Research Article

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Stability and the inverse gravimetry problem with minimal data

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Abstract: The inverse problem in gravimetry is to find a domain D inside the reference domain Ω from boundary measurements of gravitational force outside Ω. We found that about five parameters of the unknown D can be stably determined given data noise in practical situations. An ellipse is uniquely determined by five parameters. We prove uniqueness and stability of recovering an ellipse for the inverse problem from minimal amount of data which are the gravitational force at three boundary points. In the proofs, we derive and use simple systems of linear and nonlinear algebraic equations for natural parameters of an ellipse. To illustrate the technique, we use these equations in numerical examples with various location of measurements points on $\partial\Omega$. Similarly, a rectangular D is considered. We consider the problem in the plane as a model for the three-dimensional problem due to simplicity.

Keywords: Inverse problems, gravimetry, numerical solution

MSC 2010: 31A25, 35R30, 86A22

Dedicated to Prof. M. Klibanov on his 70th birthday

1 Introduction

One of the classical inverse problems, the one of gravimetry, is to find a mass density distribution μ in a domain Ω from the gravity force induced by this mass outside Ω. This problem has a long history, a rather complete theory and important applications to geophysics. One of the interesting applications is locating underground cavities/water lakes from exterior gravitational measurements. Moreover, inverse gravimetry can serve as a model for some other important inverse problems, like magnetoencephalography [1] and the linearized inverse conductivity problem [9]. Mathematical features of this inverse problem are a substantial non-uniqueness and exponential instability [5, 7, 9]. To regain uniqueness and still be useful in applications, one assumes that $\mu = f\chi_D$, $D \subset \Omega$, where f is a known constant and χ_D is the indicator function of D. Even so, for uniqueness, one needs additional geometrical assumptions on D, like star shapedness or convexity with respect to some direction. In practical situations, $\mu = f^b + f\chi_D$, where f^b is the (background) density of Ω which is not known but can be assumed to be relatively small. Contemporary gradiometers can measure gravitational force with a very high precision (up to six digits), but the contribution of unknown f^b dramatically reduces this precision by contaminating the gravity force. As a result, in practical situations, one can rigorously recover only few parameters of D. Use of powerful and sophisticated numerical algorithms cannot change the unstable nature of the problem and substantially increase the resolution [11]. An exponential instability is also a feature of inverse source and obstacle problems for more general elliptic equations [4, 6],

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so one expects a robust recovery of only few parameters of unknown objects. One of the ways to increase a number of such parameters and hence the numerical resolution is to use prospecting by stationary waves of higher wave numbers (to consider inverse problems for the Helmholtz type equations). For a recent analytic and numerical progress in this direction, we refer to [2, 8, 10, 12, 13].

In this paper, we consider a simpler two-dimensional case when one can use powerful methods of theory of one complex variable. Let D be a domain inside a disk $B_\rho = \{x : |x| < \rho\}$, $\rho < 1$, and the unit disk B_1 is contained in a domain Ω. In Section 2, we analyze stability of recovery of general $D \subset B_{0.5}$ by using a standard expansion of gradient of its external logarithmic potential into power series and deriving bounds which show that only the first three terms of this expansion can be found in a stable way. In Section 3, we derive simple algebraic equations for natural parameters of an ellipse D in terms of the coefficients of three expansion terms and use these equations to demonstrate uniqueness, Lipschitz stability and some existence results. Similarly, we handle rectangles D, with weaker results. In Section 4, by using the same algebraic equations, we support analytic results with rigorous numerical examples.

Let $x = (x_1, x_2) = x_1 + ix_2$, $\bar{x} = x_1 - ix_2$, $\frac{\partial}{\partial x} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$, $\frac{\partial}{\partial \bar{x}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$. We will consider x simultaneously as two-dimensional vectors and as elements of the field of complex numbers. In the rest of the paper, we let $f = 1$.

2 Stability and approximation

As known, the logarithmic potential of a domain D is

$$u_D(x) = \frac{1}{2\pi}\int_D \log|x-y|\,dy.$$

We have

$$u_D(x) = \frac{1}{4\pi}\int_D [\log(x-y) + \log(\bar{x}-\bar{y})]\,dy;$$

therefore,

$$\frac{\partial u_D(x)}{\partial \bar{x}} = \frac{1}{4\pi}\int_D \frac{1}{\bar{x}-\bar{y}}\,dy = \frac{1}{4\pi\bar{x}}\int_D \sum_{n=0}^{\infty}\Big(\frac{\bar{y}}{\bar{x}}\Big)^n\,dy,$$

and hence

$$\nabla u_D(x) = 2\sum_{n=0}^{\infty}\bar{A}_n\bar{x}^{-n-1}, \quad |x| = 1, \quad \text{where } A_n = \frac{1}{4\pi}\int_D y^n\,dy. \tag{2.1}$$

Denote

$$\nabla u_D(;N) = \frac{1}{2\pi}\sum_{n=N+1}^{\infty}\bar{x}^{-n-1}\int_D \bar{y}^n\,dy.$$

Lemma 2.1. *Let $D \subset B_\rho$. Then*

$$\|\nabla u_D(x;N)\|_\infty(\partial B_1) \le \frac{\rho^{N+3}}{(N+3)(1-\rho)}, \tag{2.2}$$

$$\|\nabla u_D(;N)\|_2^2(\partial B_1) \le \frac{2\pi\rho^{2N+6}}{(N+3)^2(1-\rho^2)}. \tag{2.3}$$

Proof. We have

$$|\nabla u_D(x;N)| \le \frac{1}{2\pi}\sum_{n=N+1}^{\infty}|\bar{x}|^{-n-1}\int_D |\bar{y}|^n\,dy.$$

Since $|x| = 1$ and $D \subset B_\rho$, we have

$$|\nabla u_D(;N)| \le \frac{1}{2\pi}\sum_{n=N+1}^{\infty}\int_0^{2\pi}\int_0^{\rho} r^{n+1}\,dr\,d\theta = \sum_{n=N+1}^{\infty}\frac{\rho^{n+2}}{n+2} \le \frac{\rho^{N+3}}{N+3}\sum_{n=0}^{\infty}\rho^n = \frac{\rho^{N+3}}{(N+3)(1-\rho)},$$

which proves (2.2).

Similarly, using the orthogonality of the trigonometric system and (2.1) yields

$$\|\nabla u_D(;N)\|_2^2(\partial B_1) = 4\int_0^{2\pi}\Big|\sum_{n=N+1}^{\infty}\bar{A}_n e^{i(n+1)\theta}\Big|^2 d\theta = 4\int_0^{2\pi}\sum_{n=N+1}^{\infty}|A_n|^2\,d\theta = \frac{1}{2\pi}\sum_{n=N+1}^{\infty}\Big|\int_D y^n\,dy\Big|^2$$
$$\leq \frac{1}{2\pi}\sum_{n=N+1}^{\infty}(2\pi)^2\frac{\rho^{2(n+2)}}{(n+2)^2} \leq 2\pi\frac{\rho^{2(N+3)}}{(N+3)^2}\sum_{n=0}^{\infty}\rho^{2n} = \frac{2\pi\rho^{2(N+3)}}{(N+3)^2(1-\rho^2)},$$

as in the above argument, which proves (2.3). □

Corollary 2.2. *Let ∇u^b be the gradient of the logarithmic potential of the disk B_1 of the constant density δ and $D \subset B_\rho$, $\rho < 1$. Then*

$$\frac{\|\nabla u_D(;N)\|_\infty(\partial B_1)}{\|\nabla u^b\|_\infty(\partial B_1)} \leq \frac{2\rho^{N+3}}{(N+3)\delta(1-\rho)}, \tag{2.4}$$

$$\frac{\|\nabla u_D(;N)\|_2(\partial B_1)}{\|\nabla u^b\|_2(\partial B_1)} \leq \frac{2\rho^{N+3}}{(N+3)\delta\sqrt{1-\rho^2}}. \tag{2.5}$$

As known [7], $\nabla u^b(x) = \delta\frac{x}{2|x|^2}$, so (2.4) follows from (2.2).

As above,

$$\|\nabla u^b\|_2^2(\partial B_1) = \int_0^{2\pi}\frac{\delta^2}{4}\,d\theta = \frac{\pi\delta^2}{2},$$

and (2.5) follows from (2.3).

Despite an extremely high precision of contemporary gradiometers, the omnipresent density f^b of a medium (background) cannot be identified from the exterior data due to a very substantial non-uniqueness in the inverse gravimetry problem [7]. However, background in many cases is known only approximately; it produces a contribution to the gravity force which can be included into an error in the data. The relative error in determination of the background density in a real situation can reach 0.05. If we let $\rho = 0.5$, $\delta = 0.05$ and $N = 2$, then the right-hand side in (2.4) is 0.5 and the right-hand side in (2.5) is $\frac{1}{2\sqrt{3}}$. Therefore, only the terms in (2.1) with $n = 0, 1, 2$ and accordingly the coefficients A_0, A_1, A_2 can be viewed as detectable. Since $0 < A_0$, we are given only five real numbers. Hence we arrive at the following inverse problem.

Inverse gravimetry problem with minimal data. Find a domain D from the gradient of its approximate gravity potential

$$\nabla u_{D,a}(x(j)), \quad j = 1, 2, 3, \tag{2.6}$$

where $x(1), x(2), x(3)$ are three distinct points of $\partial\Omega$ and the approximate gradient is

$$\nabla u_{D,a}(x) = 2A_0\bar{x}^{-1} + 2\bar{A}_1\bar{x}^{-2} + 2\bar{A}_2\bar{x}^{-3}.$$

3 Uniqueness and stability of an ellipse and of a rectangle

Let us write data (2.6) as the system of equations

$$\begin{pmatrix}\nabla u_{D,a}(x(1))\\ \nabla u_{D,a}(x(2))\\ \nabla u_{D,a}(x(3))\end{pmatrix} = \begin{pmatrix}2\bar{x}(1)^{-1} & 2\bar{x}(1)^{-2} & 2\bar{x}(1)^{-3}\\ 2\bar{x}(2)^{-1} & 2\bar{x}(2)^{-2} & 2\bar{x}(2)^{-3}\\ 2\bar{x}(3)^{-1} & 2\bar{x}(3)^{-2} & 2\bar{x}(3)^{-3}\end{pmatrix}\begin{pmatrix}A_0\\ \bar{A}_1\\ \bar{A}_2\end{pmatrix}; \tag{3.1}$$

its determinant is

$$8(\bar{x}(1)\bar{x}(2)\bar{x}(3))^{-3}(\bar{x}(1)-\bar{x}(2))(\bar{x}(1)-\bar{x}(3))(\bar{x}(2)-\bar{x}(3))$$

which is not zero since $x(1), x(2), x(3)$ are distinct, and hence data (2.6) uniquely determine the coefficients A_0, A_1, A_2.

Using (2.1), we have

$$\int_D 1 = 4\pi A_0, \tag{3.2}$$

$$\int_D y_1\,dy = 2\pi(A_1 + \bar{A}_1), \qquad \int_D y_2\,dy = -2i\pi(A_1 - \bar{A}_1), \tag{3.3}$$

$$\int_D (y_1^2 - y_2^2)\,dy = 2\pi(A_2 + \bar{A}_2), \quad \int_D y_1 y_2\,dy = -i\pi(A_2 - \bar{A}_2). \tag{3.4}$$

Let D be the ellipse with the center of gravity (b_1, b_2), the semi-axes a_1, a_2 $(a_2 \le a_1)$ and the angle θ, $0 \le \theta < \pi$, between the greater semi-axis and the x_1-axis. We will call $b = (b_1, b_2)$, $a = (a_1, a_2)$, θ parameters of the ellipse.

Lemma 3.1. *Let D be the ellipse with the parameters $b_1, b_2, a_1, a_2, \theta$ and A_0, A_1, A_2 solve system* (3.1) *for the gradient of the potential of the ellipse. Then*

$$b_1 = \frac{A_1 + \bar{A}_1}{2A_0}, \qquad b_2 = -i\frac{A_1 - \bar{A}_1}{2A_0}, \tag{3.5}$$

$$a_1 a_2 = 4A_0, \qquad a_1^2 - a_2^2 = \frac{4}{A_0^2}|A_0 A_2 - A_1^2|, \tag{3.6}$$

and if in addition $0 < |A_0 A_2 - A_1^2|$, then

$$\cos 2\theta = \frac{(A_2 + \bar{A}_2)A_0 - (A_1^2 + \bar{A}_1^2)}{2|A_0 A_2 - A_1^2|}, \quad \sin 2\theta = \frac{i(-(A_2 - \bar{A}_2)A_0 + A_1^2 - \bar{A}_1^2)}{2|A_0 A_2 - A_1^2|}. \tag{3.7}$$

Moreover, if $b_1, b_2, a_1, a_2, \theta$ satisfy equations (3.5), (3.6), (3.7), *D is the ellipse with these parameters and A_0^*, A_1^*, A_2^* are coefficients obtained by solving system* (3.1), *then $A_0 = A_0^*$, $A_1 = A_1^*$, $A_2 = A_2^*$.*

Proof. Since the area of the ellipse D is $\pi a_1 a_2$, from (3.2), we obtain the first equality in (3.6).

As known,

$$b_1 = \frac{\int_D y_1\,dy}{\int_D 1} = \frac{A_1 + \bar{A}_1}{2A_0}, \quad b_2 = \frac{\int_D y_2\,dy}{\int_D 1} = -i\frac{A_1 - \bar{A}_1}{2A_0}$$

when we use (3.2), (3.3). Now we introduce new orthogonal coordinates

$$Y_1 = (y_1 - b_1)\cos\theta + (y_2 - b_2)\sin\theta, \quad Y_2 = -(y_1 - b_1)\sin\theta + (y_2 - b_2)\cos\theta. \tag{3.8}$$

The point $Y_1 = a_1$, $Y_2 = 0$ corresponds to the point $y_1 = b_1 + a_1\cos\theta$, $y_2 = b_2 + a_1\sin\theta$, and the point $Y_1 = 0$, $Y_2 = a_2$ corresponds to the point $y_1 = b_1 - a_2\sin\theta$, $y_2 = b_2 + a_2\cos\theta$. Hence, in this new coordinate system, the ellipse D becomes the ellipse $D(0)$ with the semi-axes $(0, a_1)$ and $(0, a_2)$. By elementary integration,

$$\int_{D(0)} (Y_1^2 - Y_2^2)\,dY = \frac{\pi}{4}a_1 a_2(a_1^2 - a_2^2), \qquad \int_{D(0)} Y_1 Y_2\,dY = 0 \tag{3.9}$$

due to symmetry reasons. By direct calculations, from (3.8),

$$Y_1^2 - Y_2^2 = ((y_1^2 - y_2^2) + 2b_2 y_2 - 2b_1 y_1 + (b_1^2 - b_2^2))\cos 2\theta + 2(y_1 y_2 - b_1 y_2 - b_2 y_1 + b_1 b_2)\sin 2\theta,$$

so using (3.2), (3.3), (3.4), (3.5), (3.9) yields

$$\begin{aligned}
&\frac{\pi}{4}a_1 a_2(a_1^2 - a_2^2)\\
&\quad = \pi\Big(2(A_2 + \bar{A}_2) - 2\frac{(A_1 + \bar{A}_1)^2}{A_0} - 2\frac{(A_1 - \bar{A}_1)^2}{A_0} + \frac{(A_1 + \bar{A}_1)^2 + (A_1 - \bar{A}_1)^2}{A_0}\Big)\cos 2\theta\\
&\qquad + 2\pi i\Big(-(A_2 - \bar{A}_2) + \frac{A_1 - \bar{A}_1}{A_0}(A_1 + \bar{A}_1) + \frac{A_1 + \bar{A}_1}{A_0}(A_1 - \bar{A}_1) - \frac{(A_1 + \bar{A}_1)(A_1 - \bar{A}_1)}{A_0}\Big)\sin 2\theta\\
&\quad = 2\pi\frac{(A_2 + \bar{A}_2)A_0 - A_1^2 - \bar{A}_1^2}{A_0}\cos 2\theta + 2\pi i\Big(\frac{-(A_2 - \bar{A}_2)A_0 + (A_1^2 - \bar{A}_1^2)}{A_0}\Big)\sin 2\theta
\end{aligned}$$

or

$$a_1^2 - a_2^2 = 2\frac{(A_2 + \bar{A}_2)A_0 - A_1^2 - \bar{A}_1^2}{A_0^2}\cos 2\theta + 2i\Big(\frac{-(A_2 - \bar{A}_2)A_0 + (A_1^2 - \bar{A}_1^2)}{A_0^2}\Big)\sin 2\theta. \tag{3.10}$$

Similarly,

$$Y_1Y_2 = (y_1y_2 - b_2y_1 - b_1y_2 + b_1b_2)\cos 2\theta - \Big(\frac{1}{2}(y_1^2 - y_2^2) - b_1y_1 + b_2y_2 + \frac{1}{2}(b_1^2 - b_2^2)\Big)\sin 2\theta,$$

so using (3.2), (3.3), (3.4), (3.9) yields

$$\begin{aligned}
0 &= \Big(-\pi i(A_2 - \bar{A}_2) + \pi i\frac{A_1 - \bar{A}_1}{A_0}(A_1 + \bar{A}_1) + \pi i\frac{A_1 + \bar{A}_1}{A_0}(A_1 - \bar{A}_1) - \pi i\frac{A_1 + \bar{A}_1}{A_0}(A_1 - \bar{A}_1)\Big)\cos 2\theta \\
&\quad - \Big(\pi(A_2 + \bar{A}_2) - \pi\frac{A_1 + \bar{A}_1}{A_0}(A_1 + \bar{A}_1) - \pi\frac{A_1 - \bar{A}_1}{A_0}(A_1 - \bar{A}_1) + \frac{\pi}{2}\frac{(A_1 + \bar{A}_1)^2 + (A_1 - \bar{A}_1)^2}{A_0}\Big)\sin 2\theta \\
&= \pi i\frac{-(A_2 - \bar{A}_2)A_0 + A_1^2 - \bar{A}_1^2}{A_0}\cos 2\theta - \pi\frac{(A_2 + \bar{A}_2)A_0 - \frac{1}{2}(A_1 + \bar{A}_1)^2 - \frac{1}{2}(A_1 - \bar{A}_1)^2}{A_0}\sin 2\theta
\end{aligned}$$

or

$$0 = ((A_2 + \bar{A}_2)A_0 - (A_1^2 + \bar{A}_1^2))\sin 2\theta - i(-(A_2 - \bar{A}_2)A_0 + A_1^2 - \bar{A}_1^2))\cos 2\theta. \tag{3.11}$$

Letting

$$A = (A_2 + \bar{A}_2)A_0 - (A_1^2 + \bar{A}_1^2), \quad B = i(-(A_2 - \bar{A}_2)A_0 + A_1^2 - \bar{A}_1^2), \tag{3.12}$$

we can write (3.11) as $A\sin 2\theta = B\cos 2\theta$, which implies that $A^2\sin^2 2\theta = B^2\cos^2 2\theta$, and hence

$$A^2 = (A^2 + B^2)\cos^2 2\theta, \quad B^2 = (A^2 + B^2)\sin^2 2\theta. \tag{3.13}$$

By elementary calculations,

$$\begin{aligned}
A^2 + B^2 &= ((A_2 + \bar{A}_2)A_0 - (A_1^2 + \bar{A}_1^2))^2 - (-(A_2 - \bar{A}_2)A_0 + A_1^2 - \bar{A}_1^2)^2 \\
&= 4(A_2\bar{A}_2A_0^2 - A_2A_0\bar{A}_1^2 - \bar{A}_2A_0A_1^2 + A_1^2\bar{A}_1^2) = 4(A_2A_0 - A_1^2)(\bar{A}_2A_0 - \bar{A}_1^2) = 4|A_2A_0 - A_1^2|^2.
\end{aligned}$$

Using (3.10), (3.11) and the condition $0 < a_2 \le a_1$ yields (3.7). Now, from (3.10), (3.12) and (3.13), we obtain the second equality in (3.6).

To demonstrate the second statement, observe that, according to the first part of this lemma, A_0^*, A_1^*, A_2^* satisfy the same equations (3.5), (3.6), (3.7) as A_0, A_1, A_2. Since the solution to system (3.5), (3.6), (3.7) with respect to A_0, A_1, A_2 is unique, we complete the proof. □

We let $a = a_1 + ia_2$, $b = b_1 + ib_2$, $d = \cos\theta + i\sin\theta$.

Corollary 3.2. *Under the conditions of Lemma 3.1,*

$$b = \frac{A_1}{A_0}, \tag{3.14}$$

$$a^2 = \frac{4}{A_0^2}|A_0A_2 - A_1^2| + 8A_0 i, \tag{3.15}$$

and if in addition $0 < |A_0A_2 - A_1^2|$, then the direction d of the longer semi-axis satisfies the equation

$$d^2 = \frac{A_0A_2 - A_1^2}{|A_0A_2 - A_1^2|}. \tag{3.16}$$

Moreover, a, b are Lipschitz continuous with respect to data (2.6) at any $\nabla u_{D,a}$ such that $0 < A_0$, and d is Lipschitz continuous if in addition $0 < |A_0A_2 - A_1^2|$.

Proof. Since $a^2 = a_1^2 - a_2^2 + 2ia_1a_2$, (3.15) follows from (3.6). Similarly, (3.14) follows from (3.5). Also,

$$d^2 = \cos 2\theta + i\sin 2\theta = \frac{(A_2 + \bar{A}_2)A_0 - (A_1^2 + \bar{A}_1^2) - (-(A_2 - \bar{A}_2)A_0 + (A_1^2 - \bar{A}_1^2))}{2|A_0A_2 - A_1^2|}$$

due to (3.7), and we obtain (3.16).

To demonstrate local Lipschitz stability, we observe that, from the linear algebraic system (3.1) (with non-zero determinant), we have Lipschitz dependence of A_0, A_1, A_2 on data (2.6). Lipschitz dependence of a, b, d on A_0, A_1, A_2 follows from the formulae (3.15), (3.14) and (3.16). □

Observe that (3.16) implies that

$$\theta = \frac{1}{2}\arg(A_0A_2 - A_1^2),$$

where $0 \le \arg z < 2\pi$.

Theorem 3.3. *The ellipse $D \subset B_1$ is uniquely determined by data* (2.6).

Proof. Let a_1 and a_2 be the semi-axes of the ellipse D with $(a_2 \le a_1)$, (b_1, b_2) is the center of gravity, and let θ be the angle between a_1 and the x_1-coordinate.

As observed after (3.1), data (2.6) uniquely determine A_0, A_1, A_2. The center of gravity (b_1, b_2) can be uniquely found from equation (3.5). From equations (3.6), it follows that

$$a_1^2 - \frac{16A_0^2}{a_1^2} = \frac{4}{A_0^2}|A_0A_2 - A_1^2|.$$

Since the left-hand side is increasing with respect to positive a_1, we have uniqueness of a_1; then uniqueness of a_2 follows from the first equation in (3.6).

If $0 < |A_0A_2 - A_1^2|$, then, from (3.16), we have the uniqueness for the direction d. If $|A_0A_2 - A_1^2| = 0$, then the second equation in (3.6) implies that $a_1 = a_2$, and so D is a disk which is uniquely determined by its center of gravity and its area πa_1^2. □

Corollary 3.4. *Let*

$$0 < A_0, \quad |A_1| + \sqrt{2|A_0A_2 - A_1^2| + \sqrt{4|A_0A_2 - A_1^2|^2 + 16A_0^6}} < \rho A_0. \tag{3.17}$$

Then there is a unique ellipse $D \subset B_\rho$ with the potential generating the data A_0, A_1, A_2 via (2.6), (3.1).

Proof. Equations (3.5) can be written as $b = \frac{A_1}{A_0}$. Equations (3.6) are equivalent to the equalities

$$a_1 = \frac{\sqrt{2|A_0A_2 - A_1^2| + \sqrt{4|A_0A_2 - A_1^2|^2 + 16A_0^6}}}{A_0}, \quad a_2 = \frac{4A_0}{a_1}.$$

Let D be the ellipse with the parameters a_1, a_2, b and θ, satisfying equations (3.7). Hence, due to the assumptions (3.17), $|b| + a_1 < \rho$, and therefore $D \subset B_\rho$. Using the second statement of Lemma 3.1, we conclude that ∇u_D generates data (2.6). □

Observe that conditions (3.17) are sufficient but not necessary for existence of an ellipse generating the coefficients A_0, A_1, A_2. They are relatively simple and clear. More complicated necessary and sufficient conditions can be derived by requiring that the ellipse $D \subset B_\rho$ by using the same equations. In a simpler situation, when the gradient of the potential is a polynomial $A_0\bar{x}^{-1} + A_1\bar{x}^{-2}$, they are obtained in [7, p. 48]. More results with higher-order polynomials are given in [3].

Now let D be the rectangle with the center of gravity (b_1, b_2), length $2a_1$, width $2a_2$, $a_2 \le a_1$, and the angle θ, $0 \le \theta < \pi$, between the greater side and the x_1-axis.

Lemma 3.5. *Let D be the rectangle with the parameters $b_1, b_2, a_1, a_2, \theta$ and A_0, A_1, A_2. Then*

$$b_1 = \frac{A_1 + \bar{A}_1}{2A_0}, \quad b_2 = -i\frac{A_1 - \bar{A}_1}{2A_0}, \tag{3.18}$$

$$a_1a_2 = \pi A_0, \quad a_1^2 - a_2^2 = \frac{3}{A_0^2}|A_0A_2 - A_1^2|, \tag{3.19}$$

$$\cos 2\theta = \frac{(A_2 + \bar{A}_2)A_0 - (A_1^2 + \bar{A}_1^2)}{2|A_0A_2 - A_1^2|}, \quad \sin 2\theta = \frac{i(-(A_2 - \bar{A}_2)A_0 + A_1^2 - \bar{A}_1^2)}{2|A_0A_2 - A_1^2|}. \tag{3.20}$$

Moreover, if $b_1, b_2, a_1, a_2, \theta$ *satisfy equations* (3.18), (3.19), (3.20), D *is the rectangle with these parameters,* u_D *is its logarithmic potential and* A_0^*, A_1^*, A_2^* *are coefficients obtained by solving the system with* $u = u_D$*, then* $A_0 = A_0^*, A_1 = A_1^*, A_2 = A_2^*$.

Proof. As known,

$$b_1 = \frac{\int_D y_1\,dy}{\int_D 1} = \frac{A_1 + \bar{A}_1}{2A_0}, \quad b_2 = \frac{\int_D y_2\,dy}{\int_D 1} = -i\frac{A_1 - \bar{A}_1}{2A_0}$$

when we use (3.2), (3.3).

As in the proof of Lemma 3.1, in the new coordinate system (3.8), the rectangle D becomes the rectangle $D(0) = \{Y : -a_1 < Y_1 < a_1,\ -a_2 < Y_2 < a_2\}$. By elementary integration,

$$\int\limits_{D(0)} (Y_1^2 - Y_2^2)\,dY = \frac{4}{3}a_1 a_2(a_1^2 - a_2^2), \qquad \int\limits_{D(0)} Y_1 Y_2\,dY = 0 \tag{3.21}$$

due to symmetry reasons. As in the proof of Lemma 3.1, from (3.2), (3.3), (3.4), (3.18), (3.21), we obtain

$$\frac{4}{3}a_1 a_2(a_1^2 - a_2^2) = 2\pi\frac{(A_2 + \bar{A}_2)A_0 - A_1^2 - \bar{A}_1^2}{A_0}\cos 2\theta + 2\pi i\Big(\frac{-(A_2 - \bar{A}_2)A_0 + (A_1^2 - \bar{A}_1^2)}{A_0}\Big)\sin 2\theta$$

or

$$a_1^2 - a_2^2 = 3\frac{(A_2 + \bar{A}_2)A_0 - A_1^2 - \bar{A}_1^2}{2A_0^2}\cos 2\theta + 3i\Big(\frac{-(A_2 - \bar{A}_2)A_0 + (A_1^2 - \bar{A}_1^2)}{2A_0^2}\Big)\sin 2\theta. \tag{3.22}$$

Using (3.11) and the condition $0 < a_2 \le a_1$ yields (3.20). Now, from (3.22), (3.12) and (3.13), we get the second equality in (3.19).

To demonstrate the second statement, we observe that, according to the first part of this lemma, A_0^*, A_1^*, A_2^* satisfy the same equations (3.18), (3.19), (3.20) as A_0, A_1, A_2, and a solution to (3.18), (3.19), (3.20) with respect to A_0, A_1, A_2 is unique. □

Theorem 3.6. *A rectangle D which is not a square is uniquely determined by data* (2.6).

Proof. The center of gravity (b_1, b_2) can be uniquely found from equation (3.18). From equations (3.19), it follows that

$$a_1^2 - \frac{\pi^2 A_0^2}{a_1^2} = \frac{3}{A_0^2}|A_0A_2 - A_1^2|.$$

Since the left-hand side is increasing with respect to positive a_1, we have uniqueness of a_1; then uniqueness of a_2 follows from the first equation in (3.19).

If $0 < |A_0A_2 - A_1^2|$, then from (3.20), we have the uniqueness for the angle θ. If $|A_0A_2 - A_1^2| = 0$, then, from the second equation in (3.19), we have uniqueness of the angle. □

Observe that a square D cannot be uniquely determined by data (2.6). Indeed, let $b = 0$, and let $D(0)$ be a square $\{-a_1 < Y_1 < a_1,\ -a_1 < Y_2 < a_1\}$. Then, as in (3.21),

$$\int\limits_{D(0)} (Y_1^2 - Y_2^2)\,dY = \int\limits_{D(0)} Y_1 Y_2\,dY = 0. \tag{3.23}$$

We will use the substitution $y(Y)$ in (3.8). Solving for y yields $y_1 = Y_1\cos\theta - Y_2\sin\theta$, $y_2 = Y_1\sin\theta + Y_2\cos\theta$; therefore,

$$y_1^2 - y_2^2 = \cos 2\theta(Y_1^2 - Y_2^2) - 2\sin 2\theta Y_1 Y_2, \quad y_1y_2 = \frac{1}{2}\sin 2\theta(Y_1^2 - Y_2^2) + \cos 2\theta Y_1 Y_2,$$

and from (3.23), we obtain

$$\int\limits_D (y_1^2 - y_2^2)\,dy = \int\limits_D y_1y_2\,dy = 0,$$

where D is $D(0)$ in y-variables, which is the square $D(0)$ rotated by the angle θ. Using (3.4), we conclude that $A_2 = 0$ for any angle θ, which therefore cannot be uniquely determined by A_0, A_1, A_2.

4 Numerical results for an ellipse and a rectangle

In this section, we consider different numerical examples based on the location of the points on the unit circle, and we used Matlab to get numerical results.

To recover an ellipse D from our data on a unit circle, given approximation errors in Corollary 2.2 and a discussion after it, we replace (3.1) by

$$\mathbf{G} = \mathbf{BA}, \tag{4.1}$$

where

$$\mathbf{B} = \begin{pmatrix} 2\bar{x}(1)^{-1} & 2\bar{x}(1)^{-2} & 2\bar{x}(1)^{-3} \\ 2\bar{x}(2)^{-1} & 2\bar{x}(2)^{-2} & 2\bar{x}(2)^{-3} \\ 2\bar{x}(3)^{-1} & 2\bar{x}(3)^{-2} & 2\bar{x}(3)^{-3} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} A_0 \\ \bar{A}_1 \\ \bar{A}_2 \end{pmatrix}.$$

To create

$$\mathbf{G} = \begin{pmatrix} \nabla u_D(x(1)) \\ \nabla u_D(x(2)) \\ \nabla u_D(x(3)) \end{pmatrix},$$

we can use the formula for exterior potential of an ellipse with $b = 0$, $\theta = 0$,

$$\nabla u_D(x) = 2\frac{\partial}{\partial \bar{x}} u(x) = 2\overline{\frac{\partial}{\partial x} u(x)} = \frac{a_1 a_2}{\bar{x} + \sqrt{\bar{x}^2 - e^2}}, \tag{4.2}$$

where $e = \sqrt{a_1^2 - a_2^2}$, given in [7, p. 100, (4.4.4)]. We choose $a_1 = 0.5$ and $a_2 = 0.25$ to make sure that the ellipse lies inside the disk $B_{0.5}$. The function $\sqrt{x^2 - e^2}$ has branch points at $x = e$ and $x = -e$. To avoid branch cuts, we use the principal branch given by the power series as follows:

$$\sqrt{x^2 - e^2} = \sum_{n=0}^{\infty} \frac{(-1)^{2n-1}(2n)!\, e^{2n}}{2^{2n}(n!)^2(2n-1)x^{2n-1}} \approx \sum_{n=0}^{5} \frac{(-1)^{2n-1}(2n)!\, e^{2n}}{2^{2n}(n!)^2(2n-1)x^{2n-1}}, \quad e < |x|,$$

By solving the system of equations in (4.1) using singular value decomposition, we can get $\mathbf{A}$.

In order to find the center of gravity (b_1, b_2), we use equation (3.5). To obtain the semi-axes a_1 and a_2, we solve the two equations (3.6) using substitution as in the proof of Corollary 3.4. For the angle θ, we plug in the values of the coefficients in (3.7)

4.1 Recovery of an ellipse with $a_1 = 0.5$, $a_2 = 0.25$, $b = 0$ and $\theta = 0$

Example 1. In this example, we consider three almost equidistant points on the unit circle,

$$x(1) = (-0.955, -0.296), \quad x(2) = (0.654, 0.757), \quad x(3) = (0.666, -0.746).$$

Then

$$\mathbf{G} = \begin{pmatrix} -0.0615 - 0.0211i \\ 0.0383 + 0.0486i \\ 0.0392 - 0.0481i \end{pmatrix},$$

the condition number of $\mathbf{B}$ is 1.4414 and

$$\mathbf{A} = \begin{pmatrix} 0.0312 \\ -0.000105 - 0.0000434i \\ 0.00149 + 0.0000312i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (-0.00338, 0.00139), \quad a_1 = 0.50 \quad \text{and} \quad a_2 = 0.24,$$
$$\sin 2\theta = -0.0206, \quad \cos 2\theta = 0.999 \quad \text{and} \quad \theta = -0.0103.$$

Example 2. Here we have three points on the right half of the circle,

$$x(1) = (0.967, 0.256), \quad x(2) = (0.589, -0.809), \quad x(3) = (0.112, 0.994).$$

Then the condition number of **B** is 4.0879 and

$$\mathbf{A} = \begin{pmatrix} 0.0314 \\ -0.000189 - 0.000178i \\ 0.00161 + 0.0000868i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (-0.00601, 0.00568), \quad a_1 = 0.51 \quad \text{and} \quad a_2 = 0.24,$$
$$\sin 2\theta = -0.0524, \quad \cos 2\theta = 0.999 \quad \text{and} \quad \theta = -0.0262.$$

Example 3. Here we choose the points to be

$$x(1) = (0.654, 0.757), \quad x(2) = (-0.709, 0.706), \quad x(3) = (-0.540, -0.842).$$

Then the condition number of **B** is 1.8971 and

$$\mathbf{A} = \begin{pmatrix} 0.0311 \\ 0.0000128 + 0.0000302i \\ 0.00141 + 0.00000589i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (0.000410, -0.000971), \quad a_1 = 0.49 \quad \text{and} \quad a_2 = 0.25,$$
$$\sin 2\theta = -0.00416, \quad \cos 2\theta = 0.999 \quad \text{and} \quad \theta = -0.00208.$$

4.2 Recovery of an ellipse with $a_1 = 0.5$, $a_2 = 0.25$, $b = 0$ and $\theta = 0$ after adding random noise to G

4.2.1 Relative random noise of 0.01

Example 1. In this example, we consider three almost equidistant points on the unit circle,

$$x(1) = (0.112, 0.994), \quad x(2) = (0.801, -0.599), \quad x(3) = (-0.983, 0.182).$$

Then

$$\mathbf{G} = \begin{pmatrix} 0.00617 + 0.0596i \\ 0.0488 - 0.0401i \\ -0.0641 + 0.0132i \end{pmatrix}$$

and the condition number of **B** is 1.6499. We use

$$\text{noise} = \begin{pmatrix} -0.000872 - 0.000498i \\ -0.000301 + 0.000201i \\ -0.000139 + 0.0000331 \end{pmatrix}.$$

Solving linear system (4.1) with the noise added to **G** yields

$$\mathbf{A} = \begin{pmatrix} 0.0312 \\ -0.0000987 - 0.0000329i \\ 0.00169 - 0.000238i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (-0.00316, 0.00105), \quad a_1 = 0.52 \quad \text{and} \quad a_2 = 0.23,$$
$$\sin 2\theta = 0.139, \quad \cos 2\theta = 0.99 \quad \text{and} \quad \theta = 0.0701.$$

Example 2. Here we have three points on the right half of the circle,

$$x(1) = (0.416, -0.909), \quad x(2) = (0.998, 0.0584), \quad x(3) = (0.0124, 0.999).$$

Then

$$\mathbf{G} = \begin{pmatrix} 0.0234 - 0.0559i \\ 0.0656 + 0.00425i \\ 0.00068 + 0.0598i \end{pmatrix}$$

and the condition number of $\mathbf{B}$ is 3.0629. We use

$$\text{noise} = \begin{pmatrix} 0.000591 + 0.000757i \\ -0.0000803 + 0.000129i \\ 0.000165 + 0.000414i \end{pmatrix}.$$

Solving linear system (4.1) with the noise added to $\mathbf{G}$ yields

$$\mathbf{A} = \begin{pmatrix} 0.0316 \\ -0.000482 - 0.000219i \\ 0.00169 - 0.0000337i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (-0.0153, 0.00696), \quad a_1 = 0.52 \quad \text{and} \quad a_2 = 0.24,$$
$$\sin 2\theta = 0.0239, \quad \cos 2\theta = 0.999 \quad \text{and} \quad \theta = 0.0119.$$

Example 3. Here we choose the points to be

$$x(1) = (-0.921, -0.389), \quad x(2) = (0.791, 0.612), \quad x(3) = (0.857, -0.516).$$

Then

$$\mathbf{G} = \begin{pmatrix} -0.0585 - 0.0273i \\ 0.048 + 0.0409i \\ 0.0531 - 0.0352i \end{pmatrix}$$

and the condition number of $\mathbf{B}$ is 2.1317. We use

$$\text{noise} = \begin{pmatrix} 0.000179 + 0.000375i \\ -0.0000612 - 0.000708i \\ 0.000489 + 0.000287i \end{pmatrix}.$$

Solving linear system (4.1) with the noise added to $\mathbf{G}$ yields

$$\mathbf{A} = \begin{pmatrix} 0.0312 \\ -0.000102 + 0.00000741i \\ 0.00137 + 0.000233i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (-0.00327, -0.000238), \quad a_1 = 0.49 \quad \text{and} \quad a_2 = 0.25,$$
$$\sin 2\theta = -0.168, \quad \cos 2\theta = 0.986 \quad \text{and} \quad \theta = -0.0845.$$

4.2.2 Relative random noise of 0.05

Example 1. In this example, we consider three almost equidistant points on the unit circle,

$$x(1) = (-0.0875, 0.996), \quad x(2) = (-0.98, -0.199), \quad x(3) = (0.971, -0.239).$$

Then

$$\mathbf{G} = \begin{pmatrix} -0.00481 + 0.0597i \\ -0.0638 - 0.0143i \\ 0.0629 - 0.0172i \end{pmatrix}$$

and the condition number of $\mathbf{B}$ is 1.5306. We use

$$\text{noise} = \begin{pmatrix} -0.00135 + 0.00079i \\ -0.00106 - 0.000545i \\ 0.00141 - 0.00312i \end{pmatrix}.$$

Solving linear system (4.1) with the noise added to $\mathbf{G}$ yields

$$\mathbf{A} = \begin{pmatrix} 0.0318 \\ 0.000439 - 0.000656i \\ 0.00202 - 0.000406i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (0.0138, 0.0206), \quad a_1 = 0.56 \quad \text{and} \quad a_2 = 0.23,$$
$$\sin 2\theta = 0.188, \quad \cos 2\theta = 0.982 \quad \text{and} \quad \theta = 0.0946.$$

Example 2. Here we have three points on the upper half of the circle,

$$x(1) = (0.897, 0.443), \quad x(2) = (-0.997, 0.0831), \quad x(3) = (0.575, 0.818).$$

Then

$$\mathbf{G} = \begin{pmatrix} 0.0564 + 0.0307i \\ -0.0654 + 0.00605i \\ 0.0331 + 0.0517i \end{pmatrix}$$

and the condition number of $\mathbf{B}$ is 6.1187. We use

$$\text{noise} = \begin{pmatrix} 0.00172 + 0.000322i \\ -0.00095 - 0.00296i \\ 0.00122 - 0.00237i \end{pmatrix}.$$

Solving linear system (4.1) with the noise added to $\mathbf{G}$ yields

$$\mathbf{A} = \begin{pmatrix} 0.0314 \\ 0.00138 - 0.00123i \\ 0.00269 + 0.00239i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (0.0438, 0.0389), \quad a_1 = 0.70 \quad \text{and} \quad a_2 = 0.18,$$
$$\sin 2\theta = -0.68, \quad \cos 2\theta = 0.728 \quad \text{and} \quad \theta = -0.373.$$

Example 3. Here we choose the points to be

$$x(1) = (0.654, 0.757), \quad x(2) = (0.99, -0.141), \quad x(3) = (-0.765, -0.644).$$

Then

$$\mathbf{G} = \begin{pmatrix} 0.0383 + 0.0486i \\ 0.0648 - 0.0102i \\ -0.0461 - 0.0427i \end{pmatrix}$$

and the condition number of $\mathbf{B}$ is 2.5824. We use

$$\text{noise} = \begin{pmatrix} -0.00105 + 0.00331i \\ 0.00326 + 0.000814i \\ 0.00249 - 0.00145i \end{pmatrix}.$$

Solving linear system (4.1) with the noise added to **G** yields

$$\mathbf{A} = \begin{pmatrix} 0.0318 \\ 0.000622 - 0.000492i \\ 0.00198 + 0.000444i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (0.0196, 0.0155), \quad a_1 = 0.55 \quad \text{and} \quad a_2 = 0.23,$$
$$\sin 2\theta = -0.229, \quad \cos 2\theta = 0.973 \quad \text{and} \quad \theta = -0.115.$$

4.2.3 Random noise of 0.1

Example 1. In this example, we consider three almost equidistant points on the unit circle,

$$x(1) = (-0.983, 0.182), \quad x(2) = (0.967, 0.256), \quad x(3) = (0.0292, -0.999).$$

Then

$$\mathbf{G} = \begin{pmatrix} -0.0641 + 0.0132i \\ 0.0626 + 0.0183i \\ 0.0016 - 0.0598i \end{pmatrix}$$

and the condition number of **B** is 1.4951. We use

$$\text{noise} = \begin{pmatrix} -0.00276 - 0.00608i \\ -0.00249 + 0.0000232i \\ -0.00319 - 0.000509i \end{pmatrix}.$$

Solving linear system (4.1) with the noise added to **G** yields

$$\mathbf{A} = \begin{pmatrix} 0.0317 \\ -0.000249 - 0.000972i \\ 0.000735 + 0.00197i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (-0.00785, 0.0307), \quad a_1 = 0.56 \quad \text{and} \quad a_2 = 0.23,$$
$$\sin 2\theta = -0.932, \quad \cos 2\theta = 0.363 \quad \text{and} \quad \theta = -0.599.$$

Example 2. Here we have three points on the left half of the circle,

$$x(1) = (-0.454, -0.891), \quad x(2) = (-0.378, 0.926), \quad x(3) = (-0.96, 0.279).$$

Then

$$\mathbf{G} = \begin{pmatrix} -0.0257 - 0.0551i \\ -0.0212 + 0.0566i \\ -0.0619 + 0.0199i \end{pmatrix}$$

and the condition number of **B** is 4.9316. We use

$$\text{noise} = \begin{pmatrix} 0.00577 + 0.000403i \\ -0.00677 + 0.00249i \\ 0.00341 - 0.00164i \end{pmatrix}.$$

Solving linear system (4.1) with the noise added to **G** yields

$$\mathbf{A} = \begin{pmatrix} 0.0303 \\ -0.000819 - 0.00537i \\ 0.000246 - 0.0039i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (-0.0269, 0.176), \quad a_1 = 0.78 \quad \text{and} \quad a_2 = 0.16,$$
$$\sin 2\theta = 0.949, \quad \cos 2\theta = 0.266 \quad \text{and} \quad \theta = 0.604.$$

Example 3. Here we choose the points to be

$$x(1) = (-0.469, 0.884), \quad x(2) = (0.416, -0.909), \quad x(3) = (-0.765, -0.644).$$

Then

$$\mathbf{G} = \begin{pmatrix} -0.0266 + 0.0547i \\ 0.0234 - 0.0559i \\ -0.0461 - 0.0427i \end{pmatrix}$$

and the condition number of $\mathbf{B}$ is 2.2949. We use

$$\text{noise} = \begin{pmatrix} 0.00229 + 0.00366i \\ 0.00794 - 0.00114i \\ -0.00205 - 0.00314i \end{pmatrix}.$$

Solving linear system (4.1) with the noise added to $\mathbf{G}$ yields

$$\mathbf{A} = \begin{pmatrix} 0.0314 \\ -0.00217 + 0.0017i \\ 0.000724 + 0.00115i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (-0.0691, -0.0542), \quad a_1 = 0.51 \quad \text{and} \quad a_2 = 0.25,$$
$$\sin 2\theta = -0.904, \quad \cos 2\theta = 0.434 \quad \text{and} \quad \theta = -0.563.$$

In the next two sections, we use the substitution $x' = (x - b)e^{i\theta}$ to use formula (4.2) in x'-variables. To return to x-variables, we use that, by the chain rule,

$$\nabla_x u(x) = 2\overline{\frac{\partial u(x)}{\partial x}} = 2\overline{\frac{\partial u(x(x'))}{\partial x'} \frac{\partial x'}{\partial x}} = \frac{a_1 a_2}{\bar{x}' + \sqrt{\bar{x}'^2 - e^2}} e^{-i\theta}.$$

4.3 Recovery of an ellipse with $a_1 = 0.5$, $a_2 = 0.25$, $b = (0, 0)$ and $\theta = \pi/4$

Example 1. In this example, we consider three almost equidistant points on the unit circle,

$$x(1) = (0.012, 0.999), \quad x(2) = (-0.921, -0.389), \quad x(3) = (0.942, -0.335).$$

Then the condition number of $\mathbf{B}$ is 1.2342 and

$$\mathbf{A} = \begin{pmatrix} 0.0312 \\ 0.00000136 + 0.000127i \\ -0.0000284 - 0.00146i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (0.0000437, -0.00408), \quad a_1 = 0.50 \quad \text{and} \quad a_2 = 0.249,$$
$$\sin 2\theta = 0.999, \quad \cos 2\theta = -0.0191 \quad \text{and} \quad \theta = 0.773.$$

Example 2. Here we have three points on the upper half of the circle,

$$x(1) = (0.988, 0.158), \quad x(2) = (-0.997, 0.083), \quad x(3) = (0.211, 0.978).$$

Then the condition number of $\mathbf{B}$ is 2.6672 and

$$\mathbf{A} = \begin{pmatrix} 0.0311 \\ -0.0000189 - 0.00007303i \\ 0.0000346 - 0.00153i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (-0.000608, 0.00235), \quad a_1 = 0.51 \quad \text{and} \quad a_2 = 0.245,$$
$$\sin 2\theta = 0.999, \quad \cos 2\theta = 0.0227 \quad \text{and} \quad \theta = 0.763.$$

Example 3. Here we choose the points to be

$$x(1) = (0.967, 0.256), \quad x(2) = (0.211, 0.978), \quad x(3) = (-0.9801, -0.199).$$

Then the condition number of $\mathbf{B}$ is 2.5801 and

$$\mathbf{A} = \begin{pmatrix} 0.0311 \\ 0.00000231 - 0.0000169i \\ 0.0000114 - 0.00158i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (0.0000741, 0.000546), \quad a_1 = 0.512 \quad \text{and} \quad a_2 = 0.243,$$
$$\sin 2\theta = 0.999, \quad \cos 2\theta = 0.00723 \quad \text{and} \quad \theta = 0.777.$$

4.4 Recovery of an ellipse with $a_1 = 0.5$, $a_2 = 0.25$, $b = (0.125, 0.05)$ and $\theta = \pi/4$

Example 1. In this example, we consider three almost equidistant points on the unit circle,

$$x(1) = (0.897, 0.443), \quad x(2) = (0.227, -0.974), \quad x(3) = (-0.96, 0.279).$$

Then the condition number of $\mathbf{B}$ is 1.2799 and

$$\mathbf{A} = \begin{pmatrix} 0.0319 \\ 0.00396 - 0.00195i \\ 0.000125 - 0.002004i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (0.124, 0.0611), \quad a_1 = 0.506 \quad \text{and} \quad a_2 = 0.252,$$
$$\sin 2\theta = 0.988, \quad \cos 2\theta = -0.161 \quad \text{and} \quad \theta = 0.706.$$

Example 2. Here we have three points on the right half of the circle,

$$x(1) = (0.129, -0.992), \quad x(2) = (0.998, 0.058), \quad x(3) = (0.401, 0.916).$$

Then the condition number of $\mathbf{B}$ is 3.4376 and

$$\mathbf{A} = \begin{pmatrix} 0.0303 \\ 0.00473 - 0.000497i \\ -0.00008203 - 0.00303i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (0.156, 0.0164), \quad a_1 = 0.652 \quad \text{and} \quad a_2 = 0.186,$$
$$\sin 2\theta = 0.967, \quad \cos 2\theta = -0.273 \quad \text{and} \quad \theta = 0.651.$$

Example 3. Here we choose the points to be

$$x(1) = (0.988, 0.158), \quad x(2) = (-0.0875, 0.996), \quad x(3) = (-0.622, -0.783).$$

Then the condition number of $\mathbf{B}$ is 1.5457 and

$$\mathbf{A} = \begin{pmatrix} 0.0307 \\ 0.00394 - 0.00162i \\ 0.000617 - 0.00219i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (0.128, 0.0526), \quad a_1 = 0.534 \quad \text{and} \quad a_2 = 0.230,$$
$$\sin 2\theta = 0.998, \quad \cos 2\theta = 0.111 \quad \text{and} \quad \theta = 0.703.$$

4.5 Recovery of a rectangle with $2a_1 = 0.5$, $2a_2 = 0.25$, $b = 0$ and $\theta = 0$

Equivalently, we can use (4.1). To find **G**, we can use the exterior potential formula for a rectangle

$$\nabla u(x) = \frac{-1}{4\pi i}\sum_{j=1}^{4}(\overline{e(j)} - \overline{e(j-1)})(\bar{x} - \overline{y(j)})\log(\bar{x} - \overline{y(j)}), \tag{4.3}$$

where $e(j) = \frac{(\overline{y(j+1)} - \overline{y(j)})}{(y(j+1) - y(j))}$ and $y(j)$ are the vertices of the rectangle, $j = 1, 2, 3, 4$.

We choose

$$y(1) = 0.25 + 0.125i, \quad y(2) = -0.25 + 0.125i, \quad y(3) = -0.25 - 0.125i, \quad y(4) = 0.25 - 0.125i$$

to make sure that the rectangle lies inside the unit circle.

The function $\log(\bar{x} - \overline{y(j)})$ has branch points. So, to avoid branch cuts, we used the first four terms of the power series for this function as follows:

$$\log(\bar{x} - \overline{y(j)}) \approx \log(\bar{x}) - \sum_{n=0}^{4}\frac{(\overline{y(j)})^{n+1}}{\bar{x}^{n+1}(n+1)}, \qquad |y(j)| < |x|,$$

where log on the right-hand side can be any branch. Using this representation in (4.3), after summation and using the behavior at infinity, we conclude that the sum of the terms containing $\log(\bar{x})$ is zero, so we only need the single-valued function given by the partial sum of the series.

By solving the system of linear equations in (4.1), we can get **A**. To find b_1, b_2, a_1, a_2 and θ, we plug in the values of the coefficients in equations (3.18), (3.19), (3.20).

Example 1. In this example, we consider three almost equidistant points on the unit circle,

$$x(1) = (0.112, 0.994), \quad x(2) = (0.737, -0.676), \quad x(3) = (-0.997, 0.0831).$$

Then the condition number of **B** is 1.4601 and

$$\mathbf{A} = \begin{pmatrix} 0.00995 \\ -0.00000704 - 0.00000517i \\ 0.000155 - 0.00000565i \end{pmatrix}.$$

This implies that

$$\begin{gathered}(b_1, b_2) = (-0.000707, 0.000519), \quad a_1 = 0.249 \quad \text{and} \quad a_2 = 0.125, \\ \sin 2\theta = 0.0364, \quad \cos 2\theta = 0.999 \quad \text{and} \quad \theta = 0.0182.\end{gathered}$$

Example 2. Here we have three points on the upper half of the circle,

$$x(1) = (0.998, 0.0584), \quad x(2) = (-0.886, 0.465), \quad x(3) = (-0.378, 0.926).$$

Then the condition number of **B** is 4.6101 and

$$\mathbf{A} = \begin{pmatrix} 0.00996 \\ 0.000000464 + 0.0000125i \\ 0.000149 - 0.0000116i \end{pmatrix}.$$

This implies that

$$\begin{gathered}(b_1, b_2) = (0.0000466, -0.00125), \quad a_1 = 0.247 \quad \text{and} \quad a_2 = 0.127, \\ \sin 2\theta = 0.0777, \quad \cos 2\theta = 0.997 \quad \text{and} \quad \theta = 0.0389.\end{gathered}$$

Example 3. Here we choose the points to be

$$x(1) = (-0.825, -0.565), \quad x(2) = (-0.187, 0.983), \quad x(3) = (0.726, 0.688).$$

Then the condition number of $\mathbf{B}$ is 2.8999 and

$$\mathbf{A} = \begin{pmatrix} 0.00994 \\ -0.00000325 + 0.0000000805i \\ 0.000147 + 0.00000498i \end{pmatrix}.$$

This implies that

$$(b_1, b_2) = (-0.000327, -0.00000809), \quad a_1 = 0.246 \quad \text{and} \quad a_2 = 0.127,$$
$$\sin 2\theta = -0.0339, \quad \cos 2\theta = 0.999 \quad \text{and} \quad \theta = -0.0169.$$

5 Conclusion

The next goal is to extend the results onto the fundamental three-dimensional case. While it is not hard to derive an analogue of Lemma 2.1 by using spherical harmonics, proving uniqueness of an ellipsoid from the data similar to (2.1) is getting more complicated, and at present, we can only accomplish it in some particular cases. Another goal of practical importance for magnetoencephalography is to handle Maxwell systems, in particular, to find a minimal amount of useful data/sensors. Finally, we hope to adjust the methods of this paper to a linearized inverse problem of conductivity when, in addition to locating an ellipse or ellipsoid D, one expects to find also constant conductivities of D and of the background. An effective numerical algorithm in some cases was designed and tested in [14].

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