

GRADIENT ESTIMATES FOR STOKES AND NAVIER–STOKES SYSTEMS WITH PIECEWISE DMO COEFFICIENTS*

JONGKEUN CHOI[†], HONGJIE DONG[‡], AND LONGJUAN XU[§]

Abstract. We study stationary Stokes systems in divergence form with piecewise Dini mean oscillation (DMO) coefficients and data in a bounded domain containing a finite number of subdomains with $C^{1,\text{Dini}}$ boundaries. We prove that if (u, p) is a weak solution of the system, then (Du, p) is bounded and piecewise continuous. The corresponding results for stationary Navier–Stokes systems are also established, from which the Lipschitz regularity of the stationary H^1 -weak solution in dimensions $d = 2, 3, 4$ is obtained. Our results can be applied to stationary Stokes systems and Navier–Stokes systems with the second-order term $\text{div}(\tau Su)$, where $Su = \frac{1}{2}(Du + (Du)^\top)$ is the strain tensor and τ is a positive piecewise DMO scalar function.

Key words. Stokes system, piecewise Dini mean oscillation, gradient estimate

AMS subject classifications. 76D07, 35B65, 35J47

DOI. 10.1137/21M1423518

1. Introduction. In this paper, we consider stationary Stokes systems with variable coefficients

$$(1.1) \quad \begin{cases} \mathcal{L}u + \nabla p = D_\alpha f_\alpha & \text{in } \mathcal{D}, \\ \text{div } u = g & \text{in } \mathcal{D}. \end{cases}$$

The differential operator $\mathcal{L}$ is in divergence form acting on column vector valued functions $u = (u^1, \dots, u^d)^\top$ as follows:

$$(1.2) \quad \mathcal{L}u = D_\alpha (A^{\alpha\beta} D_\beta u),$$

where we use the Einstein summation convention over repeated indices. The domain $\mathcal{D}$ is bounded in $\mathbb{R}^d$, which consists of a finite number of disjoint subdomains, and the coefficients $A^{\alpha\beta} = A^{\alpha\beta}(x)$ can have jump discontinuities along the boundaries of the subdomains. As is well known, such a system is partly motivated by the study of composite materials with closely spaced interfacial boundaries. In an earlier work [31], Masliyah et al. studied the creeping flow past a solid sphere with porous shell by using the Brinkman equation for the flow field inside the fluid permeable surface layer and the Stokes equations for the flow field external to the particle. Jaiswal and Gupta [21] investigated Stokes flow over the composite sphere filled with Reiner–Rivlin liquid and coated with the porous layer. Also, the system can be used to model the motion of inhomogeneous fluids with density-dependent viscosity (see, for instance,

*Received by the editors June 1, 2021; accepted for publication (in revised form) February 25, 2022; published electronically June 16, 2022.

<https://doi.org/10.1137/21M1423518>

Funding: The work of the first author was supported by the National Research Foundation of Korea (NRF) through grant NRF-2019R1F1A1058826. The work of the second author was partially supported by the Simons Foundation through grant 709545.

[†]Department of Mathematics Education, Pusan National University, Busan 46241, Republic of Korea (jongkeun.choi@pusan.ac.kr).

[‡]Division of Applied Mathematics, Brown University, Providence, RI 02912 USA (Hongjie.Dong@brown.edu).

[§]Department of Mathematics, National University of Singapore, 119076, Singapore (ljxu311@163.com).

[25, 30, 1]). Moreover, it has a close connection to the motion of two fluids with interfacial boundaries. In [12], Dong and Kim studied L_q -estimates under the assumption that the coefficients $A^{\alpha\beta}$ are merely measurable in one direction and satisfy the bounded mean oscillation (BMO) condition in orthogonal directions. Such type of coefficients was first introduced by Krylov [24] and is called a (variably) partially BMO coefficient. For further results about Stokes systems with irregular coefficients, we refer the reader to [23, 13, 28], the work [8] for Stokes systems with partially Dini mean oscillation (DMO) coefficients, and the recent paper [22] on systems with L^∞ viscosity coefficients.

System (1.1) is also related to hydrodynamic interactions in soft matter systems. This is reduced to the study of stress (represented by Du) concentration in high-contrast composites with densely packed inclusions whose material properties differ from that of the background. In [3], Ammari et al. investigated the stress concentration of Stokes flow between adjacent circular cylinders.

There is a large body of literature concerning regularity theory for partial differential equations/systems with coefficients which satisfy some proper piecewise continuous conditions arising from the problems of composite materials. For the theory of second-order elliptic equations/systems in divergence form, $W^{1,\infty}$ - and piecewise $C^{1,\delta'}$ -estimates were obtained by Li and Vogelius [27] for elliptic equations with piecewise C^δ coefficients in a domain which consists of a finite number of disjoint subdomains with $C^{1,\mu}$ boundaries, where $0 < \mu \leq 1$ and $0 < \delta' \leq \min\{\delta, \frac{\mu}{d(1+\mu)}\}$. A similar result was proved for systems in [26], where $0 < \delta' \leq \min\{\delta, \frac{\mu}{2(1+\mu)}\}$. The results in [26] were extended by the second and third authors [17] to the system with piecewise DMO coefficients and subdomains having $C^{1,\text{Dini}}$ boundaries. They also established piecewise $C^{1,\delta'}$ -estimate for solutions under the same conditions and with $0 < \delta' \leq \min\{\delta, \frac{\mu}{1+\mu}\}$. See also [18] for the corresponding results for parabolic systems. It is important to remark that the subdomains are allowed to touch each other in [27, 26, 17, 18] and these results are independent of the distance between subdomains. For more related results, one can refer to [7, 5, 2, 16, 15, 34] and the references therein. In particular, for 2D elliptic equations, Mateu, Orobitg, and Verdera [32] derived $C^{1,\delta'}$ -regularity in each subdomain with $0 < \delta' < \min\{\delta, \mu\}$ by assuming that the determinant of the coefficients equals 1. The works [10, 11, 33] contain $C^{1,\delta'}$ -regularity results for $\delta' = \min\{\delta, \mu\}$. However, the estimates there depend on the distance between subdomains.

Inspired by the work [8, 17] mentioned above, we are interested in gradient estimates for Stokes systems with piecewise DMO coefficients. The goal of this paper consists of two aspects. We first extend the results in [17] for elliptic systems to the stationary Stokes systems (1.1). Precisely, we show in Theorem 2.4 that if the coefficients and data are of piecewise Dini mean oscillation and the boundaries of subdomains are $C^{1,\text{Dini}}$, then for every weak solution $(u, p) \in W^{1,q}(\mathcal{D})^d \times L^q(\mathcal{D})$ to (1.1), $q \in (1, \infty)$, Du and p are locally bounded and piecewise continuous. As an application, we obtain piecewise Hölder continuity for Du and p under Hölder regularity assumptions on the coefficients and the boundaries of the subdomains. We remark that the corresponding estimates are independent of the distance between subdomains so that the boundaries of more than two subdomains can touch at some points. We also prove a local $W^{1,q}$ -estimate for $W^{1,1}$ -weak solutions in Corollary 2.8 by exploiting the argument in [6, 4] combined with Theorem 2.4.

Second, we consider the stationary Navier–Stokes systems

$$\begin{cases} \mathcal{L}u + \nabla p + u^\alpha D_\alpha u = D_\alpha f_\alpha & \text{in } \mathcal{D}, \\ \operatorname{div} u = g & \text{in } \mathcal{D}. \end{cases}$$

We obtain that any $W^{1,q}$ -solution is Lipschitz and piecewise C^1 , where $q \in [d/2, \infty)$; see Theorem 2.10 for details. This result can be applied to H^1 -weak solutions to stationary Navier–Stokes systems with piecewise DMO coefficients in dimensions $d = 2, 3, 4$. Related work can be found in [19], in which the author considered the Laplace operator and proved the smoothness of every weak solution for $d = 4$, provided the data are good enough.

Let us briefly describe our arguments based on Campanato’s approach. Such an approach was used in [20, 29] and further developed in [11, 14, 8, 17]. The key point is to show that the mean oscillations of Du and p in balls vanish in a certain order as the radii of the balls go to zero. Recalling the nature of the domain and the coefficients, Du and p are discontinuous in one direction, say, x^d , which is the main challenge in this paper. We overcome this difficulty by choosing a coordinate system according to the geometry of the subdomains and then using the weak type-(1, 1) estimates obtained in [8, Lemma 3.4] to control the $L^{1/2}$ -mean oscillations of $D_{x'}u$ and the linear combinations $A^{d\beta}D_\beta u + pe_d - f_d$; see Lemma 3.1 for details. We point out that the proof in our case is more involved than that in [8] since our arguments and estimates depend on the coordinate system, and also more involved than that in [17] because of the pressure term and the divergence equation in the Stokes systems (1.1). For example, in the proof of local boundedness of Du and p (see Step 4 in the proof of Theorem 2.4), an additional difficulty appears from the pressure term on the right-hand side after the localization. For this, we adapt a delicate approximation argument and the fixed point theorem.

The rest of the paper is organized as follows. In section 2, we fix our notation, introduce function spaces and assumptions on the domain, coefficients, and data, and then state our main results, Theorem 2.4 for stationary Stokes systems, and Theorem 2.10 for stationary Navier–Stokes systems. In section 3, we provide the proofs of the main theorems.

2. Assumptions and main results. We first fix some notation and common definitions used throughout the paper. We use $x = (x', x^d)$ to denote a generic point in the Euclidean space $\mathbb{R}^d$, where $d \geq 2$ and $x' = (x^1, \dots, x^{d-1}) \in \mathbb{R}^{d-1}$. We also write $y = (y', y^d)$ and $x_0 = (x'_0, x_0^d)$, etc. For $r > 0$, we denote

$$B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r\}, \quad B'_r(x') = \{y' \in \mathbb{R}^{d-1} : |y' - x'| < r\}.$$

We often write B_r and B'_r instead of $B_r(0)$ and $B'_r(0)$, respectively. For $k \in \{1, \dots, d\}$, we use e_k to denote the k th unit vector in $\mathbb{R}^d$.

Let Ω be a domain in $\mathbb{R}^d$. For $q \in (0, \infty]$, we define

$$\tilde{L}^q(\Omega) = \{f \in L^q(\Omega) : (f)_\Omega = 0\},$$

where $(f)_\Omega$ is the average of f over Ω , i.e.,

$$(f)_\Omega = \int_\Omega f \, dx = \frac{1}{|\Omega|} \int_\Omega f \, dx.$$

For $q \in [1, \infty]$, we denote by $W^{1,q}(\Omega)$ the usual Sobolev space and by $W_0^{1,q}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in $W^{1,q}(\Omega)$, where $C_0^\infty(\Omega)$ is the set of all infinitely differentiable functions with compact support in Ω .

For $0 < \gamma < 1$, the Hölder seminorm is defined by

$$[u]_{C^\gamma(\Omega)} := \sup_{\substack{x, y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\gamma},$$

and the partial Hölder seminorm with respect to x' is defined by

$$[u]_{C_{x'}^\gamma(\Omega)} := \sup_{\substack{x, y \in \Omega \\ x' \neq y', x^d = y^d}} \frac{|u(x) - u(y)|}{|x' - y'|^\gamma}.$$

By $C^\gamma(\Omega)$ we denote the set for all bounded measurable functions u satisfying $[u]_{C^\gamma(\Omega)} < \infty$.

We say that a function $\omega : [0, 1] \rightarrow [0, \infty)$ is a Dini function if it is monotonically increasing and satisfies

$$(2.1) \quad \int_0^1 \frac{\omega(t)}{t} dt < +\infty.$$

We also say that a function f defined on Ω is Dini continuous if the function $\varrho_f : [0, 1] \rightarrow [0, \infty)$ given by

$$\varrho_f(t) = \sup_{\substack{x, y \in \Omega \\ |x - y| \leq t}} |f(x) - f(y)|$$

is a Dini function.

DEFINITION 2.1. Let $f \in L^1(\Omega)$. We say that f is of partially DMO with respect to x' in Ω if the function

$$\omega_{f, x'}(r) := \sup \oint_{B_r(x)} \left| f(y) - \oint_{B'_r(x')} f(z', y^d) dz' \right| dy$$

satisfies (2.1), where the sup is taken with respect to all $x \in \Omega$ with $B_r(x) \subset \Omega$.

Let $f_\alpha \in L^q(\Omega)^d$ and $g \in L^q(\Omega)$ with $q \geq 1$. We say that $(u, p) \in W^{1,q}(\Omega)^d \times L^q(\Omega)$ is a weak solution of

$$\begin{cases} \mathcal{L}u + \nabla p = D_\alpha f_\alpha & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega \end{cases}$$

if $\operatorname{div} u = g$ a.e. in Ω and

$$\int_\Omega A^{\alpha\beta} D_\beta u \cdot D_\alpha \phi dx + \int_\Omega p \operatorname{div} \phi dx = \int_\Omega f_\alpha \cdot D_\alpha \phi dx$$

holds for $\phi \in C_0^\infty(\Omega)^d$. We also say that $(u, p) \in W^{1,q}(\Omega)^d \times L^q(\Omega)$ is a weak solution of

$$\begin{cases} \mathcal{L}u + \nabla p + u^\alpha D_\alpha u = D_\alpha f_\alpha & \text{in } \Omega, \\ \operatorname{div} u = g & \text{in } \Omega, \end{cases}$$

if $\operatorname{div} u = g$ a.e. in Ω and

$$\int_\Omega A^{\alpha\beta} D_\beta u \cdot D_\alpha \phi dx + \int_\Omega p \operatorname{div} \phi dx - \int_\Omega u^\alpha D_\alpha u \cdot \phi dx = \int_\Omega f_\alpha \cdot D_\alpha \phi dx$$

holds for $\phi \in C_0^\infty(\Omega)^d$. When $q = 2$, we sometimes call (u, p) a H^1 -weak solution.

2.1. Assumptions on the domain. Before we state our assumptions on the domain, we recall the definition of a domain having a $C^{1,\text{Dini}}$ boundary.

DEFINITION 2.2. Let Ω be a domain in $\mathbb{R}^d$. We say that Ω has a $C^{1,\text{Dini}}$ boundary if there exist a constant $R_0 \in (0, 1]$ and a concave Dini function ϱ_0 such that the following holds. For any $x_0 = (x'_0, x_0^d) \in \partial\Omega$, there exist a C^1 function $\chi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ and a coordinate system depending on x_0 such that

$$\varrho_{\nabla_{x'}\chi}(t) \leq \varrho_0(t) \quad \text{for all } t \in [0, R_0]$$

and that in the new coordinate system, we have

$$|\nabla_{x'}\chi(x'_0)| = 0$$

and

$$(2.2) \quad \Omega \cap B_{R_0}(x_0) = \{x \in B_{R_0}(x_0) : x^d > \chi(x')\}.$$

In this paper, we always assume that $\mathcal{D}$ is a bounded domain in $\mathbb{R}^d$ containing M subdomains $D_1, \dots, D_M$ such that

- (i) $D_M = \mathcal{D} \setminus (\cup_{i=1}^{M-1} \overline{D_i})$;
- (ii) for $i, j \in \{1, \dots, M-1\}$ with $i \neq j$, we have either

$$(2.3) \quad \overline{D_i} \subset D_j \quad \text{or} \quad \overline{D_i} \cap \overline{D_j} = \emptyset;$$

- (iii) for $i \in \{1, \dots, M-1\}$, D_i has a $C^{1,\text{Dini}}$ boundary as in Definition 2.2 with the same constant R_0 and Dini function ϱ_0 .

Our assumptions on the domain, which look a bit different from those in [17], are in fact identical. Precisely, by disjointing the subdomains $D_1, \dots, D_{M-1}$, one can understand $\mathcal{D}$ as a domain containing M disjoint subdomains $\mathcal{D}_1, \dots, \mathcal{D}_M$ such that

- (i') $\mathcal{D}_M = D_M$;
- (ii') any point in $\mathcal{D}$ belongs to the boundaries of at most two of the subdomains;
- (iii') for $i \in \{1, \dots, M-1\}$, $\mathcal{D}_i$ has a $C^{1,\text{Dini}}$ boundary in an appropriate sense.

Among the above two expressions of the nature of the domain, the second is useful in describing the regularity conditions on the coefficients and data, which may have jump discontinuities across the interfacial boundaries; see section 2.2. On the other hand, the first expression is convenient to explain the regularity of the boundaries by using Definition 2.2. Because the disjointed subdomains $\mathcal{D}_i$ in the second expression may have “narrow” regions, (2.2) is not guaranteed with the same constant R_0 independent of the distance between subdomains. For example, if $M = 3$, $\mathcal{D}_1 := B_{1/2-\varepsilon}$, $\mathcal{D}_2 := B_{1/2} \setminus \overline{B_{1/2-\varepsilon}}$, and $\mathcal{D}_3 := B_1 \setminus \overline{B_{1/2}}$, then when we explain the regularity of $\partial\mathcal{D}_2$ via Definition 2.2, we need to take R_0 to be less than ε , which is the distance between $\mathcal{D}_1$ and $\mathcal{D}_3$. That is why we added “appropriate sense” to the condition (iii'). In the following, we will use the notation $\mathcal{D}_i$ introduced above to denote the subdomains.

We end this subsection with a remark that condition (2.3) can be relaxed to

$$D_i \subset D_j \quad \text{or} \quad D_i \cap D_j = \emptyset,$$

so that the boundaries of more than two subdomains touch at some points; see Remark 2.5.

2.2. Assumptions on the coefficients and data. We assume that the coefficients $A^{\alpha\beta}$ of the operator $\mathcal{L}$ in (1.2) are bounded and satisfy the strong ellipticity condition, that is, there exists $\nu \in (0, 1)$ such that

$$(2.4) \quad |A^{\alpha\beta}(x)| \leq \nu^{-1}, \quad \sum_{\alpha, \beta=1}^d A^{\alpha\beta}(x) \xi_\beta \cdot \xi_\alpha \geq \nu \sum_{\alpha=1}^d |\xi_\alpha|^2$$

for any $x \in \mathbb{R}^d$ and $\xi_\alpha \in \mathbb{R}^d$, $\alpha \in \{1, \dots, d\}$. We also assume that the coefficients and data are of piecewise DMO satisfying Definition 2.3 below in the domain $\mathcal{D}$ containing M disjoint subdomains $\mathcal{D}_1, \dots, \mathcal{D}_M$ as in section 2.1.

DEFINITION 2.3. Let $f \in L^1(\mathcal{D})$. We say that f is of piecewise DMO in $\mathcal{D}$ if there exists a Dini function ω_f such that for any $x_0 \in \mathcal{D}$ and $r \in (0, 1]$ satisfying $B_r(x_0) \subset \mathcal{D}$, we have

$$(2.5) \quad \oint_{B_r(x_0)} |f(x) - \hat{f}(x)| dx \leq \omega_f(r),$$

where $\hat{f} = \hat{f}_{x_0, r}$ is a piecewise continuous function on $B_r(x_0)$ given by

$$\hat{f}(x) = \oint_{B_r(x_0) \cap \mathcal{D}_i} f(y) dy \quad \text{if } x \in B_r(x_0) \cap \mathcal{D}_i.$$

Our definition of a function of piecewise DMO is equivalent to the definition in [17], where the piecewise mean oscillation is measured by taking the infimum over the set of all piecewise constant functions.

2.3. Main results. The main results of this paper are as follows.

THEOREM 2.4. Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^d$ containing M disjoint subdomains $\mathcal{D}_1, \dots, \mathcal{D}_M$ with $C^{1, \text{Dini}}$ boundaries as in section 2.1. Also, let $q \in (1, \infty)$ and $(u, p) \in W^{1, q}(\mathcal{D})^d \times L^q(\mathcal{D})$ be a weak solution of

$$(2.6) \quad \begin{cases} \mathcal{L}u + \nabla p = D_\alpha f_\alpha & \text{in } \mathcal{D}, \\ \operatorname{div} u = g & \text{in } \mathcal{D}, \end{cases}$$

where $f_\alpha \in L^\infty(\mathcal{D})^d$ and $g \in L^\infty(\mathcal{D})$. If $A^{\alpha\beta}$, f_α , and g are of piecewise DMO in $\mathcal{D}$ satisfying Definition 2.3, then for any $\mathcal{D}' \Subset \mathcal{D}$, we have

$$(u, p) \in W^{1, \infty}(\mathcal{D}')^d \times L^\infty(\mathcal{D}')$$

and

$$(u, p) \in C^1(\overline{\mathcal{D}_i} \cap \mathcal{D}')^d \times C(\overline{\mathcal{D}_i} \cap \mathcal{D}'), \quad i \in \{1, \dots, M\}.$$

If we further assume that there exist $\gamma_0 \in (0, 1)$ and $K > 0$ such that

$$(2.7) \quad \varrho_0(r) \leq Kr^{\frac{\gamma_0}{1-\gamma_0}}, \quad \omega_{A^{\alpha\beta}}(r) + \omega_{f_\alpha}(r) + \omega_g(r) \leq Kr^{\gamma_0}$$

for all $r \in (0, R_0]$, then

$$(u, p) \in C^{1, \gamma_0}(\overline{\mathcal{D}_i} \cap \mathcal{D}')^d \times C^{\gamma_0}(\overline{\mathcal{D}_i} \cap \mathcal{D}'), \quad i \in \{1, \dots, M\}.$$

Related to the theorem above, we have a few remarks.

Remark 2.5. Upper bounds of the L^∞ -norms and the modulus of continuity of Du and p can be found in the proof of the theorem; see section 3.1. Note that these upper bounds are independent of the distance between the subdomains. Thus our results can be applied to the case when the boundaries of more than two subdomains touch at some points.

In the middle of the proof, we also proved that for any $x_0 \in \mathcal{D}'$, there exists a coordinate system associated with x_0 such that the certain linear combinations

$$D_{x'}u \quad \text{and} \quad A^{d\beta}D_\beta u + pe_d - f_d$$

are continuous at x_0 . Moreover, if (2.7) holds, then they are Hölder continuous with the same exponent γ_0 .

Remark 2.6. Condition (2.7) holds, provided that the subdomains $\mathcal{D}_i$ have $C^{1,\gamma_0/(1-\gamma_0)}$ boundaries and that $A^{\alpha\beta}$, f_α , and g are in $C^{\gamma_0}(\overline{\mathcal{D}_i})$ for each $i \in \{1, \dots, M\}$.

Remark 2.7. By the same reasoning as in [8, Remark 2.4], one can extend the results in Theorem 2.4 to weak solutions of the system

$$\begin{cases} \mathcal{L}u + \nabla p = D_\alpha f_\alpha + f & \text{in } \mathcal{D}, \\ \operatorname{div} u = g & \text{in } \mathcal{D}, \end{cases}$$

where $f \in L^s(\mathcal{D})^d$ with $s > d$. The corresponding upper bounds of the L^∞ -norms and the modulus of continuity of Du and p can be found in Remark 3.2 at the end of section 3.1.

In the corollary below, we present the $W^{1,q}$ -estimate for $W^{1,1}$ -weak solutions, which follows from Theorem 2.4, the solvability results in [13], and the argument in Brezis [6] (see also [4, Appendix]). One may refer to the proof of [8, Theorem 2.5], where the authors proved the $W^{1,q}$ -estimate for $W^{1,1}$ -weak solutions to the Stokes system with partially DMO coefficients.

COROLLARY 2.8. *Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^d$ containing M disjoint subdomains $\mathcal{D}_1, \dots, \mathcal{D}_M$ as in section 2.1. Also, let $(u, p) \in W^{1,1}(\mathcal{D})^d \times L^1(\mathcal{D})$ be a weak solution of (2.6), where $f_\alpha \in L^q(\mathcal{D})^d$ and $g \in L^q(\mathcal{D})$ with $q \in (1, \infty)$. If $A^{\alpha\beta}$, f_α , and g are piecewise DMO in $\mathcal{D}$ satisfying Definition 2.3, then for $\mathcal{D}' \Subset \mathcal{D}$, we have*

$$(u, p) \in W^{1,q}(\mathcal{D}')^d \times L^q(\mathcal{D}')$$

with the estimate

$$\|u\|_{W^{1,q}(\mathcal{D}')} + \|p\|_{L^q(\mathcal{D}')} \leq N(\|u\|_{W^{1,1}(\mathcal{D})} + \|p\|_{L^1(\mathcal{D})} + \|f_\alpha\|_{L^q(\mathcal{D})} + \|g\|_{L^q(\mathcal{D})}),$$

where the constant N depends only on d , ν , M , R_0 , ϱ_0 , $\omega_{A^{\alpha\beta}}$, q , and $\operatorname{dist}(\partial\mathcal{D}, \mathcal{D}')$.

Remark 2.9. From Corollary 2.8, the results in Theorem 2.4 still hold under the assumption that $(u, p) \in W^{1,1}(\mathcal{D})^d \times L^1(\mathcal{D})$.

We also consider stationary Navier–Stokes systems with piecewise DMO coefficients.

THEOREM 2.10. *Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^d$ containing M disjoint subdomains $\mathcal{D}_1, \dots, \mathcal{D}_M$ with $C^{1,\operatorname{Dini}}$ boundaries as in section 2.1. Also, let $q \in (1, \infty)$ with $q \geq d/2$ and $(u, p) \in W^{1,q}(\mathcal{D})^d \times L^q(\mathcal{D})$ be a weak solution of*

$$\begin{cases} \mathcal{L}u + \nabla p + u^\alpha D_\alpha u = D_\alpha f_\alpha & \text{in } \mathcal{D}, \\ \operatorname{div} u = g & \text{in } \mathcal{D}, \end{cases}$$

where $f_\alpha \in L^\infty(\mathcal{D})^d$ and $g \in L^\infty(\mathcal{D})$. If $A^{\alpha\beta}$, f_α , and g are of piecewise DMO in $\mathcal{D}$ satisfying Definition 2.3, then for any $\mathcal{D}' \Subset \mathcal{D}$, we have

$$(u, p) \in W^{1,\infty}(\mathcal{D}')^d \times L^\infty(\mathcal{D}')$$

and

$$(u, p) \in C^1(\overline{\mathcal{D}_i} \cap \mathcal{D}')^d \times C(\overline{\mathcal{D}_i} \cap \mathcal{D}'), \quad i \in \{1, \dots, M\}.$$

If we further assume (2.7), then

$$(u, p) \in C^{1,\gamma_0}(\overline{\mathcal{D}_i} \cap \mathcal{D}')^d \times C^{\gamma_0}(\overline{\mathcal{D}_i} \cap \mathcal{D}'), \quad i \in \{1, \dots, M\}.$$

As a corollary of Theorem 2.10, in dimensions $d = 2, 3, 4$, for any H^1 -weak solution (u, p) to the stationary Navier–Stokes system with piecewise DMO coefficients, (Du, p) is locally bounded, i.e., the flow velocity u is locally Lipschitz, as the condition $2 \geq d/2$ is satisfied.

We finish this section with a remark that our results can be applied to (anisotropic) Stokes systems

$$(2.8) \quad \begin{cases} \operatorname{div}(\tau \mathcal{S}u) + \nabla p = D_\alpha f_\alpha & \text{in } \mathcal{D}, \\ \operatorname{div} u = g & \text{in } \mathcal{D} \end{cases}$$

as well as the corresponding stationary Navier–Stokes systems. Here $\tau = \tau(x)$ is a piecewise DMO (or piecewise Hölder continuous) scalar function satisfying $\nu \leq \tau \leq \nu^{-1}$ and $\mathcal{S}u = \frac{1}{2}(Du + (Du)^\top)$ is the so-called rate of deformation tensor or strain tensor. Systems in this form were considered, for example, in [25, 30, 1]. Notice that the coefficient matrix in this case is given by

$$A_{ij}^{\alpha\beta} = \frac{\tau}{2}(\delta_{ij}\delta_{\alpha\beta} + \delta_{\alpha j}\delta_{\beta i}),$$

which satisfies the Legendre–Hadamard ellipticity condition, but not the strong ellipticity condition (2.4). Hence, our results cannot be applied directly. However, by using the condition $\operatorname{div} u = g$ in $\mathcal{D}$, we have

$$\operatorname{div}(\tau \mathcal{S}u)^i = D_j \left(\frac{\tau}{2} (D_j u^i + D_i u^j) \right) = D_j \left(\frac{\tau}{2} D_j u^i + \left(\frac{\tau}{2} - \varepsilon \right) D_i u^j \right) + \varepsilon D_i g,$$

where $\varepsilon \in (0, \nu/2)$. Thus the first equation in (2.8) can be replaced with

$$(2.9) \quad \operatorname{div}(\widehat{\mathcal{S}}u) + \nabla p = D_\alpha f_\alpha - \varepsilon D_\alpha g_\alpha \quad \text{in } \mathcal{D},$$

where $g_\alpha^i = g\delta_{\alpha i}$ and the new coefficient matrix is given by

$$\widehat{A}_{ij}^{\alpha\beta} = \frac{\tau}{2}\delta_{ij}\delta_{\alpha\beta} + \left(\frac{\tau}{2} - \varepsilon \right) \delta_{\alpha j}\delta_{\beta i},$$

which is a piecewise DMO (or piecewise Hölder continuous) function satisfying the strong ellipticity condition with ellipticity constant ε . Therefore, our results hold true for (2.8).

3. Proofs of main theorems. Throughout this paper, we use the following notation.

NOTATION 3.1. For nonnegative (variable) quantities A and B , we denote $A \lesssim B$ if there exists a generic positive constant C such that $A \leq CB$. We add subscript letters like $A \lesssim_{a,b} B$ to indicate the dependence of the implicit constant C on the parameters a and b .

3.1. Proof of Theorem 2.4. We begin the proof with the following observation. Under the assumptions on the domain $\mathcal{D}$ with a scaling whose parameter depends only on d, R_0, ϱ_0 , and $\text{dist}(\partial\mathcal{D}, \mathcal{D}')$, we may suppose that for any $x_0 \in \mathcal{D}'$, there exist $C^{1,\text{Dini}}$ functions $\chi_i : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$, $i \in \{1, \dots, \ell\}$ for some $\ell < M$, and a coordinate system such that the following properties hold in the new coordinate system (called the coordinate system associated with x_0):

(A1) We have that

$$\varrho_{\nabla_{x'} \chi_i}(r) \leq \varrho_0(r)$$

for all $r \in [0, R_0]$ and $i \in \{1, \dots, \ell\}$, and that

$$\chi_0(x') < \chi_1(x') < \dots < \chi_\ell(x') < \chi_{\ell+1}(x')$$

for all $x' \in B'_1(x_0)$, where we have adopted the notation $\chi_0 \equiv x_0^d - 1$ and $\chi_{\ell+1} \equiv x_0^d + 1$.

(A2) $B_1(x_0) \subset \mathcal{D}$ and $B_1(x_0)$ is divided into $\ell + 1$ disjoint subdomains

$$\widehat{\mathcal{D}}_i := \{x \in B_1(x_0) : \chi_{i-1}(x') < x^d < \chi_i(x')\}, \quad i \in \{1, \dots, \ell + 1\}.$$

Here, in an appropriate sense one may think of $\widehat{\mathcal{D}}_i$ as $\mathcal{D}_i \cap B_1(x_0)$. Moreover,

$$x_0 \in \widehat{\mathcal{D}}_{i_0} \cup \partial\widehat{\mathcal{D}}_{i_0} \quad \text{for some } i_0 \in \{1, \dots, \ell + 1\},$$

the closest point on $\partial\widehat{\mathcal{D}}_{i_0}$ to x_0 is $(x'_0, \chi_{i_0}(x'_0))$, and $\nabla_{x'} \chi_{i_0}(x'_0) = 0'$.

Throughout this proof, we shall use the following notation and properties in the coordinate system associated with x_0 satisfying (A1) and (A2).

(B1) For $i \in \{1, \dots, \ell + 1\}$, we denote

$$\Omega_i = \{x \in B_1(x_0) : \chi_{i-1}(x'_0) < x^d < \chi_i(x'_0)\}.$$

By [17, Lemma 2.3], there exists $R_1 = R_1(R_0, \varrho_0) \in (0, R_0]$ such that for any $r \in (0, R_1]$,

$$(3.1) \quad r^{-d} |(\widehat{\mathcal{D}}_i \setminus \Omega_i) \cap B_r(x_0)| \lesssim_{d, M, \varrho_0} \varrho_1(r),$$

where ϱ_1 is a Dini function derived from ϱ_0 .

(B2) Let f be of piecewise DMO in $\mathcal{D}$ satisfying Definition 2.3 with a Dini function ω_f . For $r \in (0, R_1]$, we define piecewise continuous functions $\hat{f} = \hat{f}_{x_0, r}$ and $\bar{f} = \bar{f}_{x_0, r}$ in $B_r(x_0)$ by

$$\hat{f}(x) = \int_{\widehat{\mathcal{D}}_i \cap B_r(x_0)} f(y) \, dy \quad \text{if } x \in B_r(x_0) \cap \widehat{\mathcal{D}}_i$$

and

$$(3.2) \quad \bar{f}(x) = \int_{\widehat{\mathcal{D}}_i \cap B_r(x_0)} f(y) \, dy \quad \text{if } x \in B_r(x_0) \cap \Omega_i,$$

where $\bar{f}$ is indeed a function of x^d . Since $\hat{f} \equiv \bar{f}$ in $B_r(x_0) \cap \widehat{\mathcal{D}}_i \cap \Omega_i$, by (3.1), we have

$$\begin{aligned} \int_{B_r(x_0)} |\hat{f} - \bar{f}| \, dx &= \frac{1}{|B_r|} \sum_{i=1}^{\ell+1} \int_{(\widehat{\mathcal{D}}_i \setminus \Omega_i) \cap B_r(x_0)} |\hat{f} - \bar{f}| \, dx \\ &\lesssim \|f\|_{L^\infty(B_r(x_0))} \varrho_1(r). \end{aligned}$$

From this together with (2.5), it follows that

$$(3.3) \quad \int_{B_r(x_0)} |f - \bar{f}| dx \lesssim_{d,M,\varrho_0} \omega_f(r) + \|f\|_{L^\infty(B_r(x_0))} \varrho_1(r).$$

(B3) We set

$$U = A^{d\beta} D_\beta u + p e_d - f_d.$$

For $y \in \mathcal{D}$ and $r > 0$ with $B_r(y) \subset B_1(x_0)$, we define

$$\Phi_{x_0}(y, r) = \inf_{\Theta \in \mathbb{R}^{d \times d}} \left(\int_{B_r(y)} |(D_{x'} u, U) - \Theta|^{\frac{1}{2}} dx \right)^2,$$

where we used the subindex x_0 to indicate that the function is defined in the coordinate system associated with x_0 .

To prove Theorem 2.4, we will use the following decay estimates.

LEMMA 3.1. *Let $x_0 \in \mathcal{D}'$, $r \in (0, R_1]$, and $\gamma \in (0, 1)$. Then under the same hypotheses as those of Theorem 2.4 with an additional assumption that Du and p are locally bounded, there exists $N = N(d, \nu, M, \varrho_0, \gamma) > 0$ such that the following assertions hold:*

(i) *For any $\rho \in (0, r]$, we have*

$$(3.4) \quad \begin{aligned} \Phi_{x_0}(x_0, \rho) &\leq N \left(\frac{\rho}{r} \right)^\gamma \Phi_{x_0}(x_0, r) + N \|Du\|_{L^\infty(B_r(x_0))} (\tilde{\omega}_{A^{\alpha\beta}}(\rho) + \tilde{\varrho}_1(\rho)) \\ &\quad + N (\|f_\alpha\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))}) \tilde{\varrho}_1(\rho) \\ &\quad + N (\tilde{\omega}_{f_\alpha}(\rho) + \tilde{\omega}_g(\rho)). \end{aligned}$$

(ii) *For any $y \in B_{r/2}(x_0)$ and $\rho \in (0, r/2]$ such that $B_\rho(y) \subset \widehat{\mathcal{D}}_{i_1}$ for some $i_1 \in \{1, \dots, \ell + 1\}$, we have*

$$(3.5) \quad \begin{aligned} \Phi_{x_0}(y, \rho) &\leq N \left(\frac{\rho}{r} \right)^\gamma \Phi_y(y, r/2) + N \|Du\|_{L^\infty(B_{r/2}(y))} (\tilde{\omega}_{A^{\alpha\beta}}(\rho) + \tilde{\varrho}_1(\rho)) \\ &\quad + N (\|f_\alpha\|_{L^\infty(B_{r/2}(y))} + \|g\|_{L^\infty(B_{r/2}(y))}) (\tilde{\omega}_{A^{\alpha\beta}}(\rho) + \tilde{\varrho}_1(\rho)) \\ &\quad + N (\tilde{\omega}_{f_\alpha}(\rho) + \tilde{\omega}_g(\rho)). \end{aligned}$$

In the above, $\tilde{\omega}_\bullet$ and $\tilde{\varrho}_1$ are Dini functions derived from $\omega_\bullet$ and ϱ_1 , respectively, as formulated in (3.10).

Proof. We may assume that $x_0 = 0$ for simplicity of notation. For a given function f , we denote by $\bar{f} = \bar{f}(x^d)$ the piecewise constant function in B_r defined as in (3.2).

We first prove assertion (i). Let $\mathcal{L}_0$ be an elliptic operator given by

$$\mathcal{L}_0 u = D_\alpha (\bar{A}^{\alpha\beta} D_\beta u)$$

and set

$$u_e = u - \int_{-1}^{x^d} u_0 ds, \quad p_e = p - p_0,$$

where $u_0 = (u_0^1, \dots, u_0^d)^\top$ and p_0 are functions of x^d satisfying

$$u_0^d = \bar{g}, \quad \bar{A}^{dd} u_0 + p_0 e_d = \bar{f}_d.$$

Then (u_e, p_e) satisfies

$$\begin{cases} \mathcal{L}_0 u_e + \nabla p_e = D_\alpha F_\alpha & \text{in } B_r, \\ \operatorname{div} u_e = G & \text{in } B_r, \end{cases}$$

where $F_\alpha = (\bar{A}^{\alpha\beta} - A^{\alpha\beta})D_\beta u + f_\alpha - \bar{f}_\alpha$ and $G = g - \bar{g}$. We decompose

$$(3.6) \quad (u_e, p_e) = (v, p_1) + (w, p_2),$$

where $(v, p_1) \in W_0^{1,2}(B_r)^d \times \tilde{L}^2(B_r)$ is the unique weak solution of

$$\begin{cases} \mathcal{L}_0 v + \nabla p_1 = D_\alpha (I_{B_{r/4}} F_\alpha) & \text{in } B_r, \\ \operatorname{div} v = I_{B_{r/4}} G - (I_{B_{r/4}} G)_{B_r} & \text{in } B_r. \end{cases}$$

Here, $I_{B_{r/4}}$ is the characteristic function. Since $\bar{A}^{\alpha\beta}$ only depends on x^d , by [8, Lemma 3.4] with scaling and relabeling the coordinate axes, we have for all $t > 0$ that

$$|\{x \in B_{r/4} : |Dv(x)| + |p_1(x)| > t\}| \lesssim_{d,\nu} \frac{1}{t} \int_{B_{r/4}} (|F_\alpha| + |G|) \, dx.$$

This implies that (cf. [8, equation (4.5)])

$$(3.7) \quad \left(\int_{B_{r/4}} (|Dv| + |p_1|)^{\frac{1}{2}} \, dx \right)^2 \lesssim \int_{B_{r/4}} (|F_\alpha| + |G|) \, dx.$$

On the other hand, since (w, p_2) satisfies

$$\begin{cases} \mathcal{L}_0 w + \nabla p_2 = 0 & \text{in } B_{r/4}, \\ \operatorname{div} w = (I_{B_{r/4}} G)_{B_r} & \text{in } B_{r/4}, \end{cases}$$

by [8, equation (3.7)], we obtain

$$(3.8) \quad \left(\int_{B_{\kappa r}} |D_{x'} w - (D_{x'} w)_{B_{\kappa r}}|^{\frac{1}{2}} + |W - (W)_{B_{\kappa r}}|^{\frac{1}{2}} \, dx \right)^2 \lesssim \kappa \inf_{\Theta \in \mathbb{R}^{d \times d}} \left(\int_{B_{r/4}} |(D_{x'} w, W) - \Theta|^{\frac{1}{2}} \, dx \right)^2$$

for any $\kappa \in (0, 1/8]$, where $W = \bar{A}^{d\beta} D_\beta w + p_2 e_d$.

Now we set

$$U_e = \bar{A}^{d\beta} D_\beta u_e + p_e e_d$$

and observe that

$$(3.9) \quad D_{x'} u_e = D_{x'} u, \quad U - U_e = (A^{d\beta} - \bar{A}^{d\beta}) D_\beta u - (f_d - \bar{f}_d).$$

By (3.6)–(3.8) and the triangle inequality, we have

$$\begin{aligned}
& \left(\int_{B_{\kappa r}} |D_{x'} u_e - (D_{x'} w)_{B_{\kappa r}}|^{\frac{1}{2}} + |U_e - (W)_{B_{\kappa r}}|^{\frac{1}{2}} dx \right)^2 \\
& \lesssim \left(\int_{B_{\kappa r}} |D_{x'} w - (D_{x'} w)_{B_{\kappa r}}|^{\frac{1}{2}} + |W - (W)_{B_{\kappa r}}|^{\frac{1}{2}} dx \right)^2 \\
& \quad + \left(\int_{B_{\kappa r}} (|Dv| + |p_1|)^{\frac{1}{2}} dx \right)^2 \\
& \lesssim \kappa \inf_{\Theta \in \mathbb{R}^{d \times d}} \left(\int_{B_{r/4}} |(D_{x'} w, W) - \Theta|^{\frac{1}{2}} dx \right)^2 + \kappa^{-2d} \int_{B_{r/4}} (|F_\alpha| + |G|) dx \\
& \lesssim \kappa \inf_{\Theta \in \mathbb{R}^{d \times d}} \left(\int_{B_r} |(D_{x'} u_e, U_e) - \Theta|^{\frac{1}{2}} dx \right)^2 + \kappa^{-2d} \int_{B_r} (|F_\alpha| + |G|) dx.
\end{aligned}$$

From this, together with (3.3) and (3.9), we get

$$\begin{aligned}
\Phi_0(0, \kappa r) & \leq N_0 \kappa \Phi_0(0, r) + N_0 \kappa^{-2d} \|Du\|_{L^\infty(B_r)} (\omega_{A^{\alpha\beta}}(r) + \varrho_1(r)) \\
& \quad + N_0 \kappa^{-2d} (\|f_\alpha\|_{L^\infty(B_r)} + \|g\|_{L^\infty(B_r)}) \varrho_1(r) + N_0 \kappa^{-2d} (\omega_{f_\alpha}(r) + \omega_g(r)),
\end{aligned}$$

where $N_0 = N_0(d, \nu, M, \varrho_0) > 0$. Fix $\kappa \in (0, 1/8]$ small enough so that $N_0 \kappa^{1-\gamma} \leq 1$. Then

$$\begin{aligned}
\Phi_0(0, \kappa r) & \leq \kappa^\gamma \Phi_0(0, r) + N \|Du\|_{L^\infty(B_r)} (\omega_{A^{\alpha\beta}}(r) + \varrho_1(r)) \\
& \quad + N (\|f_\alpha\|_{L^\infty(B_r)} + \|g\|_{L^\infty(B_r)}) \varrho_1(r) + N (\omega_{f_\alpha}(r) + \omega_g(r)),
\end{aligned}$$

where $N = N(d, \nu, M, \varrho_0, \gamma) > 0$. Let $\tilde{\omega}_\bullet$ and $\tilde{\varrho}_0$ be Dini functions defined by

$$\begin{aligned}
\tilde{\omega}_\bullet(r) & = \sum_{i=1}^{\infty} \kappa^{\gamma i} (\omega_\bullet(\kappa^{-i} r) [\kappa^{-i} r < 1] + \omega_\bullet(1) [\kappa^{-i} r \geq 1]), \\
\tilde{\varrho}_1(r) & = \sum_{i=1}^{\infty} \kappa^{\gamma i} (\varrho_1(\kappa^{-i} r) [\kappa^{-i} r < 1] + \varrho_1(1) [\kappa^{-i} r \geq 1]),
\end{aligned} \tag{3.10}$$

where we used the Iverson bracket notation, i.e., $[P] = 1$ if P is true and $[P] = 0$ otherwise. By iterating and using the fact that

$$\sum_{i=1}^j \kappa^{\gamma(i-1)} \omega_\bullet(\kappa^{j-i} r) \leq \kappa^{-\gamma} \tilde{\omega}_\bullet(\kappa^j r), \quad j \in \{1, 2, \dots\},$$

we obtain

$$\begin{aligned}
\Phi_0(0, \kappa^j r) & \leq \kappa^{\gamma j} \Phi_0(0, r) + N \|Du\|_{L^\infty(B_r)} (\tilde{\omega}_{A^{\alpha\beta}}(\kappa^j r) + \tilde{\varrho}_1(\kappa^j r)) \\
& \quad + N (\|f_\alpha\|_{L^\infty(B_r)} + \|g\|_{L^\infty(B_r)}) \tilde{\varrho}_1(\kappa^j r) + N (\tilde{\omega}_{f_\alpha}(\kappa^j r) + \tilde{\omega}_g(\kappa^j r)),
\end{aligned} \tag{3.11}$$

which also obviously holds for $j = 0$. Finally, for $\rho \in (0, r]$, by taking the nonnegative integer j such that $\kappa^{j+1} < \rho/r \leq \kappa^j$ and using (3.11) with ρ in place of $\kappa^j r$, we get the desired estimate.

Next, we prove assertion (ii). For a given function f , we define

$$\hat{f} = \int_{B_\rho(y)} f(x) dx.$$

Notice from the definition of U that for any $\Theta_\beta \in \mathbb{R}^d$ and $\theta \in \mathbb{R}$, we have

$$|U - \Theta_0|^{\frac{1}{2}} \leq |(A^{d\beta} - \hat{A}^{d\beta})D_\beta u|^{\frac{1}{2}} + |\hat{A}^{d\beta}(D_\beta u - \Theta_\beta)|^{\frac{1}{2}} + |p - \theta|^{\frac{1}{2}} + |f_d - \hat{f}_d|^{\frac{1}{2}},$$

where $\Theta_0 = \hat{A}^{d\beta}\Theta_\beta + \theta e_d - \hat{f}_d$, in the coordinate system associated with x_0 . By averaging the above inequality on $B_\rho(y)$, taking the square, and using (2.5) (with the fact that $B_\rho(y)$ is contained in a subdomain), we obtain

$$\left(\int_{B_\rho(y)} |U - \Theta_0|^{\frac{1}{2}} dx \right)^2 \lesssim \left(\int_{B_\rho(y)} |D_\beta u - \Theta_\beta|^{\frac{1}{2}} + |p - \theta|^{\frac{1}{2}} dx \right)^2 + \|Du\|_{L^\infty(B_\rho(y))} \omega_{A^{\alpha\beta}}(\rho) + \omega_{f_\alpha}(\rho).$$

From this we get

$$(3.12) \quad \Phi_{x_0}(y, \rho) \lesssim \Psi(y, \rho) + \|Du\|_{L^\infty(B_\rho(y))} \omega_{A^{\alpha\beta}}(\rho) + \omega_{f_\alpha}(\rho),$$

where

$$\Psi(y, \rho) := \inf_{\substack{\theta \in \mathbb{R} \\ \Theta \in \mathbb{R}^{d \times d}}} \left(\int_{B_\rho(y)} |Du - \Theta|^{\frac{1}{2}} + |p - \theta|^{\frac{1}{2}} dx \right)^2.$$

Note that $\Psi(y, \rho)$ is independent of coordinate systems.

We now control the quantity $\Psi(y, \rho)$ in the coordinate system associated with y . Using (2.5) and the relation

$$(3.13) \quad D_d u^d = g - \sum_{i=1}^{d-1} D_i u^i,$$

we have

$$\begin{aligned} & \inf_{\theta \in \mathbb{R}} \left(\int_{B_\rho(y)} |D_d u^d - \theta|^{\frac{1}{2}} dx \right)^2 \\ & \lesssim \sum_{i=1}^{d-1} \inf_{\theta \in \mathbb{R}} \left(\int_{B_\rho(y)} |D_i u^i - \theta|^{\frac{1}{2}} dx \right)^2 + \int_{B_\rho(y)} |g - \hat{g}| dx \\ (3.14) \quad & \lesssim \Phi_y(y, \rho) + \omega_g(\rho). \end{aligned}$$

Note that

$$(3.15) \quad \sum_{j=1}^{d-1} A_{ij}^{dd} D_d u^j = U^i - \sum_{j=1}^d \sum_{\beta=1}^{d-1} A_{ij}^{d\beta} D_\beta u^j - A_{id}^{dd} D_d u^d + f_d^i, \quad i \in \{1, \dots, d-1\},$$

where, by the ellipticity condition on $A^{\alpha\beta}$, $(A_{ij}^{dd})_{i,j=1}^{d-1}$ is nondegenerate. Hence,

$$\mathcal{X} = \mathcal{Y}\mathcal{Z},$$

where

$$\begin{aligned} \mathcal{X} &= (D_d u^1, \dots, D_d u^{d-1})^\top, \quad \mathcal{Y} = ((A_{ij}^{dd})_{i,j=1}^{d-1})^{-1}, \\ \mathcal{Z} &= (\mathcal{Z}^1, \dots, \mathcal{Z}^{d-1})^\top, \quad \mathcal{Z}^i = U^i - \sum_{j=1}^d \sum_{\beta=1}^{d-1} A_{ij}^{d\beta} D_\beta u^j - A_{id}^{dd} D_d u^d + f_d^i. \end{aligned}$$

Since

$$|\mathcal{X} - \hat{\mathcal{Y}}\vartheta| \leq |(\mathcal{Y} - \hat{\mathcal{Y}})\mathcal{Z}| + |\hat{\mathcal{Y}}(\mathcal{Z} - \vartheta)| \quad \text{for all } \vartheta \in \mathbb{R}^{d-1},$$

we see that

$$\begin{aligned} & \inf_{\vartheta \in \mathbb{R}^{d-1}} \left(\int_{B_\rho(y)} |\mathcal{X} - \vartheta|^{\frac{1}{2}} dx \right)^2 \\ & \lesssim \left(\int_{B_\rho(y)} |\mathcal{Y} - \hat{\mathcal{Y}}|^{\frac{1}{2}} dx \right)^2 \|\mathcal{Z}\|_{L^\infty(B_\rho(y))} + \inf_{\vartheta \in \mathbb{R}^{d-1}} \left(\int_{B_\rho(y)} |\mathcal{Z} - \vartheta|^{\frac{1}{2}} dx \right)^2 \\ & \lesssim \Phi_y(y, \rho) + (\|Du\|_{L^\infty(B_\rho(y))} + \|f_\alpha\|_{L^\infty(B_\rho(y))}) \omega_{A^{\alpha\beta}}(\rho) + \omega_{f_\alpha}(\rho) + \omega_g(\rho) =: K_0. \end{aligned}$$

From this, together with (3.14), we get

$$\inf_{\Theta \in \mathbb{R}^{d \times d}} \left(\int_{B_\rho(y)} |Du - \Theta|^{\frac{1}{2}} dx \right)^2 \lesssim K_0.$$

By the relation

$$(3.16) \quad p = U^d - \sum_{j=1}^d \sum_{\beta=1}^d A_{dj}^{d\beta} D_\beta u^j + f_d^d,$$

we also have

$$\inf_{\theta \in \mathbb{R}} \left(\int_{B_\rho(y)} |p - \theta|^{\frac{1}{2}} dx \right)^2 \lesssim K_0.$$

Combining these inequalities, we obtain that $\Psi(y, \rho) \lesssim K_0$, which together with (3.12) gives $\Phi_{x_0}(y, \rho) \lesssim K_0$. We finish the proof of assertion (ii) by applying (3.4) with y and $r/2$ in place of x_0 and r , to bound K_0 by the right-hand side of (3.5). $\square$

We are ready to prove Theorem 2.4.

Proof of Theorem 2.4. We adapt the arguments in the proof of [17, Theorem 1.1]. Let $\tilde{\omega}_\bullet$ and $\tilde{\varrho}_1$ be the Dini functions derived from $\omega_\bullet$ and ϱ_1 , respectively, as formulated in (3.10) with a fixed $\gamma \in (0, 1)$. We denote

$$\mathcal{F}(r) = \int_0^r \frac{\tilde{\omega}_{f_\alpha}(t) + \tilde{\omega}_g(t)}{t} dt.$$

For given $y \in \mathcal{D}$ and $\rho > 0$ with $B_\rho(y) \subset B_1(x_0)$, we let $\Theta_{x_0}(y, \rho) \in \mathbb{R}^{d \times d}$ be such that

$$\Phi_{x_0}(y, \rho) = \left(\int_{B_\rho(y)} |(D_{x'}u, U) - \Theta_{x_0}(y, \rho)|^{\frac{1}{2}} dx \right)^2.$$

We divide the proof into four steps. In the first step, we will derive an a priori L^∞ -estimate for (Du, p) under the assumption that (Du, p) is locally bounded. We then obtain an estimate of the modulus of continuity of $(D_{x'}u, U)$ in the second step, from which the piecewise continuity of (Du, p) follows. In the third step, we shall derive an a priori estimate of the modulus of continuity of (Du, p) under the additional condition (2.7). In the last step, we shall show that (Du, p) is indeed locally bounded by using the technique of flattening the boundary and a fixed point argument combined with partial Schauder estimates for Stokes systems.

Step 1. Let $r \in (0, R_1]$. Note that Lemma 3.1 (i) implies

$$\lim_{i \rightarrow \infty} \Phi_{x_0}(x_0, \kappa^i r) = 0$$

for all $x_0 \in \mathcal{D}'$, where $\kappa \in (0, 1/8]$ is the constant from the proof of Lemma 3.1. Thus, using the assumption that Du and p are bounded, we have

$$\lim_{i \rightarrow \infty} \Theta_{x_0}(x_0, \kappa^i r) = (D_{x'}u(x_0), U(x_0))$$

for a.e. $x_0 \in \mathcal{D}'$ in the coordinate systems associated with x_0 satisfying (A1) and (A2). By the same iteration argument that led to [8, equation (4.10)], we have

$$(3.17) \quad |(D_{x'}u(x_0), U(x_0)) - \Theta_{x_0}(x_0, r)| \lesssim \sum_{i=0}^{\infty} \Phi_{x_0}(x_0, \kappa^i r).$$

Since

$$|\Theta_{x_0}(x_0, r)| \lesssim r^{-d} (\|D_{x'}u\|_{L^1(B_r(x_0))} + \|U\|_{L^1(B_r(x_0))}),$$

by Lemma 3.1 (i) and the fact that

$$(3.18) \quad \sum_{i=0}^{\infty} \tilde{\omega}_{\bullet}(\kappa^i r) \lesssim \int_0^r \frac{\tilde{\omega}_{\bullet}(t)}{t} dt, \quad \sum_{i=0}^{\infty} \tilde{\varrho}_1(\kappa^i r) \lesssim \int_0^r \frac{\tilde{\varrho}_1(t)}{t} dt,$$

we obtain

$$\begin{aligned} |D_{x'}u(x_0)| + |U(x_0)| &\lesssim_{d,\nu,M,\varrho_0,\gamma} \|Du\|_{L^\infty(B_r(x_0))} \int_0^r \frac{\tilde{\omega}_{A^{\alpha\beta}}(t) + \tilde{\varrho}_1(t)}{t} dt \\ &\quad + r^{-d} (\|D_{x'}u\|_{L^1(B_r(x_0))} + \|U\|_{L^1(B_r(x_0))}) \\ &\quad + (\|f_\alpha\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))}) \int_0^r \frac{\tilde{\varrho}_1(t)}{t} dt + \mathcal{F}(r). \end{aligned}$$

From this, together with the fact that

$$|Du| + |p| \lesssim_{d,\nu} |D_{x'}u| + |U| + |f_d| + |g|,$$

we get

$$\begin{aligned} |Du(x_0)| + |p(x_0)| &\leq N_0 \|Du\|_{L^\infty(B_r(x_0))} \int_0^r \frac{\tilde{\omega}_{A^{\alpha\beta}}(t) + \tilde{\varrho}_1(t)}{t} dt \\ &\quad + N_0 r^{-d} (\|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))}) \\ &\quad + N_0 (\|f_\alpha\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))}) \left(1 + \int_0^r \frac{\tilde{\varrho}_1(t)}{t} dt\right) + N_0 \mathcal{F}(r), \end{aligned}$$

where $N_0 = N_0(d, \nu, M, \varrho_0, \gamma)$. Taking $r_0 \in (0, R_1]$ sufficiently small so that

$$N_0 \int_0^{r_0} \frac{\tilde{\omega}_{A^{\alpha\beta}}(t) + \tilde{\varrho}_1(t)}{t} dt \leq \frac{1}{3^d},$$

we have

$$\begin{aligned} |Du(x_0)| + |p(x_0)| &\leq 3^{-d} \|Du\|_{L^\infty(B_r(x_0))} \\ &\quad + N_0 r^{-d} (\|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))}) \\ &\quad + N_0 (\|f_\alpha\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))}) + N_0 \mathcal{F}(r) \end{aligned}$$

for all $r \in (0, r_0]$. Note that the above inequality holds for a.e. $x_0 \in \mathcal{D}'$ and does not depend on coordinate systems. Therefore, by the same iteration argument that led to [8, equation (4.16)], we obtain the following L^∞ -estimate for Du and p :

$$(3.19) \quad \begin{aligned} & \|Du\|_{L^\infty(B_{r/2}(x_0))} + \|p\|_{L^\infty(B_{r/2}(x_0))} \\ & \leq Nr^{-d}(\|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))}) \\ & \quad + N(\|f_\alpha\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))}) + N\mathcal{F}(r), \end{aligned}$$

where $x_0 \in \mathcal{D}'$ and $r \in (0, R_1]$ with $B_r(x_0) \subset \mathcal{D}'$. In the above, N depends only on d , ν , M , ϱ_0 , $\omega_{A^{\alpha\beta}}$, and γ .

Step 2. Let $x_0 \in \mathcal{D}'$ and $r \in (0, R_1]$ with $B_r(x_0) \subset \mathcal{D}'$, and fix a coordinate system associated with x_0 satisfying (A1) and (A2). We claim that

$$(3.20) \quad \begin{aligned} & |(D_{x'}u(x_0), U(x_0)) - (D_{x'}u(y_0), U(y_0))| \\ & \lesssim r^{-d}(\|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))})\mathcal{E}(|x_0 - y_0|) \\ & \quad + (\|f_\alpha\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))})\mathcal{E}(|x_0 - y_0|) \\ & \quad + \mathcal{F}(r)\mathcal{E}(|x_0 - y_0|) + \mathcal{F}(|x_0 - y_0|) \end{aligned}$$

for any $y_0 \in B_{r/4}(x_0)$, where

$$\mathcal{E}(|x_0 - y_0|) := \left(\frac{|x_0 - y_0|}{r}\right)^\gamma + \int_0^{|x_0 - y_0|} \frac{\tilde{\omega}_{A^{\alpha\beta}}(t) + \tilde{\varrho}_1(t)}{t} dt.$$

Let $y_0 \in B_{r/4}(x_0)$ and $\rho := |x_0 - y_0|$. We consider the following two cases:

$$B_\rho(y_0) \subset \widehat{\mathcal{D}}_{i_0}, \quad B_\rho(y_0) \not\subset \widehat{\mathcal{D}}_{i_0}.$$

Case 1. $B_\rho(y_0) \subset \widehat{\mathcal{D}}_{i_0}$. By the triangle inequality, we have

$$\begin{aligned} & |(D_{x'}u(x_0), U(x_0)) - (D_{x'}u(y_0), U(y_0))|^{\frac{1}{2}} \\ & \leq |(D_{x'}u(x_0), U(x_0)) - \Theta_{x_0}(x_0, \rho)|^{\frac{1}{2}} + |(D_{x'}u(x), U(x)) - \Theta_{x_0}(x_0, \rho)|^{\frac{1}{2}} \\ & \quad + |(D_{x'}u(y_0), U(y_0)) - \Theta_{x_0}(y_0, \rho)|^{\frac{1}{2}} + |(D_{x'}u(x), U(x)) - \Theta_{x_0}(y_0, \rho)|^{\frac{1}{2}} \end{aligned}$$

for all $x \in B_\rho(x_0) \cap B_\rho(y_0)$. Taking the average over $x \in B_\rho(x_0) \cap B_\rho(y_0)$ and then taking the square, we obtain that

$$|(D_{x'}u(x_0), U(x_0)) - (D_{x'}u(y_0), U(y_0))| \lesssim I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= |(D_{x'}u(x_0), U(x_0)) - \Theta_{x_0}(x_0, \rho)| + \Phi_{x_0}(x_0, \rho), \\ I_2 &= |(D_{x'}u(y_0), U(y_0)) - \Theta_{x_0}(y_0, \rho)| + \Phi_{x_0}(y_0, \rho). \end{aligned}$$

Note that by (3.17), we have

$$I_1 \lesssim \sum_{i=0}^{\infty} \Phi_{x_0}(x_0, \kappa^i \rho).$$

It follows from Lemma 3.1 (ii) that

$$\lim_{i \rightarrow \infty} \Phi_{x_0}(y_0, \kappa^i r) = 0.$$

Then by replicating an argument similar to that used in (3.17), we obtain

$$I_2 \lesssim \sum_{i=0}^{\infty} \Phi_{x_0}(y_0, \kappa^i \rho).$$

Therefore, by Lemma 3.1, (3.18), and (3.19), we get (3.20).

Case 2. $B_\rho(y_0) \not\subset \widehat{\mathcal{D}}_{i_0}$. In this case, for simplicity of notation, we assume that $y_0 = 0$. Suppose that $0 \in \widehat{\mathcal{D}}_{i_1} \cup \partial \widehat{\mathcal{D}}_{i_1}$ for some $i_1 \in \{1, \dots, \ell + 1\}$ and denote by $\tilde{y}_0$ the closest point on $\partial \widehat{\mathcal{D}}_{i_1}$ to the origin. We also denote $\tilde{x}_0 = (x'_0, \chi_{i_0}(x'_0))$, which is the closest point on $\partial \widehat{\mathcal{D}}_{i_0}$ to x_0 . Since $|\tilde{y}_0| < \rho$ and $|\tilde{x}_0 - x_0| < 2\rho$, we have

$$(3.21) \quad |\tilde{x}_0 - \tilde{y}_0| \leq |\tilde{x}_0 - x_0| + |x_0| + |\tilde{y}_0| < 4\rho < r \leq R_1.$$

Let

$$y = \Lambda x, \quad x = \Lambda^{-1}y = \Gamma y,$$

where Λ is a $d \times d$ rotation matrix from the coordinate systems associated with x_0 to a coordinate system associated with the origin. Then by (3.21) and the same argument as in [17, pp. 2465–2466], we see that

$$|\mathbf{I} - \Gamma| \lesssim \varrho_1(4\rho),$$

where $\mathbf{I}$ is the $d \times d$ identity matrix. From the definition of $\tilde{\varrho}_1$ and (3.18), it follows that

$$(3.22) \quad |\mathbf{I} - \Gamma| \lesssim \tilde{\varrho}_1(\rho) \lesssim \int_0^\rho \frac{\tilde{\varrho}_1(t)}{t} dt.$$

Now we set

$$v(y) = \Lambda u(x), \quad \pi(y) = p(x),$$

which satisfies

$$\begin{cases} D_\alpha(\mathcal{A}^{\alpha\beta} D_\beta v) + \nabla \pi = D_\alpha F_\alpha, \\ \operatorname{div} v = G, \end{cases}$$

where

$$\mathcal{A}^{\alpha\beta}(y) = \Lambda(\Lambda^{\alpha k} \Lambda^{\beta l} A^{kl}(x))\Gamma,$$

$$(F_1, \dots, F_d)(y) = \Lambda(f_1, \dots, f_d)(x)\Gamma, \quad G(y) = g(x).$$

We also denote

$$V = \mathcal{A}^{d\beta} D_\beta v + \pi e_d - F_d.$$

By the triangle inequality, we have

$$\begin{aligned} & |(D_{x'}u(x_0), U(x_0)) - (D_{x'}u(0), U(0))|^{\frac{1}{2}} \\ & \leq |(D_{x'}u(x_0), U(x_0)) - \Theta_{x_0}(x_0, \rho)|^{\frac{1}{2}} + |(D_{x'}u(x), U(x)) - \Theta_{x_0}(x_0, \rho)|^{\frac{1}{2}} \\ & \quad + |\Gamma(D_{y'}v(0), V(0)) - \Gamma\Theta_0(0, \rho)|^{\frac{1}{2}} + |\Gamma(D_{y'}v(\Lambda x), V(\Lambda x)) - \Gamma\Theta_0(0, \rho)|^{\frac{1}{2}} \\ & \quad + |(D_{x'}u(0), U(0)) - \Gamma(D_{y'}v(0), V(0))|^{\frac{1}{2}} \\ & \quad + |(D_{x'}u(x), U(x)) - \Gamma(D_{y'}v(\Lambda x), V(\Lambda x))|^{\frac{1}{2}} \end{aligned}$$

for any $x \in B_\rho(x_0) \cap B_\rho(0)$, where $\Gamma(D_{y'}v, V) := (\Gamma D_{y'}v, \Gamma V)$. Taking the average over $x \in B_\rho(x_0) \cap B_\rho(0)$ and then taking the square, we obtain that

$$(3.23) \quad |(D_{x'}u(x_0), U(x_0)) - (D_{x'}u(0), U(0))| \lesssim J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &= |(D_{x'}u(x_0), U(x_0)) - \Theta_{x_0}(x_0, \rho)| + \Phi_{x_0}(x_0, \rho), \\ J_2 &= |(D_{y'}v(0), V(0)) - \Theta_0(0, \rho)| + \Phi_0(0, \rho), \\ J_3 &= \operatorname{ess\,sup}_{x \in B_\rho(x_0) \cap B_\rho(0)} |(D_{x'}u(x), U(x)) - \Gamma(D_{y'}v(\Lambda x), V(\Lambda x))|. \end{aligned}$$

Note that J_1 and J_2 can be estimated by Lemma 3.1 (i), (3.18), and (3.19) in the same way as in Case 1. For the estimate of J_3 , we observe that

$$D_{x'}u(x) - \Gamma D_{y'}v(\Lambda x) = D_x u(x) \mathbf{I}_0 - \Gamma D_y v(\Lambda x) \mathbf{I}_0 = D_x u(x) (\mathbf{I} - \Gamma) \mathbf{I}_0,$$

where $\mathbf{I}_0 = (I_0^{\alpha\beta})$ is a $d \times (d-1)$ matrix with

$$I_0^{\alpha\beta} = \delta_{\alpha\beta} \text{ for } \alpha, \beta = 1, \dots, d-1; \quad I_0^{d\beta} = 0 \text{ for } \beta = 1, \dots, d-1,$$

and

$$\begin{aligned} U(x) - \Gamma V(\Lambda x) &= (\delta_{d\alpha} - \Lambda^{d\alpha}) A^{\alpha\beta}(x) D_\beta u(x) \\ &\quad + p(x)(I - \Gamma)e_d + (f_1, \dots, f_d)(x)(\mathbf{I} - \Gamma)^{\cdot d}, \end{aligned}$$

where $(\mathbf{I} - \Gamma)^{\cdot d}$ is the d th column of $\mathbf{I} - \Gamma$. Hence by (3.18) and (3.22), we have

$$\begin{aligned} J_3 &\lesssim (\|Du\|_{L^\infty(B_{r/4}(x_0))} + \|p\|_{L^\infty(B_{r/4}(x_0))} + \|f_\alpha\|_{L^\infty(B_r(x_0))}) \int_0^\rho \frac{\tilde{\varrho}_1(t)}{t} dt \\ &\lesssim r^{-d} (\|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))}) \int_0^\rho \frac{\tilde{\varrho}_1(t)}{t} dt \\ &\quad + (\|f_\alpha\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))}) \int_0^\rho \frac{\tilde{\varrho}_1(t)}{t} dt + \mathcal{F}(r) \int_0^\rho \frac{\tilde{\varrho}_1(t)}{t} dt. \end{aligned}$$

Using this together with the estimates J_1 and J_2 , we get (3.20) from (3.23).

Note that the piecewise continuity of (Du, p) follows from the estimate (3.20) combined with the fact that the coefficients and data are piecewise continuous. Indeed, by using the relations (3.13), (3.15), and (3.16), and using the triangle inequality, we have that

$$|D_d u^d(x_0) - D_d u^d(y_0)| \leq |D_{x'}u(x_0) - D_{x'}u(y_0)| + |g(x_0) - g(y_0)|,$$

$$\begin{aligned} |\mathcal{X}(x_0) - \mathcal{X}(y_0)| &\lesssim_{d,\nu} |(D_{x'}u(x_0), U(x_0)) - (D_{x'}u(y_0), U(y_0))| \\ &\quad + (\|Du\|_{L^\infty(B_{r/4}(x_0))} + \|p\|_{L^\infty(B_{r/4}(x_0))}) |A^{\alpha\beta}(x_0) - A^{\alpha\beta}(y_0)| \\ &\quad + \|f_\alpha\|_{L^\infty(B_{r/4}(x_0))} |A^{\alpha\beta}(x_0) - A^{\alpha\beta}(y_0)| \\ &\quad + |D_d u^d(x_0) - D_d u^d(y_0)| + |f_\alpha(x_0) - f_\alpha(y_0)|, \end{aligned}$$

where $\mathcal{X} = (D_d u^1, \dots, D_d u^{d-1})^\top$, and

$$\begin{aligned} |p(x_0) - p(y_0)| &\lesssim_{d,\nu} |(D_{x'}u(x_0), U(x_0)) - (D_{x'}u(y_0), U(y_0))| \\ &\quad + \|Du\|_{L^\infty(B_{r/4}(x_0))} |A^{\alpha\beta}(x_0) - A^{\alpha\beta}(y_0)| + |f_\alpha(x_0) - f_\alpha(y_0)|. \end{aligned}$$

Therefore, by (3.19) and (3.20), we obtain that

$$\begin{aligned}
 & |(Du(x_0), p(x_0)) - (Du(y_0), p(y_0))| \\
 & \leq Nr^{-d} (\|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))}) (\mathcal{E}(|x_0 - y_0|) + |A^{\alpha\beta}(x_0) - A^{\alpha\beta}(y_0)|) \\
 & \quad + N (\|f_\alpha\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))}) (\mathcal{E}(|x_0 - y_0|) + |A^{\alpha\beta}(x_0) - A^{\alpha\beta}(y_0)|) \\
 & \quad + N\mathcal{F}(r) (\mathcal{E}(|x_0 - y_0|) + |A^{\alpha\beta}(x_0) - A^{\alpha\beta}(y_0)|) + N\mathcal{F}(|x_0 - y_0|) \\
 (3.24) \quad & + N|f_\alpha(x_0) - f_\alpha(y_0)| + N|g(x_0) - g(y_0)|
 \end{aligned}$$

for any $x_0, y_0 \in \mathcal{D}'$ and $r \in (0, R_1]$ satisfying $y_0 \in B_{r/4}(x_0) \subset B_r(x_0) \subset \mathcal{D}'$, which gives the piecewise continuity of (Du, p) .

Step 3. In this step, we derive the corresponding estimate of (3.24) under the additional stronger (2.7). We again let $x_0 \in \mathcal{D}'$ and $r \in (0, R_1]$ with $B_r(x_0) \subset \mathcal{D}'$ and fix a coordinate system associated with x_0 satisfying (A1) and (A2). To present the precise dependence of the constant in the estimates, we assume that

$$(3.25) \quad \varrho_0(r) \leq K_0 r^{\frac{\gamma_0}{1-\gamma_0}}, \quad \omega_{A^{\alpha\beta}}(r) \leq K_0 r^{\gamma_0}, \quad \omega_{f_\alpha}(r) + \omega_g(r) \leq K_1 r^{\gamma_0}$$

for some constants $K_0, K_1 > 0$. Thus if f_α and g are in $C^{\gamma_0}(\overline{\mathcal{D}}_i)$ for each $i \in \{1, \dots, M\}$, then K_1 can be regarded as

$$\max_{1 \leq i \leq M} \{[f_\alpha]_{C^{\gamma_0}(\overline{\mathcal{D}}_i)} + [g]_{C^{\gamma_0}(\overline{\mathcal{D}}_i)}\}.$$

From [27, Lemma 5.1] it follows that for any $r \in (0, R_1]$

$$r^{-d} |(\widehat{\mathcal{D}}_i \setminus \Omega_i) \cap B_r(x_0)| \lesssim_{d, M, K_0, \gamma_0} r^{\gamma_0} =: \varrho_1(r).$$

Hence we have

$$\tilde{\omega}_{A^{\alpha\beta}}(r) + \tilde{\varrho}_1(r) \lesssim_{d, M, K_0, \gamma_0} r^{\gamma_0}$$

and

$$\tilde{\omega}_{f_\alpha}(r) + \tilde{\omega}_g(r) \lesssim_{d, M, K_0, \gamma_0} K_1 r^{\gamma_0}.$$

Therefore by (3.24) with $\gamma = \frac{1+\gamma_0}{2}$, we conclude that

$$\begin{aligned}
 & |(Du(x_0), p(x_0)) - (Du(y_0), p(y_0))| \\
 & \leq Nr^{-d} (\|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))}) \left(\frac{|x_0 - y_0|^{\gamma_0}}{r^{\gamma_0}} + |A^{\alpha\beta}(x_0) - A^{\alpha\beta}(y_0)| \right) \\
 & \quad + N (\|f_\alpha\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))}) \left(\frac{|x_0 - y_0|^{\gamma_0}}{r^{\gamma_0}} + |A^{\alpha\beta}(x_0) - A^{\alpha\beta}(y_0)| \right) \\
 & \quad + NK_1 (|x_0 - y_0|^{\gamma_0} + |A^{\alpha\beta}(x_0) - A^{\alpha\beta}(y_0)|) \\
 (3.26) \quad & + N|f_\alpha(x_0) - f_\alpha(y_0)| + N|g(x_0) - g(y_0)|,
 \end{aligned}$$

where $N = N(d, \nu, M, K_0, \gamma_0)$. We can see from (3.26) that if x_0 and y_0 are in the same subdomain, then the estimate of the modulus of continuity of (Du, p) is established.

Step 4. In this last step, we prove the local boundedness of (Du, p) . We first observe that

$$(3.27) \quad (Du, p) \in L^q_{\text{loc}}(\mathcal{D})^d \times L^q_{\text{loc}}(\mathcal{D}) \quad \text{for any } q < \infty.$$

Indeed, since (u, p) satisfies (2.6), where the coefficients $A^{\alpha\beta}$ are of variably partially small bounded mean oscillation (variably partially BMO) satisfying [13, Assumption 2.2 (ρ) (i)] for any $\rho > 0$ and the data f_α, g are bounded, by applying a local version of [13, Theorem 2.4] combined with a bootstrap argument, we get (3.27).

Due to the regularity result in [8], where the authors proved $W^{1,\infty}$ -estimates for solutions to Stokes systems with (partially) DMO coefficients in a ball, it suffices to show that for $x_0 = (x'_0, x_0^d) \in \partial\mathcal{D}_i$, $i \in \{1, \dots, M-1\}$, there is a neighborhood of x_0 in which (Du, p) is bounded. Recall that x_0 belongs to the boundaries of at most two of the subdomains. Thus we can find a small $r_0 > 0$ and a $C^{1,\text{Dini}}$ function, say $\chi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$, such that $B_{r_0}(x_0)$ is divided into two disjoint subdomains separated by χ and $|\nabla_{x'}\chi(x'_0)| = 0$ in a coordinate system. Here, we choose r_0 small enough so that

$$(3.28) \quad |\nabla_{x'}\chi(x')| \leq \mu_0 \quad \text{if } |x' - x'_0| \leq r_0,$$

where $\mu_0 > 0$ is a constant to be chosen below. Without loss of generality, we assume that $x_0 = (0', 0)$ and $\chi(0') = 0$. For sufficiently small $\varepsilon > 0$, we let χ_ε be a standard mollification of χ with respect to x' . We also let $\phi \in C_0^\infty(B_1)$ be a smooth nonnegative function with unit integral, and define piecewise mollifications of $A^{\alpha\beta}$ by

$$A_\varepsilon^{\alpha\beta}(x) = \int_{B_\varepsilon(x_\varepsilon)} \phi_\varepsilon(x_\varepsilon - y) A^{\alpha\beta}(y) dy = \int_{B_\varepsilon} \phi_\varepsilon(y) A^{\alpha\beta}(x_\varepsilon - y) dy,$$

where $\phi_\varepsilon(x) = \varepsilon^{-d} \phi(x/\varepsilon)$ and

$$x_\varepsilon = \begin{cases} x + \lambda \varepsilon e_d & \text{if } x^d > \chi_\varepsilon(x'), \\ x - \lambda \varepsilon e_d & \text{if } x^d < \chi_\varepsilon(x'). \end{cases}$$

Here λ is large enough, say $\lambda = \mu_0 + 1$. Similarly, we define $f_{\alpha,\varepsilon}$ and g_ε . Then the piecewise mollifications are piecewise DMO in B_{r_0} with

$$\omega_{\bullet_\varepsilon}(r) \leq \omega_\bullet(r).$$

Let $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ be the weak solution in $W_0^{1,2}(B_{r_0})^d \times \tilde{L}^2(B_{r_0})$ to the problem

$$(3.29) \quad \begin{cases} D_\alpha(A_\varepsilon^{\alpha\beta} D_\beta \tilde{u}_\varepsilon) + \nabla \tilde{p}_\varepsilon = D_\alpha(f_\alpha - f_{\alpha,\varepsilon}) + D_\alpha((A_\varepsilon^{\alpha\beta} - A^{\alpha\beta}) D_\beta u), \\ \operatorname{div} \tilde{u}_\varepsilon = g - g_\varepsilon - (g - g_\varepsilon)_{B_{r_0}}. \end{cases}$$

Since $f_{\alpha,\varepsilon} \rightarrow f_\alpha$ in L^2 , $g_\varepsilon \rightarrow g$ in L^2 , and $A_\varepsilon^{\alpha\beta} \rightarrow A^{\alpha\beta}$ a.e., by the dominated convergence theorem, the right-hand sides of (3.29) go to zero in L^2 as $\varepsilon \rightarrow 0^+$. By the $W^{1,2}$ -estimate, we see that

$$\|D\tilde{u}_\varepsilon\|_{L^2(B_{r_0})} + \|\tilde{p}_\varepsilon\|_{L^2(B_{r_0})} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+,$$

and thus there is a subsequence, still denoted by $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$, such that $|D\tilde{u}_\varepsilon| + |\tilde{p}_\varepsilon| \rightarrow 0$ a.e. in B_{r_0} .

Now we set $(u_\varepsilon, p_\varepsilon) = (u - \tilde{u}_\varepsilon, p - \tilde{p}_\varepsilon) \in W^{1,2}(B_{r_0})^d \times L^2(B_{r_0})$, which satisfies

$$(3.30) \quad \begin{cases} D_\alpha(A_\varepsilon^{\alpha\beta} D_\beta u_\varepsilon) + \nabla p_\varepsilon = D_\alpha f_{\alpha,\varepsilon}, \\ \operatorname{div} u_\varepsilon = g_\varepsilon + (g - g_\varepsilon)_{B_{r_0}} \end{cases}$$

in B_{r_0} . By the same reasoning as in (3.27), it holds that

$$(Du_\varepsilon, p_\varepsilon) \in L_{\text{loc}}^q(B_{r_0})^{d \times d} \times L_{\text{loc}}^q(B_{r_0}) \quad \text{for any } q < \infty.$$

We shall prove that $(Du_\varepsilon, p_\varepsilon)$ is bounded near the origin so that (3.19) can be applied to the above system, which gives the uniform L^∞ -estimate of $(Du_\varepsilon, p_\varepsilon)$. To this end, we fix $\varepsilon > 0$ and let

$$y = \Lambda(x) = (x', x^d - \chi_\varepsilon(x')), \quad x = \Lambda^{-1}(y) = \Gamma(y) = (y', y^d + \chi_\varepsilon(y')).$$

Then $(v(y), \pi(y)) = (u_\varepsilon(x), p_\varepsilon(x))$ satisfies

$$(3.31) \quad \begin{cases} D_\alpha(\mathcal{A}^{\alpha\beta} D_\beta v) + \nabla \pi = D_\alpha F_\alpha + D_d(\pi b), \\ \operatorname{div} v = G + D_d v \cdot b \end{cases}$$

in B_{r_1} with a sufficiently small $r_1 > 0$ so that $\overline{B_{r_1}} \subset \Lambda(B_{r_0})$, where

$$\mathcal{A}^{\alpha\beta}(y) = D_l \Lambda^\beta D_k \Lambda^\alpha A_\varepsilon^{kl}(x), \quad F_\alpha(y) = D_k \Lambda^\alpha f_{k,\varepsilon}(x),$$

$$G(y) = g_\varepsilon(x) + (g - g_\varepsilon)_{B_{r_0}}, \quad b(y) = (D_1 \chi_\varepsilon(y'), \dots, D_{d-1} \chi_\varepsilon(y'), 0).$$

Note that the coefficients and data are of partially DMO in B_{r_1} except πb and $D_d v \cdot b$, which are only known to be in $L^q(B_{r_1})$ for $q < \infty$. Thus we are not able to apply the result in [8, Theorem 2.2] to (3.31) directly. To overcome this difficulty, we use the following fixed point argument.

Let η be an infinitely differentiable function in $\mathbb{R}^d$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_{r_1/2}, \quad \operatorname{supp} \eta \subset B_{r_1}.$$

Then we see that $(\eta v, \eta \pi)$ satisfies

$$(3.32) \quad \begin{cases} D_\alpha(\mathcal{A}^{\alpha\beta} D_\beta(\eta v)) + \nabla(\eta \pi) = D_\alpha \tilde{F}_\alpha + \tilde{F} + D_d(\eta \pi b), \\ \operatorname{div}(\eta v) = \tilde{G} + D_d(\eta v) \cdot b \end{cases}$$

in B_{r_1} , where

$$\tilde{F}_\alpha = \eta F_\alpha + \mathcal{A}^{\alpha\beta} D_\beta \eta v, \quad \tilde{F} = \mathcal{A}^{\alpha\beta} D_\alpha \eta D_\beta v - D_\alpha \eta F_\alpha - D_d \eta \pi b + \nabla \eta \pi,$$

$$\tilde{G} = \eta G + \nabla \eta \cdot v - D_d \eta v \cdot b.$$

For each positive integer k , let $(v^{(k)}, \pi^{(k)})$ be the weak solution in $W_0^{1,2}(B_{r_1})^d \times \tilde{L}^2(B_{r_1})$ to the problem

$$(3.33) \quad \begin{cases} D_\alpha(\mathcal{A}^{\alpha\beta} D_\beta v^{(k)}) + \nabla \pi^{(k)} = D_\alpha \tilde{F}_\alpha + \tilde{F} + D_d(\pi^{(k-1)} b), \\ \operatorname{div} v^{(k)} = \tilde{G} + D_d v^{(k-1)} \cdot b - (\tilde{G} + D_d v^{(k-1)} \cdot b)_{B_{r_1}} \end{cases}$$

in B_{r_1} , where $(v^{(0)}, \pi^{(0)}) = (0, 0)$. By applying the $W^{1,2}$ -estimate to

$$(3.34) \quad (v^{(k+1)} - v^{(k)}, \pi^{(k+1)} - \pi^{(k)})$$

and using (3.28), we have

$$(3.35) \quad \begin{aligned} & \|Dv^{(k+1)} - Dv^{(k)}\|_{L^2(B_{r_1})} + \|\pi^{(k+1)} - \pi^{(k)}\|_{L^2(B_{r_1})} \\ & \leq N_0 \|(Dv^{(k)} - Dv^{(k-1)})b\|_{L^2(B_{r_1})} + N_0 \|(\pi^{(k)} - \pi^{(k-1)})b\|_{L^2(B_{r_1})} \\ & \leq \mu_0 N_0 \|Dv^{(k)} - Dv^{(k-1)}\|_{L^2(B_{r_1})} + \mu_0 N_0 \|\pi^{(k)} - \pi^{(k-1)}\|_{L^2(B_{r_1})}, \end{aligned}$$

where the constant N_0 is independent of ε and $\{(v^{(k)}, \pi^{(k)})\}$. We take r_0 sufficiently small so that (3.28) holds with $\mu_0 = 1/(2N_0)$. Then by the fixed point theorem, there exists

$$(v^*, \pi^*) = (v_\varepsilon^*, \pi_\varepsilon^*) \in W_0^{1,2}(B_{r_1})^d \times \tilde{L}^2(B_{r_1})$$

such that as $k \rightarrow \infty$,

$$v^{(k)} \rightarrow v^* \quad \text{in } W_0^{1,2}(B_{r_1}), \quad \pi^{(k)} \rightarrow \pi^* \quad \text{in } L^2(B_{r_1})$$

and such that, in B_{r_1} ,

$$(3.36) \quad \begin{cases} D_\alpha(\mathcal{A}^{\alpha\beta} D_\beta v^*) + \nabla \pi^* = D_\alpha \tilde{F}_\alpha + \tilde{F} + D_d(\pi^* b), \\ \operatorname{div} v^* = \tilde{G} + D_d v^* \cdot b - (\tilde{G} + D_d v^* \cdot b)_{B_{r_1}}. \end{cases}$$

From (3.32) and (3.36), it follows that in B_{r_1} ,

$$\begin{cases} D_\alpha(\mathcal{A}^{\alpha\beta} D_\beta(\eta v - v^*)) + \nabla(\eta\pi - (\eta\pi)_{B_{r_1}} - \pi^*) = D_d((\eta\pi - \pi^*)b), \\ \operatorname{div}(\eta v - v^*) = D_d(\eta v - v^*) \cdot b + (\tilde{G} + D_d v^* \cdot b)_{B_{r_1}}. \end{cases}$$

Note that $D_d b = 0$ and $(\tilde{G} + D_d(\eta v) \cdot b)_{B_{r_1}} = 0$. Hence by the $W^{1,2}$ -estimate with the smallness of b , we obtain that

$$\eta v = v^*, \quad \eta\pi - (\eta\pi)_{B_{r_1}} = \pi^*.$$

Next, let $\rho_0 \in (0, r_0]$ be small enough so that

$$(3.37) \quad |\nabla_{x'} \chi(x')| \leq \mu_1 \quad \text{if } |x'| \leq \rho_0,$$

where μ_1 is a constant to be chosen below. We also let $\rho_1 \in (0, r_1]$ such that $\overline{B_{\rho_1}} \subset \Lambda(B_{\rho_0})$. Observe that $\mathcal{A}^{\alpha\beta}$, $\tilde{F}_\alpha$, and $\tilde{G}$ are partially Hölder continuous with respect to y' , $\tilde{F}_d \in L^\infty(B_{\rho_1})$, and $\tilde{F} \in L^q(B_{\rho_1})$ for all $q < \infty$. Therefore, the regularity results in [8, Theorem 2.2 (b)] are applicable to the system (3.33). Precisely, by applying [8, Theorem 2.2 (b) and Remark 2.4], combined with covering and scaling arguments, we obtain that

$$(Dv^{(1)}, \pi^{(1)}) \in L^\infty(B_\rho)^{d \times d} \times L^\infty(B_\rho) \quad \text{for all } \rho < \rho_1.$$

Moreover,

$$\mathcal{A}^{d\beta} D_\beta v^{(1)} + \pi^{(1)} e_d \in C^\delta(B_\rho)^d, \quad D_{y'} v^{(1)} \in C^\delta(B_\rho)^{d \times (d-1)},$$

from which we get

$$(Dv^{(1)}, \pi^{(1)}) \in C_{y'}^\delta(B_\rho)^{d \times d} \times C_{y'}^\delta(B_\rho) \quad \text{for all } \delta \in (0, 1).$$

Repeating this procedure, we see that

$$(Dv^{(k)}, \pi^{(k)}) \in (L^\infty(B_\rho)^{d \times d} \times L^\infty(B_\rho)) \cap (C_{y'}^\delta(B_\rho)^{d \times d} \times C_{y'}^\delta(B_\rho))$$

for any positive integer k . Hence, from the estimates in the proof of [8, Theorem 2.2 (b)] applied to (3.34) with covering and scaling arguments, we deduce that for any

$0 < s < \rho < \rho_1$,

$$\begin{aligned}
& \|Dv^{(k+1)} - Dv^{(k)}\|_{L^\infty(B_s)} + \|\pi^{(k+1)} - \pi^{(k)}\|_{L^\infty(B_s)} \\
& + (\rho - s)^\delta \left([Dv^{(k+1)} - Dv^{(k)}]_{C_{y'}^\delta(B_s)} + [\pi^{(k+1)} - \pi^{(k)}]_{C_{y'}^\delta(B_s)} \right) \\
& \leq N_1(\rho - s)^{-d} \left(\|Dv^{(k+1)} - Dv^{(k)}\|_{L^1(B_\rho)} + \|\pi^{(k+1)} - \pi^{(k)}\|_{L^1(B_\rho)} \right) \\
& \quad + N_1 \left(\|(Dv^{(k)} - Dv^{(k-1)})b\|_{L^\infty(B_\rho)} + \|(\pi^{(k)} - \pi^{(k-1)})b\|_{L^\infty(B_\rho)} \right) \\
& \quad + N_1(\rho - s)^\delta \left([(Dv^{(k)} - Dv^{(k-1)})b]_{C_{y'}^\delta(B_\rho)} + [(\pi^{(k)} - \pi^{(k-1)})b]_{C_{y'}^\delta(B_\rho)} \right) \\
& \leq \mu_0 N_0 N_1 (\rho - s)^{-d/2} \left(\|Dv^{(k)} - Dv^{(k-1)}\|_{L^2(B_{r_1})} + \|\pi^{(k)} - \pi^{(k-1)}\|_{L^2(B_{r_1})} \right) \\
& \quad + \mu_1 N_1 \left(\|Dv^{(k)} - Dv^{(k-1)}\|_{L^\infty(B_\rho)} + \|\pi^{(k)} - \pi^{(k-1)}\|_{L^\infty(B_\rho)} \right) \\
& \quad + [b]_{C_{y'}^\delta(B_{\rho_1})} N_1 (\rho - s)^\delta \left(\|Dv^{(k)} - Dv^{(k-1)}\|_{L^\infty(B_\rho)} + \|\pi^{(k)} - \pi^{(k-1)}\|_{L^\infty(B_\rho)} \right) \\
& \quad + \mu_1 N_1 (\rho - s)^\delta \left([Dv^{(k)} - Dv^{(k-1)}]_{C_{y'}^\delta(B_\rho)} + [\pi^{(k)} - \pi^{(k-1)}]_{C_{y'}^\delta(B_\rho)} \right),
\end{aligned}$$

where we used (3.28), (3.35), and (3.37) in the second inequality. Note that the constant N_1 is independent of $\{(v^{(k)}, \pi^{(k)})\}$, but it may depend on ε . By choosing ρ_0 sufficiently small, which (along with ρ_1) may depend on ε , and following a standard iteration argument, we get uniform L^∞ bounds of $Dv^{(k)}$ and $\pi^{(k)}$ in $B_{\rho_1/2}$. Thus the functions

$$Dv(y) = Dv^*(y), \quad \pi(y) = \pi^*(y),$$

and hence $Du_\varepsilon(x)$ and $p_\varepsilon(x)$ are bounded in a neighborhood of the origin with a radius depending also on ε . It is easy to check that the same argument as above still works at every point near the origin, for instance, in $B_{r_0/2}$, where r_0 is the constant from the beginning of this step, which is independent of ε . Therefore,

$$(Du_\varepsilon, p_\varepsilon) \in L^\infty(B_{r_0/2})^{d \times d} \times L^\infty(B_{r_0/2}).$$

Now we can apply the a priori estimate in Step 1 to (3.30) to get uniform L^∞ -bounds of $(Du_\varepsilon, p_\varepsilon)$, and then take the limit $\varepsilon \rightarrow 0^+$ to obtain the boundedness of the limit function (Du, p) in $B_{r_0/2}$. The theorem is proved. $\square$

We conclude the proof of Theorem 2.4 with the following remark.

Remark 3.2. As mentioned in Remark 2.7, the regularity results in Theorem 2.4 can be extended to weak solutions of

$$\begin{cases} \mathcal{L}u + \nabla p = D_\alpha f_\alpha + f & \text{in } \mathcal{D}, \\ \operatorname{div} u = g & \text{in } \mathcal{D}, \end{cases}$$

where $f \in L^s(\mathcal{D})^d$ with $s > d$. In this case, the upper bounds of the L^∞ -norm of (Du, p) and the modulus of continuity of $(D_{x'}u, U)$ can be derived as follows.

Let $x_0 \in \mathcal{D}'$ and $r \in (0, R_1]$ such that $B_r(x_0) \subset \mathcal{D}'$. Due to the solvability of the divergence equation (see, for instance, [9, Lemma 3.1]), there exist $h_\alpha \in W^{1,s}(B_r(x_0))^d$, $\alpha \in \{1, 2, \dots, d\}$, such that

$$\sum_{\alpha=1}^d D_\alpha h_\alpha = f \quad \text{in } B_r(x_0)$$

and

$$(h_\alpha)_{B_r(x_0)} = 0, \quad \|Dh_\alpha\|_{L^s(B_r(x_0))} \lesssim_{d,s} \|f\|_{L^s(B_r(x_0))}.$$

Then (u, p) satisfies

$$\begin{cases} \mathcal{L}u + \nabla p = D_\alpha(f_\alpha + h_\alpha) & \text{in } B_r(x_0), \\ \operatorname{div} u = g & \text{in } B_r(x_0), \end{cases}$$

where, by both Morrey and Poincaré inequalities,

$$r^{1-d/s}[h_\alpha]_{C^{1-d/s}(B_r(x_0))} + \|h_\alpha\|_{L^\infty(B_r(x_0))} \lesssim r^{1-d/s}\|f\|_{L^s(B_r(x_0))}.$$

Thus by the same argument as in the proof of Theorem 2.4 with a fixed $\gamma \in (1 - \frac{d}{s}, 1)$, we have

$$\begin{aligned} & \|Du\|_{L^\infty(B_{r/2}(x_0))} + \|p\|_{L^\infty(B_{r/2}(x_0))} \\ & \leq Nr^{-d}(\|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))}) \\ & \quad + N(\|f_\alpha\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))}) + N\mathcal{F}(r) + Nr^{1-d/s}\|f\|_{L^s(B_r(x_0))}, \end{aligned}$$

where $N = N(d, \nu, M, \varrho_0, \omega_{A^{\alpha\beta}}, s)$. Moreover, for $y_0 \in B_{r/4}(x_0)$, we obtain that

$$\begin{aligned} & |(D_{x'}u(x_0), U(x_0)) - (D_{x'}u(y_0), U(y_0))| \\ & \leq Nr^{-d}(\|Du\|_{L^1(B_r(x_0))} + \|p\|_{L^1(B_r(x_0))})\mathcal{E}(|x_0 - y_0|) \\ & \quad + N(\|f_\alpha\|_{L^\infty(B_r(x_0))} + \|g\|_{L^\infty(B_r(x_0))})\mathcal{E}(|x_0 - y_0|) \\ & \quad + N(\mathcal{F}(r) + r^{1-d/s}\|f\|_{L^s(B_r(x_0))})\mathcal{E}(|x_0 - y_0|) + N\mathcal{F}(|x_0 - y_0|) \\ & \quad + N\|f\|_{L^s(B_r(x_0))}|x_0 - y_0|^{1-d/s}. \end{aligned}$$

3.2. Proof of Theorem 2.10. Note that (u, p) satisfies

$$\begin{cases} \mathcal{L}u + \nabla p = D_\alpha f_\alpha + f & \text{in } \mathcal{D}, \\ \operatorname{div} u = g & \text{in } \mathcal{D}, \end{cases}$$

where $f = -u^\alpha D_\alpha u$. We consider two cases.

Case 1. $q > d$. In this case, by the Morrey-Sobolev embedding theorem, we see that $f \in L^q_{\text{loc}}(\mathcal{D})^d$. Thus the theorem follows from Remark 2.7 applied to a slightly diminished domain.

Case 2. $q \leq d$. From the first case, it suffices to improve the regularity of Du from L^q to L^s_{loc} for some $s > d$. Let $x_0 \in \mathcal{D}$. We may assume that $x_0 = 0$ and $B_1 \subset \mathcal{D}$ after translating and scaling the coordinates.

We first derive an a priori estimate for (Du, p) under the assumption that $(u, p) \in W^{1,q^*}(B_1)^d \times L^{q^*}(B_1)$, where q^* is the Sobolev conjugate of q , i.e., $q^* = dq/(d-q)$ when $q < d$ and $q^* \in (q, \infty)$ is arbitrary when $q = d$. Let η be an infinitely differentiable function in $\mathbb{R}^d$ such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } B_{1/2}, \quad \operatorname{supp} \eta \subset B_1, \quad |\nabla \eta| \lesssim_d 1.$$

We define an elliptic operator $\tilde{\mathcal{L}}$ by

$$\tilde{\mathcal{L}}u = D_\alpha(\tilde{A}^{\alpha\beta} D_\beta u),$$

where $\tilde{A}^{\alpha\beta} = \eta A^{\alpha\beta} + \nu(1 - \eta)\delta_{\alpha\beta}\mathbf{I}$. Here, ν is the constant from (2.4), $\delta_{\alpha\beta}$ is the Kronecker delta symbol, and $\mathbf{I}$ is the $d \times d$ identity matrix. Note that $\tilde{A}^{\alpha\beta}$ and $\Omega = B_1$ satisfy [13, Assumption 2.2 (ρ)] for any $\rho > 0$. Therefore, the W^{1,q^*} -estimate in [13, Theorem 2.4] is available for $\tilde{\mathcal{L}}$ on $\Omega = B_1$.

Now, for r, R with $0 < r < R \leq 1/2$, let $\zeta = \zeta_{r,R}$ be an infinitely differentiable function in $\mathbb{R}^d$ such that

$$0 \leq \zeta \leq 1, \quad \zeta \equiv 1 \text{ in } B_r, \quad \text{supp } \zeta \subset B_R, \quad |\nabla \zeta| \lesssim_d (R - r)^{-1}.$$

Then $(v, \pi) = (\zeta u, \zeta p) \in W_0^{1,q}(B_1)^d \times L^q(B_1)$ satisfies

$$(3.38) \quad \begin{cases} \tilde{\mathcal{L}}v + \nabla \pi = F + D_\alpha F_\alpha & \text{in } B_1, \\ \text{div } v = G & \text{in } B_1, \end{cases}$$

where

$$\begin{aligned} F &= D_\alpha \zeta A^{\alpha\beta} D_\beta u + \nabla \zeta p - D_\alpha \zeta f_\alpha - u^\alpha D_\alpha v + u^\alpha u D_\alpha \zeta, \\ F_\alpha &= A^{\alpha\beta} u D_\beta \zeta + \zeta f_\alpha, \quad G = \nabla \zeta \cdot u + \zeta g. \end{aligned}$$

Observe that $F_\alpha \in L^{q^*}(B_1)^d$, $G \in L^{q^*}(B_1)$, and

$$\begin{aligned} \|F\|_{L^q(B_1)} &\lesssim_{d,\nu} (R - r)^{-1} (\|Du\|_{L^q(B_R)} + \|p\|_{L^q(B_R)} + \|u\|_{L^{2q}(B_R)}^2) \\ &\quad + R(R - r)^{-1} \|f_\alpha\|_{L^{q^*}(B_R)} + \|u\|_{L^d(B_R)} \|Dv\|_{L^{q^*}(B_R)}. \end{aligned}$$

Then by the W^{1,q^*} -solvability in [13, Theorem 2.4], (3.38) also have a unique solution $(\tilde{v}, \tilde{\pi}) \in W_0^{1,q^*}(B_1)^d \times \tilde{L}^{q^*}(B_1)$, which is also in $W_0^{1,q}(B_1)^d \times \tilde{L}^q(B_1)$. By the uniqueness of $W_0^{1,q}(B_1)^d \times L^q(B_1)$ solutions, we get $(\tilde{v}, \tilde{\pi}) = (v, \pi - (\pi)_{B_1})$. By applying the W^{1,q^*} -estimate in [13, Theorem 2.4] to (3.38) and using the above inequality, we obtain that

$$\begin{aligned} &\|Dv\|_{L^{q^*}(B_1)} + \|\pi - (\pi)_{B_1}\|_{L^{q^*}(B_1)} \\ &\leq N (\|F\|_{L^q(B_1)} + \|F_\alpha\|_{L^{q^*}(B_1)} + \|G\|_{L^{q^*}(B_1)}) \\ &\leq N_0 (R - r)^{-1} (\|Du\|_{L^q(B_R)} + \|p\|_{L^q(B_R)} + \|u\|_{L^{2q}(B_R)}^2) + N_0 (R - r)^{-1} \|u\|_{L^{q^*}(B_R)} \\ &\quad + N_0 R (R - r)^{-1} \|f_\alpha\|_{L^{q^*}(B_R)} + N_0 \|g\|_{L^{q^*}(B_R)} + N_0 \|u\|_{L^d(B_R)} \|Dv\|_{L^{q^*}(B_R)}, \end{aligned}$$

where $N_0 = N_0(d, \nu, M, R_0, \varrho_0, \omega_{A^{\alpha\beta}}, q)$. Then by taking $R \in (0, 1/2]$ sufficiently small so that

$$N_0 \|u\|_{L^d(B_R)} \leq \varepsilon := \frac{1}{8},$$

we can absorb the last term on the right-hand side of (3.39) to the left-hand side. From the triangle and Hölder's inequalities, we have

$$(3.40) \quad \begin{aligned} &\|Du\|_{L^{q^*}(B_r)} + \|p\|_{L^{q^*}(B_r)} \\ &\leq \|Dv\|_{L^{q^*}(B_1)} + \|\pi - (\pi)_{B_1}\|_{L^{q^*}(B_1)} + N_1 \|\pi\|_{L^1(B_1)} \\ &\leq \|Dv\|_{L^{q^*}(B_1)} + \|\pi - (\pi)_{B_1}\|_{L^{q^*}(B_1)} + N_1 \|p\|_{L^q(B_R)}. \end{aligned}$$

Noting that $q^* \geq 2q$, it follows from (3.40) and (3.39) that

$$\begin{aligned} &\|Du\|_{L^{q^*}(B_r)} + \|p\|_{L^{q^*}(B_r)} \\ &\leq (N_0 + N_1) (R - r)^{-1} (\|Du\|_{L^q(B_R)} + \|p\|_{L^q(B_R)} + \|u\|_{L^{2q}(B_R)}^2) \\ &\quad + N_0 (R - r)^{-1} \|u\|_{L^{q^*}(B_R)} + N_0 R (R - r)^{-1} \|f_\alpha\|_{L^{q^*}(B_R)} + N_0 \|g\|_{L^{q^*}(B_R)}. \end{aligned}$$

We are ready to prove

$$(3.41) \quad Du \in L_{\text{loc}}^s(\mathcal{D})^{d \times d} \quad \text{for some } s > d.$$

From the a priori estimate (3.39) and the fixed point argument as in the proof of Theorem 2.4, one can show that $Du \in L_{\text{loc}}^{q^*}(\mathcal{D})^{d \times d}$. This yields (3.41) when $d/2 < q \leq d$ because $q^* > d$. On the other hand, if $q = d/2$, then since $Du \in L_{\text{loc}}^{q_1}(\mathcal{D})^{d \times d}$ for all $q_1 \leq d$, by applying the above regularity result again, we get (3.41). We have thus proved the regularity results in the theorem. The corresponding upper bounds of the L^∞ -norm of (Du, p) and the modulus of continuity of $(D_{x'}u, U)$ can be derived as in Remark 3.2. $\square$

REFERENCES

- [1] H. ABIDI, G. GUI, AND P. ZHANG, *On the decay and stability of global solutions to the 3D inhomogeneous Navier-Stokes equations*, Comm. Pure Appl. Math., 64 (2011), pp. 832–881.
- [2] H. AMMARI, E. BONNETIER, F. TRIKI, AND M. VOGELIUS, *Elliptic estimates in composite media with smooth inclusions: An integral equation approach*, Ann. Sci. Éc. Norm. Supér. (4), 48 (2015), pp. 453–495.
- [3] H. AMMARI, H. KANG, D. W. KIM, AND S. YU, *Quantitative Estimates for Stress Concentration of the Stokes Flow between Adjacent Circular Cylinders*, preprint, <https://arxiv.org/abs/2003.06578>, 2020.
- [4] A. ANCONA, *Elliptic operators, conormal derivatives and positive parts of functions (with an appendix by Haïm Brezis)*, J. Funct. Anal., 257 (2009), pp. 2124–2158.
- [5] E. BONNETIER AND M. VOGELIUS, *An elliptic regularity result for a composite medium with “touching” fibers of circular cross-section*, SIAM J. Math. Anal., 31 (2000), pp. 651–677, <https://doi.org/10.1137/S0036141098333980>.
- [6] H. BREZIS, *On a conjecture of J. Serrin*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 19 (2008), pp. 335–338.
- [7] M. CHIPOT, D. KINDERLEHRER, AND G. VERGARA-CAFFARELLI, *Smoothness of linear laminates*, Arch. Rational Mech. Anal., 96 (1986), pp. 81–96.
- [8] J. CHOI AND H. DONG, *Gradient estimates for Stokes systems with Dini mean oscillation coefficients*, J. Differential Equations, 266 (2019), pp. 4451–4509.
- [9] J. CHOI, H. DONG, AND D. KIM, *Conormal derivative problems for stationary Stokes system in Sobolev spaces*, Discrete Contin. Dyn. Syst., 38 (2018), pp. 2349–2374.
- [10] G. CITTI AND F. FERRARI, *A sharp regularity result of solutions of a transmission problem*, Proc. Amer. Math. Soc., 140 (2012), pp. 615–620.
- [11] H. DONG, *Gradient estimates for parabolic and elliptic systems from linear laminates*, Arch. Ration. Mech. Anal., 205 (2012), pp. 119–149.
- [12] H. DONG AND D. KIM, *L_q -estimates for stationary Stokes system with coefficients measurable in one direction*, Bull. Math. Sci., 9 (2019), 30.
- [13] H. DONG AND D. KIM, *Weighted L_q -estimates for stationary Stokes system with partially BMO coefficients*, J. Differential Equations, 264 (2018), pp. 4603–4649.
- [14] H. DONG AND S. KIM, *On C^1 , C^2 , and weak type-(1,1) estimates for linear elliptic operators*, Comm. Partial Differential Equations, 42 (2017), pp. 417–435.
- [15] H. DONG AND H. LI, *Optimal estimates for the conductivity problem by Green’s function method*, Arch. Ration. Mech. Anal., 231 (2019), pp. 1427–1453.
- [16] H. DONG AND J. XIONG, *Boundary gradient estimates for parabolic and elliptic systems from linear laminates*, Int. Math. Res. Not. IMRN, 2015 (2015), pp. 7734–7756.
- [17] H. DONG AND L. XU, *Gradient estimates for divergence form elliptic systems arising from composite material*, SIAM J. Math. Anal., 51 (2019), pp. 2444–2478, <https://doi.org/10.1137/18M1226658>.
- [18] H. DONG AND L. XU, *Gradient estimates for divergence form parabolic systems from composite materials*, Calc. Var. Partial Differential Equations, 60 (2021), 98.
- [19] C. GERHARDT, *Stationary solutions to the Navier-Stokes equations in dimension four*, Math. Z., 165 (1979), pp. 193–197.
- [20] M. GIAQUINTA, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Ann. Math. Stud. 105, Princeton University Press, Princeton, NJ, 1983.

- [21] B. R. JAISWAL AND B. R. GUPTA, *Stokes flow over composite sphere: Liquid core with permeable shell*, J. Appl. Fluid Mech., 8 (2015), pp. 339–350.
- [22] M. KOHR, S. E. MIKHAILOV, AND W. L. WENDLAND, *Layer potential theory for the anisotropic Stokes system with variable L_∞ symmetrically elliptic tensor coefficient*, Math. Methods Appl. Sci., 44 (2021), pp. 9641–9674, <https://doi.org/10.1002/mma.7167>.
- [23] M. KOHR AND W. L. WENDLAND, *Variational approach for the Stokes and Navier-Stokes systems with nonsmooth coefficients in Lipschitz domains on compact Riemannian manifolds*, Calc. Var. Partial Differential Equations, 57 (2018), 165.
- [24] N. V. KRYLOV, *Second-order elliptic equations with variably partially VMO coefficients*, J. Funct. Anal., 257 (2009), pp. 1695–1712.
- [25] O. A. LADYŽENSKAJA AND V. A. SOLONNIKOV, *The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids*, in Boundary Value Problems of Mathematical Physics and Related Questions of the Theory of Functions, 8, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 52 (1975), pp. 52–109, 218–219 (in Russian).
- [26] Y. LI AND L. NIRENBERG, *Estimates for elliptic systems from composite material*, Comm. Pure Appl. Math., 56 (2003), pp. 892–925.
- [27] Y. LI AND M. VOGELIUS, *Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients*, Arch. Ration. Mech. Anal., 153 (2000), pp. 91–151.
- [28] S. LIANG AND S. ZHENG, *Variable Lorentz estimate for stationary Stokes system with partially BMO coefficients*, Commun. Pure Appl. Anal., 18 (2019), pp. 2879–2903.
- [29] G. M. LIEBERMAN, *Intermediate Schauder theory for second order parabolic equations. IV. Time irregularity and regularity*, Differential Integral Equations, 5 (1992), pp. 1219–1236.
- [30] P.-L. LIONS, *Mathematical Topics in Fluid Mechanics. Vol. 1. Incompressible Models*, Oxford Lecture Ser. Math. Appl. 3, The Clarendon Press, Oxford University Press, New York, 1996.
- [31] J. H. MASLIYAH, G. NEALE, K. MALYSA, AND T. G. M. VAN DE VEN, *Creeping flow over a composite sphere: Solid core with porous shell*, Chem. Engrg. Sci., 42 (1987), pp. 245–253.
- [32] J. MATEU, J. OROBITG, AND J. VERDERA, *Extra cancellation of even Calderón-Zygmund operators and quasiconformal mappings*, J. Math. Pures Appl., 91 (2009), pp. 402–431.
- [33] J. XIONG AND J. BAO, *Sharp regularity for elliptic systems associated with transmission problems*, Potential Anal., 39 (2013), pp. 169–194.
- [34] J. ZHUGE, *Regularity of a transmission problem and periodic homogenization*, J. Math. Pures Appl., 153 (2021), pp. 213–247.