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GLUING NON-COMMUTATIVE TWISTOR SPACES

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Abstract

We describe a general procedure, based on Gerstenhaber–Schack complexes, for extending to quantized twistor spaces the Donaldson–Friedman gluing of twistor spaces via deformation theory of singular spaces. We consider in particular various possible quantizations of twistor spaces that leave the underlying spacetime manifold classical, including the geometric quantization of twistor spaces originally constructed by the second author, as well as some variants based on non-commutative geometry. We discuss specific aspects of the gluing construction for these different quantization procedures.

1. Introduction

1.1. *Introductory historical comments: motivations and signatures*

Twistor theory was originally put forward, in December 1963 (see [49, 53]), as a novel geometrical proposal for the description of physics, specifically attuned to Einstein's 1905 theory of special relativity. Minkowski's 1908 geometrical framework for that theory [42] was as a four-dimensional spacetime M which differs from Euclidean 4-space in that the Euclidean $(+,+,+,+)$-signature metric is replaced by a Lorentzian $(-,+,+,+)$ one, or, as we shall prefer here, a Lorentzian $(+,-,-,-)$ metric, according to which it is the time measure along timelike curves that is what is directly defined. The symmetries of Minkowski's spacetime M are given by the 10-dimensional Poincaré group.

In twistor theory, this symmetry is extended to the 15-dimensional conformal group $SO(2,4)$ of symmetries of compactified Minkowski space M^C, of topology $S^1 \times S^3$, which extends M by the incorporation of a 'light cone at infinity' $\mathcal{I}$, whose vertex is a point i representing both spatial and temporal infinity, joined to a 3-cylinder of topology $S^2 \times \mathbb{R}$ representing 'lightlike' (or null) infinity. The free-field Maxwell equations extend to M^C, as do the other massless field equations for various spins (see [47, 48]).

In the positive-definite Euclidean case, the connection with physics is less direct, making use of the concept of spacetime 'Euclideanization', which plays a role in various approaches to quantum field theory (dating back to [62]), by means of the 'trick' of allowing the time to be described by an

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imaginary parameter. The compactification now becomes the standard 1-point conformal compactification of Euclidean 4-space $\mathbb{R}^4$ to the conformal sphere S^4. The symmetry group is now $SO(5,1)$. However, passing to the common complexification of both M^C and S^4, we again obtain a complex-conformal 4-quadric $\mathbb{C}S^4$, wherein the single point i, in the compactification of S^4, now extends to the complex 3-cone $\mathbb{C}\mathcal{I}$, with vertex i.

The idea of twistor theory is to appeal to the Grassmann–Klein representation of the family of projective lines in complex projective 3-space $\mathbb{CP}^3$ as a complex 4-quadric, but where we now take this in the reverse sense, that is to say, the complexified conformal spacetime $\mathbb{C}S^4$ is to be regarded as the Klein representation of projective lines in a complex projective 3-space $\mathbb{P}\mathcal{T}$, the projective space of the complex vector space $\mathcal{T}$, referred to as twistor space. The complexifications of S^4 and M^C are identical, so in each case we get the same projective twistor space $\mathbb{P}\mathcal{T}$. However we are also interested in reality structures in the two cases, and these come out very differently, despite the fact that the same complex space $\mathbb{CP}^3$ arises as its twistor space $\mathbb{P}\mathcal{T}$ in each case. The difference lies in the way that the 'real' points of the spacetime are interpreted within $\mathbb{P}\mathcal{T}$.

Let us first consider the original Lorentzian case [49]. Here, we obtain a realization of the isomorphism between the Minkowskian conformal spacetime group $SO^+(4,2)$ and the twistor symmetry group $SU(2,2)$. The points of the compactified spacetime M^C are all represented by $\mathbb{CP}^2$s lying in a 5-real-dimensional subspace $\mathbb{P}\mathcal{N} \subset \mathbb{P}\mathcal{T}$ of topology $S^3 \times S^2$. The points of $\mathbb{P}\mathcal{N}$ themselves correspond to null straight lines—that is, light rays—in M^C, including the generators of $\mathcal{I}$. The S^2-family of light rays through a fixed point p in M^C corresponds to a $\mathbb{CP}^1$ in $\mathbb{P}\mathcal{N}$. The condition for two points of M^C to be null separated (that is, joined by a light ray) is that the $\mathbb{CP}^1$s that represent them in $\mathbb{P}\mathcal{N}$ intersect.

The points of $\mathbb{P}\mathcal{T} \smallsetminus \mathbb{P}\mathcal{N}$ also have an interpretation within the (compactifed) real spacetime M^C, but not as light rays. It is easy to interpret a point p in $\mathbb{P}\mathcal{T} \smallsetminus \mathbb{P}\mathcal{N}$ by considering, instead, the dual p^*, with respect to the $SU(2,2)$ structure, which is a $\mathbb{CP}^2$ that intersects $\mathbb{P}\mathcal{N}$ in an S^3. This S^3 can be taken to represent p via this duality. It is naturally fibered according to the Clifford–Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$, where each point $q \in S^3$ lies on an S^1 fiber which is the intersection of S^3 with the complex line joining q to p. In spacetime terms, this construction provides us with the realization of S^3 as a twisting 3-parameter family of light rays termed as 'Robinson congruence', which originally provided the name 'twistor' (see [53]).

In the case of a Riemannian (Euclideanized) spacetime S^4, where we are primarily interested only in the conformal Riemannian structure of S^4, each point of S^4 still corresponds to a $\mathbb{CP}^1$ in the $\mathbb{CP}^3$ (that is, $\mathbb{P}\mathcal{T}$), but in this Riemannian case, we have no null-separated points in the spacetime. Accordingly, none of the lines in this $\mathbb{CP}^3$ (that is, $\mathbb{P}\mathcal{T}$) can intersect, and thus, instead of the real spacetime points being determined by a subspace (that is, $\mathbb{P}\mathcal{N}$) within $\mathbb{P}\mathcal{T}$, we have a fibration

$$\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3 \to S^4$$

(see, for example, [1]).

The twistor framework for either spacetime signature, or in the complex case, becomes remarkably useful for the description of massless free fields (in regions within S^4, M^C or their common complexification $\mathbb{C}S^4$) for various spins, such as the free electromagnetic Maxwell field, which is the case of spin 1. Any such field—which we may now take to be a complex solution of the field equations (this being relevant to quantum wavefunctions)—can be separated into its right-handed (positive helicity s) and left-handed (negative helicity s) parts, where $|s|$ is the spin of the field. The quantity s, where $2s$ is necessarily an integer, s being called the helicity, is negative for left-handed helicity and positive for right-handed, and we also allow $s = 0$. The field equation for $s = 0$ is simply the wave equation, and for $|s| = 1$, we get the Maxwell equations. For $|s| = 2$, we get the description of the free gravitational field according to the weak-field linear limit of Einstein's general theory of relativity. For $s = \pm 1/2$, we get the neutrino/antineutrino equations in the limiting case of zero mass.

Explicitly, the solutions of these equations are represented very directly as simple contour integrals of holomorphic functions of a single twistor $\mathbf{z}$, referred to as twistor functions, taken to be holomorphic and homogeneous of degree $-2s-2$, but otherwise subject to no equations. Twistor functions are more properly thought of as defining elements of 1st Čech cohomology (see, for example [55], particularly Section 6.10). To describe the field in a local region of the spacetime, it is sufficient to use a 2-set Čech covering of the corresponding region in $\mathbb{P}\mathcal{T}$, which we may regard as an open 'thickening' of a $\mathbb{CP}^1$.

In the case $s = -2$, so that the twistor function's homogeneity is $+2$, we find that this construction generates the left-handed (that is, negative helicity) weak-field solutions of the full Einstein vacuum field equations (with or without a cosmological constant Λ), that is, (complex) Einstein 4-manifolds. This comes about as follows. In the linear case, locally (in the spacetime) we need consider only a 2-patch Čech covering, the twistor function being defined on their overlap. For the full nonlinear situation, we take this twistor function to define a 'gluing' of one patch to the other which differs from the identity map. A theorem of Kodaira [32] and Kodaira–Spencer [33] tells us that, so long as this displacement is not too large, we still get a 4-parameter family of $\mathbb{CP}^1$s, straddling the patches. Regarding these as defining points in a conformal 4-manifold (with conformal structure defined in terms of intersections between $\mathbb{CP}^1$s, as in the conformally flat case), we obtain, locally, completely general analytic conformal 4-manifolds which are anti-self-dual, this constraint referring to the vanishing of the self-dual part $\mathbf{W}^+$ of the Weyl curvature tensor $\mathbf{W} = \mathbf{W}^+ + \mathbf{W}^-$, where $\mathbf{W}^-$ would be its anti-self-dual part. This construction yields, locally, the most general such anti-self-dual conformal 4-manifold.

Moreover, in standard twistor space $\mathcal{T}$, we have a certain 2-form $\mathbf{I}$ referred to as he 'infinity twistor', which gives M a Euclidean metric if $\mathbf{I}$ is degenerate (that is, rank 2) and a de Sitter or anti-de Sitter one if $\mathbf{I}$ is non-degenerate. If in the above 'gluing' we preserve $\mathbf{I}$ from patch to patch, then the assigned metric is necessarily Einstein (Ricci tensor being proportional to the metric tensor) and provides us, locally, with the most general such anti-self-dual 4-space (see [49, 60]). This has become known as the 'nonlinear graviton' construction [51, 52].

The question naturally arises as to whether some sort of twistor construction might give rise to generic space-times which provide us with completely general (analytic) space-times. Even more pertinently, in the directly physical Lorentzian case, one might well regard the above construction as completely useless because in this case the decomposition $\mathbf{W} = \mathbf{W}^+ + \mathbf{W}^-$ is a complex one, where $\mathbf{W}^+$ and $\mathbf{W}^-$ are complex conjugates of one another, so that if one vanishes, so does the other, and we are restricted to conformally flat space-times.

On the other hand, one might imagine that there could be some 'non-linearization' of the twistor procedure that had enabled us to generate linearized self-dual Weyl tensors from twistor functions of homogeneity degree -6, analogously to the way that the nonlinear graviton achieved this for homogeneity $+2$. (This had been termed the 'googly problem', by analogy with a difficult bowling action in the game of cricket.) Then perhaps one might 'add the two together' in some sense so as to obtain a solution to the general problem of finding a full expression for solutions of Einstein's vacuum equations in twistor terms. However, despite many attempts, employing different types of idea, no solution has yet come close to a solution along these lines.

It should be mentioned, at this point that many researchers have adopted a different viewpoint, referred to as the use of 'ambitwistors', which involves combining a twistor with a dual twistor into a single entity. Although much significant work has been achieved along these lines (see, for example [36]), nevertheless, this must be regarded as a solution to a different kind of problem, and some of the economy that is a feature of twistor theory is lost.

Another way of thinking about our difficulty here is that spatial reflection takes twistors into dual twistors (members of the dual twistor space); moreover self-dual and anti-self-dual parts of the Weyl curvatures are interchanged upon 3-spatial reflection. In the Lorentz-signature framework, the complex conjugate of a twistor is a dual twistor, and vice versa. Accordingly, a holomorphic function of a twistor would reflect into an anti-holomorphic one, so it would seem that we need to extend the formalism to include both twistors and dual twistors if we are to preserve holomorphicity, so we appear to be driven back to ambitwistors, and thereby lose much of the economy that is inherent in the twistor formalism.

However, there is another solution, which is to appeal to the framework of *twistor quantization*, whereby holomorphicity (crucial for many expressions of twistor theory) is retained, although at the expense of the non-commutativity that is inherent in the use of differential operators. In this procedure, the complex conjugate of a twistor is replaced by a holomorphic differential operator [50], this procedure having had relevance in many of the expressions of twistor theory. Some aspects of this non-commutative algebraic approach, in the Lorentzian framework, are to appear elsewhere, under the name of 'palatial' twistor theory (see [54]), but here we go more deeply into the general structure of this procedure, mainly in the positive-definite signature situation, and explore the resulting non-commutative twistor geometry describing classical spacetimes.

1.2. *Summary and organization of the paper*

In [50] one of us introduced a quantization of twistor spaces, based on a geometric quantization procedure. This construction was motivated by the fact that extending the twistor formalism for conformally curved spacetimes involves the problem of dealing with non-analytic transformations of twistor space that mix the Z^α and the conjugate $\bar{Z}_\alpha$ coordinates. The observation that such transformation do, however, preserve Poisson brackets obtained by viewing the $\bar{Z}_\alpha$ as canonical conjugate variables of the Z^α leads naturally to considering a quantized version of twistor space, which still makes it possible to work with holomorphic functions of the Z^α, where the operator corresponding to $\bar{Z}_\alpha$ is identified with $\partial/\partial Z^\alpha$.

More recently, non-commutative deformations of twistor spaces were considered in the context of non-commutative geometry (see [6, 7, 34, 35]), obtained using the Connes–Landi θ-deformation technique [19]. These constructions are based on quantizing the Hopf fibration

$$\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3 \to S^4. \tag{1.1}$$

However, all these constructions involve a quantization of the spacetime manifold S^4 and a compatible quantization of the twistor space $\mathbb{CP}^3$ determined by the geometry of the Hopf fibration. The motivation for these non-commutative deformations lies primarily in the construction of instantons on non-commutative 4-spheres, hence the non-commutative deformation of the spacetime manifold is crucial to the purpose.

The point of view we are interested in here is different, in the sense that we are interested in quantizations of the twistor space that leave the spacetime manifold commutative.

The main focus of the paper is the gluing problem for quantized twistor spaces formulated in Part D of [54]. We use the Gerstenhaber–Schack theory of non-commutative deformations [28] to extend the Donaldson–Friedman gluing [22] of classical twistor spaces to their non-commutative counterparts. In this paper, we work primarily with Riemannian, rather than Lorentzian manifolds, in order to be able to directly compare the gluing result we discuss with the classical result of [22]. However, the general procedure we describe for the gluing of quantized twistor spaces would apply also in the Lorentzian setting in which the problem was originally formulated in [54].

Section 2 of the paper introduces some examples of non-commutative deformations of twistor spaces. In particular, we show that, in addition to the geometric quantization construction of quantized twistor spaces originally introduced by one of us in [50], other variants are possible, which have a natural interpretation in the setting of non-commutative geometry. In particular, we investigate quantizations of twistor spaces that are obtained, for an (anti)self-dual Riemannian manifold M, by imposing that M remains commutative and that the quantization of the twistor space $Z(M)$ is compatible with the Hopf fibrations relating the twistor space $Z(M)$ to M with twistor lines $\mathbb{CP}^1$ fibers and the twistor space $Z(M)$ and the sphere bundle $S(M)$ of the spinor bundle $\mathcal{S}^+(M)$ with fiber S^1. We describe the geometry of some possible quantizations obtained via these requirements. We also show the different role that the Hopf fibration plays in the geometric quantization of twistor spaces of [50], in the Lorentzian setting, and its compatibility with the quantization.

In Section 3 we focus on the main question of gluing of non-commutative twistor spaces. We present an abstract and very general procedure that applies to any chosen non-commutative deformation that can be described in terms of deformation quantization. In Section 4, we show more explicitly how the examples of twistor space quantization introduced in Section 2 fit into this general procedure.

Our main construction of Section 3 is based on a non-commutative generalization of the gluing result of [22]. Donaldson and Friedman showed in [22] that one can associate to the connected sum $M = M_1 \# M_2$ of two (anti)self-dual Riemannian 4-manifolds M_i a singular space $\tilde{Z}(M)$ obtained by first blowing up the twistor spaces $Z(M_i)$ along one of the $\mathbb{CP}^1$ fibers and then gluing together the two exceptional divisors, $\tilde{Z}(M) = \tilde{Z}(M_1) \sqcup_{E_1 \simeq E_2} \tilde{Z}(M_2)$, with $\tilde{Z}(M_i) = \mathrm{Bl}_{\mathbb{CP}^1}(Z(M_i))$. The gluing map of the exceptional divisors is determined by an orientation-reversing isometry of the tangent spaces of M_i at the points x_i where the connected sum is performed and where the respective fibers $F_{x_i} = \mathbb{CP}^1$ are blown up. The space $\tilde{Z}(M)$ obtained in this way has a normal crossing singularity along the identified exceptional divisors, which form a $\mathbb{CP}^1 \times \mathbb{CP}^1$. In [22] they then consider the question of whether the singular space $\tilde{Z}(M)$ admits an unobstructed deformation to a smooth space, and they show that, when this is the case, the resulting smooth space is the twistor space $Z(M)$ of the connected sum manifold. In particular, this ensures the existence of (anti)self-dual metrics on the connected sum. The analysis of deformations and obstructions used for the result of [22] is based on a deformation theory of spaces with normal crossings singularites developed in [27]. The main deformation result of [22] states that, if the twistor spaces $Z_i = Z(M_i)$ have unobstructed deformation theory, namely if the cohomology $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$, then the deformation theory of $\tilde{Z}(M)$ is also unobstructed.

We investigate to what extent the gluing and deformation procedure of [22] can be adapted to quantized twistor spaces. The work of Gerstenhaber and Schack [28] showed that classical Kodaira–Spencer deformation theory of complex manifolds can be subsumed as a 'commutative part' of a more general deformation theory that includes non-commutative deformations and that is governed by a parameterization of infinitesimal deformations and obstructions in terms of Hochschild cohomology.

Using this formulation of deformation theory, and starting with unobstructed non-commutative deformations of the twistor spaces $Z_i = Z(M_i)$, relative to a choice of a twistor line L_i, we show that it is possible to obtain an unobstructed non-commutative deformation of the singular space $\tilde{Z}$, subject to a compatibility condition between the choices of the cochains that define the higher terms of the deformation. In particular, if the commutative parts of the deformations of the Z_i is unobstructed, the construction recovers the gluing and deformation of [22], so that we can identify the resulting non-commutative deformation of $\tilde{Z}$ with a non-commutative deformation of the twistor space $Z(M_1 \# M_2)$ when the latter exists. If the commutative part of the deformations of the Z_i is obstructed but the non-commutative deformations are unobstructed, the resulting non-commutative deformation of $\tilde{Z}$ can be viewed as a quantized twistor space for $M_1 \# M_2$ which may exist even if the classical one does not, for instance in cases when $M_1 \# M_2$ does not carry an (anti)self-dual structure.

In Section 4, we look again at the specific examples of non-commutative deformations of twistor spaces discussed in Section 2 and we show to what extent the general gluing procedure of Section 3 applies in each case. We show that it can be applied to the original quantization of twistor spaces of [50], where it agrees with a geometric quantization of a Gompf sum of symplectic manifolds. We then show how the gluing works explicitly for the other variants of quantization of twistor spaces obtained in Section 2 from deformations of the Hopf fibration, with different geometric properties of the corresponding deformation theory.

2. Non-commutative twistor spaces

We discuss different non-commutative deformations of twistor spaces. Our primary interest is the quantized twistor space introduced by one of us in [50]. However, we also show that, if one works with (anti)self-dual Riemannian manifolds and imposes the requirements that the non-commutative deformation of the twistor space is compatible with the Hopf fibration, while leaving the spacetime manifold commutative, this can lead to a choice of somewhat different quantizations of the twistor spaces. In particular we first recall the geometric quantization of twistor space, viewed in the context of geometric quantization and deformation quantization, and then we analyze a few different variants of the construction of a quantized twistor space. We then return to discuss the geometric quantization of twistor spaces of [50] and we analyze more in detail the role of the Hopf fibration and the compatibility of the quantization with the Hopf fibration, which is different from the other cases we discuss in this section.

In order to construct these different quantizations of twistor space, we focus on the geometry of the Hopf fibration. We show that there are different ways of deforming the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ to non-commutative spaces, which result in a compatible non-commutative deformation of the Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$, and more generally of the unit sphere bundle $\mathbb{S}(\Lambda_+(M))$ of a self-dual 4-manifold M, in a way that leaves the space manifold S^4 or M commutative.

A first method we discuss is based on deforming the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ by deforming all the 2-tori of the Hopf foliation of S^3 to non-commutative tori. This deformation and the resulting deformations of $\mathbb{S}(\Lambda_+(M))$ fall within the setting of the Connes–Landi θ-deformations of non-commutative geometry. Moreover, they have a counterpart, where the non-commutative deformation can be expressed in terms of a non-commutative deformation of the fibrations $\mathbb{C}^* \hookrightarrow \mathbb{C}^2 \smallsetminus \{0\} \to \mathbb{CP}^1$ and $\mathbb{C}^* \hookrightarrow \mathbb{C}^4 \smallsetminus \{0\} \to \mathbb{CP}^3$, and which can be described in terms of the non-commutative toric deformations of Cirio–Landi–Szabo [12–14]. However, we will show that this method does not correspond to the quantization of twistor spaces introduced in [50]. Indeed, in this deformation the base $S^2 \simeq \mathbb{CP}^1$ of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ remains commutative, unlike what is expected as effect of the quantization of [50]. This results in a non-commutative sphere bundle $\mathbb{S}(\Lambda_+(M))_\theta$ that fibers with S^1-fibers over a commutative twistor space $Z(M)$ and also fibers over the commutative spacetime manifold M, with fibers the non-commutative spheres S^3_θ.

A second method is based instead on a non-commutative deformation of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ that is based on the deformation quantization method originally introduced in [3], where the compatibility of the deformation and the Hopf fibration is achieved using the construction of [45, 46]. We will show that this non-commutative deformation induces a deformation of the sphere bundle $S^3 \hookrightarrow \mathbb{S}(\Lambda_+(M)) \to M$ that leaves the self-dual 4-manifold M commutative, and a compatible non-commutative deformation of the twistor space $\mathbb{CP}^1 \hookrightarrow Z(M) \to M$. The resulting non-commutative $Z(M)_\hbar$ obtained in this way, however, is not exactly the quantization described in [50]. Indeed we show that, instead of the commutation relations $[Z^\alpha, Z^\beta] = 0$, $[\bar{Z}_\alpha, \bar{Z}_\beta] = 0$ and $[Z^\alpha, \bar{Z}_\beta] = \hbar\delta^\alpha_\beta$, in the twistor space $Z(M)_\hbar$ we obtain by this deformation method, both $[Z^\alpha, \bar{Z}_\alpha] = \hbar$ and also $[Z^\alpha, \overline{Z^\alpha}] = \hbar$, where $\bar{Z}_0 = \overline{Z^2}$, $\bar{Z}_1 = \overline{Z^3}$, $\bar{Z}_2 = \overline{Z^0}$ and $\bar{Z}_3 = \overline{Z^1}$.

This variant of the commutation relations of [50], with the additional non-trivial commutators $[Z^\alpha, \overline{Z^\alpha}] = \hbar$, also has a natural interpretation in terms of the settings described by one of us in [54]. Indeed, in this case one is considering in the deformation both the symplectic form of [50] (see (2.2) below), as well as the one discussed in Section C.6 of [54] and related to the cosmological constant.

There is also a third construction that we will discuss, which is also associated with deformation quantization methods and which produces a non-commutative twistor space that is an almost-commutative geometry (in the sense of [9]) over the spacetime manifold M. This construction has as base of the non-commutative Hopf fibration the fuzzy 2-sphere. We will also discuss briefly the properties of these resulting 'fuzzy twistor spaces'. In this case also the commutation relations between the twistor variables are as in the previous case, rather than as in [50].

By focusing on the case of $M = S^4$ we then show that, if we require the same form of compatibility with the Hopf fibration as in the previous cases, then the commutator prescription

$$[Z^\alpha, Z^\beta] = 0, \quad [\bar{Z}_\alpha, \bar{Z}_\beta] = 0, \quad [Z^\alpha, \bar{Z}_\beta] = \hbar\delta^\alpha_\beta. \tag{2.1}$$

of [50] would imply that the spacetime manifold S^4 is also deformed to a non-commutative space. In the original construction of [50], however, the role of the Hopf fibration is different from the other cases we discuss in this section, and is best understood in the original Lorentzian setting. We describe in Section 2.6 how copies of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ are embedded in the subspace $\mathbb{P}N$ of the twistor space defined by the vanishing of the signature $(+,+,-,-)$ norm $\sum_\alpha Z^\alpha \bar{Z}_\alpha$ of the $SU(2,2)$ structure on $\mathbb{CP}^3$. We then show that the geometric quantization of twistor space induces a compatible quantization of these Hopf fibrations. This different role of the Hopf fibration then suggests yet another possible variant of non-commutative deformation of twistor space, again based on the θ-deformations, applied to all the Hopf fibrations in $\mathbb{P}N$. We discuss this other variant in Section 2.7.

While our primary interest is in investigating the gluing problem for the original geometric quantization of twistor space of [50], we include the discussion of all these different non-commutative deformation methods anyway, because it seems interesting to compare how these constructions behave with respect to the gluing problem, see Section 4.

2.1. *Symplectic geometric quantization of twistor space*

We review briefly the quantization of twistor spaces originally introduced by one of us in [50], seen in terms of symplectic geometric quantization and in terms of deformation quantization. The deformation quantization viewpoint will be useful in order to relate this non-commutative deformation to the general deformation and obstruction procedure for the gluing of non-commutative twistor spaces that we introduce in Section 3.

Let (M,g) be a Riemannian manifold with an (anti)self-dual metric. Then there is an integrable almost complex structure J on the tangent bundle TZ of the twistor space $Z = Z(M) = \mathbb{S}(\Lambda_+(M)) = \mathbb{P}(\mathcal{S}^+(M))$, with $\Lambda_+(M)$ the bundle of self-dual 2-forms and $\mathcal{S}(M)$ the positive part of the spinor bundle, and Z is a three-dimensional complex manifold. The fibration $\mathbb{C}^* \hookrightarrow \mathcal{S}^+(M)_0 \to \mathbb{P}(\mathcal{S}^+(M))$, with $\mathcal{S}^+(M)_0$ the complement of the zero section, in turn determines a complex involution J on $\mathcal{S}^+(M)_0$. We denote by Z^α, $\alpha = 0,\dots,3$ and $\bar{Z}_\alpha$ the complex coordinates that are conjugate under this complex structure. We can consider the symplectic form

$$\omega = \sum_\alpha dZ^\alpha \wedge d\bar{Z}_\alpha. \tag{2.2}$$

The complex structure J determines subspaces $T^{0,1}$ and $T^{1.0}$ of the complexified $T(\mathcal{S}^+(M)_0)^{\mathbb{C}}$, spanned by vectors $v \pm iJv$. The subspace $P = T^{0,1}$, spanned by the vectors $\partial/\partial\bar{Z}_\alpha$ gives the complex polarization used for geometric quantization.

The geometric quantization procedure, associated with a symplectic manifold (X,ω), consists of two steps: the prequantization and the polarization and quantization [63]. In the prequantization stage, one considers the Hilbert space of square-integrable sections of a hermitian line bundle $\mathcal{H} = L^2(X,\mathcal{L})$, with Chern class $c_1(\mathcal{L}) = \hbar^{-1}[\omega]$, with a prequantization map that assigns to functions f on X operators on $\mathcal{H}$ of the form $-i\hbar X_f - \theta(X_f) + f$, with θ the symplectic potential and X_f the Hamiltonian vector field associated with f, so that Poisson brackets of functions are mapped, up to a factor of $i\hbar^{-1}$ to commutators of operators. Here $d - i\hbar^{-1}\theta$ is the local form of a connection on the line bundle $\mathcal{L}$. For $X = T^*\mathbb{R}^N$ with coordinates (q^k, p_k) the operators assigned to the position coordinates q^k are of the form $i\hbar\frac{\partial}{\partial p_k} + q^k$ and those associated with the momenta p_ℓ are of the form $-i\hbar\frac{\partial}{\partial q^\ell}$. The prequantization space and operators involve functions of a mixture of both positions and momenta. This can be narrowed down, through a choice of polarization, to a set of variables that separates positions and momenta and makes it possible to work with just half of the variable. The polarization P is a half-dimensional subbundle of TX. The prequantum sections of $\mathcal{L}$ are then replaced by the polarized sections, namely those that are, in the appropriate sense, covariantly constant along P.

More explicitly, in the case we are considering, we write the symplectic form (2.2) as $\omega = i\partial\bar{\partial}K$ with $K = \sum_\alpha Z^\alpha \bar{Z}_\alpha$ and we consider the symplectic potential $\theta = -i\partial K$ that vanishes on the polarization P. Thus polarized sections of the prequantum line bundle can be identified in a local chart with functions of the holomorphic coordinates Z^α. The operators corresponding in this quantization to the coordinates Z^α and $\bar{Z}_\alpha$ satisfy the commutator relations (2.1).

For our purpose of investigating the gluing of quantized twistor spaces, it is convenient to associate with this description of the quantized twistor space in terms of symplectic quantization a description in terms of deformation quantization. This can be done along the lines of Fedosov quantization [24].

In the theory of deformation quantization developed in [3], a formal deformation of $\mathcal{A}$ is a $\mathbb{C}[[t]]$-algebra obtained by assigning a $\mathbb{C}[[t]]$-linear multiplication $\alpha_t : \mathcal{A}[[t]] \times \mathcal{A}[[t]] \to \mathcal{A}[[t]]$, with $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$ with α the multiplication of $\mathcal{A}$ and $\mathbb{C}$-linear maps $\alpha_i : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, so that associativity $\alpha_t(\alpha_t(a,b),c) = \alpha_t(a,\alpha_t(b,c))$ holds.

Under the procedure of Fedosov quantization [24], given a symplectic manifold (X,ω), one considers the space $\mathcal{W} = \mathrm{Sym}^*_{\mathbb{C}}(TX)[[\hbar]]$, with the two gradings of symmetric powers and of powers of

$\hbar$, and the subspace of flat sections $\Gamma_\nabla(\mathcal{W})$ with respect to a flat connection ∇ on $\mathcal{W}$, whose potential can be recursively determined as power series with respect to both gradings (see [24, 25, 44]). Fedosov showed that every $f \in \mathcal{C}^\infty(X)[[\hbar]]$ determines uniquely a section $\rho(f) = \hat{f}$ in $\Gamma_\nabla(\mathcal{W})$ with degree zero part (in the $\mathrm{Sym}^*_{\mathbb{C}}(TX)$-grading) equal to f and that the associative product of the deformation quantization $\mathcal{C}^\infty(X)[[\hbar]]$ can be obtained by inverting this map, $f \star_\hbar g = \rho^{-1}(\hat{f} \circ \hat{g})$.

PROPOSITION 2.1 *The geometric quantization of twistor spaces of* [50] *has a compatible associated deformation quantization.*

Proof. In the case of geometric quantization on a complex manifold, with the holomorphic polarization, as in our case of twistor spaces, it is shown in [44] that the compatibility between geometric quantization and deformation quantization reduces to two conditions:

(1) For f, g holomorphic, the product $f \star_\hbar g$ is also holomorphic.
(2) The $\star_\hbar$ product of a function that is affine-linear in $\frac{\partial K}{\partial Z^\alpha}$ with a holomorphic function is still affine-linear.

Thus, it suffices to show that these two conditions are satisfied to ensure that the geometric quantization of twistor spaces can also be described in terms of an associated compatible deformation quantization. We have $K = \sum_\alpha Z^\alpha \bar{Z}_\alpha$, hence the second condition requires that the product of an affine-linear function of the $\bar{Z}_\alpha$ coordinates with a holomorphic function of the Z^α coordinates is still affine-linear in the $\bar{Z}_\alpha$. The coefficients $\frac{\partial^2 K}{\partial Z^\alpha \partial \bar{Z}_\beta} = \delta_{\alpha,\beta}$ of the symplectic form are constant, hence there are no associated curvature terms. In this case, as observed in [24], the product takes the form

$$\begin{aligned} f \star_\hbar g &= \exp\left(-\frac{i\hbar}{2}\omega^{\alpha\beta}\frac{\partial}{\partial X^\alpha}\frac{\partial}{\partial Y^\beta}\right) f(X,\hbar) g(Y,\hbar)|_{X=Y} \\ &= \sum_{k=0}^\infty \left(\frac{-i\hbar}{2}\right)^k \frac{1}{k!}\omega^{\alpha_1\beta_1}\cdots\omega^{\alpha_k\beta_k}\frac{\partial^k f}{\partial X^{\alpha_1}\cdots\partial X^{\alpha_k}}\frac{\partial^k g}{\partial X^{\beta_1}\cdots\partial X^{\beta_k}}. \end{aligned}$$

It is then clear that, if both f and g are holomorphic functions of the holomorphic coordinates Z^α then also $f \star_\hbar g$ is a holomorphic function of the Z^α and if f is affine linear in the $\bar{Z}_\alpha$ and g is a holomorphic function of the Z^α, the product still has an affine-linear dependence on the $\bar{Z}_\alpha$ variables. □

2.2. *Hopf fibrations and twistor spaces*

We discuss next some other possible quantizations of twistor spaces, which also have the property that the underlying spacetime manifold remains classical. These are obtained using different methods in non-commutative geometry (given, respectively, by θ-deformations, deformation quantization and fuzzy spaces), applied to the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. In order to describe these quantizations, we first recall a few facts regarding the role of the Hopf fibrations in the geometry of twistor spaces.

The first significant example of twistor space that illustrates the relation to the Hopf fibration is the case of $M = S^4$ with $Z(M) = \mathbb{CP}^3$ and the Hopf fibration (1.1) relating them. The twistor space construction is illustrated in this case by the commutative diagram of Hopf fibrations

$$\begin{array}{ccccc} S^1 & \xrightarrow{=} & S^1 & & \\ \downarrow & & \downarrow & & \\ S^3 & \longrightarrow & S^7 & \longrightarrow & S^4 \\ \downarrow & & \downarrow & & \downarrow{=} \\ \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^3 & \longrightarrow & S^4 = \mathbb{HP}^1 \end{array} \tag{2.3}$$

The Hopf fibration projection $S^3 \to S^2$ is given by $(z_0, z_1) \mapsto (2z_0\bar{z}_1, |z_0|^2 - |z_1|^2)$ or, in Hopf coordinates, by $(e^{i\xi_1}\cos\eta, e^{i\xi_2}\sin\eta) \mapsto (e^{i(\xi_1-\xi_2)}\sin 2\eta, \cos 2\eta)$. In fact, after the identification of S^2 with $\mathbb{CP}^1$ via the stereographic projection, the Hopf projection map is simply the restriction to $S^3 \subset \mathbb{C}^2 \smallsetminus \{0\}$ of the projection $\mathbb{C}^2 \smallsetminus \{0\} \to \mathbb{CP}^1$, $(z_0, z_1) \mapsto (z_0 : z_1)$, in the affine chart $(z_0, z_1) \mapsto z_0 z_1^{-1}$. In this form, the Hopf projection map remains of the same form $(q_0, q_1) \mapsto q_0 q_1^{-1}$ in the case of the Hopf fibration $S^3 \hookrightarrow S^7 \to S^4 \simeq \mathbb{HP}^1$, after replacing $z_i \in \mathbb{C}$ by quaternions $q_i \in \mathbb{H}$. Thus, we can equivalently consider the diagram of fibrations

$$\begin{array}{ccccc} \mathbb{C}^* & \xrightarrow{=} & \mathbb{C}^* & & \\ \downarrow & & \downarrow & & \\ \mathbb{C}^2 \smallsetminus \{0\} & \longrightarrow & \mathbb{C}^4 \smallsetminus \{0\} & \longrightarrow & \mathbb{HP}^1 \\ \downarrow & & \downarrow & & \downarrow{=} \\ \mathbb{CP}^1 & \longrightarrow & \mathbb{CP}^3 & \longrightarrow & \mathbb{HP}^1 \end{array} \tag{2.4}$$

where we identify $\mathbb{C}^2 \smallsetminus \{0\} = \mathbb{H} \smallsetminus \{0\}$ and $\mathbb{C}^4 \smallsetminus \{0\} = \mathbb{H} \times \mathbb{H} \smallsetminus \{0\}$.

Our investigation of different forms of quantization of twistor spaces starts by considering the twistor space $\mathbb{CP}^3$ and possible quantizations of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ that leave the spacetime S^4 manifold classical. This will have, in particular, the advantage that the same method can be applied to more general spacetime manifolds M, in both the Lorentzian and Euclidean setting, that admit a twistor space $Z = Z(M)$ with a corresponding fibration

$$\mathbb{CP}^1 \hookrightarrow Z(M) \to M. \tag{2.5}$$

In this more general setting, we want to consider a diagram analogous to diagram (2.3) relating the Hopf fibration of the twistor space $\mathbb{CP}^3$ to the Hopf fibrations $S^1 \hookrightarrow S^3 \to S^2$ and $S^3 \hookrightarrow S^7 \to S^4$.

More precisely, let M be an (anti)-self-dual Riemannian 4-manifold. Then the associated twistor space $Z = Z(M)$ is the sphere bundle $Z(M) = \mathbb{S}(\Lambda_+(M))$ of $\Lambda_+(M)$, where $\Lambda^2(M) = \Lambda_+(M) \oplus \Lambda_-(M)$ is the splitting of 2-forms into self-dual and anti-self-dual parts. Thus, there is a fibration $S^2 \hookrightarrow Z \to M$, see [**2**]. If $\mathcal{S}(M) = \mathcal{S}^+(M) \oplus \mathcal{S}^-(M)$ denotes the spinor bundle of M, with $\mathcal{S}^\pm(M)$ complex 2-plane bundles, then one can also describe the twistor space as the projectivized spinor bundle $Z(M) = \mathbb{P}(\mathcal{S}^+(M))$. The self-duality condition guarantees integrability of the almost complex structure [**2**], hence the twistor space is a complex manifold (in general non-Kähler, unless M is conformally equivalent to either S^4 or $\mathbb{CP}^2$ [**31**]); the embedding of the fibers $\mathbb{CP}^1 \hookrightarrow Z(M)$ is holomorphic, while the projection $Z(M) \to M$ is only a smooth map. We will use the notation

$S(M) := \mathbb{S}(\mathcal{S}^+(M))$ for the unit sphere bundle of the spinor bundle. The twistor and spinor bundles fit into the analog of diagram (2.3),

$$\begin{array}{ccccc} S^1 & \xrightarrow{=} & S^1 & & \\ \downarrow & & \downarrow & & \\ S^3 & \longrightarrow & S(M) & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\ \mathbb{CP}^1 & \longrightarrow & Z(M) & \longrightarrow & M \end{array} \tag{2.6}$$

The horizontal fibration comes from the identification $Z(M) = \mathbb{S}(\Lambda_+(M))$ and the vertical one from $Z(M) = \mathbb{P}(\mathcal{S}^+(M))$. We will also consider the associated diagram

$$\begin{array}{ccccc} \mathbb{C}^* & \xrightarrow{=} & \mathbb{C}^* & & \\ \downarrow & & \downarrow & & \\ \mathbb{C}^2 \smallsetminus \{0\} & \longrightarrow & \mathcal{S}^+(M)^0 & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\ \mathbb{CP}^1 & \longrightarrow & Z(M) & \longrightarrow & M \end{array} \tag{2.7}$$

where $\mathcal{S}^+(M)^0$ is the complement of the zero section in the spinor bundle.

2.3. *θ-deformations and toric deformations*

We discuss our first non-commutative deformation method. The Connes–Landi θ-deformation method [19, 64] associates to a compact Riemannian spin manifold (X, g) that admits an action of a torus $T^2 = U(1) \times U(1)$ by isometries a non-commutative space X_θ with $\theta \in \mathbb{R}$ a deformation parameter. Here the notion of non-commutative space is understood in the sense of spectral triples [16], a natural setting for a non-commutative formulation of Riemannian spin geometry. In this setting, the original commutative manifold (X, g) is encoded as the data $(\mathcal{C}^\infty(X), L^2(X, \mathbb{S}), \not{D})$ of its algebra of smooth functions, the Hilbert space of square-integrable spinors and the Dirac operator. A reconstruction theorem [15] shows that, conversely, a commutative spectral triple satisfying the relevant list of axioms determines a classical manifold. Given a torus action by isometries $T^2 \hookrightarrow \mathrm{Isom}(X, g)$, the algebra $\mathcal{C}^\infty(X)$ can be deformed to a non-commutative algebra, which we denote by $\mathcal{C}^\infty(X)_\theta$, obtained by decomposing smooth functions in the original algebra into Fourier modes (weighted components) with respect to the torus action, and replacing their commutative pointwise product by a non-commutative product modeled on the non-commutative 2-torus T^2_θ of modulus θ. More precisely, by viewing functions $f \in \mathcal{C}^\infty(X)$ as bounded multiplication operators on the Hilbert space $L^2(X, \mathbb{S})$, one decomposes f into components $f_{n,m}$ according to the torus action, $\alpha_{(t_1,t_2)}(f_{n,m}) = e^{2\pi i(nt_1+mt_2)} f_{n,m}$. The deformed product of $\mathcal{C}^\infty(X_\theta)$ is then defined component-wise by setting

$$f_{n,m} \star_\theta h_{k,r} = e^{\pi i \theta (nr-mk)} f_{n,m} h_{k,r}. \tag{2.8}$$

The Hilbert space and the Dirac operator of the spectral triple remain undeformed, so that one obtains an isospectral deformation $X_\theta := (\mathcal{C}^\infty(X)_\theta, L^2(X,\mathbb{S}), \not{D})x$. A reconstruction theorem for theta-deformations is proved in [8].

For our main application here we are especially interested in the case of the θ deformation of S^3 obtained by deforming all the tori T^2 in the Hopf fibration to non-commutative tori T^2_θ. This means that in the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ one considers the action of T^2 by translations on each of the tori of the Hopf foliation of S^3, translating the Hopf coordinates (ξ_1, ξ_2). The effect of the θ-deformation then transforms each T^2 in the foliation of S^3 with a non-commutative T^2_θ while maintaining the Hopf link given by the fibers over 0 and ∞ undeformed. We refer to the resulting non-commutative space as S^3_θ.

More explicitly, we represent the sphere S^3 as the unit quaternions $q \in \mathbb{H}$

$$q = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} = \begin{pmatrix} e^{i\xi_1}\cos\eta & e^{i\xi_2}\sin\eta \\ -e^{-i\xi_2}\sin\eta & e^{-i\xi_1}\cos\eta \end{pmatrix},$$

with $z_1, z_2 \in \mathbb{C}$, $|z_1|^2 + |z_2|^2 = 1$, and with (ξ_1, ξ_2, η) the Hopf coordinates

$$z_1 = x_1 + ix_2 = e^{i\xi_1}\cos\eta, \quad z_2 = x_3 + ix_0 = e^{i\xi_2}\sin\eta.$$

The θ-deformation of the 3-sphere replaces q with

$$\begin{pmatrix} U\ \cos\eta & V\ \sin\eta \\ -V^*\sin\eta & U^*\cos\eta \end{pmatrix}$$

where U, V are the generators of the non-commutative torus T^2_θ algebra, satisfying $UV = e^{2\pi i\theta}VU$. Thus, as shown in [19], the algebra describing the non-commutative space S^3_θ is generated by $\alpha = U$ cos η and $\beta = V$ sin η, satisfying the relations

$$\alpha\beta = e^{2\pi i\theta}\beta\alpha, \quad \alpha^*\beta = e^{-2\pi i\theta}\beta\alpha^*, \quad \alpha^*\alpha = \alpha\alpha^*, \quad \beta^*\beta = \beta\beta^*, \quad \alpha\alpha^* + \beta\beta^* = 1. \tag{2.9}$$

We show that quantizing the 3-sphere through the θ-deformation that renders all the Hopf tori non-commutative has the effect of generating a non-commutative deformation of the sphere bundle of the spinor bundle of a self-dual 4-manifold M, which however leaves the twistor space $Z(M)$ classical.

Proposition 2.2 *The θ-deformation S^3_θ of the 3-sphere determines a non-commutative deformation $S(M)_\theta$ of the sphere bundle of the spinor bundle $S(M) = \mathbb{S}(\mathcal{S}^+(M))$ with the property that the non-commutative $S(M)_\theta$ fits into a diagram of fibrations*

$$\begin{array}{ccccc} S^1 & \xrightarrow{=} & S^1 & & \\ \downarrow & & \downarrow & & \\ S^3_\theta & \longrightarrow & S(M)_\theta & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow{=} \\ \mathbb{CP}^1 & \longrightarrow & Z(M) & \longrightarrow & M \end{array} \tag{2.10}$$

where only the spaces S^3_θ and $S(M)_\theta$ are non-commutative and all the other spaces, including the twistor space $Z(M)$, remain classical.

Proof. If one considers the θ-deformation S^3_θ of the 3-sphere considered above, one still has the Hopf fibration, where the total space S^3_θ is non-commutative, but both the base S^2 and the fiber S^1 remain commutative. To see this, consider the $U(1)$-action on the algebra defining the non-commutative space S^3_θ given by $\alpha \mapsto \lambda\alpha$ and $\beta \mapsto \lambda^{-1}\beta$, for $\lambda \in U(1)$. This clearly preserves the defining relations. The invariant subalgebra $(S^3_\theta)^{U(1)}$, which corresponds to the base of the fibration, is generated by the elements $X = \beta\alpha$, $X^* = \alpha^*\beta^*$ and $Y = \alpha\alpha^* - \frac{1}{2}$ with the relations $XY = YX$, $YX^* = X^*Y$ and $Y^2 + XX^* = \frac{1}{4}$, hence it is the algebra of functions of a commutative two-dimensional sphere. This Hopf fibration $S^1 \hookrightarrow S^3_\theta \to S^2$ is considered from the point of view of spectral triples and Dirac operators in [**20**].

We then construct the deformation $S(M)_\theta$ by considering the fibration over the commutative manifold M, where all the fibers are obtained by replacing the commutative sphere S^3 with its θ-deformation S^3_θ. The resulting $S(M)_\theta$ can itself be regarded as a θ-deformation, where the isometric action of T^2 on $S(M)$ used for the deformation is the action that translates the Hopf tori in each fiber S^3. The defining algebra of $S(M)_\theta$ is generated by sections $\alpha(x)$, $\beta(x)$ with $x \in M$, with the relations as in the case of S^3_θ. The same argument used to show that the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ becomes the fibration $S^1 \hookrightarrow S^3_\theta \to S^2$ then shows that the invariant subalgebra of the $U(1)$ action $\alpha(x) \mapsto \lambda\alpha(x)$ and $\beta(x) \mapsto \lambda^{-1}\beta(x)$ has fibers over M given by the quotient 2-spheres and is identified with the bundle $S^2 \hookrightarrow Z(M) \to M$. Thus, we obtain the fibration $S^1 \hookrightarrow S(M)_\theta \to Z(M)$ that fits the diagram above. □

This Connes–Landi θ-deformation has an associated Cirio–Landi–Szabo toric deformation, which is obtained by considering diagram (2.7). The main idea behind this class of toric deformations is to deform algebraic tori $\mathbb{G}_m(\mathbb{C})^n = (\mathbb{C}^*)^n$ to non-commutative algebraic tori (as defined in Section 2.1 of [**12**]) rather than deforming tori $T^n = (S^1)^n$ to the usual non-commutative tori.

Corollary 2.3 *There is a toric deformation $(\mathbb{C}^2 \smallsetminus \{0\})_\theta$ that fits into a Hopf fibration $\mathbb{C}^* \hookrightarrow (\mathbb{C}^2 \smallsetminus \{0\})_\theta \to \mathbb{CP}^1$. This determines a corresponding non-commutative deformation of diagram* (2.7).

Proof. The non-commutative toric deformation of $(\mathbb{C}^2 \smallsetminus \{0\})_\theta$ is obtained by considering the toric structure and the deformation $\mathbb{C}[\sigma]_\theta$ of the algebras of the cones with the algebra $(\mathbb{C}^2 \smallsetminus \{0\})_\theta$ determined by a gluing diagram

$$0 \to (\mathbb{C}^2 \smallsetminus \{0\})_\theta \to \prod_\sigma \mathbb{C}[\sigma]_\theta \to \prod_{\sigma,\sigma'} \mathbb{C}[\sigma \cap \sigma']_\theta.$$

The gluing diagram is well defined because the algebras $\mathbb{C}[\sigma]_\theta$ are subalgebras of the same non-commutative deformation of the ring of Laurent polynomials associated with the maximal torus and the algebraic torus actions all agree. The explicit form of the relations that determine the maps $\mathbb{C}[\sigma]_\theta \to \mathbb{C}[\sigma \cap \sigma']_\theta$ is given in [**12**], p.54. The diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^2 \smallsetminus \{0\}$ determines a $\mathbb{C}^*$-action on the deformed algebra $(\mathbb{C}^2 \smallsetminus \{0\})_\theta$ with invariant subalgebra that determines a commutative $\mathbb{CP}^1$ so that the deformation fits into a Hopf fibration $\mathbb{C}^* \hookrightarrow (\mathbb{C}^2 \smallsetminus \{0\})_\theta \to \mathbb{CP}^1$. We can then consider the non-commutative space obtained from $\mathcal{S}^+(M)^0$ by deforming the $\mathbb{C}^2 \smallsetminus \{0\}$ fibers to $(\mathbb{C}^2 \smallsetminus \{0\})_\theta$ while leaving M commutative, namely a bundle of $(\mathbb{C}^2 \smallsetminus \{0\})_\theta$ algebras over M. The non-commutative space $\mathcal{S}^+(M)^0_\theta$ obtained in this way fibers over the commutative twistor space $Z(M)$ with fibers $\mathbb{C}^*$. □

2.4. *Deformation quantization of the Hopf fibration and twistor spaces*

We now consider a second type of non-commutative deformation of diagram (2.6), still based on a non-commutative deformation of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. This time, however, both the base S^2 and the total space S^3 are deformed to non-commutative spaces. We show that this deformation of the Hopf fibration gives rise to compatible non-commutative deformations of both the sphere bundle $S(M)$ of the spinor bundle and the twistor space $Z(M)$. This method is based on deformation quantization. We show that the non-commutative twistor space $Z(M)_\hbar$ constructed in this way differs from the quantization prescription of [**50**] through the presence of one additional non-trivial commutator $[Z^\alpha, \overline{Z^\alpha}] = \hbar$.

It is well known that there are difficulties in applying the formalism of deformation quantization to fibrations and principal bundles, including the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ that we are interested in here. This is discussed in detail in [**5**] (see also Remark 2.10 of [**46**]). In fact, a satisfactory very general theory of Riemannian principal bundles in non-commutative geometry was only developed very recently [**10**]. In the next example we do not consider this more sophisticated viewpoint, as we work only at the level of the algebras, not of spectral triples. We use here the construction of [**45**, **46**], based on a deformation quantization of contact manifolds. This allows us to consider a non-commutative version of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ that gives rise to compatible deformation quantizations of S^3 and S^2. The latter can be identified as a non-commutative Kähler manifold deformation of $\mathbb{CP}^1$, so that the complex manifold structure of the twistor space is maintained.

The Wick algebra is the algebra $\mathcal{A}(\mathbb{C}^2_\hbar)$ of the quantum plane $\mathbb{C}^2_\hbar$, with generators $\zeta_0, \zeta_1, \zeta_0^\dagger, \zeta_1^\dagger$ (corresponding to the two complex coordinates ζ_i and their conjugates $\overline{\zeta_i}$) and commutation relations

$$[\zeta_i, \zeta_j] = 0, \quad [\zeta_i^\dagger, \zeta_j^\dagger] = 0, \quad [\zeta_i, \zeta_j^\dagger] = \hbar\delta_{ij}. \tag{2.11}$$

This algebra can be identified (see [**45**, **46**]) with a dense subalgebra of the deformation quantization $(\mathcal{C}^\infty(\mathbb{C}^2)[[\hbar]], \star)$, with the associative product written in the Moyal form as

$$\begin{aligned} f_1 \star f_2 &:= f_1 \exp(\hbar(\overleftarrow{\partial}_\zeta \overrightarrow{\partial}_{\bar\zeta} - \overleftarrow{\partial}_{\bar\zeta} \overrightarrow{\partial}_\zeta)) f_2, \quad \text{where} \\ f_1(\overleftarrow{\partial}_\zeta \overrightarrow{\partial}_{\bar\zeta} - \overleftarrow{\partial}_{\bar\zeta} \overrightarrow{\partial}_\zeta) f_2 &:= \frac{1}{2}\sum_i (\partial_{\zeta_i} f_1 \partial_{\bar\zeta_i} f_2 - \partial_{\bar\zeta_i} f_1 \partial_{\zeta_i} f_2), \end{aligned} \tag{2.12}$$

and all the terms in the expansion are bidifferential operators. Our notation here differs slightly from [**45**, **46**], where the commutation relation of the Wick algebra is $[\xi_i, \bar\xi_j] = -2\hbar\delta_{ij}$. Thus, our generators $\zeta_i, \zeta_i^\dagger$ of the algebra are related to the generators $\xi_i, \bar\xi_i$ of [**45**, **46**] by $\zeta_i = \sqrt{2}\,\bar\xi_i$ and $\zeta_i^\dagger = \sqrt{2}\,\xi_i$. We work here with the version as in (2.11) for consistency with the commutation relations of the twistor coordinates in [**50**].

We now show that this non-commutative $\mathbb{C}^2_\hbar$, with an associated non-commutative Hopf fibration, determine compatible quantizations of the sphere bundle $S(M)$ and the twistor spaces $Z(M)$.

PROPOSITION 2.4 *The Wick algebra deformation $\mathcal{A}(\mathbb{C}^2_\hbar)$ determines non-commutative deformation quantizations of S^3 and S^2 compatible with the Hopf fibration. Given a* (*anti-*)*self-dual 4-manifold*

M with twistor space Z(M), the deformation above induces compatible deformation quantizations of S(M) and Z(M), related by a diagram

$$\begin{array}{ccccc} S^1 & \xrightarrow{=} & S^1 & & \\ \downarrow & & \downarrow & & \\ S^3_\hbar & \longrightarrow & S(M)_\hbar & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\ S^2_\hbar & \longrightarrow & Z(M)_\hbar & \longrightarrow & M \end{array} \tag{2.13}$$

Proof. The relation of the Wick algebra to the Hopf fibration was described in detail in [45, 46]. We recall the main steps here as we need them in the construction of the non-commutative twistor space.

In the Wick algebra consider the element $R^2 := \zeta_0^\dagger \star \zeta_0 + \zeta_1 \star \zeta_1^\dagger = \zeta_0 \star \zeta_0^\dagger + \zeta_1^\dagger \star \zeta_1$. In the algebra $(\mathcal{C}^\infty(\mathbb{C}^2 \smallsetminus \{0\})[[\hbar]], \star)$ this is an invertible element with a square root R (Lemma 3.1, Theorem 3.3, and p. 929 of [46]). Consider then the subalgebra $\mathcal{A}$ generated by the elements $\eta_i := \sqrt{2} R^{-1} \star \zeta_i^\dagger$ and $\eta_i^\dagger := \sqrt{2}\zeta_i \star R^{-1}$, and $\mu := -2\hbar R^{-2}$. These satisfy the relations

$$\begin{gathered} [\mu^{-1}, \eta_i] = -\eta_i, \;\; [\mu^{-1}, \eta_i^\dagger] = \eta_i^\dagger, \;\; [\eta_0, \eta_1] = 0, \\ \eta_i \star \eta_j^* - (1-\mu)\eta_j^\dagger \star \eta_i = \mu \delta_{ij}, \;\; \eta_0^\dagger \star \eta_0 + \eta_1^\dagger \star \eta_1 = 1. \end{gathered} \tag{2.14}$$

As shown in [45, 46], the algebra $\mathcal{A} =: \mathcal{A}(S^3_\hbar)$ is a dense subalgebra of a closed subalgebra $\mathcal{A}^\infty$ of $(\mathcal{C}^\infty(\mathbb{C}^2 \smallsetminus \{0\})[[\hbar]], \star)$ which is isomorphic to a deformation quantization of the 3-sphere, $\mathcal{A}^\infty \simeq \mathcal{C}^\infty(S^3)[[\mu]]$. The algebra $\mathcal{A}^\infty$ can also be characterized as the fixed point subalgebra of the flow $\rho : \mathbb{R} \to \mathrm{Aut}(\mathcal{C}^\infty(\mathbb{C}^2 \smallsetminus \{0\})[[\hbar]], \star)$ with $\rho_t \zeta_i = e^t \zeta_i$ and $\rho_t \zeta_i^\dagger = e^t \zeta_i^\dagger$, and with $\rho_t \hbar = e^{2t}\hbar$. The quantized 2-sphere at the base of the Hopf fibration is obtained by considering the algebras $\mathcal{C}^\infty(U_i)[[\hbar]]$ with $U_i \subset \mathbb{C} \smallsetminus \{0\}$ given by $\{\zeta_i \neq 0\}$. The algebra $\mathcal{A}^\infty$ admits localizations $\mathcal{A}^\infty_{U_i}$, which are the invariant subalgebras, under the flow ρ_t of $\mathcal{C}^\infty(U_i)[[\hbar]]$, and the algebra $\mathcal{A}^\infty$ is obtained as a gluing

$$\begin{array}{ccccc} & & \mathcal{A}^\infty & & \\ & \swarrow & & \searrow & \\ \mathcal{A}^\infty_{U_0} & & & & \mathcal{A}^\infty_{U_1} \\ & \searrow & & \swarrow & \\ & & \mathcal{A}^\infty_{U_0 \cap U_1} & & \end{array}$$

The elements $Z = \zeta_0^{-1} \star \zeta_1 = \eta_0^{-1} \star \eta_1$ and $W = \zeta_1^{-1} \star \zeta_0$ are defined on U_0 and U_1, respectively, and satisfy $[\mu, Z] = [\mu, Z^\dagger] = [\mu, W] = [\mu, W^\dagger] = 0$. Thus, one can consider the subalgebras of $\mathcal{A}^\infty_{U_i}$ generated, respectively, by $\mu, Z, Z^\dagger, \eta_0, \eta_0^\dagger$ and $\mu, W, W^\dagger, \eta_1, \eta_1^\dagger$, with the transition function on $U_0 \cap U_1$ given by $(\mu, W, W^\dagger, \eta_1, \eta_1^\dagger) = (\mu, Z^{-1}, (Z^\dagger)^{-1}, Z \star \eta_0, \eta_0^\dagger \star Z^\dagger)$. Here Z and W define the local coordinates on the deformed $\mathbb{CP}^1$ and η_0, η_1, which satisfy $[Z, \eta_0] = [W, \eta_1] = 0$ and $\eta_1 = Z \star \eta_0$, can be regarded as holomorphic sections of a line bundle over this deformed $\mathbb{CP}^1$. The coordinates $Z, Z^\dagger$ satisfy $[Z, Z^\dagger] = \mu(1 + Z \star Z^\dagger) \star (1 + Z \star Z^\dagger)$, which is regarded in [46] as a deformation of the Kähler metric on $\mathbb{CP}^1$ satisfying $\{z, \bar{z}\} = (1 + z\bar{z})^2$. The canonical conjugate variable of Z in this deformed $\mathbb{CP}^1$ is not $Z^\dagger$ but $(1+Z^\dagger \star Z)^{-1} \star Z^\dagger$, since these satisfy the commutation relation $[Z, (1+Z^\dagger \star Z)^{-1} \star Z^\dagger] = \mu$, see Lemma 4.2 of [**46**]. Thus, we obtain in this way a consistent deformation quantization of S^3 and $\mathbb{CP}^1$. The respective deformation parameters in this construction of [**45**, **46**] are related by $\mu = -2\hbar R^{-2}$, where $\hbar$ is the deformation parameter of $\mathbb{C}^2$ in the Wick algebra $\mathbb{C}^2_\hbar$ and μ the deformation parameter of S^3 and $\mathbb{CP}^1$. For simplicity, we will refer to all three of these deformations using the same notation $\mathbb{C}^2_\hbar$, $S^3_\hbar$ and $\mathbb{CP}^1_\hbar$, and the corresponding algebras as $\mathcal{A}(\mathbb{C}^2_\hbar)$, $\mathcal{A}(S^3_\hbar)$, and $\mathcal{A}(\mathbb{CP}^1_\hbar)$, where the dependence of the deformation parameter μ on $\hbar$ is as stated above.

Consider then a self-dual Riemannian 4-manifold M with its twistor space $Z(M) = \mathbb{S}(\Lambda_+(M)) = \mathbb{P}(\mathcal{S}^+(M))$ and with the sphere bundle $S(M) = \mathbb{S}(\mathcal{S}^+(M))$ of the spinor bundle. Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of M that trivializes both $S(M)$ and $Z(M)$. We denote by $\phi^Z_{\alpha\beta}$ and $\phi^S_{\alpha\beta}$ a set of transition functions for $Z(M)$ and $S(M)$, respectively, compatible with the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ in diagram (2.6),

$$\begin{array}{ccc} S^1 & \xrightarrow{id} & S^1 \\ \downarrow & & \downarrow \\ S^3 & \xrightarrow{\phi^S_{\alpha\beta}} & S^3 \\ \downarrow & & \downarrow \\ S^2 & \xrightarrow{\phi^Z_{\alpha\beta}} & S^2 \end{array}$$

where $\phi^S_{\alpha\beta}$ are the $SU(2)$-valued transition functions of the spinor bundle $\mathcal{S}^+(M)$ acting on the associated sphere bundle $\mathbb{S}(\mathcal{S}^+(M))$. We construct then a bundle of Wick algebras over M, seen as a deformation of the spinor bundle $S^+(M)$. Namely, we consider over each open set U_α the trivial product $U_\alpha \times \mathbb{C}^2_\hbar$ which means the algebra given by the tensor product $\mathcal{A}(U_\alpha) \otimes \mathcal{A}(\mathbb{C}^2_\hbar)$ of the Wick algebra with functions on U_α. The $SU(2)$-valued transition functions of the spinor bundle $\phi^S_{\alpha,\beta}$ determine algebra automorphisms $\phi^S_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathrm{Aut}(\mathbb{C}^2_\hbar)$ which act by mapping the generators $(\zeta_0, \zeta_1) \mapsto (\tilde\zeta_0 = a\zeta_0 + b\zeta_1, \tilde\zeta_1 = c\zeta_0 + d\zeta_1)$ and $(\zeta_0^\dagger, \zeta_1^\dagger) \mapsto (\tilde\zeta_0^\dagger = \bar a \zeta_0^\dagger + \bar b \zeta_1^\dagger, \tilde\zeta_1^\dagger = \bar c \zeta_0^\dagger + \bar d \zeta_1^\dagger)$. The $\tilde\zeta_0, \tilde\zeta_1, \tilde\zeta_0^\dagger, \tilde\zeta_0^\dagger$ are generators of the Wick algebra with $[\tilde\zeta_i, \tilde\zeta_j] = 0$, $\tilde\zeta_i^\dagger, \tilde\zeta_j^\dagger] = 0$ and $[\tilde\zeta_i, \tilde\zeta_j^\dagger] = (|a|^2 + |b|^2)\hbar\delta_{ij} = \hbar\delta_{ij}$. Thus, identifying $\mathcal{A}(U_\alpha \cap U_\beta) \otimes \mathcal{A}(\mathbb{C}^2_\hbar)$ with the algebra $\mathcal{A}(\mathcal{U}_\alpha \cap U_\beta, \mathcal{A}(\mathbb{C}^2_\hbar))$ of $\mathcal{A}(\mathbb{C}^2_\hbar)$-functions on $U_\alpha \cap U_\beta$ we obtain automorphisms $\Phi_{\alpha,\beta}(F)(x) = \phi^S_{\alpha,\beta}(x)(F(x))$. We denote by $\mathcal{S}^+(M)_\hbar$ the non-commutative space obtained in this way through the gluing with the transition functions described above,

$$0 \to \mathcal{A}(\mathcal{S}^+(M)_\hbar) \to \prod_\alpha \mathcal{A}(U_\alpha) \otimes \mathcal{A}(\mathbb{C}^2_\hbar) \to \prod_{\alpha,\beta} \mathcal{A}(U_\alpha \cap U_\beta) \otimes \mathcal{A}(\mathbb{C}^2_\hbar).$$

We obtain a non-commutative $S(M)_\hbar = \mathbb{S}(\mathcal{S}^+(M))_\hbar$ with the same construction, with fiber the algebra $\mathcal{A}(S^3_\hbar)$. Indeed, the algebra $\mathcal{A}^\infty \simeq \mathcal{C}^\infty(S^3)[[\mu]]$ is characterized as the subalgebra of $\mathcal{C}^\infty(\mathbb{C} \smallsetminus \{0\})[[\hbar]]$ invariant under the flow ρ_t and the change of variables $(\zeta_0, \zeta_1, \zeta_0^\dagger, \zeta_1^\dagger) \mapsto (\tilde\zeta_0, \tilde\zeta_1, \tilde\zeta_0^\dagger, \tilde\zeta_1^\dagger)$ commutes with the flow. We then obtain the compatible non-commutative twistor space $Z(M)_\hbar$ by considering, over the same open covering, a locally trivial bundle of algebras $\mathbb{CP}^1_\hbar$ with the transition functions induced by the transition functions $\phi^S_{\alpha,\beta}$. The algebra of $\mathbb{CP}^1_\hbar$ is characterized in

[46] as the subalgebra of $\mathcal{A}^\infty$ given by the condition $\{f \in \mathcal{A}^\infty \,|\, [\mu, f] = 0\}$. This depends on the fact that $[\mu, Z] = [\mu, W] = 0$, which in turn is determined by the commutator relations of the element R^2 with the generators of the Wick algebra, which with the commutation relations (2.11) is given by $\zeta_i \star R^2 = (R^2 + \hbar)\zeta_i$. Thus, it suffices to check that the $SU(2)$ action $(\zeta_0, \zeta_1) \mapsto (\tilde{\zeta}_0, \tilde{\zeta}_1)$ preserves these commutation relations. This is the case since the $\tilde{\zeta}_i$ are linear combinations of the ζ_i. Thus, we obtain compatible constructions of the non-commutative $S(M)_\hbar$ and $Z(M)_\hbar$ as bundles of non-commutative algebras over the commutative space M, that fit diagram (2.13). □

We now compare the non-commutative twistor space $Z(M)_\hbar$ obtained in this way with the non-commutative twistor space introduced by one of us in [50] and we show that our $Z(M)_\hbar$ has one additional non-trivial commutator relation.

PROPOSITION 2.5 *Let M be a self-dual 4-manifold and let $Z(M)_\hbar$ be the non-commutative twistor space obtained as in* Proposition 2.4. *The quantized $S(M)_\hbar$ and $Z(M)_\hbar$ differ from the quantization prescription of* [50] *by the presence of two rather than one non-trivial commutators, $[Z^\alpha, \bar{Z}_\alpha] = \hbar$ and $[Z^\alpha, \overline{Z^\alpha}] = \hbar$.*

Proof. In twistor coordinates Z^α and $\bar{Z}_\alpha$, the classical variables $\bar{Z}_\alpha$ are the twistor conjugate variables, namely $\bar{Z}_0 = \overline{Z^2}$, $\bar{Z}_1 = \overline{Z^3}$, $\bar{Z}_2 = \overline{Z^0}$, $\bar{Z}_3 = \overline{Z^1}$. The quantization of twistor space introduced in [50] is obtained by imposing the condition that the variables Z^α commute with each other, $[Z^\alpha, Z^\beta] = 0$, and also $[\bar{Z}_\alpha, \bar{Z}_\beta] = 0$, while Z^α and $\bar{Z}_\alpha$ are conjugate variables satisfying the relation $[Z^\alpha, \bar{Z}_\beta] = \hbar\delta^\alpha_\beta$. The $\hbar$ parameter can be absorbed into a rescaling of the variables, but we will consider it here explicitly as deformation parameter, to compare with the Wick algebras considered above. In the case of $M = S^4 = \mathbb{HP}^1$ we can consider the two affine charts of $\mathbb{HP}^1$ given by $(q, 1)$ and $(1, \tilde{q})$ with $q, \tilde{q} \in \mathbb{H}$ and the transition function $\tilde{q} = q^{-1}$ on the overlap $\mathbb{H} \smallsetminus \{0\}$. We write $q = z_0 + z_1 j$ with $z_0, z_1 \in \mathbb{C}$. For the construction of $\mathcal{S}^+(S^4)_\hbar$ as in Proposition 2.4, we then consider an algebra of the form $\mathcal{A}(\mathbb{C}^2) \otimes \mathcal{A}(\mathbb{C}^2_\hbar)$ for each of these two open charts. These have generators $z_i \otimes \zeta_j$ for $i, j \in \{0, 1\}$, satisfying the relations $[z_i \otimes \zeta_j, z_a \otimes \zeta_b] = 0$, $[\bar{z}_i \otimes \zeta_j^\dagger, \bar{z}_a \otimes \zeta_b^\dagger] = 0$ and $[z_0 \otimes \zeta_i, \bar{z}_1 \otimes \zeta_i^\dagger] = \hbar$. Thus, we obtain an identification of $\mathcal{S}^+(S^4)_\hbar$ with the algebra with generators $Z^\alpha = z_\alpha \otimes \zeta_\alpha$ and $\bar{Z}_\alpha = \bar{z}_\alpha \otimes \zeta_\alpha^\dagger$ with commutation relations $[Z^\alpha, Z^\beta] = 0 = [\bar{Z}_\alpha, \bar{Z}_\beta]$ and with two non-trivial commutator relations between the Z^α and the $\bar{Z}_\alpha$ given by $[Z^\alpha, \bar{Z}_\beta] = \hbar$ when either $\alpha = \beta$ or $\alpha + 2 = \beta$ mod 4. In terms of the conventional notation with twistor variables mentioned above this means both $[Z^\alpha, \bar{Z}_\alpha] = \hbar$ and $[Z^\alpha, \overline{Z^\alpha}] = \hbar$. □

2.5. *Fuzzy twistor spaces*

We mention one more possible construction that satisfies a form of the Hopf fibration compatibility and the commutativity of spacetime and which also is close to satisfying (2.1). This is related to the fuzzy sphere approximations of the 2-sphere.

Consider first the non-commutative deformation $\mathbb{C}^2_\hbar$, with generators $\zeta_0, \zeta_1, \zeta_0^\dagger, \zeta_1^\dagger$ (corresponding to the usual coordinates $z_0, z_1, \bar{z}_0, \bar{z}_1$ of the commutative case) and with the commutation relations (2.11). For a single copy of $\mathbb{C}$ the deformation $\mathbb{C}_\hbar$ with $[\zeta, \zeta^\dagger] = \hbar$ just corresponds to the quantum plane where the real coordinates satisfy $[y, x] = 2i\hbar$ and as an algebra $\mathbb{C}^2_\hbar = \mathbb{C}_\hbar \otimes \mathbb{C}_\hbar$ is a product of two such quantum planes, which we will also write as $\mathbb{C}_\hbar \times \mathbb{C}_\hbar$.

In this setting, the non-commutative deformation of the Hopf fibration is obtained by considering the Wick algebra $\mathcal{A}(\mathbb{C}^2_\hbar)$ generated by the $\zeta_0, \zeta_1, \zeta_0^\dagger, \zeta_1^\dagger$ satisfy the commutation relations (2.11) and the algebras $\mathcal{A}((\mathbb{C}^2 \setminus \{0\})_\hbar)$ and $\mathcal{A}(S^3_\hbar)$ as described in Section 2.4.

Consider then the elements

$$L_1 = \frac{1}{2}(\zeta_0\zeta_1^\dagger + \zeta_0^\dagger\zeta_1), \quad L_2 = \frac{i}{2}(\zeta_0\zeta_1^\dagger - \zeta_0^\dagger\zeta_1), \quad L_3 = \frac{1}{2}(\zeta_0^\dagger\zeta_0 - \zeta_1^\dagger\zeta_1). \tag{2.15}$$

These generate a $U(1)$-invariant subalgebra for the action $\zeta_i \mapsto \lambda\zeta_i$ and $\zeta_i^\dagger \mapsto \bar{\lambda}\zeta_i^\dagger$ and they satisfy the commutation relation $[L_a, L_b] = i\hbar\epsilon^{abc}L_c$. By regarding the subalgebra generated by the L_a as a deformed 2-sphere, one can view the inclusion of this subalgebra as a version of a deformed Hopf fibration.

Thus, we see that the non-commutative twistor spaces $Z(M)_\hbar$, obtained via deformation quantization, have an associated family of 'fuzzy twistor spaces' based on the relation between the deformed 2-sphere described here above and the fuzzy spheres S^2_N. In turn the fizzu spheres have a direct connection with deformation quantization of the 2-sphere, as discussed in [**26**]. The fuzzy spheres [**38**] of level $N = 2j$ determine an approximation of the ordinary 2-sphere S^2 by finite non-commutative spaces. These are based on decomposing the algebra of functions on the 2-sphere, seen as a $\mathcal{U}(\mathfrak{su}(2))$-module, into irreducible representations $\oplus_{\ell\geq 0}\mathcal{V}_\ell$, with $\mathcal{V}_\ell$ spanned by the spherical harmonics $\Theta_{\ell,m}$, and then truncating at some energy level $N = 2j$, by only considering $0 \leq \ell \leq 2j$. A description of the fuzzy spheres in terms of spectral triples is given in [**21**] and a precise sense in which the fuzzy spheres converge to the ordinary sphere when $N \to \infty$ is analyzed in [**57**].

The fuzzy sphere algebra S^2_N is obtained by mapping the coordinates (x_1, x_2, x_3) of the 2-sphere $S^2 \subset \mathbb{R}^3$, with $x_1^2 + x_2^2 + x_3^2 = 1$, to operators

$$X_a := \frac{1}{\sqrt{j(j+1)}} J_a,$$

where $N = 2j$ and J_a the generators of the Lie algebra $\mathfrak{su}(2)$ satisfying $[J_a, J_b] = i\epsilon^{abc}J_c$, viewed as operators acting in the $(N+1)$-dimensional representation of $SU(2)$. The normalization factor is chosen so that the sphere relation $\sum_a X_a^2 = 1$ is preserved. The map $x_a \mapsto X_a$ is not an algebra homomorphism, but it determines an isomorphism of $\star$-representations of $\mathcal{U}(\mathfrak{su}(2))$. The resulting algebra S^2_N describing the fuzzy sphere is generated by X_1, X_2, X_3 with the relation $X_1^2 + X_2^2 + X_3^2 = 1$ and the non-trivial commutation relation

$$[X_a, X_b] = \frac{1}{\sqrt{j(j+1)}} i\epsilon_{abc}X_c. \tag{2.16}$$

This algebra is in fact just the matrix algebra $M_{N+1}(\mathbb{C})$.

Under the map $x_a \mapsto X_a$ the spherical harmonics $\Theta_{\ell,m}$ are mapped to matrices $\hat{\Theta}_{\ell,m} \in M_{N+1}(\mathbb{C})$ (the fuzzy spherical harmonics), whose entries are Clebsch–Gordon coefficients, and where one retains only the harmonics with $\ell = 0, \ldots, N$. Thus, the algebra S^2_N can be equivalently described by considering the expansion $f(x) = \sum_{\ell,m} a_{\ell,m}\Theta_{\ell,m}(x)$ in spherical harmonics $\Theta_{\ell,m}$ of functions on the 2-sphere and replacing $f(x)$ with the element $\hat{f} = \sum_{\ell=0}^N \sum_m a_{\ell,m}\hat{\Theta}_{\ell,m}$ in S^2_N. For functions on S^2 that only involve modes in the spherical harmonics with $\ell \leq N$ the fuzzy sphere product is then given by $f_1 \star_{S^2_N} f_2 := \hat{f}_1 \cdot \hat{f}_2$ as product of the corresponding matrices in $M_{N+1}(\mathbb{C})$. As shown in [**26**], this product of the fuzzy sphere algebra S^2_N is related to the deformation quantization product of $S^2_\hbar$ by the relation

$$f_1 \star_{S^2_N} f_2 = \mathcal{P}_N(f_1 \star_\hbar f_2)|_{\hbar=2/(N+1)},$$

where $\mathcal{P}_N$ denotes the projection of the first $N+1$ modes $\ell = 0, \dots, N$ in the spherical harmonics and f_1, f_2 are in the range of $\mathcal{P}_N$.

The construction of fuzzy twistor spaces is similar to the construction of the non-commutative twistor spaces based on deformation quantization discussed in Section 2.4.

PROPOSITION 2.6 *The fuzzy sphere algebra* $\mathcal{A}(S^2_N) = M_{N+1}(\mathbb{C})$ *seen as a subalgebra of the Wick algebra* $\mathbb{C}^2_\hbar$ *for* $\hbar = 1/\sqrt{j(j+1)}$ *and* $N = 2j$*, determines fuzzy twistor spaces* $Z(M)_N$ *and* $S(M)_N$ *compatible with the Hopf fibration* (2.6).

Proof. The fuzzy twistor spaces $Z(M)_N$ are obtained by considering an open covering $\{U_\alpha\}$ of M that trivializes the spinor bundle $\mathcal{S}^+(M)$, with $SU(2)$-valued partition functions $\phi^S_{\alpha,\beta}$. We then consider over each U_α the algebra $\mathcal{A}(U_\alpha) \otimes \mathbb{C}^2_\hbar$ and the associated non-commutative space $\mathcal{S}^+(M)_\hbar$ obtained by gluing these algebras with the transition functions as in Proposition 2.4. For $\hbar = 1/\sqrt{j(j+1)}$ and $N = 2j$, the fuzzy twistor space is then obtained by considering the subalgebras $\mathcal{A}(U_\alpha) \otimes \mathcal{A}(S^2_N) = \mathcal{A}(U_\alpha) \otimes M_{2j+1}(\mathbb{C})$ with the transition functions $\phi^S_{\alpha,\beta}$ acting as automorphisms of the algebra $\mathcal{A}(S^2_N)$ using the $(N+1)$-dimensional representation of $SU(2)$. □

As mentioned above, for $\hbar = 1/\sqrt{j(j+1)}$ and $N = 2j$, the algebra describing the fuzzy sphere S^2_N is the matrix algebra $M_{2j+1}(\mathbb{C})$, hence the fuzzy twistor space $Z(M)_N$ is an almost-commutative geometry in the sense of [**9**]. We will discuss some of the properties of this almost-commutative geometry more in detail in the next section.

2.6. *Geometric quantization of twistor spaces and the Hopf fibration*

We return now to the original geometric quantization of twistor spaces [**50**], recalled in Section 2.1 and we discuss the role of the Hopf fibration (2.6), in comparison with the other cases introduced above.

In the Riemannian setting, the Hopf fibration is involved in the geometric quantization of the twistor space in the form of the $\mathbb{C}^*$-bundle $\mathcal{S}^+(M)_0$ over the twistor space $Z(M)$ and the complex structure J on $T(\mathcal{S}^+(M)_0)$ compatible with the complex structure on $Z(M)$, with the twistor coordinates Z^α and $\bar{Z}_\alpha$. However, in this case, the role of the Hopf fibration is more subtle than in the other forms of quantization we described in this section.

One can see this by focusing on the Riemannian case with $M = S^4$, with $Z(M) = \mathbb{CP}^3$ and $S(M) = S^7$. In this case, we can see explicitly that if the commutation prescription (2.1) is obtained as a Wick algebra deformation and we also impose the same compatibility requirements with the Hopf fibration diagram (2.6) used in the previous constructions, that would necessarily lead to a non-commutative S^4.

For $M = S^4$ we have $Z(M) = \mathbb{CP}^3$ and $S(M) = S^7$ and these spaces fit in diagram (2.3) of Hopf fibrations, or equivalently in diagram (2.4). We now require that $\mathbb{C}^4$ is quantized as a Wick algebra $\mathbb{C}^4_\hbar$, with generators $\zeta_0, \zeta_1, \zeta_2, \zeta_3$ and $\zeta_0^\dagger, \zeta_1^\dagger, \zeta_2^\dagger, \zeta_3^\dagger$ and commutation relations $[\zeta_i, \zeta_j] = 0$, $[\zeta_i^\dagger, \zeta_j^\dagger] = 0$

and $[\zeta_i, \zeta_j^\dagger] = \hbar\delta_{ij}$. This agrees with the commutators (2.1) for the non-commutative twistor space of S^4 by identifying the variables Z^α with the generators ζ_i and the variables $\bar{Z}_\alpha$ with the generators $\zeta_i^\dagger$. We also require that the resulting quantizations of $Z(S^4) = \mathbb{CP}^3$ and of $S(S^4) = S^7$ are compatible with the Hopf fibration diagrams (2.3) and (2.4). This means that the prescription for the quantization of $(\mathbb{C}^4 \smallsetminus \{0\})_\hbar$ should be compatible with the projection maps $\mathbb{C}^4 \smallsetminus \{0\} \to \mathbb{HP}^1$ and $S^7 \to \mathbb{HP}^1$.

The commutative algebra $\mathcal{A}(S^4)$ of functions on S^4 has commuting generators $\alpha, \alpha^\dagger, \beta, \beta^\dagger, x$ with relation $\alpha\alpha^\dagger + \beta\beta^\dagger + x^2 = 1$ and the projection map is given by (see Appendix A of [**35**])

$$\alpha = 2(z_0\bar{z}_2 + z_1\bar{z}_3), \quad \beta = 2(z_1 z_2 - z_0 z_3), \quad x = z_0\bar{z}_0 + z_1\bar{z}_1 - z_2\bar{z}_2 - z_3\bar{z}_3.$$

The subalgebra of $(\mathbb{C}^4 \smallsetminus \{0\})_\hbar$ generated by the elements

$$\alpha := 2(\zeta_0\zeta_2^\dagger + \zeta_1\zeta_3^\dagger), \quad \beta := 2(\zeta_1\zeta_2 - \zeta_0\zeta_3), \quad x = \zeta_0\zeta_0^\dagger + \zeta_1\zeta_1^\dagger - \zeta_2\zeta_2^\dagger - \zeta_3\zeta_3^\dagger$$

satisfies

$$\begin{aligned}
\frac{1}{4}[\alpha, \beta] &= \zeta_0\zeta_1(\zeta_2^\dagger\zeta_2 - \zeta_2\zeta_2^\dagger) - \zeta_0\zeta_1(\zeta_3^\dagger\zeta_3 - \zeta_3\zeta_3^\dagger) = 0 \\
\frac{1}{4}[\alpha, \alpha^\dagger] &= \zeta_0\zeta_0^\dagger\zeta_2^\dagger\zeta_2 - \zeta_0^\dagger\zeta_0\zeta_2\zeta_2^\dagger + \zeta_1\zeta_1^\dagger\zeta_3^\dagger\zeta_3 - \zeta_1^\dagger\zeta_1\zeta_3\zeta_3^\dagger \\
&= \hbar\zeta_2^\dagger\zeta_2 - \hbar\zeta_0^\dagger\zeta_0 + \hbar\zeta_3^\dagger\zeta_3 - \hbar\zeta_1^\dagger\zeta_1 = -\hbar x = \hbar(R_1^2 - R_0^2) \\
\frac{1}{4}[\beta, \beta^\dagger] &= \zeta_1\zeta_1^\dagger\zeta_2\zeta_2^\dagger - \zeta_1^\dagger\zeta_1\zeta_2^\dagger\zeta_2 + \zeta_0\zeta_0^\dagger\zeta_3\zeta_3^\dagger - \zeta_0^\dagger\zeta_0\zeta_3^\dagger\zeta_3 \\
&= \hbar\zeta_2\zeta_2^\dagger + \hbar\zeta_1^\dagger\zeta_1 + \hbar\zeta_3\zeta_3^\dagger + \hbar\zeta_0^\dagger\zeta_0 = \hbar(R_0^2 + R_1^2),
\end{aligned}$$

where as before we write $R_0^2 = \zeta_0^\dagger\zeta_0 + \zeta_1\zeta_1^\dagger$ and $R_1^2 = \zeta_2^\dagger\zeta_2 + \zeta_3\zeta_3^\dagger$.

$$\frac{1}{4}[\alpha, \beta^\dagger] = -[\zeta_0, \zeta_0^\dagger]\zeta_2^\dagger\zeta_3^\dagger + [\zeta_1, \zeta_1^\dagger]\zeta_2^\dagger\zeta_3^\dagger = 0$$

and $[\beta, \alpha^\dagger] = 0$ likewise. The commutators $[\alpha, \alpha^\dagger]$ and $[\beta, \beta^\dagger]$ do not simultaneously vanish, hence the subalgebra obtained in this way is also non-commutative.

Moreover, we can see that, if we adapt to the Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$ the argument used in Section 2.4 for the deformation quantization of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$, by replacing complex numbers with quaternions, we also end up with a non-commutative $\mathbb{HP}^1_\hbar$ obtained as the non-commutative $\mathbb{CP}^1_\hbar$ discussed in Section 2.4. In this case also the commutator relations (2.1) are satisfied and the same strict compatibility with the Hopf fibration used in our other constructions of quantized twistor spaces are satisfied, but this cannot be made compatible with the requirement that the spacetime manifold S^4 remains commutative.

The discussion above shows that the compatibility between the quantization of twistor spaces by commutators (2.1) and the Hopf fibration is not implemented by diagrams (2.3) and (2.6). However, a different form of compatibility with the Hopf fibration holds for the twistor quantization of [**50**]. In order to better identify the role of the Hopf fibration in the geometric quantization of the twistor space of [**50**], it is useful to look at the construction in the original setting of a Lorentzian metric, and the occurrence of the Hopf fibration of S^3 (Clifford parallels) in the Lorentzian version of twistor theory.

PROPOSITION 2.7 *The geometric quantization of (Lorentzian) twistor spaces of [50] with commutator relations (2.1) is compatible with the Hopf fibration, by viewing copies of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ embedded in the Hopf fibration $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3 \to S^4$ by first restricting the projection $\mathbb{CP}^3 \to S^4$ to $\mathbb{P}N \to S^3$ over the equatorial sphere of S^4 and then slicing $\mathbb{P}N$ with planes P in $\mathbb{CP}^3$ passing through a chosen point q in the upper half $\mathbb{P}T^+$ of $\mathbb{CP}^3 \setminus \mathbb{P}N$.*

Proof. Here, in the Lorentzian case, $\mathbb{CP}^3$ has an $SU(2,2)$ rather than an $SU(4)$ structure. The subspace $\mathbb{P}N$ of $\mathbb{CP}^3$, consisting of the element of zero $SU(2,2)$ norm, divides $\mathbb{CP}^3$ into two halves $\mathbb{P}T^{\pm}$, respectively of positive and negative norms. The S^2 fibration of $\mathbb{CP}^3$ over S^4 has $\mathbb{P}N$ over an S^3 equatorial subspace of the sphere S^4. Now, to see the Hopf fibrations, we take an arbitrary point q in the top (positive norm) half of $\mathbb{CP}^3$ and take an arbitrary $\mathbb{CP}^2$ plane P through q, of the kind which contains a projective line in the bottom half of $\mathbb{CP}^3$, so that P has positive $SU(2,2)$ norm. We find that the intersection of P with $\mathbb{P}N$ is a Hopf-fibered S^3, where the Hopf circles are the intersections of the projective lines through q with this S^3. The S^2 fibration of $\mathbb{CP}^3$ carries these Hopf fibrations down to the equatorial S^3 in the sphere S^4.

The S^4 here is not really 'physical space-time', but it may be thought of as having the physical 3-space at time $t = 0$, represented by the equatorial S^3, but where the S^4 arises when the time t evolves away from zero through pure-imaginary numbers (a so-called 'Wick rotation'). Thus, S^4 should be regarded as the conformally compactified Wick-rotated space-time.

The original importance of these Hopf fibrations to Lorentzian twistor theory (and whence the original name 'twistor') came about from the fact that the points of $\mathbb{P}N$ have an immediate physical interpretation, in terms of light rays in the Minkowskian space-time. The way that we can 'see' the points in $\mathbb{CP}^3$ (or, more directly, the planes P in $\mathbb{CP}^3$) in physical terms, is in terms of these twisted congruences of light rays in the physical 3-space, here represented as the equatorial S^3 described above.

This means that the diagram illustrating the role of the Hopf fibration in the geometric quantization of twistor space of [50] is not the one we considered in (2.3) but it arises by considering the inclusions

$$\begin{array}{ccccc} \mathbb{CP}^1 & \hookrightarrow & \mathbb{CP}^3 & \longrightarrow & S^4 \\ \uparrow{=} & & \uparrow & & \uparrow \\ \mathbb{CP}^1 & \hookrightarrow & \mathbb{P}N & \longrightarrow & S^3 \\ \uparrow & & \uparrow & & \uparrow \\ S^1 & \hookrightarrow & S^3 & \longrightarrow & S^2 \end{array} \tag{2.17}$$

where the second line is obtained by restricting the fibration $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3 \to S^4$ over the equatorial S^3 in S^4 and the third line is obtained by slicing $\mathbb{P}N$ with a plane $P = \mathbb{CP}^2$ through a chosen point q in the upper (positive norm) half of $\mathbb{CP}^3$, and correspondingly slicing the fibration of $\mathbb{P}N$ over S^3.

The submanifold $\mathbb{P}N$ of the twistor space $\mathbb{CP}^3$ is a level set $\mathbb{P}N = \{K = \sum_\alpha Z^\alpha \bar{Z}_\alpha = 0\}$ of the signature $(+,+,-,-)$ norm associated with the $SU(2,2)$ structure on $\mathbb{CP}^3$ mentioned above. The symplectic form $\omega = \sum_\alpha dZ^\alpha \wedge d\bar{Z}_\alpha$ on the twistor space satisfies $\omega = i\partial\bar{\partial}K = d(dK \circ J)$. The 1-form $\alpha = dK \circ J|_{\mathbb{P}N}$ determines the contact structure on $\mathbb{P}N$ with distribution of contact hyperplanes $\xi = \mathrm{Ker}(\alpha) = T\mathbb{P}N \cap JT\mathbb{P}N$, with J the complex structure. The geometric quantization of the twistor space $\mathbb{CP}^3$ as a symplectic manifold, that we recalled in Section 2.1, induces a compatible quantization of the contact manifold $\mathbb{P}N$. (For a general formalism for geometric quantization of symplectic manifolds with contact boundary see, for example [61].) The planes P through a point $p \in \mathbb{P}T^+$ are symplectic submanifolds and $T(P \cap \mathbb{P}N) \cap \xi$ determines the associated distribution of contact planes on the Hopf spheres $S^3 = P \cap \mathbb{P}N$. One obtains in this way a quantization of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ compatible with the geometric quantization of the twistor space. □

2.7. *Another θ-deformation*

The role of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ in the case of Lorentzian twistor spaces, described in (2.17) and Proposition 2.7, suggests then a different use of the Connes–Landi θ-deformations to obtain a non-commutative deformation of twistor spaces. Instead of deforming S^3 to the non-commutative S^3_θ in diagrams (2.3) and (2.6), as we discussed in Proposition 2.2, which gives a non-commutative $S(M)_\theta$ with commutative $Z(M)$, we can apply the same θ-deformation of the Hopf fibration, with non-commutative S^3_θ and commutative S^1 and S^2, to all the Hopf spheres $S^3 = P \cap \mathbb{P}N$ in (2.17). This gives rise to a resulting θ-deformation for the twistor space $Z(M) = \mathbb{CP}^3$, or of more general twistor spaces in the Lorentzian case. Notice that, while the spacetime manifold is Lorentzian, and Lorentzian geometry is explicitly used to identify copies of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$ inside the Hopf fibration $\mathbb{CP}^1 \hookrightarrow Z(M) \to M$, only the Riemannian structure of S^3 is used in these θ-deformations as the Lorentzian spacetime manifold remains undeformed and classical. Thus, the formalism of θ-deformations (which requires the Riemannian setting of spectral triples) can still be applied. We summarize this reasoning with the following statement, whose proof is analogous to Proposition 2.2.

Proposition 2.8 *A non-commutative Connes–Landi θ-deformation of the twistor space $Z(M) = \mathbb{CP}^3$ and of the Hopf fibration $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^3_\theta \to S^4$ can be obtained by simultaneously applying the Connes–Landi θ-deformation $S^1 \hookrightarrow S^3_\theta \to S^2$ to all the Hopf spheres $S^3 = P \cap \mathbb{P}N$ with P varying over planes in $\mathbb{CP}^3$ passing through a given point $p \in \mathbb{P}T^+$.*

There is a significant difference between a non-commutative deformation of twistor space obtained as in Proposition 2.8 and the geometric quantization of [50]. In the case discussed here, the non-commutative deformation is entirely carried by the Hopf spheres S^3_θ that deform the intersections $P \cap \mathbb{P}N = S^3$. Thus, the non-commutativity only affects the $\mathbb{P}N$ part of twistor space rather than the entire $\mathbb{P}T^\pm$ parts. Significant examples of classical spaces with non-commutative boundaries occur elsewhere, for example the non-commutative boundaries of modular curves studied in [39–41]. On the other hand, in the quantization of [50] it is the entire twistor space that is quantized through its symplectic structure, with a compatible quantization of the contact submanifold $\mathbb{P}N$.

3. Deformations and gluing

In this section, we consider the problem of gluing non-commutative twistor spaces formulated in [54], and we present a general setting based on the Gerstenhaber–Schack complex, [28], to address this question for non-commutative twistor spaces obtained via a procedure of deformation quantization.

In the commutative case, for (anti)self-dual Riemannian manifolds, the gluing of twistor spaces that corresponds to a connected sum of spacetime manifolds is analyzed in [22] in terms of Kodaira–Spencer deformation theory, in the form developed in [27] for singular spaces with normal crossings singularities, applied to the gluing along the exceptional divisors of the blowups of the twistor spaces along one of the twistor lines. Here we consider the more general deformation theory, as formulated in [28], which involves both commutative and non-commutative deformation. We formulate the problem of gluing quantized twistor spaces in terms of the deformation theory of a diagram of algebras. We first discuss the general setting and then we apply it to the different forms of quantization of twistor spaces illustrated in the previous section.

3.1. *Non-commutative deformation and obstructions*

The construction of [22] of the gluing of twistor spaces corresponding to connected sums of the underlying self-dual 4-manifolds relies essentially on the Kodaira–Spencer deformation theory for complex manifolds. Since in our setting we are dealing with non-commutative twistor spaces, we first recall here a setting, the Gerstenhaber–Schack complex, where the usual Kodaira–Spencer deformation theory can be recovered as part of a more general deformation theory of diagrams of unital associative algebras. We follow the exposition of [28] for this summary. For our purposes we restrict to the case of algebras over $\mathbb{C}$, though the setting of [28] is much more general.

In this setting, the deformation theory for a single unital associative algebra $\mathcal{A}$ over $\mathbb{C}$ is governed by its Hochschild cohomology $\oplus_n HH^n(\mathcal{A},\mathcal{A})$. Consider a deformation of an associative algebra $\mathcal{A}$, namely a $\mathbb{C}[[t]]$-algebra $\mathcal{A}[[t]]$ with $\mathbb{C}[[t]]$-linear multiplication $\alpha_t = \alpha + t\alpha_1 + t^2\alpha_2 + \cdots$ extending the multiplication α of $\mathcal{A}$ with $\mathbb{C}$-linear maps $\alpha_i : \mathcal{A}\times\mathcal{A}\to\mathcal{A}$, satisfying the associativity relation $\alpha_t(\alpha_t(a,b),c) = \alpha_t(a,\alpha_t(b,c))$. Using the notation $f\star g(a,b,c) := f(g(a,b),c) - f(a,g(b,c))$, we can rewrite the associativity constraint as a sequence of equations

$$\sum_{p+q=n,p,q>0} \alpha_p \star \alpha_q(a,b,c) = a\alpha_n(b,c) - \alpha_n(ab,c) + \alpha_n(a,bc) - \alpha_n(a,b)c, \tag{3.1}$$

where the right-hand side is the Hochschild coboundary $\delta\alpha_n(a,b,c)$. The first order term α_1 satisfies $\delta\alpha_1 = 0$, so that α_1 defines a Hochschild 2-cocycle and cocycles that differ by a coboundary determine equivalent deformations. Thus, one identifies $HH^2(\mathcal{A},\mathcal{A})$ as parameterizing the infinitesimal deformations of $\mathcal{A}$. The next condition gives $\alpha_1(\alpha_1(a,b),c) - \alpha_1(a,\alpha_1(b,c)) = \delta\alpha_2$, where the left-hand side defines a Hochschilf 3-cocycle. Thus, this constraints represents a possible obstruction to extending the infinitesimal deformation α_1 to a global deformation α_t. One can view this as a quadratic map $\Theta : HH^2(\mathcal{A},\mathcal{A}) \to HH^3(\mathcal{A},\mathcal{A})$. The condition $\Theta([\alpha_1]) = 0$ is the necessary vanishing of the primary obstruction that corresponds to the second associativity constraint. Similarly, the expressions $\sum_{p+q=n,p,q>0} \alpha_p \star \alpha_q$ define 3-cocycles and the constraints (3.1) require that all of these are coboundaries $\delta\alpha_n$, hence trivial in $HH^3(\mathcal{A},\mathcal{A})$.

The deformation theory of a single associative algebra $\mathcal{A}$ is generalized in [28] to a deformation theory of diagrams of algebras. In this setting, given a small category $\mathcal{C}$, a diagram of associative $\mathbb{C}$-algebras over $\mathcal{C}$ is a contravariant functor $\mathbb{A} : \mathcal{C}^{op} \to \mathrm{Alg}_{\mathbb{C}}$. Examples include the cases where $\mathcal{C}$ is the poset of open sets of a smooth manifold ordered by inclusion or the poset of Stein open sets of a complex manifold, seen as a category, and associated commutative algebras of smooth or holomorphic functions, respectively, with restriction maps. The formal deformations of a diagram $\mathbb{A}$ are diagrams of $\mathbb{C}[[t]]$-algebras over the same $\mathcal{C}$ that reduce modulo t to $\mathbb{A}$. This means, for every object C of $\mathcal{C}$ a deformed associative multiplication $\alpha_t^C = \alpha^C + t\alpha_1^C + t^2\alpha_2^C + \cdots$ on the corresponding algebra $\mathbb{A}^C$ and for every morphism $\phi : C \to C'$ a $\mathbb{C}[[t]]$-algebra morphism $\phi_{\mathbb{A}} : \mathbb{A}_t^{C'} \to \mathbb{A}_t^C$ of the deformed algebras, so that one obtains a diagram $\mathbb{A}_t : \mathcal{C}^{op} \to \mathrm{Alg}_{\mathbb{C}[[t]]}$. A suitable notion of equivalence of diagrams and deformations is discussed in Section 17 of [28]. A single algebra $\mathcal{A} = \mathcal{A}_{\mathbb{A}}$ can be associated with a diagram $\mathbb{A} : \mathcal{C} \to \mathrm{Alg}_{\mathbb{C}}$. It is defined as a convolution product over the diagram in the following way. As a $\mathbb{C}$-vector space $\mathcal{A}$ is spanned by elements of the form $\sum a^C \phi_{\mathbb{A},C}$, with elements $a^C \in \mathbb{A}^C$ for objects $C \in \mathrm{Obj}(\mathcal{C})$ and morphisms $\phi_{\mathbb{A},C} \in \mathrm{Mor}(\mathcal{C})$ with source $s(\phi_{\mathbb{A},C}) = C$. The convolution product is determined on the individual components by

$$(a^C \phi_{\mathbb{A},C}) \cdot (a^{C'} \psi_{\mathbb{A},C'}) = a^C \phi_{\mathbb{A},C}(a^{C'})(\phi\psi)_{\mathbb{A},C} \tag{3.2}$$

when $t(\phi) = s(\psi) = C'$ and zero otherwise, with $(\phi\psi)_{\mathbb{A},C} = \psi_{\mathbb{A},C'} \circ \phi_{\mathbb{A},C}$. The deformation theory of diagrams $\mathbb{A}$ is constructed in Section 21 of [28] in terms of a cochain complex that computes a generalization of a local cohomology for a local system over the nerve of the category $\mathcal{C}$, which is given by a Yoneda cohomology.

The subdivision $\mathcal{C}'$ of a small category $\mathcal{C}$ is a category whose simplicial nerve $\mathcal{N}(\mathcal{C}')$ is the first barycentric subdivision of the nerve $\mathcal{N}(\mathcal{C})$. The second subdivision $\mathcal{C}''$ is always a poset. The subdivision comes endowed with a functor $\mathcal{C}' \to \mathcal{C}$, hence a diagram $\mathbb{A} : \mathcal{C} \to \mathrm{Alg}_{\mathbb{C}}$ has an associated subdivision $\mathbb{A}' : \mathcal{C}' \to \mathrm{Alg}_{\mathbb{C}}$ by precomposition. We denote by $\mathcal{A}'$ and $\mathcal{A}''$ the assembled algebras associated with the subdivisions $\mathbb{A}'$ and $\mathbb{A}''$ of the diagram. One also denotes by $\mathbb{A}_{\#}$ the extension of the diagram $\mathbb{A}$ to the category $\mathcal{C}_{\#}$ where a terminator object ∞ has been added with a unique map $C \to \infty$ from every $C \in \mathrm{Obj}(\mathcal{C})$, by setting $\mathbb{A}^\infty = \mathbb{C}$ and $\mathbb{C} \to \mathbb{A}^C$ the unique homomorphism determined by the $\mathbb{C}$-algebra structure of $\mathbb{A}^C$. In the case where the small category $\mathcal{C}$ is a poset, there is an isomorphism of Hochschild homologies $HH^*(\mathbb{A},\mathbb{A}) \simeq HH^*(\mathcal{A},\mathcal{A})$ of a diagram $\mathbb{A}$ of algebras and of its associated single algebra $\mathcal{A} = \mathcal{A}_{\mathbb{A}}$. This is the 'special cohomology comparison theorem' of [29]. For more general small categories $\mathcal{C}$ a similar 'general cohomology comparison theorem' holds ([28], Section 23) which identifies the Hochschild homology $HH^*(\mathbb{A},\mathbb{A}) \simeq HH^*(\mathcal{A}'',\mathcal{A}'')$. These identifications are then used (see Section 25 of [28]) to compare the deformation theory of the diagram $\mathbb{A}$ with that of the assembled algebras $\mathcal{A}$ and $\mathcal{A}''$. Indeed, it is proved in Section 25 of [28] that the deformation theory of a diagram $\mathbb{A}$ is equivalent to the deformation theory of the single algebra $\mathcal{A}''_{\#}$ (see p. 224 of [28]).

In particular, in the case of a complex manifold X, with $\mathcal{T} = \mathcal{T}_X$ the sheaf of (germs of) holomorphic tangent vector fields, and with $\Lambda^k\mathcal{T}$ the exterior powers, a covering of X consisting of Stein open sets (or affine open sets in the case of projective algebraic varieties) closed under intersections determines a poset $\mathcal{C}$ and a diagram $\mathbb{A} : \mathcal{C} \to \mathrm{Alg}_{\mathbb{C}}$ of commutative algebras with $\mathbb{A}^U$ the ring of holomorphic functions on the open set U. Then the Hochschild homology of the diagram is given by $HH^n(\mathbb{A},\mathbb{A}) = \oplus_{\ell+k=n} H^\ell(X,\Lambda^k\mathcal{T})$, where the terms $H^\ell(X,\Lambda^k\mathcal{T})$ are identified with the terms $HH^{\ell,k}(\mathbb{A},\mathbb{A})$ defined more generally for a diagram of commutative algebras in Section 26 of [28]. These identifications $HH^{\ell,k}(\mathbb{A},\mathbb{A}) \simeq H^\ell(X,\Lambda^k\mathcal{T})$ were proved in [28] for the case where X is a smooth projective variety and $\mathcal{C}$ the poset determined by a covering of affine open sets and conjectured for the case of a complex manifold with Stein open sets. A more general setting where these identifications hold, which includes complex analytic manifolds and smooth schemes in characteristic zero is given in [58]. In particular, all the infinitesimal deformations of the diagram $\mathbb{A}$ of commutative algebras are parameterized by $HH^2(\mathbb{A},\mathbb{A})$, with an obstruction map $HH^2(\mathbb{A},\mathbb{A}) \to HH^3(\mathbb{A},\mathbb{A})$. Among these deformations, the part $HH^{1,1}(\mathbb{A},\mathbb{A}) \simeq H^1(X,\mathcal{T})$ parameterizes deformations of $\mathbb{A}$ to diagrams of commutative algebras, that is, classical deformations of the underlying manifold X, with the obstruction map $HH^{1,1}(\mathbb{A},\mathbb{A}) \to H^{2,1}(\mathbb{A},\mathbb{A})$ identified with the classical obstruction map $H^1(X,\mathcal{T}) \to H^2(X,\mathcal{T})$. These are deformations 'in the commutative direction'. The part $HH^{2,0}(\mathbb{A},\mathbb{A}) \simeq H^0(X,\Lambda^2\mathcal{T})$ of the space classifying infinitesimal deformations of $\mathbb{A}$ corresponds instead to those deformations of the diagram of commutative algebras to diagrams of *non-commutative* associative algebras, deformations 'in the non-commutative direction'.

3.2. *Classical and non-commutative deformations of twistor spaces*

We analyze here the classical and non-commutative deformation theory of the twistor spaces $Z_i = Z(M_i)$ of two (anti)self-dual Riemannian manifolds M_i and of their blowups $\tilde{Z}_i$ along a fixed twistor line. We describe the classical and non-commutative deformation theory of the gluing $\tilde{Z}$ of the blowups along the exceptional divisors in terms of the Hochschild cohomology of an associated diagram of algebras as in [**28**]. We start by showing how to associate to the gluing $\tilde{Z}(M) = \tilde{Z}(M_1) \sqcup_{E_1 \simeq E_2} \tilde{Z}(M_2)$, of the blowups $\tilde{Z}(M_i) = \mathrm{Bl}_{\mathbb{CP}^1}(Z(M_i))$ along the exceptional divisors a diagram of algebras in the sense of [**28**].

LEMMA 3.1 *The singular space $\tilde{Z}(M)$ obtained by gluing the blowups $\tilde{Z}(M_i)$ along the exceptional divisors E_i determines an associated diagram of commutative algebras $\mathbb{A}(\tilde{Z}) : \mathcal{C} \to \mathrm{Alg}_{\mathbb{C}}$, where $\mathcal{C}$ is a poset determined by a system of Stein open sets in the complement of E_i in $\tilde{Z}_i$ and pairs of Stein open sets in $\tilde{Z}_1$ and $\tilde{Z}_2$ that contain the identified exceptional divisors.*

Proof. Let $\gamma : T_{x_1}(M_1) \to T_{x_2}(M_2)$ be the orientation reversing isometry of the tangent spaces of the spacetime manifolds M_i at the points x_i where the connected sum is performed. We denote by the same symbol γ the induced identification of the exceptional divisors $\gamma : E_1 \to E_2$ of the blowups of the twistor spaces $Z(M_i)$ at the twistor lines $\mathbb{CP}^1_{x_i}$. Let $\mathcal{U}_i = \{U_{i,\alpha}\}$ be open coverings of the blown up twistor spaces $\tilde{Z}_i = \tilde{Z}(M_i)$ by Stein open sets, closed under intersections. They form a poset under inclusions. Consider then the small category $\mathcal{C}$ with objects given by those $U_{i,\alpha}$ in the coverings $\mathcal{U}_i$ with the property that $U_{i,\alpha} \cap E_i = \emptyset$ and additional objects given by pairs $(U_{1,\alpha}, U_{2,\beta})$ of open sets in these coverings such that $E_1 \cap U_{1,\alpha} \neq \emptyset$ and $E_2 \cap U_{2,\beta} \neq \emptyset$ and such that $\gamma : E_1 \cap U_{1,\alpha} \to E_2 \cap U_{2,\beta}$ is an isomorphism. Morphisms of $\mathcal{C}$ between open sets of each covering $\mathcal{U}_i$ are inclusions and morphisms between pairs $(U_{1,\alpha}, U_{2,\beta})$ and $(U_{1,\alpha'}, U_{2,\beta'})$ are pairs of inclusions $\iota_{1,\alpha,\alpha'} : U_{1,\alpha} \hookrightarrow U_{1,\alpha'}$ and $\iota_{2,\beta,\beta'} : U_{1,\beta} \hookrightarrow U_{1,\beta'}$ with the property that $\iota_{2,\beta,\beta'}|_{E_2 \cap U_{2,\beta}} \circ \gamma = \iota_{1,\alpha,\alpha'}|_{E_1 \cap U_{1,\alpha}}$. We then construct a functor $\mathbb{A} : \mathcal{C} \to \mathrm{Alg}_{\mathbb{C}}$ by assigning to objects $U_{i,\alpha}$ $\mathbb{A}^{U_{i,\alpha}} = \mathcal{A}(U_{i,\alpha})$ the algebra of holomorphic functions on $U_{i,\alpha}$ with morphisms $\mathbb{A}(\iota_{i,\alpha,\alpha'}) = \rho_{i,\alpha',\alpha} : \mathcal{A}(U_{i,\alpha'}) \to \mathcal{A}(U_{i,\alpha})$ the restriction map corresponding to the inclusion $\iota_{i,\alpha,\alpha'} : U_{i,\alpha} \hookrightarrow U_{i,\alpha'}$. To objects $(U_{1,\alpha}, U_{2,\beta})$ we assign the algebra $\mathbb{A}^{(U_{1,\alpha}, U_{2,\beta})}$ given by

$$\{(f_{1,\alpha}, f_{2,\beta}) : f_{1,\alpha} \in \mathcal{A}(U_{1,\alpha}), f_{2,\beta} \in \mathcal{A}(U_{2,\beta}), f_{2,\beta}|_{E_2 \cap U_{2,\beta}} \circ \gamma = f_{1,\alpha}|_{E_1 \cap U_{1,\alpha}}\},$$

with morphisms $(\iota_{1,\alpha,\alpha'}, \iota_{2,\beta,\beta'})$ with $\iota_{2,\beta,\beta'}|_{E_2 \cap U_{2,\beta}} \circ \gamma = \iota_{1,\alpha,\alpha'}|_{E_1 \cap U_{1,\alpha}}$ in $\mathcal{C}$ mapped to the restriction maps $\rho_{\alpha',\beta',\alpha,\beta} : \mathbb{A}^{(U_{1,\alpha'}, U_{2,\beta'})} \to \mathbb{A}^{(U_{1,\alpha}, U_{2,\beta})}$. □

The general construction of the assembled algebra $\mathcal{A}_{\mathbb{A}}$ associated with a diagram and the special cohomology comparison theorem of [28] then give the following.

COROLLARY 3.2 *The deformation theory of the diagram $\mathbb{A}(\tilde{Z})$ is equivalent to the deformation theory of a single algebra generated by elements of the form $f_{i,\alpha}\rho_{i,\alpha',\alpha}$ and $(f_{1,\alpha}, f_{2,\beta})\,\rho_{\alpha',\beta',\alpha,\beta}$ with the convolution product*

$$f_{i,\alpha}\rho_{i,\alpha',\alpha}\cdot f_{i,\alpha'}\rho_{i,\alpha'',\alpha'} = f_{i,\alpha}\rho_{i,\alpha',\alpha}(f_{i,\alpha'})\,\rho_{i,\alpha'',\alpha}$$

$$(f_{1,\alpha}, f_{2,\beta})\,\rho_{\alpha',\beta',\alpha,\beta}\cdot(f_{1,\alpha'}, f_{2,\beta'})\,\rho_{\alpha'',\beta'',\alpha',\beta'} = (f_{1,\alpha}\rho_{\alpha',\alpha}(f_{1,\alpha'}), f_{2,\beta}\rho_{\beta',\beta}(f_{2,\beta'}))\,\rho_{\alpha'',\beta'',\alpha,\beta}$$

and zero otherwise.

The computation of the Hochschild cohomology that governs the deformation theory of the diagram $\mathbb{A}(\tilde{Z})$ then gives the following result that recovers the Donaldson–Friedman deformation theory of the singular space $\tilde{Z}$ as the part of the deformation theory of the diagram $\mathbb{A}(\tilde{Z})$ that corresponds to deformations ‘in the commutative direction’.

THEOREM 3.3 *The commutative part of the deformation theory of the diagram $\mathbb{A}(\tilde{Z})$ recovers the Donaldson–Friedman deformation theory of the singular space $\tilde{Z}$.*

Proof. Near the normal crossings singular locus, the gluing $\tilde{Z} = \tilde{Z}_1 \sqcup_{E_1 \overset{\gamma}{\simeq} E_2} \tilde{Z}_2$ is locally described by $\{z_0 z_1 = 0\} \subset \mathbb{C}^4$ and we can assume that the open coverings $\mathcal{U}_i$ of $\tilde{Z}_i$ are chosen so that this local description holds for each $U_{1,\alpha} \sqcup_{U_{1,\alpha}\cap E_1 \simeq E_2 \cap U_{2,\beta}} U_{2,\beta}$. Thus, we can view the algebras $\mathcal{A}_{\alpha,\beta} := \mathcal{A}(U_{1,\alpha}, U_{2,\beta})$ as copies of the algebra associated with $V = \{z_0 z_1 = 0\} \subset \mathbb{C}^4$. In this case, the Hochschild cohomology is computed by André–Quillen cohomology, namely the decomposition $HH^n(\mathcal{A},\mathcal{A}) = \oplus_r HH^{n-r,r}(\mathcal{A},\mathcal{A})$ satisfies $HH^{n-r,r}(\mathcal{A},\mathcal{A}) \simeq T^{n-r,r}(\mathcal{A})$, where the André–Quillen cohomology $T^{i,j}(\mathcal{A})$ is the jth cohomology group of $\mathrm{Hom}_{\mathcal{A}}(\Lambda^i \mathbb{L}_A, A)$ for the derived exterior power $\Lambda^i \mathbb{L}_A$ of the cotangent complex, see [37] Section 3.5. The terms $T^{i,1}(\mathcal{A})$ correspond to the terms defined as $\mathbb{T}^i_V$ of [22] and are identified with the piece $HH^{n-1,1}(\mathcal{A},\mathcal{A}) = T^{n-1,1}(\mathcal{A})$ of $HH^n(\mathcal{A},\mathcal{A})$. In terms of deformation theory, the term $HH^{1,1}(\mathcal{A},\mathcal{A})$ of the second Hochschild cohomology parameterizes the infinitesimal deformations in the ‘commutative direction’, while the term $HH^{2,0}(\mathcal{A},\mathcal{A})$ of the second Hochschild cohomology represents the infinitesimal deformations of $\mathcal{A}$ in the ‘non-commutative direction’. The obstruction map for the classical deformations is given by the component $\Phi : HH^{1,1}(\mathcal{A},\mathcal{A}) \to HH^{2,1}(\mathcal{A},\mathcal{A})$ of the overall obstruction map $\Phi : HH^2(\mathcal{A},\mathcal{A}) \to HH^3(\mathcal{A},\mathcal{A})$. This corresponds to the obstruction map $\Phi : \mathbb{T}^1_V = T^{1,1}(\mathcal{A}) \to \mathbb{T}^2_V = T^{2,1}(\mathcal{A})$ considered in [22]. To see then that this identification holds not only at the local level of the algebras $\mathcal{A}_{\alpha,\beta}$ but also globally for $\tilde{Z}$, we can use the fact that the Hochschild cohomology $HH^*(\mathbb{A},\mathbb{A})$ for diagrams of algebras and the pieces $HH^{r,n-r}(\mathbb{A},\mathbb{A})$ of the decomposition can be computed in terms of two filtrations (that truncate the first rows or columns, respectively) on a double complex $C^{*,*}(\mathbb{A},\mathbb{A})e(r)$, with $e(r)$ the idempotent that determines the $(r, n-r)$ piece, which has the Hochschild differential on the vertical direction and the simplicial differential of the nerve of the category $\mathcal{C}$ in the horizontal direction and building blocks given by $\prod_{\dim\sigma=p} C^{*+r}(\mathbb{A}^{c\sigma},\mathbb{A}^{c\sigma})e(r) \otimes_{\mathbb{A}^{c\sigma}} \mathbb{A}^{d\sigma}$ where a p-simplex $\sigma : [p] \to \mathcal{C}$ is a covariant functor from the category $[p] = \{0 < \cdots < p\}$ to $\mathcal{C}$ and $c\sigma = \sigma(0)$ and $d\sigma = \sigma(p)$. These filtrations determine a spectral sequence converging to $HH^*(\mathbb{A},\mathbb{A})$, see

Sections 21–26 of [28]. For $\mathbb{A}(\tilde{Z})$ one obtains in this way the deformation theory $\Phi : \mathbb{T}^1_{\tilde{Z}} \to \mathbb{T}^2_{\tilde{Z}}$ of [22] from the component $\Phi : HH^{1,1}(\mathbb{A}(\tilde{Z}), \mathbb{A}(\tilde{Z})) \to HH^{2,1}(\mathbb{A}(\tilde{Z}), \mathbb{A}(\tilde{Z}))$ of the deformation theory $\Phi : HH^2(\mathbb{A}(\tilde{Z}), \mathbb{A}(\tilde{Z})) \to HH^3(\mathbb{A}(\tilde{Z}), \mathbb{A}(\tilde{Z}))$ of the diagram of algebras. □

The result above shows, in particular, that even when the deformation theory $\Phi : \mathbb{T}^1_{\tilde{Z}} \to \mathbb{T}^2_{\tilde{Z}}$ of [22] for the gluing $\tilde{Z}$ of the blowups of the twistor spaces $Z(M_i)$ is obstructed, it may still be possible to obtain an unobstructed deformation theory in the 'non-commutative direction', that is, for the infinitesimal deformations in $HH^{2,0}(\mathbb{A}(\tilde{Z}), \mathbb{A}(\tilde{Z}))$. This means that, in such cases, even if the singular $\tilde{Z}$ cannot be deformed commutatively to the smooth twistor space $Z(M)$ for the connected sum $M = M_1 \# M_2$ (for instance if M does not admit a (anti)self-dual structure) one still has a non-commutative twistor space $Z(M)_\hbar$ obtained as a deformation in the non-commutative direction of $\tilde{Z}$.

We also need a consistency relation between the choices of the non-commutative deformations on the twistor spaces $Z_i = Z(M_i)$ and on the glued $\tilde{Z}$. This can be obtained by first showing that the deformations of the twistor spaces Z_i induce deformations of the blown up twistor spaces $\tilde{Z}_i$ and then by identifying a compatibility condition between the deformations of the $\tilde{Z}_i$ and the deformation of $\tilde{Z}$.

We first recall the following setting. As shown in [4], given a diagram of algebras

$$\begin{array}{ccc} A & \xrightarrow{\phi_1} & A_1 \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \psi_1} \\ A_2 & \xrightarrow{\psi_2} & A_3 \end{array} \tag{3.3}$$

such that

$$0 \to A \overset{\phi_1 \oplus \phi_2}{\to} A_1 \oplus A_2 \overset{\psi_1 - \psi_2}{\to} A_3 \to 0 \tag{3.4}$$

is an exact sequence of A-bimodules, with the properties that the maps $A \to A_i$ are flat epimorphisms, then there is a long Mayer–Vietoris exact sequence for Hochschild homology

$$\cdots \to HH_n(A, A) \to HH_n(A_1, A_1) \oplus HH_n(A_2, A_2) \to HH_n(A_3, A_3) \to \cdots$$

We cannot apply this to directly to the case of the spaces $\tilde{Z}_i$ and $\tilde{Z}$ and their local models near the normal crossings singularity $E_1 \simeq E_2$, because the algebra homomorphisms ϕ_i in the corresponding diagram do not satisfy the flatness hypothesis. Thus, we cannot compare directly the deformation classes and the obstructions for $\tilde{Z}_i$ and $\tilde{Z}$ through the Mayer–Vietoris sequence. However, there is still a long exact sequence of Hochschild cohomology that we can use to compare these deformations.

Let $\mathcal{A}_{\alpha,\beta}$ be one of the algebras describing the geometry of $\tilde{Z}$ near the normal crossings singularity in the diagram of algebras $\mathbb{A}(\tilde{Z})$ and let $\mathcal{A}_{\alpha,1}$ and $\mathcal{A}_{\beta,2}$ be algebras in the diagrams $\mathbb{A}(\tilde{Z}_1)$ and $\mathbb{A}(\tilde{Z}_2)$, respectively, describing the geometry near the exceptional divisor E_i. For simplicity of notation we drop the subscripts α, β and we just refer to these algebras as $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2$.

LEMMA 3.4 *Let $\gamma_i \in HH^2(\mathcal{A}_i, \mathcal{A}_i)$ for $i = 1, 2$ be unobstructed deformation classes, $\Phi(\gamma_i) = 0 \in HH^3(\mathcal{A}_i, \mathcal{A}_i)$, and let $\gamma \in HH^2(\mathcal{A}, \mathcal{A})$ be a deformation class that is also unobstructed, $\Phi(\gamma) = 0 \in$*

$HH^3(\mathcal{A},\mathcal{A})$. There are epimorphisms $\phi_i : \mathcal{A} \to \mathcal{A}_i$ that induce a long exact sequence of Hochschild cohomology

$$\cdots \to HH^n(\mathcal{A},\mathcal{A}) \to HH^n(\mathcal{A},\mathcal{A}_1) \oplus HH^n(\mathcal{A},\mathcal{A}_2) \to HH^n(\mathcal{A},\mathcal{A}_3) \to \cdots \tag{3.5}$$

and morphisms $HH^n(\mathcal{A}_i,\mathcal{A}_i) \to HH^n(\mathcal{A},\mathcal{A}_i)$. The image $c_i(\gamma)$ of γ in $Z^2(\mathcal{A},\mathcal{A}_i)$ is a 2-cocycle that extends to a 2-cocycle in the complex $(Z^2(\mathcal{A},\mathcal{A}_i)[[t]],\delta)$ where $\delta c = \delta_i c + [m_\gamma, c]$ with δ_i the Hochschild differential of $C^*(\mathcal{A},M)$ for the bimodule $M = \mathcal{A}_i$ and m_γ the deformed multiplication on $\mathcal{A}$ determined by the unobstructed deformation class $\gamma \in HH^2(\mathcal{A},\mathcal{A})$. Similarly, the image $c(\alpha_i)$ of γ_i in $Z^2(\mathcal{A},\mathcal{A}_i)$ is a 2-cocycle that extends to a 2-cocycle in the complex $(Z^2(\mathcal{A},\mathcal{A}_i)[[t]],\delta)$.

Proof. In our case the geometry near the normal crossings singularity can be described as the locus $\{z_1 z_2 = 0\}$ with $\{z_1 = 0\}$ and $\{z_2 = 0\}$ the two components and $\{z_1 = z_2 = 0\}$ the intersection. The corresponding algebras $\mathcal{A},\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$ then fit into a diagram (3.3) which satisfies the exactness of the associated sequence of $\mathcal{A}$-modules (3.4) and the epimorphism condition, which can also be stated as the condition that $\mathcal{A}_i \otimes_{\mathcal{A}} \mathcal{A}_i \simeq \mathcal{A}_i$. The short exact sequence (3.4) of $\mathcal{A}$-modules induces a long exact sequence (3.5) of Hochschild cohomology (see [**28**], p. 36). Moreover, the Hochschild cohomology is a contravariant functor in the algebra, hence the homomorphisms $\phi_i : \mathcal{A} \to \mathcal{A}_i$ induce homomorphisms $\phi_i^* : HH^n(\mathcal{A}_i,\mathcal{A}_i) \to HH^n(\mathcal{A},\mathcal{A}_i)$. Consider an unobstructed deformation class $\gamma \in HH^2(\mathcal{A},\mathcal{A})$. The condition $\Phi(\gamma) = 0 \in HH^3(\mathcal{A},\mathcal{A})$, ensuring that all obstructions vanish, is the condition that the left-hand side of (3.1) are all coboundaries for all n. A homomorphism $\phi : M \to N$ of $\mathcal{A}$-modules induces a morphism $C^n(\mathcal{A},M) \to C^n(\mathcal{A},N)$ by composition, mapping a multi-linear map $f : \mathcal{A} \times \cdots \times \mathcal{A} \to M$ by to the multi-linear map $\phi \circ f : \mathcal{A} \times \cdots \times \mathcal{A} \to N$, which is a cochain map. We still denote by γ a 2-cocycle representing the deformation of $\mathcal{A}$ and by $\phi_i(\gamma)$ its image in $Z^2(\mathcal{A},\mathcal{A}_i)$. The 3-cochains $\Phi(\gamma)_n$ in the left-hand side of (3.1) are similarly mapped to 3-cochains $\phi_i \circ \Phi(\gamma)_n$ in $C^3(\mathcal{A},\mathcal{A}_i)$. We need to check that these cochains are the cochains that determine the extensibility condition of the cocycle $\phi_i(\gamma) \in Z^2(\mathcal{A},\mathcal{A}_i)$ to a 2-cocycle in the complex $(Z^*(\mathcal{A},\mathcal{A}_i)[[t]],\delta_\gamma)$. This extensibility condition is discussed in [**56**]. A 2-cocycle $\upsilon \in Z^2(\mathcal{A},M)$ extends to a 2-cocycle υ_t to $(Z^*(\mathcal{A},M)[[t]],\delta_\gamma)$ iff υ_t is determined by a choice of a collection $\upsilon_n \in C^2(\mathcal{A},M)$ satisfying the property that all the obstructions 3-cochains

$$\omega_n(\upsilon) := \sum_{p+q=n,\, p>0} \upsilon_q \star \gamma_p$$

are coboundaries, where

$$\upsilon_q \star \gamma_p(a,b,c) = \upsilon_q(\gamma_p(a,b),c) - \upsilon_q(a,\gamma_p(b,c)).$$

In particular, for $\upsilon_i = \phi_i(\gamma) \in C^2(\mathcal{A},\mathcal{A}_i)$ we have $\omega_n(\upsilon_i) = \phi_i(\omega_n(\gamma))$, hence if γ is an unobstructed deformation of $\mathcal{A}$ with $[\omega_n(\gamma)] = 0 \in HH^3(\mathcal{A},\mathcal{A})$ we also have that $\upsilon_i = \phi_i(\gamma)$ is a cocycle that extends to $(Z^*(\mathcal{A},\mathcal{A}_i)[[t]],\delta_\gamma)$. Moreover, the cocycles $\upsilon_i = \phi_i(\gamma) \in C^2(\mathcal{A},\mathcal{A}_i)$ determined the same class $[\psi_1(\upsilon_1)] = [\psi_2(\upsilon_2)] \in HH^3(\mathcal{A},\mathcal{A}_3)$ by the long exact sequence. The case for the contravariant functoriality $\phi_i^* : HH^n(\mathcal{A}_i,\mathcal{A}_i) \to HH^n(\mathcal{A},\mathcal{A}_i)$ is similar: if γ_i is an unobstructed deformation of $\mathcal{A}_i$ then the 2-cocycle $\phi_i^*(\gamma_i)$ extends to $(Z^*(\mathcal{A},\mathcal{A}_i)[[t]],\delta_\gamma)$. □

We can then propose as compatibility condition between the deformations of the algebras $\mathcal{A}_i$ and of $\mathcal{A}$ as the condition that the 2-cocycles in $Z^2(\mathcal{A},\mathcal{A}_i)$ obtained in this way define the same class, $[c(\gamma_i)] = [c_i(\gamma)] \in HH^2(\mathcal{A},\mathcal{A}_i)$.

We still need to discuss how an unobstructed deformation theory for the twistor spaces $Z_i = Z(M_i)$ determines an unobstructed deformations of their blowups $\tilde{Z}_i = \mathrm{Bl}_{\mathbb{CP}^1_{x_i}}(Z_i)$. In fact, this issue is already discussed in [22], although only deformations in the 'commutative direction' $HH^{1,1}(\mathbb{A},\mathbb{A})$ are considered there with obstruction map $\Phi : HH^{1,1}(\mathbb{A},\mathbb{A}) \to HH^{2,1}(\mathbb{A},\mathbb{A})$. We show here how the argument needs to be modified in our setting to account for the full non-commutative deformation theory in $HH^2(\mathbb{A},\mathbb{A})$ with obstruction map $\Phi : HH^2(\mathbb{A},\mathbb{A}) \to HH^3(\mathbb{A},\mathbb{A})$.

PROPOSITION 3.5 *For $i = 1, 2$, let (Z_i, L_i) be the pairs of the twistor spaces $Z_i = Z(M_i)$ of (anti)self-dual Riemannian 4-manifolds M_i and the twistor lines $L_i = \mathbb{CP}^2_{x_i}$ over chosen points $x_i \in M_i$. Let γ_i be unobstructed non-commutative deformations of the pairs (Z_i, L_i). These determine compatible unobstructed non-commutative deformations of Z_i and of the blowups $\tilde{Z}_i$.*

Proof. Let $\gamma_i^{1,1}$ be the Hodge component in $HH^{1,1}(\mathbb{A}(Z_i),\mathbb{A}(Z_i))$. Since we are assuming that γ_i involves a non-trivial deformation in the non-commutative direction, we know $\gamma_i^{1,1} \neq 0$. The $\gamma_i \in HH^2(\mathbb{A}(Z_i),\mathbb{A}(Z_i))$ satisfy $\Phi(\gamma_i) = 0 \in HH^3(\mathbb{A}(Z_i),\mathbb{A}(Z_i))$ hence the $\gamma_i^{1,1} \in HH^{1,1}(\mathbb{A}(Z_i),\mathbb{A}(Z_i))$ also satisfy $\Phi(\gamma_i^{1,1}) = 0 \in HH^{2,1}(\mathbb{A}(Z_i),\mathbb{A}(Z_i))$. We can view the non-commutative deformation as being parameterized by a non-trivial holomorphic skew multivector field, $\gamma_i \in H^0(Z_i, \Lambda^2\mathcal{T}_{Z_i}))$, with the obstruction vanishing in $H^0(Z_i, \Lambda^3\mathcal{T}_{Z_i}))$. As in [22], we denote by $\mathcal{T}_{Z_i,L_i}$ the sheaf of holomorphic vector fields on Z_i that are tangent to L_i along L_i. These are related to $\mathcal{T}_{Z_i}$ by the short exact sequence of sheaves $0 \to \mathcal{T}_{Z_i,L_i} \to \mathcal{T}_{Z_i} \to \nu_i \to 0$, with ν_i the normal bundle of L_i in Z_i. Holomorphic vector fields on Z_i that preserve L_i extend to holomorphic vector fields on the blowup $\tilde{Z}_i = \mathrm{Bl}_{L_i}(Z_i)$, hence deformations of the pair (Z_i, L_i) classified by elements in $H^0(Z_i, \Lambda^2\mathcal{T}_{Z_i,L_i}))$ with obstructions in $H^0(Z_i, \Lambda^3\mathcal{T}_{Z_i,L_i}))$ determine corresponding deformations of $\tilde{Z}_i = \mathrm{Bl}_{L_i}(Z_i)$. Since Z_i is a three-dimensional complex manifold, sections in $H^0(Z_i, \Lambda^3\mathcal{T}_{Z_i}))$ are spanned as $\mathcal{A}(Z_i)$-module by $\partial_{z_0} \wedge \partial_{z_1} \wedge \partial_{z_2}$, for local coordinates (z_0, z_1, z_2). Since L_i is a line, this means that along L_i the vector fields in $\mathcal{T}_{Z_i,L_i}$ are generated as $\mathcal{A}(Z_i)$-module by ∂_z with z a local coordinate on the line L_i, hence the exterior powers vanish along L_i, which means that sections of $\Lambda^3\mathcal{T}_{Z_i,L_i}$ are locally of the form $f(z_0, z_1, z_2)\,\partial_{z_0} \wedge \partial_{z_1} \wedge \partial_{z_2}$ where f is in the ideal of functions vanishing along the line L_i. Similarly, if $\partial_{z_i} \wedge \partial_{z_j}$ with $i<j$ is a local basis for sections of $\Lambda^2\mathcal{T}_{Z_i}$, we can see sections of $\Lambda^2\mathcal{T}_{Z_i,L_i}$ as satisfying a vanishing condition along L_i. The obstructions $\omega_n(\gamma_i)$ of a section of $\Lambda^2\mathcal{T}_{Z_i,L_i}$ are given by

$$\omega_n(\gamma_i) = \sum_{\ell+k=n,\, \ell>0} \gamma_{i,\ell} \star \gamma_{i,k},$$

for a collection of Hochschild 2-cochains $\{\gamma_{i,k}\}_{k\in\mathbb{N}}$ for the pair (Z_i, L_i). These determine Hochschild 3-cocycles of the deformation theory of (Z_i, L_i) which we can identify with sections in $H^0(Z_i, \Lambda^3\mathcal{T}_{Z_i,L_i}))$. Thus, we obtain that unobstructed non-commutative deformations of the pairs (Z_i, L_i) determine compatible unobstructed non-commutative deformations of Z_i and of the blowup $\tilde{Z}_i$. □

In this section, we have focused primarily on the Riemannian case, in order to compare our deformation and gluing procedure for non-commutative twistor spaces, based on the Gerstenhaber–Schack complex, with the deformation and gluing theory of classical twistor spaces of Donaldson–Friedman, which is formulated in the Riemannian context. It is important to stress, though, that the Gerstenhaber–Schack approach to deformations and the associated obstruction theory does not require the Riemannian assumption and can be applied very generally to non-commutative twistor spaces, either Riemannian ot Lorentzian, described in terms of deformation quatizations. Other forms of non-commutative deformations, such as those based on the Connes–Landi θ-deformations, however, have an underlying Riemannian assumption, since they are based on the spectral triples formalism, which at present is not fully developed in the Lorentzian case. On the other hand, in the case of the original quantization of twistor spaces of [**50**] the Gerstenhaber–Schack formalism described in this section applies in both Riemannian and Lorentzian setting, and provides a general setting for the gluing problem described in Section D of [**54**]. We discuss these specific cases more in detail in the next section.

4. Gluing quantized twistor spaces

The deformation and gluing procedure described above is very general in the sense that it applies in any setting where a quantization of twistor spaces is constructed using a deformation quantization procedure. The specific quantizations of twistor spaces that we discussed in Section 2, however, have additional structure such as the geometric quantization, the θ-deformation, the deformation quantization of the Hopf fibration, and the almost commutative geometry. Thus, it is better for each of these cases to analyze how a gluing procedure works that accounts for these additional structures.

4.1. *Gluing of geometric quantizations*

We start with our main object of interest, which is the geometric quantization of twistor spaces constructed by one of us in [**50**]. We have shown in Section 2.1 that these quantized twistor spaces can be seen as deformation quantizations, through the Fedosov relation [**24**] between geometric and deformation quantization. We can then apply the construction we presented in Section 3.2.

We can proceed as described in the previous section to construct a non-commutative twistor space for the connected sum $M = M_1 \# M_2$, given the quantizations of the twistor spaces $Z(M_i)$. If these quantizations are obtained using the geometric quantization method of [**50**], then we want to check that, if a classical unobstructed deformation Z_t exists of the singular gluing $\tilde{Z}$ of the blowups $\tilde{Z}_i$ of the twistor spaces $Z(M_i)$, then the gluing of the quantized twistor spaces can be performed in a way that gives rise of a geometric quantization of the deformation Z_t.

PROPOSITION 4.1 *Let M_i be two (anti)self-dual Riemannian manifolds with $Z_i = Z(M_i)$ their twistor spaces. Under the connected sum $M = M_1 \# M_2$ operation, a gluing of the geometric quantizations of the Z_i's is determined by the geometric quantization of a Gompf symplectic sum of of the $X_i = \mathcal{S}^+(M_i)_0$ that fiber over Z_i with $\mathbb{C}^*$ fibers.*

Proof. As in Section 2.1, we consider the symplectic form $\omega_i = \sum_\alpha dZ_i^\alpha \wedge d\bar{Z}_{i,\alpha}$ on $X_i := \mathcal{S}^+(M_i)_0$, and we consider $\tilde{X}_i = \tilde{\mathcal{S}}^+(M_i)_0$, the pullback of the $\mathbb{C}^*$-bundle $\mathcal{S}^+(M_i)_0$ along the projection map $\tilde{Z}_i \to Z_i$ from the blowup $\tilde{Z}_i = \mathrm{Bl}_{L_{x_i}}(Z_i)$ of a twistor line L_{x_i} in Z_i. The singular space $\tilde{Z}$ obtained by the gluing of the complex manifolds $\tilde{Z}_i$ along their exceptional divisors $\tilde{Z} = \tilde{Z}_i \cup_{E_1 \simeq E_2} \tilde{Z}_2$ corresponds to a gluing $\tilde{X} = \tilde{X}_1 \cup_{V_1 \simeq V_2} X_2$, with V_i the real codimension two symplectic submanifold of $\tilde{X}_i$ given by the preimage of the exceptional divisor E_i, which is a singular symplectic variety with a normal crossings singularity. The singular space $\tilde{Z}$ satisfies the d-semistable condition, namely the normal bundles ν_i of E_i inside $\tilde{Z}_i$ are such that $\nu_1 \otimes \nu_2$ is the trivial line bundle. Thus, the Gompf symplectic sum construction of [**30**] applies to the pairs $(\tilde{X}_i, V_i, \tilde{\omega}_i)$ and gives a one-parameter deformation family, in the form of a nearly regular symplectic fibration $(\mathcal{X}, \omega, \pi : \mathcal{X} \to \mathbb{C})$ with $\pi^{-1}(0) = \tilde{X}$. For $t \neq 0$, the restriction ω_t of ω to $X_t = \pi^{-1}(t)$ is non-degenerate, and is a smoothing of $\tilde{X}$. Thus, we can regard the geometric quantization of X_t as the quantized twistor space resulting from the gluing of the quantized twistor spaces of the manifolds M_i. If the classical deformation theory of $\tilde{Z}$ is unobstructed, so that we have a smooth deformation Z_t of $\tilde{Z}$, then this construction can be done compatibly with the $\mathbb{C}^*$-fibrations $X_i \to Z_i$ so that $X_t = \mathcal{S}^+(M_t)_0$ provides such a deformation, where M_t is the connected sum 4-manifold $M_1 \# M_2$ endowed with an (anti)self-dual metric g_t for which $Z_t = Z(M_t)$ is the twistor space. □

4.2. *Gluing of deformations of the Hopf fibration*

In the case of the deformation quantization of the Hopf fibration and the associated quantization of twistor spaces discussed in Section 2.4 the question is whether the gluing procedure described in Section 3.2 maintains the compatibility with the Hopf fibration. Since in the Riemannian setting the Hopf fibration $Z(M) \to M$ with fibers the twistor lines assumes the existence of an (anti)self-dual structure on M, we can work under the hypothesis that the underlying commutative deformation theory of the $Z(M_i)$ is unobstructed and there is a resulting twistor space $Z(M)$, where $M = M_1 \# M_2$ has an (anti)self-dual structure, obtained as classical deformation of the singular $\tilde{Z}$ as in [**22**].

Under this assumption, we can identify the result of the non-commutative deformation of $\tilde{Z}$ of Section 3.2 with a non-commutative deformation of $Z(M)$. We need to check that, if the non-commutative deformations of the $Z(M_i)$ are chosen to be deformations as in Section 2.4, obtained via a non-commutative deformation of the Hopf fibration $S^1 \hookrightarrow S^3 \to \mathbb{CP}^1$, then the resulting non-commutative deformation of $Z(M)$ is also of this form. We can view this as the non-commutative analog of the argument of [**22**] showing that the classical deformation of the singular space $\tilde{Z}$ is indeed the twistor space of $M = M_1 \# M_2$, hence in particular it has an associated Hopf fibration. Indeed the result of [**22**] for the classical deformation will directly imply the compatibility of the non-commutative deformations.

PROPOSITION 4.2 *Let $Z_{i,\hbar}$ be non-commutative deformations of the twistor spaces $Z_i = Z(M_i)$ with compatible non-commutative deformations $S_{i,\hbar}$ of $S_i = S(M_i)$, obtained as in* Proposition 2.4 *that fit in the Hopf fibrations diagram* (2.13). *Let Z_t be a classical deformation of the singular space $Z_0 = \tilde{Z}$ obtained by gluing the blowups of Z_i at a twistor line along the exceptional divisors. Then the deformations $Z_{i,\hbar}$ and $S_{i,\hbar}$ and Z_t determine compatible non-commutative deformations $\tilde{Z}_\hbar$, $\tilde{S}_\hbar$ and $Z_{t,\hbar}$ and $S_{t,\hbar}$ that satisfy the same compatibility with the Hopf fibration as in* (2.13).

Proof. As in [**22**], notice that the set of $\mathbb{CP}^1$ lines in the blowup $\tilde{Z}_i = \mathrm{Bl}_{\mathbb{CP}^1_{x_i}}(Z_i)$ is parameterized by $M_1 \setminus \{x_i\} \cup \mathbb{P}(T_{x_i}(M_i))$, that is, the real blowup $\tilde{M}_i = \mathrm{Bl}_{x_i}(M_i)$, with $\mathbb{P}(T_{x_i}(M_i)) \simeq \mathbb{RP}^3$. The set of $\mathbb{CP}^1$ lines in the singular space is similarly parameterized by the gluing of these real blowups along the exceptional divisors $P_i := \mathbb{P}(T_{x_i}(M_i)) \simeq \mathbb{RP}^3$, which we denote by $\tilde{M} = \tilde{M}_i \sqcup_{P_1 \simeq P_2} \tilde{M}_2$. Thus, we can construct a space $\tilde{S}$ obtained from the singular space $\tilde{Z}$ by the Hopf fibration diagram

$$\begin{array}{ccccc} S^1 & \xrightarrow{=} & S^1 & & \\ \downarrow & & \downarrow & & \\ S^3 & \longrightarrow & \tilde{S} & \longrightarrow & \tilde{M} \\ \downarrow & & \downarrow & & \downarrow = \\ \mathbb{CP}^1 & \longrightarrow & \tilde{Z} & \longrightarrow & \tilde{M} \end{array} \tag{4.1}$$

where the map $\tilde{Z} \to \tilde{M}$ has fiber over $x \in \tilde{M}$ the $\mathbb{CP}^1$ line in $\tilde{Z}$ that the point x parameterizes, and the space $\tilde{S}$ is obtained by building over each $\mathbb{CP}^1$ line in $\tilde{Z}$ a 3-sphere S^3 via the Hopf fibration.

This allows us to apply the construction of the compatible non-commutative deformations of Proposition 2.4 to the pair $\tilde{S}$, $\tilde{Z}$, compatibly with the non-commutative deformation of the Hopf fibration, which we represent as the diagram of non-commutative spaces

$$\begin{array}{ccccc} S^1 & \xrightarrow{=} & S^1 & & \\ \downarrow & & \downarrow & & \\ S^3_\hbar & \longrightarrow & \tilde{S}_\hbar & \longrightarrow & \tilde{M} \\ \downarrow & & \downarrow & & \downarrow = \\ \mathbb{CP}^1_\hbar & \longrightarrow & \tilde{Z}_\hbar & \longrightarrow & \tilde{M} \end{array} \tag{4.2}$$

We then consider an unobstructed one-parameter deformation Z_t of the singular space $Z_0 = \tilde{Z}$ to the twistor space $Z(M)$ of the connected sum manifold $M = M_1 \# M_2$. We denote by M_t the (anti)self-dual structure on M that is the smoothing of $\tilde{M}$ with local form $xy = t$ near the normal crossings singular locus of $\tilde{M}$. Then, as shown in [**22**], the set of lines of Z_t is parameterized by the points of M_t; all lines have the correct normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ and a fixed-point-free antiholomorphic involution leaving the lines invariant, hence they satisfy the characterization of twistor spaces and can be identified with $Z_t = Z(M_t)$. Thus, we also have an associated $S_t = S(M_t)$ that fits in the Hopf fibrations diagram (2.6). We can then apply the same construction of Proposition 2.4 on all of the pairs (S_i, Z_i), $(\tilde{S}_i, \tilde{Z}_i)$, $(\tilde{S}, \tilde{Z})$, (S_t, Z_t) and obtain corresponding non-commutative deformations obtained by deforming the Hopf fibration. The compatibility between all of these non-commutative deformations comes from the compatibilities of the underlying commutative spaces parameterizing lines in Z_i, $\tilde{Z}_i$, $\tilde{Z}$ and Z_t. □

4.3. *Gluing of θ-deformations*

In the case of the θ-deformations (as well as the case of the fuzzy twistor spaces that we discuss below in Section 4.4) the gluing procedure can be handled in a different way that does not require relying on the non-commutative deformation theory of Section 3.2.

PROPOSITION 4.3 *Suppose given unobstructed classical one-parameter deformation Z_t of the singular space $\tilde{Z}$ to the twistor space $Z(M)$ of the connected sum $M = M_1 \# M_2$. This determines an associated family of θ-deformations $S(M_t)$ compatible with a θ-deformation $\tilde{S}_\theta$ of a fibration $\tilde{S}$ over the singular $\tilde{Z}$ and θ-deformations $S(M_i)_\theta$ obtained as in* Section 2.3.

Proof. In the case of the θ-deformations of Section 2.3 it is only the spaces $S(M)_\theta = \mathbb{S}(\mathcal{S}^+(M))_\theta$ that is deformed to a non-commutative space, while the twistor space $Z(M)$ itself remains commutative. In this case, the gluing and deformation theory of $Z(M_i)$ and $Z(M_2)$ remains the same as in the setting of [**22**], with the twistor space $Z(M_1 \# M_2)$ obtained from a commutative deformation of the singular space $\tilde{Z}$ built from unobstructed commutative deformations of the $Z(M_i)$. Thus, in order to obtain compatible non-commutative deformations of the $S(M_i)_\theta$ that glue to a non-commutative deformation of $S(M)_\theta$, we need to show how to associate to the choice of an unobstructed deformation of $Z(M_i)$ and the non-commutative θ-deformations $S(M_i)_\theta$ a resulting deformation of the singular space $\tilde{Z}$ with an associated θ-deformation $S(\tilde{Z})$ such that the deformation of $\tilde{Z}$ to $Z(M)$ yields the desired θ-deformation $S(M)_\theta$.

As in Proposition 4.2, we parameterize lines in $\tilde{Z}_i$ by the real blowp $\tilde{M}_i$ and lines in $\tilde{Z}$ by the resulting gluing $\tilde{M}$, and the construct compatible fibrations $\tilde{S}_i$ and $\tilde{S}$ that fit the Hopf fibration diagram (4.1). This leads to an associated construction of a θ-deformation $\tilde{S}_\theta$ that fits the non-commutative Hopf fibration diagram

$$
\begin{array}{ccccc}
S^1 & \xrightarrow{=} & S^1 & & \\
\downarrow & & \downarrow & & \\
S^3_\theta & \longrightarrow & \tilde{S}_\theta & \longrightarrow & \tilde{M} \\
\downarrow & & \downarrow & & \downarrow{=} \\
\mathbb{CP}^1 & \longrightarrow & \tilde{Z} & \longrightarrow & \tilde{M}
\end{array}
\tag{4.3}
$$

with $\tilde{Z}$ corresponding to a $U(1)$-invariant subalgebra of the algebra $\mathcal{A}(\tilde{S}_\theta)$ of the θ-deformation, as in the cases of the θ-deformations $S(M_i)_\theta$ obtained as in Section 2.3. The compatibility between the θ-deformations $\tilde{S}_\theta$ and the $S(M_i)_\theta$, for $i = 1, 2$ is provided by the fact that the parameterizing families of lines of $\tilde{Z}$ in the complement of the glued exceptional divisors agree with those of Z_i, namely with $M_i \smallsetminus \{x_i\}$ so that the θ-deformations of the 3-spheres over each of these $\mathbb{CP}^1$-lines agree for $S(M_i)$ and $\tilde{S}$.

We then proceed again as in Proposition 4.2, by considering the Z_t and compatible S_t that fit the Hopf fibration diagrams (2.6). The characterization of [**22**] of Z_t as twistor spaces of the (anti)self-dual structure M_t on the connected sum spacetime manifold then identifies the S_t constructed in this way with $S(M_t)$.

This means that we can then compatibly build a family of θ-deformations $S(M_t)_\theta$ that fit into the Hopf fibrations diagram (2.10). □

The gluing of the θ-deformations of Proposition 2.8 is more interesting, since in this case the twistor spaces $Z(M_i)$ themselves are deformed to non-commutative spaces $Z(M_i)_\theta$ via a θ-deformation of the respective subspaces $\mathbb{P}N_i \subset Z(M_i)$ along the Hopf spheres S^3_θ.

Proposition 4.4 *The gluing $\tilde{Z}$ of the blowups $\tilde{Z}(M_i) = \mathrm{Bl}_{\mathbb{CP}^1_{x_i}}(Z(M_i))$ of the twistor spaces $Z(M_i)$ along the exceptional divisors E_i admits a θ-deformation $\tilde{Z}_\theta$ compatible with the θ-deformations $Z(M_i)_\theta$ of* Proposition 2.8.

Proof. Since in the θ-deformations of Proposition 2.8 only the subspace $\mathbb{P}N$ of the twistor space is deformed to a non-commutative space, we can assume that the points $x_i \in M_i$ are chosen so that the fibers $\mathbb{CP}^1_{x_i}$ are contained in the respective $\mathbb{P}N_i \subset Z(M_i)$. The Hopf spheres that are θ-deformed to obtain the non-commutative $Z(M_i)_\theta$ are the intersections $S^3_P = P \cap \mathbb{P}N_i$, with a family of planes passing through a chosen point in the positive norm part $\mathbb{P}T^+$ of the twistor space. The planes P cut out Hopf circles $S^1_{x_i,P}$ in the fiber $\mathbb{CP}^1_{x_i}$. Since the non-commutative deformation involves the individual Hopf spheres S^3_P, we can restrict our attention to a single sphere. Thus, instead of working with the gluing $\tilde{Z}$ of the blowups $\tilde{Z}(M_i) = \mathrm{Bl}_{\mathbb{CP}^1_{x_i}}(Z(M_i))$ along their exceptional divisors, we can consider Hopf spheres $S^3_{P_i} \subset \mathbb{P}N_i$ and their real blowups along the Hopf circles $S^1_{x_i,P_i}$. The resulting singular space S is a gluing of two 3-spheres $S^3_{P_i}$ along a torus T^2, which can be identified with the boundary of a tubular neighborhood of one of the Hopf circles $S^1_{x_i,P_i}$. In Hopf coordinates, we can identify such a tubular neighborhood with values $0 \le \eta \le \epsilon$ for some $\epsilon > 0$ and $(\xi_1, \xi_2) \in T^2$. Thus, the resulting space can be regarded as the gluing of two copies of S^3 along one of the tori T^2 that are θ-deformed to non-commutative tori in the deformation to S^3_θ. While the gluing S of the two 3-spheres along the torus is not a smooth manifold, but has a normal crossing singularity along the torus, it is still possible to define a θ-deformation S_θ, since the deformation happens along the individual tori that are deformed to non-commutative tori. These are either the torus along which the gluing is performed or the other Hopf tori in each of the to S^3. The resulting θ deformation is obtained by considering the algebra of functions that are smooth on each of the two S^3 and that have matching values on the torus where the gluing is performed. The non-commutative θ-deformed product on this algebra of functions is then defined as in (2.8), which has the effect of deforming all the individual Hopf tori in S to non-commutative tori. In order to view the θ-deformation S_θ as a spectral triple, one can take as Hilbert space and Dirac operator the direct sum of the respective ones on the two copies of S^3. This is analogous to the spectral triple construction used in the gluing of copies of smooth manifolds into fractal configurations, see [**11**, **23**]. □

In this case, because of the very explicit nature of the non-commutative deformation in terms of Hopf tori, we have not used the description of deformations in terms of Hochschild cohomology. Notice, however, that a description of the deformation and obstruction theory for the non-commutative 3-spheres S^3_θ has been discussed, in a more general setting of non-commutative deformations of 3-spheres, by Connes and Dubois–Violette in [**18**]. For the relation of deformation quantization and isospectral deformations see also [**59**].

4.4. *Gluing of fuzzy twistor spaces*

The fuzzy twistor spaces introduced in Section 2.5 provide an example of quantization of twistor spaces to which the general deformation theory approach described in Section 3.2 does not directly apply, due to a rigidity property.

Proposition 4.5 *The fuzzy twistor spaces are rigid, in the sense that they do not admit any deformations that maintain the underlying spacetime manifold M commutative.*

Proof. It is shown in Section 16 of [28] that the Hochschild cohomology and the set of equivalence classes of deformations are a Morita invariant. Namely, if two unital associative algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, then there is an isomorphism $HH^*(\mathcal{A},\mathcal{A}) \to HH^*(\mathcal{B},\mathcal{B})$ that preserves the cup product and the graded Lie bracket. A bijection between the set of equivalence classes of deformations is then obtained in the following way. There is a finitely generated projective right $\mathcal{A}$-module $\mathcal{E}$ such that $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$; hence, we can identify $\mathcal{B} = eM_N(\mathcal{A})e$, for some idempotent $e \in M_N(\mathcal{A})$ with $M_N(\mathcal{A})eM_N(\mathcal{A}) = M_N(\mathcal{A})$, by regarding $\mathcal{E}$ as a summand of a free module of rank N. Given a deformation $\mathcal{A}_t$ of $\mathcal{A}$, there is a corresponding deformation $M_N(\mathcal{A}_t)$ of $M_N(\mathcal{A})$ and an idempotent e_t in $M_N(\mathcal{A}_t)$, with constant term equal to e. Then, $e_tM_N(\mathcal{A}_t)e_t$ determines a deformation of $\mathcal{B}$ that is Morita equivalent to $\mathcal{A}_t$.

For a Fréchet algebra of smooth functions on a compact smooth manifold, $\mathcal{A} = \mathcal{C}^\infty(M)$, the behavior of the Hochschild homology was analyzed in [17]. As discussed in [43], the continuous and smooth deformation theories of $\mathcal{C}^\infty(M)$ are governed by the same cohomology $HH^*_{\mathrm{cont}}(\mathcal{C}^\infty(M),\mathcal{C}^\infty(M)) \simeq HH^*_{\mathrm{smooth}}(\mathcal{C}^\infty(M),\mathcal{C}^\infty(M)) \simeq H^0(M,\Lambda^*\mathcal{T}_M)$, where $H^0(M,\Lambda^*\mathcal{T}_M)$ denotes the space of global sections of the exterior algebra of the tangent bundle of M, that is, the skew multivector fields on M. The infinitesimal deformations up to equivalence can then be seen as the elements of the second Hochschild cohomology, that is, the sections in $H^0(M,\Lambda^2\mathcal{T}_M)$, while the obstructions live in this third cohomology, identified with $H^0(M,\Lambda^3\mathcal{T}_M)$. These correspond only to deformations of the spacetime manifold M in the 'non-commutative direction'. Thus, all these deformations violate the requirement that spacetime itself remains commutative. □

In this case, however, a gluing of fuzzy twistor spaces that corresponds to the connected sum of the underlying spacetime manifolds can be performed without passing through the deformation and obstruction theory discussed in Section 3.2.

PROPOSITION 4.6 *The orientation reversing isometry $\gamma : T_{x_1}M_1 \to T_{x_2}M_2$ at the points $x_i \in M_i$ where the connected sum $M = M_1 \# M_2$ is performed determines a gluing of the fuzzy twistor spaces of M_i to a fuzzy twistor space of M.*

Proof. Consider the manifolds with boundary M_i', obtained by removing a small ball U_i near the points x_i, with boundary $\partial M_i' \simeq S^3$. We can consider cylindrical ends $S^3 \times [0,\epsilon)$ attached to M_i' and a metric (without self-duality property) interpolating smoothly between the metric of M_i' to a cylindrical metric on a smaller interval, built using the metric on the tangent space at x_i. The orientation reversing isometry $\gamma : T_{x_1}M_1 \to T_{x_2}M_2$ determines a gluing map that identifies these cylindrical ends. Assuming that over a slightly larger ball $U_i' \subset M_i$ containing x_i the almost commutative geometry of the fuzzy space is a product $\mathcal{A}(U_i') \otimes \mathcal{A}(S^2_N)$, we can use the same $SU(2)$-valued gluing map, seen as an automorphism of $\mathcal{A}(S^2_N)$ through the $(N+1)$-dimensional representation of $SU(2)$, to glue together the non-commutative spaces $\mathcal{A}(S^2_N)$ over the cylindrical ends. The resulting non-commutative space is still an almost commutative geometry, in the general form of [9], where the underlying commutative geometry is by construction M, with non-commutative fiber S^2_N; hence, it provides a model for a fuzzy twistor space for M, regardless of the existence of an (anti)self-dual metric on the connected sum. □

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