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Moufang patterns and geometry of information

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Dedicated to Don Zagier

Abstract: Technology of data collection and information transmission is based on various mathematical models of encoding. The words "Geometry of information" refer to such models, whereas the words "Moufang patterns" refer to various sophisticated symmetries appearing naturally in such models.

In this paper, we show that the symmetries of spaces of probability distributions, endowed with their canonical Riemannian metric of information geometry, have the structure of a commutative Moufang loop. We also show that the F–manifold structure on the space of probability distribution can be described in terms of differential 3–webs and Malcev algebras. We then present a new construction of (non-commutative) Moufang loops associated to almost–symplectic structures over finite fields, and use them to construct a new class of code loops with associated quantum error–correcting codes and networks of perfect tensors.

Keywords: Probability distributions, convex cones, Moufang loops, quasigroups, Malcev algebras, error–correcting codes, asymptotic bound, code loops, perfect tensors, tensor networks, CRSS quantum codes.

1. Introduction and summary

This paper can be roughly subdivided into two parts: Sections 2–5 and Sections 6–7.

The words "Geometry of information" in the first part refer to models of databases subject to noise – *probability distributions on finite sets.* The same words in the second part refer to theory of *error–correcting codes.*

The introductory subsections of each part define mathematical structures, describing symmetries of relevant geometries: Commutative Moufang Loops

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in the first part, and (virtually) noncommutative Moufang Loops in the second part.

Here are some more details.

In Sec. 2, we recall the definitions of symmetric quasigroups and CH–quasigroups, describe their relations to commutative Moufang loops, and we summarise their role in algebraic geometry, as in [25], in particular in the case of the set of algebraic points of a cubic curve in the projective plane. In Sec. 3, we give the differentiable version of Moufang loops, in the form of Malcev algebras [24], which generalise to loops the relation between Lie algebras and Lie groups.

The main new results of this paper start in Sec. 4, where we consider spaces of probability distributions on finite sets, endowed with a family of canonical Riemannian metrics. We consider symmetries of the space of probabilities given by automorphisms of order two that are boundary limits of the reflections of geodesics about the center. We show that these automorphisms define a composition law on the set of points that is an abelian symmetric quasigroup.

In Sec. 5, we consider the structure of F–manifold on the space of probability distribution, previously discussed in [8] and [11]. Using a formulation in terms of differential 3–webs, we show the compatibility between the quasigroup structure, which determines a family of (bundles) of Malcev algebras, and the family of F–structures, whose flat structures are obtained from the Chern connections of the family of differential 3–webs.

In Sec. 6, we recall some definitions and properties of (noncommutative) Moufang loops, from [14] and [20]. We also recall some notions and results about classical error–correcting codes, from [26, 27]. Then we introduce code loops, as defined in [14] (see also [12]) and further studied in [20].

In Sec. 7, we develop a construction of code loops based on a generalisation to the almost–symplectic case of the symplectic quantisation in positive characteristic of [16] and [17]. In particular, we extend to the case of code loops results of [19] on the construction of quantum error–correcting codes and perfect tensors from isotropic and Lagrangian subspaces, in the symplectic quantisation case.

More precisely, at the beginning of Sec. 7, we recall in more details the construction of code loops of [14] and [20] and its formulation in terms of doubly even binary codes. In Sec. 7.1, we recall some definitions and results on quantum error-correcting codes. In Sec. 7.2, we recall the result of [19] on the symplectic CRSS algorithm constructing quantum codes from classical codes that are isotropic subspaces of a symplectic vector space, via the quantisation procedure of [16] and [17].

Similarly, in Sec. 7.3, we recall the result of [19] on the construction of perfect tensors from Lagrangian subspaces in general position with respect to the Darboux decomposition. In Sec. 7.4, we show how the results summarised in Sec. 7.2 and 7.3 extend to the case of characteristic 2 that was not treated in [19].

In Sec. 7.5, we introduce a new construction of code loops based on almost-symplectic vector spaces over finite fields. We describe the Moufang condition for these loops in terms of the construction of [15] of free Moufang loops in the variety generated by code loops. In Section 7.6, we prove that the symplectic CRSS algorithm for constructing quantum codes from classical codes extends to the case of almost–symplectic code loops. In Sec. 7.7, we show that the construction of perfect tensors from Lagrangians also extends to the case of almost–symplectic code loops, when the almost-symplectic structure is locally conformally symplectic. This condition ensures that a version of the Darboux decomposition can still be obtained, hence the general position property of Lagrangians that gives the perfect tensor condition. In Sec. 7.7.1, we discuss networks of perfect tensors and the associated entanglement entropy function. We also recall the equivalence of categories between loops and Latin square designs and the subcategory of central Latin square designs that corresponds to Moufang loops.

Finally, in Sec. 7.7.4, we show that the construction of perfect tensors obtained in Sec. 7.7 determines a tensor network on a subgraph of the graph associated to the Latin square design of the almost-symplectic code loop. In Sec. 7.8, we recall the formalisms of chamber systems associated to loops and their Latin square designs and their relation to buildings, and formulate some questions on the possible construction of tensor networks on these chamber systems and buildings and their possible holographic properties.

2. Quasigroups, commutative Moufang loops and algebraic varieties

2.1. Symmetric quasigroups

As in [25], Ch. 1, we start with considering a set E with binary composition law $\circ : E \times E \to E, (x, y) \mapsto x \circ y$. Such a structure will be called *a symmetric quasigroup* if the triple relation $L(x, y, z) : x \circ y = z$ is $\mathbf{S}_3$–invariant.

A symmetric quasigroup $(E, \circ)$ as above is called *abelian*, if for any element $u \in E$ the composition law $(x, y) \mapsto u \circ (x \circ y)$ turns E into an abelian group with identity u.

Finally, a symmetric quasigroup $(E, \circ)$ is called a *CH–quasigroup*, if any subset of E of cardinality 3 generates an abelian subquasigroup. (An explanation of CH in this definition will be given below).

2.2. Commutative Moufang loops (CML)

By definition, a CML is a set E endowed with a commutative binary composition law $* : E \times E \to E : (x, y) \mapsto x * y$, with identity $u \in E$ and left inverse map $E \to E : x \mapsto x^{-1}$. The main additional constraints below were called "weak associativity" relations in Def. 1.4 of [25]:

$$x * (x * y) = (x * x) * y, (x * y) * (x * z) = (x * x) * (y * z),$$

$$(x * y) * (x * z) = ((x * x) * y) * z.$$

The *associative centre* of a CML $(E, *)$ is the subset

$$Z(E) := \{x \in E \,|\, x * (y * z) = (x * y) * z \ for\ all\ y, x \in E\}.$$

Together with induced multiplication, $Z(E)$ is an associative subloop, and therefore an abelian group. The quotient loop $E/Z(E)$ is a CML of exponent 3:

$$x^{*3} := x * (x * x) = (x * x) * x = u.$$

Loops of exponent 3 form a subcategory of all CML's.

2.3. Connections between quasigroups and Moufang loops

There are several natural ways to make CH–quasigroups, resp. CML's, objects of categories, by defining morphisms between them. Then connections between objects of these two categories must naturally become functors. But in this subsection, we will neglect morphisms.

Proposition 2.1. *(i) Let $(E, \circ)$ be a CH–quasigroup, and $u \in E$ its element. Then E endowed with composition law $(x, y) \mapsto x * y := u \circ (x \circ y)$ is a CML with identity u. Different choices of u lead to isomorphic CML's.*

(ii) Let $(E, *)$ be a CML, and c an element of its associative centre $Z(E)$. Then E with composition law $x \circ y := c * x^{-1} * y^{-1}$ is a CH–quasigroup.

2.4. Quasigroups and loops in algebraic geometry

Appearance of the simplest CML's in algebraic–geometric setup was motivated in [25] by smooth cubic curves in a projective plane $\mathbf{P}^2_K$ over a field K. The set E of K–points of such a curve X forms a CML with composition law $x * y = u \circ (x \circ y)$ as in Prop. 2.1 (i) above, if $u + x + y$ is the intersection cycle of X with a projective line $\mathbf{P}^1_K \subset \mathbf{P}^2_K$.

Further developments led to a theory of such structures, necessary for their applicability to higher–dimensional cubic hypersurfaces.

Below we will argue that finitely generated loops of this type naturally act as symmetries of spaces of probability distributions of finite sets. Therefore, it is worth considering posets in groupoids and the related thin categories of such loops, morphisms in which are embeddings: see [9], subsections 5.1 and 5.2.

3. Analytical commutative Moufang loops and Malcev algebras

In his work [24], I. A. Malcev considered Moufang loops endowed with an additional structure of (local) differentiable or (real) analytic variety, with which the Moufang composition is compatible. He has introduced and studied the induced structures upon tangent vector bundles, generalising the relationship between Lie groups and Lie algebras.

Below we will sketch this theory. Our exposition relies mainly upon [40], [32], [33], and [37].

Definition 3.1 (Def. 4.2 in [33]). *A vector space $\mathcal{T}$ over a field, endowed with an antisymmetric bilinear composition map*

$$[,]^\mu : L \otimes L \to L, \quad x \otimes y \mapsto [x,y]^\mu$$

is called a Malcev algebra, it satisfies the identity

$$(1)\quad [[x,y]^\mu,[x,z]^\mu]^\mu = [[[x,y]^\mu,z]^\mu,x]^\mu + [[[y,z]^\mu,x]^\mu,x]^\mu + [[[z,x]^\mu,x]^\mu,y]^\mu.$$

Remark 3.2. *(i) We replaced the notation $[,]$ of [33] by our $[,]^\mu$ in order to distinguish it from the usual Lie brackets for vector fields. Moreover, in our applications, the ground field will be mostly real or complex numbers.*

(ii) The same definition is given in Sec. 2 of [40], but our $[x,y]^\mu$ is denoted there xy, or $x.y$, or $(x.y)$ (cf. (1))).

From the operadic perspective, Nagy's notation [33] is more consistent.

More general (not necessarily commutative) Moufang loops will also be discussed in the following sections of this paper. These are sets $\mathcal{L}$ endowed with a binary composition law with identity and inverse element, that satisfies a weak associativity relation (Moufang identity): see Definition 6.1 below.

Proposition 3.3. *Let $(\mathcal{L}, *)$ be a Moufang loop with identity e, endowed with compatible structure of real analytic (or smooth) variety.*

Then, the tangent bundle $\mathcal{TL}$ carries a natural structure of a bundle of Malcev algebras $(\mathcal{T}_e\mathcal{L}, [\cdot,\cdot])$, that describe the "first order approximation" to the Moufang composition law $*$.

For a proof, see the first pages of [40], or [37]. In the case of a CML the tangent Malcev algebra is an abelian Lie algebra.

4. Symmetries of cones of probability distributions on finite sets

4.1. Spaces of probability distributions

The setup on which we focus our attention now is briefly described in Sec. 1 and 4 of [11] (see also [8]). Much more details a reader can find in the primary sources [43] and [29]. See also [31].

Briefly, let X be a finite set, and R^X the space of real–valued functions $X \to \mathbf{R}$. A classical probability distribution on X consists of functions $X \ni x \mapsto p_x \in \mathbf{R}$ such that all p_x are non–negative, and $\sum_x p_x = 1$. Thus, the space of such distributions is a simplex Δ_X of dimension $\operatorname{card} X - 1$, with the set of vertices that can be canonically identified with X. It is convenient also to consider the open simplex ${}^\circ\Delta_X$ consisting of points (p_x) with all $p_x > 0$.

${}^\circ\Delta_X$ is a convex domain in R^X, if $\operatorname{card} X \geq 2$. Here we understand convex domains in the sense of the Def. 7 in Ch. I, Sec. 5 of [43].

We will also consider the family of functions $X \ni x \mapsto q_x$, with $q_x > 0$ for all x. This is a convex cone in R^X in the sense of Def.1 in Ch. 1, Sec. 1 of [43]. This is the *cone fitted onto the convex domain* ${}^\circ\Delta_X$ in the sense of Def. 9 in Ch. I, Sec. 5 of [43].

4.2. Symmetries

Let S_X be the group of all permutations of X. We will write the left action $S_X \times X \to X$ as $(s, x) \mapsto s(x)$. By linearity, it extends to the left action $S_X \times {}^\circ\Delta_X \to {}^\circ\Delta_X$.

The central construction of [43], surveyed also in [8], [9], establishes that this space carries a family of canonical Riemannian metrics. One element of such a family is defined, for example, by a choice of its *center*: a point $c \in {}^{\circ}\Delta_X$. As soon as c is chosen, the geodesics with respect to this metric are segments of real lines in Δ_X containing c and one of the vertices x. It is important to keep in mind that the distance from c to the intersection point of such a real line with interior part of the boundary face of Δ_X is infinite.

Moreover, each such geodesic then defines an automorphism of order two t_x of metric space ${}^{\circ}\Delta_X$, such that $t_x(c) = c$ and the respective geodesic is t_x–invariant. This action naturally extends to Δ_X.

However, action of t_x on other vertices induces generally a non–trivial permutation of them.

Proposition 4.1. *The family of maps above satisfies identities*

$$(t_x t_y t_z)^2 = id \tag{2}$$

where id is the identical map $X \to X$.

Proof. Intuitively, the symmetries t_x for $x \in X$ can be considered as *boundary limits* of involutions s_c, defined for every point $c \in {}^{\circ}\Delta_X$. Along any geodesic, passing through c, this involution acts a "mirror symmetry": it preserves the Riemannian distance between c and a variable point d, but reverses the direction from c to d, so that $s_c^2 = id$.

We will skip here an easy reasoning, showing that this intuition works, and relations (2) indeed follow from relations

$$(s_c s_d s_e)^2 = id \tag{3}$$

The relations (3) themselves constitute the content of a classification theory: see [25], Ch. 1, and more recent publications [23], [42]. □

Now consider a general (pseudo)–Riemannian manifold M endowed with a family of isometric involutions s_c, $c \in M$, satisfying relations (3).

Define the multiplication law $* : M \times M \to M$ by

$$c * d := s_c(d) \tag{4}$$

Theorem 4.2. *The set of points of* M *endowed with composition law* $*$ *is an abelian symmetric quasigroup.*

The proof is rather straightforward, and we omit it.

This statement can be considered as a bridge between Moufang loops and symmetries of spaces of probability distributions. In fact, as is shown in [25], the following family of identities holds. Let $(E, *)$ be a CML, $c \in Z(E)$. Define maps $t_x : E \to E$ by $t_x(y) := cx^{-1} * y^{-1}$. These maps satisfy relations (1).

5. *F*–manifolds, 3–webs, and Malcev algebras

5.1. *F*–identity

Consider a linear space (or a sheaf of linear spaces) $\mathcal{T}$, endowed with two bilinear operations: commutative and associative binary composition $\circ$ and Lie bracket $[,]$.

Define the Poisson tensor $P : \mathcal{T} \otimes \mathcal{T} \otimes \mathcal{T} \to \mathcal{T}$ by

$$P_X(Z, W) := [X, Z \circ W] - [X, Z] \circ W - Z \circ [X, W].$$

By definition, F–identity is the following constraint upon $(\circ, [,])$ ([8]):

$$P_{X \circ Y}(Z, W) = X \circ P_Y(Z, W) + Y \circ P_X(Z, W). \tag{5}$$

Here we would like to understand a connection between F–identity and main identities defining Malcev algebras.

From the first sight, Malcev's identities differ from F–identity: they impose linear relations upon operadic monomials of two and three variables (cf. (1)), whereas (5) consists of operadic monomials in four variables. Following [32] and [33] we will show, how to overcome this obstacle.

5.2. Differential 3–webs

Let M be a smooth manifold of dimension $2r$, $(r \geq 1)$, $\mathcal{T}M$ its tangent bundle. A (differentiable) *3–web on* M is defined as a family of three foliations on M of rank r, encoded by their tangent subbundles:

$$\textit{horizontal tangents } \mathcal{T}^h M \subset \mathcal{T}M,$$

$$\textit{vertical tangents } \mathcal{T}^v M \subset \mathcal{T}M,$$

$$\textit{transversal tangents } \mathcal{T}^t M \subset \mathcal{T}M,$$

such that *the direct sum of any two different members of this family coincides with* $\mathcal{T}M$.

As in [32], denote by H, resp. V, T, the projection operator $\mathcal{T}M \to \mathcal{T}M$ with kernel $\mathcal{T}^v M + \mathcal{T}^t M$, resp. $\mathcal{T}^h M + \mathcal{T}^t M$, $\mathcal{T}^h M + \mathcal{T}^v M$.

From the definition, it follows that $H^2 = H, V^2 = V$, $T^2 = T$. Moreover, one easily sees that there exists a unique operator J on $\mathcal{T}M$ such that $J^2 = Id$, $HJ + JH = J$, and J induces an isomorphism between $\mathcal{T}^h M$ and $\mathcal{T}^v M$.

Such a pair is called an $\{H, J\}$–structure on M, and it carries exactly the same information as needed for a description of a 3–web on M. The following results (Theorems 3.2 and 3.4 in [32]) play the crucial role in the following:

Proposition 5.1. *(i) Consider a manifold M with an $\{H, J\}$–structure. Then there exists a unique covariant derivation ∇ on M such that $\nabla H = \nabla J = 0$, and that the torsion tensor $t(X, Y) := \nabla_{HX} Y - \nabla_{VY} X - [HX, HY] = 0$ for any X, Y.*

(ii) This $\{H, J\}$–structure comes from a 3–web, that is, respective distributions are integrable, if and only if the following conditions are satisfied:

$$Vt(HX, HY) = Ht(VX, VY) = 0,$$

$$JHt(HX, HY) + Vt(JHX, JHY) = 0. \tag{6}$$

We do not reproduce the proof here.

The covariant derivative ∇ defined in Prop. 5.1, is called the *canonical connection*, or *Chern connection* of the respective $\{H, J\}$–structure.

Let now M be a space of probability distributions. As was shown in Sec. 4, it has a family of structures of CMLs, and thus of quasigroups. The latter one produces a family of (bundles) of Malcev algebras.

Theorem 5.2. *In this setup, M admits a family of 3–webs, whose Chern connections define compatible flat structures of the respective family of F–structures on M, in the sense of [8], Sec. 2.3.*

A key observation proving this is the formula $X \circ Y = [X, [Y, C]]$ in Sec. 2.3 of [8], its comparison with (6) above. For additional information, see [1], [7].

6. General Moufang loops and codes

6.1. Non(necessarily)commutative Moufang loops (ML)

We keep the notation $*$ for binary multiplication, but add or change other essential notations and conditions (cf. [14] and references therein).

Definition 6.1. *(i) A loop $\mathcal{L}$ is a set with a binary composition law $* : \mathcal{L} \times \mathcal{L} \to \mathcal{L}, (a, b) \mapsto a * b$, endowed by two–sided unit, denoted 1 if it will not lead to confusion, and such that for each elements $a, b \in \mathcal{L}$ the equations $a * x = b$ and $y * a = b$ have a unique solution in $\mathcal{L}$, denoted a^{-1}.*

(ii) A loop $(\mathcal{L}, *)$ is called Moufang, if any triple $(a, b, c) \in \mathcal{L}^3$ satisfies the "near–associativity" relation

$$(a * b) * (c * a) = a * ((b * c) * a).$$

(iii) The operations commutator $[a, b]$ and associator $[a, b, c]$ in a Moufang loop $\mathcal{L}$ are defined as follows:

$$[a, b] := (a * b) * ((b * a)^{-1}), \quad [a, b, c] := ((a * b) * c) * ((a * (b * c))^{-1}).$$

Definition 6.2. *Let $(\mathcal{L}, *)$ be a Moufang loop.*

(i) The Moufang centre $C(\mathcal{L})$ is the set of all elements $a \in \mathcal{L}$ such that $[a, b] = 1$ for each $b \in \mathcal{L}$.

(ii) The nucleus $N(\mathcal{L})$ of $\mathcal{L}$ is the set of all elements $a \in \mathcal{L}$ such that for any $b, c \in \mathcal{L}$ we have

$$[a, b, c] = [b, a, c] = [b, c, a] = 1.$$

(iii) The centre $Z(\mathcal{L})$ is defined as $N(\mathcal{L}) \cap C(\mathcal{L})$.

One can easily check that the nucleus $N(\mathcal{L})$ is a subgroup of $\mathcal{L}$, and the centre $Z(\mathcal{L})$ is an abelian subgroup.

Let now p be a prime. We denote by $\mathcal{L}_p$ the set of elements of $\mathcal{L}$ whose order is a power of p. The torsion subloop of a Moufang loop $\mathcal{L}$ is the direct product of the $\mathcal{L}_p$ over all primes p (see [20]). It is also shown in [20] that, if $\mathcal{L}$ is a Moufang loop such that $\mathcal{L}/Z(\mathcal{L})$ is an abelian group, then for all $p > 3$, the subloop $\mathcal{L}_p$ is a group. A Moufang loop $\mathcal{L}$ is caled a p–loop if every element of $\mathcal{L}$ has order a power of p. For a Moufang loop, the order of any element divides the order of the loop, hence a Moufang loop of order a power of p is a finite p–loop, $\mathcal{L} = \mathcal{L}_p$, hence in particular a group, if $p > 3$ (see Theorem A of [20]).

6.2. Error–correcting codes

The family of codes with which we deal in this paper can be described as follows (see [27] and references therein).

Let $q \geq 2$ be an integer, and A (or A_q) a finite set of cardinality q (alphabet). A sequence (α_i) of elements of A, $i = 1, 2, \ldots, n$, is called a *word of length* n. By definition, a code C is a non–empty subset $C \subset A^n$.

Define the *Hamming distance* between two words (α_i) and (α'_i) of the same length as

$$d((\alpha_i), (\alpha'_i)) := \text{card}\,\{i \in (1, \ldots, n) \,|\, \alpha_i \neq \alpha'_i\}.$$

Given such a code C, we denote by $n(C)$ the common length of all words in C, by $d(C)$ the minimal distance between two different words in C, and by $k(C)$ the number $[\log_q \text{card}\,(C)]$. The quadruple of integers $[n, k, d]_q$ defines a finite family of codes $C \subset A_q^n$, for which $d = d(C)$ and $k = k(C)$.

A code C becomes a kind of "dictionary of an artificial language" as soon as one ascribes to words in C some meanings, "semantics". Finite sequences of code words are "sentences".

If then information encoded by such a sentence ought to be transmitted, say, by broadcasting, it might become distorted. The idea of error–correcting codes consists in imagining that noise in such a channel, with large probability, distorts only rare letters in code words. Hence, if the distance between two different code words is big enough, one can recognise the distorted letters and to correct them. We must pay for it by using code words of larger length that is strictly necessary for encoding relevant information.

For this reason the number $R(C) := k(C)/n(C)$ is called the *transmission rate* of C, and the number $\delta(C) := d(C)/n(C)$ is called the *relative minimal distance* (between code words) of C.

The words "geometry of information" in this setup refer to the geometry of the set of *code points* $P_C := (R(C), \delta(C)) \in [0, 1]^2$.

6.2.1. Unstructured vs structured codes If sets of code words are endowed with additional data/restrictions, we call generally the respective A_q–codes "structured" ones.

Two most studied classes of structured codes are the following ones:

(i) *Linear codes.* Here $A_q := \mathbf{F}_q$, finite field of cardinality q, and $C \subset \mathbf{F}_q^n$ are $\mathbf{F}_q$–linear subspaces.

(ii) *Algebraic–geometric codes.* For the same class of alphabets, one can consider $\mathbf{F}_q$–points in affine (or projective) $\mathbf{F}_q$–schemes with a chosen coordinate system.

As we will see below, Moufang symmetries generally become visible in special structured codes. See also an unusual setup of [39].

6.3. Code loops

Code loops were originally constructed (see [14] and also [34], [35]) from a family of $\mathbf{F}_2$–linear codes (including the Golay code), as a generalization of extensions given by cocycles. In this setting, one considers a linear code $C \subset \mathbf{F}_2^n$ and a function $\theta : C \times C \to \mathbf{F}_2$. If the function θ satisfies the cocycle identity

$$\theta(v, w) - \theta(u + v, w) + \theta(u, v + w) - \theta(u, v) = 0,$$

then one obtains the Heisenberg group $H(C, \theta) := C \ltimes_\theta \mathbf{F}_2$, with multiplication

$$(v, x) \star_\theta (w, y) = (v + w, x + y + \theta(v, w)).$$

(see further details below). This multiplication is associative, but generally noncommutative.

In order to remake $C \ltimes_\theta \mathbf{F}_2$ into a nonassociative loop with identity element $(0, 0)$, one replaces the cocycle identity by the *twisted cocycle identity*

$$\theta(v, w) - \theta(u + v, w) + \theta(u, v + w) - \theta(u, v) = \delta(u, v, w),$$

where twisting $\delta : C \times C \times C \to \mathbf{F}_2$ is a certain function.

Theorem A of [20] implies that, for any $p > 3$, Moufang p–loops $\mathcal{L}$ (defined as recalled at the end of Section 6.1) that are obtained as central extensions of a code $C \subset \mathbf{F}_q^n$, with $q = p^r$, by the center $Z(\mathcal{L}) = \mathbf{F}_q$ are in fact *groups*. Thus, all these cases fall within the framework of usual construction of central extensions of groups, as in Section 6.1. However, for $p = 2$ and $p = 3$ one has interesting non-associative code loops. To better compare these cases to the setting of Section 6.1, we will recall, at the beginning of Sec. 7, the general construction of [20, 21] of Moufang loops obtained as central extensions of Frattini type, before introducing our new, more general construction in Sec. 7.5.

6.4. Geometry of information: asymptotic bounds for error–correcting codes

We return here to the definition of code points at the end of subsection 6.2 above. Fix cardinality q of an alphabet, and consider a class of error–correcting codes Cod_q with this alphabet. Denote by cp the map $Cod_q \to [0, 1]^2$ sending $C \in Cod_q$ to P_C. The *multiplicity* of a code point x is defined as the cardinality of $cp^{-1}(x)$.

Definition 6.3. *A continuous function $\alpha_q(\delta), \delta \in [0,1]$ is called the asymptotic bound for the family Cod_q, if it satisfies the following conditions:*

(i) The set of code points of infinite multiplicity is exactly the set of rational points (R, δ) satisfying $R \leq \alpha_q(\delta)$.

(ii) Code points of finite multiplicity all lie above the asymptotic bound and all are isolated: a sufficiently small open neighbourhood of each such point contains no other code points.

Theorem 6.4. *Asymptotic bounds exist*

(i) For unstructured codes.

(ii) For linear codes over $\mathbf{F}_q$ where q is a power of prime.

(iii) For certain classes of algebraic–geometric codes over finite fields.

In its present final form this theorem was proved in [26].

Moufang symmetries appear not directly in this geometric picture, but rather in various formalisms motivated by theoretical physics and based on the vision of an asymptotic bound as a phase transition curve, in a classical or quantum version.

7. Moufang loops, almost symplectic structures, and quantum codes

In the papers [3], [4] an algorithm was introduced, producing quantum codes from self–orthogonal classical codes. We will call it the CRSS algorithm.

In [19] the CRSS algorithm associating quantum codes to self–orthogonal classical codes was reformulated geometrically in terms of the canonical quantisation of symplectic spaces over finite fields of [16], via representations of Heisenberg groups.

The main result of this section is a new construction of code loops based on canonical symplectic quantization over finite fields, adapted to an almost–symplectic case. We also show that the code loops obtained in this way have associated CRSS quantum codes determined by isotropic subspaces, and perfect tensors associated to Lagrangians.

To compare our approach with previous constructions of code loops, it is important to note the following. The code loops described in [14], as well as the more general construction in [20], [21] of Moufang loops $\mathcal{L}$ that are central extensions

$$0 \to Z \to \mathcal{L} \to C \to 0,$$

of abelian groups $Z = Z(\mathcal{L})$ and $C = \mathcal{L}/Z(\mathcal{L})$, rely on introducing a notion of "cubic symplectic structure" (see [20], [21] for more details). In the case

where $Z(\mathcal{L}) \simeq \mathbf{Z}/p\mathbf{Z}$, Moufang loops $\mathcal{L}$ obtained in this way are called *Frattini extensions*.

Results of [20] on the Moufang p–loops imply that, for $p > 3$, loops $\mathcal{L}$ obtained as central extensions of a code $C \subset \mathbf{F}_p^n$ by $Z(\mathcal{L}) = \mathbf{F}_p$ are in fact always groups.

However, for $p = 2$ and $p = 3$ one has interesting non–associative code loops. In the case of $p = 2$ it follows from [20], [21], and [6] that all the code loops obtained through the "cubic symplectic structures" of [20] can be realised by doubly even codes as in the construction of [14]. This means that C is a binary linear code $C \subset \mathbf{F}_2^n$ that is *doubly even*, namely the weight $|v| = \#\{v_i = 1\} = v_1 + \cdots + v_n$ of the code words (the number of ones in the word) is divisible by 4, and the twisted cocycle θ that gives the code loop (see Section 6.3 above) has twisting function

$$\delta(u, v, w) = |u\&v\&w| \mod 2,$$

where $u\&v := (u_1v_1, \ldots, u_nv_n)$ denotes the logical AND operation, with

$$\theta(v, w) + \theta(w, v) = \frac{1}{2}|v\&w| \mod 2$$

$$\theta(v, v) = \frac{1}{4}|v| \mod 2.$$

In the construction we present in this section, the twisted cocycle is an almost-symplectic structure (as we discuss more precisely below). For $p > 2$, our construction is a special case of the central extensions mentioned above, but in the case $p = 2$ they are different, following the setting of [17] for symplectic quantization in characteristic 2.

Question Let $\mathcal{L}$ be a code Moufang loop corresponding to a doubly even code V. Describe the set $\mathrm{Cod}(\mathcal{L})$ of all doubly even codes W such that the corresponding Moufang loop is isomorphic to $\mathcal{L}$.

7.1. Quantum codes

As in subsubsection 6.2.1 above, consider an alphabet of cardinality $q = p^r$ endowed with a structure of $\mathbf{F}_q$–linear space of dimension n. Call $\mathcal{V}_q = \mathbf{C}^q$ a *single q–ary qubit* space. In the case $q = 2$ we refer to it just as the *single qubit space*. Then $\mathcal{V}_q^{\otimes n}$ is the space of n q–ary qubits. A *quantum error* on the vector space $\mathcal{V}_q^{\otimes n}$ is a linear operator E of the form $E = E_1 \otimes \cdots \otimes E_n$.

Definition 7.1. *Consider on the space* $\mathcal{V}_q^n = (\mathbf{C}^q)^{\otimes n}$ *of* n q*–ary qubits the orthonormal basis* $|a\rangle$*, parameterized by vectors* $a = (a_1, \ldots, a_n) \in \mathbf{F}_q^n$*. On* $\mathcal{V}_p = \mathbf{C}^p$ *the bit flip and phase flip operators* T *and* R *are defined in this basis as*

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & & & & \\ & \xi & & & \\ & & \xi^2 & & \\ & & & \ddots & \\ & & & & \xi^{p-1} \end{pmatrix},$$

where ξ *a* p*–th root of unity,* $\xi^p = 1$*.*

For any pair of vectors $a = (a_1, \ldots, a_n), b = (b_1 \ldots, b_n) \in \mathbf{F}_q^n$*, the error operators* $E_{a,b}$ *on the space* $\mathcal{V}_q^n = (\mathbf{C}^q)^{\otimes n}$ *is defined in the above basis as*

$$E_{a,b} = T_a R_b = (T_{a_1} \otimes \cdots \otimes T_{a_n})(R_{b_1} \otimes \cdots \otimes R_{b_n}),$$

for $a = (a_1, \ldots, a_n), b = (b_1 \ldots, b_n) \in \mathbf{F}_q^n$*, with*

$$T_{a_i} := T^{a_{i1}} \otimes \cdots \otimes T^{a_{ir}}, \quad R_{b_j} := R^{b_{j1}} \otimes \cdots \otimes R^{b_{jr}}.$$

Here we look at $\mathbf{F}_q$ *as an* r*–dimensional vector space over* $\mathbf{F}_p$*, with* $a_i = (a_{i\ell})_{\ell=1}^r$ *and* $b_j = (b_{j,\ell})_{\ell=1}^r$*, with entries* $a_{i\ell}, b_{j,\ell} \in \{0, \ldots, p-1\}$ *in the exponents of the corresponding powers of the bit flip and phase flip operators* T *and* R*.*

The bit flip and phase flip operators satisfy $T^p = R^p = \mathrm{id}$ and the commutation relation $TR = \xi RT$. The error operators $E_{a,b}$ define a linear basis of $M_{q^n \times q^n}(\mathbf{C})$, orthonormal with respect to the inner product $\langle A, B\rangle = \mathrm{Tr}(A^* B)$. In particular, they generate all possible quantum errors on $\mathcal{V}^{\otimes n}$.

An $[[n, k, d]]_q$ *quantum error–correcting code* is a subspace $\mathcal{C} \subset \mathcal{V}^{\otimes n}$, spanned by vectors $|a\rangle$ with a in a subspace of dimension k over $\mathbf{F}_q$. So it can correct $\leq d-1$ errors. This means that, for every error operator of the form $E = E_1 \otimes \cdots \otimes E_n$, with $\omega(E) = \mathrm{card}\{i \,:\, E_i \neq I\} < d$, the orthogonal projection $P_{\mathcal{C}}$ onto $\mathcal{C}$ in $\mathcal{V}^{\otimes n}$ satisfies

$$P_{\mathcal{C}} E P_{\mathcal{C}} = \lambda_E \, P_{\mathcal{C}}.$$

The definition of quantum error–correcting codes in [3], [4] describes the same family of codes:

Definition 7.2. *A quantum error–correcting code is a subspace* $\mathcal{C} \subset \mathcal{V}^{\otimes n}$ *given by a joint eigenspace of the operators* $E_{a,b}$ *in an abelian subgroup* $\mathcal{S}$ *of the group* $\mathcal{G}_n = \{\xi^i E_{a,b},\, a, b \in \mathbf{F}_q^n,\, 0 \leq i \leq p-1\}$*.*

7.2. The symplectic CRSS algorithm

The CRSS (Calderbank–Rains–Shor–Sloane) algorithm of [3, 4] constructs quantum codes from classical self–orthogonal error correcting codes. In [19] this construction was generalised using symplectic quantisation over finite fields, so that the original self–orthogonal case is recovered as a special case of this geometric construction. We refer to this version as the *symplectic CRSS algorithm*. We recall here briefly the results of Sections 2 and 3 of [19] where this construction is presented.

The construction of [19] relies on the functorial geometric quantisation of symplectic vector spaces over a finite field (of characteristic $p > 2$) developed in [16].

Definition 7.3. *Let $q = p^r$ with p odd. A symplectic vector space (V, ω) consists of a finite dimensional vector space (V, ω) of dimension $2n$ over $k = \mathbf{F}_q$, together with a symplectic form ω, namely a function $\omega : V \times V \to k$ that is antisymmetric $\omega(u, v) = -\omega(v, u)$ and non-degenerate, namely for all $u \neq 0$ in V there exists a $v \in V$ with $\omega(u, v) = 1$, and that is moreover closed, namely it satisfies the cocycle condition*

$$d\omega(u, v, w) = \omega(v, w) - \omega(u + v, w) + \omega(u, v + w) - \omega(u, v) = 0.$$

The Heisenberg group Heis(V, ω) is the central extension

$$0 \to k \to \mathit{Heis}(V, \omega) \to V \to 0$$

determined by the cocycle ω.

The multiplication in $\mathrm{Heis}(V, \omega)$ is given by

$$(v, x) \cdot (w, y) = (v + w, x + y + \frac{1}{2}\omega(v, w)).$$

The cocycle condition ensures the associativity of this multiplication law. The center of the Heisenberg group is $Z(\mathrm{Heis}(V, \omega)) = \{(0, x) : x \in k\}$. The cocycle condition $d\omega = 0$ is the vanishing of the Hochschild differential.

The choice of a central character $\chi : k \to \mathbf{C}^*$ determines an irreducible complex representation $\mathcal{H}_\chi$ of the Heisenberg group $\mathrm{Heis}(V, \omega)$, the *Heisenberg representation* π_χ.

As described in [16], the choice of a Lagrangian subspace $L \subset V$ determines a model for the Heisenberg representation

$$\pi_{L,\chi} : \mathrm{Heis}(V, \omega) \to \mathrm{GL}(\mathcal{H}_{(V,L,\omega,\chi)}),$$

where $\mathcal{H}_{(V,L,\omega,\chi)}$ is the subspace of the space $\mathbf{C}[\mathrm{Heis}(V,\omega)]$ of complex valued functions on the set $\mathrm{Heis}(V,\omega) \simeq V \times k$ that satisfy

$$f((0,x)\cdot(w,y)) = \chi(x)\, f(v,y), \quad \forall x \in k,\ \forall (w,y) \in V \times k,$$

$$f((v,0)\cdot(w,y)) = f(w,y), \quad \forall (v,0) \in L,\ \forall (w,y) \in V \times k.$$

Here $\mathrm{Heis}(V,\omega)$ acts upon on $\mathcal{H}_{(V,L,\omega,\chi)}$ by right translations:

$$(\pi_{L,\chi}(v,x)\, f)(w,y) = f((w,y)\cdot(v,x)).$$

This version is modelled on the usual construction of the quantum mechanical Hilbert space that identifies the position and momentum representations with a Lagrangian subspace L and its dual space $L^\vee$.

The further enrichment adds to this an *orientation on the Lagrangian*, replacing L with a pair $L^o = (L, o_L)$ of a Lagrangian subspace $L \subset V$ and a non–zero vector $o_L \in \Lambda^{top} L$. It determines *intertwining isomorphisms* $T_{L_1^o, L_2^o} : \mathcal{H}_{(V,L_1,\omega,\chi)} \to \mathcal{H}_{(V,L_2,\omega,\chi)}$ producing a trivialisation of the bundle of the Heisenberg representation models $\mathcal{H}_{(V,L,\omega,\chi)}$ over the space of oriented Lagrangians. This expresses the functoriality of the geometric quantisation of [16].

The symplectic form ω determines a Darboux decomposition of the symplectic space V into a sum of $\mathbf{F}_q^2$ subspaces, which in turn, by functoriality of the geometric quantisation, determines a tensor product decomposition of $\mathcal{H}_\chi$ into copies of the space $\mathbf{C}^q$ of a single q–ary qubit.

Example 7.4. *In the case of $(\mathbf{F}_q^{2n}, \omega)$ with ω the standard Darboux form, the Heisenberg group $Heis(\mathbf{F}_q^{2n}, \omega)$ representation with the central character determined by a p–th root of unity ξ with $\xi^p = 1$, is given by the error operators $E_{ab} = T_a R_b$ of Definition 7.1.*

This example is the key to the relation between Heisenberg group representations (that is, the functorial geometric quantization of [16]) and the construction of quantum error correcting codes. The main result is summarized as follows.

Proposition 7.5. *[19]. Let (V,ω) be a $2n$–dimensional symplectic vector space over $\mathbf{F}_q$ with q odd. An isotropic subspace $C \subset V$ of dimension k determines an abelian subgroup of the Heisenberg group $Heis(V,\omega)$, that is, mutually diagonalisable error operators in the corresponding representation $\mathcal{H}_\chi(V,\omega)$. Each joint eigenspace of C in $\mathcal{H}_\chi(V,\omega)$ gives a quantum code $\mathcal{C}_C$ associated to the classical code C.*

We refer to the assignment $C \mapsto \mathcal{C}_C$ as the symplectic CRSS algorithm. The original construction of [3, 4] for self–orthogonal classical codes is a special case of this symplectic construction, see Sections 2 and 3 of [19] for more details.

7.3. Perfect tensors

Let $\mathcal{V} = \mathbf{C}^q$ be the single q–ary qubit space. An m-tensor is an element $T \in \mathcal{V}^{\otimes m}$. We write such a tensor, in the standard basis $\{|a\rangle\}_{a=(a_1,\dots,a_m)\in \mathbf{F}_q^m}$ of Definition 7.1, as $T = (T_{a_1,\dots,a_m})$ with m indices a_i. We assume that $\mathcal{V}$ is endowed with an inner product to identify it with its dual. This means that we can raise and lower indices of T: after raising j indices we can identify T with an element in $Hom(\mathcal{V}^{\otimes j}, \mathcal{V}^{\otimes(m-j)})$. We refer to such an identification as a $(j, m-j)$-splitting (bipartition) of the indices of the tensor T.

Definition 7.6. *A perfect tensor T is a tensor in $\mathcal{V}^{\otimes m}$, such that, for any $j \leq m/2$, all resulting splittings of the set of indices are isometries*

$$T : \mathcal{V}^{\otimes j} \to \mathcal{V}^{\otimes(m-j)}.$$

A perfect tensor determines a perfect code that encodes one q-ary qubit to $m-1$ q-ary qubits. These quantum codes realize maximal entanglement across bipartitions. Tensor networks obtained by contracting legs of an arrangement of perfect tensors along a tessellation of a hyperbolic space have been considered in the context of the AdS/CFT holographic correspondence in string theory as discretizations of the bulk geometry that produce entangled boundary states, in such a way that the entanglement entropy on the boundary is expressible in terms of geodesic lengths in the bulk, according to the Ryu–Takayanagi conjecture. We refer the reader to [38] for this role of perfect tensors and tensor networks. This has become a prominent area of research in models of AdS/CFT holography.

Here we just recall a result of [19] that shows how perfect tensors can be constructed from the geometric quantization of [16] of symplectic spaces over finite fields of characteristic $p > 2$, through the geometry of Lagrangian subspaces.

Consider a symplectic vector space (V, ω) over $\mathbf{F}_q$ and a Lagrangian subspace $L \subset V$ (necessarily of dimension $\dim(V)/2$). Let $V = \oplus_i V_i$ be the Darboux decomposition of (V, ω), with $V_i \simeq \mathbf{F}_q^2$, and let $\mathcal{H} = \otimes_i \mathcal{H}_i$ be the corresponding decomposition of the irreducible Heisenberg representation in q–ary qubits. A choice of splitting of the indices of a tensor T in $\mathcal{H}$ corresponds to a decomposition $V = W \oplus W'$, given by a partition of the pieces of the Darboux decomposition.

We will say that a Lagrangian L is in *general position* with respect to a decomposition $V = W \oplus W'$ if the intersections $L \cap W$ and $L \cap W'$ with the pieces of the decomposition are as small as possible.

Proposition 7.7. *[19]. A Lagrangian L that is in general position with respect to the Darboux decomposition $V = \oplus_i V_i$, determines a symplectomorphism $\psi_L : \bar{W} \to W'$ for a given splitting $V = W \oplus W'$ as above into half-dimensional pieces, with $(\bar{W}, \bar{\omega}) = (W, -\omega)$ the dual symplectic space. The corresponding map $\mathcal{H}(\psi_L) : \mathcal{H}(W)^\vee \to \mathcal{H}(W')$ under the quantization functor of [16] is a perfect tensor*

$$T_L \in \mathcal{H}(W) \otimes \mathcal{H}(W') = \mathcal{H}(V).$$

One of our main goals in the rest of this section will be a generalisation of Propositions 7.5 and 7.7 to code loops. To this purpose, we first have to discuss the case of characteristic 2, which was not considered in [19], since the most interesting code loops arise in characteristic 2.

7.4. Geometric quantisation in characteristic 2

The functorial quantisation of symplectic vector spaces over finite fields requires a separate treatment for the case of characteristic $p = 2$, for which we recall the setting of [17].

Consider a finite field $\mathbf{F}_{2^r}$. We will identify it with residue field $\mathcal{O}_K/\mathbf{m}_K = \mathbf{F}_{2^r}$ of an unramified extension K of degree r of $\mathbf{Q}_2$. More precisely, let $\mathcal{O}_K \subset K$ the ring of integers and $\mathbf{m}_K$ the maximal ideal. Consider the ring $R = \mathcal{O}_K/\mathbf{m}_K^2$. Let $(\tilde{V}, \tilde{\omega})$ be a free R–module endowed with a symplectic form. The $\mathbf{F}_{2^r}$–vector space $V = \tilde{V}/\mathbf{m}_K$ is endowed with a R–valued non–degenerate skew–symmetric form determined by $\omega = 2\tilde{\omega}$. In the following, when we say that (V, ω) a symplectic vector space over $\mathbf{F}_{2^r}$, we mean a pair obtained as described here, with an R–valued form ω.

Definition 7.8. *A polarization of the symplectic form $\tilde{\omega}$ is a bilinear form $\tilde{\beta} : \tilde{V} \times \tilde{V} \to R$ with $\tilde{\beta}(\tilde{v}, \tilde{w}) - \tilde{\beta}(\tilde{w}, \tilde{v}) = \tilde{\omega}(\tilde{v}, \tilde{w})$.*

Note that bilinearity implies the cocycle condition

$$\tilde{\beta}(\tilde{v}, \tilde{w} + \tilde{u}) - \tilde{\beta}(\tilde{v}, \tilde{w}) - \tilde{\beta}(\tilde{v} + \tilde{w}, \tilde{u}) + \tilde{\beta}(\tilde{w}, \tilde{u}) = 0.$$

Setting $\beta = 2\tilde{\beta}$ induces an R–valued cocycle on V with $\beta(v, w) - \beta(w, v) = \omega(v, w)$.

Definition 7.9. *The Heisenberg group in the characteristic* 2 *case is the extension*

$$0 \to R \to \mathit{Heis}(V, \beta) \to V \to 0$$

determined by the cocycle β *as above, with multiplication*

$$(v, r) \star (w, s) = (r + s + \beta(v, w), v + w).$$

The choice of a character $\chi : R \to \mathbf{C}^*$ determines an irreducible complex representation $\mathcal{H}_\chi(V, \beta)$ of $\mathrm{Heis}(V, \beta)$.

Following [17] we also consider the realisations of this representation, associated to choices of Lagrangians. Here we need to use its enriched version: *enhanced Lagrangians.*

Definition 7.10. *Let* (V, ω) *be a symplectic vector space over* $\mathbf{F}_{2^r}$*, in the sense described above. An enhanced Lagrangian consists of a pair* (L, α) *where* $L \subset V$ *is a Lagrangian subspace and* $\alpha : L \to R$ *satisfies*

$$\alpha(v + w) - \alpha(v) - \alpha(w) = \beta(v, w).$$

This datum $\alpha : L \to R$ defines a section of the projection $\mathrm{Heis}(V, \beta) \to V$ over $L \subset V$ by $\tau : v \mapsto (v, \alpha(v))$ which satisfies $\tau(v+w) = (v+w, \alpha(v+w)) = (v+w, \alpha(v)+\alpha(w)+\beta(v, w)) = \tau(v) \star \tau(w)$, for $v, w \in L$. The corresponding realization $\mathcal{H}_{(V,L,\beta,\chi)}$ of the Heisenberg representation $\pi_{\chi,L}$ is given by the subspace of $\mathbf{C}[\mathrm{Heis}(V, \beta)]$ of functions with

$$f((0, x) \cdot (w, y)) = \chi(x)\, f(w, y), \quad \forall x \in k,\ \forall (w, y) \in V \times k,$$

$$f(\tau(v) \cdot (w, y)) = f(w, y), \quad \forall v \in L,\ \forall (w, y) \in V \times k,$$

with the action of $\mathrm{Heis}(V, \beta)$ by right translations, see [17] for more details.

In this case again one can consider an isotropic subspace $C \subset V$. Since $\omega|_C \equiv 0$, the polarization function β restricted to C is symmetric. Given as above a function $\alpha : C \to R$ with $\alpha(v + w) - \alpha(v) - \alpha(w) = \beta(v, w)$ for all $v, w \in C$, the section $\tau : C \to \mathrm{Heis}(V, \beta)$, $\tau(v) = (v, \alpha(v))$, determines an abelian subgroup of $\mathrm{Heis}(V, \beta)$, since $\beta(v, w) = \beta(w, v)$ on C.

Proposition 7.5 admits then the following version in characteristic 2 case.

Proposition 7.11. *Let* (C, α) *be a pair of an isotropic subspace of* (V, ω) *and an enhancement function* $\alpha : C \to R$ *satisfying*

$$\alpha(v + w) - \alpha(v) - \alpha(w) = \beta(v, w),$$

for β a polarization of ω. A common eigenspace of all the operators $\pi_{\chi,L}(\tau(C))$ on the space $\mathcal{H}_{(V,L,\beta,\chi)}$ defines a quantum error–correcting code

$$\mathcal{C}_{C,\alpha} \subset \mathcal{H}_{(V,L,\beta,\chi)}.$$

The assignment $(C,\alpha) \mapsto \mathcal{C}_{(C,\alpha)}$ gives the symplectic CRSS algorithm for $p = 2$.

In the following subsection, we use this setting to obtain a new construction of code loops, given by extensions

$$0 \to R \to \mathcal{L} \to C \to 0,$$

where $C \subset \mathbf{F}_{2^r}^n$ is a linear code endowed with an almost–symplectic structure, and $R = \mathcal{O}_K/\mathbf{m}_K^2$ as above.

7.5. Code loops and almost symplectic structures

We now pass from the setting of Heisenberg groups to that of code loops by replacing symplectic structures with almost–symplectic structures. Our code loops are a direct natural generalisation of Heisenberg groups, when the symplectic form is no longer required to be closed and is therefore replaced by an almost–symplectic form.

Definition 7.12. *An almost symplectic structure on a finite dimensional vector space V over $\mathbf{F}_q$, with q odd, is a non-degenerate skew-symmetric form $\omega : V \times V \to \mathbf{F}_q$. Namely ω satisfies*

(i) $\omega(u,v) = -\omega(v,u)$, with $\omega(u,0) = \omega(0,u) = 0$,

(ii) for any $u \neq 0$ in V, there is some $v \in V$ satisfying $\omega(u,v) \neq 0$.

The form ω is not required to be closed and has a nontrivial coboundary $d\omega = \delta$, given by

$$d\omega(u,v,w) = \omega(v,w) - \omega(u+v,w) + \omega(u,v+w) - \omega(u,v) = \delta(u,v,w).$$

The nontrivial Hochschild coboundary $d\omega = \delta$ is exactly what in the literature on code loops is usually referred to as the "twisted cocycle" condition, with "twisting" δ (see Section 6.3 above). We prefer to use here the coboundary terminology for consistency with the usual case of almost–symplectic structures on manifolds.

We focus here especially on the case of characteristic $p = 2$. In this case, we proceed as in the symplectic case of [17] recalled above.

Definition 7.13. *For $q = 2^r$, consider as above the ring $R = \mathcal{O}_K/\mathbf{m}_K^2$, where $\mathbf{F}_{2^r} = \mathcal{O}_K/\mathbf{m}_K$. An almost–symplectic vector space (V, ω) is defined as in Definition 7.12, with the almost–symplectic form $\omega : V \times V \to R$. A polarisation of the almost–symplectic form is a function $\beta : V \times V \to R$ satisfying the relation*

$$\beta(u, v) - \beta(v, u) = \omega(u, v),$$

with

$$d\beta(u, v, w) = \beta(v, w) - \beta(u + v, w) + \beta(u, v + w) - \beta(u, v) = \gamma(u, v, w),$$

Then

$$\delta(u, v, w) = \gamma(u, v, w) + \gamma(w, v, u).$$

A polarisation $\beta(u, v) - \beta(v, u) = \omega(u, v)$ is normalised if it satisfies $\beta(v, 0) = 0$ for all $v \in V$.

Since $\beta(u, v) - \beta(v, u) = \omega(u, v)$ and $\omega(0, v) = \omega(v, 0) = 0$, for each polarisation we have $\beta(v, 0) = \beta(0, v)$.

Remark 7.14. *Unlike the symplectic case recalled in the previous subsections, in the almost–symplectic setting ω and β are not multilinear, as that would imply the cocycle condition (the vanishing of δ and γ).*

Definition 7.15. *We define the following functions that measure lack of linearity of β in the left/right variable:*

$$\gamma_\ell(u, v, w) := \beta(u + v, w) - \beta(u, w) - \beta(v, w)$$

$$\gamma_r(u, v, w) := \beta(u, v + w) - \beta(u, w) - \beta(v, w),$$

so that we can write

$$\gamma(u, v, w) = \gamma_r(u, v, w) - \gamma_\ell(u, v, w),$$

and similarly for $\delta_\ell(u, v, w)$ and $\delta_r(u, v, w)$, measuring the lack of linearity of ω.

The code loops we consider here are obtained as follows.

Definition 7.16. *The almost–symplectic code loops $\mathcal{L}(V, \omega)$ and $\mathcal{L}(V, \beta)$ over $\mathbf{F}_q$ are defined as follows.*

(i) If q is odd, such a loop is an extension

$$0 \to \mathbf{F}_q \to \mathcal{L}(V,\omega) \to V \to 0,$$

where (V,ω) is an almost–symplectic vector space over $\mathbf{F}_q$.

(ii) If $q = 2^r$, it is an extension

$$0 \to R \to \mathcal{L}(V,\beta) \to V \to 0,$$

where (V,ω) is an almost–symplectic vector space (V,ω) with polarization β over $\mathbf{F}_{2^r}$.

The non–associative multiplication is given, in the first case, by

$$(u,x)\star(v,y) = (u+v, x+y+\frac{1}{2}\omega(u,v)),$$

and in the second case by

$$(u,x)\star(v,y) = (u+v, x+y+\beta(u,v)).$$

The case with q odd can be seen as a special case of existing construction of loops described in [20] and [21]. Thus, we focus on the case of characteristic 2, which is different. We start with a characterization of the Moufang condition for the loops $\mathcal{L}(V,\beta)$.

The following argument was suggested to us by the referee, based on the construction of free Moufang loops in the variety generated by code loops of [15].

We note that a Moufang loop of the form $\mathcal{L} = \mathcal{L}(V,\beta)$ is a particular case of the Moufang loop from the variety $\mathcal{E}_4$, given by the set of identities

$$\begin{aligned} &x^8 = [x^2,y] = (x^2,y,z) = ([x,y],z,t) = \\ &(x,y,z)^2 = [[x,y],z] = [x,y]^2 = 1\,. \end{aligned} \tag{7}$$

Let $\phi : \mathcal{L}(V,\beta) \to V$ be the epimorphism from Definition 7.16 and let $\{v_i' \,|\, i = 1,\dots,n\}$ be a subset of $\mathcal{L}(V,\beta)$ such that $\{v_i = \phi(v_i') \,|\, i = 1,\dots,n\}$ is a basis of V. We denote by $\mathcal{L}'$ a subloop of $\mathcal{L}(V,\beta)$ generated by $\{v_i' \,|\, i = 1,\dots,n\}$. We have $\mathcal{L}(V,\beta) = \mathcal{L}'$ iff $\mathrm{Ker}(\phi) = A(\mathcal{L}(V,\beta))$, where $A(\mathcal{L})$ is the minimal normal subloop of a loop $\mathcal{L}$ such that $\mathcal{L}/A(\mathcal{L})$ is an elementary abelian group of exponent two. The restriction of ϕ to $\mathcal{L}'$ is an epimorphism. In general, it is not true that $\mathcal{L}(V,\beta) = \mathcal{L}' \times Z_1$, for some subgroup of $\mathrm{Ker}(\phi)$. However, we can find an epimorphism $\alpha : (\mathbf{Z}/4\mathbf{Z})^m \times \mathcal{L}' \to \mathcal{L}(V,\beta)$ with minimal m, such that the restriction of α to $\mathcal{L}'$ is ϕ.

Thus, we can reduce the problem of describing the loops of the form $\mathcal{L}(V,\beta)$ to the problem of describing the free loops in the variety $\mathcal{E}_4$. Let F_n denote a free loop with n generators in the variety $\mathcal{E}_4$.

7.5.1. Construction of F_n Let $\{x_1,\dots,x_n\}$ be a set of free generators of the free loop F_n. Let $\dim_{\mathbf{F}_2} V = n$. We define an abelian group $Z = \oplus_{i=1}^{n}(\mathbf{Z}/4\mathbf{Z})z_i \oplus V\wedge V \oplus V\wedge V\wedge V$. We can consider $V\wedge V\oplus V\wedge V\wedge V$ as an $\mathbf{F}_2$-vector space with a basis $\{w_{ij} = v_i\wedge v_j\,;\, u_{ijk} = v_i\wedge v_j\wedge v_k \,|\, 1\le i<j<k\le n\}$, for some fixed basis $\{v_1,\dots,v_n\}$ of V over $\mathbf{F}_2$.

We denote by x_σ the elements $x_\sigma := (\cdots(x_{i_1}x_{i_2})\cdots)x_{i_s} \in F_n$ and we take $v_\sigma := v_{i_1} + v_{i_2} + \cdots + v_{i_s}$, for a set of indices $\sigma = \{i_1 < \cdots < i_s\}$. We then define a cocycle $\tau : V\times V \to Z$ by setting

$$\tau(v_\sigma, v_\mu) := \Big(\prod_{i\in\sigma\cap\mu} z_i\Big)\Big(\prod_{i>j,i\in\sigma,j\in\mu} w_{ij}\Big)\Big(\prod_{i\in\sigma,j<k\in\mu} u_{ijk}\Big).$$

Theorem 7.17. *The loop $\mathcal{L}(V,\tau)$ is Moufang and is isomorphic to a free loop F_n in the variety $\mathcal{E}_4$.*

Proof. First we check that τ is a Moufang cocycle. Let N be a subloop of $\mathcal{L} = \mathcal{L}(V,\tau)$ generated by $\{z_1^2,\dots,z_n^2\}$. It is clear that $\mathcal{I} \simeq \mathbf{F}_2^n$ and $\mathcal{L}/N \simeq E_n$, where E_n is a free Moufang loop on n generators in the variety $\mathcal{E}$ defined in [15]. Let $f(x,y,z) = ((xy)(zx))(x((yz)x))^{-1}$. Then $\mathcal{L}$ is a Moufang loop iff $f(\mathcal{L},\mathcal{L},\mathcal{L}) = 1$. We showed that $f(\mathcal{L},\mathcal{L},\mathcal{L}) \subseteq N$. On the other hand, if $\mathcal{I} = V\wedge V \oplus V\wedge V\wedge V \subset Z \subset \mathcal{L}$, then $\mathcal{L}/\mathcal{I} \simeq (\mathbf{Z}/8\mathbf{Z})^n$ is a group. Thus, $f(\mathcal{L},\mathcal{L},\mathcal{L}) \subset N\cap\mathcal{I} = 1$ hence $\mathcal{L}$ is a Moufang loop. Then, since F_n is a free loop in the variety $\mathcal{E}_4$ and $\mathcal{E}\subset\mathcal{E}_4$, there exists an epimorphism $\lambda : F_n \to E_n$ with kernel $\mathrm{Ker}(\lambda) \simeq \mathbf{F}_2^n$ generated by $x_1^4,\dots,x_n^4$. Thus, we have $\mathrm{Ker}(\lambda)\simeq N$ and $|F_n| = |\mathcal{L}(V,\tau)|$. Since F_n is a free loop, there exists an epimorphism $\mu : F_n \to \mathcal{L}(V,\tau)$, so that we obtain $F_n \simeq \mathcal{L}(V,\tau)$. □

Corollary 7.18. *Let $\mathcal{L}(V,\beta)$ be a Moufang loop with n generators obtained through the construction of Definition 7.16. Then $|\mathcal{L}(V,\beta)| \le 2^t$ where $t = n(n^2+8)/3$.*

We can then obtain a characterization of loops $\mathcal{L}(V,\beta)$ that satisfy the Moufang condition, using the previous results. Indeed, Theorem 7.17 allows us to describe all Moufang loops of the form $\mathcal{L}(V,\beta)$ for a fixed choice of a $\mathbf{F}_2$-vector space V and a natural number.

We can assume that $\mathcal{L}(V,\beta) \neq Q\times\mathcal{L}'$, where for $\phi : \mathcal{L}(V,\beta)\to V$ as above, we have $\phi(\mathcal{L}') = V$ and $\phi(Q) = 0$. In this case, we say that $\mathcal{L}(V,\beta)$ is *irreducible*.

We can then construct a non-splitting extension $\mathbf{F}_n$ of F_n in the following way. If $P = (\mathbf{Z}/4\mathbf{Z})^m \times F_n$, with $m = (n-1)n(n+1)/6$, let $\{t_{ij}; t_{ijk} \,|\, 1 \leq i < j < k \leq n\}$ be a basis of the free $\mathbf{Z}/4\mathbf{Z}$-module $(\mathbf{Z}/4\mathbf{Z})^m$. We denote by J a normal subloop of P generated by $\{w_{ij}t_{ij}; u_{ijk}t_{ijk} \,|\, 1 \leq i < j < k \leq n\}$. By construction we have $\mathbf{F}_n = P/J$. We then have the following characterization of the Moufang property for loops $\mathcal{L}(V, \beta)$.

Proposition 7.19. *Let $\mathcal{L}(V, \beta)$ be an irreducible Moufang loop obtained as in Definition 7.16. Then there exists an epimorphism $\mu : \mathbf{F}_n \to \mathcal{L}(V, \beta)$ with central kernel.*

Proof. Let $\phi : \mathcal{L}(V, \beta) \to V$ be an epimorphism with $\{\phi(v_i') \,|\, i = 1, \ldots, n\}$ a basis of V. We denote by $\mathcal{L}'$ a subloop of $\mathcal{L}(V, \beta)$ generated by the set $\{v_i' \,|\, i = 1, \ldots, n\}$. Since F_n is a free loop, there exists an epimorphism $\pi : F_n \to \mathcal{L}'$. Let

$$W = \mathrm{Ker}(\phi) \cap \mathcal{L}' = (\mathbf{Z}/4\mathbf{Z})^s \times (\mathbf{Z}/2\mathbf{Z})^t \subseteq R = (\mathbf{Z}/4\mathbf{Z})^r \,.$$

By the structure of F_n we get that $s \leq n$ and $t \leq m = (n-1)n(n+1)/6$. There exists a subgroup T of $\mathrm{Ker}(\phi)$ such that $W \subset T = (\mathbf{Z}/4\mathbf{Z})^{s+t}$ and an epimorphism $\mu : \mathbf{F}_n \to \mathcal{L}' \cdot W$. If $\mathcal{L}' \cdot W \neq \mathcal{L}(V, \beta)$, then the loop $\mathcal{L}(V, \beta)$ would be reducible. □

7.6. Quantum codes from almost–symplectic code loops

We come now to extending the result of Proposition 7.5 to our construction of loops $\mathcal{L}(V, \beta)$. To this purpose, we first need to recall the appropriate notion of linear representations of loops, then we need to introduce isotropic subspaces, and then obtain from them the respective CRSS quantum codes.

7.6.1. Linear representations of loops A notion of linear representations of loops was developed in [22]. It is closely related to the Eilenberg notion of representation for non–associative algebras [13].

Given a loop $\mathcal{L}$ and a vector space $\mathcal{H}$ over a field F, left and right composition maps are defined as $\ell, \rho : \mathcal{L} \to Aut(\mathcal{H})$, which we write simply as

$$\ell_a(h) = a \star h, \quad \rho_a(h) = h \star a.$$

These maps should satisfy $a \star (h + h') = a \star h + a \star h'$, $(h + h') \star a = h \star a + h' \star a$, $a \star (\lambda h) = \lambda\, a \star h$, $(\lambda h) \star a = \lambda\, h \star a$, for all $a \in \mathcal{L}$, $h, h' \in \mathcal{H}$, $\lambda \in F$. One defines on $\mathcal{L} \times \mathcal{H}$ the multiplication

$$(a, h) \star (b, h') = (a \star b, a \star h' + h \star b).$$

We also define the associator

$$[a, b, h] = (a \star b) \star h - a \star (b \star h)$$

for $a, b \in \mathcal{L}$ and $h \in \mathcal{H}$.

Over a field F, one can associate to a loop $\mathcal{L}$ the non–associative algebra $F[\mathcal{L}]$, the analog of the associative group algebra for groups. The maps ℓ, ρ of a representation of $\mathcal{L}$ on an F–vector space $\mathcal{H}$ extend by linearity to $F[\mathcal{L}]$, in the sense of representations of non-associative algebras [13].

If the loop $\mathcal{L}$ satisfies the Moufang identity, then the maps $\ell, \rho : \mathcal{L} \to Aut(\mathcal{H})$ of a representation of $\mathcal{L}$ must satisfy the following conditions (see [22]): the associator $[a, b, h]$ is skew-symmetric for all $a, b \in F[\mathcal{L}]$ and $h \in \mathcal{H}$; the identities $h\star(b\star(a\star b)) = ((h\star b)\star a)\star b$ and $((a\star b)\star a)\star h = a\star(b\star(a\star h))$ hold, for all $a, b \in F[\mathcal{L}]$ and all $h \in \mathcal{L}$.

7.6.2. Isotropic subspaces As above, denote by V a vector space over $\mathbf{F}_{2^r}$, put $R = \mathcal{O}_K/\mathbf{m}_K^2$ where $\mathcal{O}_K/\mathbf{m}_K = \mathbf{F}_{2^r}$, and assume that V is endowed with an almost–symplectic structure $\omega : V \times V \to R$ with normalised polarisation β.

Definition 7.20. *An isotropic subspace $C \subset V$ is a linear subspace where the almost symplectic form vanishes identically, $\omega|_C = 0$. A polarisable subspace $P \subset V$ is a linear subspace for which there is an enhancement function $\alpha : P \to R$ satisfying*

$$\alpha(u + v) - \alpha(u) - \alpha(v) = \beta(u, v), \quad \forall u, v \in P.$$

A polarized subspace is a pair (P, α) satisfying the condition above.

The polarization relation is just the Hochschild coboundary relation $\beta = d\alpha$, hence it implies $\gamma|_P = d\beta|_P = 0$.

Definition 7.21. *A polarized subspace (P, α) determines a section $\tau : P \to \mathcal{L}(V, \beta)$ of the projection $\mathcal{L}(V, \beta) \to V$, with image $\tau(P) \subset \mathcal{L}(V, \beta)$ a subgroup of the loop $\mathcal{L}(V, \beta)$. If P is also isotropic, then $\tau(P) \subset \mathcal{L}(V, \beta)$ is an abelian subgroup.*

Proof. The section $\tau : P \to \mathcal{L}(V, \beta)$ is constructed as in the symplectic case of [17], by taking $\tau(v) = (v, \alpha(v))$ for $v \in P$. This satisfies

$$(v, \alpha(v)) \star (w, \alpha(w)) = (v + w, \alpha(v) + \alpha(w) + \beta(v, w)) = (v + w, \alpha(v + w)).$$

This multiplication is associative since $d\beta|_P = 0$. On an isotropic subspace the polarization β is symmetric, hence the resulting multiplication is also commutative. □

7.6.3. CRSS quantum codes from almost–symplectic loops We consider here almost–symplectic loops $\mathcal{L}(V,\omega)$ in characteristic $p>2$ and $\mathcal{L}(V,\beta)$ in characteristic $p=2$, as in Definition 7.16. We simply write $\mathcal{L}$ for the loop when both cases are considered.

Let $\mathcal{H}=\mathbf{C}[\mathcal{L}]$ be the complex vector space of complex valued functions on $\mathcal{L}$, endowed with the left and right composition maps $\ell,\rho:\mathcal{L}\to Aut(\mathcal{H})$, as in Section 7.6.1, given by the left and right action of $\mathcal{L}$ on itself extended by linearity. We write $|a\rangle$ with $a\in\mathcal{L}$ for the canonical basis of $\mathcal{H}$.

Given a character $\chi: Z(\mathcal{L})\to\mathbf{C}^*$ (that is, a character $\chi: R\to\mathbf{C}^*$ for $p=2$ or $\chi:\mathbf{F}_q\to\mathbf{C}^*$ for $p>2$), let $\mathcal{H}_\chi\subset\mathcal{H}$ be the subspace of functions $f:\mathcal{L}\to\mathbf{C}$ that transform like $\ell_{(0,x)}f(u,y)=\chi(x)f(u,y)$, for $x\in Z(\mathcal{L})$ and $(u,y)\in\mathcal{L}$.

When $p=2$, a polarized isotropic subspace (C,α) is a pair of an isotropic subspace $C\subset V$ together with an enhancement function as in Definition 7.20 above. When $p>2$ let $C\subset V$ be an isotropic subspace. In the following, for simplicity, we will refer to both cases simply as "an isotropic subspace", with the function α implicitly understood in the characteristic 2 case.

Theorem 7.22. *An isotropic subspace $C\subset V$ determines a commuting family of error operators $\chi(\tau(v))E_v$, with $v\in C$, and an associated error correcting quantum code $\mathcal{C}_C\subset\mathcal{H}_\chi$ given by a joint eigenspace of these operators. The assignment $C\mapsto\mathcal{C}_C$ is the almost–symplectic CRSS algorithm.*

Proof. The left composition map $\ell:\mathcal{L}\to Aut(\mathcal{H}_\chi)$ induces a representation $\pi:\tau(C)\to Aut(\mathcal{H}_\chi)$ of the abelian subgroup $\tau(C)\subset\mathcal{L}$, as in Proposition 7.21. We can write the operators on $\mathcal{H}_\chi$ obtained in this way as $\pi(v,\tau(v))=\chi(\tau(v))E_v$, and regard them as a commuting family of error operators on $\mathcal{H}_\chi$. A common eigenspace of the $\chi(\tau(v))E_v$ in $\mathcal{H}_\chi$ gives a subspace $\mathcal{C}_C\subset\mathcal{H}_\chi$ that is the CRSS quantum code associated to the classical code $C\subset V$ through the code loop $\mathcal{L}$. □

7.7. Locally conformally symplectic structures and perfect tensors

We now discuss how to generalize Proposition 7.7 to the case of code loops.

In general, in the almost–symplectic case, the fact that $\delta=d\omega\neq 0$ means that we do not have a Darboux decomposition of (V,ω), hence ω by itself does not determine an explicit identification of of $\mathcal{H}_\chi$ with a tensor product of q–ary qubits. Thus, in the almost–symplectic setting one needs to consider special cases, such as an analog of the conformally flat almost–symplectic structures on manifolds, from which a decomposition of the space into 2–dimensional Darboux pieces can still be obtained.

When a decomposition into a product of qubits is given, one can again use as in [19] Lagrangians in general position with respect to this decomposition (enhanced Lagrangians (L, α) in the case of characteristic 2) to obtain perfect tensors through the same kind of CRSS construction described above.

We focus here in particular on a case modelled on manifolds with locally conformally symplectic structures, for which a Darboux theorem holds, see [36].

Definition 7.23. *Let V be a finite dimensional vector space over $\mathbf{F}_q$.*

An 1–form is given by a function $\theta : V \to A$, and a 2–form is given by a function $\omega : V \times V \to A$, where $A = \mathbf{F}_q$, if q is odd, and $A = R$, if q is even.

Define the wedge product $\theta \wedge \omega$ as the function of three arguments

$$(\theta \wedge \omega)(u, v, w) := \theta(u)\,\omega(v, w) + \theta(w)\,\omega(u, v).$$

This definition is compatible with defining the wedge product of two 1–forms θ_1, θ_2 as

$$(\theta_1 \wedge \theta_2)(v, w) := \theta_1(v)\theta_2(w) - \theta_1(w)\theta_2(v),$$

through the expected relation

$$d(\theta_1 \wedge \theta_2) = d\theta_1 \wedge \theta_2 - \theta_1 \wedge d\theta_2.$$

Definition 7.24. *Let V be a vector space over $\mathbf{F}_q$. An almost–symplectic form ω on V is called a locally conformally symplectic structure if there is a closed 1–form θ such that*

$$d\omega = \theta \wedge \omega.$$

Moreover, θ and ω must be homogeneous with respect to scalar multiplication on V.

Consider an almost symplectic vector space (V, ω) over $\mathbf{F}_q$, and the associated loop $\mathcal{L}$ (that is, $\mathcal{L}(V, \omega)$ for characteristic $p > 2$ and $\mathcal{L}(V, \beta)$ in characteristic $p = 2$). Let $L \subset V$ be a Lagrangian with respect to ω (an enhanced Lagrangian (L, α) for $p = 2$) and let $\tau(L) \subset \mathcal{L}$ be the resulting subloop, with $\tau(L) = \{(v, 0) \,|\, v \in L\}$ for $p > 2$ and $\tau(L) = \{(v, \alpha(v)) \,|\, v \in L\}$ for $p = 2$. By Proposition 7.21 we know that $\tau(L)$ is in fact an abelian subgroup.

Let $\mathcal{H}(V, L, \omega) \subset \mathbf{C}[\mathcal{L}]$ be the subspace of functions $f(u, x)$ that are invariant under the action of $\ell(\tau(L))$, through the left composition map ℓ of the loop representation. Let $\mathcal{H}_\chi(V, L, \omega)$ be the subset of functions that also transform as $\ell_{(0,y)} f(u, x) = \chi(y) f(u, x)$, under a character $\chi : Z(\mathcal{L}) \to \mathbf{C}^*$ (that is, $\chi : \mathbf{F}_q \to \mathbf{C}^*$ for $p > 2$ and $\chi : R \to \mathbf{C}^*$ for $p = 2$).

Proposition 7.25. *A locally conformally symplectic structure* (V,ω) *over* $\mathbf{F}_q$ *determines a decomposition into qubits,* $\mathcal{H} \simeq \otimes_i \mathcal{H}_i$ *with* $\mathcal{H}_i \simeq \mathbf{C}^q$*, of* $\mathcal{H} = \mathcal{H}_\chi(V, L, \omega)$.

Proof. Since θ is homogeneous, the closedness $d\theta = 0$ means that $d\theta(u,v) = \theta(v) - \theta(u+v) + \theta(u) = 0$, that is, θ is linear. Thus, we can decompose the vector space V into the kernel $K = Ker(\theta)$ and a one–dimensional complement, $V = K \oplus \mathbf{F}_q$, satisfying the condition $d\omega|_K \equiv 0$.

Since ω is non–degenerate, one can find a pair of vectors u, v in K such that $\omega(u,v) \neq 0$. Since ω is closed on K, one can then decompose K into this two-dimensional subspace and a complement $W = \{w \in K \,|\, \omega(u,w) = \omega(v,w) = 0\}$. One can proceed in the same way by restricting ω to W, and obtain in this way a decomposition of K into subspaces $K_i \simeq \mathbf{F}_q^2$, with $K \simeq \oplus_i K_i \oplus \mathbf{F}_q$. This provides an overall decomposition of $V \simeq \oplus_i V_i$ with $V_i \simeq \mathbf{F}_q^2$. The direct sum $V = \oplus_i V_i$ with $\omega_i = \omega|_{V_i}$ gives a corresponding decomposition of the complex vector space $\mathcal{H} = \otimes_i \mathcal{H}_i$ with each $\mathcal{H}_i \simeq \mathbf{C}^q$ a single qubit space. □

We refer to the decomposition of the locally conformally symplectic space (V,ω) obtained in this way and the corresponding decomposition of $\mathcal{H}$ into qubits as the *Darboux decomposition.*

We thus obtain the generalisation of Proposition 7.7 to the case of code loops, by the same argument as in [19].

Theorem 7.26. *A Lagrangian L that is in general position with respect to the Darboux decomposition of the locally conformally symplectic structure, determines a perfect tensor in* $\mathcal{H}$.

7.7.1. Networks of perfect tensors We now show that the construction of perfect tensors associated to almost–symplectic code loops with a locally conformally symplectic structure and a Lagrangian in general position, as in Theorem 7.26 can be used to construct networks of perfect tensors associated to certain combinatorial structures that arise from the relation between Moufang loops and Latin square designs.

In particular, we use this construction of networks of perfect tensors to show that the Latin square designs obtained from our code loops have an associated information–theoretic entropy functional.

We will first introduce tensor networks and the associated entanglement entropy. We will then review the relation between Moufang loops and Latin square designs and present our construction of networks of perfect tensors. We then conclude the section with some questions on the construction of tensor networks on chamber systems and on their universal 2–covers, when the latter are buildings.

7.7.2. Tensor networks and entanglement entropy A tensor network is a pattern of contraction of indices of tensors. This can be stated more precisely as follows.

We will encode combinatorics of finite graphs by identifying each such graph G with a quadruple $G = (F, V, \partial, j)$, where F is the set of flags (half–edges), V the set of vertices, ∂ the boundary map $\partial : F \to V$ that identifies the root vertex of each flag, and j is the structure involution $j : F \to F$, $j^2 = id$, that describes how half–edges are glued together into edges of G. Using the physics terminology, we call *internal edges* those pairs $e = (f, f')$ with $f \neq f'$ and $f' = j(f)$, and *external edges* the flags f that are fixed by the involution: $j(f) = f$.

Much more details can be found in [2] Sec. 1, in particular, a description of morphisms of graphs and other information, which we will use below without repeating the definitions.

Definition 7.27. *A tensor network $(G, \mathcal{H}, T)$ consists of a finite graph G as above, without multiple edges, where the vertices $v \in V$ are decorated by pairs $(\mathcal{H}_v, T^{(v)})$ of a complex vector space $\mathcal{H}_v = (\mathbf{C}^q)^{\otimes \deg(v)}$, for some $q = p^r > 0$ a power of some prime p, with $\deg(v)$ the valence of the vertex, and a $T^{(v)} \in \mathcal{H}_v$.*

We can view such $T^{(v)}$ as a tensor $T^{(v)} = (T^{(v)})_{i_1,\dots,i_{\deg(v)}}$, with indices $i_f \in \mathbf{F}_q$, labelled by the flags $f \in F$ with $\partial(f) = v$. An edge $e = (f, f')$, $f' = j(f)$, with $\partial e = \{v, v'\}$ corresponds to a contraction of indices of the tensors $T^{(v)}$ and $T^{(v')}$ of the form

$$\sum_{i_f, i'_{f'} \in \mathbf{F}_q} \delta^{i_f, i'_{f'}} \, T^{(v)}_{i_1,\dots,i_{\deg(v)}} \, T^{(v')}_{i'_1,\dots,i'_{\deg(v')}},$$

with δ^{ij} the Kronecker delta function. The internal edges of G are called the bonds of the tensor networks. The external edges of the graph G correspond to indices of the tensors that remain non–contracted. We call them the dangling legs of the tensor networks. The graph G is called the support of the tensor network.

Definition 7.28. *Let G be a finite connected graph. A cut–set of G is such a subset $C \subset E_{in}(G)$ of the set of internal edges, that if all the edges $e \in C$ are cut, the graph G is split into exactly two non–empty connected components, $G \smallsetminus C = G_{C,1} \sqcup G_{C,2}$.*

Lemma 7.29. *Let G be a finite connected graph and let $E_{in}(G)$ and $E_{ext}(G)$ be the sets of internal and external edges of G. A tensor network $\mathcal{T} = (G, \mathcal{H}, T)$ computes an entangled state $|\psi_{\mathcal{T}}\rangle$ in the space $\mathcal{H}_{\mathcal{T}} = (\mathbf{C}^q)^{\otimes |E_{ext}(G)|}$,*

with $|E_{ext}(G)|$ the number of external edges of the graph G. In the case where $E_{ext}(G) = \emptyset$, this computation just gives a complex number, the amplitude $\alpha_{\mathcal{T}}$. Given a cut-set C, one obtains entangled states $|\psi_{C,i}\rangle$ in $(\mathbf{C}^q)^{\otimes|C|}$, associated to the restrictions of the tensor network to the components $G_{C,i}$, satisfying $|\alpha_{\mathcal{T}}| = |\langle\psi_{C,1}, \psi_{C,2}\rangle|$.

Proof. Consider the standard basis $|a_1 \dots a_N\rangle$ of the space $(\mathbf{C}^q)^{\otimes N}$ of N q-ary qubits, with $a = (a_1, \dots, a_N) \in \mathbf{F}_q^N$ and $a_i \in \mathbf{F}_q$, and $|a_1 \dots a_N\rangle = |a_1\rangle \otimes \cdots \otimes |a_N\rangle$. At each vertex $v \in V(G)$ we obtain an entangled state

$$|\psi_v\rangle = \sum_{a_1,\dots,a_{\deg(v)} \in \mathbf{F}_q} T^{(v)}_{a_1,\dots,a_{\deg(v)}} |a_1 \dots a_{\deg(v)}\rangle,$$

obtained as a superposition of the pure states $|a_1\rangle \otimes \cdots \otimes |a_{\deg(v)}\rangle$. Contracting two tensors $T^{(v)}$ and $T^{(v')}$ along an edge e with $\partial(e) = \{v, v'\}$ gives rise to an entangled state that is a superposition of the pure states associated to the remaining dangling legs at the two vertices,

$$|\psi_e\rangle = \sum_{a_i, b_j \in \mathbf{F}_q} \delta^{a_f, b_{f'}} T^{(v)}_{a_1,\dots,a_{\deg(v)}} T^{(v')}_{b_1,\dots,b_{\deg(v')}} |\hat{a}^{(f)}, \hat{b}^{(f')}\rangle,$$

where $\hat{a}^{(f)} = (a_1, \dots, \hat{a}_f, \dots, a_{\deg(v)})$ and $\hat{b}^{(f')} = (b_1, \dots, \hat{b}_{f'}, \dots, b_{\deg(v')})$, and $\hat{a}_f$ and $\hat{b}_{f'}$ means that this entry in the vector has been removed. In a similar way, performing the contractions of the tensor indices along the edges of the graph G gives rise to an entangled state $|\psi_G\rangle$ that is a superposition of the pure states associated to the dangling legs of G

$$|\psi_G\rangle = \sum_{c_1,\dots,c_N \in \mathbf{F}_q} \tau_{c_1,\dots,c_N} |c_1 \dots, c_N\rangle,$$

where $N = \#E_{ext}(G)$ is the number of external edges. The coefficients $\tau_{c_1,\dots,c_N}$ are computed by performing all the contraction of indices across all the internal edges of the graph G.

If G has no external edges, each edge $e \in C$, seen as a pair $e = (f_1, f_2)$ with $f_2 = j(f_1)$ and $\partial(f_i) \in G_{C,i}$, endows both components $G_{C,i}$ with an external edge, so that the total number of such edges is $|E_{ext}(G_{C,i})| = |C|$, for both $i = 1, 2$. One can then consider the states $|\psi_{C,i}\rangle$ computed by the tensor network as above. The amplitude $\alpha_{\mathcal{T}}$ is obtained from these by contracting the indices corresponding to the pairs (f_1, f_2). □

A tensor network $\mathcal{T} = (G, \mathcal{H}, T)$ with the associated entangled state $|\psi_{\mathcal{T}}\rangle$ in $\mathcal{H}_{\mathcal{T}} = (\mathbf{C}^q)^{\otimes|E_{ext}(G)|}$ as above determines a corresponding density matrix,

written in bra-ket notation as

$$\rho = \frac{1}{\langle \psi_{\mathcal{T}} | \psi_{\mathcal{T}} \rangle} \, |\psi_{\mathcal{T}}\rangle \, \langle \psi_{\mathcal{T}}|.$$

Given a partition $A \sqcup B$ of the set of external edges of G, we can consider

$$\rho_A = \mathrm{Tr}_B(\rho)$$

with $\mathrm{Tr}_B : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A$, so that ρ_A is obtained from ρ by tracing out (contracting the indices of) the dangling legs in B.

Definition 7.30. *The entanglement entropy of the tensor network $\mathcal{T} = (G, \mathcal{H}, T)$ is then given by the assignment*

$$A \mapsto S_{\mathcal{T}}(A) := Tr(\rho_A \log \rho_A),$$

for $A \subset E_{ext}(G)$ ranging over all subsets of external edges.

In the case of a connected graph G with no external edges, the entanglement entropy of the tensor network $\mathcal{T} = (G, \mathcal{H}, T)$ is given by the assignment

$$A_i \mapsto S_{\mathcal{T},C,i}(A_i) := Tr(\rho_{C,A_i} \log \rho_{C,A_i}),$$

for C ranging over cut–sets and $A_i \subset E_{ext}(G_{C,i})$ ranging over all subsets of external edges of the components $G_{C,i}$, and with $\rho_{C,A_i} = Tr_{C \smallsetminus A_i}(\rho_{C,i})$ where $\rho_{C,i}$ is the density matrix associated to the entangled state $|\psi_{C,i}\rangle$.

7.7.3. Moufang loops and Latin square designs We recall here briefly some notions from combinatorial designs and the geometry of buildings, closely related to loops. We refer the reader to [5, 18, 30] for more details.

Definition 7.31. *A Latin square design is a pair $\mathcal{D} = (P, A)$.*

Here P is a set of $3N$ points, represented as a disjoint union $P = P_1 \sqcup P_2 \sqcup P_3$ of three subsets of cardinality N.

A is a family of subsets of P, called lines, with the property that each line in A contains exactly 3 points, one from each of the three subsets P_i, and such that any two points from two different subsets P_i belong to exactly one line in A.

The Latin square of the design $\mathcal{D}$ is the $N \times N$ – matrix with entries corresponding to the N^2 lines in A and with (x_1, x_2)–entry equal to x_3 if the line containing $x_1 \in P_1$ and $x_2 \in P_2$ has $x_3 \in P_3$ as the third point. The order of a Latin square is the number N of points of each type.

Latin square designs form a category with objects $\mathcal{D} = (P, A)$ and morphisms $\mathcal{D} \to \mathcal{D}'$ given by a triple of maps $\alpha_i : P_i \to P_i'$ such that, if (x_1, x_2, x_3) is a line in A then $(\alpha_1(x_1), \alpha_2(x_2), \alpha_3(x_3))$ is a line in A'.

Given a loop $\mathcal{L}$, the *Thomsen design* $\mathcal{D}(\mathcal{L})$ has set of points $P = \mathcal{L}_1 \sqcup \mathcal{L}_2 \sqcup \mathcal{L}_3$, three copies of $\mathcal{L}$ labelled $i = 1, 2, 3$, and set of lines $A = \{(x_1, x_2, x_3) \mid (x_1 \star x_2) \star x_3 = 1 \in \mathcal{L}\}$. Conversely, given any Latin square design $\mathcal{D}$, there is a loop $\mathcal{L}(\mathcal{D})$ with this property, the *Thomsen loop* of $\mathcal{D}$. The Thomsen loop assignment $\mathcal{D} \mapsto \mathcal{L}(\mathcal{D})$ is functorial and gives an equivalence of categories between the category of Latin square designs and the category of loops, where objects are loops $\mathcal{L}$ and morphisms are isotopisms, namely triples of maps $(\alpha, \beta, \gamma) : \mathcal{L} \to \mathcal{L}'$ satisfying $\alpha(x) \star' \beta(y) = \gamma(x \star y)$ for all $x, y \in \mathcal{L}$, see Theorem 3.4 of [18].

An *automorphism* of a Latin square design $\mathcal{D} = (P, A)$ is a permutation of P that sends lines to lines. A central automorphism τ_x of $\mathcal{D}$, *centered* at a point $x \in P$ is an automorphism that fixes x and exchanges the remaining two points on each line in A containing x (see Section 3.2 of [18]).

Definition 7.32. *A central Latin square design is a design that admits a central automorphism at every point $x \in P$.*

The Thomsen functor restricted to this subcategory gives an equivalence between the category of central Latin square designs and the category of *Moufang loops* (Theorem 3.11 of [18]).

A subdesign $\mathcal{D}' = (P', A')$ of a Latin square design $\mathcal{D} = (P, A)$ consists of sets $P' \subseteq P$ and $A' \subseteq A$ of points and lines that form a Latin square design.

Any non–empty set of lines in $\mathcal{D}$ is contained in a unique minimal subdesign. This is referred to as the subdesign generated by the given set of lines.

Lemma 7.33. *Consider the almost–symplectic code loops $\mathcal{L}(V, \omega)$, if characteristic p is odd, or $\mathcal{L}(V, \beta)$ if $p = 2$, as in Definition 7.16.*

The Thomsen design $\mathcal{D}(\mathcal{L}(V, \omega))$, resp. $\mathcal{D}(\mathcal{L}(V, \beta))$ has an associated graph $G = G_{\mathcal{L}(V,\omega)}$, resp. $G = G_{\mathcal{L}(V,\beta)}$, describing how points of the design are connected by lines, with $\operatorname{card} V(G) = 3N$ and $\operatorname{card} E(G) = 3N^2$, where $N = q^{2n+1}$ for $q = p^r$ with p odd and $N = q^{2n+2}$ for $q = 2^r$.

The choice of an isotropic subspace $C \subset V$ with $\dim_{\mathbf{F}_q} C = k$ determines a subdesign $\mathcal{D}(\tau(C))$ and a subgraph $G_{\tau(C)}$ with $3q^k$ vertices and $3q^{2k}$ edges. For $p = 2$ any pair of intersecting lines in $\mathcal{D}(\tau(C))$ generate a subdesign of order 2.

Proof. We can identify as sets $\mathcal{L}(V, \omega) \simeq V \times \mathbf{F}_q$ and $\mathcal{L}(V, \beta) \simeq V \times R$, hence $\operatorname{card} \mathcal{L}(V, \omega) = q^{2n+1}$, where $2n = \dim_{\mathbf{F}_q} V$, and $\operatorname{card} \mathcal{L}(V, \beta) = 2^{2nr+2r}$, where

$\dim_{\mathbf{F}_{2^r}} V = 2n$. The Thomsen design $\mathcal{D}(\mathcal{L}(V,\omega))$ has P consisting of three copies of $\mathcal{L}(V,\omega)$, marked with labels $i = 1, 2, 3$, and set of lines

$$A = \{((u,x)_1, (v,y)_2, (w,z)_3) \mid u+v+w = 0, \\ x+y+z+\frac{1}{2}\omega(u,v)+\frac{1}{2}\omega(u+v,w) = 0\}.$$

The characteristic 2 case is similar: the Thomsen design $\mathcal{D}(\mathcal{L}(V,\beta))$ has P consisting of three labelled copies of $\mathcal{L}(V,\beta)$ and set of lines

$$A = \{((u,x)_1, (v,y)_2, (w,z)_3) \mid u+v+w = 0, \\ x+y+z+\beta(u,v)+\beta(u+v,w) = 0\}.$$

The order of the corresponding Latin square is $N = q^{2n+2}$ in the characteristic 2 case, with $q = 2^r$ and $N = q^{2n+1}$ in characteristic $p > 2$. Given a point $(u,x)_i$ in P, the panel $\Pi_{(u,x)_i}$ of lines through the point $(u,x)_i$ contains N lines, each containing two other points. Two panels $\Pi_{(u,x)_i}$ and $\Pi_{(v,y)_j}$ with types $i \neq j$ intersect in a single line. Thus we can form a graph G with set of vertices $V = P$ of uniform valence $2N$ and a single edge between any two points with types $i \neq j$. The number of edges is $\operatorname{card} E = 3N^2$. The construction for the subgraph $G_{\tau(C)}$ is analogous.

In the case of characteristic 2, the subspace C, seen as an abelian group is an elementary abelian 2–group and so is its image $\tau(C) \subset \mathcal{D}(\mathcal{L}(V,\beta))$. As shown in Lemma 4.3 of [30], the condition that any pair of intersecting lines generate a subdesign of order 2 is equivalent to the property that the associated loop is an elementary abelian 2–group, hence the property holds in this case. □

7.7.4. Networks of perfect tensors Consider the almost-symplectic code loops $\mathcal{L}(V,\omega)$, in characteristic p odd, or $\mathcal{L}(V,\beta)$ in characteristic $p = 2$, as in Definition 7.16. We assume in both cases that the almost-symplectic form ω is a locally conformally symplectic structure as in Definition 7.24. Let L (respectively, (L,α)) be a Lagrangian (respectively, enhanced Lagrangian) that is in general position with respect to the Darboux decomposition of the conformally symplectic structure, and let T_L be the associated perfect tensor, as in Theorem 7.26.

We now construct a tensor network associated to the design $\mathcal{D}(\mathcal{L}(V,\omega))$ or $\mathcal{D}(\mathcal{L}(V,\beta))$ and its subdesign $\mathcal{D}(\tau(L))$, for the chosen Lagrangian.

Proposition 7.34. *Let (V,ω) be an almost–symplectic vector space with $\dim_{\mathbf{F}_q} V = 2n$ and ω locally conformally symplectic. The choice a Lagrangian*

L in general position with respect to the Darboux decomposition gives rise to a network of perfect tensors $(G, \mathcal{H}, T)$ with support $G \subset G_{\tau(L)}$ a uniform subgraph with $V(G) = V(G_{\tau(L)})$ and valence $2n$, and with $\mathcal{H} = \mathcal{H}(V)$ with $T \in \mathcal{H}$ the perfect tensor $T = T_L$.

Proof. We write here $\mathcal{L}$ for either $\mathcal{L}(V, \omega)$ or $\mathcal{L}(V, \beta)$ in odd/even characteristic. Consider as in Lemma 7.33 the graph $G_{\mathcal{L}}$ with the subgraph $G_{\tau(L)}$. As support of the tensor network we consider a subgraph $G \subset G_{\tau(L)}$ with the same set of q^n vertices and with the set of edges obtained as follows. Each vertex in $G_{\tau(L)}$ has valence $\operatorname{card} \tau(L) = 2q^n$, with the corolla of the vertex identified with the line segments connecting a point $u_i = (u, \alpha(u))_i$ of $\mathcal{D}(\tau(L))$ to the remaining two points on each line in the panel Π_{u_i}.

Consider now the set of lines in Π_{u_i} that contain the points v_j with $i \neq j$ with $v = u + e_r$, $r = 1, \dots, n$ where $\{e_r\}_{r=1,\dots,n}$ is the standard basis of vectors in $\mathbf{F}_q^n \simeq L$ with 1 in the r–th entry and 0 elsewhere. Consider as set of edges $E(G)$ the corresponding edges of $E(G_{\tau(L)})$ connecting the points u_i and $(u + e_r)_j$, for $\{e_r\}_{r=1,\dots,n}$. Each vertex in G has valence $2n$. Let T be a perfect tensor in $\mathcal{H} = (\mathbf{C}^q)^{\otimes 2n}$. We write $T = T_{\ell_1,\dots,\ell_{2n}}$ with indices labelled by vectors $\ell = (\ell_1, \dots, \ell_{2n}) \in \mathbf{F}_q^{2n} \simeq V$, in the Darboux basis, so that we have a given splitting of this set of indices into two subsets $\ell = (\ell_1, \dots, \ell_n, \ell'_1, \dots, \ell'_n)$.

We assign to each vertex u_i of G a copy of the space $\mathcal{H}$ with the tensor T, so that the indices of T correspond to the $2n$ legs of the corolla of u_i where we identify the two subsets of indices with the legs connecting u_i to $(u + e_r)_j$ and to $(u + e_r)_k$, respectively, with j, k the two remaining types with (i, j, k) a cyclic permutation of $(1, 2, 3)$. This fixes an identification of the indices of the tensor with the set of half edges at each vertex. Each edge then corresponds to a contraction of the corresponding indices of the copies of the tensor at the adjacent vertices. □

Note that, while the subgraph $G \subset G_{\tau(L)}$ has exponentially lower connectivity (valence $2n$ rather than $2q^n$) than $G_{\tau(L)}$, we can still interpret the perfect tensor as encoding the rest of the geometry of $G_{\tau(L)}$, through the contribution of the $T_\ell = T_{\ell_1,\dots,\ell_n,\ell'_1,\dots,\ell'_n}$ to the entangled state associated to the corolla of a vertex u_i in G, given by

$$|\psi_{u_i}\rangle = \sum_\ell T_\ell \, |\ell\rangle,$$

where $|\ell\rangle$ is the standard basis of $\mathcal{H}$ as in Definition 7.1.

7.8. Tensor networks on chamber systems and buildings

In addition to the Latin square designs associated to loops, discussed in the previous subsections, there are other related combinatorial structures.

Definition 7.35. *A chamber system of type I on a set Ω is a family $\{\rho_i\}_{i\in I}$ of equivalence relations on Ω satisfying the following conditions:*

(i) if $\omega \sim_i \omega'$ and $\omega \sim_j \omega'$, for some $i \neq j \in I$, then $\omega = \omega'$;

(ii) the I-graph with vertex set Ω and edges $e_{\omega,\omega'}$, for $\omega \sim_i \omega'$ for some ρ_i, is connected.

Given a subset $J \subset I$ a residue of type J is a connected component of the J–graph.

The number of colors card I is the rank of the chamber system.

A graph $\Delta = (\mathcal{V}, \mathcal{E}, \phi)$ with vertex set $\mathcal{V}$, edge set $\mathcal{E}$ and an assignment of edge colors $\phi : \mathcal{E} \to I$ is a chamber system if the monochromatic subgraphs Δ_i with vertex set V and edge set $\phi^{-1}(i)$ are a disjoint union of complete subgraphs with at least two vertices each (see Section 15.5 of [18]). The set $V = \Omega$ is the set of *chambers*, the connected components of the monochromatic subgraphs are the *panels* of the chamber system. *Galleries* are paths in Δ.

A *Latin chamber system* is a chamber system of rank 3 where any two panels of different colors intersect in a unique chamber.

A Latin square design determines a *Latin chamber system*. This has Ω given by the set of the N^2 cells of the Latin square (labelled (a, b) with $a, b = 1, \ldots, N$), with three equivalence relations: (1) $(a, b) \sim_{\rho_1} (a', b')$ if $a = a'$ (same row); (2) $(a, b) \sim_{\rho_2} (a', b')$ if $b = b'$ (same column); (3) $(a, b) \sim_{\rho_3} (a', b')$ if these cells contain the same symbol. There is only one rank 2 residue of each type $J \subset I$ with card $J = 2$, which consists of an $N \times N$ grid containing all the cells. The set of chambers of a Latin chamber system is the set of lines of the corresponding Latin square design. The set of panels is the set of points of the Latin square design, as a panel is given by the set of lines that contain a given point.

A chamber system is simply 2–connected if it is connected and each closed path (gallery) is 2–homotopic to the trivial one. The latter condition means that any closed path can be reduced to the trivial path through a sequence of replacements of subgalleries lying in rank 2 residues by other galleries within the same residue. In particular, buildings are simply 2–connected. Given a collection $\mathcal{C}$ of closed walks in a graph Δ, a $\mathcal{C}$–covering $\tilde{\Delta} \to \Delta$ is a covering such that every closed walk in $\mathcal{C}$ lifts to a closed walk in $\tilde{\Delta}$. A universal $\mathcal{C}$–cover exists (see Section I.1.2.3 of [41]). A 2-covering of a chamber system is

a $\mathcal{C}$-covering of the edge-labelled graph Δ of the chamber system with respect to the collection $\mathcal{C}$ of all closed walks (closed galleries) in rank 2 residues (see Chapter 10 of [41]).

As shown in Proposition 4.2 of [30], a Latin chamber system Δ has universal 2–cover that is a building if and only if the corresponding loop is a group. This is the case, for example, for all the code loops obtained by considering an isotropic subspace $C \subset V$ of an almost-symplectic (V, ω) as above.

This implies that a choice of an isotropic subspace $C \subset V$ of an almost-symplectic (V, ω) determines a chamber system that has a building as universal 2-cover. We can then formulate the following question. We use the notation $\mathcal{B}_C$ for the building obtained in this way.

Question *Can the geometric construction of CRSS quantum codes and of perfect tensors of Theorems 7.22 and 7.26 be used to construct tensor networks on the buildings $\mathcal{B}_C$ that satisfy a form of the Ryu–Takayanagi conjecture?*

Note that one should not expect in general to have good holographic properties for tensor networks on these classes of buildings, and it is likely that only special cases will satisfy some form of Ryu–Takayanagi conjecture, relating entanglement entropy on the boundary to geodesic lengths in the bulk. Indeed, it is expected that the $CAT(-1)$ rather than $CAT(0)$ property may be required for a Ryu–Takayanagi conjecture to hold. However, even in the absence of these stronger holographic properties, an entanglement entropy associated to chamber systems obtained from loop codes and their perfect tensors would show that there are interesting entangled states capturing various aspects of the geometry of the chamber system and its building universal 2–cover. Properties of tensor networks on buildings are a topic currently under active investigation in the context of the holographic AdS/CFT correspondence.

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