNR or M increases to infinity the CRB decreases to zero at the rate of $\frac{1}{\mathrm{SNR}}$ and the much faster rate of $\frac{1}{M^3}$, correspondingly. This establishes the asymptotic consistency in M of the ML estimator in (3) for the signal model in (2) [12].

III. Proposed Single-Sample DOA Estimation Method

Asymptotic performance characteristics of estimators (ML DoA estimators, in our case) can be both illuminating and misleading when it comes down to field deployment [13]. For the purposes of this present work, we are interested in the development of DoA estimators that operate effectively with very few data samples (one actually, herein), small number of antenna elements, and low SNR. To accomplish this objective, we turn to modern advances in linear algebra and exploit their power.

Consider a single received data sample from a ULA $\mathbf{x} = [x_1, x_2, \cdots, x_M]^T$ and let $\Re\{\cdot\}$, $\Im\{\cdot\}$ denote the real and imaginary-part extractor operator, correspondingly. For a "window" parameter D, $2 \leq D < M$, we construct the side-by-side block Hankel matrix[1] $\mathbf{X} \in \mathbb{R}^{D \times 2W}$ in (5) (top of the next page) where $W \triangleq M - D + 1$.

Subsequently, we carry out rank k, $1 \leq k < min\{D, 2W\}$, decomposition of $\mathbf{X}$ by executing standard SVD to produce its corresponding low-rank representation as

$$\mathbf{Y} = \mathbf{U}_{D \times k} \mathbf{\Sigma}_{k \times k} \mathbf{V}_{k \times 2W}^T. \tag{6}$$

For k less than the full rank of $\mathbf{X}$, it is worth noting that $\mathbf{Y}$ in (6) is not in general a side-by-side block Hankel matrix anymore. We next convert $\mathbf{Y}$ to the closest in the Frobenius norm sense side-by-side block Hankel matrix

$$\tilde{\mathbf{Y}}_{D \times 2W} = [\mathcal{H}_{mean}(\mathbf{Y}_{\Re})_{D \times W}, \mathcal{H}_{mean}(\mathbf{Y}_{\Im})_{D \times W}] \tag{7}$$

where $\mathcal{H}_{mean}(\cdot)$ denotes the matrix operator that replaces all anti-diagonals entries of a matrix by their corresponding mean value, $\mathbf{Y}_{\Re} \triangleq [\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_W]$, and $\mathbf{Y}_{\Im} \triangleq [\mathbf{y}_{W+1}, \mathbf{y}_{W+2}, \cdots, \mathbf{y}_{2W}]$. Thereafter, we directly extract the filtered antenna-array sample $\tilde{\mathbf{y}} \in \mathbb{C}^M$ from $\tilde{\mathbf{Y}}$ as the first-column, last-row readout from each constituent Hankel matrix as seen below ($j \triangleq \sqrt{-1}$),

$$\tilde{\mathbf{y}} = [\tilde{y}_{1,1} + j\tilde{y}_{1,W+1}, \tilde{y}_{2,1} + j\tilde{y}_{2,W+1}, \cdots, \tilde{y}_{D,1} + j\tilde{y}_{D,W+1}, \tilde{y}_{D,2} + j\tilde{y}_{D,W+2}, \cdots, \tilde{y}_{D,W} + j\tilde{y}_{D,2W}]^T \in \mathbb{C}^M. \tag{8}$$

Finally, we perform single-sample DoA estimation by carrying out response-vector matched-filter energy scanning on the SVD-filtered data point $\tilde{\mathbf{y}}$ over the $[-90^\circ, 90^\circ)$ continuum,

$$\hat{\theta}_{SVD} = \arg \max_{\phi \in [-90^0, 90^\circ)} \left|\mathbf{s}(\phi)^H \tilde{\mathbf{y}}\right|^2. \tag{9}$$

The proposed algorithm for single-sample DoA estimation is summarized in Fig. 1.

Proposed Algorithm: Side-by-Side Block Hankel SVD Method

Input: Single received antenna array sample $\mathbf{x} \in \mathbb{C}^M$.

1: Form side-by-side block Hankel matrix $\mathbf{X} \in \mathbb{R}^{D \times 2(M-D+1)}$ by (5) for chosen "window" parameter $D \in \{2, \cdots, M-1\}$.

2: For rank k, $1 \leq k < min\{D, 2(M-D+1)\}$, decompose $\mathbf{X}$ to $\mathbf{Y} = \mathbf{U}_{D \times k} \mathbf{\Sigma}_{k \times k} \mathbf{V}_{k \times 2W}^T$ by (6).

3: Transform $\mathbf{Y}$ to side-by-side block Hankel matrix $\tilde{\mathbf{Y}}$ by (7).

4: Extract $\tilde{\mathbf{y}}$ from $\tilde{\mathbf{Y}}$ by (8).

5: Estimate $\hat{\theta}$ by (9).

Output: $\hat{\theta}$.

Fig. 1: Summary of proposed single-sample DoA estimator based on SVD decomposition of side-by-side block Hankel matrix.

[1] A matrix is called Hankel if each anti-diagonal has elements of constant value [14].

$$\mathbf{X}_{D\times 2W} = \left[\mathbf{X}_{\Re} \triangleq \begin{bmatrix} \Re\{x_1\} & \Re\{x_2\} & \cdots & \Re\{x_W\} \\ \Re\{x_2\} & \Re\{x_3\} & \dots & \Re\{x_{W+1}\} \\ \Re\{x_3\} & \Re\{x_4\} & \cdots & \Re\{x_{W+2}\} \\ \vdots & \vdots & \ddots & \vdots \\ \Re\{x_D\} & \Re\{x_{D+1}\} & \cdots & \Re\{x_M\} \end{bmatrix} \mathbf{X}_{\Im} \triangleq \begin{bmatrix} \Im\{x_1\} & \Im\{x_2\} & \cdots & \Im\{x_W\} \\ \Im\{x_2\} & \Im\{x_3\} & \cdots & \Im\{x_{W+1}\} \\ \Im\{x_3\} & \Im\{x_4\} & \cdots & \Im\{x_{W+2}\} \\ \vdots & \vdots & \ddots & \vdots \\ \Im\{x_D\} & \Im\{x_{D+1}\} & \cdots & \Im\{x_M\} \end{bmatrix} \right]. \quad (5)$$

IV. Numerical Studies

In this section, we examine and illustrate via extensive numerical studies and comparisons the performance of the newly proposed single-sample DoA estimator against the ML estimator for the signal model in (2) with Nyquist inter-element distance set to half of the signal wavelength ($d = \frac{\lambda}{2}$.) In all numerical studies presented herein, the window and low-rank parameters D and k of the algorithm (see Fig. 1) are genie-assisted chosen on a sample-by-sample basis. We recall that subspace-type DoA estimators for the single-sample case are not defined.

First, we investigate the behavior of the two estimators in conjunction with the CRB for estimators of $sin\theta$ given in (4). In Fig. 2, we plot the mean-square error of the estimate $\hat{sin\theta}$ as the number of antenna elements M varies from 4 to 32 for SNR values -2 dB, 0 dB, and 2dB (Fig. 2(a), 2(b), and 2(c), respectively.) We observe the significant superiority of the proposed SVD-filtering method over ML; see for example, the system case scenario $M = 16$ or 32 antenna elements and SNR$=$ 0dB in Fig 2(b). We also observe that the MSE of the proposed estimator drops significantly below the Cramér–Rao bound, which signals that the large gains in MSE come at the expense of being biased, as anticipated.

How much biased the proposed estimator is and how its bias compares numerically to the bias of the ML estimator for the studied finite M values is examined in Fig. 3 directly in the context of the squared bias metric. Fig. 3 repeats the study of Fig. 2 and plots instead $E\{\hat{\theta}-\theta\}^2$. The significant superiority of the proposed SVD-filtering scheme over ML is arguably surprising. Finally, for completeness purposes, in Fig. 4 we plot the mean-square error $E\{(\hat{\theta}-\theta)^2\}$ as M varies from 4 to 32 for SNR values -2 dB, 0 dB, and 2dB in parts 4(a), 4(b), and 4(c), respectively.

V. Conclusion

In this paper, we considered the fundamental problem of direction-of-arrival estimation of one signal in white Gaussian noise based on *a single sample* collected by a narrowband antenna array as it is often required in modern, highly mobile, networked robotic platforms that operate autonomously in diverse environments. We presented a novel, simple and practical one-sample DoA estimator based on side-by-side block Hankel matrix singular-vector decompositions and numerically compared its performance against the Maximum-Likelihood (ML) one-sample estimator in studies of varying-size uniform-linear-antenna arrays and varying SNR values. We observed below Cramér–Rao lower bound (CRB) mean-square error performance of the proposed estimator and very significant gains in both mean-square error and bias over ML estimation.

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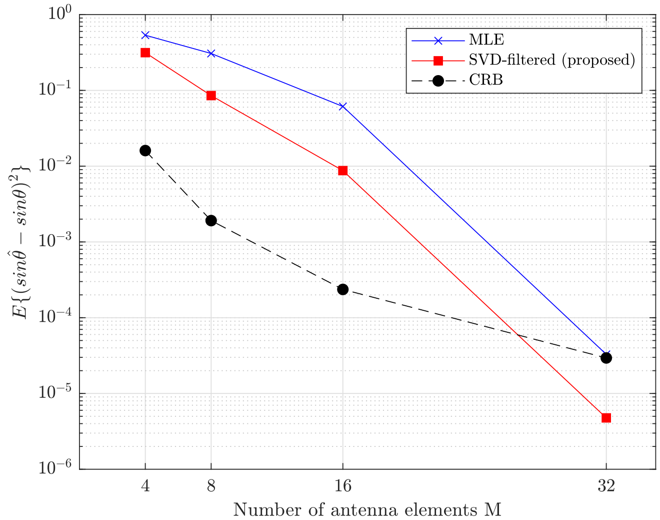
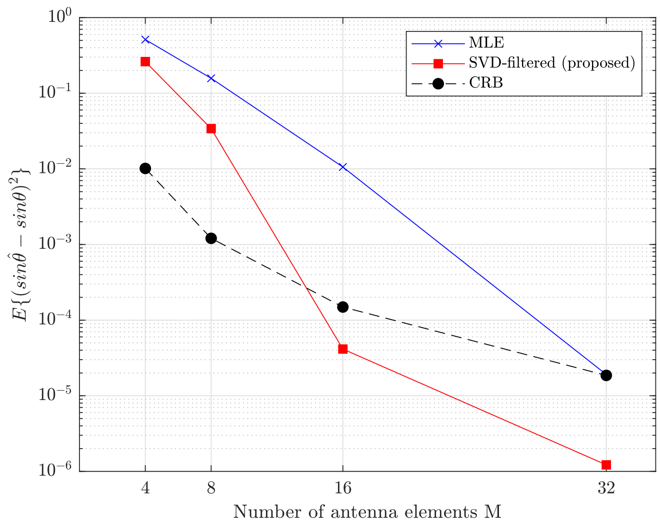
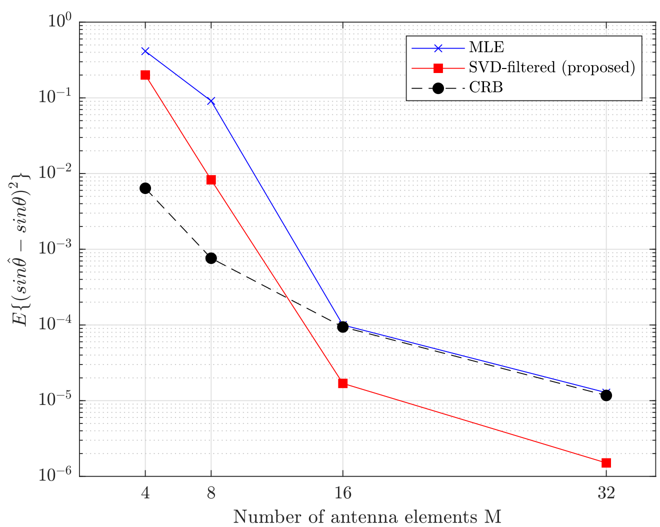


(a)

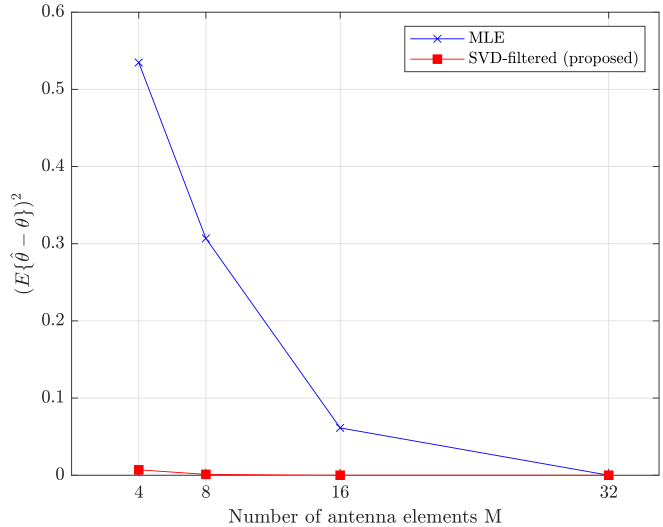


(b)

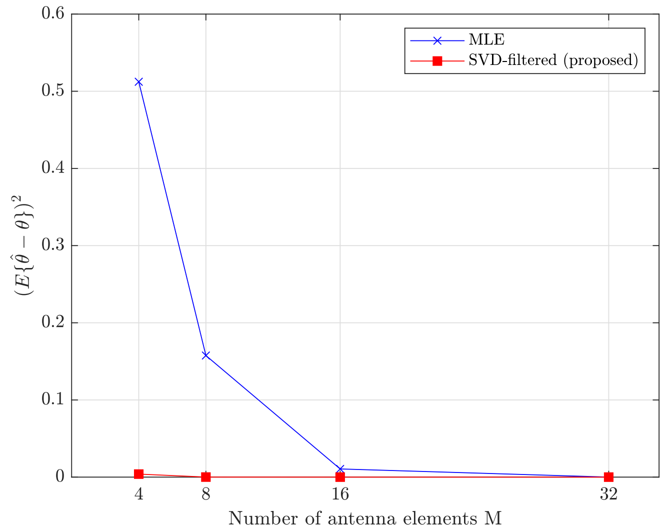


(c)

Fig. 2: Mean-square error of $sin\hat{\theta}$ estimates versus number of antenna elements: (a) SNR = -2 dB, (b) SNR = 0 dB, and (c) SNR = 2 dB.



(a)



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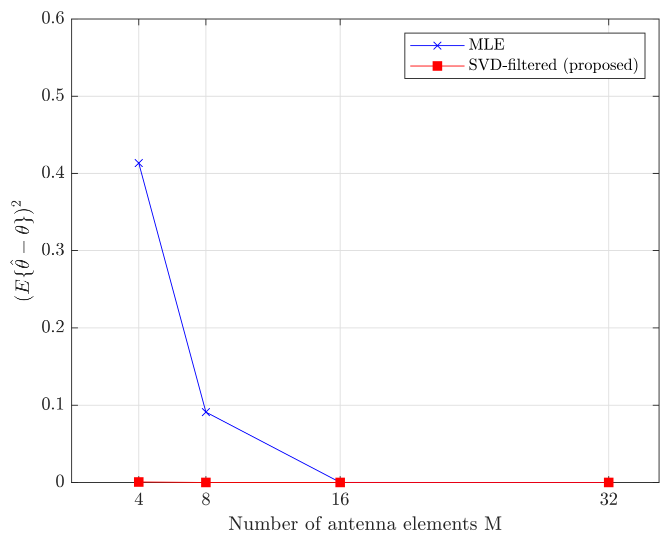

(c)

Fig. 3: Squared bias versus number of antenna elements: (a) SNR = -2 dB, (b) SNR = 0 dB, and (c) SNR = 2 dB.

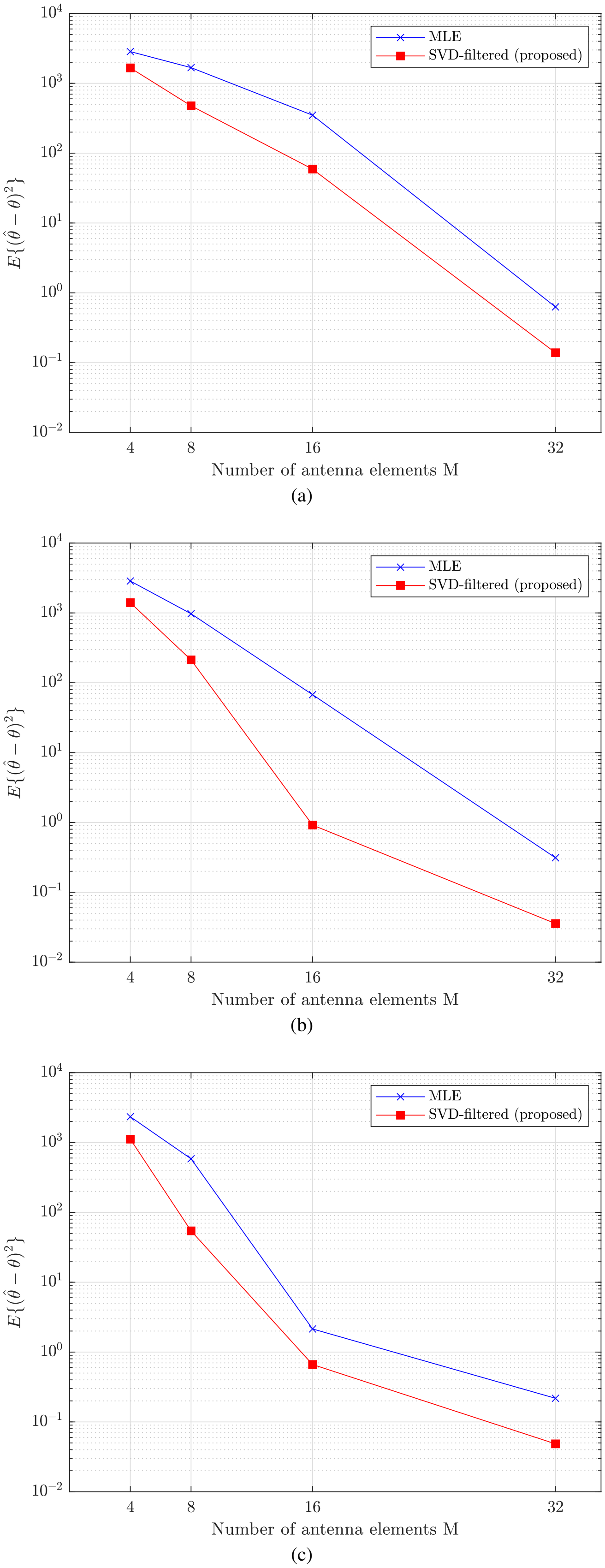

Fig. 4: Mean-square error of $\hat{\theta}$ estimates versus number of antenna elements: (a) SNR = -2 dB, (b) SNR = 0 dB, and (c) SNR = 2 dB.