

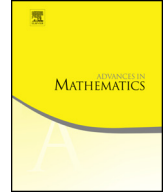


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Déviissage and localization for the Grothendieck spectrum of varieties



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ABSTRACT

We introduce a new perspective on the K -theory of exact categories via the notion of a CGW -category. CGW -categories are a generalization of exact categories that admit a Quillen Q -construction, but which also include examples such as finite sets and varieties. By analyzing Quillen's proofs of déviissage and localization we define $ACGW$ -categories, an analogous generalization of abelian categories for which we prove theorems akin to déviissage and localization. In particular, although the category of varieties is not quite $ACGW$, the category of reduced schemes of finite type is; applying déviissage and localization allows us to calculate a filtration on the K -theory of schemes of finite type. As an application of this theory we construct a comparison map showing that the two authors' definitions of the Grothendieck spectrum of varieties are equivalent.

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1. Introduction

On August 16, 1964, Grothendieck wrote to Serre of a conjectured category of motives. Such a category (called $\mathbf{M}(k)$) would encode schemes up to decomposition (by cutting out subvarieties), but would itself be an abelian category capturing the *cohomological* structures involved.

The sad truth is that for the moment I do not know how to define the abelian category of motives, even though I am beginning to have a rather precise yoga for this category. For example, for any prime $\ell \neq p$, there is an exact functor T_ℓ from $\mathbf{M}(k)$ into the category of finite-dimensional vector spaces over \mathbb{Q} on which the pro-group $\mathrm{Gal}(\overline{k}_i/k_i)_i$ acts, where k_i runs over subextensions of finite type of k and \overline{k}_i is the algebraic closure of k_i in \overline{k} ; this functor is faithful but not, of course, fully faithful. . . I will not venture to make any general conjecture on the above homomorphism; I simply hope to arrive at an actual construction of the category of motives via this kind of heuristic considerations, and this seems to me to be an essential part of my “long run program.” [3, p 174-175]

Grothendieck’s letter proposes several other properties of this conjectured category, and discusses his attempts at the construction. Since then, there have been many other approximations to construct this category—for an overview see, for example, [9]—but all fall short of the ideal.

Grothendieck’s approach begins with the construction of a “ K -group” of varieties. These days, this is known as the Grothendieck ring of varieties, denoted $K_0(\mathbf{Var}_k)$. It is generated by isomorphism classes of k -varieties, $[X]$, subject to the relations that $[X] = [Z] + [X \setminus Z]$ for closed inclusions $Z \hookrightarrow X$. Kontsevich, following Drinfeld [7], calls this the ring of “poor man’s motives.” He notes that any reasonable abelian category of motives, \mathcal{M}_k , will have a map $K_0(\mathbf{Var}_k) \rightarrow K_0(\mathcal{M}_k)$. For example, in [6, Thm. 4], Gillet and Soulé show that there is a group homomorphism $K_0(\mathbf{Var}_k) \rightarrow K_0(\mathbf{M}_\sim)$ where \mathbf{M} is the category of (pure) motives associated to the equivalence relation \sim . It is thus useful to understand $K_0(\mathbf{Var}_k)$ in a deep way in order to learn more about how motives should work. It is even better to understand how it behaves in relation to abelian categories.

We move toward such an understanding in this paper. Before doing so, we rephrase the question. The Grothendieck group of an abelian category is a shadow of the much richer structure of Quillen’s higher algebraic K -theory [10]. Thus there should in fact exist a map on higher algebraic K -theory spectra $K(\mathbf{Var}_k) \rightarrow K(\mathcal{M}_k)$ provided that one can define the objects in the map. It is currently far beyond the state of the art to attempt to understand the right-hand side. However, the authors separately have come up with models for the left [2,16]. Under these constructions the category of varieties behaves very similarly to an abelian category, and one may be tempted to conjecture that from some novel perspective the category of varieties would “become” abelian.

Our goal in this paper is to construct such a perspective. This has the added benefit of putting all objects of interest on the same footing. Our perspective begins with thinking of sequences $Z \hookrightarrow X \leftarrow X \setminus Z$ as our “exact sequences.” It turns out that with this perspective one can execute nearly all constructions that one enjoys in abelian categories: kernels, cokernels, localizations, etc. The main insight is that we should not think of these constructions algebraically, but in a kind of diagrammatic calculus, where one of the arrows points the opposite way that one would expect. Such diagrammatic calculi are, of course, the foundation of Grothendieck’s seminal Tohoku paper [5].

While we do not develop the general theory of homological algebra of these types of categories,¹ one can ask which K -theoretic theorems hold. Pondering the fundamental theorems of Quillen’s algebraic K -theory, we come to the following desiderata for the construction of K -theories of geometric and algebraic objects:

- (1) The categorical machinery should somehow encompass *both* the category of varieties with its “exact sequences” defined above, and Quillen’s exact categories [10, p. 92].
- (2) Dévissage should hold: Given an inclusion of categories $\mathcal{A} \subset \mathcal{B}$ such that everything in \mathcal{B} can be “broken up” into objects in \mathcal{A} , there should be an equivalence $K(\mathcal{A}) \simeq K(\mathcal{B})$.
- (3) Localization should hold: given two such categories $\mathcal{A} \subset \mathcal{B}$, one should be able to produce a localized category \mathcal{B}/\mathcal{A} as one can with abelian categories. One would also like a localization sequence

$$K(\mathcal{A}) \longrightarrow K(\mathcal{B}) \longrightarrow K(\mathcal{B}/\mathcal{A})$$

as in [10, Thm. 5].

In this paper we show that there *is* such a categorical structure, and we are able to satisfy the requirements listed above. Moreover, it has the correct “yoga”: we are able to not only make the theorems work, but also Quillen’s original proofs. Although this does not get us much closer to understanding the conjectural category of motives, it does

¹ We hope to develop this in future work; it appears to be that homological algebra ends up working almost identically to the classical theory.

provide us with a new perspective and concrete technical tools. The perspective could be summarized as follows: varieties, together with the exact sequences above, behave *almost* like abelian categories and one should work with this structure for as long as possible before passing to abelian categories. As will be shown below, this perspective is extremely fruitful when discussing algebraic K -theory, and we expect it to be more useful generally.

The fundamental notion introduced in this paper is that of a *CGW-category*. It is essentially a category equipped with two subclasses of maps, \mathcal{M} and \mathcal{E} (to be thought of as analogous to admissible monomorphisms and admissible epimorphisms in exact categories), together with distinguished squares that tell us how objects are built. In all examples we know, the horizontal and vertical morphisms need not compose in the category, and therefore we situate the classes \mathcal{M} and \mathcal{E} in a double category. With this minimal amount of data we define K -theory following the classical constructions due to Quillen (Sect. 4) or Waldhausen (Sect. 7). We show that the resulting K -theory spaces have the correct group of components in Theorem 4.3. CGW-categories satisfy requirement (1) above: they encompass varieties *and* exact categories.

Of course, as in the case of exact categories, additional structure is required to prove these theorems. To this end we introduce the definition of an ACGW-category, which is meant to be a sort of “abelian” version of a CGW-category. The category of reduced schemes of finite type is such a category, with the category of varieties sitting inside it as a full subcategory. Roughly, an ACGW-category is a category that formally satisfies all of the properties that open and closed sets do (the complement of a closed set is open, you can intersect closed sets and union open sets, etc). Using this definition we prove the first main theorem of the paper:

Theorem 1.1 (*Dévissage*). *Let \mathcal{A}, \mathcal{B} be ACGW-categories with $\mathcal{A} \subset \mathcal{B}$ satisfying certain technical conditions. Suppose every $B \in \mathcal{B}$ has a finite filtration B_i such that the difference between B_i and B_{i-1} lies in \mathcal{A} . Then $K(\mathcal{A}) \simeq K(\mathcal{B})$.*

Here “difference between” could mean a quotient or a complement; for the precise statement see Theorem 6.2. The definition of ACGW-category has a number of requirements, but these requirements are satisfied by the motivating examples of the category of reduced schemes of finite type, polytopes [16], finite sets, and abelian categories.

The formal similarities between ACGW-categories and abelian categories suggest that other theorems in algebraic K -theory can be extended to the CGW case. Quillen’s other major tool in algebraic K -theory is the localization theorem, which relates the K -theories of two abelian categories \mathcal{A}, \mathcal{B} with the K -theory of their quotient category \mathcal{B}/\mathcal{A} . A very similar theorem holds for ACGW-categories:

Theorem 1.2 (*Localization*). *Let \mathcal{C} be an ACGW category and \mathcal{A} a sub-ACGW-category of \mathcal{C} satisfying certain technical conditions. Then there is a localization ACGW-category $\mathcal{C} \backslash \mathcal{A}$ such that*

$$K(\mathcal{A}) \longrightarrow K(\mathcal{C}) \longrightarrow K(\mathcal{C} \backslash \mathcal{A})$$

is a homotopy fiber sequence.

For a more precise statement of this theorem, see Theorem 8.6.

An interesting observation about the proofs of these theorems is how closely they follow Quillen's original proofs. The category of varieties really does “behave like” an exact category, in the sense that many of the motions that are necessary to prove theorems have direct analogs in the category of varieties. (In fact, the category of varieties lacks only “pushouts” to behave like an abelian category; this is why switching to reduced schemes of finite type is necessary. For more detail on this, see Section 5.)

We expect there to be substantial applications of the dévissage and localization theorems. The main application that we discuss in this paper is a comparison of models for the K-theory of varieties that both authors have constructed. Surprisingly, this theorem seems to use every bit of K-theoretic machinery the authors have developed: assemblers, cofiber sequences in K-theory, and the dévissage and localization theorems. All combine to give the following theorem.

Theorem 1.3 (Comparison). *Let $K^C(\mathbf{Var}^n)$ denote the K-theory of the SW-category \mathbf{Var}^n defined in [2], and let $K^Z(\mathbf{Var}^n)$ denote the K-theory of the assembler \mathbf{Var}^n defined in [16]. Then there is a zig-zag of weak equivalences*

$$K^C(\mathbf{Var}^n) \xrightarrow{\sim} \bullet \xleftarrow{\sim} K^Z(\mathbf{Var}^n).$$

For a more detailed statement of this theorem, see Theorem 9.1.

Each of the models constructed has its own strengths, and this theorem allows us to pass between models to exploit these. We expect a more general theorem relating Waldhausen-style K-theory to assembler style K-theory to be true, but we leave that for future work.

Whether this new perspective leads to a new theory of motives or not is unclear; however, the striking behavioral similarities between varieties and abelian categories was too beautiful to leave unexplored.

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2. CGW-categories

This section contains the main definition of the paper: the definition of a CGW-category. Because exact categories all embed into abelian categories, the data of exact sequences is defined using universal properties in this abelian category: if

$$X \longrightarrow Y \longrightarrow Z$$

is an exact sequence, X is the kernel of $Y \longrightarrow Z$ and Z is the cokernel of $X \longrightarrow Y$. However, if we instead discard this “ambient” abelian category, and think of the exact sequences as simply data to be manipulated (as per Quillen’s original definition [10]), a simple observation comes to light: there is no intrinsic reason why admissible monomorphisms and admissible epimorphisms must compose. It is simply necessary that we encode their relationships to one another.

An efficient way to encode this kind of structure is using the formalism of double categories. We thus begin by recalling the definition of a double category, as well as establishing some notation for working with double categories. The notion of double categories goes back to [4]. We do not include the complete definition; for the reader interested in a more in-depth introduction, see for example [8, Section II.6].

Definition 2.1. A *double category* \mathcal{C} is an internal category in **Cat**. More concretely, a double category consists of a pair of categories, denoted $\mathcal{E}_{\mathcal{C}}$ and $\mathcal{M}_{\mathcal{C}}$, which have the same objects. We denote morphisms in $\mathcal{M}_{\mathcal{C}}$ by \rightharpoonup and morphisms in $\mathcal{E}_{\mathcal{C}}$ by $\circ\rightarrow$. This pair is endowed with a collection of squares, called *distinguished squares*. These are denoted

$$\begin{array}{ccc} A & \xrightarrow{f'} & B \\ g' \circ\downarrow & \square & \downarrow g \\ C & \xrightarrow{f} & D \end{array}.$$

In each distinguished square, $f, f' \in \mathcal{M}_{\mathcal{C}}$ and $g, g' \in \mathcal{E}_{\mathcal{C}}$. The squares satisfy compositional axioms, which say in effect that gluing two squares horizontally or vertically gives another distinguished square. In addition, if f and f' are both isomorphisms then for any g, g' either both of the following squares exist, or neither does:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \circ\downarrow & \square & \downarrow g' \\ C & \xrightarrow{f'} & D \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{f^{-1}} & A \\ g' \circ\downarrow & \square & \downarrow g \\ D & \xrightarrow{f'^{-1}} & C \end{array}$$

We sometimes write $\mathcal{C} = (\mathcal{E}_\mathcal{C}, \mathcal{M}_\mathcal{C})$. When \mathcal{C} is clear from context we omit the subscripts from the notation.

Example 2.2. Let \mathcal{A} be any category, and \mathcal{E} and \mathcal{M} two subcategories containing all isomorphisms in \mathcal{A} . We can define a double category structure $(\mathcal{E}, \mathcal{M})$ by letting the objects be the objects of \mathcal{C} , the horizontal morphisms be given by \mathcal{M} and the vertical morphisms by \mathcal{E} . We let distinguished squares be any subset of the commutative squares in \mathcal{A} which satisfies appropriate closure conditions.

In most cases of interest, the double categories we work with arise as in Example 2.2, so it is useful to introduce language for these categories.

Definition 2.3. If a double category $(\mathcal{E}, \mathcal{M})$ arises from a situation as in Example 2.2, we say that \mathcal{A} is an *ambient* category for $(\mathcal{E}, \mathcal{M})$. In such a case the identity functor gives a natural isomorphism of categories $\text{iso } \mathcal{E} \rightarrow \text{iso } \mathcal{M}$.

CGW-categories will be double categories equipped with extra data. Most of the data involves the specification of the existence of certain distinguished squares. We define certain categories that come up repeatedly in these specifications.

Definition 2.4. Let $\mathcal{C} = (\mathcal{E}, \mathcal{M})$ be a double category. We write $\text{Ar}_\square \mathcal{E}$ for the category whose objects are morphisms $A \circ \rightarrow B$ in \mathcal{E} , and where

$$\text{Hom}_{\text{Ar}_\square \mathcal{E}}(A \circ \xrightarrow{g} B, A' \circ \xrightarrow{g'} B') = \left\{ \begin{array}{c} \text{distinguished} \\ \text{squares} \end{array} \quad \begin{array}{ccc} A & \xrightarrow{\quad} & A' \\ g \downarrow & \square & \downarrow g' \\ B & \xrightarrow{\quad} & B' \end{array} \right\}$$

We have an analogous category $\text{Ar}_\square \mathcal{M}$. Every 2-cell in \mathcal{C} appears uniquely as a morphism in $\text{Ar}_\square \mathcal{E}$ and $\text{Ar}_\square \mathcal{M}$.

Now let \mathcal{D} be any ordinary category. We write $\text{Ar}_\Delta \mathcal{D}$ for the category whose objects are morphisms $A \rightarrow B$ in \mathcal{D} , and where

$$\text{Hom}_{\text{Ar}_\Delta \mathcal{D}}(A \xrightarrow{f} B, A' \xrightarrow{f'} B') = \left\{ \begin{array}{c} \text{commutative} \\ \text{squares} \end{array} \quad \begin{array}{ccc} A & \xrightarrow{\cong} & A' \\ f \downarrow & & \downarrow f' \\ B & \longrightarrow & B' \end{array} \right\}.$$

We now come to the definition of a CGW-category.

Definition 2.5. A *CGW-category* $(\mathcal{C}, \phi, c, k)$ is a double category $\mathcal{C} = (\mathcal{E}, \mathcal{M})$, an isomorphism of categories $\phi: \text{iso } \mathcal{M} \rightarrow \text{iso } \mathcal{E}$ which is the identity on objects, and equivalences of categories

$$k: \text{Ar}_{\square} \mathcal{E} \longrightarrow \text{Ar}_{\Delta} \mathcal{M} \quad \text{and} \quad c: \text{Ar}_{\square} \mathcal{M} \longrightarrow \text{Ar}_{\Delta} \mathcal{E}$$

which satisfy:

(Z) \mathcal{C} contains an object \emptyset which is initial in both \mathcal{E} and \mathcal{M} .

(I) If $f: A \rightharpoonup B$ is any isomorphism in \mathcal{M} then all four of the following squares are distinguished:

$$\begin{array}{cccc} \begin{array}{ccc} A & \xrightarrow{f} & B \\ \phi(f) \downarrow & \square & \downarrow 1_B \\ B & \xrightarrow{1_B} & B \end{array} & \begin{array}{ccc} A & \xrightarrow{1_A} & A \\ 1_A \downarrow & \square & \downarrow \phi(f) \\ A & \xrightarrow{f} & B \end{array} & \begin{array}{ccc} A & \xrightarrow{f} & B \\ 1_A \downarrow & \square & \downarrow \phi(f^{-1}) \\ A & \xrightarrow{1_A} & A \end{array} & \begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \phi(f) \downarrow & \square & \downarrow 1_A \\ B & \xrightarrow{f^{-1}} & A \end{array} \end{array}$$

(M) Every morphism in the categories \mathcal{E} and \mathcal{M} is monic.

(K) For every $g: A \rightharpoonup B$ in \mathcal{E} , the codomains of g and $k(g)$ are equal. We write $k(g)$ as $A^{k/g} \xrightarrow{g^k} B$. There exists a (unique up to unique isomorphism) distinguished square

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & A \\ \downarrow & \square & \downarrow g \\ A^{k/g} & \xrightarrow{g^k} & B \end{array}$$

Dually, for every $f: A \rightharpoonup B$ in \mathcal{M} the codomains of f and $c(f)$ are equal; we write $c(A \xrightarrow{f} B) = A^{c/f} \xrightarrow{f^c} B$. There exists a (unique up to unique isomorphism) distinguished square

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & A^{c/f} \\ \downarrow & \square & \downarrow f^c \\ A & \xrightarrow{f} & B \end{array}$$

(A) For any objects A and B there exist distinguished squares

$$\begin{array}{ccc} \emptyset & \xrightarrow{\quad} & A \\ \downarrow & \square & \downarrow \\ B & \xrightarrow{\quad} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} \emptyset & \xrightarrow{\quad} & B \\ \downarrow & \square & \downarrow \\ A & \xrightarrow{\quad} & X \end{array}.$$

As isomorphisms can be considered to be “both e-morphisms and m-morphisms” we will generally draw them as plain arrows.

When it is clear from context, we write $A^{k/B}$ or A^k instead of $A^{k/g}$ (and analogously for c). When ϕ , c and k are clear from context we omit them from the notation. When \mathcal{C} has an ambient category \mathcal{A} and ϕ is the identity functor, we omit ϕ from the notation.

The definition of a CGW-category is symmetric with respect to m-morphisms and e-morphisms. This duality is highly versatile and allows us to get symmetric results about e-morphisms and m-morphisms with no extra work.

Remark 2.6. Axiom (A) is used only to show that $K_0(\mathcal{C})$ is an abelian group. Thus if in some case such a property is not necessary this axiom can be dropped and the rest of the analysis will still hold.

Functors of CGW-categories must preserve all structure in sight.

Definition 2.7. A *CGW-functor* of CGW-categories is a double functor $F: (\mathcal{E}, \mathcal{M}) \rightarrow (\mathcal{E}', \mathcal{M}')$ which commutes with c and k . More concretely, F is a CGW-functor if the following two diagrams commute:

$$\begin{array}{ccc} \text{Ar}_{\square} \mathcal{E} & \xrightarrow{k} & \text{Ar}_{\triangle} \mathcal{M} \\ \text{Ar}_{\square} F \downarrow & & \downarrow \text{Ar}_{\triangle} F \\ \text{Ar}_{\square} \mathcal{E}' & \xrightarrow{k'} & \text{Ar}_{\triangle} \mathcal{M}' \end{array} \qquad \begin{array}{ccc} \text{Ar}_{\square} \mathcal{M} & \xrightarrow{c} & \text{Ar}_{\triangle} \mathcal{E} \\ \text{Ar}_{\square} F \downarrow & & \downarrow \text{Ar}_{\triangle} F \\ \text{Ar}_{\square} \mathcal{M}' & \xrightarrow{c'} & \text{Ar}_{\triangle} \mathcal{E}' \end{array}$$

The fact that c and k take distinguished squares to commutative triangles means that distinguished squares are *equifibered* (the vertical arrows have equal “kernels” given by k) and *equicofibered* (the horizontal arrows have equal “cokernels” given by c). By Axiom (K), c and k are mutual inverses on objects.

We now prove some technical consequences of the axioms.

Lemma 2.8. *For any A , the morphism $f: \emptyset \rightarrowtail A$ has $f^c = 1_A$. Dually, the morphism $f: \emptyset \twoheadrightarrow A$ has $f^k = 1_A$.*

The following lemma is the most important of the technical results. It states that e-morphisms and m-morphisms can be *commuted past* one another using distinguished squares. This is what will allow the Q -construction in Section 4 to work.

Lemma 2.9. *For any diagram $A \xrightarrow{f} B \xrightarrow{g} C$ there is a unique (up to unique isomorphism) distinguished square*

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow & \square & \downarrow g \\
 D & \xrightarrow{\quad} & C
 \end{array}$$

The analogous statement holds for any diagram $A \circ \xrightarrow{f} B \xrightarrow{g} C$.

Proof. As the categories \mathcal{M} and \mathcal{E} are symmetric in the definition of a CGW-category it suffices to check the first part. Given a diagram as in the statement of the lemma, we can apply c to the first morphism to obtain a diagram

$$A^{c/f} \circ \xrightarrow{f^c} B \circ \xrightarrow{g} C.$$

This diagram represents a morphism $(A^{c/f} \circ \xrightarrow{f^c} B) \circ \xrightarrow{g} (A^{c/f} \circ \xrightarrow{gf^c} C)$ in $\text{Ar}_\Delta \mathcal{E}$. Applying c^{-1} to this morphism produces a distinguished square

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow & \square & \downarrow g \\
 (A^{c/f})^k / gf^c & \xrightarrow{\quad} & C
 \end{array},$$

where we have used that c and k are inverses on objects.

To check that this distinguished square is unique, suppose we are given any other such square

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow & \square & \downarrow g \\
 D & \xrightarrow{f'} & C
 \end{array}$$

Applying c to this square produces a morphism

$$(A^{c/f} \circ \xrightarrow{f^c} B) \circ \xrightarrow{g} (D^{c/f'} \circ \xrightarrow{f'^c} C) \in \text{Ar}_\Delta \mathcal{E}.$$

Since the square is distinguished, we must have $A^{c/f} \cong D^{c/f'}$; if we choose $D^{c/f'} = A^{c/f}$ the codomain of the above morphism becomes $A^{c/f} \circ \xrightarrow{gf^c} C$. Thus any such distinguished square is mapped by c to the original diagram; since c is an equivalence of categories, the square must be canonically isomorphic to the square produced above. \square

Lemma 2.10. *Given any composition*

$$C \xrightarrow{\quad} B \xrightarrow{\quad} A$$

there is an induced map $B^{c/A} \circrightarrow C^{c/A}$ such that the triangle

$$\begin{array}{ccc} B^{c/A} & \xrightarrow{\quad} & C^{c/A} \\ & \searrow & \swarrow \\ & A & \end{array}$$

commutes.

Proof. We begin by applying the equivalence of categories given by k^{-1} from Axiom (K). Since $k^{-1} = c$ on objects, we have the induced diagram

$$\begin{array}{ccc} C^{c/B} & \xrightarrow{h} & C^{c/A} \\ \downarrow & \square & \downarrow \\ B & \xrightarrow{\quad} & A \end{array}$$

We now apply the equivalence given by c to produce the diagram

$$B^{c/A} = (C^{c/B})^{c/h} \circrightarrow C^{c/A} \circrightarrow A. \quad \square$$

We conclude this section with a pair of definitions that will be useful in later sections.

Definition 2.11. Let $\mathcal{C} = (\mathcal{E}, \mathcal{M}, \phi, c, k)$ be a CGW-category. A *CGW-subcategory* is a sub-double category $\mathcal{A} \subseteq \mathcal{C}$ such that $(\mathcal{A}, \phi|_{\mathcal{A}}, c|_{\mathcal{A}}, k|_{\mathcal{A}})$ is also a CGW-category.

Definition 2.12. We say that a CGW-subcategory \mathcal{A} of a CGW-category $(\mathcal{C}, \phi, c, k)$ is *closed under subobjects* if for any morphism $B \twoheadrightarrow C \in \mathcal{M}$, if $C \in \mathcal{A}$ then $B \in \mathcal{A}$. We say that \mathcal{A} is *closed under quotients* if for any morphism $B \circrightarrow C \in \mathcal{E}$, if $C \in \mathcal{A}$ then $B \in \mathcal{A}$. We say that \mathcal{A} is *closed under extensions* if for every distinguished square

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \downarrow & \square & \downarrow \\ C & \xrightarrow{\quad} & D \end{array}$$

if A , B and C are in \mathcal{A} then so is D .

3. Examples

In this section we give several motivating examples of CGW-categories.

Example 3.1. Let \mathcal{A} be an exact category. Let (\mathcal{C}, c, k) be given by

$$\mathcal{E} = \{\text{admissible epimorphisms}\}^{\text{op}} \quad \text{and} \quad \mathcal{M} = \{\text{admissible monomorphisms}\};$$

Define ϕ to be the identity on objects and *inversion* on morphisms. The distinguished squares are stable squares: those squares that are both pushouts and pullbacks in \mathcal{A} . The equivalence k is given by mapping every admissible epimorphism to its kernel; the equivalence c is given by taking every admissible monomorphism to its cokernel.

We check the axioms explicitly.

- (Z) The zero object is initial in \mathcal{M} and terminal in \mathcal{E} , so it is initial in both \mathcal{M} and \mathcal{E} .
- (I) This follows directly from the definition.
- (M) This holds by definition.
- (K) k and c give the correct equivalences, since distinguished squares are both equifibered (since they are pullbacks) and equicofibered (since they are pushouts).
- (A) This holds with $X = A \oplus B$.

Thus an exact category gives rise to a CGW-category. However, there are examples of CGW-categories which are not exact.

Example 3.2. Consider the category \mathbf{FinSet}_* of based finite sets. We define a CGW-category (\mathcal{C}, c, k) by setting

$$\mathcal{M} = \{\text{injections}\} \quad \text{and} \quad \mathcal{E} = \left\{ f: A \longrightarrow B \mid f|_{f^{-1}(B \setminus \{*\})} \text{ is a bijection} \right\}^{\text{op}}.$$

The distinguished squares are the pushout squares; these are also all pullback squares. The equivalence ϕ is defined, as in the previous example, by taking inverses. Define k by taking $f: A \longrightarrow B$ to $f^{-1}(*) \hookrightarrow A$. Define c by taking $g: A \hookrightarrow B$ to $B \twoheadrightarrow B/g(A)$, with the elements not in the image of g mapping to themselves, and everything else mapping to the basepoint.

That axioms (Z), (I), (M), and (A) are satisfied is direct from the definition. The distinguished squares are pullback squares in the underlying category; therefore in a distinguished square the preimages of the basepoint of the two vertical maps are isomorphic. This proves half of (K). Dually, the complements of the two injections horizontally are also isomorphic, since g is injective away from the basepoint.

One of the advantages of CGW-categories is the observation that the contravariance in the \mathcal{E} -direction is not necessary. All of the following examples come equipped with an ambient category, so we omit mention of ϕ .

Example 3.3. Consider the category \mathbf{FinSet} . We define a CGW-category (\mathcal{C}, c, k) by setting

$$\mathcal{E} = \mathcal{M} = \{\text{injections}\}.$$

The distinguished squares are the pushout squares; since all morphisms are injections, they are also pullback squares. The equivalences c and k are given by taking any injection $A \hookrightarrow B$ to the inclusion $B \setminus A \hookrightarrow B$.

That axioms (Z), (I), (M), and (A) are satisfied is direct from the definition. Since distinguished squares are pushouts, the complements of the images in the horizontal maps are isomorphic; the same holds dually for the vertical maps. Thus (K) holds.

We can also improve the intuition from the finite sets example to get a CGW-category structure on the category of varieties.

Example 3.4. Let $\mathcal{C} = \mathbf{Var}$

$$\mathcal{E} = \{\text{open immersions}\} \quad \text{and} \quad \mathcal{M} = \{\text{closed immersions}\}.$$

We let both c and k take a morphism to the inclusion of the complement. The distinguished squares

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \downarrow & \square & \downarrow f \\ C & \xrightarrow{\quad g \quad} & D \end{array}$$

are the pullback squares in which $\text{im } f \cup \text{im } g = D$. Axiom (Z) is satisfied by the empty variety. Axiom (I) holds by definition. Axiom (M) is verified by noting that open and closed immersions satisfy base change in the category of varieties. Axiom (A) holds by setting $X = A \amalg B$. To see that Axiom (K) holds, consider a distinguished square

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \downarrow & \square & \downarrow \\ C & \xrightarrow{\quad} & D \end{array}.$$

By definition, $D \setminus C \cong B \setminus A$, since the image of B in D contains the complement of the image of C . The dual statement for e-morphisms holds as well.

The CGW-category of varieties includes into the larger category of reduced schemes of finite type via a CGW-functor:

Example 3.5. Let \mathbf{Sch}_{rf} be the category of reduced schemes of finite type, with morphisms the compositions of open and closed immersions. We define the \mathcal{E} -morphisms to be the open immersions and the \mathcal{M} -morphisms to be the closed immersions. The distinguished squares are those squares

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \circ \downarrow & \square & \downarrow \circ g \\ C & \xrightarrow{\quad} & D \end{array}$$

for which $D = \text{im } f \cup \text{im } g$ and which are pullbacks in the category of schemes.

We can also restrict attention just to *smooth* varieties.

Example 3.6. The category $\mathbf{Var}_{/k}^{sm}$ of smooth varieties can be given a CGW-structure. We set the m-morphisms to be closed immersions with smooth complements, and the e-morphisms to be open immersions with smooth complements. Thus $\mathbf{Var}_{/k}^{sm}$ is a sub-CGW-category (but not a full sub-CGW-category) of $\mathbf{Var}_{/k}$.

4. The K -theory of a CGW-category

We are now ready to define the K -theory of a CGW-category. The construction exactly follows Quillen's Q -construction [10] for exact categories. After the introduction of the definition, the rest of the section is taken up by noting some useful technical results and providing the standard presentation for the group $K_0(\mathcal{C})$.

Definition 4.1. For a CGW-category $(\mathcal{C}, \phi, c, k)$ we define

$$K(\mathcal{C}) = \Omega|Q\mathcal{C}|,$$

where $Q\mathcal{C}$ is the category with

objects: the objects of \mathcal{C} ,

morphisms: morphisms $A \longrightarrow B$ are equivalence classes of diagrams

$$A \circ \xrightarrow{f} X \xrightarrow{g} B,$$

where $f \in \mathcal{E}$ and $g \in \mathcal{M}$. Two diagrams

$$A \circ \xrightarrow{f} X \xrightarrow{g} B \quad \text{and} \quad A \circ \xrightarrow{f'} X' \xrightarrow{g'} B$$

are considered equivalent if there exists a diagram

$$\begin{array}{ccccc} & & X & & \\ & f \nearrow & \downarrow \cong & \nwarrow g & \\ A & & & & B \\ & f' \searrow & \downarrow & \swarrow g' & \\ & & X' & & \end{array}$$

where the left-hand triangle commutes in \mathcal{E} and the right-hand triangle commutes in \mathcal{M} . The functor ϕ is implicitly being used to place the vertical isomorphism in both \mathcal{E} and \mathcal{M} simultaneously.

composition: defined using Lemma 2.9. More concretely, given two equivalence classes of diagrams represented by

$$A \circ \xrightarrow{f} X \xrightarrow{g} B \quad \text{and} \quad B \circ \xrightarrow{f'} Y \xrightarrow{g'} C$$

there exists a unique (up to unique isomorphism) distinguished square

$$\begin{array}{ccc} X & \xrightarrow{g} & B \\ f'' \downarrow & \square & \downarrow f' \\ Z & \xrightarrow{g''} & Y \end{array}.$$

The composition of the two diagrams is defined to be the class of diagrams represented by

$$A \circ \xrightarrow{f''f} Z \xrightarrow{g'g''} C.$$

The basepoint is taken to be \emptyset .

Remark 4.2. Although we have defined K -theory for CGW categories, the K -theory of a double category is defined for any double category satisfying Lemma 2.9.

As with any definition of K -theory, the first step is to check that it gives the desired group on K_0 .

Theorem 4.3. $K_0(\mathcal{C})$ is the free abelian group generated by objects of \mathcal{C} , modulo the relation that for any distinguished square

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ \downarrow & \square & \downarrow g \\ D & \xrightarrow{f} & C \end{array}$$

we have $[D] + [B] = [A] + [C]$.

Proof. There are two ways to proceed. One could prove this by showing that $K(\mathcal{C})$ is equivalent to some variant of the S_\bullet construction, and proceeding from there, or one could mimic Quillen's original proof that $\pi_1(BQC) = K_0(\mathcal{C})$ for exact categories. We opt for the latter, again to emphasize the analogy with exact categories.

We follow a more modern version of the proof (see, e.g. [15, Proposition IV.6.2]).

The morphisms $\emptyset \xrightarrow{=} \emptyset \twoheadrightarrow A$ form a maximal tree in BQC . By [15, Lemma IV.3.4], the fundamental group $\pi_1(BQC)$ is generated by the morphisms of BQC , modulo the relations $[\emptyset \xrightarrow{=} \emptyset \twoheadrightarrow A] = 1$ and $[f] \cdot [g] = [f \circ g]$ for composable morphisms in QC . We proceed by a series of reductions to get the set of generators and relations in the theorem. In what follows we let $[A \circ \rightarrow X \twoheadrightarrow B]$ denote the equivalence class of a morphism $A \rightarrow B$ in $\pi_1(BQC)$. The notation $[A \twoheadrightarrow B]$ corresponds to the morphism $[A \xrightarrow{=} A \twoheadrightarrow B]$ and similarly $[A \circ \rightarrow B]$ corresponds to $[A \circ \rightarrow B \xrightarrow{=} B]$.

From the definition of Q we have $[B \twoheadrightarrow C][A \twoheadrightarrow B] = [A \twoheadrightarrow C]$. In particular, since $[\emptyset \twoheadrightarrow X] = 1$ in $\pi_1(BQC)$ for all objects X , $[A \twoheadrightarrow B] = 1$ for all m-morphisms.

We begin by noting that by definition

$$[A \twoheadrightarrow B][D \circ \rightarrow A] = [D \circ \rightarrow A \twoheadrightarrow B].$$

Now consider $[B \circ \rightarrow C][A \twoheadrightarrow B]$. By Lemma 2.9 there exists a distinguished square

$$\begin{array}{ccc} A & \twoheadrightarrow & B \\ \downarrow & \square & \downarrow \\ D & \twoheadrightarrow & C \end{array}$$

which implies the relation

$$[B \circ \rightarrow C][A \twoheadrightarrow B] = [D \twoheadrightarrow C][A \circ \rightarrow D]$$

via the composition relation. Each distinguished square produces such a relation. Since all morphisms in \mathcal{M} are equal to the identity, this reduces to the equation

$$[B \circ \rightarrow C] = [A \circ \rightarrow D]$$

for all distinguished squares. We have now shown that $\pi_1(BQC)$ has as generators the morphisms of \mathcal{E} , with relations induced by composition and distinguished squares.

Since

$$[\emptyset \circ \rightarrow A_1][A_1 \circ \rightarrow A_2] = [\emptyset \circ \rightarrow A_2]. \quad (4.4)$$

$\pi_1(BQC)$ is generated by the elements $[\emptyset \circ \rightarrow A]$, which we abbreviate to $[A]$. This expression also eliminates the composition relation. We can substitute for both sides in the relations induced by the distinguished squares to get

$$[B]^{-1}[C] = [A]^{-1}[D].$$

This gives the desired presentation of $K_0(\mathcal{C})$.

It remains to check that $K_0(\mathcal{C})$ is abelian; in other words, that $[A][B] = [B][A]$. The relations imposed by the squares in Axiom (A) state that

$$[A][B] = [X] = [B][A],$$

as desired. \square

The rest of this section is devoted to some technical lemmas exploring the properties of this Q -construction. The first identifies the isomorphisms in $Q\mathcal{C}$ via their components.

Lemma 4.5. *If $\alpha: A \rightarrow B$ is an isomorphism inside $Q\mathcal{C}$ for a CGW-category \mathcal{C} represented by*

$$A \circ \xrightarrow{f} X \xrightarrow{g} B$$

then both f and g are isomorphisms in \mathcal{C} .

Proof. Suppose that the inverse of α is represented by

$$B \circ \xrightarrow{f'} Y \xrightarrow{g'} A.$$

Then the composition is represented by a diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & X & \xrightarrow{g} & B \\ & & \circ & & \circ \\ & f'' \downarrow & & \square & \downarrow f' \\ & Z & \xrightarrow{g''} & Y & \\ & & & & \downarrow g' \\ & & & & A \end{array}$$

Since this is equivalent to 1_A , $f''f$ is an isomorphism. Since f'' is monic and f is its right inverse, it must be an isomorphism; thus f is an isomorphism. Doing the composition in reverse, we see that g has a right inverse and thus must also be an isomorphism. \square

The next lemma illustrates that we can think of a morphism in $Q\mathcal{C}$ as a set of “layers” inside \mathcal{M} . This allows us to think about the Q -construction in CGW-categories analogously to the way that Quillen originally thought about exact categories in [10].

Lemma 4.6. *For any CGW-category \mathcal{B} and any $B \in \mathcal{B}$, the category $Q\mathcal{B}_{/B}$ is equivalent to the category $L_B\mathcal{B}$ with*

objects: *diagrams $B_1 \twoheadrightarrow B_2 \twoheadrightarrow B$ in \mathcal{B} ,*

morphisms: *commutative diagrams*

$$\begin{array}{ccccc} B_1 & \xrightarrow{\quad} & B_2 & \xrightarrow{\quad} & B \\ \uparrow & & \downarrow & & \parallel \\ B'_1 & \xrightarrow{\quad} & B'_2 & \xrightarrow{\quad} & B \end{array}$$

In particular, $Q\mathcal{B}/_B$ is a preorder for any B .

Proof. It suffices to prove the first part of the lemma; the second follows from the definition of $L_B\mathcal{B}$ and axiom (M).

We define a functor $\kappa: Q\mathcal{B}/_B \rightarrow L_B\mathcal{B}$. An object of $Q\mathcal{B}/_B$ is a diagram $B_1 \circ^g B_2 \xrightarrow{\quad} B$. We send this to the diagram $B_1^k \xrightarrow{g^k} B_2 \xrightarrow{\quad} B$. Seeing that this extends to a functor is a bit more complicated. Suppose that

$$B_1 \circ^g B_2 \xrightarrow{f} B \quad \text{and} \quad B'_1 \circ^{g'} B'_2 \xrightarrow{f'} B$$

are two objects of $Q\mathcal{B}/_B$, and suppose that we are given a morphism between them. This morphism consists of an object $C \in \mathcal{B}$ and a diagram

$$\begin{array}{ccccc} B_1 & \circ^{h'} & C & \xrightarrow{\quad} & B'_1 \\ & \searrow g & \downarrow h & \square & \downarrow g' \\ & & B_2 & \xrightarrow{\quad} & B'_2 \\ & & \searrow f & & \downarrow f' \\ & & & & B \end{array}$$

Applying c^{-1} to the upper-left triangle, this diagram corresponds to a unique diagram

$$\begin{array}{ccccc} B_1^{k/C} & \xrightarrow{\quad} & C & \xrightarrow{\quad} & B'_1 \\ \downarrow h'' & \square & \downarrow h & \square & \downarrow g' \\ B_1^{k/B_2} & \xrightarrow{\quad} & B_2 & \xrightarrow{\quad} & B'_2 \\ & & \searrow f & & \downarrow f' \\ & & & & B \end{array}$$

Applying k , this time to the two distinguished squares on the top, gives us a unique diagram

$$\begin{array}{ccccccc}
 B_1^{k/B'_2} = (B_1^{k/B_2})^{k/h''} & \twoheadrightarrow & B_1^{k/B_2} & \twoheadrightarrow & B_2 & \twoheadrightarrow & B'_2 \\
 & & & & \searrow f & & \downarrow f' \\
 & & & & & & B
 \end{array}$$

This can be rearranged into a diagram

$$\begin{array}{ccccc}
 B_1^{k/B_2} & \twoheadrightarrow & B_2 & \xrightarrow{f} & B \\
 \uparrow & & \downarrow & & \parallel \\
 B_1^{k/B'_2} & \twoheadrightarrow & B'_2 & \xrightarrow{f'} & B
 \end{array}$$

as desired.

The inverse equivalence is given by sending a diagram $B_1 \twoheadrightarrow B_2 \twoheadrightarrow B$ to $B_1^{c/B_2} \twoheadrightarrow B_2 \twoheadrightarrow B$. By Axiom (K) these two functors give inverse equivalences. \square

We now give several examples of K -theories of CGW-categories.

Example 4.7. We consider the examples from Section 3.

Example 3.1: When (\mathcal{C}, c, k) arises from an exact category \mathcal{A} , $BQC = BQA$, so $K(\mathcal{C}) = K(\mathcal{A})$.

Example 3.2: The simplicial set BQC is an edgewise subdivision of the S -construction for the Waldhausen category \mathbf{FinSet}_* with injections as the cofibrations (for a more in-depth discussion, see Theorem 7.8). Thus

$$K(\mathcal{C}) := \Omega BQ\mathbf{FinSet}_* \simeq K^{\text{Wald}}(\mathbf{FinSet}_+) \simeq \Omega^\infty \Sigma^\infty S^0$$

where the last equivalence is by Barrat-Priddy-Quillen [1].

Example 3.3: In this case we also have $K(\mathcal{C}) \simeq \Omega^\infty \Sigma^\infty S^0$. Indeed, there is an equivalence of CGW-categories between (\mathbf{FinSet}, c, k) and $(\mathbf{FinSet}_*, c, k)$ from Example 3.2 given as follows. An injection $[i] \hookrightarrow [j]$ considered as an element of $\mathcal{E} \subset \mathbf{FinSet}$ corresponds to an injection $[i]_+ \hookrightarrow [j]_+$ in \mathbf{FinSet}_* . An injection $u: [i] \hookrightarrow [j]$ considered as an element of $\mathcal{M} \subset \mathbf{FinSet}$ corresponds to a surjection $[j]_+ \twoheadrightarrow [i]_+$ by taking $m \in [j]$ to $u^{-1}(m)$ and the rest of $[j]$ to the distinguished basepoint.

Example 3.4: $K(\mathbf{Var})$ is equivalent to the K -theory of varieties defined in [2]; for a more detailed discussion, see Section 7.

5. ACGW-categories

A CGW-category behaves like an exact category. In order to create categories that are analogous to abelian categories (with the goal of proving Quillen’s dévissage and localization) we need to assume some extra conditions. The extra conditions amount to the requirement that certain “pushout-like” objects exist and are compatible with c and k ; in geometric settings this corresponds to certain gluings of objects.

Definition 5.1. An *enhanced* double category is a double category \mathcal{C} with *two* notions of 2-cell, called the *distinguished* and *pseudo-commutative* squares. These are required to satisfy the property that forgetting either of the sets of squares produces a double category, and all distinguished squares are pseudo-commutative. We denote distinguished squares with \square and pseudo-commutative squares with \circlearrowright .

We write $\text{Ar}_{\circlearrowright} \mathcal{M}$ for the category whose objects are morphisms in \mathcal{M} and whose morphisms are pseudo-commutative squares in \mathcal{C} . We write $\text{Ar}_{\times} \mathcal{M}$ for the category whose objects are morphisms in \mathcal{M} and whose morphisms are pullback squares in \mathcal{M} . The category $\text{Ar}_{\square} \mathcal{M}$ is a subcategory of $\text{Ar}_{\circlearrowright} \mathcal{M}$ and $\text{Ar}_{\triangle} \mathcal{M}$ is a subcategory of $\text{Ar}_{\times} \mathcal{M}$ (since all morphisms in \mathcal{M} are monic).

Remark 5.2. The term “pseudo-commutative” is inspired by the role that commutative squares play in the case when we are discussing abelian categories. Consider an abelian category \mathcal{A} , and the associated CGW-category \mathcal{C} . The distinguished squares in \mathcal{C} are the stable squares. However, the commutative squares in \mathcal{A} also play a role in the following sense. In an abelian category, every morphism $f: A \rightarrow B$ can be factored as $A \rightarrow \text{im } f \hookrightarrow B$, an epic followed by a monic. This means that in \mathcal{C} , any diagram of the form

$$X \circ \longrightarrow Z \longleftarrow Y,$$

which represents a monic followed by an epic, can be completed to a square in an essentially unique way. This square will *not* necessarily be distinguished, but it is still important. This completion is the “mixed pullback” that we define in the next definition.

Before we define a pre-ACGW-category we need one extra helper-definition; this is necessary because, although monomorphisms always behave well with respect to pullbacks, they do not always behave well with respect to pushouts.

Definition 5.3. Let \mathcal{C} be a category in which all morphisms are monic, and let

$$C \longleftarrow A \longrightarrow B$$

be a diagram in \mathcal{C} . The *restricted pushout* of this diagram is the initial object (if it exists) in the category of commutative squares

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & X \end{array}$$

which are also pullback squares; in other words, it is cones X under the diagram such that $A \cong B \times_X C$. As usual, a morphism between diagrams is a natural transformation in which all components are equal to the identity except at X . We denote restricted pushouts by $B \star_A C$.

The important intuition behind this definition lies in the following example:

Example 5.4. Let \mathcal{C} be the category of sets and injections. Then \mathcal{C} does not contain all pushouts, as for example the diagram

$$A \longleftarrow \emptyset \longrightarrow A$$

does not have a pushout for any nonempty set A ; this is because the map $A \amalg A \rightarrow A$ is not a monomorphism. However, the restricted pushout of this diagram exists and, as expected, will be isomorphic to $A \amalg A$.

We are now ready to define pre-ACGW-categories:

Definition 5.5. A *pre-ACGW-category* $(\mathcal{C}, \phi, c, k)$ is an enhanced double category \mathcal{C} which is a CGW-category when the pseudo-commutative squares are forgotten, and in which the following extra axioms are satisfied:

- (P) \mathcal{M} and \mathcal{E} are closed under pullbacks.
- (U) The functors c and k extend to equivalences of categories

$$c: \operatorname{Ar}_{\circlearrowleft} \mathcal{M} \longrightarrow \operatorname{Ar}_{\times} \mathcal{E} \quad \text{and} \quad k: \operatorname{Ar}_{\circlearrowleft} \mathcal{E} \longrightarrow \operatorname{Ar}_{\times} \mathcal{M}.$$

These are compatible in the sense that for any diagram $A \rightrightarrows C \leftarrow B$ there exists a unique isomorphism

$$\varphi: (A \times_C B^k)^{c/A} \longrightarrow (A^c \times_C B)^{k/B}$$

such that the square

$$\begin{array}{ccccc} (A \times_C B^k)^{c/A} & \xrightarrow{\varphi} & (A^c \times_C B)^{k/B} & \rightrightarrows & B \\ \downarrow \circlearrowleft & & \circlearrowleft & & \downarrow \circlearrowleft \\ A & \rightrightarrows & & & C \end{array}$$

is a pseudo-commutative square.

We write $A \circledast_C B \stackrel{\text{def}}{=} (A^c \times_C B)^{k/B} \cong (A \times_C B^k)^{c/A}$, so that we have a “mixed pullback square”

$$\begin{array}{ccc} A \circledast_C B & \xrightarrow{\quad} & B \\ \circ \downarrow & \circlearrowleft & \downarrow \circ \\ A & \xrightarrow{\quad} & C \end{array}$$

(S) Suppose that we are given a pullback square

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\quad} & A \\ \downarrow \Upsilon & \searrow \ell & \downarrow \Upsilon \\ B & \xrightarrow{\quad} & C \end{array}$$

in \mathcal{M} . Then $X \stackrel{\text{def}}{=} A \star_{A \times_C B} B$ exists. The induced commuting square

$$\begin{array}{ccc} X^{c/C} & \circ \longrightarrow & B^{c/C} \\ \circ \downarrow & & \downarrow \circ \\ A^{c/C} & \circ \longrightarrow & (A \times_C B)^{c/C} \end{array}$$

(constructed using Lemma 2.10) is a restricted pushout.

The dual of this statement also holds.

Given a pre-ACGW-category $(\mathcal{C}, \phi, c, k)$, a *pre-ACGW-subcategory* \mathcal{D} is a sub-double category \mathcal{D} of \mathcal{C} (under *both* double category structures in \mathcal{C}) such that $(\mathcal{D}, \phi|_{\mathcal{D}}, c|_{\mathcal{D}}, k|_{\mathcal{D}})$ is also a pre-ACGW-category. We say that \mathcal{D} is *full* if the vertical (resp. horizontal) category of \mathcal{D} is a full subcategory of the vertical (resp. horizontal) category of \mathcal{C} .

Definition 5.6. An *ACGW-category* is a pre-ACGW-category $(\mathcal{C}, \phi, c, k)$ such that the following condition holds:

(PP) Restricted pushouts exist in \mathcal{M} . These are compatible with cokernels, in the sense that a restricted pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow g' \\ C & \xrightarrow{f'} & B \star_A C \end{array}$$

induces an isomorphism $A^{c/B} \xrightarrow{\cong} C^{c/(B \star_A C)}$. In addition, restricted pushouts are compatible with distinguished squares in the sense that given a diagram

$$\begin{array}{ccccc} C & \longleftarrow & A & \longrightarrow & B \\ \downarrow \circ & & \downarrow \circ & & \downarrow \circ \\ C' & \longleftarrow & A' & \longrightarrow & B' \end{array}$$

there is an induced map $B \star_A C \rightarrow B' \star_{A'} C'$ such that the two induced squares are distinguished. These maps are compatible with compositions of distinguished squares.

The dual statement for e-morphisms holds as well.

The definition of \star implies that it behaves functorially like a pushout, in the sense that given a diagram

$$C \longleftarrow A \xrightarrow{f_1} B \xrightarrow{f_2} B'$$

it follows that $(f_2 f_1)' = f_2' f_1'$.

Example 5.7. Let \mathcal{A} be an abelian category. Then \mathcal{A} defines an ACGW-category for which \mathcal{M} is the category of monomorphisms, \mathcal{E} is the opposite category of the epimorphisms, distinguished squares are stable squares and pseudo-commutative squares are commutative squares. Here, the “mixed pullback” of a diagram

$$A \twoheadrightarrow B \longleftarrow \circ C$$

is the factorization of the morphism $A \rightarrow C$ into an epic followed by a monic.

Axiom (S) translates to the following observation. Assuming that we are working in Mod_R , let C be an R -module, and A and B be submodules of C . Then $A \times_C B$ is $A \cap B$. Then $X = A + B$, and the square

$$\begin{array}{ccc}
 C/(A \cap B) & \longrightarrow & C/B \\
 \downarrow & & \downarrow \\
 C/A & \longrightarrow & C/X
 \end{array}$$

is a pullback square.

(PP) corresponds to the fact that an abelian category has all pushouts of monics, and such pushouts preserve cokernels.

Example 5.8. The category **Var** is a pre-ACGW-category. Here we define the pseudo-commutative squares to be the pullback squares.

We check the axioms in turn. Axiom (P) holds because varieties are closed under pullbacks. In order to check Axiom (U) it suffices to check that given a variety X and an open subvariety U and a closed subvariety Z , we have

$$Z \setminus (Z \cap (X \setminus U)) \cong U \cap ((X \setminus Z) \cap U).$$

This is true because it is true in the underlying topological spaces, where each one is simply $Z \times_X U$. Axiom (S) holds because it holds in the underlying topological spaces.

Counterexample 5.9. The CGW-category $\mathbf{Var}_{/k}^{sm}$ is *not* a pre-ACGW-category, since it is possible that the intersection of smooth subvarieties is not smooth. This means that the m-morphisms are not closed under pullbacks.

Example 5.10. The category \mathbf{Sch}_{rf} is an ACGW-category, with the pseudo-commutative squares being pullback squares. With this definition we can consider **Var** a pre-ACGW-subcategory of the ACGW-category \mathbf{Sch}_{rf} . That Axioms (P), (U), and (S) hold follows identically as for the case of varieties.

Thus it remains to check Axiom (PP), in particular that \star -products exist. The pushout of schemes along open immersions produces a square of open immersions by the definition of a scheme; the pushout of schemes along closed immersion produces a square of closed immersions of schemes by [11, Corollary 3.9]. These are *not* pushouts in the categories of closed/open immersions; these are pushouts in the entire category of schemes. That this satisfies the conditions of (PP) follows from the universal property of pushouts.

We now consider an example that will be used in Section 9.

Example 5.11. Let G be a discrete group, and consider the category $\mathbf{FinSet} \wr G$, with

objects: finite sets, and

morphisms: $S \longrightarrow T$ is a pair of functions $(f: S \longrightarrow T, f': S \longrightarrow G)$.

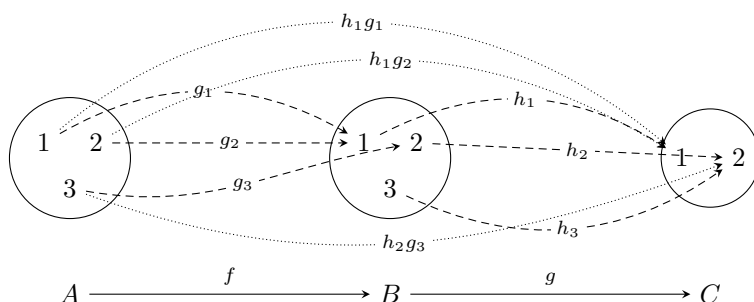
A composition of morphisms $(f, f'): S \rightarrow T$ and $(g, g'): T \rightarrow U$ is given by the pair consisting of $g \circ f$ and the composition

$$S \longrightarrow \Gamma_f \subseteq S \times T \cong T \times S \xrightarrow{g' \times f'} G \times G \xrightarrow{\mu} G,$$

where μ is the composition in G .

More informally, we think of a morphism $S \rightarrow T$ in $\mathbf{FinSet} \wr G$ as a map of finite sets $S \rightarrow T$ together with a decoration by elements of G on each element of S . When we compose two such morphisms, we decorate each element by the multiplication of the two elements that it was decorated with in the composition: the decoration of the original element in the first morphism, and the decoration of its image in the second morphism. The swap in the definition is necessary because composition of morphisms acts on the left, rather than the right.

This can be demonstrated with the following picture:



Here, $A, B, C \in \mathbf{FinSet} \wr G$ are sets, illustrated by the elements in the circles. The dashed lines above morphisms f and g illustrate where elements map under f and g , together with decorations. The labeled dotted lines are the data of gf .

When G is trivial $\mathbf{FinSet} \wr G \cong \mathbf{FinSet}$. By forgetting the decoration we get a functor $\mathbf{FinSet} \wr G \rightarrow \mathbf{FinSet}$.

We define an ACGW-structure on $\mathbf{FinSet} \wr G$ by declaring the e-morphisms and m-morphisms to both be all maps which are injective on the underlying sets, and declare a square to be distinguished if it commutes in the ambient category and if it is distinguished when mapped down to \mathbf{FinSet} . This makes $\mathbf{FinSet} \wr G$ an ACGW-category.

We finish this section with a couple of technical lemmas which will be useful later.

Lemma 5.12. *Let \mathcal{C} be a pre-ACGW category. Given a diagram*

$$\begin{array}{ccccc}
 C & \circ \longrightarrow & B & \rhd \longrightarrow & A \\
 \downarrow & & \downarrow & \circ & \downarrow \\
 C' & \circ \longrightarrow & B' & \rhd \longrightarrow & A'
 \end{array} \tag{5.13}$$

where $C \cong C' \times_{B'} B$ there exists a cube

$$\begin{array}{ccccc}
 C & \rhd \longrightarrow & D & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & B & \rhd \longrightarrow & A & \\
 \downarrow & & \downarrow & & \downarrow \\
 C' & \rhd \longrightarrow & D' & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & B' & \rhd \longrightarrow & A' &
 \end{array}$$

where the top and bottom squares are distinguished, the left and right squares are pull-backs, and the front and back face are pseudo-commutative.

The statement with the roles of e -morphisms and m -morphisms swapped also holds.

Proof. Let

$$D = (C^{k/B})^{c/A} \quad \text{and} \quad D' = ((C')^{k/B'})^{c/A'}$$

Applying c^{-1} to the left-hand square in (5.13) produces a diagram

$$\begin{array}{ccccc}
 C^{k/B} & \rhd \longrightarrow & B & \rhd \longrightarrow & A \\
 \downarrow & \circ & \downarrow & \circ & \downarrow \\
 (C')^{k/B'} & \rhd \longrightarrow & B' & \rhd \longrightarrow & A'
 \end{array}$$

which corresponds, under c , to the pullback square on the right of the cube. Lemma 2.9 shows that the squares on the top and bottom of the cube must be distinguished. To finish the proof of the lemma it remains to check that the back face of the cube is distinguished. To prove this it suffices to check that, after applying c to the m -morphisms in the diagram, it corresponds to a pullback square. This is a straightforward diagram chase using the fact that all morphisms are monic. \square

Lemma 5.14. *Let \mathcal{C} be a pre-ACGW category. In any pseudo-commutative square*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \circ \downarrow & \circ & \downarrow \circ \\ C & \xrightarrow{f'} & D \end{array}$$

if f' is an isomorphism, so is f .

Proof. Apply k vertically. This produces a pullback square

$$\begin{array}{ccc} A^k & \xrightarrow{(f')^k} & B^k \\ \downarrow & & \downarrow \\ C & \xrightarrow{f} & D \end{array}$$

Since f' is an isomorphism, $(f')^k$ must be, as well. Thus the pseudo-commutative square is mapped to an isomorphism inside $\text{Ar}_\times \mathcal{M}$; in particular, both horizontal morphisms in the pseudo-commutative square must be isomorphisms. Thus f is an isomorphism, as desired. \square

6. Dévissage

We can now prove a direct analog to Quillen's dévissage [10, Theorem 5.4]. Analogously to the case of exact and abelian categories, the K -theory of an ACGW-category is defined to be the K -theory of the underlying CGW-category.

As the definition of “creation of colimits” appears to differ from context to context we include the definition needed for the next theorem here:

Definition 6.1. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ creates restricted pushouts if for every diagram

$$B \longleftarrow A \longrightarrow C$$

in \mathcal{C} , if

$$F(B) \longleftarrow F(A) \longrightarrow F(C)$$

has a restricted pushout in \mathcal{D} , then there exists a $D \in \mathcal{C}$ such that D is the pushout of the original diagram, and $F(D)$ is the pushout of its image under F .

Theorem 6.2. *Let \mathcal{A} be a full pre-ACGW-subcategory of the pre-ACGW-category $(\mathcal{B}, \phi, c, k)$, closed under subobjects and quotients (see Definition 2.12), such that the*

inclusion $\mathcal{A} \cap \mathcal{E} \longrightarrow \mathcal{E}$ creates restricted pushouts. Suppose that for all objects $B \in \mathcal{B}$ there is a sequence

$$\emptyset = B_0 \rightrightarrows B_1 \rightrightarrows \cdots \rightrightarrows B_n = B$$

such that B_{i-1}^{c/B_i} is in \mathcal{A} for all $i = 1, \dots, n$. Then the inclusion functor $\mathcal{A} \hookrightarrow \mathcal{B}$ induces an equivalence $K(\mathcal{A}) \longrightarrow K(\mathcal{B})$.

Proof. The proof proceeds exactly as in [10]. Let $\iota: \mathcal{A} \longrightarrow \mathcal{B}$ be the inclusion of \mathcal{A} into \mathcal{B} . We would like ι to give a homotopy equivalence

$$BQA \xrightarrow{BQ\iota} BQB.$$

By Quillen's Theorem A it is enough to show that $Q\iota_{/B}$ is contractible for any $B \in \mathcal{B}$. Since \mathcal{A} is closed under subobjects, $Q\iota_{/B}$ is the full subcategory of $QB_{/B}$ of those objects

$$A_1 \circ \longrightarrow B_2 \rightrightarrows B$$

where $A_1 \in \mathcal{A}$. By Lemma 4.6, $QB_{/B}$ is a preorder, and thus $Q\iota_{/B}$ is also a preorder.

By the hypothesis of the theorem, there exists a sequence

$$\emptyset = B_0 \rightrightarrows B_1 \rightrightarrows \cdots \rightrightarrows B_n = B$$

with $B_{i-1}^{c/B_i} \in \mathcal{A}$ for all $i = 1, \dots, n$. We prove that $Q\iota_{/B_n}$ is contractible by induction on n .

We have $B_1 \in \mathcal{A}$; in this case $Q\iota_{/B_1}$ is contractible, since it has the terminal object $B_1 \circ \rightrightarrows B_1 \rightrightarrows B_1$.

To prove the inductive step it suffices to show that for any $h: B \rightrightarrows B'$ with $B^c \in \mathcal{A}$ the map $Q\iota_{/B} \longrightarrow Q\iota_{/B'}$ induced by postcomposition is a homotopy equivalence. Let $L_B^A \mathcal{B}$ be the full subcategory of $L_B \mathcal{B}$ containing those objects $B_1 \rightrightarrows B_2 \rightrightarrows B$ where $B_1^{c/B_2} \in \mathcal{A}$. By Lemma 4.6 it suffices to check that the functor $\iota: L_B^A \mathcal{B} \longrightarrow L_{B'}^A \mathcal{B}$ induced by postcomposition with h is a homotopy equivalence.

Let $B_1 \rightrightarrows B_2 \rightrightarrows B'$ be any object of $L_{B'}^A \mathcal{B}$. We have the diagram

$$\begin{array}{ccccc} B_1 \times_{B'} B & \xrightarrow{g'} & B_2 \times_{B'} B & \rightrightarrows & B \\ \downarrow & & \downarrow & & \downarrow h \\ B_1 & \rightrightarrows & B_2 & \rightrightarrows & B' \end{array}$$

where both squares are pullback squares. We define functors

$$r: L_{B'}^A \mathcal{B} \longrightarrow L_B^A \mathcal{B} \quad r(B_1 \rightrightarrows B_2 \rightrightarrows B') = B_1 \times_{B'} B \xrightarrow{g'} B_2 \times_{B'} B \rightrightarrows B.$$

$$s: L_{B'}^{\mathcal{A}} \mathcal{B} \longrightarrow L_B^{\mathcal{A}} \mathcal{B} \quad s(B_1 \rhd B_2 \rhd B') = B_1 \times_{B'} B \rhd B_2 \rhd B'.$$

If s is well-defined (so $(B_1 \times_{B'} B)^{c/B_2} \in \mathcal{A}$) then so is r , because $(B_1 \times_{B'} B)^{c/g'}$ is a subobject of $(B_1 \times_{B'} B)^{c/B_2}$. Thus we just need to check that s is well-defined.

First, by Axiom (U) there exists a map $(B_2 \times_{B'} B)^{c/B_2} \rhd B^{c/B'}$; since $B^{c/B'} \in \mathcal{A}$, it follows that $(B_2 \times_{B'} B)^{c/B_2}$ must be, as well. Now by Axiom (S), $(B_1 \times_{B'} B)^{c/B_2} \cong B_1^{c/B_2} \star_{Y^{c/B_2}} (B_2 \times_{B'} B)^{c/B_2}$, where $Y = B_1 \star_{B_1 \times_{B'} B} (B_2 \times_{B'} B)$, which exists by Axiom (S); since the inclusion $\mathcal{A} \cap \mathcal{E} \longrightarrow \mathcal{E}$ creates restricted pushouts, if each component of this pushout is in \mathcal{A} , then so is $(B_1 \times_{B'} B)^{c/B_2}$. By assumption $B_1^{c/B_2} \in \mathcal{A}$ and by the above $(B_2 \times_{B'} B)^{c/B_2} \in \mathcal{A}$, so Y^{c/B_2} is also in \mathcal{A} (as \mathcal{A} is closed under subobjects). Thus $(B_1 \times_{B'} B)^{c/B_2} \in \mathcal{A}$, and s is well-defined, as desired.

Redrawing the above diagram, we have the following diagram:

$$\begin{array}{ccccc}
 B_1 & \xrightarrow{\quad} & B_2 & \xrightarrow{\quad} & B' \\
 \uparrow & & \parallel & & \parallel \\
 B_1 \times_{B'} B & \xrightarrow{\quad} & B_2 & \xrightarrow{\quad} & B' \\
 \parallel & & \uparrow & & \parallel \\
 B_1 \times_{B'} B & \xrightarrow{\quad} & B_2 \times_{B'} B & \xrightarrow{\quad} & B' \\
 & & & & \parallel \\
 & & & & \iota r
 \end{array}
 \quad
 \begin{array}{c}
 1_{L_{B'}^{\mathcal{A}} \mathcal{B}} \\
 \parallel \\
 \downarrow \\
 s \\
 \parallel \\
 \uparrow \\
 \iota r
 \end{array}$$

The upper row of squares gives a natural transformation $1_{L_{B'}^{\mathcal{A}} \mathcal{B}} \Rightarrow s$; the lower row gives a natural transformation $\iota r \Rightarrow s$. Since natural transformations realize to homotopies, we see that ιr is homotopic to the identity on $L_{B'}^{\mathcal{A}} \mathcal{B}$. On the other hand, $r\iota$ is equal to the identity on $L_B^{\mathcal{A}} \mathcal{B}$, so these produce a homotopy equivalence of spaces, as desired. \square

We can now apply this theorem to compare the K -theory of varieties to the K -theory of reduced schemes of finite type.

Example 6.3. We use the dual of Theorem 6.2 to prove that $K(\mathbf{Var}) \simeq K(\mathbf{Sch}_{r.f.})$.

\mathbf{Var} is a subcategory of $\mathbf{Sch}_{r.f.}$ closed under subobjects and quotients; the inclusion $\mathbf{Var} \cap \mathcal{M} \longrightarrow \mathcal{M}$ creates pushouts since the pushout of varieties along closed immersions is a variety [11, Cor. 3.9]. To apply the theorem we must show that for every reduced scheme of finite type X there exists a filtration

$$X_0 \circ \longrightarrow X_1 \circ \longrightarrow \cdots \circ \longrightarrow X_n = X$$

such that $X_i \setminus X_{i-1}$ is a variety for all i . Since X is of finite type there exists a finite cover of X by affine opens U_1, \dots, U_n ; each of these is reduced since X is and separated because each is affine. We then define

$$X_i = \bigcup_{j=1}^i U_j.$$

This gives a finite open filtration of X ; it remains to show that $X_i \setminus X_{i-1}$ is a variety for all i . Note that $X_i \setminus X_{i-1} = U_i \setminus \bigcup_{j=1}^{i-1} (U_j \cap U_i)$. This is reduced, separated and of finite type, and is thus a variety, as desired.

7. Relationship with the S_\bullet -construction

In this section we relate our Q -construction to a variation of the S_\bullet -construction of Waldhausen [14]. We will show that the Q -construction is equivalent to the construction defined for \mathbf{Var}_k in [2]. As the S_\bullet -construction applied to an abelian category is not abelian, it is unreasonable to expect that in all cases it will be possible to iterate the construction. However, as the S_\bullet -construction for ACGW-categories produces a CGW-category, it is possible to iterate it twice. It turns out that this is sufficient to prove a cofiber sequence and the relationship to the Q -construction.

Remark 7.1. In the interest of keeping this section short and readable, we do not state definitions or results in the full generality that would be analogous to Waldhausen's exposition. Instead, we restrict attention to the special cases of interest to us.

We begin by presenting the definition of the S_\bullet construction for CGW-categories.

Definition 7.2. Let \mathcal{C} be a CGW-category. Define $S_n\mathcal{C}$ to be the simplicial set with n simplices $S_n\mathcal{C}$ given by diagrams in the double category \mathcal{C}

$$\begin{array}{ccccccc}
 C_{00} & \rightrightarrows & C_{01} & \rightrightarrows & C_{02} & \rightrightarrows & \cdots \rightrightarrows C_{0(n-1)} & \rightrightarrows & C_{0n} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \circ & & \circ & & \circ & & \circ \\
 & & C_{11} & \rightrightarrows & C_{12} & \rightrightarrows & \cdots \rightrightarrows & C_{1(n-1)} & \rightrightarrows & C_{1n} \\
 & & & & & & & & & \uparrow \\
 & & & & & & & & & \circ \\
 & & & & & & & & & \vdots \\
 & & & & & & & & & \circ \\
 & & & & & & & & & C_{nn}
 \end{array}$$

such that

- (1) $C_{ii} = \emptyset$ for all i , and
- (2) Every subdiagram

$$\begin{array}{ccc}
 C_{ki} & \xrightarrow{\quad} & C_{kl} \\
 \uparrow & \square & \uparrow \\
 C_{ji} & \xrightarrow{\quad} & C_{jl}
 \end{array}$$

for $k < j$ and $i < l$ is distinguished.

The face and degeneracies are defined as in the usual S_{\bullet} -construction: the i th face map is deleting the i th row and i th column, and the degeneracies are given by repetition. (For more on the traditional S_{\bullet} -construction, see [14, Section 1.3]; for a more explicit description of how this works in the case of varieties, see the \tilde{S}_{\bullet} -construction in [2, Definition 3.31].)

Remark 7.3. The arrow directions in the diagram are chosen to agree with existing examples.

Example 7.4. When $\mathcal{C} = \mathbf{Var}$, then $S\mathcal{C}$ is exactly the \tilde{S}_{\bullet} construction of [2, Def. 3.31].

Definition 7.5. Given a CGW category $(\mathcal{C}, \mathcal{M}, \mathcal{E})$ define

$$K^S(\mathcal{C}) := \Omega|S\mathcal{C}|$$

Remark 7.6. When \mathcal{C} is, for example, an exact category this agrees with Waldhausen's S_{\bullet} -construction by Corollary 2 following [14, Lem. 1.4.1].

Remark 7.7. In [2], the author introduced the \tilde{S}_{\bullet} construction, which is a version of the Waldhausen construction that works on SW -categories [2, Defn 3.23]. These categories are meant to encode cutting and pasting, just as CGW categories do. In fact, in that paper there are three notions of such categories that appear: 1. pre-subtractive category, 2. subtractive categories and 3. SW -categories. Pre-subtractive are closely related to CGW-categories; they are categories where one can define a higher geometric object that encodes cutting and pasting. Subtractive categories correspond to ACGW-categories: certain pushouts and pullbacks are required to exist. Finally, SW -categories, like Waldhausen categories, are allowed to have weak equivalences other than isomorphisms. Subtractive categories satisfy the axioms for ACGW-categories, and in this case the corresponding S_{\bullet} constructions are equivalent and, in fact, equal; in such situations we will say that the ACGW-category *arises from a subtractive category*. An ACGW-category where the distinguished squares are *cartesian* in the underlying category \mathcal{A} , is an SW -category, and we may use the full machinery of SW -categories. This is true, for example, for $\mathbf{Sch}_{rf,/k}$ and \mathbf{FinSet}_{*} .

As expected, this new definition of K -theory is equivalent to the original one.

Theorem 7.8. *Let $(\mathcal{C}, \mathcal{M}, \mathcal{E})$ be a CGW category. Then there is a weak equivalence of topological spaces*

$$K^S(\mathcal{C}) \xrightarrow{\sim} K(\mathcal{C})$$

induced by a map of simplicial sets $S.\mathcal{C} \rightarrow Q\mathcal{C}$.

The equivalence above is one of topological spaces, not of infinite loop spaces or spectra. While in many cases the equivalences are equivalences of infinite loop spaces, that statement is not true in this generality (for example, smooth varieties cannot be delooped in the way described in [2, Sec. 5] since it relies on the existence of pushouts). We hope to address deloopings in future work.

In order to make the proof of Theorem 7.8 as formally similar to the classical “ $S. = Q$ ” theorem due to Waldhausen ([14, Sect 1.9]) we introduce the following definition.

Definition 7.9. Let $(\mathcal{C}, \mathcal{M}, \mathcal{E})$ be a CGW-category. We define $iS_n\mathcal{C}$ to be a category with

objects: Elements of $S_n\mathcal{C}$

morphisms: A collection of isomorphisms $f_{ij}: C_{ij} \rightarrow C'_{ij}$ in \mathcal{M} such that the diagrams

$$\begin{array}{ccc} C_{ik} & \xrightarrow{\quad} & C_{lk} \\ f_{ik} \downarrow & & \downarrow f_{lk} \\ C'_{ik} & \xrightarrow{\quad} & C'_{lk} \end{array} \qquad \begin{array}{ccc} C_{ij} & \circ \longrightarrow & C_{ik} \\ \phi(f_{ij}) \downarrow & & \downarrow \phi(f_{ik}) \\ C'_{ij} & \circ \longrightarrow & C'_{ik} \end{array}$$

commute in \mathcal{M} and \mathcal{E} , respectively.

Remark 7.10. We could have also used the isomorphisms in \mathcal{E} in the above definition. The isomorphism ϕ guarantees the resulting definition is categorically equivalent to the one above.

Proof of Theorem 7.8. The definitions are designed to make this statement work exactly as in Waldhausen [14, Sect 1.9]. Let $iQ\mathcal{C}$ be the double category where vertical morphisms are isomorphisms in $Q\mathcal{C}$ and horizontal morphisms are morphisms in $Q\mathcal{C}$. Taking the nerve in the horizontal direction, we obtain a simplicial category $iQ.\mathcal{C}$. There is an equivalence $|Q\mathcal{C}| \xrightarrow{\sim} |iQ.\mathcal{C}|$ given by Waldhausen’s Swallowing Lemma [14, Lem. 1.6.5].

Similarly, let $\mathbf{sd} iS.\mathcal{C}$ be the simplicial category we obtain from edgewise subdividing the $S.$ -construction (for an introduction and proof of the properties of edgewise subdivision see [12, App. 1]). There is now a functor

$$\mathbf{sd} iS.\mathcal{C} \longrightarrow iQ.\mathcal{C}$$

defined as in [14, Sect. 1.9]. It is a level-wise categorical equivalence, and thus induces a weak equivalence of bisimplicial sets, by the usual realization lemma (see, e.g. [13, Lem. 5.1]).

Altogether we have

$$|iS.C| \xleftarrow{\cong} |\mathbf{sd} \, iS.C| \xrightarrow{\sim} |iQ.C| \xleftarrow{\sim} |QC|$$

where the first map, a homeomorphism, is given by [12, Prop. A.1].

Finally, we have the commutative diagram

$$\begin{array}{ccccc} |S.C| & \xleftarrow{\cong} & |\mathbf{sd} S.C| & \longrightarrow & |QC| \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ |iS.C| & \xleftarrow{\sim} & |\mathbf{sd} iS.C| & \xrightarrow{\sim} & |iQC| \end{array}$$

where we know that all of the indicated arrows are weak equivalences, and so the remaining arrow is a weak equivalence. The composite across the top $|S.C| \longrightarrow |QC|$ is thus a weak equivalence. Upon taking loop spaces this gives the statement of the theorem. \square

As a corollary we can now show that Dévissage works for SW-categories that are the ambient categories of pre-ACGW-categories.

Corollary 7.11. *Let \mathcal{A} and \mathcal{C} be pre-ACGW-categories satisfying the conditions of Theorem 6.2. Then the map*

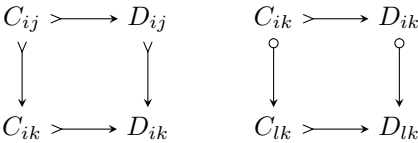
$$K^S(\mathcal{A}) \longrightarrow K^S(\mathcal{C})$$

is an equivalence. In particular, if \mathcal{A} and \mathcal{C} are constructed from SW-categories [2] then the induced maps on K -theories of the SW-categories are also an equivalence.

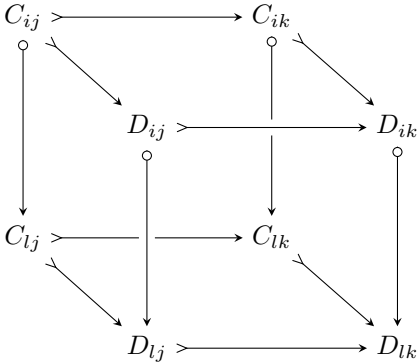
We now use Waldhausen's approach to define relative K -theory (Definition 7.14) and prove a homotopy fiber sequence between the relative K -theory and ordinary K -theories (Proposition 7.15, analogous to [14, Prop. 1.5.5]). These will be needed in Section 9 to prove that the previous constructions of the K -theory of varieties are equivalent.

Definition 7.12. Let \mathcal{A} be an ACGW-category. We define a CGW-structure on $S_n \mathcal{A}$. We give $S_n \mathcal{A}$ distinguished families of \mathcal{M} and \mathcal{E} morphisms as follows.

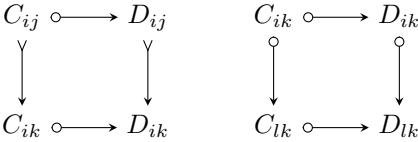
\mathcal{M} -morphisms: A collection of maps $f_{ij}: C_{ij} \twoheadrightarrow D_{ij}$ in \mathcal{M} such that



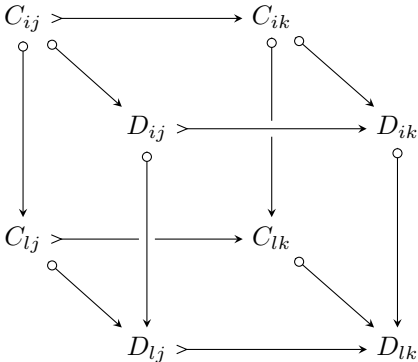
are in $\text{Ar}_\times \mathcal{M}$ and $\text{Ar}_\circ \mathcal{M}$, respectively. We visualize these as cubes



\mathcal{E} -morphisms: A collection of maps $g_{ij}: C_{ij} \longrightarrow D_{ij}$ in \mathcal{E} such that



are in $\text{Ar}_\circ \mathcal{E}$ and $\text{Ar}_\times \mathcal{E}$, respectively. We visualize these as cubes



Distinguished squares: Let $C_{..}, D_{..}, E_{..}, F_{..}$ denote objects in $S_n\mathcal{A}$. A distinguished square consists of \mathcal{M} -morphisms $C_{..} \rightrightarrows D_{..}$, $E_{..} \rightrightarrows F_{..}$ and \mathcal{E} -morphisms $C_{..} \circ \rightarrow E_{..}$, $D_{..} \circ \rightarrow F_{..}$ such that each

$$\begin{array}{ccc} C_{ij} & \rightrightarrows & D_{ij} \\ \circ \downarrow & & \circ \downarrow \\ E_{ij} & \rightrightarrows & F_{ij} \end{array}$$

is distinguished

The functors ϕ, c, k : The isomorphism ϕ is induced from the isomorphisms on \mathcal{A} . The functors c, k are defined pointwise. The fact that the resulting squares are as described is guaranteed by Definition 5.5, Axiom (U).

We now describe the enhanced double category structure on $S_n\mathcal{A}$.

Enhanced Structure: The enhanced double category structure on $S_n\mathcal{A}$, we define pseudo-commutative squares pointwise. That is, let $C_{..}, D_{..}, E_{..}, F_{..}$ denote objects in $S_n\mathcal{A}$. An element of $\text{Ar}_{\circlearrowleft} S_n\mathcal{A}$ is given by $C_{..} \rightrightarrows D_{..}$ and $E_{..} \circ \rightarrow F_{..}$ and $C_{..} \circ \rightarrow E_{..}$ and $D_{..} \rightrightarrows F_{..}$ such that each

$$\begin{array}{ccc} C_{ij} & \rightrightarrows & D_{ij} \\ \circ \downarrow & & \circ \downarrow \\ E_{ij} & \rightrightarrows & F_{ij} \end{array}$$

is in $\text{Ar}_{\circlearrowleft} \mathcal{M}$. The 2-cells $\text{Ar}_{\times} \mathcal{M}_{S_n\mathcal{A}}$, $\text{Ar}_{\circlearrowleft} \mathcal{E}_{S_n\mathcal{A}}$ and $\text{Ar}_{\times} \mathcal{E}_{S_n\mathcal{A}}$ are defined similarly.

With the definitions above, the following is tedious, but straightforward. Indeed, the definitions were chosen to make this lemma true.

Lemma 7.13. $S_n\mathcal{A}$, with the structure from Definition 7.12, satisfies all of the axioms of a CGW-category except for Axiom (A). In particular, the $S_{..}$ -construction can be applied to $S_{..}\mathcal{A}$.

Using this we can define the relative $S_{..}$ -construction.

Definition 7.14. A pair $(\mathcal{B}, \mathcal{A})$ of an ACGW-category \mathcal{B} and a sub-ACGW-category \mathcal{A} is *good* if \mathcal{A} is full and if for every isomorphism $B \xrightarrow{\cong} B'$ in \mathcal{B} , B is in \mathcal{A} if and only if B' is. For a good pair $(\mathcal{B}, \mathcal{A})$, define $S_n(\mathcal{B}, \mathcal{A})$ via the pullback

$$\begin{array}{ccc} S_n(\mathcal{B}, \mathcal{A}) & \hookrightarrow & S_{n+1}\mathcal{B} \\ \downarrow & & \downarrow d_0 \\ S_n\mathcal{A} & \hookrightarrow & S_n\mathcal{B}. \end{array}$$

In other words, $S_n(\mathcal{B}, \mathcal{A})$ is the full subcategory of those objects C_{\bullet} in $S_{n+1}\mathcal{B}$ in which $C_{ij} \in \mathcal{A}$ for all $i > 0$.

The category $S_n(\mathcal{B}, \mathcal{A})$ inherits the structure of a CGW-category. The *relative K-theory* of $(\mathcal{B}, \mathcal{A})$ is defined to be

$$K(\mathcal{B}, \mathcal{A}) := \Omega[S_{\bullet}(\mathcal{B}, \mathcal{A})]$$

To conclude the section we prove an analog of additivity for the Q -construction and use it to construct a homotopy fiber sequence relating relative K -theory to the K -theory of the component categories.

Proposition 7.15 (*Additivity and Cofiber sequence*). *Let $(\mathcal{B}, \mathcal{A})$ be a good pair which arises from a subtractive category and a full subtractive subcategory. Then there exists a weak equivalence*

$$QS_n(\mathcal{B}, \mathcal{A}) \xrightarrow{\sim} Q\mathcal{B} \times QS_n\mathcal{A}.$$

Moreover, the following is a homotopy fiber sequence after geometric realization:

$$Q\mathcal{B} \longrightarrow QS_{\bullet}(\mathcal{B}, \mathcal{A}) \longrightarrow QS_{\bullet}\mathcal{A}.$$

Proof. For any object C_{\bullet} in $S_n\mathcal{B}$, write $C_{\bullet-1, -1}$ for the object in $S_{n-1}\mathcal{B}$ containing all elements with positive indices. When $C_{\bullet} \in S_n(\mathcal{B}, \mathcal{A})$, $C_{\bullet-1, -1}$ can be considered to lie in $S_n\mathcal{A}$.

There are functors

$$\begin{array}{ccc} S_n(\mathcal{B}, \mathcal{A}) & \xrightarrow{F'} & \mathcal{B} \\ C_{\bullet} & \longmapsto & C_{0,0} \end{array} \quad \text{and} \quad \begin{array}{ccc} S_n(\mathcal{B}, \mathcal{A}) & \xrightarrow{F''} & S_n\mathcal{A} \\ C_{\bullet} & \longmapsto & C_{\bullet-1, \bullet-1} \end{array}$$

which induce a map $f: QS_n(\mathcal{B}, \mathcal{A}) \rightarrow Q\mathcal{B} \times QS_n\mathcal{A}$. This map is a coretraction, where the reverse map is constructed using the subtractive structure of \mathcal{B} . These fit into a commutative diagram

$$\begin{array}{ccccc}
 Q\mathcal{B} & \longrightarrow & QS_n(\mathcal{B}, \mathcal{A}) & \longrightarrow & QS_n\mathcal{A} \\
 \parallel & & \downarrow f & & \parallel \\
 Q\mathcal{B} & \longrightarrow & Q\mathcal{B} \times QS_n\mathcal{A} & \longrightarrow & QS_n\mathcal{A}
 \end{array}$$

in which the bottom row is a homotopy fiber sequence (in fact a trivial fiber sequence), $Q\mathcal{B} \rightarrow QS_n\mathcal{A}$ is constant, and $QS_n\mathcal{A}$ is connected. Thus, by [13, Prop. 5.2], to prove that the geometric realization of

$$Q\mathcal{B} \longrightarrow QS_n(\mathcal{B}, \mathcal{A}) \longrightarrow QS_n\mathcal{A}$$

is a homotopy fiber sequence it suffices to check that f is a weak equivalence; thus the second part of the proposition follows from the first.

Consider the following commutative diagram:

$$\begin{array}{ccc}
 QS_n(\mathcal{B}, \mathcal{A}) & \xrightarrow{f} & Q\mathcal{B} \times QS_n\mathcal{A} \\
 \wr \uparrow & & \uparrow \wr \\
 S_n(\mathcal{B}, \mathcal{A}) & \xrightarrow{f'} & S_n\mathcal{B} \times S_n\mathcal{A}
 \end{array}$$

The vertical arrows are weak equivalences by Theorem 7.8 and [13, Lemma. 5.1]. Thus f is a weak equivalence if and only if f' is. By assumption, \mathcal{A} and \mathcal{B} arise from SW -categories, and as S -constructions for ACGW-categories that arise from SW -categories agree by definition, f' is a weak equivalence by [2, Proposition 5.3]. Thus f' must also be a weak equivalence, and the proposition follows. \square

Remark 7.16. In fact, the assumption that \mathcal{A} and \mathcal{B} arise from SW -categories can be significantly weakened; the only assumption necessary is that axioms (A) and (PP) hold sufficiently functorially. In order to check this it is necessary to check that all steps in the proofs of [2, Theorem 4.5, Proposition 5.3] work analogously in ACGW-categories. However, as this would significantly disrupt the flow of this paper (and not add significantly to understanding) we omit this more general result here; instead, we restrict solely to the case in which it is needed later in the paper.

8. Localization of ACGW-categories

In this section we state the new definition necessary to state the localization theorem. The goal of a localization theorem is to identify the homotopy cofiber of the map $K(\mathcal{A}) \rightarrow K(\mathcal{C})$ induced by the inclusion of a sub-CGW-category. In order to prove the cleanest version of the theorem it is necessary to make extra assumptions about the structure of \mathcal{A} and \mathcal{C} , and thus passage to ACGW-categories is necessary.

In addition, in order to ensure that objects in \mathcal{A} can be worked with easily, we assume some nice closure properties on \mathcal{A} (similar to the closure properties assumed by Quillen).

Let $\mathcal{C} = (\mathcal{E}, \mathcal{M})$ be an ACGW-category, and let \mathcal{A} be a full ACGW-subcategory closed under subobjects, quotients and extensions, as defined in Definition 2.12. The first step towards stating localization is identifying the CGW-category whose K -theory we hope to be the cofiber.

The idea of the localized category is to define morphisms $A \rightarrow B$ to be morphisms defined from a “dense subset” to a “dense subset,” where “dense” is defined to be subobjects/quotients whose cokernel/kernel are in the subcategory \mathcal{A} . This is motivated by the definition of monomorphism/epimorphism in an abelian quotient category \mathcal{C}/\mathcal{A} , where (for example), a monomorphism $A \rightarrow B$ in the quotient category is a diagram

$$A \hookrightarrow A' \hookrightarrow B' \twoheadrightarrow B$$

where the cokernel of $A' \hookrightarrow A$ and the kernel of $B' \twoheadrightarrow B$ are both in \mathcal{A} . We can commute the monomorphism and epimorphism past one another (in an epic-monic factorization) to instead write this as a diagram

$$A \hookrightarrow A' \twoheadrightarrow B'' \hookrightarrow B$$

where the cokernel of $A' \hookrightarrow A$ and the kernel of $A' \twoheadrightarrow B''$ must be in \mathcal{A} . Two such diagrams are equivalent when they have a “common refinement” on which they are identical. This is exactly the definition of an m-morphism in the localized category, except that we are allowed to “reduce the size of A ” by both an m-morphism and an e-morphism.

Definition 8.1. Let $A \twoheadrightarrow B$ be a morphism in \mathcal{M} . We write $\twoheadrightarrow\bullet$ if $A^c \in \mathcal{A}$. We define $\bullet\rightarrow$ analogously.

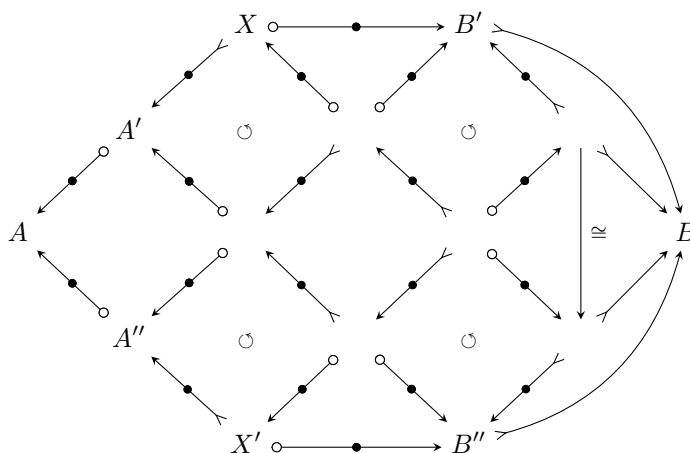
Let $\mathcal{C}\backslash\mathcal{A}$ be the double category with

objects: the objects of \mathcal{C} ,

m-morphisms: A morphism $A \twoheadrightarrow B$ is an equivalence class of diagrams in \mathcal{C}

$$A \leftarrow \bullet \circ A' \leftarrow \bullet \prec X \circ \bullet \rightarrow B' \twoheadrightarrow B.$$

If there exists a diagram in \mathcal{C}



then the two formal compositions around the outside are considered equivalent. The right-most square with the isomorphism in the middle is the same square that determines when two morphisms in QC are equivalent.

Composition is defined via a similar type of diagram, commuting the different types of morphisms past one another.

e-morphisms: A morphism $A \circ \rightarrow B$ is an equivalence class of diagrams in \mathcal{C}

$$A \leftarrow \bullet \leftarrow A' \leftarrow \bullet \leftarrow X \rightarrow \bullet \rightarrow B' \circ \rightarrow B.$$

The equivalence relation between these is defined to be the dual condition to the condition on m-morphisms.

distinguished squares: The distinguished squares are generated by the distinguished squares in \mathcal{C} and axiom (I). For a more detailed description, see Appendix A.

In this section we will often be working with morphisms in $\mathcal{C} \setminus \mathcal{A}$ as represented by diagrams in \mathcal{C} . As these categories have the same objects this can get confusing. To help with this, we denote morphisms in \mathcal{C} by arrows with straight shafts, and morphisms in $\mathcal{C} \setminus \mathcal{A}$ by morphisms with wavy shafts. We can thus say that an m-morphism $A \rightarrow B$ in $\mathcal{C} \setminus \mathcal{A}$ is represented by a diagram

$$A \leftarrow \bullet \leftarrow A' \leftarrow \bullet \leftarrow X \circ \rightarrow B' \rightarrow B$$

in \mathcal{C} .

We define $c: \text{Ar}_{\square} \mathcal{M} \rightarrow \text{Ar}_{\Delta} \mathcal{E}$ on objects by $c(A \rightarrow B) = c_{\mathcal{C}}(B' \rightarrow B)$, and $k: \text{Ar}_{\square} \mathcal{E} \rightarrow \text{Ar}_{\Delta} \mathcal{M}$ by $k(A \rightarrow B) = k_{\mathcal{C}}(B' \circ \rightarrow B)$.

There is a functor of double categories $s: \mathcal{C} \rightarrow \mathcal{C} \setminus \mathcal{A}$ which takes each object to itself and takes every morphism to itself.

Remark 8.2. As currently defined, $\mathcal{C} \backslash \mathcal{A}$ does not have the structure of a CGW-category, as we cannot prove that the definitions of c and k give equivalences of categories. Proving that such a structure exists appears to require a development of a theory of a left calculus of fractions for a double category. As this is far beyond the scope of this paper, we state as a condition of the localization theorem that $\mathcal{C} \backslash \mathcal{A}$ extends to a CGW-category in a fashion compatible with the CGW-structure on \mathcal{C} and the functor $s: \mathcal{C} \rightarrow \mathcal{C} \backslash \mathcal{A}$ and show that this works for our relevant examples. In Proposition A.1 we show that as long as c and k give equivalences of categories, $\mathcal{C} \backslash \mathcal{A}$ is a well-defined CGW-category. In future work we hope to simplify this condition.

If $\mathcal{C} \backslash \mathcal{A}$ is a CGW-category then by definition the functor s is a CGW-functor.

Before turning to the main theorem we revisit the example of the localization of an abelian category in detail, as the above definition is by no means easy to understand.

Example 8.3. Let \mathcal{C} be an abelian category and \mathcal{A} a Serre subcategory, considered as ACGW-categories. Then we claim that $\mathcal{C} \backslash \mathcal{A}$ is exactly the abelian category \mathcal{C}/\mathcal{A} , considered as an ACGW-category. First, consider the monics. A morphism in \mathcal{C} is monic in \mathcal{C}/\mathcal{A} exactly when it can be represented by a zigzag

$$X \xleftarrow{s} Z \xrightarrow{f} Y$$

where the kernel and cokernel of s are in \mathcal{A} , and when the kernel of f is in \mathcal{A} . Writing both s and f in an epic-monic factorization and switching to the notation of CGW-categories, such a monic can be represented by a zigzag

$$X \leftarrow \bullet \prec X' \circ \bullet \rightarrow Z \leftarrow \bullet \circ Y' \rightrightarrows Y.$$

As \mathcal{C} is abelian, e-morphisms are closed under pullbacks (i.e., epimorphisms are closed under pushouts in \mathcal{C}), and thus this representation is equivalent to the representation

$$X \leftarrow \bullet \prec X' \leftarrow \bullet \circ X' \times_Z Y' \circ \bullet \rightarrow Y' \rightrightarrows Y.$$

Using Lemma 2.10 we can swap the order of the two arrows on the left half, to produce a representation

$$X \leftarrow \bullet \circ X'' \leftarrow \bullet \prec X' \times_Z Y' \circ \bullet \rightarrow Y' \rightrightarrows Y,$$

as desired. Given that we can also reverse this construction, we see that the monics (and, analogously, the epics) are as represented.

Since \mathcal{C}/\mathcal{A} is abelian it immediately follows that $\mathcal{C} \backslash \mathcal{A}$ must be a CGW-category.

Before we state the main theorem, we need some auxillary definitions.

Definition 8.4. Let V be an object in $\mathcal{C} \setminus \mathcal{A}$. The category \mathcal{I}_V^m has as its objects pairs (N, ϕ) , where $N \in \mathcal{C}$ and $\phi: sN \dashrightarrow V$ is an isomorphism in $\mathcal{C} \setminus \mathcal{A}$. A morphism $(N, \phi) \rightarrow (N', \phi')$ is an equivalence class of diagrams $g: N \xleftarrow{g_e} Y \xrightarrow{g_m} N'$ (where diagrams are allowed to differ by an isomorphic choice of Y) such that $\phi's(g) = \phi$. Here, $s(g)$ is considered as an isomorphism in $\mathcal{C} \setminus \mathcal{A}$. Composition is defined using mixed pullbacks.

The category \mathcal{I}_V^e is defined analogously with the roles of m-morphisms and e-morphisms swapped.

If \mathcal{I}_V^m is filtered for all V we say that \mathcal{A} is *m-well-represented* in \mathcal{C} . Dually, if \mathcal{I}_V^e is filtered for all V we say that \mathcal{A} is *e-well-represented* in \mathcal{C} .

We think of \mathcal{I}_V^m as the category of representatives inside \mathcal{C} of an isomorphism class of objects in $\mathcal{C} \setminus \mathcal{A}$. When this category is filtered it means that representatives of V can always be chosen compatibly, at least in the m-morphism direction.

Definition 8.5. Suppose that for every diagram

$$A \rightrightarrows B \circ \rightarrow C$$

in \mathcal{C} there exists a pseudo-commutative square

$$\begin{array}{ccc} A' & \rightrightarrows & B \\ \parallel & \circlearrowleft & \downarrow \\ A' & \rightrightarrows & C \end{array}$$

such that $A' \rightrightarrows B$ factors through $A \rightrightarrows B$. Then we say that \mathcal{A} is *m-negligible* in \mathcal{C} . If the same statement holds with the m-morphisms and e-morphisms swapped, we say that \mathcal{A} is *e-negligible* in \mathcal{C} .

Negligibility is a “dual” notion to well-representability. Whereas well-representability states that representatives can always be compatibly combined, negligibility says that certain representatives can be ignored. If \mathcal{A} is m-negligible in \mathcal{C} this means that we never have to think about e-components of morphisms inside QC ; all such morphisms can be represented (up to pseudo-commutative square) purely as an m-morphism.

We are now ready to state the CGW version of localization.

Theorem 8.6. Suppose that \mathcal{C} is an ACGW-category and \mathcal{A} is a sub-ACGW-category satisfying the following conditions:

- (W) \mathcal{A} is m-well-represented or m-negligible in \mathcal{C} and \mathcal{A} is e-well-represented or e-negligible in \mathcal{C} .

(CGW) $\mathcal{C} \backslash \mathcal{A}$ is a CGW-category.

- (E) For two diagrams $A \leftarrow \bullet \circ X \rightarrow \bullet B$ and $A \leftarrow \bullet \circ X' \rightarrow \bullet B$ which represent the same morphism in $\mathcal{C} \backslash \mathcal{A}$ there exists an e -morphism $C \circ \bullet \rightarrow B$ and an isomorphism $\alpha: X \otimes_B C \rightarrow X' \otimes_B C$ such that the induced diagram

$$\begin{array}{ccc} A & \xleftarrow{\bullet} \circ & X \otimes_B C \\ \uparrow \circ & \searrow \alpha & \downarrow \bullet \\ X' \otimes_B C & \xrightarrow{\bullet} & C \end{array}$$

commutes. The same statement holds with e -morphisms and m -morphisms swapped.

Then the sequence

$$K(\mathcal{A}) \longrightarrow K(\mathcal{C}) \longrightarrow K(\mathcal{C} \backslash \mathcal{A})$$

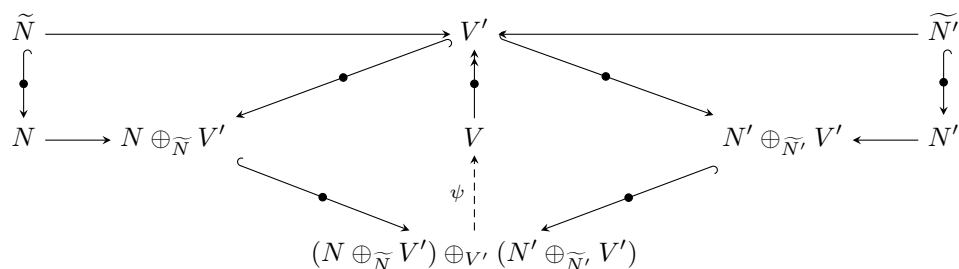
is a homotopy fiber sequence.

We postpone the proof of Theorem 8.6 until Section 10. As mentioned in Remark 8.2, in order for condition (CGW) to hold it suffices to check that c and k (as defined on objects) extend to equivalences of categories. In this section we focus on two applications of the theorem.

The first application is a sanity check, showing that in the case of an abelian category the theorem is the same as Quillen's localization [10, Theorem 5.5].

Example 8.7. Continuing Example 8.3, we show that Theorem 8.6 applies in this example; thus Theorem 8.6 is truly a generalization of Quillen's localization theorem.

Consider condition (W); we will show that \mathcal{A} is both m - and e -well-represented in \mathcal{C} . By symmetry it suffices to check that \mathcal{I}_V^m is filtered. An object $(N, \phi) \in \mathcal{I}_V^m$ is an object $N \in \mathcal{C}$ together with a $\text{mod-}\mathcal{A}$ -isomorphism $N \rightarrow V$; a morphism $(N, \phi) \rightarrow (N', \phi')$ is a morphism $g: N \rightarrow N'$ in \mathcal{C} such that $\phi's(g) = \phi$. Suppose that we are given two morphisms $g, g': (N, \phi) \rightarrow (N', \phi')$. Then the morphism $N' \rightarrow N'/\text{im}(g - g')$ is a $\text{mod-}\mathcal{A}$ -isomorphism which equalizes g and g' ; thus \mathcal{I}_V^m has coequalizers. Now suppose that we are given two objects (N, ϕ) and (N', ϕ') in \mathcal{I}_V^m . Choosing representatives appropriately, these give a diagram in \mathcal{C}



where the bulleted arrows represented mod- \mathcal{A} -isomorphisms. The object $((N \oplus_{\tilde{N}} V') \oplus_{V'} (N' \oplus_{\tilde{N}'} V'), \psi)$ then represents an object under both (N, ϕ) and (N', ϕ') . Thus \mathcal{I}_V^m is filtered, as desired.

It remains to check (E). This is simply the fact that for any two morphisms $A \rightarrow B$ in \mathcal{C} which map to the same isomorphism in \mathcal{C}/\mathcal{A} there is a quotient of B (by an object in \mathcal{A}) on which they are equal—in other words, this is the observation that if g and g' represent the same isomorphism in \mathcal{C}/\mathcal{A} then $\text{im}(g - g')$ is in \mathcal{A} .

The second example is the case of reduced schemes of finite type of bounded dimension; we will be using this example in Section 9 to compare different models of the K -theory of varieties.

Example 8.8. Let \mathbf{Sch}_{rf}^d be the category of reduced schemes of finite type over k which are at most d -dimensional. As mentioned in Example 5.10, \mathbf{Sch}_{rf} is an ACGW-category; since morphisms can only increase the dimension of a scheme it follows directly that \mathbf{Sch}_{rf}^d is also an ACGW-category.

We claim that Theorem 8.6 applies for $\mathbf{Sch}_{rf}^{d-1} \subseteq \mathbf{Sch}_{rf}^d$. We check the conditions in turn.

First, consider condition (W). We claim that \mathbf{Sch}_{rf}^{d-1} is m-well-represented and e-negligible in \mathbf{Sch}_{rf}^d . Here, an isomorphism in $\mathbf{Sch}_{rf}^d \setminus \mathbf{Sch}_{rf}^{d-1}$ is (the germ of) an isomorphism between open subsets whose complements are at most $d - 1$ -dimensional. Thus when considering an isomorphism we can discard all irreducible components of dimension less than d . In addition, we can assume that all d -dimensional components are smooth and consider isomorphisms to be birational isomorphisms. To check that \mathbf{Sch}_{rf}^d is m-well-represented it suffices to check that for any two representatives of a birational isomorphism there exists a common dense open subset on which they are defined. This is clearly true.

To check that \mathbf{Sch}_{rf}^{d-1} is e-negligible in \mathbf{Sch}_{rf}^d we note that for any diagram $A \circ \bullet \bullet B \rhd \bullet \bullet C$ if we take the nonsingular locus of the d -dimensional irreducible components of C and intersect it with the image of A we get exactly the desired subset, as all that the inclusion $B \rhd \bullet \bullet C$ can add is either (a) disjoint components of dimension less than d or (b) components of dimension less than d that intersect d -dimensional com-

ponents. In case (b) the intersections are singular in C , so when we remove them we produce exactly the desired morphism.

We now check condition (CGW). Proposition A.1 states that for $\mathcal{C} \backslash \mathcal{A}$ to be a CGW-category we are only required to show that c and k are well-defined equivalences of categories; the other axioms follow directly from the definitions. In $\mathbf{Sch}_{rf}^d \backslash \mathbf{Sch}_{rf}^{d-1}$ all objects are canonically isomorphic to the disjoint union of their d -dimensional connected components, so it suffices to consider these examples. By definition, both the e-morphisms and m-morphisms in $\mathbf{Sch}_{rf}^d \backslash \mathbf{Sch}_{rf}^{d-1}$ are birational isomorphisms of the domain with a subset of the components of the codomain. Both c and k simply take the components not hit by the morphism. Consider taking each object to its set of connected components; from the definition of the distinguished squares (see Appendix A) a square in $\mathbf{Sch}_{rf}^d \backslash \mathbf{Sch}_{rf}^{d-1}$ is distinguished if and only if the produced square in the category of finite sets is distinguished. The fact that c and k are equivalences of categories thus follows from the fact that they are induced from c and k on the category **FinSet**.

It remains to check condition (E). Since \odot in \mathbf{Sch}_{rf}^d is simply intersection of schemes the condition as stated follows by the same argument as the negligibility condition above. To check the condition with m-morphisms and e-morphisms reversed, let A_d be the d -dimensional irreducible components of A . Then $A_d \succ \bullet \bullet X \times_A X'$ exists, and the maps $A_d \succ \bullet \bullet X \circ \bullet \bullet B$ and $A_d \succ \bullet \bullet X' \circ \bullet \bullet B$ are equal inside the (ordinary) category of schemes (since they must be equal on a dense open subset, as they are equivalent in $\mathbf{Sch}_{rf}^d \backslash \mathbf{Sch}_{rf}^{d-1}$). Factoring this morphism as $A_d \circ \bullet \bullet C \succ \bullet \bullet B$ gives the desired object C .

We now observe that, by the equivariant Barratt–Priddy–Quillen theorem,

$$K(\mathbf{Sch}_{rf}^d \backslash \mathbf{Sch}_{rf}^{d-1}) \simeq \bigoplus_{\alpha \in B_n} \Omega^\infty \Sigma^\infty B\mathrm{Aut}(\alpha).$$

Here, B_n is the set of birational isomorphism classes of schemes of dimension d , and $\mathrm{Aut}(\alpha)$ is the group of birational automorphisms of a representative of the class.

9. A comparison of models

In this section we compare both authors' models for $K(\mathbf{Var}_{/k})$. Write $K^C(\mathbf{Var}_{/k})$ for the K -theory of varieties defined as in [2], and let $K^Z(\mathbf{Var}_{/k})$ denote the model in [16]. We then have the following comparison theorem.

Theorem 9.1. $K^C(\mathbf{Var}_{/k})$ is weakly equivalent to $K^Z(\mathbf{Var}_{/k})$.

The rest of this section focuses on the proof of the theorem. For conciseness we fix the base field k and omit it from the notation. To prove the theorem we construct an auxiliary SW-category \mathbf{Sch}_{rfw} and show that there are weak equivalences

$$K^C(\mathbf{Var}) \xrightarrow{\sim} K^C(\mathbf{Sch}_{rf}) \xrightarrow{\sim} K^C(\mathbf{Sch}_{rfw}) \xleftarrow{\sim} K^Z(\mathbf{Var}).$$

Recall that \mathbf{Sch}_{rf} is the ACGW-category of reduced schemes of finite type (Example 5.10). By an abuse of notation, we also write \mathbf{Sch}_{rf} for the SW-category of reduced schemes of finite type (see Remark 7.7). The left-hand map is an equivalence by Corollary 7.11, so we focus on the zig-zag on the right.

Remark 9.2. Constructing the weak equivalence on the right (and checking that it is, in fact, a weak equivalence) is a relatively straightforward exercise in simplicial objects (see Proposition 9.13), and has been known to the authors for several years. The most difficult part of this proof is actually checking that the middle map (which is induced by an inclusion of SW-categories) is a weak equivalence on K -theory. In Waldhausen categories, this is analogous to the following question: suppose that \mathcal{C} is a Waldhausen category in which the weak equivalences do not satisfy the Extension axiom [14, p. 327]. Let \mathcal{C}' be the Waldhausen category with the same underlying category and cofibrations as \mathcal{C} together with the minimal set of weak equivalences that includes all weak equivalences in \mathcal{C} and satisfies Extension. Does the natural functor $\mathcal{C} \rightarrow \mathcal{C}'$ induce a weak equivalence on K -theory? The authors could not find an answer to this question, but the current example on schemes produces an interesting example where the answer is “yes.”

Definition 9.3. We define a new SW-category \mathbf{Sch}_{rfw} . Its underlying category is \mathbf{Sch}_{rf} , the category of reduced schemes of finite type. We define the structure maps by setting

cofibrations: the open immersions, and

complement maps: the closed immersions, and

weak equivalences: those morphisms $f: X \rightarrow Y$ such that there exists a stratification

$$\emptyset = Y_0 \xrightarrow{cl} Y_1 \xrightarrow{cl} \dots \xrightarrow{cl} Y_n = Y$$

of Y by closed immersions such that for all i , the induced map $f_i: X \times_Y (Y_i \setminus Y_{i-1}) \rightarrow Y_i \setminus Y_{i-1}$ is an isomorphism.

Remark 9.4. This is equivalent to the statement that there is a corresponding filtration X_i on X such that $f_i: X_i \setminus X_{i-1} \rightarrow Y_i \setminus Y_{i-1}$ is an isomorphism. We sometimes use the condition in this form.

We state the relevant definitions on the assembler-side of the equivalence.

Definition 9.5. The assembler \mathbf{Var} (resp. \mathbf{Sch}_{rf}) has as objects the varieties (resp. reduced schemes of finite type), with morphisms the locally closed immersions. The topology on \mathbf{Var} (resp. \mathbf{Sch}_{rf}) is generated by the coverage consisting of pairs $\{Y \hookrightarrow X, X \setminus Y \hookrightarrow X\}$, where $Y \hookrightarrow X$ is a closed immersion.

The inclusion of assemblers $\mathbf{Var} \rightarrow \mathbf{Sch}_{rf}$ includes an equivalence of K -theories by [16, Theorem B], as every reduced scheme of finite type has a finite disjoint cover by varieties.

As the proof of Theorem 9.1 has many parts, we begin by presenting the basic outline. This will reduce the proof to showing that certain morphisms are equivalences on K -theory, and the rest of the section will focus on each of those maps in turn.

Outline of proof for Theorem 9.1. The category of reduced schemes of finite type comes equipped with a filtration by dimension. This filtration is inherited by \mathbf{Sch}_{rf} and \mathbf{Sch}_{rfw} , and the inclusion $\mathbf{Sch}_{rf} \rightarrow \mathbf{Sch}_{rfw}$ is compatible with this filtration. Note that

$$K^C(\mathbf{Sch}_{rf}) = \operatorname{hocolim}_n K^C(\mathbf{Sch}_{rf}^n),$$

and similarly for $K^Z(\mathbf{Sch}_{rf})$ and $K^C(\mathbf{Sch}_{rfw})$. Thus to show the theorem it suffices to show that there exist equivalences $K^C(\mathbf{Sch}_{rf}^n) \rightarrow K^C(\mathbf{Sch}_{rfw}^n)$ and $K^Z(\mathbf{Sch}_{rf}^n) \rightarrow K^C(\mathbf{Sch}_{rfw}^n)$ for all n which are compatible with the inclusions on the filtrations.

Proposition 9.13 constructs a map $K^Z(\mathbf{Sch}_{rf}^n) \rightarrow K^C(\mathbf{Sch}_{rfw}^n)$ which is an equivalence for all n . The map $K^C(\mathbf{Sch}_{rf}^n) \rightarrow K^C(\mathbf{Sch}_{rfw}^n)$ is induced by the identity map on the underlying categories (as both \mathbf{Sch}_{rf} and \mathbf{Sch}_{rfw} have the same underlying SW-category; they differ only in their choice of weak equivalences).

Our proof proceeds by induction on n . When $n = 0$, $\mathbf{Sch}_{rf}^0 = \mathbf{Sch}_{rfw}^0$, so the K -theories of these are equal. We now assume that the natural inclusion $K^C(\mathbf{Sch}_{rf}^{n-1}) \rightarrow K^C(\mathbf{Sch}_{rfw}^{n-1})$ is an equivalence. Consider the following diagram:

$$\begin{array}{ccccc} K^C(\mathbf{Sch}_{rf}^{n-1}) & \xrightarrow{\sim} & K^C(\mathbf{Sch}_{rfw}^{n-1}) & \xleftarrow{\sim} & K^Z(\mathbf{Sch}_{rf}^{n-1}) \\ \downarrow i & & \downarrow i' & & \downarrow i \\ K^C(\mathbf{Sch}_{rf}^n) & \xrightarrow{g} & K^C(\mathbf{Sch}_{rfw}^n) & \xleftarrow{\sim} & K^Z(\mathbf{Sch}_{rf}^n) \\ \downarrow & & \downarrow & & \downarrow \\ K^C(\mathbf{Sch}_{rf}^n, \mathbf{Sch}_{rf}^{n-1}) & \xrightarrow{g'} & K^C(\mathbf{Sch}_{rfw}^n, \mathbf{Sch}_{rfw}^{n-1}) & \xleftarrow{f} & K^Z((\mathbf{Sch}_{rf}^n/i)_\bullet) \end{array} \quad (9.6)$$

The columns in this diagram are homotopy fiber sequences. The column on the right is produced by [16, Theorem C], the other two columns are produced by [2, Prop. 5.5]. The maps between the columns are given below. Since the columns are homotopy fiber sequences of loop spaces, f must be a weak equivalence by the five lemma. The map g is a weak equivalence if and only if g' is, so we focus on proving that g' is a weak equivalence.

In Definitions 9.12, 9.14, 9.16, and 9.18 we show that there exists a category \mathcal{D} and morphisms

$$\lambda: K^C(\mathbf{Sch}_{rf}^n, \mathbf{Sch}_{rf}^{n-1}) \longrightarrow K^C(\mathcal{D}), \quad (9.7)$$

$$\beta: K^C(\mathbf{Sch}_{rfw}^n, \mathbf{Sch}_{rfw}^{n-1}) \longrightarrow K^C(\mathcal{D}), \text{ and} \quad (9.8)$$

$$\rho: K^Z((\mathbf{Sch}_{rf}^n/i)_\bullet) \longrightarrow K^C(\mathcal{D}) \quad (9.9)$$

making the following diagram commute:

$$\begin{array}{ccccc}
 K^C(\mathbf{Sch}_{rf}^n, \mathbf{Sch}_{rf}^{n-1}) & \xrightarrow{g'} & K^C(\mathbf{Sch}_{rfw}^n, \mathbf{Sch}_{rfw}^{n-1}) & \xleftarrow{\ell} & K^Z((\mathbf{Sch}_{rf}^n/i).) \\
 & \searrow \lambda & \downarrow \beta & \swarrow \rho & \\
 & & K^C(\mathcal{D}) & &
 \end{array}$$

Here, the top row is the bottom row of (9.6). The map β is a weak equivalence by Proposition 9.17. Thus we see that g' is an equivalence if and only if λ is; that λ is an equivalence is exactly the conclusion of Proposition 9.19. Thus g' is an equivalence, and the inductive step is complete. \square

We now turn our attention to filling in the details of the proof above. We begin by checking that \mathbf{Sch}_{rfw} is well-defined.

Lemma 9.10. *Let $X, Y, Z \in \mathbf{Sch}_{rfw}$ and suppose $X \rightarrow Y$ and $Y \rightarrow Z$ are weak equivalences. Then $X \rightarrow Z$ is a weak equivalence.*

Proof. Recall that $X \rightarrow Y$ being a weak equivalence is the statement that there is a stratification

$$\emptyset = Y_0 \rhd Y_1 \rhd \cdots \rhd Y_n$$

such that $X \times_Y (Y_i \setminus Y_{i-1}) \xrightarrow{\cong} Y_i \setminus Y_{i-1}$. Similarly for $Y \rightarrow Z$. We must produce a new stratification of Z , call it Z'_i , such that $X \times_Z (Z'_i \setminus Z'_{i-1}) \xrightarrow{\cong} (Z'_i \setminus Z'_{i-1})$. We do this by stratifying each $(Z_i \setminus Z_{i-1})$ in turn, using the stratification of Y , and gluing these together.

The problem thus reduces to the following. Given $Y_1 \rhd Y_2$ and $Z_1 \rhd Z_2$ with an isomorphism $\varphi: Y_2 \setminus Y_1 \xrightarrow{\cong} Z_2 \setminus Z_1$, and a further stratification $Y_{1,0} \rhd \cdots \rhd Y_{1,n} = Y_2$, produce a corresponding stratification for $Z_1 \rhd Z_2$. To do this, define $Z_{1,i} = Z_2 \setminus \varphi(Y_2 \setminus Y_{1,i})$. One checks that

$$\begin{aligned}
 Z_{1,i} \setminus Z_{1,i-1} &= (Z_2 \setminus \varphi(Y_2 \setminus Y_{1,i})) \setminus (Z_2 \setminus (Y_{1,i-1})) \\
 &= \varphi(Y_2 \setminus Y_{1,i-1}) \setminus \varphi(Y_2 \setminus Y_{1,i}) \cong \varphi(Y_i \setminus Y_{i-1}) \quad \square
 \end{aligned}$$

Lemma 9.11. *\mathbf{Sch}_{rfw} is an SW-category.*

Proof. For this we only need to check the axioms of SW-categories that apply to weak equivalences [2, Defn. 3.24], which are wholly analogous to [14, p. 326]. First, the isomorphisms are certainly contained in w . Second, we must check that subtraction respects weak equivalences. That is, if we have a commutative square with sides as indicated:

$$\begin{array}{ccc}
 X & \hookrightarrow & X' \\
 \downarrow \wr & & \downarrow \wr \\
 Y & \hookrightarrow & Y'
 \end{array}$$

then there is a weak equivalence $X' \setminus X \rightarrow Y' \setminus Y$ making the induced square commute. Thus, we need a stratification on $Y' \setminus Y$. Since we are subtracting off Y , the stratification of Y will not come into play. Define the stratification to be

$$\emptyset = (Y' \setminus Y) \times_{Y'} Y'_0 \hookrightarrow (Y' \setminus Y) \times_{Y'} Y'_1 \hookrightarrow \cdots \hookrightarrow Y' \setminus Y.$$

Finally, we must check that in a diagram as below, where all the horizontal maps are cofibrations and the squares are pullbacks, the induced map between pushouts is a weak equivalence:

$$\begin{array}{ccccc}
 X'' & \hookleftarrow & X & \hookrightarrow & X' \\
 \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
 Y'' & \hookleftarrow & Y & \hookrightarrow & Y'
 \end{array}$$

Since $X' \rightarrow Y'$ is a weak equivalence, $X \rightarrow Y$ is trivially so: a stratification $Y'_i \twoheadrightarrow Y'$ pulls back to one on Y , $Y \times_{Y'} Y'_i \rightarrow Y$. A similar statement also holds for $X'' \rightarrow Y''$.

It suffices to consider the case where both $X' \rightarrow Y'$ and $X'' \rightarrow Y''$ are given by two step stratifications. Let these be $Y'_1 \twoheadrightarrow Y'$ and $Y''_1 \twoheadrightarrow Y''$. Denote the two induced stratifications on Y by $Y_1^{(1)} \twoheadrightarrow Y$ and $Y_1^{(2)} \twoheadrightarrow Y$ so that $Y_1^{(1)} = Y \times_{Y'} Y'_1$ and $Y_1^{(2)} = Y \times_{Y''} Y''_1$. We now consider the three-step stratification

$$Y''_1 \amalg_{Y_1^{(2)} \times_Y Y_1^{(1)}} Y'_1 \twoheadrightarrow Y'' \amalg_{Y_1^{(1)}} Y'_1 \twoheadrightarrow Y'' \amalg_Y Y'$$

One verifies that

$$\begin{aligned}
 (Y'' \amalg_{Y_1^{(1)}} Y'_1) \setminus (Y''_1 \amalg_{Y_1^{(2)} \times_Y Y_1^{(1)}} Y'_1) &\cong (Y'' \setminus Y''_1) \\
 (Y'' \amalg_Y Y') \setminus (Y'' \amalg_{Y_1^{(1)}} Y'_1) &\cong (Y' \setminus Y'_1) \quad \square
 \end{aligned}$$

We now define our second helper-category, \mathcal{D} .

Definition 9.12. Let \mathcal{D} be the category with

objects: finite disjoint unions of smooth n -dimensional varieties, written $\coprod_{i \in I} X_i$, where each X_i is irreducible,

morphisms: $\coprod_{s \in S} X_s \rightarrow \coprod_{t \in T} Y_t$ are maps of sets $f: S \rightarrow T$ together with birational isomorphisms $X_s \rightarrow Y_{f(s)}$.²

composition: induced by composition of set maps together with the composition of birational isomorphisms.

The category \mathcal{D} has a forgetful functor to **FinSet** induced by mapping $\coprod_{s \in S} X_s$ to S .

We put an ACGW-structure on \mathcal{D} by declaring all morphisms with injective underlying maps of sets to be both e-morphisms and m-morphisms, and by setting the distinguished (resp. commutative) squares to be the squares that become distinguished (resp. commutative) in the ACGW-structure on **FinSet**; the forgetful functor then becomes a functor of ACGW-categories.

The SW-structure on \mathcal{D} is given by

cofibrations: morphisms whose underlying set map is injective,

complement maps: the same as the cofibrations, and

weak equivalences: isomorphisms.

With these definitions, the S_\bullet -construction gives equal structures for the K -theory of \mathcal{D} considered as a CGW- or an SW-category.

The ACGW-category \mathcal{D} is equivalent to a disjoint union of categories of the form **FinSet** \wr G , for G a group of birational automorphisms (see Example 5.11).

The main work of this section goes into proving Propositions 9.13 and 9.19 which together immediately imply Theorem 9.1.

Proposition 9.13. *For $n \leq \infty$,*

$$K^Z(\mathbf{Sch}_{rf}^n) \simeq K^C(\mathbf{Sch}_{rfw}^n),$$

induced by taking each tuple of varieties in \mathbf{Sch}_{rf}^n to their disjoint union.

Proof. For conciseness of notation, we give the proof for the case $n = \infty$ and omit the n from the notation. The proof works identically for all finite n . Throughout this proof we freely use the notation and definitions of [16].

We construct a functor of simplicial categories $F: \mathcal{W}(\mathbf{Sch}_{rf}^{\vee \bullet}) \rightarrow wS.\mathbf{Sch}_{rfw}$ which has a levelwise right adjoint. Thus the functor is levelwise a homotopy equivalence, and we get an equivalence on the geometric realizations of the simplicial categories. This equivalence produces an equivalence $K^Z(\mathbf{Sch}_{rf})_1 \rightarrow K^C(\mathbf{Sch}_{rfw})_1$, and (since these are both Ω -spectra above level 1) an equivalence of K -theories.

² Here, by “birational isomorphism” we mean an equivalence class of maps, rather than a specific map which is a birational isomorphism.

The functor is defined in the following manner. $\mathcal{W}(\mathbf{Sch}_{rf}^{\vee m})$ is the full subcategory of $\mathcal{W}(\mathbf{Sch}_{rf})^m$ consisting of those objects with disjoint indexing sets. We will thus refer to objects of $\mathcal{W}(\mathbf{Sch}_{rf}^{\vee m})$ as tuples $(\{A_{1i}\}_{i \in I_1}, \dots, \{A_{mi}\}_{i \in I_m})$ in $\mathcal{W}(\mathbf{Sch}_{rf})^m$ and simply ensure that at all stages the indexing sets are disjoint. Let $F_m(A_1, \dots, A_m)$ be the functor $X: \widetilde{\mathbf{Ar}}[m] \rightarrow \mathbf{Sch}_{rfw}$ given by

$$X_{i,j} = \coprod_{k=i+1}^j \coprod_{\ell \in I_k} A_{k\ell},$$

with morphisms given by the natural inclusions into the coproduct. A morphism of tuples gives a natural transformation of functors, each component of which is a weak equivalence in \mathbf{Sch}_{rfw} , so F_m is well-defined. The simplicial maps in $\mathcal{W}(\mathbf{Sch}_{rf}^{\vee \bullet})$ are induced by maps on the indexing sets, so these commute with the simplicial structure maps in $wS.\mathbf{Sch}_{rfw}$. Thus F_\bullet is a simplicial functor.

It remains to check that F_m has a right adjoint. Given a diagram $X: \widetilde{\mathbf{Ar}}[m] \rightarrow \mathbf{Sch}_{rf}$, we define $G_m(X)$ to have as its i -th component $\{X_{0i} \setminus X_{0(i-1)}\}_{\{i\}}$.

We define the unit of the adjunction by taking each $\{A_{ji}\}_{i \in I_j}$ to $\{\coprod_{i \in I_j} A_{ji}\}_{\{j\}}$; this is a valid morphism in $\mathcal{W}(\mathbf{Sch}_{rf})$, so gives a valid morphism in $\mathcal{W}(\mathbf{Sch}_{rf})^m$, with the indexing sets disjoint by definition.

Now consider $F_m \circ G_m$. This takes a functor $X: \widetilde{\mathbf{Ar}}[m] \rightarrow \mathbf{Sch}_{rf}$ to the functor $X': \widetilde{\mathbf{Ar}}[m] \rightarrow \mathbf{Sch}_{rf}$, where

$$X'_{ij} = \coprod_{k=i+1}^j X_{ij} \setminus X_{i(j-1)}.$$

There is a natural weak equivalence $X' \rightarrow X$ by simply mapping each component to itself. This gives the counit of the adjunction and completes the proof of the proposition. \square

We can now define the map β (9.8).

Definition 9.14. To define a map $K^C(\mathbf{Sch}_{rfw}^n, \mathbf{Sch}_{rfw}^{n-1}) \rightarrow K^C(\mathcal{D})$ it suffices to define for all r , a map $|wS.^{(r)}S.(\mathbf{Sch}_{rfw}^n, \mathbf{Sch}_{rfw}^{n-1})| \rightarrow |iS.^{(r)}\mathcal{D}|$. In order to construct such a map it suffices to construct a partial functor $b_\bullet: S.(\mathbf{Sch}_{rfw}^n, \mathbf{Sch}_{rfw}^{n-1}) \rightarrow \mathcal{D}$ defined on the subcategories of closed immersions, open immersions, and weak equivalences, as long as this functor is compatible with the simplicial structure maps and takes objects in the S -construction to objects in the S -construction. An object of $S_m(\mathbf{Sch}_{rfw}^n, \mathbf{Sch}_{rfw}^{n-1})$ is a diagram

$$\begin{array}{ccccccc}
 & & & & & & X_{m,m} \\
 & & & & & & \downarrow \\
 & & & & & & \vdots \\
 & & & & & & \downarrow \\
 & & & X_{2,2} & \xrightarrow{o} & \cdots & \xrightarrow{o} & X_{2,m} \\
 & & & \downarrow & & & & \downarrow \\
 & X_{1,1} & \xrightarrow{o} & X_{1,2} & \xrightarrow{o} & \cdots & \xrightarrow{o} & X_{1,m} \\
 & \downarrow & & \downarrow & & & & \downarrow \\
 Y_0 & \xrightarrow{o} & Y_1 & \xrightarrow{o} & Y_2 & \xrightarrow{o} & \cdots & \xrightarrow{o} & Y_m
 \end{array}$$

in which each $X_{i,j} \in \mathbf{Sch}_{rfw}^{n-1}$. We define b_m to take this diagram to the tuple containing the irreducible n -dimensional components of the nonsingular points of Y_m (indexed over the set of irreducible n -dimensional components of Y_m).

Lemma 9.15. *The partial functor b_\bullet is well-defined and induces a map on K -theory.*

Proof. First, suppose that X and Y are irreducible and n -dimensional. Then a weak equivalence $X \xrightarrow{\sim} Y$ is, by definition, a birational isomorphism. In particular, this means that under b_\bullet , all weak equivalences in $S.(\mathbf{Sch}_{rfw}^n, \mathbf{Sch}_{rfw}^{n-1})$ are taken to isomorphisms in \mathcal{D} . An open embedding $X \xrightarrow{o} Y$ is also a birational isomorphism; a closed embedding is an honest isomorphism, unless we allow X to have dimension less than n ; in that case, X is taken to the empty tuple in \mathcal{D} . Thus a diagram in the S -construction of $S.(\mathbf{Sch}_{rfw}^n, \mathbf{Sch}_{rfw}^{n-1})$ is taken to a diagram with injective underlying maps of sets both vertically and horizontally (decorated with birational isomorphisms); the pushout condition translates to the analogous pushout condition on the underlying diagram of sets. The weak equivalence direction is mapped to morphisms which are isomorphisms of the underlying maps of sets, decorated with birational isomorphisms. This is exactly the S -construction applied to \mathcal{D} , and thus each b_m is well-defined. The simplicial structure maps never fully get rid of one of the Y 's in the bottom row of the diagram; since all of the horizontal maps in the diagram above are birational isomorphisms (as the complements have dimension strictly less than n) the partial functor is well-defined, and induces a map on K -theory. \square

The map induced by b_\bullet is β .

We now consider the map ρ .

Definition 9.16. The map $\rho: K^Z((\mathbf{Sch}_{r_f}^n/i).) \rightarrow K^C(\mathcal{D})$ is defined by a composition of two maps. The first map is the map $K^Z((\mathbf{Sch}_{r_f}^n/i).) \rightarrow K^Z(\mathcal{D})$, defined by taking each irreducible scheme of dimension n to its birational isomorphism type. The second map $K^Z(\mathcal{D}) \rightarrow K^C(\mathcal{D})$ is induced by the map $K^Z(\mathcal{D}) \rightarrow K(\mathrm{SC}(\mathcal{D})) \rightarrow K^C(\mathcal{D})$, where the first map is the natural transformation taking $K^Z(\mathcal{D})$ to the Waldhausen K -theory of the Waldhausen category $\mathrm{SC}(\mathcal{D})$ (defined in [17, Theorem 2.1, Proposition 2.6]), and the second map is induced by an equivalence of K -theories taking an object in $\mathrm{SC}(\mathcal{D})$ (which is a tuple of tuples of birational isomorphism classes) to the “flattened” tuple (indexed by the disjoint union of the indexing sets of the tuples). In this map, the key observation is that cofibrations in $\mathrm{SC}(\mathcal{D})$ are cofibrations in the SW-category \mathcal{D} , while cofiber maps in $\mathrm{SC}(\mathcal{D})$ are *opposites* of complement maps in \mathcal{D} : a cofiber map in $\mathrm{SC}(\mathcal{D})$ is induced by a map selecting a subset of the indexing set, and this is exactly a description of the opposite of a map in \mathcal{D} .

Proposition 9.17. *The map ρ is a weak equivalence, and the diagram*

$$\begin{array}{ccc} K^Z((\mathbf{Sch}_{r_f}^n/i).) & \xrightarrow{f} & K^C(\mathbf{Sch}_{r_{fw}}^n, \mathbf{Sch}_{r_{fw}}^{n-1}) \\ & \searrow \rho & \downarrow \beta \\ & & K^C(\mathcal{D}) \end{array}$$

commutes. The map β is therefore also a weak equivalence.

Proof. The map f is a weak equivalence because it is a map induced on homotopy cofibers by a pair of weak equivalences. The map ρ is a weak equivalence by [16, Proposition 7.1] (where it is the map p).

We now check the commutativity of the diagram. The map f is defined analogously to the natural transformation in [17, Proposition 2.6], designed to make this triangle commute. The map β is defined analogously to p in [16, Proposition 7.1], again designed to make this diagram commute. In particular, both of these compositions take all objects in simplicial levels higher than 0 (in the original categories) to the empty set, and take the objects in simplicial level 0 to a tuple of birational isomorphism classes of varieties (with morphisms given by permutations of these decorated by birational isomorphisms). The indexing set of each tuple is the set of irreducible components of the varieties, so this diagram does, indeed, commute.

Thus, by 2-of-3, β is also a weak equivalence. \square

Since β is a weak equivalence, g is a weak equivalence if and only if λ is. Thus it remains to consider the map λ .

Definition 9.18. The map $\lambda: K^C(\mathbf{Sch}_{r_f}^n, \mathbf{Sch}_{r_f}^{n-1}) \rightarrow K^C(\mathcal{D})$ is defined to be the composition

$$K^C(\mathbf{Sch}_{rf}^n, \mathbf{Sch}_{rf}^{n-1}) \longrightarrow K^C(\mathbf{Sch}_{rfw}^n, \mathbf{Sch}_{rfw}^{n-1}) \xrightarrow{\beta} K^C(\mathcal{D}).$$

Proposition 9.19. *The map λ is a weak equivalence.*

Proof. Let $\lambda': \mathbf{Sch}_{rf}^n \setminus \mathbf{Sch}_{rf}^{n-1} \rightarrow \mathcal{D}$ be the CGW-functor taking each variety of dimension n to the set of birational isomorphism classes of its irreducible components. This is actually an equivalence of categories (on the level of CGW-categories) with the inverse equivalence given by choosing a representative in each birational isomorphism class and taking an object in \mathcal{D} to the disjoint union of its representatives. Thus λ' is a weak equivalence.

Consider the following diagram:

$$\begin{array}{ccccc} \Omega|iS.S.(\mathbf{Sch}_{rf}^n, \mathbf{Sch}_{rf}^{n-1})| & \xrightarrow{\sim} & \Omega|QS.(\mathbf{Sch}_{rf}^n, \mathbf{Sch}_{rf}^{n-1}) & \xrightarrow{\sim} & \Omega|Q(\mathbf{Sch}_{rf}^n \setminus \mathbf{Sch}_{rf}^{n-1})| \\ \lambda \downarrow & & & & \downarrow \lambda' \\ \Omega|iS.\mathcal{D}| & \xrightarrow{\sim} & & & \Omega|Q\mathcal{D}| \end{array}$$

The leftmost two horizontal maps are given by the natural transformation described in the proof of Theorem 7.8 for the comparison between the Q -construction and the S -construction. The right-hand map in the top row is a weak equivalence by Theorem 8.6. Thus, by 2-of-3, λ is a weak equivalence. \square

10. Proof of Theorem 8.6

The goal of this section is to prove Theorem 8.6. The idea of the proof is to use Quillen's Theorem B [10, Theorem B] applied to the functor Qs . There are therefore two steps to the proof: proving that the theorem applies to Qs , and proving that the fiber agrees with $K(\mathcal{A})$.

Let $i: \mathcal{A} \rightarrow \mathcal{C}$ be the inclusion functor. Then Qi factors as

$$Q\mathcal{A} \xrightarrow{M \mapsto (M, 1_\emptyset)} Qs_{\emptyset/} \xrightarrow{(N, u) \mapsto N} Q\mathcal{C}$$

If Theorem B applies to Qs then its fiber is $Qs_{\emptyset/}$. In this case, to show that the fiber agrees with $K(\mathcal{A})$ it suffices to check that the left-hand map in this factorization is a weak equivalence. We see that the theorem is thus a direct consequence of the following two propositions:

Proposition 10.1. *The inclusion $Q\mathcal{A} \rightarrow Qs_{\emptyset/}$ is a homotopy equivalence.*

Proposition 10.2. *Quillen's Theorem B applies to the functor Qs . More concretely, for any $u: V \rightarrow V'$ in $Q(\mathcal{C} \setminus \mathcal{A})$, the induced functor $u^*: Qs_{V'/} \rightarrow Qs_{V/}$ is a homotopy equivalence.*

The rest of this section is taken up with the proof of these two propositions. We write $\mathcal{C} = (\mathcal{E}, \mathcal{M})$ and $\mathcal{A} = (\mathcal{E}_{\mathcal{A}}, \mathcal{M}_{\mathcal{A}})$. We begin by analyzing how morphisms in $\mathcal{C} \setminus \mathcal{A}$ and $Q(\mathcal{C} \setminus \mathcal{A})$ work.

Lemma 10.3. $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{A}}$ satisfies 1-of-3, in the sense that $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{A}}$ are subcategories of \mathcal{M} and \mathcal{E} , respectively, and given any composable morphisms $f, g \in \mathcal{M}$ (resp. \mathcal{E}), if $gf \in \mathcal{M}_{\mathcal{A}}$ (resp. $\mathcal{E}_{\mathcal{A}}$) then so are f and g .

Proof. We prove this for $\mathcal{M}_{\mathcal{A}}$; the result for $\mathcal{E}_{\mathcal{A}}$ follows by duality.

Suppose that we are given $f: A \rightrightarrows B$ and $g: B \rightrightarrows C$ in \mathcal{C} . This corresponds to a diagram

$$\begin{array}{ccccc}
 & & \emptyset & \xrightarrow{\quad} & B^{c/g} \\
 & & \downarrow & & \downarrow \\
 & & & \square & \\
 \emptyset & \xrightarrow{\quad} & A^{c/f} & \xrightarrow{\quad} & A^{c/gf} \\
 \downarrow & & \downarrow & & \downarrow \\
 & & & \square & \\
 A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & C
 \end{array}$$

The lower-left square exists by the definition of $A^{c/f}$; the lower-right square exists by applying k^{-1} to the bottom row; the upper square exists because $(A^{c/f})^{c/A^{c/gf}} \cong B^{c/g}$ by the definition of c . Consider the upper square; since \mathcal{A} is closed under subobjects, quotients and extensions, $A^{c/gf}$ is in \mathcal{A} if and only if $A^{c/f}$ and $B^{c/g}$ are. Thus, if f and g are in $\mathcal{M}_{\mathcal{A}}$ so is gf (showing that $\mathcal{M}_{\mathcal{A}}$ is a subcategory) and if gf is in $\mathcal{M}_{\mathcal{A}}$ then f and g must be, as well. \square

Lemma 10.4. The categories $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}$ satisfy the following properties:

- (a) The subcategories $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}$ are preserved under pullbacks and mixed pullbacks along morphisms in \mathcal{E} and \mathcal{M} .
- (b) Pullback squares and mixed pullback satisfy 3-of-4: if three of the morphisms in a square are in $\mathcal{M}_{\mathcal{A}}$ or $\mathcal{E}_{\mathcal{A}}$, the fourth must be as well.

Proof. We first prove (a). Suppose that we have a square

$$\begin{array}{ccc}
 A & \xrightarrow{f'} & B \\
 \downarrow & \circlearrowleft & \downarrow \\
 C & \xrightarrow{f} & D
 \end{array} .$$

We want to show that if f is in $\mathcal{M}_{\mathcal{A}}$, so is f' . Applying c to this diagram produces a pullback square

$$\begin{array}{ccc} A^{c/f'} & \longrightarrow & B \\ \downarrow & & \downarrow \\ C^{c/f} & \longrightarrow & D \end{array}$$

By definition, $C^{c/f'} \in \mathcal{A}$; thus, since \mathcal{A} is closed under quotients, $A^{c/f} \in \mathcal{A}$, as desired. The other proofs of closure under pullbacks follow analogously.

We turn our attention to (b). To check 3-of-4, consider a square as above where we know that $A \twoheadrightarrow B$ is in $\mathcal{M}_{\mathcal{A}}$ and $B \twoheadrightarrow D$ is in $\mathcal{E}_{\mathcal{A}}$. Because $\mathcal{E}_{\mathcal{A}}$ is closed under pullbacks, it follows that $A^{c/f} \twoheadrightarrow C^{c/f'}$ is also in $\mathcal{E}_{\mathcal{A}}$. Thus we have a distinguished square

$$\begin{array}{ccc} \emptyset & \twoheadrightarrow & A^{c/f} \\ \downarrow & \square & \downarrow \\ (A^{c/f})^{k/c} & \twoheadrightarrow & C \end{array}$$

in which we know everything but C is in \mathcal{A} . Since \mathcal{A} is closed under extensions, $C \in \mathcal{A}$ as well. The other forms of 3-of-4 follow analogously. \square

This proposition implies that we can identify the isomorphisms in $\mathcal{C} \setminus \mathcal{A}$ in the following manner:

Lemma 10.5. *An m-morphism in $\mathcal{C} \setminus \mathcal{A}$ represented by a diagram*

$$A \leftarrow \bullet \circ A' \leftarrow \bullet \circ X \circ \bullet \rightarrow B' \twoheadrightarrow B$$

is an isomorphism if and only if $B' \twoheadrightarrow B$ is in $\mathcal{M}_{\mathcal{A}}$; the dual statement holds for e-morphisms.

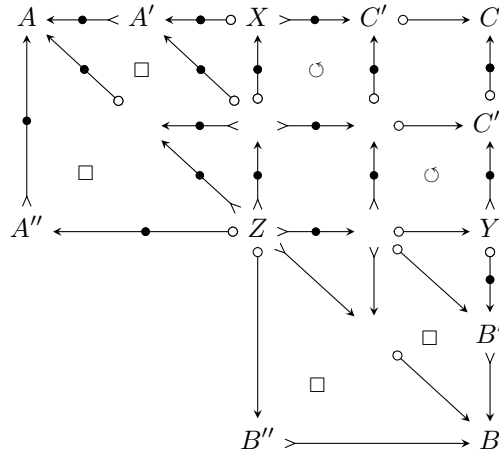
Any morphism $u: A \rightarrow B$ in $Q(\mathcal{C} \setminus \mathcal{A})$ can be represented by a diagram

$$A \leftarrow \bullet \circ A' \leftarrow \bullet \circ X \circ \bullet \rightarrow B' \twoheadrightarrow B$$

in \mathcal{C} . Such a diagram represents an isomorphism if and only if $X \circ \bullet \rightarrow B'$ is in $\mathcal{E}_{\mathcal{A}}$ and $B' \twoheadrightarrow B$ is in $\mathcal{M}_{\mathcal{A}}$.

Proof. If $B' \twoheadrightarrow B$ is in $\mathcal{M}_{\mathcal{A}}$ then the given diagram represents an isomorphism by definition (by reversing the composition for the inverse). Conversely, if an m-morphism has an inverse then tracing through the definition of composition and using Lemmas 10.3 and 10.4 gives that $B' \twoheadrightarrow B$ must be in $\mathcal{M}_{\mathcal{A}}$.

A morphism $A \rightarrow B$ in $Q(\mathcal{C} \setminus \mathcal{A})$ is represented by a composition of an e-morphism $A \circ \twoheadrightarrow C$ and an m-morphism $C \twoheadrightarrow B$. We can represent these by the top and right side of the following diagram:



The rest of the diagram shows that the composition around the bottom is an equivalent representation of this morphism; its construction liberally uses the previous lemmas and results about CGW-categories in Section 2.

Since morphisms in $Q(\mathcal{C} \setminus \mathcal{A})$ are isomorphisms exactly when both components are isomorphisms (by Lemma 4.5), the composition is an isomorphism if and only if the morphisms $C \circ \rightarrow C'$ and $B' \twoheadrightarrow B$ are isomorphisms, meaning that they are in $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}$, respectively. If this is the case then $Z \circ \rightarrow B''$ and $B'' \twoheadrightarrow B$ are in $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}$, respectively, and this represents an isomorphism. Conversely, if this is an isomorphism then we must have $Z \circ \rightarrow B''$ and $B'' \twoheadrightarrow B$ in $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}$; tracing through and using that $\mathcal{E}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}$ satisfy 1-of-3 we obtain the converse. \square

We turn our attention to proving Proposition 10.1.

Definition 10.6. Let $V \in Q(\mathcal{C} \setminus \mathcal{A})$, and let \mathcal{F}_V be the full subcategory of $Q_{s_{V/}}$ of those objects $(M, u: V \rightarrow sM)$ in which u is an isomorphism.

Proposition 10.1 is the $V = \emptyset$ case of the following:

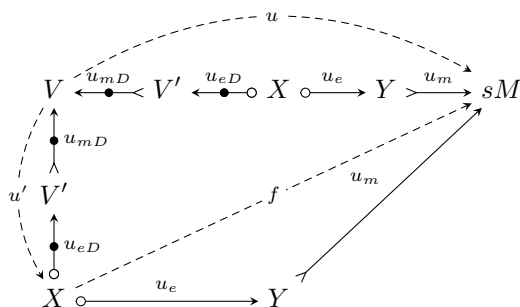
Proposition 10.7. The inclusion $\iota_V: \mathcal{F}_V \rightarrow Q_{s_{V/}}$ is a homotopy equivalence for all $V \in Q(\mathcal{C} \setminus \mathcal{A})$.

Proof. By [10, Theorem A], it suffices to check that for all $(M, u) \in Q_{s_{V/}}$, the category $\iota_V/(M, u)$ is contractible for all (M, u) . By the dual of [10, Proposition 3, Corollary 2] it suffices to check that it is a cofiltered category. By Lemma 10.5, u can be represented by a diagram

$$V \xleftarrow{u_m D} V' \xleftarrow{u_e D} X \xrightarrow{u_e} Y \xrightarrow{u_m} sM.$$

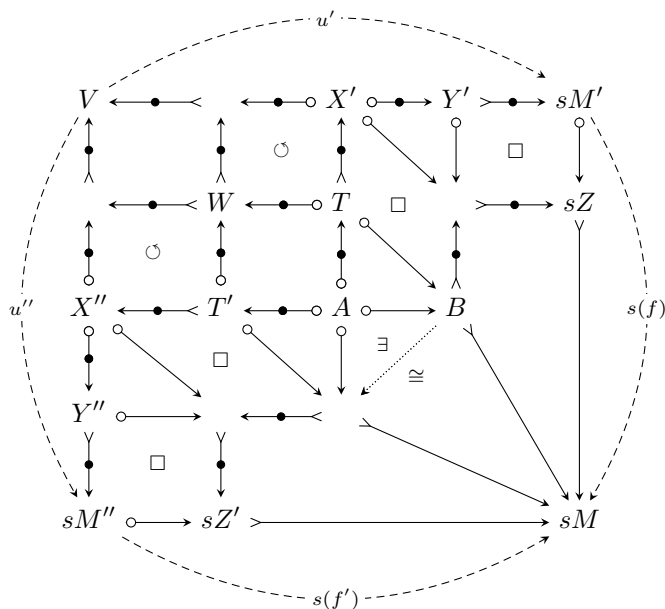
An object of $\iota_V/(M, u)$ is a triple (u', M', f) of an isomorphism $u': V \rightarrow sM'$ in \mathcal{F}_V together with a morphism $f: M' \rightarrow M$ in \mathcal{QC} such that $s(f)u' = u$. A morphism $(u', M', f) \rightarrow (u'', M'', f')$ is a morphism $g: M' \rightarrow M''$ in \mathcal{QC} such that $f'g = f$. In particular, there is a faithful forgetful functor to \mathcal{QC}/M ; since by Lemma 4.6 this is a preorder, so is $\iota_V/(M, u)$. All it remains to check is that it is nonempty and that any two objects have a common object above them.

To see that $\iota_V/(M, u)$ is nonempty, consider the following diagram in \mathcal{C} :



This represents an object of $\iota_V/(M, u)$ as desired.

Now suppose that we are given two different objects of $\iota_V/(M, u)$; we want to show that there is an object mapping to both of them. Suppose that the two objects are given by $(u': V \rightarrow sM', f: M' \rightarrow M)$ and $(u'': V \rightarrow sM'', f': M'' \rightarrow M)$. Writing these in terms of their representations we get the outside of the following diagram; it is possible to complete the outside to the diagram on the inside because $s(f)u' = s(f')u''$.



Consider the object represented by

$$(A \circ \bullet \rightarrow W \succ \bullet \rightarrow V, A \circ \rightarrow B \succ \rightarrow M).$$

This is a well-defined morphism of $\iota_V/(M, u)$. This comes with a morphism to (u', f) given by the formal composition

$$A \circ \bullet \rightarrow T \succ \bullet \rightarrow X' \circ \bullet \rightarrow Y' \succ \bullet \rightarrow M'$$

and an analogous morphism to (u'', f') . Thus $\iota_V/(M, u)$ is cofiltered, as desired. \square

We now turn our attention to Proposition 10.2; this proof is quite complicated and will take the rest of this section. In order to prove that u^* is a homotopy equivalence for all u it suffices to show that it is true for the morphisms $\emptyset \circ \rightarrow V$ and $\emptyset \succ \rightarrow V$. Since all of the conditions of the theorem are symmetric in m-morphisms and e-morphisms, it suffices to prove this for $\emptyset \succ \rightarrow V$; we focus on this case for the rest of this proof. The key idea of the proof is to construct a category \mathcal{H}_N and functors $P_{(N, \phi)}: \mathcal{H}_N \rightarrow \mathcal{F}_V$ and $k_N: \mathcal{H}_N \rightarrow Q\mathcal{A}$ such that the diagram

$$\begin{array}{ccccc} \mathcal{H}_N & \xrightarrow{P_{(N, \phi)}} & \mathcal{F}_V & \hookrightarrow & Qs_{V/} \\ k_N \downarrow & & & & \downarrow u^* \\ Q\mathcal{A} & \xrightarrow{\cong} & \mathcal{F}_{\emptyset} & \hookrightarrow & Qs_{\emptyset/} \end{array} \quad (10.8)$$

commutes up to homotopy. We will then show that k_N and $P_{(N, \phi)}$ are both homotopy equivalence. From this Proposition 10.2 follows by 2-of-3 and Proposition 10.7.

We thus turn our attention to constructing \mathcal{H}_N , k_N and $P_{(N, \phi)}$.

Definition 10.9. The category \mathcal{H}_N has as objects equivalence classes of diagrams

$$M \xleftarrow{\bullet \circ h_e} X \xrightarrow{\bullet \circ h_m} N,$$

where two diagrams are allowed to differ by an isomorphic choice of X . A morphism

$$(M \xleftarrow{\bullet \circ h_e} X \xrightarrow{\bullet \circ h_m} N) \longrightarrow (M' \xleftarrow{\bullet \circ h'_e} X' \xrightarrow{\bullet \circ h'_m} N)$$

is a diagram $M \xrightarrow{j} M_1 \xrightarrow{i} M'$ such that there exists a map $\tilde{h}_m: X \succ \rightarrow X'$ such that the triangle on the left commutes and the square on the right

$$\begin{array}{ccc} X & \xrightarrow{\tilde{h}_m} & X' \\ & \searrow h_m & \swarrow h'_m \\ & N & \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\tilde{h}_m} & X' \\ jh_e \downarrow & \circlearrowleft & \downarrow h'_e \\ M_1 & \xrightarrow{i} & M' \end{array}$$

is a pseudo-commutative square. Composition works via composition in QC ; using the following diagram we see that it is well-defined:

$$\begin{array}{ccccc} X & \xrightarrow{\tilde{h}_m} & X' & \xrightarrow{\tilde{h}'_m} & X'' \\ jh_e \downarrow & \circlearrowleft & \downarrow h'_e & & \downarrow h''_e \\ M_1 & \xrightarrow{i} & M' & \circlearrowleft & \\ \downarrow & \square & \downarrow j' & & \\ \bullet & \xrightarrow{\quad} & M_2 & \xrightarrow{i'} & M'' \end{array}$$

The functor $k_N: \mathcal{H}_N \rightarrow Q\mathcal{A}$ takes $M \xleftarrow{h_e} X \xrightarrow{\quad} N$ to X^{k/h_e} . A morphism is taken to the representation

$$X^{k/h_e} \longrightarrow X^{k/jh_e} \xrightarrow{\tilde{h}_m} X',$$

where the first map is obtained by applying c^{-1} .

Definition 10.10. Let (N, ϕ) be an object of \mathcal{I}_V^m (Definition 8.4). We define $P_{(N, \phi)}: \mathcal{H}_N \rightarrow \mathcal{F}_V$ by letting it take every object $M \xleftarrow{\bullet} X \xrightarrow{\bullet} N$ to the composition

$$V \xrightarrow{\phi^{-1}} sN \xleftarrow{\bullet} sX \xrightarrow{\bullet} sM$$

in $\mathcal{F}_V \subseteq Qs_{V/}$. As both \mathcal{H}_N and \mathcal{F}_V have as morphisms the morphisms of QC , the functor is defined to take a morphism to the morphism represented by the same data.

Lemma 10.11. $P_{(N, \phi)}$ is a well-defined functor.

Proof. Checking that $P_{(N, \phi)}$ is well-defined on objects is straightforward from the definition. Suppose that we are given a morphism in \mathcal{H}_N as defined in Definition 10.9. We must show that this produces a well-defined morphism in \mathcal{F}_V ; from the definition the produced morphism in $Qs_{V/}$ is an isomorphism, so it suffices to show that a morphism in \mathcal{H}_N gives a well-defined morphism in $Qs_{V/}$. For this to be true it suffices to check that the morphisms represented by

$$N \xleftarrow{\bullet} X \xrightarrow{\bullet} M \xrightarrow{\bullet} M_1 \xrightarrow{\bullet} M'$$

and

$$N \leftarrow \bullet \prec X' \circ \bullet \rightarrow M'$$

are equivalent in $Q(\mathcal{C} \setminus \mathcal{A})$. This is true because they are equivalent isomorphisms inside the m -morphisms of $\mathcal{C} \setminus \mathcal{A}$ via the following diagram:

$$\begin{array}{ccccc} N & \leftarrow \bullet \prec & X & \circ \bullet \rightarrow & M_1 & \succ \bullet \rightarrow & M' \\ & \swarrow \bullet & \downarrow \bullet & \circ & \downarrow \bullet & \parallel & \\ & & X' & \circ \bullet \rightarrow & M' & & \end{array}$$

where the marked square is pseudo-commutative from the definition of a morphism in \mathcal{H}_N . That $P_{(N,\phi)}$ respects composition follows directly from the definition, since composition in both $Q_{S_{V/}}$ and \mathcal{H}_N is defined using composition in QC . \square

We begin our analysis by showing that (10.8) commutes up to homotopy.

Lemma 10.12. *In (10.8) the composition around the top and the composition around the bottom are homotopic.*

Proof. Consider an object $M \xleftarrow{\bullet \circ}^{h_e} X \xrightarrow{\bullet \circ} N$ in \mathcal{H}_N . Under the composition around the top it is mapped to

$$\emptyset \xrightarrow{\quad} V \xrightarrow{\quad \phi^{-1} \quad} sN \leftarrow \bullet \prec sX \circ \bullet \rightarrow sM;$$

this is equivalent to the representation

$$\emptyset \xrightarrow{\quad} sM.$$

Around the bottom this is mapped to $\emptyset \xrightarrow{\quad} X^{k/h_e}$. There is a natural map $h_e^k: X^{k/h_e} \xrightarrow{\quad} M$ which induces a morphism between these in $Q_{S_{\emptyset/}}$, so we just need to check that this gives a natural transformation. To see that this transformation is natural, suppose that we are given a morphism

$$(M \xleftarrow{\bullet \circ}^{h_e} X \xrightarrow{\bullet \circ}^{h_m} N) \longrightarrow (M' \xleftarrow{\bullet \circ}^{h'_e} X' \xrightarrow{\bullet \circ}^{h'_m} N)$$

represented by $M \xrightarrow{\quad j \quad} M_1 \xrightarrow{\quad i \quad} M'$. Consider the following diagram in \mathcal{C} :

$$\begin{array}{ccccc}
 X^{k/h_e} & \circlearrowright & X^{k/jh_e} & \rightharpoonup & (X')^{k/h'_e} \\
 \downarrow h_e^k & & \downarrow (jh_e)^k & & \downarrow h'^k_e \\
 M & \circlearrowright & M_1 & \rightharpoonup & M'
 \end{array}$$

The left-hand square exists and is distinguished by the definition of k . The right-hand square exists and commutes by the condition on morphisms in \mathcal{H}_N ; this is exactly k applied to the pseudo-commutative square. After applying s to the diagram and considering the outer corners as objects under \emptyset , we see that this diagram exactly corresponds to a naturality square for functors $\mathcal{H}_N \rightarrow Qs_{\emptyset/}$, as desired. \square

It remains to show that k_N and $P_{(N,\phi)}$ are homotopy equivalences. We begin with k_N ; however, before we can prove that k_N is a homotopy equivalence we must develop some theory.

Definition 10.13. Let \mathcal{J}_N be a skeleton of the full subcategory of $\mathcal{M}_{/N}$ containing those morphisms $A \twoheadrightarrow N$ such that $A^c \in \mathcal{A}$. The category \mathcal{J}_N has a terminal object: 1_N .

Definition 10.14. Let \mathcal{H}'_N be the full subcategory of \mathcal{H}_N containing those objects where h_m is an isomorphism; in particular, each object in \mathcal{H}'_N can be uniquely represented by an e-morphism $N \circ \rightarrow X$. For any m-morphism $i: M \twoheadrightarrow N$ we define the functor $\rho_i: \mathcal{H}'_N \rightarrow \mathcal{H}'_M$ by sending the e-morphism $N \circ \xrightarrow{f} X$ to the e-morphism $M \circ \rightarrow \tilde{X}$, where $M \circ \rightarrow \tilde{X}$ is determined by the following distinguished square:

$$\begin{array}{ccc}
 M & \xrightarrow{i} & N \\
 \downarrow & & \downarrow f \\
 \tilde{X} & \xrightarrow{\quad} & X
 \end{array}$$

Given a morphism represented by $X \circ \rightarrow X_1 \twoheadrightarrow X'$ in \mathcal{H}'_N , this is mapped to the morphism represented by $\tilde{X} \circ \rightarrow \tilde{X}_1 \twoheadrightarrow \tilde{X}'$, where \tilde{X}_1 is defined by the distinguished square

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow{\quad} & X \\
 \downarrow & & \downarrow \\
 \tilde{X}_1 & \xrightarrow{\quad} & X_1
 \end{array}$$

Lemma 10.15. Let $i: (J \twoheadrightarrow N) \rightarrow (I \twoheadrightarrow N)$ be a morphism in \mathcal{J}_N . Then the diagram

$$\begin{array}{ccc} \mathcal{H}'_I & \xrightarrow{\rho_i} & \mathcal{H}'_J \\ & \searrow k'_I \quad \swarrow k'_J & \\ & Q\mathcal{A} & \end{array}$$

commutes up to natural isomorphism.

Proof. Consider the object $I \circ_{\bullet\bullet}^h M$ in \mathcal{H}'_I . Its image under k'_I is $I^{k/h}$. For the other composition, we consider the distinguished square

$$\begin{array}{ccc} J & \xrightarrow{i} & I \\ \circ \downarrow h' & \square & \downarrow \circ h \\ M' & \xrightarrow{\quad} & M \end{array}$$

h is mapped to h' , and then to $J^{k/h'}$. Since $I^{k/h}$ and $J^{k/h'}$ are the kernels in a natural distinguished square, they are naturally isomorphic, as desired. \square

Consider the functor $F: \mathcal{H}_N \rightarrow \mathcal{J}_N$ defined by sending each class $[M \leftarrow \bullet \circ X \rightarrow \bullet \rightarrow N]$ to $X \rightarrow \bullet \rightarrow N$, assuming that this representative is chosen so that this morphism is in \mathcal{J}_N . Note that for each class this representative is unique.

Lemma 10.16. \mathcal{H}_N is fibered over \mathcal{J}_N .

Proof. For any $i: I \rightarrow N \in \mathcal{J}_N$, $F^{-1}(i)$, the fiber over i , is isomorphic to \mathcal{H}'_I . The category $F_{i/}$ has as its objects the solid part of the diagram

$$\begin{array}{ccccc} M' & \dashleftarrow \circ & I & \xrightarrow{i} & N \\ \downarrow \text{Y} & & \downarrow \text{Y} & \nearrow & \\ M & \longleftarrow \circ & X & & \end{array}$$

The functor taking such a diagram to $I \circ \rightarrow M'$ is the right adjoint to the inclusion $\mathcal{H}'_I = F^{-1}(i) \rightarrow F_{i/}$.

Thus \mathcal{H}_N is preferred over \mathcal{J}_N . To check that it is fibered it suffices to check that this right adjoint is compatible with composition in the following sense. For any $j: (I \xrightarrow{i} N) \rightarrow (I' \xrightarrow{i'} N)$ in \mathcal{J}_N we get an induced functor $j^*: F^{-1}(i') \rightarrow F^{-1}(i)$ defined by the composition

$$\left(M \longleftarrow \circ I' \xrightarrow{i'} N \right) \longmapsto \left(\begin{array}{ccccc} M & \xleftarrow{\quad} & I & \xrightarrow{i} & N \\ \downarrow & & \square & \downarrow j & \nearrow i' \\ M' & \longleftarrow \circ & I' & & \end{array} \right) \longmapsto \left(M' \longleftarrow \circ I \xrightarrow{i} N \right).$$

We must show that for any composable j and k , $(kj)^*$ is naturally isomorphic to j^*k^* . This is true because completing a formal composition to a distinguished square is unique up to unique isomorphism. As both j^*k^* and $(kj)^*$ are obtained by completing a formal composition

$$M \longleftarrow \circ I'' \xleftarrow{k} I' \xleftarrow{j} I$$

to a distinguished square, they are naturally isomorphic. \square

We are now ready to prove that k_N is a homotopy equivalence.

Lemma 10.17. k_N is a homotopy equivalence.

Proof. We begin by checking that $k'_N \stackrel{\text{def}}{=} k_N|_{\mathcal{H}'_N}$ is a homotopy equivalence. Let T be an object in QA ; it suffices to check that k'_N/T is contractible for all T . An object of k'_N/T is a triple $(M, h_e: N \circ \rightarrow M, u: N^k \rightarrow T)$ with $u \in QA$. Let \mathcal{C}' be the full subcategory of k'_N/T consisting of those morphisms u which can be represented purely by an e-morphism.

Represent u as $X^k \xrightarrow{i} Y \circ \rightarrow T$, and consider the following diagram:

$$\begin{array}{ccccc} N^k & \xrightarrow{h_e^k} & M & \xleftarrow{h_e} \circ & N \\ \downarrow i & & \downarrow & \circlearrowleft & \parallel \\ Y & \xrightarrow{(h_e^k)'} & M \star_{N^k} Y & \xleftarrow{k^{-1}((h_e^k)')} \circ & N \\ \downarrow j & & & & \\ T & & & & \end{array}$$

(A dashed arrow labeled u points from N^k to T .)

Here, the upper-left square is produced by condition (PP). We claim that the map taking (M, h_e, u) to $(M \star_{N^k} Y, h'_e, j)$ is a functor which produces a retraction from k'_N/T to \mathcal{C}' . To check that this is functorial, consider a morphism in k'_N/T . This is represented by a diagram

$$\begin{array}{ccccccc}
 T & \xleftarrow{j} & \circ & Y & \xleftarrow{i} & N^{k/h_e} & \xrightarrow{\quad} & M \\
 & \searrow j' & & \downarrow & & \downarrow & & \downarrow h_e \\
 & & \circ & Y' & \xleftarrow{\quad} & N^{k/jh_e} & \xrightarrow{\quad} & M_1 \\
 & & & & & \downarrow & & \downarrow \\
 & & & & & N^{k/h'_e} & \xrightarrow{\quad} & M' \\
 & & & & & & & \downarrow h'_e \\
 & & & & & & & N
 \end{array}$$

\square \square \circ $=$

where the morphism is considered to go from the object represented by the diagram around the top to the object represented by the diagram around the bottom. This diagram produces a map $M \star_{N^{k/h_e}} Y \circ \rightarrow M_1 \star_{N^{k/jh_e}} Y' \twoheadrightarrow M' \star_{N^{k/h'_e}} Y'$ by the functoriality conditions in (PP). This is compatible with composition by Lemma 5.12 and the definition of morphism composition in QA . This functor also comes with a natural transformation from the identity produced by the map $M \twoheadrightarrow M \star_{N^k} Y$. Thus k'_N/T is homotopy equivalent to \mathcal{C}' . The category \mathcal{C}' has an initial object $(N, 1_N, \emptyset \circ \rightarrow T)$, so it is contractible. Thus k'_N/T is contractible for all T , and so k'_N is a homotopy equivalence.

We have now shown that k'_N is a homotopy equivalence. By 2-of-3, in order to show that k_N is a homotopy equivalence it suffices to check that the inclusion $\mathcal{H}'_N \rightarrow \mathcal{H}_N$ is a homotopy equivalence.

Since k'_n is a homotopy equivalence, by Lemma 10.15 we see that ρ_i is a homotopy equivalence for all $i \in \mathcal{J}_N$. Thus, since \mathcal{H}_N is fibered over \mathcal{J}_N , by [10, Theorem B, Cor], for all $I \twoheadrightarrow N$, \mathcal{H}'_I is homotopy equivalent to the homotopy fiber of F . However, since \mathcal{J}_N is contractible it follows that the inclusion $\mathcal{H}'_I \rightarrow \mathcal{H}_N$ is a homotopy equivalence. In particular, taking the m-morphism to be the identity on N gives the desired result. \square

We now turn our attention to $P_{(N,\phi)}$.

We will need two different proofs for this functor, depending on whether \mathcal{A} is m-negligible or m-well-represented in \mathcal{C} .

Lemma 10.18. *If \mathcal{A} is m-negligible in \mathcal{C} then $P_{(N,\phi)}$ is a homotopy equivalence.*

Proof. We prove this using Theorem A. An object of \mathcal{F}_V is an isomorphism $V \xrightarrow{\psi} sA$. We will show that $(P_{(N,\phi)})_{A/}$ is contractible. We can fix representatives for ϕ and ψ such that an object of $(P_{(N,\phi)})_{A/}$ is represented by a diagram

$$\begin{array}{ccccccc}
 & & \phi^{-1} & & & & \\
 & \swarrow \text{dashed} & & \searrow \text{dashed} & & & \\
 V & \leftarrow \bullet \circ V' & \leftarrow \bullet < Z & \succ \bullet \rightarrow N' & \circ \rightarrow N & & \\
 & & \downarrow Y & & \uparrow \wedge & & \\
 & & A' & & X & & \\
 & & \downarrow \circ & & \downarrow \circ & & \\
 & & A & \leftarrow \bullet \circ M' & \leftarrow \bullet < M & & \\
 & \nwarrow \text{dashed} & & \swarrow \text{dashed} & & & \\
 & & \psi & & & &
 \end{array} \quad (10.19)$$

where the dashed arrows commute inside $Q(\mathcal{C} \setminus \mathcal{A})$. V', Z, N', A' are all fixed by our choice of representatives; the only part of the diagram that is allowed to change are the bottom and rightmost rows. A representative of an object is well-defined up to unique isomorphism, since both the right-hand column (an object in \mathcal{H}_N) and the bottom row are well-defined up to unique isomorphism. The maps $M \succ \rightarrow M'$ and $M' \circ \rightarrow A$ must also be in $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{A}}$, respectively, since $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{E}_{\mathcal{A}}$ are closed under 2-of-3 by Lemma 10.3. (This follows by computing a representative of the composition and noting that since its components are in $\mathcal{M}_{\mathcal{A}}$ (resp. $\mathcal{E}_{\mathcal{A}}$) the two maps across the bottom are.)

A morphism $(M/A) \rightarrow (M'/A)$ of $(P_{(N,\phi)})/A$ is a diagram

$$\begin{array}{ccccccc}
 V & \leftarrow \bullet \circ V' & \leftarrow \bullet < Z & \succ \bullet \rightarrow N' & \circ \rightarrow N & \xlongequal{\quad} & N \\
 & & \downarrow Y & & \uparrow \wedge & & \uparrow \wedge \\
 & & A' & & X & \leftarrow \bullet < & \widehat{X} \\
 & & \downarrow \circ & & \downarrow \circ & & \downarrow \circ \\
 A & \leftarrow \bullet \circ M' & \leftarrow \bullet < M & \leftarrow \bullet < M_1 & \leftarrow \bullet \circ \widehat{M} & &
 \end{array}$$

where the morphism $\widehat{M} \rightarrow A$ in $Q\mathcal{C}$ is given by the composition across the bottom.

Let \mathcal{D} be the full subcategory of $(P_{(N,\phi)})/A$ of those objects which can be represented by a diagram where the morphism $X \circ \bullet \rightarrow M$ is the identity. Given any object represented by (10.19) there is a well-defined morphism given by

$$\begin{array}{ccccccc}
 V & \leftarrow \bullet \circ V' & \leftarrow \bullet < Z & \succ \bullet \rightarrow N' & \circ \rightarrow N & \xlongequal{\quad} & N \\
 & & \downarrow Y & & \uparrow \wedge & & \uparrow \wedge \\
 & & A' & & X & \xlongequal{\quad} & X \\
 & & \downarrow \circ & & \downarrow \circ & & \parallel \\
 A & \leftarrow \bullet \circ M' & \leftarrow \bullet < M & \xlongequal{\quad} M & \leftarrow \bullet \circ X & &
 \end{array}$$

which is natural in our object (since the choice of X is unique up to unique isomorphism). This shows that \mathcal{D} is a retractive subcategory of $(P_{(N,\phi)})/A$, and is thus homotopy equivalent to it.

A morphism inside \mathcal{D} is represented by a diagram

$$\begin{array}{ccccccc} V & \leftarrow \bullet \circ & V' & \leftarrow \bullet < & Z & > \bullet \rightarrow & N' \circ \bullet \rightarrow N = N \\ & & \swarrow & & & & \uparrow \\ & & A' \circ \bullet \rightarrow A & \leftarrow \bullet \circ & M' & \leftarrow \bullet < & M \leftarrow \bullet < \widehat{M} \end{array}$$

The only important information here is the lower-right-hand side. Thus we will think of morphisms in \mathcal{D} as represented by diagrams

$$N \leftarrow \bullet < M > \bullet \rightarrow M' \circ \bullet \rightarrow A$$

which are equivalent inside $\mathcal{C} \backslash \mathcal{A}$. Since all morphisms in \mathcal{M} are monic, such a morphism (if it exists) is unique; thus \mathcal{D} is a preorder. To show that \mathcal{D} is contractible we will show that it is nonempty and cofiltered.

Given two objects

$$N \leftarrow \bullet < M > \bullet \rightarrow M' \circ \bullet \rightarrow A \quad \text{and} \quad N \leftarrow \bullet < \widetilde{M} > \bullet \rightarrow \widetilde{M}' \circ \bullet \rightarrow A$$

we know that they are equivalent inside $\mathcal{C} \backslash \mathcal{A}$ if there exists a diagram $X \circ \bullet \rightarrow Y \bullet \rightarrow N$ such that precomposition by this diagram sends these to diagrams which are equivalent in \mathcal{C} . However, since \mathcal{A} is m-negligible in \mathcal{C} we see that such a diagram exists if and only if such a diagram exists with the e-component equal to the identity. Picking such a morphism $Y \bullet \rightarrow N$ we see that the object represented by

$$N \leftarrow \bullet < Y \times_N M > \bullet \rightarrow M' \circ \bullet \rightarrow A$$

maps to both of these objects. Thus \mathcal{D} is cofiltered.

To see that \mathcal{D} is nonempty, consider the diagram

$$Z \bullet \rightarrow N' \circ \bullet \rightarrow N$$

given by the chosen representative for ϕ . Since \mathcal{A} is m-negligible in \mathcal{C} there exists an m-morphism $M \bullet \rightarrow N$ such that $M \odot_N N' \cong M$ and $M \bullet \rightarrow N'$ factors through $Z \bullet \rightarrow N'$. Then the diagram

$$N \leftarrow \bullet < M > \bullet \rightarrow A' \circ \bullet \rightarrow A$$

gives a well-defined object of \mathcal{D} . Thus \mathcal{D} is nonempty and cofiltered, and therefore contractible. \square

If \mathcal{A} is m-negligible in \mathcal{C} we are now done. Thus we can now assume that \mathcal{A} is m-well-represented in \mathcal{C} .

Consider a diagram $N \xleftarrow{g_e} X \xrightarrow{g_m} N'$ which we denote g . We define the functor $g_*: \mathcal{H}_N \rightarrow \mathcal{H}_{N'}$ by

$$\begin{array}{ccc} M \xleftarrow{\bullet} Y \xrightarrow{\bullet} N & \longmapsto & \begin{array}{ccc} Y \otimes_N X \xrightarrow{\bullet} X \xrightarrow{g_m} N' \\ \downarrow \circlearrowleft & \circlearrowleft & \downarrow g_e \\ M \xleftarrow{\bullet} Y \xrightarrow{\bullet} N & & \end{array} \\ & \longmapsto & M \xleftarrow{\bullet} Y \otimes_N X \xrightarrow{\bullet} N' \end{array}$$

This is functorial because pseudo-commutative squares compose.

Lemma 10.20. *There is a natural transformation $k_N \Rightarrow k_{N'} g_*$.*

Proof. We have

$$k_N(M \xleftarrow{h_e} Y \xrightarrow{\bullet} N) = Y^{k/M}.$$

On the other hand,

$$k_{N'} g_*(M \xleftarrow{h_e} Y \xrightarrow{\bullet} N) = (Y \otimes_N X)^{k/M}.$$

The map $Y \otimes_N X \rightarrow M$ factors through $Y \rightarrow M$, so (by Lemma 2.10) there is a functorially induced map $Y^{k/M} \rightarrow (Y \otimes_N X)^{k/M}$. This map gives the natural transformation. To check that this is actually natural, consider a map $(M \xleftarrow{\bullet} Y \xrightarrow{\bullet} N) \rightarrow (M' \xleftarrow{\bullet} Y' \xrightarrow{\bullet} N)$ represented by $M \rightarrow M_1 \rightarrow M'$. We must show that the square

$$\begin{array}{ccccc} Y^{k/M} & \xrightarrow{\circlearrowleft} & Y^{k/M_1} & \xrightarrow{\circlearrowleft} & (Y')^{k/M'} \\ \downarrow & & & & \downarrow \\ (Y \otimes_N X)^{k/M} & \xrightarrow{\circlearrowleft} & (Y \otimes_N X)^{k/M_1} & \xrightarrow{\circlearrowleft} & (Y' \otimes_N X)^{k/M'} \end{array}$$

commutes in $Q\mathcal{A}$. To show this it suffices to show that there exists a map $Y^{k/M_1} \rightarrow (Y \otimes_N X)^{k/M_1}$ such that the left-hand square is distinguished and the right-hand square commutes. The map exists and makes the right-hand square commute by Lemma 2.10. To check that the left-hand square is distinguished it suffices to check that given any diagram

$$A \rightarrow B \rightarrow C \rightarrow D$$

the square

$$\begin{array}{ccc} B^{k/C} & \longrightarrow & A^{k/C} \\ \downarrow & & \downarrow \\ B^{k/D} & \longrightarrow & A^{k/D} \end{array}$$

is distinguished. This follows directly from the definition of c and k . \square

Since k_N and $k_{N'}$ are both homotopy equivalences, we get the following corollary:

Corollary 10.21. *g_* is a homotopy equivalence.*

Consider the functor $H: \mathcal{I}_V^n \rightarrow \mathbf{Cat}$ sending (N, ϕ) to \mathcal{H}_N and $g: (N, \phi) \rightarrow (N, \phi')$ to g_* .

Lemma 10.22. *There is an isomorphism of categories*

$$\tilde{H}: \operatorname{colim}_{\mathcal{I}_V^n} H \longrightarrow \mathcal{F}_V$$

induced by $P_{(N, \phi)}: \mathcal{H}_N \rightarrow \mathcal{F}_V$.

Proof. We first check that \tilde{H} is well-defined. To prove this it suffices to check that for $g: (N, \phi) \rightarrow (N', \phi')$,

$$P_{(N', \phi')} g_* = P_{(N, \phi)}.$$

Since morphisms in \mathcal{H}_N are defined to be morphisms in QC satisfying extra conditions, and since both $P_{(N, \phi)}$ and g_* do not change any of the representation data in the morphism, if the two sides agree on objects they must also agree on morphisms. $P_{(N, \phi)}$ maps an object $(M \leftarrow \bullet \circ X \rightarrow \bullet N)$ to the composition

$$V \xrightarrow{\phi^{-1}} sN \leftarrow \bullet \circ sX \rightarrow \bullet sM,$$

while $P_{(N', \phi')} g_*$ maps it to the composition

$$V \xrightarrow{\phi'^{-1}} sN' \leftarrow \bullet \circ sY \rightarrow \bullet sN \leftarrow \bullet \circ sX \rightarrow \bullet sM.$$

However, since $\phi' s(g) = \phi$, these two compositions represent equivalent diagrams (since after being considered inside $\mathcal{C} \setminus \mathcal{A}$, all g_* does is compose with g) and thus the left and right sides agree on objects. Therefore the functors $P_{(N, \phi)}$ produce a valid cone under H and \tilde{H} is well-defined.

It now remains to show that it is, in fact, an isomorphism of categories.

First we show that \tilde{H} is surjective on objects; in other words, that for every $(M, u: V \xrightarrow{\cong} sM)$ in \mathcal{F}_V there exists an (N, ϕ) and an object (M', h) in \mathcal{H}_N such that $P_{(N, \phi)}(M', h) = (M, u)$. To do this, let $(N, \phi) = (M, u^{-1})$ and let $(M', h) = (M, M = M = M)$. Thus \tilde{H} is surjective on objects.

Now consider injectivity. Since \mathcal{I}_V^m is filtered, it suffices to check that each individual $P_{(N, \phi)}$ is injective on objects. Suppose that

$$P_{(N, \phi)}(M, h) = P_{(N, \phi)}(M', h').$$

We must show that there exists $g: (N, \phi) \rightarrow (N', \phi')$ in \mathcal{I}_V^m such that $g_*(M, h) = g_*(M', h')$. Note, that by definition in order for this to hold we must have $M = M'$ and $s(h) = s(h')$. The fact that such a g exists is implied by condition (E); in fact, this g will be represented by a morphism where the m-component is the identity. Thus \tilde{H} is injective on objects.

We now consider morphisms. As before, we consider surjectivity first. Consider a morphism $g: (M, u) \rightarrow (M', u')$ in \mathcal{F}_V . This is given by a morphism $g: M \rightarrow M'$ in QC such that $s(g)u = u'$ in $Q(\mathcal{C} \setminus \mathcal{A})$. Since both u and u' are isomorphisms, $s(g)$ must be as well; thus it is represented by a diagram $M \circ \bullet \rightarrow X \triangleright \bullet \rightarrow M'$. Consider the distinguished square

$$\begin{array}{ccc} M & \xrightarrow{h_m} & X' \\ g_e \downarrow \bullet & \square & \bullet \downarrow h_e \\ X & \xrightarrow{g_m} & M \end{array}$$

where the composition around the bottom is given by the components of g . Since all distinguished squares are pseudo-commutative, this defines a morphism

$$(M, M \triangleright \bullet \xrightarrow{h} {}_m X') \xrightarrow{f} (M', M' \xleftarrow{h_e} \bullet \circ X')$$

in $\mathcal{H}_{X'}$. Note that $s(g_e)u = s(h_e^{-1})u'$. Thus $P_{(X, s(g_e)u)}(f) = g$, as desired.

Now consider injectivity. As before, it suffices to consider a single $P_{(N, \phi)}$ and show that it is faithful. Suppose that $P_{(N, \phi)}(g) = P_{(N, \phi)}(g')$. By definition,

$$g, g': (M \leftarrow \bullet \circ X \rightarrow \bullet \rightarrow N) \longrightarrow (M' \leftarrow \bullet \circ X' \rightarrow \bullet \rightarrow N)$$

are given by morphisms $\tilde{g}, \tilde{g}': M \rightarrow M'$ in QC satisfying the diagram in Definition 10.9. For $P_{(N, \phi)}(g) = P_{(N, \phi)}(g')$ we must have $\tilde{g} = \tilde{g}'$; however, in this case we must have g and g' equal as well. Thus \tilde{H} is injective on morphisms, and we are done. \square

We are now ready to finish:

Proposition 10.23. *If \mathcal{A} is m -well-represented in \mathcal{C} then $P_{(N, \phi)}$ is a homotopy equivalence.*

Proof. [10, Proposition 3, Corollary 1] states the following: given any filtered category \mathcal{C} and a functor $F: \mathcal{C} \rightarrow \mathbf{Cat}$ such that for all $f: A \rightarrow B \in \mathcal{C}$, $F(f)$ is a homotopy equivalence. Then the induced map $F(A) \rightarrow \operatorname{colim}_{\mathcal{C}} F$ is a homotopy equivalence for all $A \in \mathcal{C}$.

Applying this to the functor H , we get that the map $H(N, \phi) \rightarrow \operatorname{colim}_{\mathcal{I}_V^m} H \cong \mathcal{F}_V$ is a homotopy equivalence for all $(N, \phi) \in \mathcal{I}_V^m$. By definition this is exactly $P_{(N, \phi)}: \mathcal{H}_N \rightarrow \mathcal{F}_V$, and we are done. \square

Appendix A. Checking that $\mathcal{C} \setminus \mathcal{A}$ is a CGW-category

In this appendix we check as much as possible that the definition of $\mathcal{C} \setminus \mathcal{A}$ gives a well-defined CGW-category. More concretely, it is necessary to check that the m -morphisms and e -morphisms give well-defined categories, that the distinguished squares compose correctly, that ϕ exists, that c and k are equivalences of categories, and that axioms (Z), (I), (M), (K), and (A) hold.

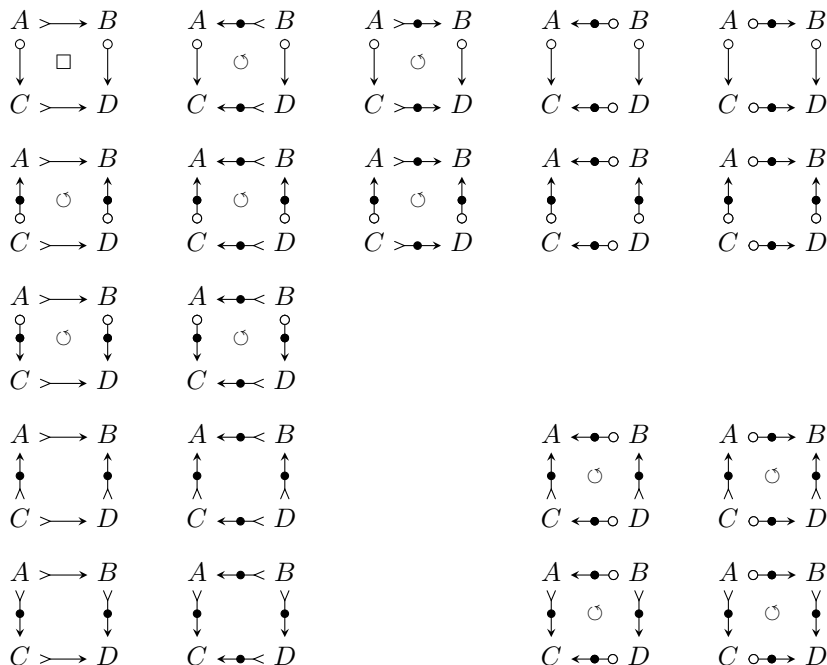
Proposition A.1. *Let \mathcal{C} be an ACGW-category, \mathcal{A} a full ACGW-subcategory closed under subobjects, quotients, and extensions. Then $\mathcal{C} \setminus \mathcal{A}$ is a well-defined CGW-category assuming that the following condition holds:*

(Ex) *The definitions of c and k in Definition 8.1 give equivalences of categories, in the sense that there exist equivalences of categories $k: \operatorname{Ar}_{\square} \mathcal{E} \rightarrow \operatorname{Ar}_{\triangle} \mathcal{M}$ and $c: \operatorname{Ar}_{\square} \mathcal{M} \rightarrow \operatorname{Ar}_{\triangle} \mathcal{E}$ which agree with the given definitions on objects.*

The rest of this appendix is a proof of this proposition.

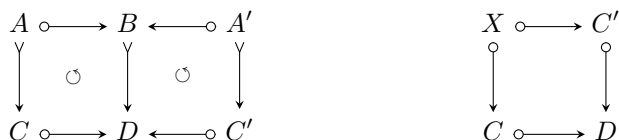
As the definition of $\mathcal{C} \setminus \mathcal{A}$ is symmetric with respect to e -morphisms and m -morphisms it suffices to focus on proving only half of each statement; the other half will follow by symmetry.

We first begin with a somewhat more explicit definition of the distinguished squares in $\mathcal{C} \setminus \mathcal{A}$. These are generated by the following types of squares:

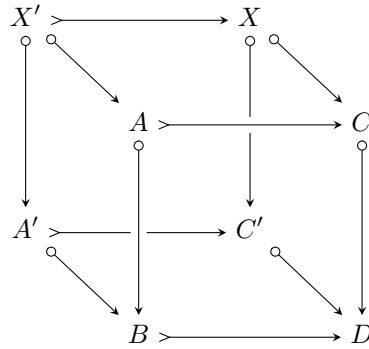


We now prove a series of lemmas about how different types of squares in \mathcal{C} interact. The common consequence of all of these lemmas is that the given squares fit into a cube with opposite sides of the same “type” (be that pseudo-commutative squares, distinguished squares, or simply squares that commute inside \mathcal{E} or \mathcal{M}). We do not worry about which arrows have c or k in \mathcal{A} ; the properties of \mathcal{A} ensure that whenever such an arrow is “pulled back”, the pullback also has c or k in \mathcal{A} .

Lemma A.2. *Given two diagrams in \mathcal{C}*



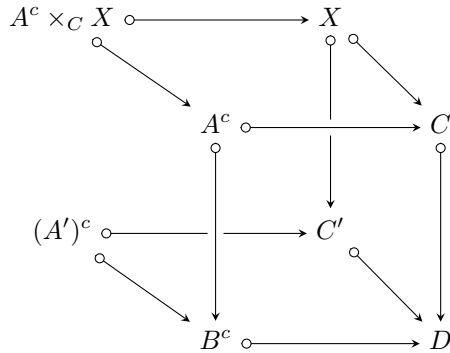
we can assemble these into a cube



in which all faces with mixed morphisms are pseudo-commutative. If $ABCD$ was originally distinguished, then $X'A'XC'$ will be, as well.

An analogous statement with the roles of e -morphisms and m -morphisms swapped also holds.

Proof. Apply c to the left-hand diagram. This turns both of the squares into pullback squares in \mathcal{E} (by definition). We can then form the following diagram:



To prove the main statement of the lemma it suffices to show that a morphism $A^c \times_C X \rightarrow (A')^c$ exists and makes the back face into a pullback. To show the last statement it suffices to show that if $A^c \rightarrow B^c$ is an isomorphism then this morphism is also an isomorphism. This is a straightforward diagram chase using the fact that all solid faces in the above diagram are pullbacks and all morphisms in \mathcal{E} are monic. \square

As a corollary we can see that assembling distinguished squares and pullbacks commute:

Corollary A.3. Suppose that we are given a diagram

$$A \rightrightarrows B \circ \rightarrow C \longleftarrow \circ D.$$

The two diagrams

$$\begin{array}{ccc}
 A \rightrightarrows B & \longleftarrow \circ & B \times_C D \\
 \downarrow \circ & \square & \downarrow \circ \\
 X \rightrightarrows C & \longleftarrow \circ & D
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccccc}
 A \otimes_B (B \times_C D) & \rightrightarrows & B \times_C D & \circ \rightrightarrows & D \\
 \downarrow \circ & & \downarrow \circ & & \downarrow \circ \\
 A & \rightrightarrows & B & \circ \rightrightarrows & C
 \end{array}$$

fit into a cube

$$\begin{array}{ccccc}
 A & \xrightarrow{\quad} & B & & \\
 \downarrow \circ & \searrow & \downarrow \circ & \searrow & \\
 & X & \xrightarrow{\quad} & C & \\
 \uparrow \circ & \uparrow & \uparrow \circ & \uparrow & \\
 A \otimes_B (B \times_C D) & \xrightarrow{\quad} & B \times_C D & & \\
 \downarrow \circ & \searrow & \downarrow \circ & \searrow & \\
 & W & \xrightarrow{\quad} & D & \\
 \uparrow \circ & \uparrow & \uparrow \circ & \uparrow & \\
 & X & \xrightarrow{\quad} & C & \\
 \downarrow \circ & \searrow & \downarrow \circ & \searrow & \\
 & B & \xrightarrow{\quad} & C & \\
 \uparrow \circ & \uparrow & \uparrow \circ & \uparrow & \\
 A & \xrightarrow{\quad} & B & &
 \end{array}$$

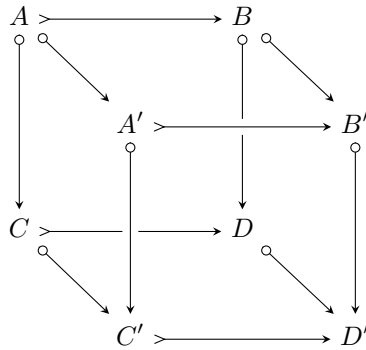
in which the top and bottom face are distinguished squares, the front and the back face are pseudo-commutative squares, and the right and left face are commutative in \mathcal{E} with the right-hand face a pullback.

We now prove a “complement” to Lemma 5.12: instead of assuming that a commutative square in \mathcal{E} is attached to the back of a pseudo-commutative square, we assume that it is attached to the front:

Lemma A.4. Suppose that we are given a diagram

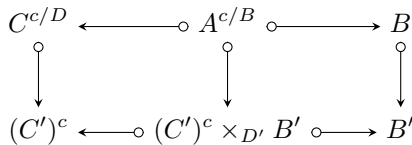
$$\begin{array}{ccccc}
 A \rightrightarrows B & \circ \rightrightarrows & B' \\
 \downarrow \circ & \circ & \downarrow \circ \\
 C \rightrightarrows D & \circ \rightrightarrows & D'
 \end{array}$$

Then this diagram assembles into a cube



where the front, back, and top faces are pseudo-commutative and the bottom face is distinguished. If the right-hand square is a pullback then the top face will also be distinguished. The dual statement also holds.

Proof. Define C' so that the bottom face of the cube is a distinguished square. Define $A' = C' \otimes_{D'} B'$. By definition this produces a diagram where the front face is pseudo-commutative and the bottom face is distinguished. It therefore suffices to check that there exists a morphism $A \circ \rightarrow A'$ such that the left face commutes in \mathcal{E} and the top face is pseudo-commutative. To prove this it suffices to check that there exists a morphism $A^{c/B} \circ \rightarrow B' \times_{D'} (C')^c$ such that in the diagram



the left-hand square commutes and the right-hand square is a pullback. This follows directly from the definitions. \square

We are now ready to turn our attention to proving that $\mathcal{C} \backslash \mathcal{A}$ is a CGW-category.

The m-morphisms form a well-defined category

The m-morphisms in $\mathcal{C} \backslash \mathcal{A}$ are defined to be equivalence classes of diagrams

$$A \xleftarrow{\bullet} A' \xleftarrow{\bullet} X \xrightarrow{\bullet} B' \xrightarrow{\bullet} B.$$

The equivalence relation is generated by the following types of diagrams (up to isomorphism), where the red diagram is declared to be equivalent to the blue diagram:

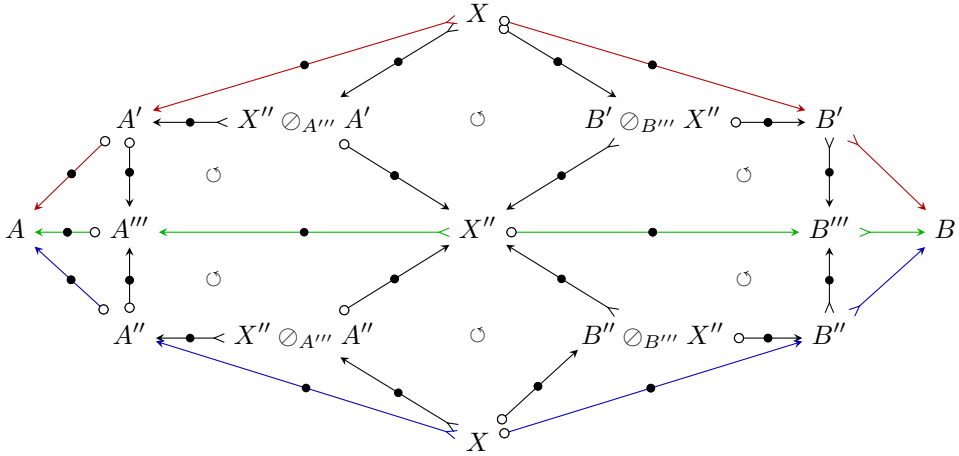
(A.5)

(For interpretation of the colors in the figures, the reader is referred to the web version of this article.) The relation defined between m-morphisms is a formal composition of two such relations, one inverse to another. Thus to show that the relation is well-defined we must check that if we are given two such relations built on top of one another, then either they compose to a single one, or that we can “pull back” two such relations.

Let us consider the first such case. Suppose that we are given two such diagrams, one relating $A \leftarrow \bullet \bullet A' \leftarrow \bullet \bullet X \circ \bullet \bullet B' \rightrightarrows B$ to $A \leftarrow \bullet \bullet A'' \leftarrow \bullet \bullet X' \circ \bullet \bullet B'' \rightrightarrows B$, and one relating $A \leftarrow \bullet \bullet A'' \leftarrow \bullet \bullet X' \circ \bullet \bullet B'' \rightrightarrows B$ to $A \leftarrow \bullet \bullet A''' \leftarrow \bullet \bullet X'' \circ \bullet \bullet B''' \rightrightarrows B$. We can rearrange this data into the following diagram, where the first formal composition is in red, the second is in blue, and the third is in green:

By regrouping the pseudo-commutative squares, we see that the red composition is equivalent to the green composition, as desired.

To show the second case, consider the following diagram, which shows that red and blue are both equivalent to green:

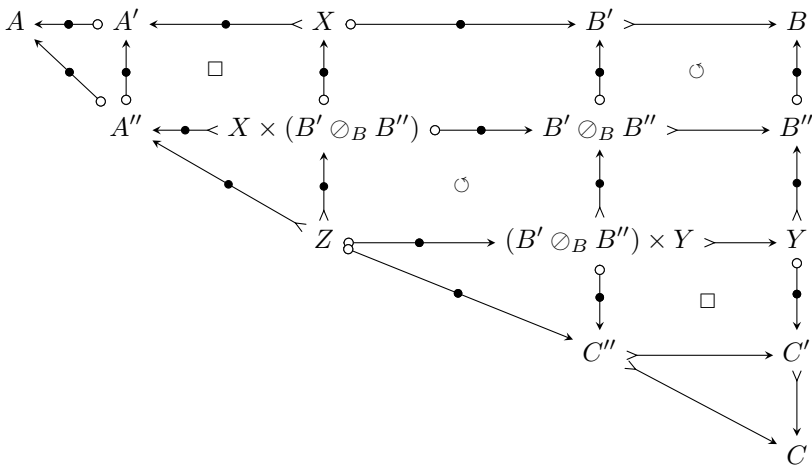


Then the composition

$$A \xleftarrow{\bullet \circ} A' \times_{A'''} A'' \xleftarrow{\bullet \circ} ((B' \times_{B'''} B'') \otimes_{B'''} X'') \otimes_{X''} (X'' \otimes_{A'''} (A' \times_{A'''} A'')) \\ \circ \bullet \rightarrow B' \times_{B'''} B'' \xrightarrow{\bullet} B$$

is equivalent to both the red and the blue, completing the desired picture. Putting these together shows that the relation defined on m-morphisms is an equivalence relation, as desired.

Now we can work with the definition of the m-morphisms directly. Given two morphisms $A \multimap B$ and $B \multimap C$ their composition is defined to be represented by the diagonal in the following square:



Here, $Z = (X \times (B' \circ_B B'')) \circ_{B' \circ_B B''} ((B' \circ_B B'') \times Y)$ and A'' and C'' are uniquely determined by the distinguished squares they are in.

To check that this is well-defined, it suffices to check that given a diagram as in (A.5) and a morphism represented as one of $\leftarrow \bullet \circ$, $\leftarrow \bullet \prec$, $\circ \bullet \rightarrow$ or $\succ \bullet \rightarrow$ the composition (resp. precomposition) with the red morphism and the composition (resp. precomposition) with the blue morphism are equivalent. We check the case of composing with a morphism represented by $\leftarrow \bullet \circ$; all of the other cases are analogous. This is a straightforward diagram chase, using Lemma A.2 to push the diagram showing the equivalence of the two representations along the composition; the only nontrivial part is ensured by Lemma 5.12.

We need to check that composition is associative. As a morphism is a formal composition of four arrows, it suffices to check that compositions of those component arrows are associative. It is not necessary to worry about which morphisms have kernel/cokernel in \mathcal{A} , since that is preserved by the definition of composition; all we are checking is associativity. Thus our definition of morphism is symmetric in e-morphism and m-morphism. In addition, since both \mathcal{E} and \mathcal{M} are closed under pullbacks, by standard arguments about span categories we know that when all three morphisms are e-morphisms or all three morphisms are m-morphisms composition is associative. Thus it remains to consider the case of 2 m-morphisms and 1 e-morphism or 1 m-morphism and 2 e-morphisms. By symmetry again it suffices to consider this second case, and, in fact, it suffices to consider the case when the m-morphism is directed covariantly with the composition.

Now there are 12 cases left (three positions for the m-morphism and four directions in which the e-morphisms can point). Most of these have only a single composition, so associativity holds automatically for these. The remaining three cases are $\succ \rightarrow \circ \rightarrow \circ \rightarrow$, $\succ \rightarrow \leftarrow \circ \leftarrow \circ$ and $\succ \rightarrow \circ \rightarrow \leftarrow \circ$. The first and second of these give associative compositions because distinguished and pseudo-commutative squares work correctly with respect to composition. Thus the last case is the only one of interest, which directly follows from Corollary A.3. The fact that the two different compositions assemble into a cube implies that they are equivalent in $\mathcal{C} \setminus \mathcal{A}$.

Distinguished squares compose correctly This is true by definition.

There exists a ϕ We must show that the subcategory of m-isomorphisms is isomorphic to the category of e-isomorphisms by a functor which takes objects to themselves. To construct this functor, use Lemma 2.9 to change a representation of an m-isomorphism as

$$A \leftarrow \bullet \circ A' \leftarrow \bullet \prec X \circ \bullet \rightarrow B' \succ \bullet \rightarrow B$$

to

$$A \leftarrow \bullet \prec A'' \leftarrow \bullet \circ X \succ \bullet \rightarrow B'' \circ \bullet \rightarrow B,$$

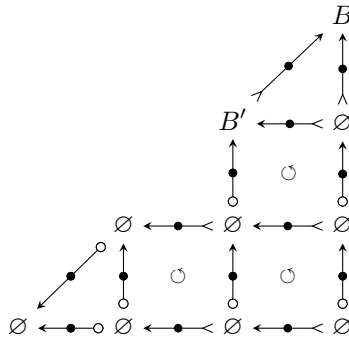
which gives a representation of an e-isomorphism. Since distinguished squares are unique up to unique isomorphism, this is an isomorphism of categories.

Axiom (Z) We must check that \emptyset is initial in \mathcal{M} .

There exists a morphism $\emptyset \twoheadrightarrow B$ for any B by simply taking the representation where all but the last morphism are the identity. We must now check that this morphism is unique. Suppose that we are given any diagram

$$\emptyset \leftarrow \bullet \circ \emptyset \leftarrow \bullet \prec \emptyset \circ \bullet \rightarrow B' \twoheadrightarrow B.$$

We must have $B' \in \mathcal{A}$ for this diagram to be valid. The diagram



shows that the two are equivalent. Thus \emptyset is horizontally initial.

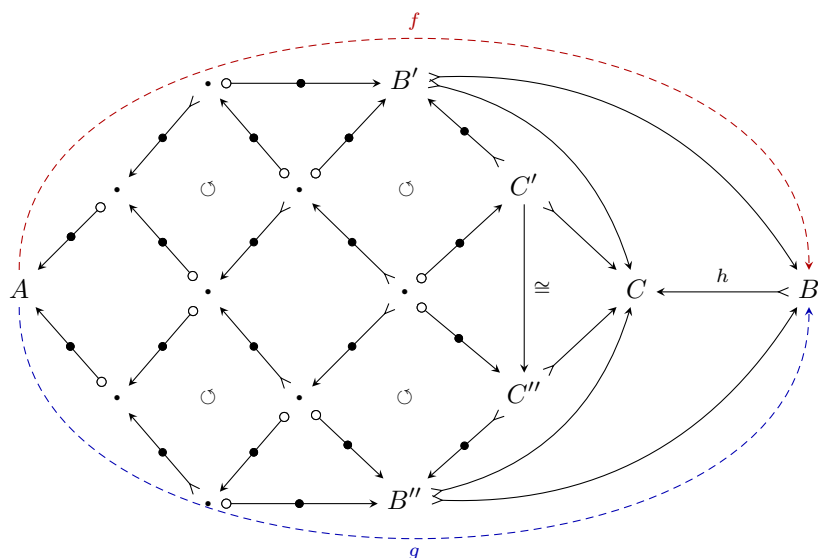
Axiom (I) The m-morphisms which are isomorphisms are exactly those morphisms of the form

$$A \leftarrow \bullet \circ \leftarrow \bullet \prec \circ \bullet \rightarrow \twoheadrightarrow \bullet B.$$

Using this description and the listing of different kinds of distinguished squares we can construct each of the required squares by hand.

Axiom (M) It suffices to check this for the m-morphisms of $\mathcal{C} \setminus \mathcal{A}$; the statement for the e-morphisms will follow by symmetry. Thus we want to check that if we are given two morphisms $f, g: A \twoheadrightarrow B$ and a morphism $h: B \twoheadrightarrow C$ in $\mathcal{C} \setminus \mathcal{A}$ then if $hf = hg$ then $f = g$. All morphisms in \mathcal{M} are equal, up to isomorphism, to ones represented by diagrams $\bullet \twoheadrightarrow \bullet$. Thus it suffices to assume that h is of this form. This means that the compositions hf and hg are computed simply by composing the last m-morphism components.

The fact that $hf = hg$ implies that for any choice of representatives for f and g , the following diagram exists:



To show that $f = g$ it suffices to check that there exist maps $C' \twoheadrightarrow B$ and $C'' \twoheadrightarrow B$ such that the triangle

$$\begin{array}{ccc} C' & & \\ \cong \downarrow & \searrow & \\ C'' & \nearrow & B \end{array}$$

commutes. Setting these maps to be the evident ones generated by the above diagram, we see that the given triangle must commute, as it commutes after postcomposition with h and h is monic.

Axiom (K) As before, we prove this only for c ; the result for k follows by symmetry.

Let $f: A \twoheadrightarrow B$ be a morphism. Given a representative

$$A \leftarrow \bullet \circ A' \leftarrow \bullet \circ X \circ \bullet \rightarrow B' \twoheadrightarrow B$$

of f , we can conclude that $c(f) \cong (B')^c \circ \rightarrow B$. Thus if we can show that a distinguished square as desired exists for this representative, we will be done. The following diagram shows that this is the case

$$\begin{array}{ccccccc} \emptyset & \leftarrow \bullet \circ & \emptyset & \leftarrow \bullet \circ & \emptyset & \circ \bullet \rightarrow & \emptyset \twoheadrightarrow (B')^c \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \leftarrow \bullet \circ & A' & \leftarrow \bullet \circ & X & \circ \bullet \rightarrow & B' \twoheadrightarrow B \end{array}$$

as it is a composition of squares which are distinguished in $\mathcal{C} \backslash \mathcal{A}$.

Axiom (A) This holds because it holds inside \mathcal{C} and all distinguished squares in \mathcal{C} are also distinguished in $\mathcal{C} \backslash \mathcal{A}$. \square

References

- [1] M. Barratt, S. Priddy, On the homology of non-connected monoids and their associated groups, *Comment. Math. Helv.* 47 (1972) 1–14, MR 0314940.
- [2] J.A. Campbell, The K -theory spectrum of varieties, *Trans. Am. Math. Soc.* 371 (11) (2019) 7845–7884, MR 3955537.
- [3] A. Grothendieck, P. Colmez, J.P. Serre, Grothendieck–Serre Correspondence, American Mathematical Society, 2004.
- [4] C. Ehresmann, Catégories structurées, *Ann. Sci. Éc. Norm. Supér. (3)* 80 (1963) 349–426, MR 0197529 (33 #5694).
- [5] A. Grothendieck, Sur quelques points d’algèbre homologique, i, *Tohoku Math. J. (2)* 9 (2) (1957) 119–221.
- [6] H. Gillet, C. Soulé, Descent, motives and K -theory, *J. Reine Angew. Math.* 478 (1996) 127–176.
- [7] M. Kontsevich, Notes on motives in finite characteristic, in: *Algebra, Arithmetic, and Geometry: In Honor of Yu. I. Manin. Vol. II*, in: *Progr. Math.*, vol. 270, Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 213–247, MR 2641191.
- [8] T. Leinster, Structures in higher-dimensional category theory, arXiv:math/0109021.
- [9] J.S. Milne, Motives—Grothendieck’s dream, in: *Open Problems and Surveys of Contemporary Mathematics*, in: *Surv. Mod. Math.*, vol. 6, Int. Press, Somerville, MA, 2013, pp. 325–342, MR 3204952.
- [10] D. Quillen, Higher algebraic K -theory: I, in: *Algebraic K-Theory, I: Higher K-Theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, Springer, 1973, pp. 85–147.
- [11] K. Schwede, Gluing schemes and a scheme without closed points, in: *Recent Progress in Arithmetic and Algebraic Geometry*, in: *Contemp. Math.*, vol. 386, Amer. Math. Soc., 2005, pp. 157–172.
- [12] G. Segal, Configuration-spaces and iterated loop-spaces, *Invent. Math.* 21 (1973) 213–221, MR 0331377.
- [13] F. Waldhausen, Algebraic K -theory of generalized free products. I, II, *Ann. Math. (2)* 108 (1) (1978) 135–204, MR 0498807.
- [14] F. Waldhausen, Algebraic K -theory of spaces, in: *Algebraic Geometry and Topology (New Brunswick, N.J., 1983)*, in: *Lecture Notes in Mathematics*, vol. 1126, Springer, 1985.
- [15] C.A. Weibel, The K -Book: An Introduction to Algebraic K -Theory, *Graduate Studies in Mathematics*, vol. 145, American Mathematical Society, Providence, RI, 2013, MR 3076731.
- [16] I. Zakharevich, The K -theory of assemblers, *Adv. Math.* 304 (2017) 1176–1218, MR 3558230.
- [17] I. Zakharevich, On K_1 of an assembler, *J. Pure Appl. Algebra* 221 (7) (2017) 1867–1898, MR 3614978.