Chapter 5
Quantum Operads

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Dedicated to C. N. Yang

Abstract The most standard description of symmetries of a mathematical structure produces *a group*. However, when the definition of this structure is motivated by physics, or information theory etc., the respective symmetry objects might become more sophisticated such as quasigroups, loops, quantum groups. In this paper, we introduce and study quantum symmetries of very general categorical structures: *operads*. Its initial motivation were spaces of probability distributions on finite sets. We also investigate here how structures of quantum information, such as quantum states and some constructions of quantum codes, are algebras over operads.

Keywords Information spaces · Moufang loops · Monoidal categories · Operads · Monoids · Magmas

> ... esta selva selvaggia e aspra e forte
> che nel pensier rinova la paura ...
> Dante Alighieri, Inferno, Canto 1

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5.1   Introduction and Brief Survey

The common definition of symmetries of a structure given on a set S (in the sense of Bourbaki) is *the group of bijective maps $S \rightarrow S$ compatible with this structure*.

But in fact, symmetries of various structures related to storing and transmitting information such as *information spaces* are naturally embodied in various classes of *loops* such as *Moufang loops*, non-associative analogs of groups (cf. [4]).

Here are some representative examples: quasigroups, loops, and Moufang loops.

A *quasigroup* is a set $\mathcal{L}$ together with binary composition law

$$* : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}: \quad (x_1, x_2) \mapsto x_1 * x_2 =: x_3, \tag{5.1}$$

such that any two elements among $\{x_1, x_2, x_3\}$ uniquely determine the third one.

A *loop* is a quasigroup with two-sided identity $e \in \mathcal{L}$: it means that $e * x = x * e = x$ for any $x \in \mathcal{L}$.

Finally, a *Moufang loop* is a quasigroup whose compostion law satisfies the "near-associativity" condition

$$(x_1 * x_2) * (x_3 * x_4) = x_1 * ((x_2 * x_3) * x_4). \tag{5.2}$$

The idea of symmetry embodied in a group is closely related to classical physics, in a very definite sense, going back at least to Archimedes. When quantum physics started to replace classical, it turned out that classical symmetries must also be replaced by their quantum versions. For a short history of this evolution, see pp. 1–4 of [7] and also [6, 8, 9]. As a result, the mathematical theory of *quantum groups* emerged.

In this paper we suggest to apply the formalism of quantisation on the operadic level [2] to symmetries of information spaces. The motivation of our use of adjective "quantum" in [2] was sometimes too intuitive, but the tools developed in [13], furnish a very precise and well axiomatized framework for this.

The general conception of "Quantum Operad" introduced and studied here was also inspired by the introduction of quantum error-correcting codes with a Moufang loop action: see [4], Sec. 6.

We are pleased to dedicate our paper to C. N. Yang, who led the breakthrough studies of gauge symmetries in quantum field theory.

5.2   Quantum Structures in Symmetric Monoidal Categories

5.2.1   Monoidal (=Tensor) Categories V ([13], Sec. 2.2, 2.3)

Data: multiplication $\otimes$ of objects, with identity object $\mathbf{1}$ and natural isomorphisms

$$\alpha_{A,B;C} : (A \otimes B) \otimes C \rightarrow A \otimes (B \otimes C), \quad \rho_A : A \otimes \mathbf{1} \rightarrow A, \quad \lambda_A : \mathbf{1} \otimes A \rightarrow A.$$

5.2.2   Symmetric Monoidal Categories

Additional *twist* isomorphisms $\tau_{A,B} : A \otimes B \to B \otimes A$, with $\tau_{A,B}\tau_{B,A} = \mathrm{id}_{A\otimes B}$, plus many commutative diagrams.

5.2.3   Magmas, Comagmas, Bimagmas, Associativity and Commutativity for (co, bi)magmas in Symmetric Monoidal Categories ([13], Sec. 2.4)

Basic data for a magma: an object A with multiplication morphism $\nabla : A \otimes A \to A$.

Basic data for a comagma: an object A with comultiplication morhism $\Delta : A \to A \otimes A$.

Basic data for a bimagma: a triple (A, ∇, Δ) as above such that the "bimagma diagram" (2.4) ([13], p. 49) commutes.

(Co, bi)-magmas in a symmetric monoidal category V are themselves objects of respective categories. Morphisms in them are those morphisms in V, which are compatible with respective basic data.

Unitality and counitality structures for a magma (A, ∇) (resp. comagma (A, Δ)) are respectively the morphisms $\eta : \mathbf{1} \to A$ or $\varepsilon : A \to \mathbf{1}$ subject to additional restrictions.

5.2.4   Monoids, Comonoids, Bimonoids, and Hopf Algebras in Symmetric MonoIdal Categories ([13], Def. 2.7)

They are essentially (co, bi)magmas with additional (co,bi)associativity restrictions.

5.2.5   Quantum Quasigroups ([13], Sec. 3.1)

A quantum quasigroup (A, ∇, Δ) is a bimagma, for which both *left composite* and *right composite* morphisms are invertible:

$$A \otimes A \xrightarrow{\Delta \otimes \mathrm{id}_A} A \otimes A \otimes A \xrightarrow{\mathrm{id}_A \otimes \nabla} A \otimes A \ ,$$

$$A \otimes A \xrightarrow{\mathrm{id}_A \otimes \Delta} A \otimes A \otimes A \xrightarrow{\nabla \otimes \mathrm{id}_A} A \otimes A \ .$$

These morphisms are sometimes called *fusion operators* or *Galois operators*.

5.2.6   Quantum Loops

A *quantum loop* in V is a biunital bigmagma $(A, \nabla, \Delta, \eta, \varepsilon)$ such that (A, ∇, Δ) is a quantum quasigroup.

5.2.7   Functoriality ([13], Prop. 3.4)

Any symmetric monoidal functor

$$F : (V, \otimes, \mathbf{1}_V) \to (W, \otimes, \mathbf{1}_W)$$

sends quantum quasigroups (resp. quantum loops) in V to quantum quasigroups (resp. quantum loops) in W.

5.2.8   Magmas etc. in the Categories of Sets with Direct Product

According to [13], beginning of Sec. 3.3, in such categories comultiplication in a counital comagma is always the respective diagonal embedding. As a corollary, we see that quantum loops and counital quantum quasigroups in such caregories are cocommutative and coassociative.

As a result, we see, that in such a category counital quantum quasigroups are equivalent to classical quasigroups, and quantum loops are equivalent to classical loops ([13], Prop. 3.11).

5.3   Monoidal Categories of Operads

5.3.1   Graphs and Their Categories

Our basic definition of graphs as quadruples (F, V, ∂, j) and their categories is explained in [2], Sec. 1.1, p. 251. There F, resp. V, are called the sets of *flags*, resp. *vertices*, and structure maps ∂, resp. j are called *boundary maps*, resp. *involutions*. Usually one flag is a pair consisting of flag as such, and a *label*, that should be defined separately. *Geometric realization* of a graph is the quotient set of the disjoint union of semi-intervals $(0, 1/2]$ labeled with flags of this graph, modulo equivalence relation, in which 0-points of a flag is glued to $1/2$ of another flag, if these flags are related by the boundary relation, or structure involution.

Depending on the context and/or type of labelling of τ, elements of F_τ might be called *flags, leaves, tails* ... In the study of magmatic operad [3] and the relevant

binary trees, vertices of the relevant corollas are called *nodes*, non-root flags are called *left child, right child* etc. We will try to attach all such "heteronyms" to our basic terminology of [2].

Below the most typical labeling of our graphs will be (see details in [2], Sec. 1.3.2 a) and 1.3.2 e), pp. 257–259):

 (i) *Orientation.*
(ii) *Cyclic labeling.*

To give an orientation and cyclic labeling of corolla is essentially the same as to define it as a *planar* graph: corolla, embedded into an oriented real affine plane, with labeling compatible with its orientation.

Graphs endowed with various labelings form categories, upon which the operation of disjoint union ⊔ defines a monoidal structure: see [2], Sec. 1.2.4, pp. 254–255. Our central objects of study are initially defined only for connected graphs. Therefore, introducing this monoidal product, we must first take care of "empty" (or partially empty) graphs and explain details of their functoriality. The paper [2] is interspersed with subsections directly or indirectly motivated by this necessity.

For the purposes of this paper, the most important graphs are labelled *trees* and *forests*—disjoint unions of trees, forming *"selva selvaggia e aspra e forte"*.

5.3.2 Operads and Categories of Operads (See [2], Sec. 1.6, p. 262)

We recall here the first definition of operads in [2], 1.6 (I), and morphisms of operads as in [2], Sec. 1.6.1.

First of all, we fix a symmetric monoidal category of labelled graphs Γ with disjoint union as the monoidal structure, and a symmetric monoidal *ground category* $(\mathcal{G}, \otimes)$, satisfying a part of conditions 1.4a–f in [2], p. 259.

 (i) *An operad is a tensor functor between two monoidal categories $A : (\Gamma, \sqcup) \rightarrow (\mathcal{G}, \otimes)$ that sends any grafting morphism to an isomorphism.*
(ii) *A morphism between two operads is a functor morphism.*

Denote this category of operads by $\Gamma\mathcal{G}OPER$.

5.3.3 Operads and Collections as Symmetric Monoidal Categories

Following [2], Sec. 1.8, we will introduce now the monoidal "white product" of two operads $A, B : (\Gamma, \sqcup) \rightarrow (\mathcal{G}, \otimes)$ by the formula

$$A \circ B(\sigma) := A(\sigma) \otimes B(\sigma)$$

extended to morphisms in a straightforward way.

Clearly, $(\Gamma\mathcal{G}OPER, \circ)$ is a symmetric monoidal category.

An important related notion is that of *collection*. Starting with Γ as above, denote by ΓCOR its subcategory, whose objects are corollas in Γ, and morphisms between them are isomorphisms.

Combining it with the ground category $(\mathcal{G}, \otimes)$ as above, we can introduce the category $\Gamma\mathcal{G}COLL$ of ΓG-collections: its objects are functors $A_1 : \Gamma COR \to \mathcal{G}$, and morphisms are natural transformations between these functors.

The restriction of white product $\circ$ to $\Gamma\mathcal{G}COLL$ defines on it the structure of symmetric monoidal category. If $(\mathcal{G}, \otimes)$ has an identity object $\mathbf{1}$, then the collection $\mathbf{1}_{coll}$ sending each corolla to $\mathbf{1}$ and each isomorphism of corollas to the identical isomorphism of $\mathbf{1}$, is the identity collection.

5.3.4   Operads as Monoids

We briefly describe here a construction by B. Vallette [14], reproduced in [2], Appendix A, Subsection 5.

We will have to use here a stronger labeling of graphs in Γ than just orientation. Besides orientation, connected objects of Γ must admit a continuous real-valued function such that it decreases whenever one moves in the direction of orientation along each flag. Such graphs are called *directed ones* (see [2], Sec. 1.3.2 b).

A graph τ is called a *two-level graph*, if it is oriented, and if there exists a partition of its vertices $V_\tau = V_\tau^1 \sqcup V_\tau^2$ with the following properties:

(i) *Tails at V_τ^1 are all inputs of τ, and tails at V_τ^2 are all outputs of τ.*
(ii) *All edges in E_τ go from V_τ^1 to V_τ^2.*

For any two ΓG-collections A^1, A^2 define their product as

$$(A^2 \boxtimes_c A^1)(\sigma) := \mathrm{colim}(\otimes_{v \in V_\tau^1} A^1(\tau_v)) \otimes (\otimes_{v \in V_\tau^2} A^2(\tau_v)).$$

Here colim is taken over the category of morphisms from two level graphs to σ.

Theorem 5.1 *The product $\boxtimes_c$ is a monoidal structure on collections, and operads are monoids in the respective monoidal category.*

5.3.4.1   Freely Generated Operads

For any $\Gamma\mathcal{G}$-collection A_1 one can define another collection $\mathcal{F}(A_1)$ together with a canonical structure of operad on it, and for any operad A each morphism of collections $A_1 \to A$ extends to a morphism of operads $f_A : \mathcal{F}(A_1) \to A$.

We can imagine $\mathcal{F}(A_1)$ as the operad freely generated by the collection A_1.

5.3.5   Comonoids in Operadic Setup

We will now introduce a category OP of operads given *together with their presentations* ([2], Sec. 2.4). We start with Γ and $\mathcal{G}$ as above.

One object of OP is a family (A, A_1, i_A), where A is a $\Gamma\mathcal{G}$-operad, A_1 is a $\Gamma\mathcal{G}$-collection, such that $f_A : \mathcal{F}(A_1) \to A$ is surjective.

Define on OP a product $\odot$ by the formula

$$(A, A_1, i_A) \odot (B, B_1, i_B) = (C, C_1, i_C),$$

in which $C_1 := A_1 \circ B_1$ (cf. 2.3 above), $C :=$ the minimal suboperad, containing the image $(i_A \circ i_B)(A_1 \circ B_1) \subset A \circ B$, and i_C is the restriction of $I_A \circ i_B$ on $A_1 \circ B_1$.

Theorem 5.2 (See [2], Sec. 2.4)

(i)   The product $\odot$ defines on OP a structure of symmetric monoidal category.
(ii)   The category OP is endowed with the functor of inner cohomomorphisms

$$\underline{cohom}_{OP} : OP^{op} \times OP \to OP$$

so that we can identify, functorially with respect to all arguments,

$$\mathrm{Hom}_{OP}(A, C \odot B) = \mathrm{Hom}_{OP}(\underline{cohom}_{OP}(A, B), C)$$

(iii)   Therefore, one can define canonical coassociative comultiplication morphisms

$$\Delta_{A,B,C} : \underline{cohom}_{OP}(A, C) \to \underline{cohom}_{OP}(A, B) \odot \underline{cohom}_{OP}(B, C).$$

Corollary 5.1 *For any A, the coendomorphism operad*

$$\underline{coend}_{OP} A := \underline{cohom}_{OP}(A, A)$$

is a comagma in the sense of 5.2.3 above.

5.3.6   The Magmatic Operad (See [3])

Below we give a brief survey of some definitions and results from [3], sometimes slightly changing terminology and notation.

Here objects of our basic symmetric monoidal category $(\Gamma, \sqcup)$ will be disjoint unions of oriented trees with the following additional labeling: *for each tree, its outgoing flags (or leaves) are cyclically ordered*. Corollas in it are one-vertex graphs

with one root and at least two leaves. Connected objects can be obtained from a union of disjoint corollas by grafting each root of a corolla to one of leaves of another corolla. Morphisms are compatible with labeling.

An algebra over magmatic operad is a family $(\mathscr{A}, *)$ consisting of a set $\mathscr{A}$ with binary composition law $* : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$.

Thus, corollas in the magmatic category correspond to products

$$(x_1 * ((x_2) * \ldots (\ldots(x_n)))\ldots),$$

and generally, connected graphs in it correspond to monomials of generic arguments with all possible arrangements of brackets.

5.3.7 Quasigroup Monomials and Planar Trees

Monomials that can be obtained by iteration of binary multiplication $*$ as in (5.1) correspond to *planar* trees: see 5.3.1 above for discussion of planar corollas. Below, discussing quasigroups in general, and Moufang loops in particular, we will consider connected planar trees and quasigroup monomials as encoding each other in this way.

5.4 Moufang Loops and Operads

5.4.1 Moufang Monomials and Their Encoding by Labeled Graphs

We will start with comparing mathematical structures of two types: *labeled graphs*, and *Moufang monomials*.

The words *loop monomials* will refer to the following class of objects. Let $(\mathcal{L}, *)$ be a Moufang loop in the sense of [4], Definition 5.1.1. Let $(x_1, \ldots, x_n) \in \mathcal{L}$. We can produce new elements of $\mathcal{L}$ from this sequence by applying to them iterated multiplication $*$.

The basic examples are

$$x_1 * x_2, \tag{5.3}$$

$$(x_1 * x_2) * (x_3 * x_4). \tag{5.4}$$

We will encode the monomial (5.3) by a *cyclically labeled oriented corolla* with one vertex and three flags, exactly one of which is the *output*. The bridge from (5.3) to this corolla might be imagined as an enrichment of it by additional labeling: $x_1 * x_2$

at the output, and x_1, x_2 at two other flags, such that $(x_1, x_2, x_1 * x_2)$ corresponds to the given cyclic labeling.

Since (5.4) can be obtained from (5.3) by iteration and variables change, we must explain, how such an iteration is encoded on the level of labeled graphs. The answer is obvious: it corresponds to *graftings* of certain outputs to certain inputs, so that these outputs in the enriched picture become the inputs of the respective iteration.

In this way, (5.4) becomes encoded by an *oriented and cyclically labeled tree*, with four ordered inputs, two edges, three vertices, and one output.

5.4.2 Passage to Moufang Operad: Basic Identity

According to [4], Def. 5.1.1, the Moufang loops are defined as structures $(\mathcal{L}, *)$ satisfying the "near-associativity" relations

$$(x_1 * x_2) * (x_3 * x_4) = x_1 * ((x_2 * x_3) * x_4) \qquad (5.5)$$

The r.h.s. of (5.5) is, in turn, encoded by an oriented and cyclically labeled tree, with four ordered inputs (the same ones as in (5.4)), two edges, and four vertices.

5.4.3 Moufang Collections (See [2], Sec. 1.5, pp. 259–261)

Call a *Moufang corolla* an oriented cyclically ordered connected graph with one output, and morphims are *isomorphisms* between them. They are objects of category $\mathcal{MCOR}$ (particular case of categories ΓCOR above). Clearly, $\mathcal{MCOR}$ is a groupoid.

Choose a symmetric monoidal ground category $(\mathcal{G}, \otimes)$, and define respective Moufang collections.

5.4.4 Latin Square Designs and Their Encoding by Graphs

Let $\mathcal{D} = (P, L)$ be a Latin square design as in [4], Def. 6.8.2.1.

Denote by $G^0(\mathcal{D})$ the graph, defined by the following family of data (see [2], p.251):

Vertices $V_{G^0(\mathcal{D})}$ are lines of $\mathcal{D}$:

$$V_{G^0(\mathcal{D})} := L.$$

Flags $F_{G^0(\mathcal{D})}$ are pairs $(p, l) \in P \times L$ such that $p \in l$.

The boundary map $\partial_{G^0(\mathcal{D})} : F_{G^0(\mathcal{D})} \to V_{G^0(\mathcal{D})}$ *sends each* (p, l) *to* l.

The involution $j_{G^0(\mathcal{D})} : F_{G^0(\mathcal{D})} \to F_{G^0(\mathcal{D})}$ *sends* (p, l) *to* (p', l'), *if* $p \neq p'$ *and* $l = l'$.

5.4.4.1 Simplest Examples

Using notation from [4], Def. 6.8.2.1, we see that three simplest examples correspond to cases $N := \operatorname{card} L = 0, 1, 2$.

The case $N = 0$ is degenerate: the respective designs and graphs are empty, and usually are included in consideration only for categorical reasons.

The case $N = 1$ produces a *corolla*: the graph with one vertex and three flags, and boundary map sending each flag to this vertex. The involution map is identical one.

5.4.5 From Loops to Latin Square Designs

Consider an ML $\mathcal{L}$. Produce three labelled copies of $\mathcal{L}$: $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, with pairwise empty intersections. Define the design (P, L) by putting $P := \mathcal{L}_1 \sqcup \mathcal{L}_2 \sqcup \mathcal{L}_3$ and defining a line as such triple of points (x_1, x_2, x_3), $x_i \in \mathcal{L}_i$ that $(x_1 * x_2) * x_3 = 1 \in \mathcal{L}$. In this last formula we implicitly forget labels $1, 2, 3$ and consider Moufang multiplication in $\mathcal{L}$.

5.5 Operadic Structures on Quantum States

In this section we show that the operadic structures associated to classical probabilities on finite sets, introduced in [10], extend to *non-unital operads on quantum states*.

5.5.1 Operads of Classical and Quantum Probabilities

We first recall the main operadic structures of classical probabilities, as introduced in [10], which we follow for the exposition in this subsection. It was observed in [1] and [10] that classical probabilities on finite sets come endowed with an operad structure that describes the combination of independent subsystems.

Namely, denote by $\mathcal{P}$ the operad in *Sets* with objects $\mathcal{P}(n) = \Delta_n$, the simplex of probabilities on the finite set $\{1, \ldots, n\}$,

$$\Delta_n = \{P = (p_i)_{i=1}^n \mid p_i \geq 0, \ \sum_{i=1}^n p_i = 1\},$$

and with composition operations

$$\gamma : \mathcal{P}(n) \times \mathcal{P}(k_1) \times \cdots \times \mathcal{P}(k_n) \to \mathcal{P}(k_1 + \cdots + k_n)$$

given by the composition of probabilities of independent subsystems

$$\gamma(P; P_1, \ldots, P_n) = (p_r \, p_{r, j_r})_{r=1,\ldots,n, \, j_r=1,\ldots,k_r}$$

for $P_r = (p_{r, j_r})_{j_r=1}^{k_r}$ and $P = (p_r)_{r=1}^n$.

5.5.1.1 Averages as an Algebra Over the Operad $\mathcal{P}$

Making explicit in this setup the general definitions from Sects. 5.3.2–5.3.4 above, we see that an algebra A over the operad $\mathcal{P}$ in a symmetric monoidal category is a family of morphisms

$$\alpha : \mathcal{P}(n) \otimes A^{\otimes n} \to A,$$

satisfying associativity and unitality conditions, and compatibility with the symmetric groups actions.

The set of non-negative real numbers $\boldsymbol{R}_+$ can be seen as a category with a single object and morphisms $x \in \boldsymbol{R}_+$ and as an algebra (in the category of small categories) over the operad $\mathcal{P}$ with the simple operations

$$\alpha(P; x_1, \ldots, x_n) = \sum_i p_i x_i,$$

for $P = (p_i)_{i=1}^n$ and $x_i \in \boldsymbol{R}_+$.

5.5.1.2 A_∞-Operad and Entropy

As was shown in [10], there is another operadic structure in the setting of classical probability distributions over finite sets.

Let $\mathcal{T}$ be the A_∞-operad of planar rooted tress. We say that a collection of n-ary information measures S_n for $n \in \boldsymbol{N}$, satisfies the *coherence condition*, if for any $n > m$, the n-th entropy functional S_n agrees with S_m, when $n - m$ of the variables

are vanishing. In other words, assume that among the probabilities $(p_1, \ldots, p_n)$ the only non-zero ones are $(p_{i_1}, \ldots, p_{i_m})$ for some $i_1 < i_2 < \cdots < i_m$. Then coherence condition means that

$$S_n(p_1, \ldots, p_n) = S_m(p_{i_1}, \ldots, p_{i_m}).$$

We can now determine on $\mathbf{R}_+$ the structure of algebra over the operad $\mathcal{T}$ with the operations

$$\alpha(\tau, x_1, \ldots, x_n) = \min \left\{ \sum_{i=1}^{n} p_i x_i - \frac{1}{\beta} S_\tau(p_1, \ldots, p_n) \mid P = (p_i) \in \Delta_n \right\},$$

where $\tau \in \mathcal{T}(n)$ is a planar rooted tree with n leaves, and $\beta > 0$ is a thermodynamic parameter (inverse temperature). The n-ary entropy functional $S_\tau(p_1, \ldots, p_n)$ associated to the tree τ is uniquely determined by the branching structure of the tree τ and the collection of coherent entropies S_n.

5.5.2 Classical Probabilities from Quantum States

Let $\mathcal{M}^{(n)}$ denote the convex set of $n \times n$-density matrices (quantum states),

$$\mathcal{M}^{(n)} = \{\rho \in M_{n \times n}(\mathbf{C}) \mid \rho^* = \rho, \ \rho \geq 0, \ \mathrm{Tr}(\rho) = 1\}.$$

Positivity condition means here, that $\rho = a^* a$ for some $a \in M_{n \times n}(\mathbf{C})$, hence $Spec(\rho) \subset \mathbf{R}_+$. In a fixed basis, the diagonal density matrices form a copy of the simplex Δ_n embedded in $\mathcal{M}^{(n)}$.

There are two classical probability distributions naturally associated to a quantum state as follows.

Definition 5.1 Given $\rho \in \mathcal{M}^{(n)}$, let

$$\Lambda = (\lambda_i)_{i=1}^n \quad \text{with} \quad \lambda_i \in Spec(\rho),$$

be the set of eigenvalues of ρ, sorted in non-increasing order, and let

$$P = (p_i)_{i=1}^N \quad \text{with} \quad p_i = \rho_{ii}$$

be the list of the diagonal entries of ρ.

Definition 5.2 Given two non-increasing sequences $A = \{a_1, \ldots, a_n\}$ and $C = \{c_1, \ldots, c_n\}$ with $\sum_{i=1}^N a_i = \sum_{i=1}^N c_i$, one says that A majorises C, or $A \succ C$, if for all $1 \leq k \leq N$ one has $\sum_{i=1}^k a_i \geq \sum_{i=1}^k c_i$.

Lemma 5.1 *The Shannon information of the two classical probability distributions* $\Lambda = \Lambda(\rho)$ *and* $P = P(\rho)$ *is related by* $S(P) \geq S(\Lambda)$.

Proof By Schur lemma, the sequence Λ of eigenvalues of the hermitian matrix ρ majorises the sequence P of its diagonal entries, when both are sorted in non-increasing order. It is well known that for probabilities $\Lambda \succ P$ is equivalent to the existence a bistochastic matrix B such that $P = B\Lambda$. The Shannon entropy is monotonically non-decreasing under bistochastic matrices, so $S(B\Lambda) \geq S(\Lambda)$.

$\square$

The eigenvalues probability $\Lambda = \Lambda(\rho)$ determines the information content of the quantum probability ρ, since in the von Neumann entropy

$$S(\rho) = \text{Tr}(\rho \log \rho)$$

the term $\log \rho$ is defined via the spectral theorem, so that we have

$$S(\rho) = S(\Lambda) = -\sum_i \lambda_i \log \lambda_i,$$

the Shannon entropy of the classical probability Λ.

We will show in the next subsections that these two classical probabilities $P(\rho)$ and $\Lambda(\rho)$ determine two non-unital operad structures on the space of quantum states. The operad obtained using $P(\rho)$ has better properties and directly agrees with the operad of classical probabilities recalled in Sect. 5.5.1 above when restricted to $\Delta_n \subset \mathcal{M}^{(n)}$.

5.5.3 Non-unital Operads

In the unital case, one can equivalently describe an operad $\mathcal{O}$ through the composition laws

$$\gamma : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n) \to \mathcal{O}(k_1 + \cdots + k_n),$$

with the associativity conditions (and the symmetricity and unitality conditions in the respective cases), or else one can describe $\mathcal{O}$ through insertion operations

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n + m - 1).$$

For $1 \leq j \leq a$ and $b, c \geq 0$, with $X \in O(a)$, $Y \in O(b)$, and $Z \in O(c)$, these insertions are subject to the conditions

$$(X \circ_j Y) \circ_i Z = \begin{cases} (X \circ_i Z) \circ_{j+c-1} Y & 1 \leq i < j \\ X \circ_j (Y \circ_{i-j+1} Z) & j \leq i < b + j \\ (X \circ_{i-b+1} Z) \circ_j Y & j + b \leq i \leq a + b - 1. \end{cases}$$

The composition laws γ satisfying the associativity condition can be obtained from the insertions $\circ_i$ through

$$\gamma(X, Y_1, \ldots, Y_n) = (\cdots (X \circ_n Y_n) \circ_{n-1} Y_{n-1}) \cdots \circ_1 Y_1).$$

While these two descriptions of operads are equivalent in the unital case, they give rise to two different versions of the notion of non-unital operad, see the discussion in [11]. Indeed, non-unital operads defined through the operations $\circ_i$ are also non-unital operads with the composition operations γ, but the converse no longer holds, so the first class of non-unital operads is more restrictive.

We will show below that the non-unital operads of quantum states belong to the more restrictive class, according to the stronger notion of non-unital operad as in [11].

5.5.4   The Q_P-Operad of Quantum States

We show here that the operad $\mathcal{P}$ of classical probabilities on finite sets extends to a compatible but non-unital operad Q_P on quantum states.

Definition 5.3 For $n \geq 1$ denote by $Q_P(n) = \mathcal{M}^{(n)}$ the convex set of density matrices, endowed with the composition laws

$$\gamma_P : Q_P(n) \times Q_P(k_1) \times \cdots \times Q_P(k_n) \to Q_P(k_1 + \cdots + k_n) :$$

$$\gamma_P(\rho; \rho_1, \ldots, \rho_n) = \gamma(P(\rho); \rho_1, \ldots, \rho_n) = \begin{pmatrix} p_1 \rho_1 & & & \\ & p_2 \rho_2 & & \\ \vdots & & \cdots & \vdots \\ & & & p_n \rho_n \end{pmatrix}.$$

Lemma 5.2 *The action of the symmetric group Σ_n on $\mathcal{M}^{(n)}$ given by $\sigma(\rho) = \sigma \rho \sigma^*$ is compatible with the action by permutation of the coordinates on classical probabilities. It acts on the two probability distributions $P(\rho)$ and $\Lambda(\rho)$ by*

$$P(\sigma \rho \sigma^*) = \sigma^* P(\rho) \quad and \quad \Lambda(\sigma \rho \sigma^*) = \Lambda(\rho).$$

Proof By realising the set of classical probability distributions $\Delta_n \subset \mathcal{M}^{(n)}$ as set of diagonal density matrices in a chosen basis, we see that $\sigma\rho\sigma^*$ permutes the entries by σ^*. The diagonal entries of ρ can be obtained as $\rho_{ii} = \mathrm{Tr}(\pi_i \rho)$ with π_i the i-th 1-dimensional projection in the chosen basis, and $\mathrm{Tr}(\pi_i \sigma\rho\sigma^*) = \mathrm{Tr}(\sigma^* \pi_i \sigma\rho) = \mathrm{Tr}(\pi_{\sigma^{-1}(i)}\rho) = \rho_{\sigma^{-1}(i)\sigma^{-1}(i)}$. So we have $P(\sigma\rho\sigma^*) = \sigma^* P(\rho)$. In the case of $\Lambda(\rho)$, since this distribution is defined after choosing an order in which to list the eigenvalues of ρ, such as non-increasing order, we have $\Lambda(\rho) = \Lambda(\sigma\rho\sigma^*)$, since both matrices have the same spectrum. $\qquad\square$

Proposition 5.1 *The convex sets $\mathbf{Q}_P(n)$ with the composition operations γ_P of Definition 5.3 determine a non-unital symmetric operad $\mathbf{Q}_P$ that restricts to the unital operad $\mathcal{P}$ on classical probabilities $\Delta_n \subset \mathcal{M}^{(n)}$.*

Proof It is clear by the definition of the composition operations γ that they agree with the composition operations of the operad $\mathcal{P}$ when restricted to classical probabilities $\Delta_n \subset \mathcal{M}^{(n)}$. We need to check that they satisfy the associativity and symmetry axioms on the larger set $\mathcal{M}^{(n)}$ of quantum states.

The associativity condition is given by the identities

$$\gamma(\gamma(\rho^{(m)}; \rho^{(n_1)}, \ldots, \rho^{(n_m)}); \rho^{(r_{1,1})}, \ldots, \rho^{(r_{1,n_1})}, \ldots, \rho^{(r_{m,1})}, \ldots, \rho^{(r_{m,n_m})}) =$$

$$\gamma(\rho^{(m)}; \gamma(\rho^{(n_1)}; \rho^{(r_{1,1})}, \ldots, \rho^{(r_{1,n_1})}), \ldots, \gamma(\rho^{(n_m)}; \rho^{(r_{m,1})}, \ldots, \rho^{(r_{m,n_m})})),$$

for $\rho^{(m)} \in Q(m)$, $\rho^{(n_i)} \in Q(n_i)$, $i = 1, \ldots, m$, and $\rho^{(r_{i,\ell_i})} \in Q(r_{i,\ell_i})$ with $\ell_i = 1, \ldots, n_i$. The left-hand-side is

$$\gamma\left(\left(\begin{matrix} \rho^{(m)}_{11}\rho^{n_1} & & & \\ & \rho^{(m)}_{22}\rho^{n_2} & & \\ \vdots & & \ddots & \vdots \\ & & & \rho^{(m)}_{mm}\rho^{n_m} \end{matrix}\right); \rho^{(r_{1,1})}, \ldots, \rho^{(r_{1,n_1})}, \ldots, \rho^{(r_{m,1})}, \ldots, \rho^{(r_{m,n_m})}\right) =$$

$$\left(\begin{matrix} \rho^{(m)}_{11}\rho^{n_1}_{11}\rho^{(r_{1,1})} & & & & & \\ & \ddots & & & & \\ & & \rho^{(m)}_{11}\rho^{n_1}_{n_1 n_1}\rho^{(r_{1,n_1})} & & & \\ & & & \ddots & & \\ & & & & \rho^{(m)}_{mm}\rho^{n_1}_{11}\rho^{(r_{m,1})} & \\ & & & & & \ddots \\ & & & & & & \rho^{(m)}_{mm}\rho^{n_1}_{n_m n_m}\rho^{(r_{m,n_m})} \end{matrix}\right),$$

which agrees with the right-hand-side

$$\gamma\left(\rho^{(m)};\begin{pmatrix}\rho_{11}^{n_1}\rho^{(r_{1,1})} & & \\ & \ddots & \\ & & \rho_{n_1 n_1}^{n_1}\rho^{(r_{1,n_1})}\end{pmatrix},\ldots,\begin{pmatrix}\rho_{11}^{n_m}\rho^{(r_{m,1})} & & \\ & \ddots & \\ & & \rho_{n_m n_m}^{n_m}\rho^{(r_{m,n_m})}\end{pmatrix}\right).$$

The compatibility with the symmetric group action for Q_P is obtained directly from Lemma 4.4.2. Indeed, symmetric property of an operad is expressed by the following two identities, for permutations $\sigma_i \in \Sigma_{n_i}$ and $\sigma \in \Sigma_m$. The first condition is

$$\gamma_P(\sigma(\rho); \rho_{\sigma^{-1}(1)}, \ldots, \rho_{\sigma^{-1}(m)}) = \widetilde{\sigma}(\gamma_P(\rho; \rho_1, \ldots, \rho_m)),$$

where on the right-hand-side $\widetilde{\sigma} \in \Sigma_{n_1+\cdots+n_m}$ is the permutation that splits the set of indices into blocks of n_i indices and permutes the blocks by σ. The second symmetric group condition is

$$\gamma_P(\rho; \sigma_1(\rho_1), \ldots, \sigma_m(\rho_m)) = \widehat{\sigma}(\gamma_P(\rho; \rho_1, \ldots, \rho_m)),$$

where on the right-hand-side $\widehat{\sigma} \in \Sigma_{n_1+\cdots+n_m}$ is the permutation that acts on the i-th block of n_i indices as the permutation σ_i.

In the first case, we have

$$\gamma_P(\sigma(\rho); \rho_{\sigma^{-1}(1)}, \ldots, \rho_{\sigma^{-1}(m)}) = \gamma_P(\sigma^{-1}P(\rho); \rho_{\sigma^{-1}(1)}, \ldots, \rho_{\sigma^{-1}(m)}),$$

which is the same as $\widetilde{\sigma}\gamma_P(\rho; \rho_1, \ldots, \rho_m))\widetilde{\sigma}^*$ in $\mathcal{M}^{(n_1+\cdots+n_m)}$. A similar argument proves the second relation.

The operad is non-unital. Indeed, the unit axiom is only satisfied for $\rho = 1 \in Q(1)$ with $\gamma_P(1; \rho) = \rho$, but it fails when $\rho_i = 1 \in Q(1)$, where the composition gives instead $\gamma_P(\rho; 1, \ldots, 1) = P(\rho)$. The unit axiom $\gamma_P(\rho; 1, \ldots, 1) = \rho$ is satisfied on the subset $\Delta_n \subset \mathcal{M}^{(n)}$ of classical probabilities, where the operad agrees with the unital operad $\mathcal{P}$. $\square$

Proposition 5.2 *The composition laws γ_P of the non-unital operad Q_P are induced by insertion operations $\circ_i : Q_P(n) \times Q_P(m) \to Q_P(n + m - 1)$, hence the operad Q_P is also a non-unital operad in the stronger sense.*

Proof For the composition operations γ_P to be obtained from insertions $\circ_i$, we need to have

$$\gamma_P(\rho; \rho_1, \ldots, \rho_n) = (\cdots(\rho \circ_n \rho_n)\cdots \circ_1 \rho_1),$$

for $\rho \in \mathcal{M}^{(n)}$ and $\rho_i \in \mathcal{M}^{(n_i)}$. We can define morphisms

$$\circ_i : Q_P(n) \times Q_P(m) \to Q_P(m + n - 1)$$

as the operations $(\rho, \rho') \mapsto \rho \circ_i \rho'$ that take a density matrix $\rho \in \mathcal{M}^{(n)}$ and replace the i-th row and columns with m rows and m columns, respectively, where all the entries outside of the $m \times m$-block around the diagonal are zero, and the diagonal block is given by the matrix $\rho_{ii} \rho'$. Clearly this implies that the repeated application of these insertions performed in the order $(\cdots (\rho \circ_n \rho_n) \cdots \circ_1 \rho_1)$ produces exactly the matrix $\gamma_P(\rho; \rho_1, \ldots, \rho_n)$. □

5.5.5 The Q_Λ-Operad of Quantum States

We can construct, in a very similar way, another operad of quantum states, using the classical probabilities $\Lambda(\rho)$ instead of $P(\rho)$. The resulting operad has slightly different properties, coming from the choice of an ordering of the eigenvalues.

Definition 5.4 For $n \geq 1$ let $Q_\Lambda(n) = \mathcal{M}^{(n)}$ be the convex set of density matrices, endowed with the composition laws

$$\gamma_\Lambda : Q_\Lambda(n) \times Q_\lambda(k_1) \times \cdots \times Q_\Lambda(k_n) \to Q_\Lambda(k_1 + \cdots + k_n)$$

$$\gamma_\Lambda(\rho; \rho_1, \ldots, \rho_n) = \gamma(\Lambda(\rho); \rho_1, \ldots, \rho_n) = \begin{pmatrix} \lambda_1 \rho_1 & & & \\ & \lambda_2 \rho_2 & & \\ \vdots & & \ddots & \vdots \\ & & & \lambda_n \rho_n \end{pmatrix}$$

with λ_i the eigenvalues of ρ listed in non-increasing order.

Proposition 5.3 *The convex sets $Q_\Lambda(n)$ with the composition operations γ_Λ of Definition 5.4 determine a non-unital non-symmetric operad Q_Λ. The composition laws γ_Λ are induced by insertion operations $\circ_i : Q_\Lambda(n) \times Q_\Lambda(m) \to Q(n+m-1)$.*

Proof The argument is completely analogous to Propositions 5.1 and 5.2. The associativity requirement for the composition laws γ_Λ follows as in Proposition 5.1, using the fact that

$$Spec \begin{pmatrix} \lambda_1 \rho^{n_1} & & & \\ & \lambda_2 \rho^{n_2} & & \\ \vdots & & \ddots & \vdots \\ & & & \lambda_m \rho^{n_m} \end{pmatrix} = \bigcup_i \lambda_i \, Spec(\rho^{n_i}).$$

The operad Q_Λ is non-unital for the same reason as Q_P, namely $\gamma_\Lambda(\rho; 1\ldots, 1) = \Lambda(\rho) \neq \rho$. Even in the case of diagonal matrices these differ in general by a permutation.

The operad is non-symmetric, because we must choose an ordering of the eigenvalues in $\Lambda(\rho)$, for example, non-increasing ordering. This breaks the symmetric group action in the first of the two identities, making Q_Λ non-symmetric.

The insertion operations $\circ_i$ are as in Proposition 5.2, but the central $m \times m$ block is now of the form $\lambda_i \rho'$ □

5.5.6 Trees of Projective Quantum Measurements

We consider the range of compositions of insertion maps of the operad Q_P and quantum channels associated to these ranges.

First observe that the image of the insertion map

$$\circ_i : Q_P(n) \times Q_P(m) \to Q_P(n + m - 1)$$

consists of the set of those density matrices $\rho \in \mathcal{M}^{(n+m-1)}$ that are block diagonal with one $(n-1) \times (n-1)$-block and one $m \times m$-block. Moreover, all block diagonal density matrices are in the image of some composition of insertion maps. These are quantum states that decompose nontrivially into disjoint states with orthogonal ranges. On the other hand, a block diagonal density matrix can be obtained in more than one way through a composition of insertion maps.

Operators on density matrices are described by quantum channels, namely quantum measurements realized by completely positive maps. A particular class of such operators consists of *projective quantum measurements*.

In the following we work with a finite dimensional Hilbert space $\mathcal{H}$ of dimension N with a chosen orthonormal basis, and we denote as before by $\mathcal{M}^{(N)}$ the set of density matrices, written in the chosen basis.

Definition 5.5 A projective quantum measurement is a family $\Pi = \{\Pi_i\}_{i=1}^n$ of projectors $\Pi_i^* = \Pi_i = \Pi_i^2$ in a finite dimensional Hilbert space $\mathcal{H}$ of dimension N, that are mutually orthogonal, $\Pi_i \Pi_j = \delta_{ij} \Pi_i$, and satisfy the condition $\sum_i \Pi_i = 1$. The outcome of the projective measurement Π on a quantum state given by density matrices $\rho \in \mathcal{M}^{(N)}$ is

$$\rho_i = \frac{\Pi_i \rho \Pi_i}{\mathrm{Tr}(\Pi_i \rho)} \quad \text{with probability} \quad p_i = \mathrm{Tr}(\Pi_i \rho).$$

The projective quantum channel Π then maps $\rho \mapsto \Pi(\rho) = \sum_i p_i \rho_i$.

The range in $\mathcal{M}^{(N)}$ of a composition of insertion maps is specified by assigning a decomposition $N = k_1 + \cdots + k_n$ and the locus $\mathcal{M}_{k_1,\ldots,k_n} \subset \mathcal{M}^{(N)}$ of density

matrices that are block-diagonal in the chosen basis, with n blocks of size k_i. The following is immediate by construction.

Lemma 5.3 *Let Π be the projective measurement $\Pi = \{\Pi_i\}_{i=1}^n$, where Π_i is the orthogonal projection onto the span of the i-th subset of k_i basis elements, given by a quantum channel. It maps $\Pi : \mathcal{M}^{(N)} \to \mathcal{M}_{k_1,\ldots,k_n}$, assigning to ρ the block-diagonal density matrix $\Pi(\rho) = \sum_i p_i \rho_i$.*

Just as elements of $\mathcal{M}_{k_1,\ldots,k_n}$ can be obtained in different ways as compositions of insertion operations of the operad $\mathcal{Q}_P$ of quantum states, with different tree structures, the quantum channel $\Pi : \mathcal{M}^{(N)} \to \mathcal{M}_{k_1,\ldots,k_n}$ can also be realized through different tree structures.

Let τ be a planar rooted tree with n leaves labeled by the non-negative integers k_i. We view the tree τ as oriented from the leaves toward the root. We assign to the root vertex v_0, the identity projector $\Pi^{(v_0)} = 1$. Let v be any vertex in the tree, and consider the set of incoming edges e at v, with $v_e = s(e)$ their source vertices. Let $\{\Pi^{(v_e)}\}_{t(e)=v}$ be a set of orthogonal projections with $\sum_{e:t(e)=v} \Pi^{(s(e))} = \Pi^{(v)}$. Let Π_i denote the resulting projectors at the leaves. We write, for $t(e) = v$,

$$\rho^{(s(e))} = \frac{\Pi^{(s(e))} \rho^{(v)} \Pi^{(s(e))}}{\mathrm{Tr}(\Pi^{(s(e))} \rho^{(v)})},$$

with $\rho^{(v_0)} = \rho$.

Consider the quantum measurement Π_τ that assigns to a density matrix $\rho \in \mathcal{M}^{(N)}$, with $N = k_1 + \cdots + k_n$ outcomes ρ_i^τ with probabilities p_i^τ where

$$p_i^\tau = \prod_w \mathrm{Tr}(\Pi_{i_w}^{(w)} \rho^{(w)})$$

with the product over the vertices on the directed path from the i-th leaf to the root of τ, with i_w indicating the direction at the vertex w along this path and the ρ_i^τ are obtained by repeatedly computing $\rho^{(s(e))}$ from $\rho^{(t(e))}$ along the path connecting the i-th leaf to the root.

Lemma 5.4 *All the quantum channels Π_τ obtained in this way, are just the same quantum channel $\Pi : \mathcal{M}^{(N)} \to \mathcal{M}_{k_1,\ldots,k_n}$.*

Proof This can be seen very easily by writing

$$\prod_w \mathrm{Tr}(\Pi_{i_w}^{(w)} \rho^{(w)}) = \prod_\ell \frac{\mathrm{Tr}(\Pi^{(w_\ell)} \Pi^{(w_{\ell-1})} \cdots \Pi^{(w_0)} \rho)}{\mathrm{Tr}(\Pi^{(w_{\ell-1})} \cdots \Pi^{(w_0)} \rho)} = \prod_\ell \frac{\mathrm{Tr}(\Pi^{(w_\ell)} \rho)}{\mathrm{Tr}(\Pi^{(w_{\ell-1})} \rho)} = \mathrm{Tr}(\Pi_i \rho)$$

with Π_i the projection at the leaf, since we have $\Pi_j^{(s(e))} \Pi_e^{(t(e))} = \Pi_j^{(s(e))}$ as $\Pi_j^{(s(e))}$ is a projection onto a subspace of the range of $\Pi_e^{(t(e))}$. We similarly obtain $\rho_i^\tau = \rho_i = \frac{\Pi_i \rho \Pi_i}{\mathrm{Tr}(\Pi_i \rho)}$. $\qquad\qquad\square$

5.5.7 Entropy Functionals

Consider a family of quantum entropy functionals

$$S_n : \mathcal{M}^{(n)} \to \mathbf{R}$$

satisfying the consistency condition that S_n restricts to S_k for $k < n$ over any copy of $\mathcal{M}^{(k)}$ embedded in $\mathcal{M}^{(n)}$ as density matrices with a set of $n - k$ vanishing eigenvalues.

Examples of such consistent collections of entropy functionals include the von Neumann entropy

$$\mathcal{N}(\rho) = -\mathrm{Tr}(\rho \log \rho),$$

or, for a real parameter $q > 0$ with $q \neq 1$, the quantum Rényi entropy

$$Ry_q(\rho) = \frac{1}{1 - q} \log \mathrm{Tr}(\rho^q)$$

and the quantum Tsallis entropy

$$Ts_q(\rho) = \frac{1}{1 - q}(\mathrm{Tr}(\rho^q) - 1).$$

We obtain a family of quantum entropies associated to trees in the following way, as a direct generalization of the entropy functionals S_τ for classical probabilities constructed in [10].

Proposition 5.4 *A tree τ with n leaves labeled by integers $k_i \geq 1$, together with a coherent family $\{S_n\}$ of quantum entropies, determines an entropy functional*

$$S_\tau : \mathcal{M}^{(N)} \to \mathbf{R}$$

for $N = k_1 + \cdots + k_n$.

Proof In the case where τ is a corolla with a single root vertex and n leaves, we set

$$S_\tau(\rho) = S(P) + \sum_i p_i S(\rho_i)$$

with $p_i = \mathrm{Tr}(\Pi_i \rho)$, resp. $\rho_i = \frac{\Pi_i \rho \Pi_i}{\mathrm{Tr}(\Pi_i \rho)}$, are the probabilities, resp. density matrices, at the leaves. In the case of the von Neumann entropy, by the extensivity property, this is the same as $\mathcal{N}(\Pi^\tau(\rho)) = \mathcal{N}(\sum_i p_i \rho_i)$. Inductively assuming that S_τ is constructed for all trees with less than n leaves, consider the subtrees τ_j, $j = 1, \ldots, m$, attached at the root vertex v_0, each with a set L_j of leaves, card $L_j < n$.

By the construction of the quantum channel Π^τ we have a system Π_j of orthogonal projections with $\sum_j \Pi_j = 1$ associated to the incoming edges e_j at the root vertex, and probabilities $p_j = \mathrm{Tr}(\Pi_j \rho)$ and density matrices $\rho_j = \frac{\Pi_j \rho \Pi_j}{\mathrm{Tr}(\Pi_j \rho)}$ associated to the root vertices v_j of the subtrees τ_j. We then set

$$S_\tau(\rho) = S(P) + \sum_j p_j S_{\tau_j}(\rho_j),$$

where $S(P)$ is the Shannon entropy of the classical probability $P = (p_j)$ and S_{τ_j} are the entropy functionals inductively constructed for the subtrees τ_j with less than n leaves. This suffices to determine S_τ uniquely. $\qquad\square$

Remark 5.1 In the case of the von Neumann entropy, the extensivity property and the identification of all the quantum channels Π^τ with the quantum channel $\Pi :$ $\mathcal{M}^{(N)} \to \mathcal{M}_{k_1,\ldots,k_n}$, seen in Lemma 5.4 above, imply that $\mathcal{N}_\tau(\rho) = \mathcal{N}(\sum_i p_i \rho_i) =$ $S(P) + \sum_i p_i \mathcal{N}(\rho_i)$ for all τ. This is not the case for non-extensive entropies like Rényi and Tsallis.

5.5.8 A_∞-Operad of Quantum Channels

We now investigate structures on quantum states that generalize the operations based on classical probabilities that we recalled in Sect. 5.5.1.2.

Definition 5.6 A quantum channel $\Phi : \mathcal{M}^{(N)} \to \mathcal{M}^{(N)}$ is a trace preserving completely positive map. It is well known that any such map can be represented, in a non-unique way, in Kraus form, namely as

$$\Phi(\rho) = \sum_i A_i \rho A_i^*$$

for a collection $\{A_i\}$ of operator satisfying the condition $\sum_i A_i^* A_i = 1$.

The projective quantum channels considered in the previous subsections are those for which the operators A_i are mutually orthogonal projectors.

We can construct more general quantum channels associated to rooted trees. We consider a finite dimensional complex Hilbert space $\mathcal{H}$ of dimension N. All operators here will be linear operators on $\mathcal{H}$.

Definition 5.7 Let τ be a planar rooted tree with n leaves. We consider τ oriented from the leaves to the root. A tree quantum channel C_A^τ is an assignment of operators

$A = \{A_e\}_{e \in E(\tau)}$ to the edges of τ, satisfying the condition that at each vertex v

$$\sum_{e \,:\, t(e)=v} A_e^* A_e = 1.$$

Lemma 5.5 *The tree quantum channels C_A^τ of Definition 5.7 are quantum channel as in Definition 5.6, acting on density matrices $\rho \in \mathcal{M}^{(N)}$ by*

$$C_A^\tau(\rho) = \sum_{i=1}^n A_{e_{i,1}} \cdots A_{e_{i,m_i}} \, \rho \, A_{e_{i,m_i}}^* \cdots A_{e_{i,1}}^*,$$

where the sum is over the leaves of τ and for each $i = 1, \ldots, n$ we consider the oriented path $e_{i,1}, \ldots, e_{i,m_i}$ from the i-th leaf to the root, $s(e_{i,1}) = v_i$, $t(e_{i,j}) = s(e_{i,j+1})$, $t(e_{i,m_i}) = v_0$, the root vertex.

Proof This is essentially the Kraus form of a quantum channel, since we have

$$\sum_{i=1}^n A_{e_{i,m_i}}^* \cdots A_{e_{i,1}}^* \, A_{e_{i,1}} \cdots A_{e_{i,m_i}} = 1.$$

Indeed, write $A_v := A_{e_1} \cdots A_{e_m}$ for the composition of the operators A_e along the oriented path from the vertex v to the root v_0. So we write the above as $\sum_i A_i^* A_i$. Starting at the leaves and considering the adjacent vertices, we can rewrite the sum as

$$\sum_v \sum_{i \,:\, t(e_i)=v} A_v^* A_{e_{i,1}}^* \, A_{e_{i,1}} A_v = \sum_v A_v^* A_v,$$

where the set $\{i \,:\, t(e_i) = v\}$ is non-empty only for the vertices v adjacent to the leaves. This reduces by one the length of the path. Thus, we obtain inductively that the normalisation $\sum_v A_v^* A_v = 1$ holds, with the condition $\sum_{e \,:\, t(e)=v} A_e^* A_e = 1$ implying that it holds for length one. $\qquad\square$

Theorem 5.3 *The tree quantum channels C_A^τ form an A_∞-operad QC.*

Proof Consider $\mathbf{Z}$-modules $QC(n) := \mathrm{span}_{\mathbf{Z}}\{C_A^\tau \mid \tau \in \mathcal{T}(n)\}$ where $\mathcal{T}(n)$ is the A_∞-operad of planar rooted trees. The operadic composition laws

$$\gamma_{QC} : QC(n) \otimes QC(k_1) \otimes \cdots \otimes QC(k_n) \to QC(k_1 + \cdots + k_n)$$

are given by

$$\gamma_{QC}(C_A^\tau; C_{A_1}^{\tau_1}, \ldots, C_{A_n}^{\tau_n}) = C_{A \cup \{A_1, \ldots, A_n\}}^{\gamma_{\mathcal{T}}(\tau; \tau_1, \ldots, \tau_n)}.$$

The associativity, unity, and symmetric properties of QC follow directly from the same properties of the operad $\mathcal{T}$. The DG-structure of QC is also inherited from the DG-structure of the A_∞-operad $\mathcal{T}$, with the differential given by edge contractions

$$dC_A^\tau = \sum_{\tau':\tau=\tau'/e} \epsilon \, C_{A'}^{\tau'},$$

where $\epsilon = (-1)^{\ell(e)}$ with $\ell(e)$ the number of edges below and to the left of e in τ with respect to the planar structure.

The collection of operators A' on a tree τ' with $\tau = \tau'/e$ agrees with A on all edges e' with $t(e') \neq t(e), s(e)$ and is of the following form on the remaining edges. Let E_t be the set of edges e' of τ such that $t(e') = t(e)$ in τ' and E_s the set of edges e' in τ such that $t(e') = s(e)$ in τ'. The set $E_t \cup E_s$ consists of all the edges of τ with the same target vertex v in τ that is split into two vertices $s(e), t(e)$ in τ'. Thus, in C_A^τ we have the relation $\sum_{e' \in E_t \cup E_s} A_e^* A_e = 1$.

Let $B_t := \sum_{e' \in E_t} A_{e'}^* A_{e'}$ and $B_s := \sum_{e' \in E_t} A_{e'}^* A_{e'}$. These are positive operators, namely $\langle Bv, v \rangle \geq 0$ for all $v \in \mathcal{H}$. Put $N_s = \operatorname{card} E_s$ and consider the operators $\frac{1}{N_s} B_t$ and $A_{e'}^* A_{e'} + \frac{1}{N_s} B_t$ for $e' \in E_s$. These are also positive operators. Thus, we can write $B_s = A^* A$ and $A_{e'}^* A_{e'} + \frac{1}{N_s} B_t = \widetilde{A}_{e'}^* \widetilde{A}_{e'}$ for some operators A and $\widetilde{A}_{e'}$.

We then take $A'_e := A$ and $A'_{e'} := \widetilde{A}_{e'}$ for $e' \in E_s$. This completes the description of A' on τ' in a way that still satisfies the conditions at vertices

$$\sum_{e':t(e')=s(e)} A_{e'}'^* A_{e'}' = \sum_{e' \in E_s} (A_{e'}^* A_{e'} + \frac{1}{N_s} B_t) = \sum_{e' \in E_s} A_{e'}^* A_{e'} + \sum_{e' \in E_t} A_{e'}^* A_{e'} = 1,$$

$$\sum_{e':t(e')=t(e)} A_{e'}'^* A_{e'}' = A_e'^* A_e' + \sum_{e' \in E_t} A_{e'}^* A_{e'} = \sum_{e' \in E_s} A_{e'}^* A_{e'} + \sum_{e' \in E_t} A_{e'}^* A_{e'} = 1.$$

Note that, in order to make the composition operations compatible with the differential of the DG-structure, they should also include appropriate signs, as specified for instance in [15]. We will omit the details as they are exactly the same as in the original case of the A_∞-operad $\mathcal{T}$. □

Taking formal linear combinations of quantum channels, as in the definition of QC above, has the advantage of being able to define the differential and DG-structure described in Theorem 5.3. However, it is somewhat unnatural, since the positivity property of quantum channels is lost in the linear combinations. It is therefore more natural in this context to consider only convex combinations. This leads to a variant of the operad QC of tree quantum channels.

Definition 5.8 Let QC^+ be given by $QC^+(n) = $ convex span$\{C_A^\tau \mid \tau \in \mathcal{T}(n)\}$. The $QC^+(n)$ are convex sets rather than $\mathbf{Z}$-modules (or vector spaces). The composition

operations are the same as the γ_{QC} (without signs),

$$\gamma_{QC^+}(C_A^\tau; C_{A_1}^{\tau_1}, \ldots, C_{A_n}^{\tau_n}) = C_{A \cup \{A_1, \ldots, A_n\}}^{\gamma_{\mathcal{T}}(\tau; \tau_1, \ldots, \tau_n)}.$$

Then QC^+ defined in this way is an operad, though it no longer has a DG-structure.

Proposition 5.5 *The convex set $\mathcal{M}^{(N)}$ has the structure of an algebra over the operad QC^+.*

Proof The operations $\alpha : QC^+(n) \otimes \mathcal{M}^{(N)\otimes n} \to \mathcal{M}^{(N)}$ are given by a slight modification of the action of tree quantum channels of Lemma 5.5,

$$\alpha(C_A^\tau; \rho_1, \ldots, \rho_n) = \sum_{i=1}^n p_i \widetilde{\rho}_i,$$

$$\widetilde{\rho}_i = \frac{A_{\gamma_i} \rho_i A_{\gamma_i}^*}{\mathrm{Tr}(A_{\gamma_i}^* A_{\gamma_i} \rho_i)} \quad \text{and} \quad p_i = \mathrm{Tr}(A_{\gamma_i}^* A_{\gamma_i}),$$

where $A_{\gamma_i} = A_{e_{i,1}} \cdots A_{e_{i,m_i}}$, along the oriented path $\gamma_i = e_{i,1}, \ldots, e_{i,m_i}$ from the i-th leaf to the root. $\qquad\qquad\square$

5.6 Operads and Almost-Symplectic Quantum Codes

In this section we continue our investigation of operadic structures in quantum information, by revisiting a construction of quantum codes from (not always Moufang) loops obtained from almost-symplectic vector spaces over finite fields that we developed in [4]. Here we need to consider a slightly different definition of almost-symplectic structure over finite fields with respect to the one used in [4]. The choice we consider here is better because it allows for an operadic composition, but the construction of the associated quantum codes is then less well behaved. We show that the space of almost-symplectic structures (in the sense we consider here) is an algebra over operad modelled on May's little square operad, and the set of data defining the associated quantum codes is a partial-algebra over the same operad.

5.6.1 Rational and Binary Little Square Operads

The little square operad [12] provides a characterization of topological spaces that are 2-fold loop spaces. Little n-cube operads [12] similarly characterize n-fold loop spaces.

We consider first a sub-operad where we impose an additional condition on the linearly scaled versions of the unit square in the operad, namely that they have corners located at rational points in the unit square, namely that the scaling is affected by linear functions with rational coefficients.

Let $\mathcal{I} = [0, 1]$ be the unit interval, with $\mathcal{J} = (0, 1)$ its interior. Let $\mathcal{I}_Q = \mathcal{I} \cap Q$ and $\mathcal{J}_Q = \mathcal{J} \cap Q$. Thus, $\mathcal{I}^2$ is the unit square with its interior $\mathcal{J}^2$ and $\mathcal{I}_Q^2$ and $\mathcal{J}_Q^2$ are the respective sets of rational points. A rational little square is a function

$$c : \mathcal{I}_Q^2 \to \mathcal{I}_Q^2, \quad c = (c^1, c^2), \quad c^i(t) = (y_i - x_i)t + x_i \ \ \forall t \in \mathcal{I}_Q,$$

for some $x_i, y_i \in \mathcal{I}_Q$. An n-tuple $\langle c_1, \ldots, c_n \rangle$ of rational little squares has disjoint interiors if $c_i(\mathcal{J}_Q^2) \cap c_j(\mathcal{J}_Q^2) = \emptyset$ for $i \neq j$.

The n-th object of the rational little square operad $C_2^Q(n)$ is the space of n-tuples $\langle c_1, \ldots, c_n \rangle$ of rational little squares with disjoint interiors. For $n = 0$, the space $C_2^Q(0)$ consists of a unique function $\emptyset \to \mathcal{I}^2$.

Let $\sqcup^n \mathcal{I}^2$ denote the disjoint union $\mathcal{I}^2 \sqcup \cdots \sqcup \mathcal{I}^2$ of n copies of $\mathcal{I}^2$. By identifying $\langle c_1, \ldots, c_n \rangle$ with a function $c_1 \sqcup \cdots \sqcup c_n : \sqcup^n \mathcal{I}^2 \to \mathcal{I}^2$, the set $C_2^Q(n)$ is endowed with the topology induced by the compact-open topology on the set of continuous maps from $\sqcup^n \mathcal{I}^2$ to $\mathcal{I}^2$.

The operad compositions

$$\circ_i : C_2^Q(n) \times C_2^Q(m) \to C_2^Q(n + m - 1)$$

of $c = \langle c_1, \ldots, c_n \rangle \in C_2^Q(n)$ and $c' = \langle c'_1, \ldots, c'_m \rangle \in C_2^Q(m)$ are determined by the diagrams

$$
\begin{array}{ccc}
\sqcup^{n+m-1} \mathcal{I}_Q^2 & \xrightarrow{\ c \circ_i c'\ } & \mathcal{I}_Q^2 \\[2mm]
& \searrow{\scriptstyle id_{i-1} \sqcup c' \sqcup id_{n-i}} \qquad \nearrow{\scriptstyle c} & \\[2mm]
& \sqcup^n \mathcal{I}_Q^2 &
\end{array}
$$

The unit of the rational little square operad is the identity map $id : \mathcal{I}_Q^2 \to \mathcal{I}_Q^2$ in $C_2^Q(1)$. The action on $c = \langle c_1, \ldots, c_n \rangle$ of a permutation σ in the symmetric group Σ_n is given by

$$c\sigma := \langle c_{\sigma(1)}, \ldots, c_{\sigma(n)} \rangle,$$

permuting the labels of the little squares.

5.6.1.1 Binary Little Square Operad

We then consider subspaces $C_2^{\mathscr{F}_2}(n) \subset C_2^Q(n)$ given by those rational little squares $c = \langle c_1, \ldots, c_n \rangle$ with disjoint interiors, with the property that the endpoints of each $c_i(I^2)$ are rational points of I^2 that lie on the square grid of length 2^{-N}, for some $N \geq 0$.

   We refer to the parallel grid of length 2^{-N} in the unit square I^2 as *the N-grid*.

Lemma 5.6 *The subspaces $C_2^{\mathscr{F}_2}(n) \subset C_2^Q(n)$, with the induced composition operations, determine an operad $C_2^{\mathscr{F}_2}$, the "binary little square operad".*

Proof Under the operad composition operations of C_2^Q, the compositions $c \circ_i c'$ of two binary little squares is still a binary little square, so $C_2^{\mathscr{F}_2}$ is a sub-operad of C_2^Q. ∎

5.6.1.2 Strict Binary Little Squares

Given a binary little square $c \in C_2^{\mathscr{F}_2}(n)$, with $c = \langle c_1, \ldots, c_n \rangle$, let $N_c \in \mathbf{N}$ be the smallest natural number such that the corners of all the $c_i(I^2)$ are at vertices of the N_c-grid of size 2^{-N_c}.

Definition 5.9 A binary little square c is "strict" if every row and column of the N_c-grid has at least one square that is not contained in the union $\cup_{i=1}^n c_i(I^2)$.

   Consider the sub-spaces $C_2^{\mathscr{F}_2,s}(n) \subset C_2^{\mathscr{F}_2}(n)$ consisting of binary little squares that are strict.

Lemma 5.7 *The $C_2^{\mathscr{F}_2,s}(n)$ with the induced composition operations, determine a sub-operad of $C_2^{\mathscr{F}_2}$.*

Proof We need to check that the operad compositions preserve the strict property of little squares. Given $c \in C_2^{\mathscr{F}_2,s}(n)$ and $c' \in C_2^{\mathscr{F}_2,s}(m)$, the endpoints of the regions $(c \circ_i c')_j(I^2)$, $j = 1, \ldots, n+m-1$ are on a grid of size $2^{-N_{c \circ_i c'}}$, with $N_{c \circ_i c'} \geq N_c$. Suppose there is a row R (or column) of the $N_{c \circ_i c'}$-grid that is completely contained in the region $\cup_{j=1}^{n+m-1}(c \circ_i c')_j(I^2)$. If R does not intersect the region $c_i(I^2)$, then it is contained in the union of the $c_j(I^2)$ with $j \neq i$. Since this region has all sides along the N_c-grid, this implies that there must be in fact a row of the N_c-grid, containing R, that is contained in $\cup_{j \neq i} c_j(I^2)$, but this is not possible because c is strict. Thus, R must intersect the region $c_i(I^2)$. This means that, within the region $c_i(I^2)$ a row of the $N_{c \circ_i c'}$-grid is completely contained in the union of the linearly scaled images of the $c_j'(I^2)$, but this in turn implies that in I^2 a row of the $N_{c'}$-grid must be contained in the union of the $c_j'(I^2)$, which cannot happen because c' is strict. ∎

5.6.2 Binary Little Square Operads and Almost Symplectic Spaces

We consider here a class of (not always Moufang) loops that we previously introduced and investigated in [4]. These are obtained from almost-symplectic structures on vector spaces over a finite field of characteristic 2. We recall the basic setting from [4]. However, the notion of almost-symplectic form we consider here is somewhat different from [4]: this will allow for better properties with respect to operadic composition, but will in turn have worse properties with respect to representations of the resulting loops, hence the construction of quantum codes considered in [4] will not directly extend to this setting.

Let $\mathscr{F} = \mathscr{F}_{2^r}$ be a finite field of characteristic two and let V be a finite dimensional vector space over $\mathscr{F}$. Let K be an unramified extension of Q_2 with residue field $O_K / \boldsymbol{m}_K = \mathscr{F}$. Consider the ring $R = O_K / \boldsymbol{m}_K^2$ and a free R-module $\widetilde{V}$ with $V = \widetilde{V} / \boldsymbol{m}_K$. Consider functions $\widetilde{\omega} : \widetilde{V} \times \widetilde{V} \to R$ with $\omega = 2\widetilde{\omega}$ the induced function $\omega : V \times V \to R$. Note that these functions are not linear as they do not satisfy the Hochschild cocycle condition of symplectic forms, namely we require that

$$d\omega(u, v, w) = \omega(v, w) - \omega(u + v, w) + \omega(u, v + w) - \omega(u, v) \neq 0 \,.$$

A pair (V, ω) as above is an almost-symplectic space if ω is non-degenerate, in the sense that for any $u \in V$ there is some $v \in V$ with $(u, v) \neq (0, 0)$, such that $\omega(u, v) \neq 0$. (Note that with this definition the almost-symplectic structure ω does *not* satisfy $\omega(0, v) = \omega(u, 0) = 0$, for all $u, v \in V$.) The almost-symplectic structure ω has a polarization $\beta : V \times V \to R$ satisfying $\beta(u, v) - \beta(v, u) = \omega(u, v)$. The loop $\mathcal{L}(V, \beta)$ associated to the data (V, β) is the extension

$$0 \to R \to \mathcal{L}(V, \beta) \to V \to 0$$

with non-associative multiplication

$$(x, u) \star (y, v) = (x + y + \beta(u, v), u + v) \,.$$

The non-associativity is a consequence of the fact that (V, ω) is almost-symplectic and not symplectic, hence the Hochshild coboundary of β is nonzero,

$$d\beta(u, v, w) = \beta(v, w) - \beta(u + v, w) + \beta(u, v + w) - \beta(u, v) = \gamma(u, v, w) \neq 0 \,.$$

The Moufang condition for the loop $\mathcal{L}(V, \beta)$ is equivalent to an identity for $\gamma = d\beta$, see [4]. We do not necessarily require here the condition that the loop $\mathcal{L}(V, \beta)$ is Moufang.

We focus in particular on the special case where $\mathscr{F} = \mathscr{F}_2$ and $R = \boldsymbol{Z}/4\boldsymbol{Z}$. We can identify the map $\omega = 2\widetilde{\omega} : V \times V \to R$, where $V = \mathscr{F}_2^N$, as a subdivision

of the square $I^2 = [0, 1] \times [0, 1]$ into the N-grid of $2^N \times 2^N$ sub-squares of side 2^{-N}, which we can label by the pairs $(u, v) \in V \times V$. Each square is colored white or black according to whether $\omega(u, v) = 0$ or not. The property that ω is almost-symplectic (that is, non-degenerate) is equivalent to the fact that in every row and column of the subdivided square there is at least one black sub-square.

Let $\mathcal{V}^{\mathscr{F}_2}$ be the space of almost symplectic finite dimensional vector spaces over $\mathscr{F}_2$, topologized as a subset of the space of maps $\cup_N Maps(\mathscr{F}_2^N \times \mathscr{F}_2^N, R)$, where $\mathcal{V}^{\mathscr{F}_2}$ consists of those maps that satisfy the non-degenerate condition above, with range $2R \subset R$, since $\omega = 2\widetilde{\omega}$.

Theorem 5.4 *The space $\mathcal{V}^{\mathscr{F}_2}$ of almost symplectic finite dimensional vector spaces over $\mathscr{F}_2$ is an algebra over the operad $C_2^{\mathscr{F}_2, s}$.*

Proof To realize $\mathcal{V}^{\mathscr{F}_2}$ as an algebra over $C_2^{\mathscr{F}_2, s}$ we need operations

$$C_2^{\mathscr{F}_2, s}(n) \times \underbrace{\mathcal{V}^{\mathscr{F}_2} \times \cdots \times \mathcal{V}^{\mathscr{F}_2}}_{n-\text{times}} \to \mathcal{V}^{\mathscr{F}_2},$$

that assign to an n-tuple $(V_1, \omega_1), \ldots, (V_n, \omega_n)$ and a strict binary little square $c = \langle c_1, \ldots, c_n \rangle$ a new almost-symplectic space $(V, \omega) = \gamma(c; (V_1, \omega_1), \ldots, (V_n, \omega_n))$, compatibly with the composition operations in the operad $C_2^{\mathscr{F}_2, s}$. We construct (V, ω) in the following way. For $V_i = \mathscr{F}_2^{N_i}$, we consider the regions $\omega_i^{-1}(0)$ with sides on the N_i-grid in I^2. We take each of these copies of I^2 subdivided in the subsquares of the N_i-grid, with those in $\omega_i^{-1}(0)$ colored white and the others colored black, and we scale it linearly so as to fit, respectively, into the $c_i(I^2)$ regions of I^2 determined by the binary little square $c = \langle c_1, \ldots, c_n \rangle$. We color black the outside of $\cup_i c_i(I^2)$ in I^2. Let $N \in \mathbf{N}$ be the smallest integer such that all the resulting contours in I^2 separating the black and the white colored areas are on the N-grid. We then set $V = \mathscr{F}_2^N$ and we assign the values of $\omega(v, w)$ according to the color of the corresponding square. The fact that the binary little square is strict implies that ω is non-degenerate.   □

5.6.3 Colored p-ary Little Squares

Let now $q = p^r$ be some prime power with $p > 2$. We construct an operad that generalizes the strict binary little squares operad considered in the case of characteristic 2.

For simplicity, as in the case of characteristic 2, we restrict to the case of $\mathscr{F}_p$. We then define the p-ary N-grid in the unit square I^2 to the parallel grid of size p^{-N}, with $p^N \times p^N$ sub-squares. We just refer to it as the N-grid when the choice of p is understood.

Definition 5.10 A colored p-ary little square is a decomposition of the unit square I^2 into p regions $\mathcal{R}_0, \ldots, \mathcal{R}_{p-1}$ with disjoint interiors $\mathcal{J}(\mathcal{R}_i) \cap \mathcal{J}(\mathcal{R}_j) = \emptyset$, for $i \neq j$, with $\mathcal{J}(\mathcal{R})$ denoting the interior of a region $\mathcal{R}$. With the property that, for each $i = 0, \ldots, p - 1$, and for some integers $n_i, N_i \in \mathbb{N}$, there is a rational little square $c^{(i)} = \langle c_1^{(i)}, \ldots, c_{n_i}^{(i)} \rangle$ in $C_2^Q(n_i)$, with endpoints on a p-ary N_i-grid, such that $\mathcal{R}_i = \cup_{j=1}^{n_i} c_j^{(i)}(I^2)$. The colored p-ary little square is strict if, for $i = 0$, the little square $c^{(0)}$ is strict, namely no row or column of the p-ary N_0-grid is completely contained in $\cup_{j=1}^{n_0} c_j^{(0)}(I^2)$. We denote by $C_2^{\mathscr{F}_p}(n_0, \ldots, n_{p-1})$ the set of colored p-ary little squares as above and we take

$$C_2^{\mathscr{F}_p}(n) := \bigcup_{n_1, \ldots, n_{p-1}} C_2^{\mathscr{F}_p}(n, n_1, \ldots, n_{p-1}).$$

The set of strict colored p-ary little squares $C_2^{\mathscr{F}_{p,s}}(n)$ is similarly defined.

We denote the (strict) colored p-ary little squares in $C_2^{\mathscr{F}_p}(n)$ (or $C_2^{\mathscr{F}_{p,s}}(n)$, respectively) with the notation

$$I^2(\mathcal{R}_0, c^{(0)}; \mathcal{R}) := \left\{ (\mathcal{R}_0, c^{(0)} = \langle c_1^{(0)}, \ldots, c_n^{(0)} \rangle), (\mathcal{R}_j, c^{(j)})_{j=1,\ldots,p-1} \right\},$$

with $\mathcal{R} := \{(\mathcal{R}_j, c^{(j)})\}_{j=1,\ldots,p-1}$. The composition operations

$$\circ_i : C_2^{\mathscr{F}_p}(n) \times C_2^{\mathscr{F}_p}(m) \to C_2^{\mathscr{F}_p}(n + m - 1)$$

are determined by taking

$$I^2(\mathcal{R}_0, c^{(0)}; \mathcal{R}) \circ_i I^2(\mathcal{R}_0', c'^{(0)}; \mathcal{R}')$$

to be given by the decomposition of the unit square I^2 into the regions

$$I^2 = \mathcal{B} \cup \bigcup_{j \neq 0} \mathcal{R}_j \cup \bigcup_{\ell \neq i} c_\ell^{(0)}(I^2),$$

where the region $\mathcal{B}$ is obtained by linearly scaling the square $I^2 = \cup_j \mathcal{R}_j'$ subdivided into the regions of the second colored p-ary little square and placing it in place of the region $c_i^{(0)}(I^2)$.

Lemma 5.8 With the composition operations above the $C_2^{\mathscr{F}_p}(n)$ (respectively, $C_2^{\mathscr{F}_{p,s}}(n)$) form an operad.

Proof The new regions $\mathcal{R}''_\ell$ with $\ell = 1, \ldots, p-1$ of the composed $\mathcal{I}^2(\mathcal{R}_0, c^{(0)}; \mathcal{R}) \circ_i \mathcal{I}^2(\mathcal{R}'_0, c'^{(0)}; \mathcal{R}')$ are given by

$$\mathcal{R}''_\ell = \mathcal{R}_\ell \cup c'^{(0)}_i(\mathcal{R}'_\ell)$$

while the $\mathcal{R}''_0$ region of the composed $\mathcal{I}^2(\mathcal{R}_0, c^{(0)}; \mathcal{R}) \circ_i \mathcal{I}^2(\mathcal{R}'_0, c'^{(0)}; \mathcal{R}')$ is given by

$$\mathcal{R}''_0 = \bigcup_{j \neq i} c^{(0)}_j(\mathcal{I}^2) \cup c'^{(0)}_i(\mathcal{R}'_0) \,.$$

Thus we see that these composition operations are still the same composition operations of the operad C^Q_2, acting on the $c^{(0)}$ little squares while maintaining all the rest of the data unaffected. The strict condition is preserved under composition by the same argument as in Lemma 5.7. □

5.6.3.1 Operads and Almost-Symplectic Structures Over $\mathscr{F}_p$

An almost-symplectic vector space over $\mathscr{F}_p$ is a pair (V, ω) of a finite dimensional vector space over $\mathscr{F}_p$ and a function $\omega : V \times V \to \mathscr{F}_p$ which is non-degenerate, in the sense that for all $u \in V$ there is some v with $(u, v) \neq (0, 0)$ such that $\omega(u, v) \neq 0$. and with nontrivial Hochschild coboundary

$$d\omega(u, v, w) = \omega(v, w) - \omega(u + v, w) + \omega(u, v + w) - \omega(u, v) \neq 0 \,.$$

In this case again the non-vanishing of the Hochschild coboundary implies that ω cannot be a linear map.

Note that here also the notion of almost symplectic structure we are considering is different from [4], hence the construction of representations and quantum codes described there does not apply directly to this case.

In this case, the (not necessarily Moufang) loop $\mathcal{L}(V, \omega)$ associated to the almost symplectic structure (V, ω) is obtained as the extension

$$0 \to \mathscr{F}_p \to \mathcal{L}(V, \omega) \to V \to 0$$

with the non-associative multiplication given by

$$(x, u) \star (y, v) = (x + y + \frac{1}{2}\omega(u, v), u + v) \,.$$

As shown in [4], the Moufang condition for $\mathcal{L}(V, \omega)$ is expressed as an identity satisfied by the function $d\omega$.

Let $\mathcal{V}^{\widetilde{\mathscr{F}_p}}$ denote the space of almost symplectic structures (V, ω) over $\mathscr{F}_p$.

Theorem 5.5 *The space $\mathcal{V}^{\mathscr{F}_p}$ is an algebra over the operad $C_2^{\mathscr{F}_{p,s}}$ of strict colored p-ary little squares.*

Proof The argument is similar to Theorem 5.4, except for the coloring of the regions. Given an n-tuple of almost symplectic spaces (V_i, ω_i) and a strict colored p-ary little square $I^2(\mathcal{R}_0, c^{(0)}; \mathcal{R}) \in C_2^{\mathscr{F}_{p,s}}(n)$, we form a new

$$(V, \omega) = \gamma(I^2(\mathcal{R}_0, c^{(0)}; \mathcal{R}); (V_1, \omega_1), \dots, (V_n, \omega_n))$$

by associating to each (V_i, ω_i), with $V_i = \mathscr{F}_p^{N_i}$, a p-ary N_i-grid in I^2 subdivided into regions $\mathcal{R}_\ell = \omega_i^{-1}(\ell)$ for $\ell \in \mathscr{F}_p$. This determines a colored p-ary little square $I^2(\mathcal{R}_{i,0}, c^{i,(0)}; \mathcal{R}_i)$ associated to each (V_i, ω_i), which is strict because ω_i is non-degenerate. We then compose these according to the composition

$$\gamma(I^2(\mathcal{R}_0, c^{(0)}; \mathcal{R}); I^2(\mathcal{R}_{1,0}, c^{1,(0)}; \mathcal{R}_1), \dots, I^2(\mathcal{R}_{n,0}, c^{n,(0)}; \mathcal{R}_n))$$

$$\gamma : C_2^{\mathscr{F}_{p,s}}(n) \times C_2^{\mathscr{F}_{p,s}}(k_1) \times \cdots \times C_2^{\mathscr{F}_{p,s}}(k_n)) \to C_2^{\mathscr{F}_{p,s}}(k_1 + \cdots + k_n)$$

obtained from repeated application of the compositions $\circ_i$ in the operad $C_2^{\mathscr{F}_{p,s}}$. This results in a new colored p-ary little square with regions $\mathcal{R}_\ell''$, which is also strict. This in turn defines the resulting almost symplectic space (V, ω) which has $V = \mathscr{F}_p^N$, with N the smallest natural number such that the subdivision into regions $\mathcal{R}_\ell''$ of the resulting colored p-ary little square is along the p-ary N-grid, and $\omega(u, v) = \ell$ for $(u, v) \in \mathcal{R}_\ell''$. This ω is non-degenerate because the little square is strict. □

5.6.4 Operad Partial-Action on Quantum Codes

A vector space $V = \mathscr{F}_q^N$, determines a corresponding complex vector space $(\boldsymbol{C}^q)^{\otimes N}$, representing a system of N q-ary qubits, endowed with the canonical basis given by the $|v\rangle$, labelled by the vectors $v \in V$.

We consider here the case of $\mathscr{F}_p$ with $p > 2$. The argument can be adapted to the case of characteristic 2 along the lines discussed in [4]. Let $\mathcal{L} = \mathcal{L}(V, \omega)$ be the loop obtained from an almost-symplectic (V, ω) over $\mathscr{F}_p$ as recalled above. It acts on $\mathcal{H} = \boldsymbol{C}[\mathcal{L}(V, \omega)]$ by left and right multiplication $((x, u) \star f)(y, v) = f((x, u) \star (y, v))$ and $(f \star (x, u))(y, v) = f((y, v) \star (x, u))$. This gives rise to a loop representation, which is the product $\mathcal{L} \times \mathcal{H}$ endowed with the (non-associative) multiplication

$$((x, u), f) \star ((y, v), f') = ((x, u) \star (y, v), (x, u) \star f' + f \star (y, v)).$$

Consider the following sets Let $\mathcal{S}_1^\omega = \omega^{-1}(0) \cap \omega^{-1}(0)^\tau = \{(u, v) \in V \times V \mid \omega(u, v) = \omega(v, u) = 0\}$. Given a subset $\Omega \subset V \times V$, we write

$$\Omega_{ij} = \pi_{ij}^{-1}\Omega \subset V^L = \underbrace{V \times \cdots \times V}_{L-\text{times}},$$

with π_{ij} the projection onto the product of the i-th and j-th components. For $L \geq 2$ let $\mathcal{S}_L \subset V^L$,

$$\mathcal{S}_L^\omega = \bigcap_{1 \leq i < j \leq L} (\mathcal{S}_1^\omega)_{ij} .$$

Let $\mathcal{S} = \cup_{(V,\omega) \in \mathcal{V}^{\mathcal{F}_p}} \cup_{L \geq 1} \mathcal{S}_L^\omega$.

Given a character $\chi : \mathcal{F}_p \to \mathbf{C}^*$, let $\mathcal{H}_\chi \subset \mathcal{H}$ be the subspace of functions $f(x, u)$ that transform by $((x', 0) \star f)(x, u) = \chi(x') f(x, u)$. An element $\underline{u} = (u_1, \ldots, u_L) \in \mathcal{S}_L$ determines a set $\{\chi(x_1)E_{u_1}, \ldots, \chi(x_L)E_{u_L}\}_{x_i \in \mathcal{F}_p}$ of mutually commuting operators on $\mathcal{H}_\chi$. This determines a quantum code $Q_{\chi,\underline{u}}^\lambda \subset \mathcal{H}$ given by a common eigenspace of these operators with eigenvalue λ. We refer to the quantum codes obtained in this way as "almost-symplectic quantum codes".

The operad action on almost-symplectic structures described in Theorem 5.5 induces a partial-action of the same operad on the data that determine these quantum codes. The notion of partial-action of an operad (partial-algebra over an operad) was introduced in [5].

Proposition 5.6 *The space $\mathcal{S}$ is a partial-algebra over the operad $C_2^{\mathcal{F}_{p,s}}$ of strict colored p-ary little squares.*

Proof In $\mathcal{S}^n = \underbrace{\mathcal{S} \times \cdots \times \mathcal{S}}_{n-\text{times}}$ consider the subspace

$$\mathcal{S}_0^n = \{(\underline{u}^{(1)}, \ldots, \underline{u}^{(n)}) \in \mathcal{S}_{L_1}^{\omega_1} \times \cdots \times \mathcal{S}_{L_n}^{\omega_n} \mid (\widetilde{\underline{u}}^{(1)}, \ldots, \widetilde{\underline{u}}^{(n)}) \in \mathcal{S}_{L_1 + \cdots + L_n}^\omega\}$$

for $(V, \omega) = \gamma(\mathcal{I}(\mathcal{R}_0, c^{(0)}; \mathcal{R}); (V_1, \omega_1), \ldots, (V_n, \omega_n))$ the operad action of Theorem 5.3.4, and with $(\widetilde{\underline{u}}_i)$ the vector in V obtained from the vectors $\underline{u}_i \in V_i$ under this composition. Then the operad action on $\mathcal{V}^{\mathcal{F}_p}$ induces a partial-action $\gamma : C_2^{\mathcal{F}_{p,s}}(n) \times \mathcal{S}_0^n \to \mathcal{S}$. $\qquad\square$

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