Global L_p Estimates for Kinetic Kolmogorov–Fokker–Planck Equations in Nondivergence Form

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Abstract

We study the degenerate Kolmogorov equations (also known as kinetic Fokker–Planck equations) in nondivergence form. The leading coefficients a^{ij} are merely measurable in t and satisfy the vanishing mean oscillation condition in x, v with respect to some quasi-metric. We also assume the boundedness and uniform nondegeneracy of a^{ij} with respect to v. We prove global a priori estimates in weighted mixed-norm Lebesgue spaces and solvability results. We also show an application of the main result to the Landau equation. Our proof does not rely on any kernel estimates.

1. Introduction

Let $d \geqq 1$, $\mathbb{R}^d$ be a Euclidean space of points $(x_1, \ldots, x_d)$, and for $T \in (-\infty, \infty]$ we denote $\mathbb{R}^d_T = (-\infty, T) \times \mathbb{R}^{d-1}$. By z we denote the triple (t, x, v), where $t \in \mathbb{R}$, and $x, v \in \mathbb{R}^d$.

In this paper, we study kinetic Kolmogorov–Fokker–Planck (KFP) operator in nondivergence form given by

$$Pu = \partial_t u - v \cdot D_x u - a^{ij}(z) D_{v_i v_j} u.$$

Here the coefficients $a(z) = (a^{ij}(z), i, j = 1, \ldots, d)$ are bounded measurable and uniformly nondegenerate. When the coefficients a^{ij} are independent of x and v, we denote P by P_0. This operator appears in the theory of diffusion processes [39], mathematical finance [33], and kinetic equations of plasma. In particular, the linearized Landau equation near Maxwellian can be rewritten as a Cauchy problem

$$Pf + b \cdot D_v f + cf = h, \quad f(0, \cdot, \cdot) = f_0(\cdot, \cdot), \tag{1.1}$$

where b is a vector-valued function (see, for instance, [24]), and c is a bounded function.

The goal of this article is to prove a priori estimates and the unique solvability in the weighted mixed-norm spaces $S_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$ (see Section 2), which generalize the ultraparabolic Sobolev space S_p (see, for instance, [8]). We do this under a relaxed VMO type assumption, which appears to be new. In particular, our coefficients a^{ij} are merely measurable in time and VMO in the x, v variables with respect to some quasi-distance (see Assumption 2.3). This assumption is analogous to the VMO_x condition from the theory of nondegenerate parabolic PDEs with discontinuous coefficients (see [15,27]).

The kinetic Kolmogorov–Fokker–Planck equations have been studied extensively [1,4–8,10–14,18,21–23,31–35,40], including nonlocal equations [11,21, 23,32]. Here we discuss the results of the Sobolev space theory. The interior S_p-estimate under a VMO assumption with respect to all variables was established in [8] for nondivergence equations, and in [31] for divergence form equations. The interior estimate in the ultraparabolic Morrey spaces was proved in [35]. The first global S_p estimate was discovered in [9] for the constant coefficients a^{ij} case. For a^{ij} independent of (x, v), the a priori estimates can be found in [12]. The first global S_p estimate for the variable coefficients a^{ij} case was established by the authors of [10]. Assuming the uniform continuity of the leading coefficients, they proved the a priori estimate of $D_v^2 u$ on a sufficiently small strip $[-T, T] \times \mathbb{R}^{2d}$. To the best of our knowledge, no solvability results in ultraparabolic Sobolev spaces were presented in these works except for the case of constant coefficients, and the operators considered are more general than P. On the other hand, when the coefficients are regular enough, say Hölder continuous (in the appropriate sense), the unique solvability is established in [13] (see also [2]) by studying the fundamental solution to a KFP operator. It seems that the first results in the weighted mixed-norm space belong to [32]. The authors proved the a priori estimates and unique solvability in the $L_p((0, T), L_q(\mathbb{R}^{2d}))$ spaces with the power weight in time (see Theorem 8.1 of [32]) assuming that the functions a^{ij} are uniformly continuous with respect to some quasi-metric. They also studied quasilinear kinetic KFP equations. In the case of the zero initial condition, the main result of this article covers Theorem 8.1 in [32] because we work with spaces that have a Muckenhoupt weight in the (t, v) variables.

The current paper generalizes the aforementioned results in several directions. First, our assumption on the coefficients a^{ij} is weaker than the ones presented in the literature. Second, we prove the a priori estimates in the weighted mixed-norm space where each direction $t, v_1, \ldots, v_d$ has a different Muckenhoupt weight. Third, we also discover the a priori estimate of $D_v(-\Delta_x)^{1/6} u$ for $u \in S_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}_T, w)$, which appears to be new. Finally, we show that the constant on the right-hand side in the S_p estimate (2.8) (see also (2.9)) grows polynomially as the lower eigenvalue bound of the coefficient matrix a^{ij} decreases. This fact is crucial in the application to the linearized Landau equation near Maxwellian (1.1). In particular, the matrix of the leading coefficients a^{ij} has a lower eigenvalue bound of order n^{-3} on the annulus $(0, T) \times \Omega \times \{|v| \sim n\}, n \in \{1, 2, \ldots\}$ (see [24, Lemma 2.4]), where $\Omega \subset \mathbb{R}^3$ is a domain. Then, one can use the S_p estimate (see (2.9)) to obtain an upper bound of the L_p-norm of $D_v^2 f$. We elaborate on this in Remark 2.9; see also [14,16,24].

We comment on the ideas of the proof. To prove the main result, one needs to, first, work with the model equation

$$P_0 u + \lambda u = f, \tag{1.2}$$

where the coefficients a^{ij} are independent of x and v. Recall that in the unmixed and unweighted case, the a priori estimates for Eq. (1.2) were already proved in [9] and [12] by using the estimates of fundamental solutions. We give a new proof of these a priori estimates and establish a unique solvability result for Eq. (1.2) in weighted mixed-norm spaces. As a corollary, we obtain pointwise estimates of

$$(D_v^2 u)_T^\#, \quad ((-\Delta_x)^{1/3} u)_T^\#, \quad (D_v(-\Delta_x)^{1/6} u)_T^\#$$

in the case of variable coefficients a^{ij}, which are similar to those in Lemma 6.2.2 of [27] (see Lemma 7.1). Here the superscript # stands for the sharp function with respect to some quasi-distance. By designing a family of maximal functions and using a variant of the Hardy–Littlewood and Fefferman–Stein theorems (see Corollary 3.2), we prove the a priori estimate of Theorem 2.6 (i), which also implies the uniqueness part of Theorem 2.6. To prove the existence of solutions, we use the density argument as well as the argument of Section 8 of [15].

The novelty of this work lies in the fact that we do not use an analytic expression of the fundamental solution of the operator $P_0 + \lambda$. We use a kernel-free approach, which can be found in papers by N. V. Krylov, the first named author, D. Kim, and others (see, for example, [15,27]). Such a method is useful in developing the solvability theory in Sobolev spaces for second-order operators whose fundamental solutions do not have an explicit form. The reader can find examples of such equations in [7] and [28]. Interestingly, the authors of [11,12] also proved a variant of the Fefferman–Stein theorem (see Theorem 2.11 of [11]). In the same papers, they use the Fefferman–Stein inequality to prove the a priori estimates of $D_v^2 u$ and $(-\Delta_x)^{1/3} u$ in the L_p spaces. However, the main difference is that, instead of estimating $(D_v^2 u)_T^\#$ and $((-\Delta_x)^{1/3} u)_T^\#$ in terms of the maximal functions of $D_v^2 u$, $(-\Delta_x)^{1/3} u$ and the right-hand side of Eq. (1.2) like we do, these authors used a variant of the Stampacchia interpolation theorem (see [11, Theorem 2.12] and [12, Theorem 2.4]), which they derived from the Fefferman–Stein inequality. One more difference with our work is that the argument of [12] involves kernel estimates. The method in the present paper allows us to further treat the VMO coefficients by incorporating the perturbation in the mean oscillation estimates. It would be interesting to find out if the methods of [8], [10,12,31,32] could be used to prove Theorem 2.6.

The L_p theory of KFP equations developed in this article can be used in mathematical theory of plasma and filtering of signals. For example, in [14], the present authors and Yan Guo showed the well-posedness and higher regularity of the linear Landau equation with the specular reflection boundary condition by applying the results of the current paper. The crucial difficulty in this problem is that the presence of the boundary condition forces one to work with Kolmogorov type equations with 'rough in time' coefficients, which is why Theorem 2.6 and Corollary 2.8 are useful for such equations. See the details in Section 2 of [14]. Motivated by such

an example, we plan to study kinetic KFP equations with rough coefficients in divergence form in suitable Sobolev spaces and equations in nondivergence form in Morrey-Campanato spaces. In addition, the a priori estimates for KFP equation might be useful in developing the solvability theory for its stochastic counterparts which arise in filtering of diffusion processes and interacting particle systems. For discussion and related studies, see [34,40], and [20]. We plan to further investigate the stochastic KFP equation in subsequent papers.

The paper is organized as follows: in the next section, we introduce the notation and assumptions and state the main result of the article. In Section 3, we prove some auxiliary results, including variants of the Hardy–Littlewood and Fefferman–Stein theorems for the maximal and sharp functions defined with respect to an ultraparabolic quasi-distance. In Section 4, we prove Theorem 2.6 for Eq. (1.2) with $P = P_0$ and $p = 2$. We extend this result to $p \in (1, \infty)$ in Section 5. In Section 6, we prove Theorem 2.6 with $P = P_0$. In the last section, we prove Theorem 2.6.

2. Notation and Statement of the Main Results

2.1. Notation and assumptions

For $r > 0$ and $x_0 \in \mathbb{R}^d$, denote

$$B_r(x_0) = \{\xi \in \mathbb{R}^d : |\xi - x_0| < r\}, \quad B_r = B_r(0).$$

For $r, R > 0$ and $z_0 \in \mathbb{R}^{1+2d}$, we set

$$Q_{r,R}(z_0) = \{z : -r^2 < t - t_0 < 0, |v - v_0| < r, |x - x_0 + (t - t_0)v_0|^{1/3} < R\},$$
$$\widetilde{Q}_{r,R}(z_0) = \{z : |t - t_0| < r^2, |v - v_0| < r, |x - x_0 + (t - t_0)v_0|^{1/3} < R\},$$
$$Q_r(z_0) = Q_{r,r}(z_0), \qquad \widetilde{Q}_r(z_0) = \widetilde{Q}_{r,r}(z_0),$$
$$Q_{r,R} = Q_{r,R}(0), \quad \widetilde{Q}_{r,R} = \widetilde{Q}_{r,R}(0), \quad Q_r = Q_r(0), \quad \widetilde{Q}_r = \widetilde{Q}_r(0).$$

For an open set G, by $C_b(G)$, we denote the space of all bounded continuous functions on G. By $C_0^\infty(\mathbb{R}^d)$, we denote the space of all smooth compactly supported functions on $\mathbb{R}^d$.

We say that a function w is a weight on $\mathbb{R}^d$ if w is nonnegative, locally integrable, and $w > 0$ almost everywhere. For $p > 1$, we write $w \in A_p(\mathbb{R}^d)$ if w is a weight on $\mathbb{R}^d$ such that

$$[w]_{A_p(\mathbb{R}^d)} := \sup_{x_0 \in \mathbb{R}^d, r > 0} \left(\fint_{B_r(x_0)} w(x)\,dx \right) \left(\fint_{B_r(x_0)} w^{-1/(p-1)}(x)\,dx \right)^{p-1} < \infty.$$

Remark 2.1. For $\alpha \in (-d, d(p-1))$, $w(x) = |x|^\alpha$ is an A_p weight in $\mathbb{R}^d$ (see the details in Example 7.1.7 in [19]).

Furthermore, for any numbers $p, r_1, \ldots, r_d, q > 1$, $K \geq 1$, and a function w on $\mathbb{R}^{1+d}$, we write $[w]_{q,r_1,\ldots,r_d} \leqq K$ if there exist weights $w_i, i = 0, \ldots, d$, on $\mathbb{R}$ such that

$$w(t, v) = w_0(t) \prod_{i=1}^{d} w_i(v_i) \tag{2.1}$$

and

$$[w_0]_{A_q(\mathbb{R})}, [w_i]_{A_{r_i}(\mathbb{R})} \leqq K, \quad i = 1, \ldots, d. \tag{2.2}$$

By $L_{p,r_1,\ldots,r_d,q}(G, w)$, $L_{p;r_1,\ldots,r_d}(G, |x|^{\alpha} \prod_{i=1}^{d} w_i(v_i))$, $\alpha \in (-1, p-1)$, we denote the spaces of all Lebesgue measurable functions on $\mathbb{R}^{1+2d}$ such that

$$\|f\|_{L_{p,r_1,\ldots,r_d,q}(G,w)}$$
$$= \left| \int_{\mathbb{R}} \left| \ldots \right| \int_{\mathbb{R}} \left| \int_{\mathbb{R}^d} |f|^p(z) 1_G(z) \, dx \right|^{\frac{r_1}{p}} w_1(v_1) dv_1 \right|^{\frac{r_2}{r_1}}$$
$$\ldots w_d(v_d) dv_d \Big|^{\frac{q}{r_d}} w_0(t) dt \Big|^{\frac{1}{q}}, \tag{2.3}$$

$$\|f\|_{L_{p;r_1,\ldots,r_d}(G,|x|^{\alpha} \prod_{i=1}^{d} w_i(v_i))}$$
$$= \left| \int_{\mathbb{R}} \ldots \left| \int_{\mathbb{R}} \left| \int_{\mathbb{R}^{d+1}} |f|^p(z) 1_G(z) |x|^{\alpha} \, dx dt \right|^{\frac{r_1}{p}} w_1(v_1) dv_1 \right|^{\frac{r_2}{r_1}}$$
$$\ldots w_d(v_d) dv_d \Big|^{\frac{1}{r_d}}. \tag{2.4}$$

For the discussion of basic properties of weighted mixed-norm Lebesgue spaces, see [6]. We write $u \in S_{p,r_1,\ldots,r_d,q}(G, w)$ if

$$u, \partial_t u - v \cdot D_x u, D_v u, D_v^2 u \in L_{p,r_1,\ldots,r_d,q}(G, w).$$

The $S_{p,r_1,\ldots,r_d,q}(G, w)$-norm is defined as

$$\|u\|_{S_{p,r_1,\ldots,r_d,q}(G,w)} = \|u\|_{L_{p,r_1,\ldots,r_d,q}(G,w)} + \|D_v u\|_{L_{p,r_1,\ldots,r_d,q}(G,w)}$$
$$+ \|D_v^2 u\|_{L_{p,r_1,\ldots,r_d,q}(G,w)} + \|\partial_t u - v \cdot D_x u\|_{L_{p,r_1,\ldots,r_d,q}(G,w)}.$$

If $w \equiv 1$, we drop w from the above notation. In addition, if $p = r_1 = \ldots = r_d = q$, we replace the subscripts $p, r_1, \ldots, r_d, q$ with a single number p. Replacing $L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$ with $L_{p;r_1,\ldots,r_d}(\mathbb{R}_T^{1+2d}, |x|^{\alpha} \prod_{i=1}^{d} w_i(v_i))$, we define the space $S_{p;r_1,\ldots,r_d}(\mathbb{R}_T^{1+2d}, |x|^{\alpha} \prod_{i=1}^{d} w_i(v_i))$.

For any $s \in (0, 1)$ and $u \in L_p(\mathbb{R}^d)$, by $(-\Delta_x)^s u$ we understand the distribution given by

$$((-\Delta_x)^{1/3} u, \phi) = (u, (-\Delta_x)^{1/3} \phi), \quad \phi \in C_0^{\infty}(\mathbb{R}^d).$$

Furthermore, if $s \in (0, 1/2)$ and u is regular enough, say,

$$u \text{ is of Lipschitz class on } \mathbb{R}^d, \quad |u(x)| \leqq K,$$

then, the following pointwise formula hold:

$$(-\Delta_x)^s u(x) = c_{d,s} \int_{\mathbb{R}^d} \frac{u(x) - u(x+y)}{|y|^{d+2s}} \, dy,$$

where $c_{d,s}$ is the constant depending only on d and s. See, for instance, the discussion in Section 2 of [37].

For $c > 0$, denote

$$\rho_c(z, z_0) = \max\{|t - t_0|^{1/2}, c^{-1}|x - x_0 + (t - t_0)v_0|^{1/3}, |v - v_0|\}.$$

For a Lebesgue measurable set A, by $|A|$ we denote its Lebesgue measure. For a function $f \in L_1(A)$, we denote

$$(f)_A = \fint_A f \, dz = |A|^{-1} \int_A f \, dz$$

provided that $|A| < \infty$. For $c > 0$ and $T \in (-\infty, \infty]$, the maximal and sharp functions are defined as follows:

$$\mathbb{M}_{c,T} f(z_0) = \sup_{r > 0, z_1 \in \overline{\mathbb{R}_T^{1+2d}} : z_0 \in Q_{r,cr}(z_1)} \fint_{Q_{r,rc}(z_1)} |f(z)| \, dz, \quad \mathcal{M}_T f := \mathbb{M}_{1,T} f,$$

$$f_T^\#(z_0) = \sup_{r > 0, z_1 \in \overline{\mathbb{R}_T^{1+2d}} : z_0 \in Q_r(z_1)} \fint_{Q_r(z_1)} |f(z) - (f)_{Q_r(z_1)}| \, dz.$$

For $n \in \{0, 1, 2, \ldots\}$ and a sufficiently regular function u on $\mathbb{R}^{2d}$, by $D_x^n u$ we denote the set of all partial derivatives of order n in the x variable. We define $D_v^n u$, $D_x^m D_x^n u$ in the same way.

By $N = N(\cdots)$ and $\theta = \theta(\cdots)$ we mean constants depending only on the parameters inside the parenthesis. The constants N and θ might change from line to line. Sometimes, when it is clear what parameters N or θ depend on, we omit them.

We impose the following assumptions on the coefficients:

Assumption 2.2. The coefficients $a(z) = (a^{ij}(z), i, j = 1, \ldots, d)$ are bounded measurable functions such that for some $\delta \in (0, 1)$,

$$\delta|\xi|^2 \le a^{ij}(z)\xi_i\xi_j \le \delta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, z \in \mathbb{R}^{1+2d}.$$

Assumption 2.3. (γ_0) There exists $R_0 > 0$ such that for any z_0 and $R \in (0, R_0]$,

$$\mathrm{osc}_{x,v}(a, Q_r(z_0)) \le \gamma_0,$$

where

$$\mathrm{osc}_{x,v}(a, Q_r(z_0))$$
$$= \fint_{(t_0 - r^2, t_0)} \fint_{D_r(z_0,t) \times D_r(z_0,t)} |a(t, x_1, v_1) - a(t, x_2, v_2)| \, dx_1 dv_1 dx_2 dv_2 \, dt,$$

and

$$D_r(z_0, t) = \{(x, v) : |x - x_0 + (t - t_0)v_0|^{1/3} < r, |v - v_0| < r\}.$$

Here is an example of a somewhat stronger condition which can be viewed as a $VMO_{x,v}$ condition with respect to the anisotropic distance $|x - x'|^{1/3} + |v - v'|$. Let $\omega : [0, \infty) \to [0, \infty)$ be an increasing function such that $\omega(0+) = 0$. Assume that

$$\operatorname{osc}'_{x,v}(a, r)$$
$$:= \sup_{t,x,v} r^{-8d} \int_{x_1,x_2 \in B_{r^3}(x)} \int_{v_1,v_2 \in B_r(v)} |a(t, x_1, v_1) \qquad (2.5)$$
$$- a(t, x_2, v_2)| \, dx_1 dx_2 \, dv_1 dv_2 \leqq \omega(r).$$

Note that since

$$\operatorname{osc}_{x,v}(a, Q_r(z_0)) \leqq \operatorname{osc}'_{x,v}(a, r),$$

the condition (2.5) implies Assumption 2.3 (γ_0) for any $\gamma_0 \in (0, 1)$.

In the present article, we consider the equation

$$Pu + b^i D_{v_i} u + (c + \lambda)u = f \qquad (2.6)$$

and for $-\infty < S < T \leqq \infty$, the Cauchy problem

$$Pu + b^i D_{v_i} u + cu = f, \quad u(S, \cdot) = 0. \qquad (2.7)$$

Assumption 2.4. The functions $b = (b^i, i = 1, \ldots, d)$ and c are bounded measurable on $\mathbb{R}^{1+2d}$ and they satisfy the condition

$$|b| + |c| \leqq L$$

for some constant $L > 0$.

2.2. Main result

Definition 2.5. Let $T \in (-\infty, \infty]$. A function $u \in S_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{2d}, w)$ is a solution to Eq. (2.6) if the identity

$$\partial_t u - v \cdot D_x u = a^{ij} D_{v_i v_j} u - b^i D_{v_i} u - (c + \lambda)u$$

holds in $L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{2d}, w)$. We define a solution in the space

$$S_{p;r_1,\ldots,r_d}(\mathbb{R}_T^{2d}, |x|^\alpha \prod_{i=1}^d w_i(v_i))$$

in a similar way.

We say that

$$u \in S_{p,r_1,\ldots,r_d,q}((S, T) \times \mathbb{R}^{2d}, w)$$

is a solution to Eq (2.7) if there exists $\tilde{u} \in S_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$ such that $\tilde{u} = u$ on $(S, T) \times \mathbb{R}^{2d}$, $\tilde{u} = 0$ on $(-\infty, S) \times \mathbb{R}^{2d}$, and $P\tilde{u} + b^i D_{v_i} \tilde{u} + c\tilde{u} = f$ on $(S, T) \times \mathbb{R}^{2d}$.

Theorem 2.6. *Let* $p, r_1, \ldots, r_d, q > 1$, $K \geqq 1$ *be numbers,* $T \in (-\infty, \infty]$, *and* $w_i, i = 0, 1, \ldots, d$, *be weights satisfying* (2.2), *and* w *be a weight defined by* (2.1). *Let Assumptions* 2.2 *and* 2.4 *hold. There exist constants*

$$\beta = \beta(d, p, r_1, \ldots, r_d, q, K) > 0, \quad \gamma_0 = \delta^\beta \widetilde{\gamma}_0(d, p, r_1, \ldots, r_d, q, K) > 0$$

such that if Assumption 2.3 (γ_0) *holds, then, the following assertions are valid.*

(i) *There exist constants*

$$\begin{aligned} \theta &= \theta(d, p, r_1, \ldots, r_d, q, K) \quad and \\ \lambda_0 &= \delta^{-\theta} R_0^{-2} \widetilde{\lambda}_0(d, p, r_1, \ldots, r_d, q, K, L) \geqq 1 \end{aligned}$$

such that for any $\lambda \geqq \lambda_0$ *and any* $u \in S_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$, *one has*

$$\begin{aligned} \lambda \|u\| &+ \lambda^{1/2} \|D_v u\| + \|D_v^2 u\| + \|(-\Delta_x)^{1/3} u\| + \|D_v(-\Delta_x)^{1/6} u\| \\ &+ \|\partial_t u - v \cdot D_x u\| \leqq N\delta^{-\theta} \|Pu + b^i D_{v_i} u + cu + \lambda u\|, \quad (2.8) \end{aligned}$$

where $R_0 \in (0, 1)$ *is the constant in Assumption* 2.3 (γ_0),

$$\| \cdot \| = \| \cdot \|_{L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)}, \quad and \quad N = N(d, p, r_1, \ldots, r_d, q, K).$$

In addition, for any $f \in L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$, *Eq.* (2.6) *has a unique solution* $u \in S_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$.

(ii) *For any numbers* $-\infty < S < T < \infty$ *and* $f \in L_{p,r_1,\ldots,r_d,q}((S, T) \times \mathbb{R}^{2d}, w)$, *Eq.* (2.7) *has a unique solution* $u \in S_{p,r_1,\ldots,r_d,q}((S, T) \times \mathbb{R}^{2d}, w)$. *In addition,*

$$\begin{aligned} \|u\| &+ \|D_v u\| + \|D_v^2 u\| + \|(-\Delta_x)^{1/3} u\| + \|D_v(-\Delta_x)^{1/6} u\| \\ &+ \|\partial_t u - v \cdot D_x u\| \leqq N\|f\|, \end{aligned}$$

where $\| \cdot \| = \| \cdot \|_{L_{p,r_1,\ldots,r_d,q}((S,T)\times\mathbb{R}^{2d}, w)}$ *and*

$$N = N(d, \delta, p, r_1, \ldots, r_d, q, K, L, T - S).$$

(iii) *The assertions (i) and (ii) hold with* $S_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$ *replaced with* $S_{p;r_1,\ldots,r_d}(\mathbb{R}_T^{1+2d}, |x|^\alpha \prod_{i=1}^d w_i(v_i))$, *where* $\alpha \in (-1, p - 1)$. *Furthermore, the constants* $\beta, \gamma_0, \theta, \lambda_0, N$ *must be modified as follows: one needs to take into account the dependence on* α *and remove the dependence on* q.

Remark 2.7. The reason why we included the term $D_v(-\Delta_x)^{1/6}$ in the a priori estimate (2.8) is the following. In the proof of Theorem 2.6 with $P = P_0$, $L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$ replaced with $L_p(\mathbb{R}^{1+2d})$ and $p \in (1, 2)$ (see Theorem 5.1), we use an a priori bound of the $L_{p/(p-1)}(\mathbb{R}^{1+2d})$ norm of $D_v(-\Delta_x)^{1/6}u$ to prove (2.8) for $\|D_v^2 u\|_{L_p(\mathbb{R}^{1+2d})}$. See page 46. It turns out that the former inequality can be obtained along the lines of the proof of the a priori estimate of $(-\Delta_x)^{1/3}u$.

The next result is a direct corollary of Theorem 2.6 (i).

Corollary 2.8. *There exist constants*

$$\beta = \beta(d, p, r_1, \ldots, r_d, q, K) > 0, \quad \gamma_0 = \delta^\beta \widetilde{\gamma}_0(d, p, r_1, \ldots, r_d, q, K) > 0,$$

and

$$\theta = \theta(d, p, r_1, \ldots, r_d, q, K) > 0,$$

such that if Assumption 2.3 *(γ_0) holds, then, for any $u \in S_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$ and $\lambda \geqq 0$,*

$$\|u\|_{S_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)} + \|(-\Delta_x)^{1/3}u\| + \|D_v(-\Delta_x)^{1/6}u\|$$
$$\leqq N\delta^{-\theta}(\|Pu + b^i D_{v_i}u + cu + \lambda u\| + R_0^{-2}\|u\|), \tag{2.9}$$

where $\| \cdot \| = \| \cdot \|_{L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)}$, $N = N(d, p, r_1, \ldots, r_d, q, K, L)$, and $R_0 \in (0, 1)$ is the constant from Assumption 2.3 *(γ_0). Furthermore, the same holds if we replace $S_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$ with $S_{p;r_1,\ldots,r_d}(\mathbb{R}_T^{1+2d}, |x|^\alpha \prod_{i=1}^d w_i(v_i))$, where $\alpha \in (-1, p-1)$, and modify the constants as suggested in Theorem* 2.6 *(iii).*

Proof. We only need to consider the case when $\lambda \in (0, \lambda_0)$, where λ_0 is the constant in Theorem 2.6. Then, by (2.8) and the triangle inequality, (2.9) holds with the term $(\lambda_0 - \lambda)\|u\|$ in place of $R_0^{-2}\|u\|$. Replacing $\lambda_0 - \lambda$ with λ_0 and using the explicit expression of λ_0, we prove (2.9).   □

In the next remark, we explain how Corollary 2.8 can be applied to the linearized Landau equation. We also show why it is useful to know how the constant on the right-hand side of (2.9) depends on the lower eigenvalue bound δ of the matrix a^{ij}.

Remark 2.9. Let $d = 3$. Here we show how Corollary 2.8 can be used to estimate the S_p-norm of the solution to the linearized Landau equation near Maxwellian (1.1) in the case of the Coulomb interaction. See the details in [16]. Due to Lemma 2.4 of [24], for Eq. (1.1), there exist constants $\mu_1, \mu_2 > 0$ such that

$$\mu_1(1 + |v|)^{-3}|\xi|^2 \leqq a^{ij}(z)\xi_i\xi_j \leqq \mu_2(1 + |v|)^{-1}|\xi|^2, \quad \forall z \in \mathbb{R}^7, \forall \xi \in \mathbb{R}^3. \tag{2.10}$$

In addition, we assume that the coefficients a^{ij} are Hölder continuous with respect to the quasi-distance

$$d(z, z') = \max\{|t - t_0|^{1/2}, |x - x_0 - (t - t_0)v_0|^{1/3}, |v - v_0|\}.$$

This means that a^{ij} are bounded functions, and there exists a constant $\kappa \in (0, 1)$ such that

$$\sup_{z,z' \in (0,T) \times \mathbb{R}^6 : z \neq z'} \frac{|a^{ij}(z) - a^{ij}(z')|}{d^\kappa(z, z')} < \infty, \quad i, j = 1, 2, 3.$$

Equation (1.1) with the above assumptions arises naturally when one tries to prove the existence and uniqueness of the solution to the (nonlinear) Landau equation

near Maxwellian (see [16,24]). Note that, in this case, Assumption 2.3 (γ_0) holds for any $\gamma_0 \in (0, 1)$ with

$$R_0 \sim \gamma_0^{1/\kappa}. \tag{2.11}$$

Next, assuming that f_0 is a sufficiently regular function and replacing f with $f - f_0\phi$, where $\phi = \phi(t)$ is an appropriate cutoff function, we reduce Eq. (1.1) to the Cauchy problem

$$Pf + b \cdot \nabla_v f + cf = \eta, \quad f(0, \cdot, \cdot) \equiv 0.$$

Then, localizing (2.9) and using (2.10) and (2.11), for any $n \in \{1, 2, \ldots, \}$, we obtain

$$\begin{aligned}
&\|f\|_{S_2((0,T)\times\mathbb{R}^3\times\{n<|v|<n+1\})} \\
&\leq Nn^\theta \||\eta| + |f|\|_{L_2((0,T)\times\mathbb{R}^3\times\{n-1/2<|v|<n+3/2\})},
\end{aligned} \tag{2.12}$$

where N and θ are positive numbers independent of n and T. If the original initial value and h decay fast enough at infinity, by using the energy estimate for the Landau equation, one can show that $f, \eta \in L_2((0, T) \times \mathbb{R}^6, |v|^{\theta'})$ for some $\theta' > \theta + 1 > 0$. This combined with (2.12) gives

$$\|f\|_{S_2((0,T)\times\mathbb{R}^3\times\{n<|v|<n+1\})} \leq Nn^{\theta-\theta'} \||\eta| + |f|\|_{L_2((0,T)\times\mathbb{R}^6,|v|^{\theta'})},$$

and, hence, $f \in S_2((0, T) \times \mathbb{R}^6)$. By a Sobolev type embedding theorem for S_p spaces (see Theorem 2.1 in [35]), the above gives

$$\||f| + |\nabla_v f|\|_{L_{7/3}((0,T)\times\mathbb{R}^6)} \leq N\||\eta| + |f|\|_{L_2((0,T)\times\mathbb{R}^6,|v|^{\theta'})}.$$

Similarly, one can show that f belongs to a weighted S_2 and $L_{7/3}$ spaces. Then, by using a bootstrap type argument, we conclude that $f \in S_p((0, T) \times \mathbb{R}^6)$ for any $p \in [2, \infty)$. If p is large enough, by using a Morrey type embedding theorem for the S_p spaces (see Theorem 2.1 in [35]), one concludes that $f, \nabla_v f$ are Hölder continuous with respect to the quasi-distance d, which is crucial in the proof of the uniqueness of solutions (see, for example, [24, Lemma 8.2]). As mentioned in Section 1, the present authors used a similar argument to show the higher regularity of a finite energy weak solution to the linear Landau equation with the specular reflection boundary condition (see [14]). In particular, near the boundary, one can reduce such an equation to an equation of the KFP type with the coefficients $L_\infty((0, T), C_{x,v}^{\varkappa/3,\varkappa}(\mathbb{R}^6))$, $\varkappa \in (0, 1]$, where $C_{x,v}^{\varkappa/3,\varkappa}(\mathbb{R}^6)$ is the space of bounded functions u such that

$$\sup_{(x_i,v_i)\in\mathbb{R}^6:(x_1,v_1)\neq(x_2,v_2)} \frac{|u(x_1, v_1) - u(x_2, v_2)|}{(|x_1 - x_2|^{1/3} + |v_1 - v_2|)^\varkappa} < \infty.$$

In this case, again, Assumption 2.3 (γ_0) holds for any $\gamma_0 \in (0, 1)$ with R_0 given by (2.11).

Remark 2.10. The assertion (ii) is derived from (i) in the standard way (see, for example, Theorem 2.5.3 of [27]). We will not mention this in the sequel.

Remark 2.11. From Theorem 2.6 (i), we can derive the corresponding results for elliptic equations when the coefficients and data are independent of t. See, for instance, the proof of [26, Theorem 2.6]. The idea is that one can view an elliptic equation as a steady state parabolic equation.

Remark 2.12. It would be interesting to study Eq. (1.1) with singular drift under the conditions similar to those considered in [25,30,36,38] (see also the references therein). An interesting result concerning a Langevin type SDE with the drift in the form $b(t, x, v) = D_x F(x) + G(v)$ was established in [38].

3. Auxiliary Results

The following lemma is a variant of Lemma 2.3 of [12].

Lemma 3.1. *Let $c \geqq 1, r > 0$ be numbers. Then, the following assertions hold:*

(i) *For any $z, z_0 \in \mathbb{R}^{1+2d}$,*

$$\rho_c(z, z_0) \leqq 2\rho_c(z_0, z).$$

(ii) *For any $z, z_0, z_1 \in \mathbb{R}^{1+2d}$,*

$$\rho_c(z, z_0) \leqq 2(\rho_c(z, z_1) + \rho_c(z_1, z_0)).$$

(iii) *Denote $\widehat{\rho}_c(z, z_0) = \rho_c(z, z_0) + \rho_c(z_0, z)$. Then, $\widehat{\rho}_c$ is a (symmetric) quasi-distance.*

(iv) *Denote $\widehat{Q}_{r,cr}(z_0) = \{z \in \mathbb{R}^{1+2d} : \widehat{\rho}_c(z, z_0) < r\}$. Then,*

$$\widehat{Q}_{r,cr}(z_0) \subset \widetilde{Q}_{r,cr}(z_0) \subset \widehat{Q}_{3r,3cr}(z_0).$$

(v) *For any $T \in (-\infty, \infty]$, the triple $(\overline{\mathbb{R}^{1+2d}_T}, \widehat{\rho}_c, dz)$ (with the induced topology if $T < \infty$) is a space of homogeneous type with parameters independent of c.*

Proof. (i) It suffices to show that

$$c^{-1}|x - x_0 + (t - t_0)v_0|^{1/3} \leqq 2\rho_c(z_0, z).$$

By the triangle inequality, we have

$$|x - x_0 + (t - t_0)v_0|^{1/3} \leqq |x_0 - x + (t_0 - t)v|^{1/3} + |t - t_0|^{1/3}|v - v_0|^{1/3}$$
$$\leqq |x_0 - x + (t_0 - t)v|^{1/3} + \rho_c(z_0, z).$$

Multiplying both sides of the above inequality by c^{-1} and using the fact that $c \geqq 1$, we prove the claim.

(ii) As in (i), we only need to show that the inequality holds with the left-hand side replaced with $c^{-1}|x - x_0 + (t - t_0)v|^{1/3}$. By the triangle inequality,

$$|x - x_0 + (t - t_0)v_0|^{1/3}$$
$$\leq |x - x_1 + (t - t_1)v_1|^{1/3} + |x_1 - x_0 + (t - t_0)v_0 - (t - t_1)v_1|^{1/3}$$
$$\leq c\rho_c(z, z_1) + c\rho_c(z_1, z_0) + |t - t_1|^{1/3}|v_0 - v_1|^{1/3}.$$

By Young's inequality,

$$|t - t_1|^{1/3}|v_0 - v_1|^{1/3} \leq (2/3)\rho_c(z, z_1) + (1/3)\rho_c(z_1, z_0).$$

This combined with the fact that $c \geq 1$ yields the desired estimate.

(iii) This assertion follows from (ii).

(iv) The claim is a direct corollary of (i).

(v) We only need to check the doubling property. For any $z \in \overline{\mathbb{R}_T^{1+2d}}$, the assertion (iv) of this lemma gives

$$\mu := \frac{|\widehat{Q}_{2r,2cr}(z) \cap \mathbb{R}_T^{1+2d}|}{|\widehat{Q}_{r,cr}(z) \cap \mathbb{R}_T^{1+2d}|} \leq \frac{|\widetilde{Q}_{2r,2cr}(z)|}{|\widetilde{Q}_{r/3,cr/3}(z) \cap \mathbb{R}_T^{1+2d}|}.$$

Since $z \in \overline{\mathbb{R}_T^{1+2d}}$, we have

$$Q_{r/3,cr/3}(z) \subset \widetilde{Q}_{r/3,cr/3}(z) \cap \mathbb{R}_T^{1+2d},$$

and, hence,

$$\mu \leq N(d). \tag{3.1}$$

The claim is proved. □

Denote

$$f_{c,T}^{\#}(z_0) = \sup_{r>0, z_1 \in \mathbb{R}_T^{1+2d}: z_0 \in Q_{r,cr}(z_1)} \fint_{Q_{r,cr}(z_1)} |f(z) - (f)_{Q_{r,cr}(z_1)}| \, dz.$$

Corollary 3.2. *Let $c \geq 1$, $K \geq 1$, $p, q, r_1, \ldots, r_d > 1$ be numbers, $T \in (-\infty, \infty]$, and $f \in L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$, where w is a weight such that*

$$[w]_{q,r_1,\ldots,r_d} \leq K.$$

Then, the following assertions hold:

(i) *(Hardy–Littlewood type theorem)*

$$\|\mathbb{M}_{c,T} f\|_{L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d},w)}$$
$$\leq N(d, p, q, r_1, \ldots, r_d, K)\|f\|_{L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d},w)}.$$

(ii) *(Fefferman–Stein type theorem)*

$$\|f\|_{L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d},w)} \leq N(d, p, q, r_1, \ldots, r_d, K)\|f_{c,T}^{\#}\|_{L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d},w)}.$$

Proof. We only present the proof of the assertion in the case when $T < \infty$. The case when $T = \infty$ is treated similarly. We extend f to be zero for $t > T$.

(i) Denote

$$\widehat{\mathbb{M}}_{c,T} f(z_0) = \sup_{r>0,z_1\in\overline{\mathbb{R}}_T^{1+2d}:z_0\in\widehat{Q}_{r,cr}(z_1)} \fint_{\widehat{Q}_{r,cr}(z_1)\cap\mathbb{R}_T^{1+2d}} |f(z)|\,dz,$$

where $\widehat{Q}_{r,cr}(z_1)$ is defined in Lemma 3.1. By Lemma 3.1 (iv), for any $r > 0$ and $z_1 \in \mathbb{R}_T^{1+2d}$,

$$\fint_{Q_{r,cr}(z_1)} |f(z)|\,dz \leqq \frac{|\widehat{Q}_{3r,3cr}|}{|Q_{r,cr}|} \fint_{\widehat{Q}_{3r,3cr}(z_1)\cap\mathbb{R}_T^{1+2d}} |f(z)|\,dz$$

$$\leqq \frac{|\widetilde{Q}_{3r,3cr}|}{|Q_{r,cr}|} \fint_{\widehat{Q}_{3r,3cr}(z_1)\cap\mathbb{R}_T^{1+2d}} |f(z)|\,dz$$

$$= N(d) \fint_{\widehat{Q}_{3r,3cr}(z_1)\cap\mathbb{R}_T^{1+2d}} |f(z)|\,dz,$$

and by this

$$\mathbb{M}_{c,T} f(z) \leqq N(d)\widehat{\mathbb{M}}_{c,T} f(z), \quad \forall z \in \mathbb{R}_T^{1+2d}. \tag{3.2}$$

We recall that $\widehat{\mathbb{M}}_{c,T} f$ is a maximal function on a space of homogeneous type $(\overline{\mathbb{R}}_T^{1+2d}, \widehat{\rho}_c, dz)$ (see Lemma 3.1 (v)). Then, by the weighted Hardy–Littlewood theorem (see [3]), for any $\omega_0 \in A_p(\mathbb{R}^{1+d})$,

$$\int_{\mathbb{R}_T^{1+2d}} |\widehat{\mathbb{M}}_{c,T} f(z)|^p \omega_0(t, v)\,dz$$

$$\leqq N(d, p, [\omega_0]_{A_p(\mathbb{R}^{d+1})}) \int_{\mathbb{R}_T^{1+2d}} |f(z)|^p \omega_0(t, v)\,dz.$$

By a variant of the Rubio de Francia extrapolation theorem (see, for example, Theorem 7.11 of [17] and also [29]),

$$\|\widehat{\mathbb{M}}_{c,T} f\|_{L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d},w)} \leqq N\|f\|_{L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d},w)}.$$

Now the assertion follows from (3.2).

(ii) Let

$$\widehat{f}_{c,T}^{\#}(z_0) = \sup_{r>0,z_1\in\overline{\mathbb{R}}_T^{1+2d}:z_0\in\widehat{Q}_{r,cr}(z_1)} \fint_{\widehat{Q}_{r,cr}(z_1)\cap\mathbb{R}_T^{1+2d}} |f(z) - (f)_{\widehat{Q}_{r,cr}(z_1)\cap\mathbb{R}_T^{1+2d}}|\,dz,$$

$$\widetilde{f}_{c,T}^{\#}(z_0) = \sup_{r>0,z_1\in\overline{\mathbb{R}}_T^{1+2d}:z_0\in\widetilde{Q}_{r,cr}(z_1)} \fint_{\widetilde{Q}_{r,cr}(z_1)\cap\mathbb{R}_T^{1+2d}} |f(z) - (f)_{\widetilde{Q}_{r,cr}(z_1)\cap\mathbb{R}_T^{1+2d}}|\,dz.$$

Clearly, $\widehat{f}_{c,T}^{\#}$ is a sharp function defined on the space of homogeneous type $(\overline{\mathbb{R}}_T^{1+2d}, \widehat{\rho}_c, dz)$. By the generalized Fefferman–Stein theorem for spaces of

homogeneous type (see Theorem 2.3 of [15]) combined with the extrapolation argument as above (see Theorem 7.11 of [17]), we get

$$\|f\| \leqq N\|\widehat{f}_{c,T}^{\#}\|.$$

Therefore, it suffices to show that for any $z_0 \in \mathbb{R}_T^{1+2d}$,

$$\widehat{f}_{c,T}^{\#}(z_0) \leqq N(d) f_{c,T}^{\#}(z_0). \tag{3.3}$$

We prove this inequality in two steps.

Step 1 First, we show that

$$\widehat{f}_{c,T}^{\#}(z_0) \leqq N(d) \widetilde{f}_{c,T}^{\#}(z_0). \tag{3.4}$$

We fix any cylinder $\widehat{Q}_{r,cr}(z_1)$ containing z_0 such that $z_1 \in \overline{\mathbb{R}_T^{1+2d}}$. By Lemma 3.1 (iv) and the doubling property (see (3.1)),

$$\fint_{\widehat{Q}_{r,cr}(z_1) \cap \mathbb{R}_T^{1+2d}} |f(z) - (f)_{\widehat{Q}_{r,cr}(z_1) \cap \mathbb{R}_T^{1+2d}}| \, \mathrm{d}z$$

$$\leqq \frac{|\widetilde{Q}_{r,cr}(z_1) \cap \mathbb{R}_T^{1+2d}|^2}{|\widehat{Q}_{r,cr}(z_1) \cap \mathbb{R}_T^{1+2d}|^2} \fint_{\widetilde{Q}_{r,cr}(z_1) \cap \mathbb{R}_T^{1+2d}} \fint_{\widetilde{Q}_{r,cr}(z_1) \cap \mathbb{R}_T^{1+2d}} |f(z) - f(z')| \, \mathrm{d}z \, \mathrm{d}z'$$

$$\leqq N(d) \frac{|\widetilde{Q}_{r,cr}(z_1) \cap \mathbb{R}_T^{1+2d}|^2}{|\widehat{Q}_{3r,3cr}(z_1) \cap \mathbb{R}_T^{1+2d}|^2} \fint_{\widetilde{Q}_{r,cr}(z_1) \cap \mathbb{R}_T^{1+2d}} |f(z) - (f)_{\widetilde{Q}_{r,cr}(z_1) \cap \mathbb{R}_T^{1+2d}}| \, \mathrm{d}z$$

$$\leqq N(d) \fint_{\widetilde{Q}_{r,cr}(z_1) \cap \mathbb{R}_T^{1+2d}} |f(z) - (f)_{\widetilde{Q}_{r,cr}(z_1) \cap \mathbb{R}_T^{1+2d}}| \, \mathrm{d}z, \tag{3.5}$$

which implies (3.4).

Step 2 We claim that

$$\widetilde{f}_{c,T}^{\#}(z_0) \leqq N(d) f_{c,T}^{\#}(z_0). \tag{3.6}$$

For the sake of simplicity, we assume that $T = 0$. Let $\widetilde{Q}_{r,rc}(z_1)$ be a cylinder containing z_0 such that $z_1 \in \overline{\mathbb{R}_0^{1+2d}}$. Note that if $t_1 < -r^2$, one has

$$\widetilde{Q}_{r,cr}(z_1) \subset Q_{2r,2cr}(t_1 + r^2, x_1 - r^2 v_1, v_1) \subset \mathbb{R}_0^{1+2d},$$

and then,

$$\fint_{\widetilde{Q}_{r,cr}(z_1) \cap \mathbb{R}_0^{1+2d}} |f(z) - (f)_{\widetilde{Q}_{r,cr}(z_1) \cap \mathbb{R}_0^{1+2d}}| \, \mathrm{d}z$$

$$\leqq N(d) \fint_{Q_{2r,2cr}(t_1 + r^2, x_1 - r^2 v_1, v_1)} |f(z) - (f)_{Q_{2r,2cr}(t_1 + r^2, x_1 - r^2 v_1, v_1)}| \, \mathrm{d}z$$

$$\leqq N(d) f^{\#}(z_0).$$

Next, if $t_1 \geqq -r^2$, then,

$$Q_{r,cr}(z_1) \subset \widetilde{Q}_{r,cr}(z_1) \cap \mathbb{R}_0^{1+2d} \subset \bar{Q}_{2r,2cr}(0, x_1 + t_1 v_1, v_1).$$

By this,

$$
\begin{aligned}
&\fint_{\widetilde{Q}_{r,cr}(z_1)\cap\mathbb{R}_0^{1+2d}} |f(z) - (f)_{\widetilde{Q}_{r,cr}(z_1)\cap\mathbb{R}_0^{1+2d}}| \, dz \\
&\quad \leq N(d) \fint_{Q_{2r,2cr}(0,x_1+t_1v_1,v_1)} |f(z) - (f)_{Q_{2r,2cr}(0,x_1+t_1v_1,v_1)}| \, dz \\
&\quad \leq N(d) f_{c,T}^{\#}(z_0),
\end{aligned}
$$

which proves the claim. Thus, (3.3) holds, and the lemma is proved.   □

Corollary 3.3. *Let $p > 1$, $K \geq 1$, $c \geq 1$ be numbers, $T \in (-\infty, \infty]$ and ω be an A_p weight on the space of homogeneous type $(\overline{\mathbb{R}_T^{1+2d}}, \rho_c, dz)$ (see Lemma 3.1 (v)) such that its A_p constant is bounded by K. Then, for any $f \in L_p(\mathbb{R}_T^{1+2d}, \omega)$,*

(i) $\|\mathbb{M}_{c,T} f\|_{L_p(\mathbb{R}^{1+2d}, \omega)} \leq N(d, p, K)\|f\|_{L_p(\mathbb{R}_T^{1+2d}, \omega)}$,

(ii) $\|f\|_{L_p(\mathbb{R}^{1+2d}, \omega)} \leq N(d, p, K)\|f_{c,T}^{\#}\|_{L_p(\mathbb{R}_T^{1+2d}, \omega)}$.

Proof. As we pointed out in the proof of Corollary 3.2, the aforementioned inequalities hold with

$$
\mathbb{M}_{c,T} f, \, f_{c,T}^{\#} \quad \text{replaced with} \quad \widehat{\mathbb{M}}_{c,T} f, \, \widehat{f}_{c,T}^{\#},
$$

respectively. To conclude the validity of the assertions, we invoke (3.2), (3.4), and (3.6).   □

Lemma 3.4. *Let $p \in (1, \infty)$, $r > 0$, $c \geq 1$, $\alpha \in (-1, p - 1)$ be numbers, and $T \in (-\infty, \infty]$. Then, $|x|^\alpha$ is an A_p weight on the space of homogeneous type $(\overline{\mathbb{R}_T^{1+2d}}, \widehat{\rho}_c, dz)$, and, furthermore, for any $r > 0$, $z_0 \in \overline{\mathbb{R}_T^{1+2d}}$,*

$$
\begin{aligned}
\mathcal{A}(r, z_0) = &\fint_{\widehat{Q}_{r,cr}(z_0)\cap\overline{\mathbb{R}_T^{1+2d}}} |x|^\alpha \, dz \\
&\times \left(\fint_{\widehat{Q}_{r,cr}(z_0)\cap\overline{\mathbb{R}_T^{1+2d}}} |x|^{-\alpha/(p-1)} \, dz \right)^{p-1} \leq N(d, \alpha, p).
\end{aligned}
\tag{3.7}
$$

Proof. Allowing a constant N in (3.7) to depend on d, we may replace $\widehat{Q}_{r,cr}(z_0) \cap \overline{\mathbb{R}_T^{1+2d}}$ with $\widetilde{Q}_{r,cr}(z_0) \cap \overline{\mathbb{R}_T^{1+2d}}$ (see the argument in (3.5)). Furthermore, in the case $T < \infty$, we may assume that $T = 0$ and replace $\widetilde{Q}_{r,cr}(z_0) \cap \overline{\mathbb{R}_0^{1+2d}}$ with $Q_{r,cr}(z_0)$ in the expression for $\mathcal{A}(r, z_0)$. This follows from the fact that if $t_0 \leq -r^2$, then

$$
\widetilde{Q}_{r,cr}(z_0) \subset Q_{2r,2cr}(t_0 + r^2, x_0 - r^2 v_0, v_0) \subset \mathbb{R}_0^{1+2d},
$$

and, otherwise,

$$
\widetilde{Q}_{r,cr}(z_0) \cap \mathbb{R}_0^{1+2d} \subset Q_{2r,2cr}(0, x_0 + t_0 v_0, v_0).
$$

We may assume that $|x_0| + |v_0| > 0$. Denote $X_0 = r^{-3}x_0$, $V_0 = r^{-1}v_0$. By the argument of the previous paragraph and a change of variables, we get

$$\mathcal{A}(r, z_0) \leq N(d) \int_0^1 \fint_{B_{c^3}(X_0+tV_0)} |x|^\alpha \, dx dt$$
$$\times \left(\int_0^1 \fint_{B_{c^3}(X_0+tV_0)} |x|^{-\alpha/(p-1)} \, dx dt \right)^{p-1}.$$

We will consider two cases:

$$A : |X_0 + tV_0| > 3c^3, \ \forall t \in (0, 1), \quad B : \exists \tau \in (0, 1) : |X_0 + \tau V_0| \leq 3c^3.$$

Case A Note that for any $t \in (0, 1)$, $x \in B_{c^3}(X_0 + tV_0)$, one has

$$|x| \leq |x - X_0 - tV_0| + |X_0 + tV_0| \leq (4/3)|X_0 + tV_0|,$$
$$|x| \geq |X_0 + tV_0| - |x - X_0 - tV_0| \geq (2/3)|X_0 + tV_0|,$$

and hence,

$$\mathcal{A}(r, z_0) \leq N(\alpha, d) \int_0^1 |X_0 + tV_0|^\alpha \, dt \left(\int_0^1 |X_0 + tV_0|^{-\alpha/(p-1)} \, dt \right)^{p-1}$$
$$\leq N(\alpha, d) \int_0^1 |\omega' + t\omega|^\alpha \, dt \left(\int_0^1 |\omega' + t\omega|^{-\alpha/(p-1)} \, dt \right)^{p-1},$$

where $\omega = (|X_0|^2 + |V_0|^2)^{-1/2} V_0$, $\omega' = (|X_0|^2 + |V_0|^2)^{-1/2} X_0$.

Next, if $|\omega'| \geq 2|\omega|$, we have

$$\frac{1}{2}|\omega'| \leq |\omega' + t\omega| \leq 2|\omega'|,$$

which gives

$$\mathcal{A}(r, z_0) \leq N(d, \alpha, p).$$

If

$$|\omega'| < 2|\omega|,$$

we decompose $\omega' = \omega'_\perp + \omega'_{||}$, where $\omega'_\perp$ is perpendicular to ω, and $\omega'_{||}$ is parallel to ω. Then, for some $\lambda \in (-2, 2)$,

$$|\lambda + t|^2 |\omega|^2 \leq |\omega' + t\omega|^2 = |\omega'_\perp|^2 + |\lambda + t|^2 |\omega|^2 \leq |\omega|^2 (4 + |\lambda + t|^2).$$

Then, in the case $\alpha \in [0, p-1)$,

$$\mathcal{A}(r, z_0) \leq N(\alpha, d) \left(\int_0^1 |\lambda + t|^{-\alpha/(p-1)} \, dt \right)^{p-1} \leq N(\alpha, d, p).$$

The case $\alpha \in (-1, 0)$ is handled in the same way.

Case B Observe that for any $t \in (0, 1)$, $x \in B_{c^3}(X_0 + t V_0)$,

$$c^{-1}|x - (t - \tau)V_0|^{1/3} \leq c^{-1}|x - X_0 - t V_0|^{1/3} + c^{-1}|X_0 + \tau V_0|^{1/3} \leq 3,$$

and then,

$$\{(t, x) : t \in (0, 1), x \in B_{c^3}(X_0 + t V_0)\}$$
$$\subset \{(t, x) : t \in (\tau - 1, \tau + 1), B_{(3c)^3}((t - \tau)V_0)\}.$$

Thus, we may assume $X_0 = 0$, in addition, by shifting in the time variable, we may assume $\tau = 0$.

Next, if $|V_0| \leq 3c^3$, then for any $t \in (-1, 1)$, we have

$$B_{c^3}(t V_0) \subset B_{4c^3}(0),$$

which implies (3.7) in view of Remark 2.1. We now consider the case $|V_0| > 3c^3$. We denote

$$\kappa = \frac{2c^3}{|V_0|}$$

and note that for $|t| \leq \kappa$,

$$B_{c^3}(t V_0) \subset B_{3c^3}(0),$$

while for $\kappa < |t| < 1$ and $x \in B_{c^3}(t V_0)$, one has

$$\frac{t}{2}|V_0| \leq |x| \leq 2t|V_0|.$$

Then, in the case $\alpha \in [0, p - 1)$,

$$\int_{-1}^{1} \fint_{B_{c^3}(t V_0)} |x|^\alpha \, dx dt \leq N(d) \int_{-\kappa}^{\kappa} \fint_{B_{2c^3}(0)} |x|^\alpha \, dx dt$$
$$+ (2V_0)^\alpha \int_{-1}^{1} |t|^\alpha \, dt \leq N(d, \alpha)|V_0|^\alpha,$$

$$\int_{-1}^{1} \fint_{B_{c^3}(t V_0)} |x|^{-\alpha/(p-1)} \, dx dt \leq N(d) \int_{-\kappa}^{\kappa} \fint_{B_{2c^3}(0)} |x|^{-\alpha/(p-1)} \, dx$$
$$+ (|V_0|/2)^{-\alpha/(p-1)} \int_{-1}^{1} |t|^{-\alpha/(p-1)} \, dt \leq N(d, \alpha, p)|V_0|^{-\alpha/(p-1)},$$

and, thus, the estimate (3.7) is valid. The case $\alpha \in (-1, 0)$ is handled in the same way. The lemma is proved. $\square$

Corollary 3.5. *Let $p, r_1, \ldots, r_d > 1$, $c \geq 1$, $\alpha \in (-1, p - 1)$ be numbers, $T \in (-\infty, \infty]$, and w_j, $j = 1, \ldots, d$, be weights satisfying (2.2). Then,*

$$\|\mathbb{M}_{c,T} f\| \leq N\|f\|, \quad \|f\| \leq N\|f_{c,T}^\#\|,$$

where

$$\| \cdot \| = \| \cdot \|_{L_{p;r_1,\dots,r_d}(\mathbb{R}^{1+2d}_T, |x|^\alpha \prod_{i=1}^d w_i(v_i))},$$
$$N = N(d, p, r_1, \dots, r_d, K, \alpha),$$

and $L_{p;r_1,\dots,r_d}(\mathbb{R}^{1+2d}_T, |x|^\alpha \prod_{i=1}^d w_i(v_i))$ *is defined in* (2.4).

Proof. First, we claim that for any $\omega \in A_p(\mathbb{R}^d)$, the function $|x|^\alpha \omega(v)$ is an A_p weight on the space of homogeneous type $(\overline{\mathbb{R}^{1+2d}_T}, \rho_c, dz)$. This follows from Lemma 3.4 and the fact that in the A_p condition for $\omega(v)|x|^\alpha$ (cf. (3.7)), the integral is factored into a product of the integral over t, x and the integral over v. Then, by Corollary 3.3,

$$\int_{\mathbb{R}^{1+2d}_T} |\mathbb{M}_{c,T} f|^p |x|^\alpha \omega(v)\, dz \leqq N \int_{\mathbb{R}^{1+2d}_T} |f|^p |x|^\alpha \omega(v)\, dz,$$
$$\int_{\mathbb{R}^{1+2d}_T} |f|^p |x|^\alpha \omega(v)\, dz \leqq N \int_{\mathbb{R}^{1+2d}_T} |f^\#_{c,T}|^p |x|^\alpha \omega(v)\, dz.$$

The assertion now follows from a variant of the Rubio de Francia extrapolation theorem (see, for example, Theorem 7.11 of [17] and also [29]). □

Lemma 3.6. *Let* $p \in [1, \infty]$, $T \in (-\infty, \infty]$, *and* $u \in S_{p,loc}(\mathbb{R}^{1+2d}_T)$. *For any* $z_0 \in \mathbb{R}^{1+2d}_T$, *denote*

$$\widetilde{z} = (r^2 t + t_0, r^3 x + x_0 - r^2 t v_0, rv + v_0), \quad \widetilde{u}(z) = u(\widetilde{z}),$$
$$Y = \partial_t - v \cdot D_x, \quad \widetilde{P} = \partial_t - v \cdot D_x - a^{ij}(\widetilde{z}) D_{v_i v_j}.$$

Then,

$$Y\widetilde{u}(z) = r^2 Y u(\widetilde{z}), \quad \widetilde{P}\widetilde{u}(z) = r^2 (Pu)(\widetilde{z}).$$

Proof. The assertion is a direct consequence of the following calculations:

$$\partial_t \widetilde{u}(z) = r^2 (\partial_t u)(\widetilde{z}) - r^2 v_0 \cdot (D_x u)(\widetilde{z}),$$
$$v \cdot D_x \widetilde{u}(z) = r^3 v \cdot (D_x u)(\widetilde{z}) = r^2 (rv + v_0) \cdot (D_x u)(\widetilde{z}) - r^2 v_0 \cdot (D_x u)(\widetilde{z}),$$
$$D_{v_i v_j} \widetilde{u}(z) = r^2 (D_{v_i v_j} u)(\widetilde{z}).$$

The lemma is proved. □

4. S_2-Estimate for the Model Equation

The goal of this section is to prove Theorem 4.1, which is a version of Theorem 2.6 in the case when $p = 2$, $b \equiv 0$, $c \equiv 0$, $w = 1$, and the coefficients a^{ij} are independent of x and v. Here is the outline of the proof. First, we prove the a priori estimate for smooth and compactly supported functions in the x, v variables by taking the Fourier transform in the x, v variables and reducing the equation to

a first-order PDE. Furthermore, we use a limiting argument to extend the result to the space $S_2(\mathbb{R}_T^{1+2d})$. Then we prove the denseness of $(P_0 + \lambda)C_0^\infty(\mathbb{R}^{1+2d})$ in $L_2(\mathbb{R}^{1+2d})$ by using a localized version of the L_2-estimate. The above ingredients yield the desired existence and uniqueness result.

Theorem 4.1. *The following assertions hold:*

(i) *For any number $\lambda \geqq 0$, $T \in (-\infty, \infty]$, and $u \in S_2(\mathbb{R}_T^{1+2d})$, one has*

$$
\begin{aligned}
\lambda\|u\| + \lambda^{1/2}\|D_v u\| &+ \|D_v^2 u\| + \|(-\Delta_x)^{1/3}u\| \\
&+ \|D_v(-\Delta_x)^{1/6}u\| \leqq N(d)\delta^{-1}\|P_0 u + \lambda u\|,
\end{aligned}
\tag{4.1}
$$

where $\|\cdot\| = \|\cdot\|_{L_2(\mathbb{R}_T^{1+2d})}$.

(ii) *For any $\lambda > 0$, $T \in (-\infty, \infty]$, and $f \in L_2(\mathbb{R}_T^{1+2d})$, Eq. (1.2) has a unique solution $u \in S_2(\mathbb{R}_T^{1+2d})$.*

(iii) *For any finite numbers $S < T$ and $f \in L_2((S, T) \times \mathbb{R}^{2d})$, the Cauchy problem (2.7) with $P = P_0$, $b \equiv 0$, and $c \equiv 0$ has a unique solution $u \in S_2((S, T) \times \mathbb{R}^{2d})$. In addition,*

$$
\begin{aligned}
\|u\|_{L_2((S,T)\times\mathbb{R}^{2d})} &+ \|D_v u\|_{L_2((S,T)\times\mathbb{R}^{2d})} + \|D_v^2 u\|_{L_2((S,T)\times\mathbb{R}^{2d})} \\
&+ \|(-\Delta_x)^{1/3}u\|_{L_2((S,T)\times\mathbb{R}^{2d})} + \|D_v(-\Delta_x)^{1/6}u\|_{L_2((S,T)\times\mathbb{R}^{2d})} \\
&+ \|\partial_t u - v \cdot D_x u\|_{L_2((S,T)\times\mathbb{R}^{2d})} \leqq N(d, T-S)\delta^{-1}\|f\|_{L_2((S,T)\times\mathbb{R}^{2d})}.
\end{aligned}
$$

Lemma 4.2. *Let $\lambda > 0$ be finite, $T \in (-\infty, \infty]$, and $h \in C_b(\mathbb{R}_T^{1+2d})$, and $f \in L_\infty((-\infty, T), C_b(\mathbb{R}^{2d})) \cap L_2(\mathbb{R}_T^{1+2d})$ be functions satisfying $D_\xi h \in C_b(\mathbb{R}^{1+2d})$, $\partial_t h \in L_\infty((-\infty, T), C_b(\mathbb{R}^{2d})) \cap L_2(\mathbb{R}_T^{1+2d})$, and the equation*

$$
\partial_t h + a^{ij}(t)\xi_i\xi_j h + k_i D_{\xi_i} h + \lambda h = f.
$$

Then, one has

$$
\begin{aligned}
\lambda\|h\|_{L_2(\mathbb{R}_T^{1+2d})} &+ \||\xi|^2 h\|_{L_2(\mathbb{R}_T^{1+2d})} + \||k|^{2/3}h\|_{L_2(\mathbb{R}_T^{1+2d})} \\
&+ \||k|^{1/3}\xi h\|_{L_2(\mathbb{R}_T^{1+2d})} \leqq N(d)\delta^{-1}\|f\|_{L_2(\mathbb{R}_T^{1+2d})}.
\end{aligned}
$$

Proof. In this proof, we take $N = N(d)$. By the method of characteristics, we have

$$
\begin{aligned}
&h(t, k, \xi) \\
&= \int_{-\infty}^{t} e^{-\lambda(t-t')} e^{-\int_{t'}^{t} a^{jl}(s)(k_j(s-t)+\xi_j)(k_l(s-t)+\xi_l)\,ds} f(t', k, k(t'-t)+\xi)\,dt'.
\end{aligned}
$$

Note that by the parabolicity condition and the Minkowski inequality,

$$
\|h(t, \cdot, \cdot)\|_{L_2(\mathbb{R}^{2d})} \leqq \int_{-\infty}^{t} e^{-\lambda(t-t')}\|f(t', \cdot, \cdot)\|_{L_2(\mathbb{R}^{2d})}\,dt'.
$$

Then, by Young's inequality, we get

$$
\lambda\|h\|_{L_2(\mathbb{R}_T^{1+2d})} \leqq \|f\|_{L_2(\mathbb{R}_T^{1+2d})}.
$$

Furthermore, the parabolicity condition gives

$$
\int_{t'}^t a^{jl}(s)(k_j(s-t)+\xi_j)(k_l(s-t)+\xi_l)\,ds \geqq \delta \int_{t'}^t |k(s-t)+\xi|^2\,ds
$$
$$
= \delta(t-t')(|k|^2(t-t')^2/3 - k\cdot\xi(t-t') + |\xi|^2) \tag{4.2}
$$
$$
\geqq (\delta/24)(t-t')(|k|^2(t-t')^2 + |\xi|^2).
$$

By this and the Minkowski inequality, we get

$$
\||k|^{2/3}h(t,k,\cdot)\|_{L_2(\mathbb{R}^d)}
$$
$$
\leqq \int_{-\infty}^t |k|^{2/3} e^{-(\delta/24)(t-t')^3|k|^2}\|f(t',k,\cdot)\|_{L_2(\mathbb{R}^d)}\,dt'.
$$

Furthermore, by Young's inequality, and the change of variables $t = \delta s^3 |k|^2$,

$$
\||k|^{2/3}h(\cdot,k,\cdot)\|_{L_2(\mathbb{R}_T^{1+d})}
$$
$$
\leq |k|^{2/3}\Big(\int_0^\infty e^{-(\delta/24)s^3|k|^2}\,ds\Big)\|f(\cdot,k,\cdot)\|_{L_2(\mathbb{R}_T^{1+d})}
$$
$$
\leqq N\delta^{-1/3}\Big(\int_0^\infty t^{-2/3}e^{-t/24}\,dt\Big)\|f(\cdot,k,\cdot)\|_{L_2(\mathbb{R}_T^{1+d})}.
$$

Integrating the above inequality over $k \in \mathbb{R}^d$, we prove the estimate of $|k|^{2/3}h$.

Next, by the Cauchy–Schwartz inequality and (4.2),

$$
\||\xi|^2 h(t,\cdot,\cdot)\|_{L_2(\mathbb{R}_T^{2d})}^2 \leqq \int_{\mathbb{R}_T^{1+2d}} I_1(z)I_2(z)\,dz,
$$

where

$$
I_1(z) = \int_{-\infty}^t |\xi|^2 e^{-(\delta/24)(t-t')|\xi|^2}\,dt' \leqq N\delta^{-1},
$$
$$
I_2(z) = \int_{-\infty}^t |\xi|^2 e^{-(\delta/24)(t-t')(|k|^2(t-t')^2+|\xi|^2)} f^2(t',k,k(t'-t)+\xi)\,dt'.
$$

Furthermore, by the change of variables $\xi \to k(t'-t)+\xi$ and the Fubini theorem, we get

$$
\int_{\mathbb{R}_T^{1+2d}} I_2(z)\,dz
$$
$$
\leqq \int_{\mathbb{R}_T^{1+2d}} \int_{-\infty}^t |\xi - k(t'-t)|^2 e^{-\frac{\delta}{72}(t-t')(|k|^2(t-t')^2+|\xi|^2)} f^2(t',k,\xi)\,dt'\,dz
$$
$$
\leq 2\int_{\mathbb{R}^{2d}}\Big(\int_0^\infty (|\xi|^2+|k|^2 t^2)e^{-\frac{\delta}{72}(|k|^2 t^3+|\xi|^2 t)}\,dt\Big)\Big(\int_{-\infty}^T f^2(t,k,\xi)\,dt\Big)\,dk d\xi
$$
$$
\leqq N\delta^{-1}\int_{\mathbb{R}_T^{1+2d}} f^2(z)\,dz.
$$

Finally, the estimate of $|k|^{1/3}\xi h(z)$ follows from the estimates of $|k|^{2/3}h(z)$ and $|\xi|^2 h(z)$, and the Cauchy–Schwartz inequality. $\quad\square$

Corollary 4.3. *Let $u(z)$ be a function such that*

- *there exists some $r > 0$ such that for any $t \in (-\infty, T)$, $u(t, \cdot, \cdot)$ is compactly supported on $B_r \times B_r$,*
- $u, \partial_t u, D_x u, D_v u, D_v^2 u \in C_b(\mathbb{R}_T^{1+2d}) \cap L_2(\mathbb{R}_T^{1+2d})$.

Then, (4.1) holds.

Proof. Let $f = P_0 u + \lambda u$. For a smooth integrable function $\zeta(x, v)$, by $\hat{\zeta}(k, \xi)$, we denote its Fourier transform in the x, v variables. Then, one has

$$\partial_t \hat{u} + a^{ij}(t)\xi_i \xi_j \hat{u} + k_i D_{\xi_i} \hat{u} + \lambda \hat{u} = \hat{f}.$$

Combining Lemma 4.2 with Parseval's identity, we prove the assertion.   □

To generalize the estimate (4.1) for the class of functions $S_2(\mathbb{R}_T^{1+2d})$, we prove the following approximation result.

Lemma 4.4. *For any $u \in S_2(\mathbb{R}_T^{1+2d})$, there exists a sequence of functions $u_n, n \geq 1$, such that*

(a) *for any $t \in (-\infty, T)$, $u_n(t, \cdot)$ is compactly supported on $B_{2n^3} \times B_{2n}$,*
(b) *for any $j, k, l \in \{0, 1, 2, \ldots\}$, $\partial_t^j D_x^k D_v^l u_n \in C_b(\mathbb{R}_T^{1+2d}) \cap L_2(\mathbb{R}_T^{1+2d})$,*
(c) $\lim_{n \to \infty} \|u_n - u\|_{S_2(\mathbb{R}_T^{1+2d})} = 0$.

Proof. Let $\eta \in C_0^\infty((0, \infty) \times \mathbb{R}^{2d})$ be a function with unit integral. For $\varepsilon > 0$ and $h \in L_{1,\mathrm{loc}}(\mathbb{R}^{1+2d})$, we denote

$$h_{(\varepsilon)}(t, x, v) = \int h(t - \varepsilon^2 t', x - \varepsilon^{1/2} x', v - \varepsilon v')\eta(t', x', v')\, dx' dv' dt'.$$

Furthermore, let $\phi \in C_0^\infty(B_2 \times B_2)$ be a function such that $\phi = 1$ on $B_1 \times B_1$ and denote

$$\phi_r(z) = \phi(x/r^3, v/r), \quad u_{\varepsilon,r}(z) = u_{(\varepsilon)}(z)\phi_r(x, v), \quad u_n = u_{n^{-1},n},$$

so that $a)$ and $b)$ are clearly satisfied.

We now prove $c)$. Clearly, $Au_n \to Au$ for $A = I, D_v, D_v^2$ in $L_2(\mathbb{R}_T^{1+2d})$ as $n \to \infty$. To prove the convergence of the transport term, we first, note that for any $z \in \mathbb{R}_T^{1+2d}$,

$$(\partial_t - v \cdot D_x)(u_{(n^{-1})} - u) = h_n + g_n, \tag{4.3}$$

where

$$h_n(z) = (\partial_t u - v \cdot D_x u)_{(n^{-1})}(z) - (\partial_t u - v \cdot D_x u)(z)$$

and

$$g_n(z) = -n^{-1/2} \int u(t - n^{-2}t', x - n^{-1/2}x', v - n^{-1}v')v'$$
$$\cdot D_x \eta(t', x', v')\, dx' dv' dt'.$$

Furthermore, $h_n \to 0$ in $L_2(\mathbb{R}_T^{1+2d})$ as $n \to \infty$ and by the Minkowski inequality,

$$\|g_n\|_{L_2(\mathbb{R}_T^{1+2d})} \leqq Nn^{-1/2}\|u\|_{L_2(\mathbb{R}_T^{1+2d})}. \tag{4.4}$$

Therefore, from (4.3) we obtain $\|(\partial_t - v \cdot D_x)(u_{(n-1)} - u)\|_{L_2(\mathbb{R}_T^{1+2d})} \to 0$ as $n \to \infty$. To prove the desired convergence, we write

$$(\partial_t - v \cdot D_x)(u_n - u_{(n-1)}) = (\phi_n - 1)(\partial_t u - v \cdot D_x u)_{(n-1)}$$
$$+ (\phi_n - 1)g_n - u_{(n-1)}v \cdot D_x\phi_n =: A_n + B_n + C_n.$$

We have

$$\|C_n\|_{L_2(\mathbb{R}_T^{1+2d})} \leqq Nn^{-2}\|u\|_{L_2(\mathbb{R}_T^{1+2d})},$$

and by (4.4),

$$\|B_n\|_{L_2(\mathbb{R}_T^{1+2d})} \leqq Nn^{-1/2}\|u\|_{L_2(\mathbb{R}_T^{1+2d})}.$$

Furthermore, note that

$$|A_n| \leqq N(d)|\phi_n - 1|M_t M_x M_v(|\partial_t u - v \cdot D_x u|),$$

where M_t, M_x, and M_v are the Hardy–Littlewood maximal function with respect to the t, x, and v variables. Then, by the Hardy–Littlewood inequality and the dominated convergence theorem,

$$\|A_n\|_{L_2(\mathbb{R}_T^{1+2d})} \to 0$$

as $n \to \infty$. Thus, $(\partial_t - v \cdot D_x)(u_n - u) \to 0$ in $L_2(\mathbb{R}_T^{1+2d})$ as $n \to \infty$. The lemma is proved. □

Proof of Theorem 4.1 (i). Let $u_n, n \geqq 1$, be a sequence from Lemma 4.4. Then, by Corollary 4.3,

$$\lambda\|u_n\|_{L_2(\mathbb{R}_T^{1+2d})} + \lambda^{1/2}\|D_v u_n\|_{L_2(\mathbb{R}_T^{1+2d})} + \|D_v^2 u_n\|_{L_2(\mathbb{R}_T^{1+2d})}$$
$$+ \|(-\Delta_x)^{1/3}u_n\|_{L_2(\mathbb{R}_T^{1+2d})} + \|D_v(-\Delta_x)^{1/6}u_n\|_{L_2(\mathbb{R}_T^{1+2d})} \tag{4.5}$$
$$\leqq N\delta^{-1}\|P_0 u_n\|_{L_2(\mathbb{R}_T^{1+2d})}$$

Passing to the limit in the above inequality as $n \to \infty$ and using Lemma 4.4, we prove (4.1) for u, $D_v u$, and $D_v^2 u$.

Next, we fix any $\phi \in C_0^\infty(\mathbb{R}_T^{1+2d})$. Since $u_n \to u$ in $L_2(\mathbb{R}_T^{1+2d})$, for $A = (-\Delta_x)^{1/3}, D_v(-\Delta_x)^{1/6}$, we have

$$\left|\int_{\mathbb{R}_T^{1+2d}} (Au)\phi \, dz\right| \leqq \|\phi\|_{L_2(\mathbb{R}_T^{1+2d})} \varlimsup_{n\to\infty} \|Au_n\|_{L_2(\mathbb{R}_T^{1+2d})}.$$

Now the desired estimates for $(-\Delta_x)^{1/3}u$ and $D_v(-\Delta_x)^{1/6}$ follow from this, (4.5), and Lemma 4.4. □

Next we prove the existence part by using a density argument. We first show localized L_2 estimates.

Lemma 4.5. *Let* $\lambda \geq 0$ *and* $r_1, r_2, R_1, R_2 > 0$ *be numbers such that* $r_1 < r_2$, *and* $R_1 < R_2$. *Let* $u \in S_{2,loc}(\mathbb{R}_0^{1+2d})$ *and denote* $f = P_0 u + \lambda u$. *Then, the following assertions hold:*

(i) $\delta^{-2}(r_2 - r_1)^{-1}\|D_v u\|_{L_2(Q_{r_1,R_1})} + \|D_v^2 u\|_{L_2(Q_{r_1,R_1})}$

$$\leq N(d)\delta^{-1}\|f\|_{L_2(Q_{r_2,R_2})} + N(d)\delta^{-4}((r_2 - r_1)^{-2} \tag{4.6}$$

$$+ r_2(R_2 - R_1)^{-3})\|u\|_{L_2(Q_{r_2,R_2})}.$$

(ii) *Denote* $C_r = (-r^2, 0) \times \mathbb{R}^d \times B_r$. *Then, we have*

$$\delta^{-2}(r_2 - r_1)^{-1}\|D_v u\|_{L_2(C_{r_1})} + \|D_v^2 u\|_{L_2(C_{r_1})}$$

$$\leq N(d)\delta^{-1}\|f\|_{L_2(C_{r_2})} + N(d)\delta^{-4}(r_2 - r_1)^{-2}\|u\|_{L_2(C_{r_2})}.$$

Proof. The method is standard and can be found in Lemma 2.4.4 of [27]. In this proof, N is a constant depending only on d.

(i) Let $\zeta \in C_{loc}^\infty(\mathbb{R})$ be a function such that $\zeta = 0$ if $t \geq 1$, and $\zeta = 1$ if $t \leq 0$. Denote $\hat{r}_0 = r_1$, $\hat{R}_0 = R_1$,

$$\hat{r}_n = r_1 + (r_2 - r_1)\sum_{k=1}^n 2^{-k}, \quad \hat{R}_n = R_1 + (R_2 - R_1)\sum_{k=1}^n 2^{-k},$$

$$\eta_n(t, v) = \zeta\left(2^{2(n+1)}(r_2 - r_1)^{-2}(-\hat{r}_n^2 - t)\right)\zeta\left(2^{(n+1)}(r_2 - r_1)^{-1}(|v| - \hat{r}_n)\right),$$

and

$$\phi_n(z) = \eta_n(t, v)\,\zeta\left(2^{3(n+1)}(R_2 - R_1)^{-3}(|x| - \hat{R}_n^3)\right).$$

Note that η_n and ϕ_n are smooth functions.
Denote

$$Q(n) = Q_{\hat{r}_n, \hat{R}_n}$$

and observe that in $\mathbb{R}_0^{1+2d}$, ϕ_n vanishes outside $Q(n+1)$, and $\phi_n = 1$ on $Q(n)$. Furthermore, in $\mathbb{R}_0^{1+2d}$, we have

$$(P_0 + \lambda)(u\phi_n) = f\phi_n + u(P_0\phi_n) - 2(aD_v\phi_n) \cdot D_v u.$$

Then by (4.1),

$$\|D_v^2 u\|_{L_2(Q(n))} \leq \|D_v^2(u\phi_n)\|_{L_2(Q(n+1))} \leq N\delta^{-1}\|f\|_{L_2(Q_{r_2,R_2})}$$

$$+ N\delta^{-1}(\delta^{-1}2^{2n}(r_2 - r_1)^{-2} + 2^{3n}r_2(R_2 - R_1)^{-3})\|u\|_{L_2(Q(n+1))}$$

$$+ N\delta^{-2}2^n(r_2 - r_1)^{-1}\|D_v u\|_{L_2(Q(n+1))}.$$

By the interpolation inequality,

$$\|D_v^2 u\|_{L_2(Q(n))} + (1 + \delta^{-2}(r_2 - r_1)^{-1})\|D_v u\|_{L_2(Q(n))}$$

$$\leq N\delta^{-1}\|f\|_{L_2(Q_{r_2,R_2})} + N\delta^{-4}(2^{2n}(r_2-r_1)^{-2}$$
$$+ 2^{3n}r_2(R_2-R_1)^{-3})\|u\|_{L_2(Q(n+1))} + 2^{-4}\|D_v^2 u\|_{L_2(Q(n+1))}.$$

We multiply the above inequality by 2^{-4n} and sum up with respect to $n = 0, 1, 2, \ldots$. We get

$$\|D_v^2 u\|_{L_2(Q_{r_1,R_1})} + \sum_{n=1}^{\infty} 2^{-4n}\|D_v^2 u\|_{L_2(Q(n))} + \delta^{-2}(r_2-r_1)^{-1}\|D_v u\|_{L_2(Q_{r_1,R_2})}$$

$$\leqq N\delta^{-1}\|f\|_{L_2(Q_{r_2,R_2})} + N\delta^{-4}((r_2-r_1)^{-2} + r_2(R_2-R_1)^{-3})\|u\|_{L_2(Q_{r_2,R_2})}$$

$$+ \sum_{n=1}^{\infty} 2^{-4n}\|D_v^2 u\|_{L_2(Q(n))}.$$

Canceling the same sum on both sides of the above inequality, we prove the lemma. (ii) To prove the claim we substitute $R_2 = 2R_1$ in (4.6) and pass to the limit as $R_1 \to \infty$. This assertion is proved. □

Lemma 4.6. *For any number $\lambda \geqq 0$, the set $(P_0 + \lambda)C_0^{\infty}(\mathbb{R}^{1+2d})$ is dense in $L_2(\mathbb{R}^{1+2d})$.*

Proof. Proof by contradiction. If the claim does not hold, then there exists a function $u \in L_2(\mathbb{R}^{1+2d})$ that is not identically zero and such that for any $\zeta \in C_0^{\infty}(\mathbb{R}^{1+2d})$,

$$\int (P_0\zeta + \lambda\zeta)u \, dz = 0.$$

Hence,

$$-\partial_t u + v \cdot D_x u - a^{ij}(t)D_{v_i v_j}u + \lambda u \equiv 0$$

in the sense of distributions. Let $w(t, x, v) = u(-t, -x, v)$. We regularize w by using a mollification argument from Lemma 4.4. For $\varepsilon > 0$ and a function $h \in L_{1,\text{loc}}(\mathbb{R}^{1+2d})$, we denote

$$h_{(\varepsilon)}(z) = \int h(t, x - \varepsilon^{1/2}x', v - \varepsilon v')\eta(x', v') \, dx' dv',$$

where $\eta \in C_0^{\infty}(\mathbb{R}^{2d})$ and $\int \eta \, dx dv = 1$. Then, the function $w_{(\varepsilon)}$ satisfies the equation

$$\partial_t w_{(\varepsilon)} - v \cdot D_x w_{(\varepsilon)} - a^{ij}(t)D_{v_i v_j}w_{(\varepsilon)} + \lambda w_{(\varepsilon)} = g_{\varepsilon}, \qquad (4.7)$$

where

$$g_{\varepsilon} = -\varepsilon^{1/2} \int u(t, x - \varepsilon^{1/2}x', v - \varepsilon v')v' \cdot D_x\eta(x', v') \, dx' dv',$$

which satisfies

$$\|g_{\varepsilon}\|_{L_2(\mathbb{R}^{1+2d})} \leqq N\varepsilon^{1/2}\|u\|_{L_2(\mathbb{R}^{1+2d})}. \qquad (4.8)$$

Furthermore, by Eq. (4.7) and (4.8), $\partial_t w_{(\varepsilon)} \in L_{2,\text{loc}}(\mathbb{R}^{1+2d})$, and, hence, $w_{(\varepsilon)} \in S_{2,\text{loc}}(\mathbb{R}^{1+2d})$. Then, by Lemma 4.5, (4.7), and (4.8), for any $r > 0$,

$$
\begin{aligned}
\|D_v w_{(\varepsilon)}\|_{L_2(Q_r)} &\leqq N(d,\delta)(r\|g_\varepsilon\|_{L_2(Q_{2r})} + r^{-1}\|w_{(\varepsilon)}\|_{L_2(Q_{2r})}) \\
&\leqq N(\varepsilon^{1/2}r + r^{-1})\|w\|_{L_2(\mathbb{R}^{1+2d})}.
\end{aligned}
\tag{4.9}
$$

Therefore, $D_v w \in L_{2,\text{loc}}(\mathbb{R}_0^{1+2d})$, and, passing to the limit as $\varepsilon \to 0$ in (4.9), we get

$$
\|D_v w\|_{L_2(Q_r)} \leqq N r^{-1}\|w\|_{L_2(\mathbb{R}^{1+2d})}.
$$

Finally, passing to the limit as $r \to \infty$, we conclude that $D_v w \equiv 0$, and, hence, $w \equiv 0$ in $\mathbb{R}_0^{1+2d}$. By a translation in the time coordinate, we see that w and thus u are identically equal to 0 in $\mathbb{R}^{1+2d}$, which gives a contradiction. The lemma is proved. $\square$

Proof of Theorem 4.1 (ii). First, we consider the case when $T = \infty$. The assertion follows from the a priori estimate (i) and the denseness of $(P_0 + \lambda)C_0^\infty(\mathbb{R}^{1+2d})$ in $L_2(\mathbb{R}^{1+2d})$ (see Lemma 4.6).

When $T < \infty$, we note that the uniqueness holds by the a priori estimate of the assertion (i) of this theorem. To prove the existence, note that the equation

$$
P_0 u + \lambda u = f 1_{t < T}
$$

has a unique solution $\widetilde{u} \in S_2(\mathbb{R}^{1+2d})$. We conclude that $u := \widetilde{u} 1_{t<T}$ is a $S_2(\mathbb{R}_T^{1+2d})$ solution of Eq. (1.2). $\square$

5. S_p-Estimate for the Model Equation

Here we generalize Theorem 4.1 for $p \in (1, \infty)$. We follow the argument in Chapter 4 of [27]. To derive an estimate of the sharp functions of $(-\Delta_x)^{1/3}u$, $D_v(-\Delta_x)^{1/6}u$, and $D_v^2 u$ for $u \in S_p(\mathbb{R}_T^{1+2d})$ (see Proposition 5.7), we split u into the P_0-caloric part and the remainder. The latter is estimated in Lemma 5.2 and the former - in Proposition 5.3. Throughout this section, the matrix-valued function a is independent of x, v.

Theorem 5.1. *Let $p > 1$ be a number. The following assertions hold:*

(i) *For any number $\lambda \geqq 0$ and $u \in S_p(\mathbb{R}_T^{1+2d})$, one has*

$$
\begin{aligned}
\lambda\|u\| + \lambda^{1/2}\|D_v u\| + \|D_v^2 u\| &+ \|(-\Delta_x)^{1/3}u\| \\
&+ \|D_v(-\Delta_x)^{1/6}u\| \leqq N(d,p)\delta^{-\theta}\|P_0 u + \lambda u\|,
\end{aligned}
\tag{5.1}
$$

where $\theta = \theta(d) > 0$ and $\|\cdot\| = \|\cdot\|_{L_p(\mathbb{R}_T^{1+2d})}$.

(ii) *For any $\lambda > 0$, $T \in (-\infty, \infty]$ and $f \in L_p(\mathbb{R}_T^{1+2d})$, Eq. (1.2) has a unique solution $u \in S_p(\mathbb{R}_T^{1+2d})$.*

(iii) *For any finite numbers $S < T$ and $f \in L_p((S,T) \times \mathbb{R}^{2d})$, the Cauchy problem* (2.7) *with $P = P_0$, $b \equiv 0$, and $c \equiv 0$ has a unique solution $u \in S_p((S,T) \times \mathbb{R}^{2d})$. In addition,*

$$\|u\|_{L_p((S,T)\times\mathbb{R}^{2d})} + \|D_v u\|_{L_p((S,T)\times\mathbb{R}^{2d})} + \|D_v^2 u\|_{L_p((S,T)\times\mathbb{R}^{2d})}$$
$$+ \|(-\Delta_x)^{1/3} u\|_{L_p((S,T)\times\mathbb{R}^{2d})} + \|D_v(-\Delta_x)^{1/6} u\|_{L_p(\mathbb{R}_T^{1+2d})}$$
$$\leqq N(d, p, T-S)\delta^{-\theta}\|f\|_{L_p((S,T)\times\mathbb{R}^{2d})},$$

where $\theta = \theta(d) > 0$.

We note that in the next lemma, f is assumed to be compactly supported only in the t and v variables. The reason for such a choice will be clear when we estimate the P_0-caloric part. See the proof of Proposition 5.3.

Lemma 5.2. *Let $R \geq 1$ be a number and $f \in L_2(\mathbb{R}^{1+2d})$ vanish outside $(-1,0) \times \mathbb{R}^d \times B_1$. Let $u \in S_2((-1,0) \times \mathbb{R}^{2d})$ be the unique solution to the Cauchy problem (see Definition 2.5 and Theorem 4.1 (iii))*

$$P_0 u = f, \quad u(-1, \cdot) = 0. \tag{5.2}$$

Then, there exists a number $\theta = \theta(d) > 0$ such that

$$\||u| + |D_v u| + |D_v^2 u|\|_{L_2((-1,0)\times B_{R^3}\times B_R)}$$
$$\leqq N(d)\delta^{-1} \sum_{k=0}^{\infty} 2^{-k(k-1)/4} R^{-k} \|f\|_{L_2(Q_{1,2^{k+1}R/\delta^2})}, \tag{5.3}$$

$$(|(-\Delta_x)^{1/3} u|^2)_{Q_{1,R}}^{1/2} \leqq N(d)\delta^{-\theta} R^{-2} \sum_{k=0}^{\infty} 2^{-2k} (f^2)_{Q_{1,2^k R/\delta^2}}^{1/2}, \tag{5.4}$$

and

$$(|D_v(-\Delta_x)^{1/6} u|^2)_{Q_{1,R}}^{1/2} \leqq N(d)\delta^{-\theta} R^{-1} \sum_{k=0}^{\infty} 2^{-k} (f^2)_{Q_{1,2^k R/\delta^2}}^{1/2}.$$

Proof. It is possible to obtain the estimates of this lemma by using the fast decay of the fundamental solution of the operator P_0, which can be written down explicitly (see, for example, [10,12]). Instead of invoking the integral representation of the solution to Eq. (5.2), we decompose it into a sum of functions supported on dyadic shells and exploit the global L_2-estimate of Theorem 4.1 and the scaling property of the operator P_0.

In this proof, N is a constant depending only on d.

Estimate of u, $D_v u$, and $D_v^2 u$. Denote

$$f = f_0 + \sum_{k=1}^{\infty} f_k := f 1_{\{x \in B_{(2R/\delta^2)^3}\}} + \sum_{k=1}^{\infty} f 1_{\{x \in B_{(2^{k+1}R/\delta^2)^3} \setminus B_{(2^k R/\delta^2)^3}\}}.$$

By Theorem 4.1, for $k = 0, 1, 2, \ldots$, there exists a unique solution $u_k \in S_2((-1,0) \times \mathbb{R}^{2d})$ to (5.2) with f_k in place of f, which satisfies

$$\||u_k| + |D_v u_k| + |D_v^2 u_k|\|_{L_2((-1,0)\times\mathbb{R}^{2d})} \leq N\delta^{-1} \|f_k\|_{L_2((-1,0)\times\mathbb{R}^{2d})}. \quad (5.5)$$

By the same theorem, for $A = I$, D_v, and D_v^2, we have

$$\lim_{n\to\infty} \sum_{k=0}^{n} A u_k = A u \quad \text{in } L_2((-1,0) \times \mathbb{R}^{2d}).$$

Now we take a sequence of functions $\zeta_j = \zeta_j(x, v) \in C_0^\infty(B_{(2^{j+1}R/\delta^2)^3} \times B_{2^{j+1}R/\delta^2})$, $j = 0, 1, 2, \ldots$, such that $\zeta_j = 1$ on $B_{(2^{j+1/2}R/\delta^2)^3} \times B_{2^{j+1/2}R/\delta^2}$ and

$$|\zeta_j| \leq 1, \quad |D_v\zeta_j| \leq N2^{-j}R^{-1}\delta^2,$$
$$|D_v^2\zeta_j| \leq N2^{-2j}R^{-2}\delta^4, \quad |D_x\zeta_j| \leq N2^{-3j}R^{-3}\delta^6.$$

For $k \geq 1$ and $j = 0, 1, \ldots, k-1$, we set $u_{k,j} = u_k\zeta_j$, which satisfies

$$P_0 u_{k,j} = u_k P_0 \zeta_j - 2(a D_v \zeta_j) \cdot D_v u_k$$

because $f_k\zeta_j \equiv 0$. By Theorem 4.1,

$$\||u_{k,j}| + |D_v u_{k,j}| + |D_v^2 u_{k,j}|\|_{L_2((-1,0)\times\mathbb{R}^{2d})}$$
$$\leq N\delta^{-1}\||u_k P_0\zeta_j| + |a||D_v\zeta_j||D_v u_k|\|_{L_2((-1,0)\times\mathbb{R}^{2d})},$$

which by the properties of ζ_j and the fact that $|a^{ij}| < \delta^{-1}$ implies that

$$\||u_k| + |D_v u_k| + |D_v^2 u_k|\|_{L_2((-1,0)\times B_{(2^j R/\delta^2)^3} \times B_{2^j R/\delta^2})}$$
$$\leq N2^{-j}R^{-1}\||u_k| + |D_v u_k|\|_{L_2((-1,0)\times B_{(2^{j+1}R/\delta^2)^3} \times B_{2^{j+1}R/\delta^2})}. \quad (5.6)$$

By an induction argument, (5.5), and (5.6), for $k \geq 1$ we get

$$\||u_k| + |D_v u_k| + |D_v^2 u_k|\|_{L_2((-1,0)\times B_{R^3} \times B_R)}$$
$$\leq N^k 2^{-k(k-1)/2} R^{-k}\delta^{-1} \|f_k\|_{L_2((-1,0)\times\mathbb{R}^{2d})}$$
$$\leq N2^{-k(k-1)/4} R^{-k}\delta^{-1} \|f\|_{L_2(Q_{1,2^{k+1}R/\delta^2})}. \quad (5.7)$$

To conclude (5.3), we use (5.5) with $k = 0$, (5.7), and the triangle inequality.

Estimate of $(-\Delta_x)^{1/3}u$. Note that $u\zeta_0$ satisfies

$$P_0(u\zeta_0) = f\zeta_0 + u(P_0\zeta_0) - 2(a D_v \zeta_0) \cdot D_v u.$$

By Theorem 4.1 and (5.3),

$$\|(-\Delta_x)^{1/3}(u\zeta_0)\|_{L_2((-1,0)\times\mathbb{R}^{2d})} + \|D_v(-\Delta_x)^{1/6}(u\zeta_0)\|_{L_2((-1,0)\times\mathbb{R}^{2d})}$$
$$\leq N\delta^{-1} \sum_{k=0}^{\infty} 2^{-k(k-1)/4} R^{-k} \|f\|_{L_2(Q_{1,2^{k+1}R/\delta^2})}. \quad (5.8)$$

It remains to estimate the commutator term

$$\|(-\Delta_x)^{1/3}(u\zeta_0) - \zeta_0(-\Delta_x)^{1/3}u\|_{L_2(Q_{1,R})}. \tag{5.9}$$

Since $\zeta_0 = 1$ in $B_{(2^{1/2}R/\delta^2)^3} \times B_{2^{1/2}R/\delta^2}$, for any $z \in Q_{1,R}$, we have

$$|(-\Delta_x)^{1/3}(u\zeta_0) - \zeta_0(-\Delta_x)^{1/3}u|(z)$$
$$\leq N \int_{|y|>(2^{3/2}-1)R^3} |u|(t, x + y, v)|y|^{-d-2/3} \, dy.$$

By Lemma A.1, (5.9) is bounded by

$$N R^{-2} \sum_{j=0}^{\infty} 2^{-2j-3dj/2}\|u\|_{L_2(Q_{1,2^j R})}.$$

By (5.3), the above sum is further bounded by

$$N\delta^{-1}R^{-2} \sum_{j=0}^{\infty} 2^{-2j-3dj/2} \sum_{k=0}^{\infty} 2^{-k(k-1)/4}(2^j R)^{-k}\|f\|_{L_2(Q_{1,2^{k+j+1}R/\delta^2})},$$

which gives (5.4) by a straightforward computation with a change of order of summations.

Estimate of $D_v(-\Delta_x)^{1/6}u$. By (5.8), it suffices to estimate the $L_2(Q_{1,R})$ norm of

$$A = (-\Delta_x)^{1/6}D_v(u\zeta_0) - \zeta_0(-\Delta_x)^{1/6}D_vu,$$

which is bounded above by

$$A_1 + A_2 := |(-\Delta_x)^{1/6}(uD_v\zeta_0)| + |(-\Delta_x)^{1/6}(\zeta_0D_vu) - \zeta_0(-\Delta_x)^{1/6}D_vu|.$$

For any $z \in Q_{1,R}$,

$$A_1(z) \leq N\delta^2 \int_{|y|>(2^{3/2}-1)R^3} |u|(t, x + y, v)|y|^{-d-1/3} \, dy.$$

Then, arguing as above, we get

$$\|A_1\|_{L_2(Q_{1,R})} \leq N(d)R^{-1}\delta \sum_{k=0}^{\infty} 2^{-k-3dk/2}\|f\|_{L_2(Q_{1,2^k R/\delta^2})}. \tag{5.10}$$

Furthermore, by Lemma A.1,

$$\|(-\Delta_x)^{1/6}(\zeta_0D_vu) - \zeta_0(-\Delta_x)^{1/6}D_vu\|_{L_2(Q_{1,R})}$$
$$\leq N R^{-1} \sum_{j=0}^{\infty} 2^{-j-3dj/2}\|D_vu\|_{L_2(Q_{1,2^j R})}.$$

By a computation similar to the one for (5.9), we get

$$\|A_2\|_{L_2(Q_{1,R})} \leq N(d)R^{-1}\delta^{-1} \sum_{k=0}^{\infty} 2^{-k-3dk/2}\|f\|_{L_2(Q_{1,2^k R/\delta^2})}.$$

Combining this with (5.10), we prove the desired estimate. □

Proposition 5.3. *Let $r > 0$, $v \geqq 2$ be numbers, $T \in (-\infty, \infty]$, $z_0 \in \overline{\mathbb{R}_T^{1+2d}}$, and $u \in S_{2,loc}(\mathbb{R}_T^{1+2d})$. Assume that $P_0 u = 0$ in $(t_0 - v^2 r^2, t_0) \times \mathbb{R}^d \times B_{vr}(v_0)$. Then, there exist constants $N(d) > 0$ and $\theta = \theta(d) > 0$ such that*

$$J_1 := \left(|(-\Delta_x)^{\frac{1}{3}} u - ((-\Delta_x)^{\frac{1}{3}} u)_{Q_r(z_0)}|^2 \right)^{1/2}_{Q_r(z_0)}$$

$$\leqq N v^{-1} \delta^{-\theta} (|(-\Delta_x)^{\frac{1}{3}} u|^2)^{1/2}_{Q_{vr}(z_0)},$$

$$J_2 := \left(|D_v(-\Delta_x)^{\frac{1}{6}} u - (D_v(-\Delta_x)^{\frac{1}{6}} u)_{Q_r(z_0)}|^2 \right)^{1/2}_{Q_r(z_0)}$$

$$\leqq N v^{-1} \delta^{-\theta} (|D_v(-\Delta_x)^{\frac{1}{6}} u|^2)^{1/2}_{Q_{vr}(z_0)}$$

$$+ N v^{-1} \delta^{-\theta} \sum_{k=0}^{\infty} 2^{-2k} (|(-\Delta_x)^{\frac{1}{3}} u|^2)^{1/2}_{Q_{vr,2^k vr}(z_0)},$$

$$J_3 := \left(|D_v^2 u - (D_v^2 u)_{Q_r(z_0)}|^2 \right)^{1/2}_{Q_r(z_0)}$$

$$\leqq N v^{-1} \delta^{-\theta} (|D_v^2 u|^2)^{1/2}_{Q_{vr}(z_0)} + N v^{-1} \delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} (|(-\Delta_x)^{\frac{1}{3}} u|^2)^{1/2}_{Q_{vr,2^k vr}(z_0)}.$$

5.1. Proof of Proposition 5.3

We follow the scheme of Chapter 4 of [27]. Thanks to the Poincaré inequality, to handle the P_0-caloric function, it suffices to estimate its Hölder norm. We do this by using Caccioppoli type estimate combined with the localized version of Theorem 4.1. See Lemma 4.5. The proof of Proposition 5.3 is given at the end of this subsection after a series of lemmas.

The following lemma contains one of the key estimates of the proof.

Lemma 5.4. (Caccioppoli type estimate) *Let $u \in S_{2,loc}(\mathbb{R}_0^{1+2d})$ be a function such that*

$$P_0 u = 0 \quad in \ Q_1.$$

Then, for any numbers $0 < r < R \leqq 1$,

$$\|D_x u\|_{L_2(Q_r)} \leqq N(d, r, R) \delta^{-7} \|u\|_{L_2(Q_R)}.$$

Proof. By modifying u outside Q_1, we may certainly assume that u is compactly supported. By taking the standard mollification with respect to x and then taking the limit, we may assume without loss of generality that $(-\Delta_x)^\beta u \in S_2(\mathbb{R}_0^{1+2d})$ for any $\beta \geqq 0$. Let $r < r_1 < r_2 < R$ be numbers, $\psi \in C_0^\infty(\mathbb{R}^d)$ vanish outside B_1 and $\eta \in C_0^\infty(\mathbb{R}^{1+d})$ vanish outside $(-1, 1) \times B_1$. We set $\phi_1(t, v) = \eta(t/r_1^2, v/r_1)$, $\phi_2(x) = \psi(x/r_2^3)$ and denote

$$\phi(z) = \phi_1(t, v) \phi_2(x).$$

Throughout the proof, a constant N depends only on d, r, and R.

Note that $u\phi$ satisfies the equation

$$P_0(u\phi) = u P_0\phi - 2(a D_v\phi) \cdot D_v u \tag{5.11}$$

on $\mathbb{R}_0^{1+2d}$ because $\phi P_0 u \equiv 0$. Then, by Theorem 4.1 applied to (5.11), one has

$$\|(-\Delta_x)^{1/3}(u\phi)\|_{L_2(\mathbb{R}_0^{1+2d})}$$
$$\leqq N\delta^{-1}\|u P_0\phi\|_{L_2(\mathbb{R}_0^{1+2d})} + N\delta^{-1}\|(a D_v\phi) \cdot D_v u\|_{L_2(\mathbb{R}_0^{1+2d})}.$$

Applying Lemma 4.5 (i) with r_1 in place of r_1, $R_1 = r_2$, and with R in place of r_2 and R_2 and using the fact that $|a^{ij}| < \delta^{-1}$, from the above inequality we get

$$\|\phi_1(-\Delta_x)^{1/3}(u\phi_2)\|_{L_2(\mathbb{R}_0^{1+2d})} \leqq N\delta^{-4}\|u\|_{L_2(Q_R)}. \tag{5.12}$$

Furthermore, the function $w = (-\Delta_x)^{1/3}(u\phi)$ solves the equation

$$P_0 w = (-\Delta_x)^{1/3}[u(P_0\phi)] - 2(a D_v\phi_1) \cdot D_v(-\Delta_x)^{1/3}(u\phi_2) \tag{5.13}$$

on $\mathbb{R}_0^{1+2d}$. Due to Theorem 4.1 (i) applied to (5.13),

$$\|(-\Delta_x)^{2/3}(u\phi)\|_{L_2(\mathbb{R}_0^{1+2d})} \leqq N\delta^{-1}(\|(-\Delta_x)^{1/3}(u P_0\phi)\|_{L_2(\mathbb{R}_0^{1+2d})}$$
$$+ \|(a D_v\phi_1) \cdot D_v(-\Delta_x)^{1/3}(u\phi_2)\|_{L_2(\mathbb{R}_0^{1+2d})}) =: J_1 + J_2. \tag{5.14}$$

By (5.12),

$$J_1 \leqq N\delta^{-6}\|u\|_{L_2(Q_R)}. \tag{5.15}$$

Next we consider the term J_2. Observe that $f = (-\Delta_x)^{1/3}(u\phi_2)$ satisfies the equation

$$P_0 f = -(-\Delta_x)^{1/3}[(v \cdot D_x\phi_2)u] \quad \text{on } (-1, 0) \times \mathbb{R}^d \times B_1.$$

By Lemma 4.5 (ii) and the assumption $|a^{ij}| < \delta^{-1}$,

$$J_2 \leqq N\delta^{-3}\|\eta(-\Delta_x)^{1/3}(u\phi_2)\|_{L_2(\mathbb{R}_0^{1+2d})} + N\|\eta v_i(-\Delta_x)^{1/3}(u D_{x_i}\phi_2)\|_{L_2(\mathbb{R}_0^{1+2d})}.$$

Here $\eta \in C^\infty(\mathbb{R}^{1+d})$ is a function of (t, v) such that $\eta = 1$ on $(-r_2^2, r_2^2) \times B_{r_2}$ and $\eta = 0$ outside $(-r_3^2, r_3^2) \times B_{r_3}$, where $r_2 < r_3 < R$. Then, using (5.12), we get

$$J_2 \leqq N\delta^{-7}\|u\|_{L_2(Q_R)}. \tag{5.16}$$

By (5.14)–(5.16), we obtain

$$\|(1 - \Delta_x)^{2/3}(u\phi)\|_{L_2(\mathbb{R}_0^{1+2d})}$$
$$\leqq N(d)\|u\phi\|_{L_2(\mathbb{R}_0^{1+2d})} + N(d)\|(-\Delta_x)^{2/3}(u\phi)\|_{L_2(\mathbb{R}_0^{1+2d})}$$
$$\leqq N\delta^{-7}\|u\|_{L_2(Q_R)}.$$

We assume that $\phi_1 = 1$ on $(-r^2, r^2) \times B_r$ and $\phi_2 = 1$ on B_{r^3}. We conclude the proof as follows:

$$\|D_x u\|_{L_2(Q_r)} \leq \|D_x(u\phi)\|_{L_2(\mathbb{R}_0^{1+2d})}$$
$$\leq N(d)\|(1 - \Delta_x)^{2/3}(u\phi)\|_{L_2(\mathbb{R}_0^{1+2d})} \leq N\delta^{-7}\|u\|_{L_2(Q_R)}.$$

The lemma is proved. $\square$

We also need the following nonlocal estimates.

Lemma 5.5. *Let* $r \in (0, 1)$ *be a number and* $u \in S_{2,loc}(\mathbb{R}_0^{1+2d})$. *We denote* $f = P_0 u$ *and assume that* $f = 0$ *in* $(-1, 0) \times \mathbb{R}^d \times B_1$. *Then,*

(i) $\|D_x u\|_{L_2(Q_r)} \leq N\delta^{-4} \sum_{k=0}^{\infty} 2^{-k}(|(-\Delta_x)^{1/3}u|^2)_{Q_{1,2^k}}^{1/2},$

(ii) $\|D_x u\|_{L_2(Q_r)} \leq N\delta^{-4} \sum_{k=0}^{\infty} 2^{-2k}(|(-\Delta_x)^{1/6}u|^2)_{Q_{1,2^k}}^{1/2},$ (5.17)

where $N = N(d, r)$.

Proof. (i) By mollifying u in the x variable, we may assume that u is sufficiently regular in x. Fix some number $R \in (r, 1)$. Let $\eta \in C_0^{\infty}(\mathbb{R}^{1+2d})$ be a function such that $\eta = 1$ on $\tilde{Q}_r$ and η vanishes outside $\tilde{Q}_R$. In this proof, we assume that $N = N(d, r, R)$.

We decompose

$$\eta^2 D_x u = \eta^2 \mathcal{R}_x(-\Delta_x)^{1/2}u = \eta(Lu + \text{Comm}),$$

where $\mathcal{R}_x = D_x(-\Delta_x)^{-1/2}$ is the Riesz transform,

$$Lu = \mathcal{R}_x(-\Delta_x)^{1/6}(\eta(-\Delta_x)^{1/3}u),$$
$$\text{Comm} = \eta\mathcal{R}_x(-\Delta_x)^{1/2}u - \mathcal{R}_x(-\Delta_x)^{1/6}(\eta(-\Delta_x)^{1/3}u).$$

Thus, to prove the claim we only need to show that (5.17) holds if we replace the left-hand side with $\|Lu\|_{L_2(Q_R)} + \|\text{Comm}\|_{L_2(Q_R)}$.

Estimate of Lu. Denote

$$w = (-\Delta_x)^{1/3}u.$$

Then, since $P_0 u = 0$ in $(-1, 0) \times \mathbb{R}^d \times B_1$, the function ηw satisfies the equation

$$P_0(\eta w) = wP_0\eta - 2(aD_v\eta) \cdot D_v w \quad \text{on } \mathbb{R}_0^{1+2d}.$$ (5.18)

By the fact that $\mathcal{R}_x$ is an isometry in $L_2(\mathbb{R}^d)$, the interpolation inequality, and Theorem 4.1, we have

$$\|Lu\|_{L_2(\mathbb{R}_0^{1+2d})} \leq N(d)\|\eta w\|_{L_2(\mathbb{R}_0^{1+2d})} + N(d)\|(-\Delta_x)^{1/3}(\eta w)\|_{L_2(\mathbb{R}_0^{1+2d})}$$
$$\leq N\delta^{-2}\|w\|_{L_2(Q_R)} + N\delta^{-1}\|(aD_v\eta) \cdot D_v w\|_{L_2(Q_R)}.$$

By the fact that $P_0 w = 0$ in $(-1, 0) \times \mathbb{R}^d \times B_1$, Lemma 4.5 (i), and Assumption 2.2, the last term is bounded by

$$N(d, r, R)\delta^{-4} \|w\|_{L_2(Q_1)}.$$

Estimate of Comm. Denote

$$A = \mathcal{R}_x(-\Delta_x)^{1/6} = D_x(-\Delta_x)^{-1/3}$$

and observe that

$$\text{Comm} = \eta A w - A(\eta w).$$

It is well known that for any Schwartz function ϕ on $\mathbb{R}^d$,

$$(-\Delta_x)^{-1/3}\phi = \mathcal{K} * \phi,$$

where

$$\mathcal{K}(x) = N \int_0^\infty t^{1/3} p(t, x) \frac{dt}{t}$$

and

$$p(t, x) = (4\pi t)^{-d/2} e^{-|x|^2/(4t)}.$$

Furthermore, by the change of variables $t \to \frac{t}{|x|^2}$ for any $x \in \mathbb{R}^d \setminus \{0\}$, we have

$$
\begin{aligned}
D_x \mathcal{K}(x) &= N x \int_0^\infty t^{-2/3-d/2} e^{-|x|^2/(4t)} \frac{dt}{t} \\
&= N x |x|^{-(d+4/3)} \int_0^\infty t^{-2/3-d/2} e^{-1/(4t)} \frac{dt}{t} = N x |x|^{-(d+4/3)}.
\end{aligned}
$$

Therefore, for any $z \in \mathbb{R}_0^{1+2d}$,

$$|\text{Comm}(z)| \leq N \int_{\mathbb{R}^d} |w(t, x - y, v)| \, |\eta(t, x, v) - \eta(t, x - y, v)| \, |y|^{-(d+1/3)} \, dy.$$

We split the above integral into two parts. The first part, I_1, is the integral over $|y| \leq 2$, and the remainder is denoted by I_2. First, by the mean-value theorem,

$$|I_1(z)| \leq N \int_{|y|<2} |w(t, x - y, v)| \, |y|^{-d+2/3} \, dy, \quad z \in \mathbb{R}_0^{1+2d}.$$

Then by the Minkowski inequality,

$$
\begin{aligned}
\|I_1\|_{L_2(Q_R)} &\leq N \int_{|y|<2} \|w(\cdot, \cdot - y, \cdot)\|_{L_2(Q_R)} \, |y|^{-d+2/3} \, dy \\
&\leq N \|w\|_{L_2(Q_{1,2})} \int_{|y|<2} |y|^{-d+2/3} \, dy \leq N \|w\|_{L_2(Q_{1,2})}.
\end{aligned}
$$

Furthermore, since η vanishes outside Q_R, for any $z \in Q_R$,

$$I_2(z) = N|\eta(z)| \int_{|y|>2} |w(t, x - y, v)| \, |y|^{-(d+1/3)} \, dy.$$

By virtue of Lemma A.1,

$$\|I_2\|_{L_2(Q_R)} \leqq N \sum_{k=0}^{\infty} 2^{-k} (w^2)_{Q_{1,2^k}}^{1/2}.$$

Thus, the commutator term is less than the right-hand side of (5.17).

(ii) The proof is almost the same as of (i). Let us point out some minor differences. This time, we denote

$$\begin{aligned}
\widetilde{L}u &= \mathcal{R}_x(-\Delta_x)^{1/3}(\eta(-\Delta_x)^{1/6}u), \\
\widetilde{A} &= \mathcal{R}_x(-\Delta_x)^{1/3} = D_x(-\Delta_x)^{-1/6}, \quad \widetilde{w} = (-\Delta_x)^{1/6}u, \\
Comm &= \eta\mathcal{R}_x(-\Delta_x)^{1/2}u - \mathcal{R}_x(-\Delta_x)^{1/3}(\eta(-\Delta_x)^{1/6}u) = \eta\widetilde{A}\widetilde{w} - \widetilde{A}(\eta\widetilde{w}).
\end{aligned}$$

Note that $\eta\widetilde{w}$ satisfies Eq. (5.18) with w replaced with $\widetilde{w}$. Then, by Theorem 4.1 and Lemma 4.5 (i),

$$\|\widetilde{L}u\|_{L_2(\mathbb{R}_0^{1+2d})}$$
$$\leqq N(d)\delta^{-1}(\|\widetilde{w}P_0\eta\|_{L_2(\mathbb{R}_0^{1+2d})} + \|(aD_v\eta) \cdot D_v\widetilde{w}\|_{L_2(\mathbb{R}_0^{1+2d})}) \leqq N\delta^{-4}\|\widetilde{w}\|_{L_2(Q_1)}.$$

Furthermore, as above

$$|Comm| \leqq N \int_{\mathbb{R}^d} |\widetilde{w}(t, x - y, v)| \, |\eta(t, x, v) - \eta(t, x - y, v)| \, |y|^{-(d+2/3)} \, dy.$$

Finally, we repeat the last paragraph of the proof of (i). The lemma is proved. □

Lemma 5.6. *Let $R \in (1/2, 1)$ be a number and $u \in S_{2,loc}(\mathbb{R}_0^{1+2d})$ be a function such that $P_0u = 0$ in $(-1, 0) \times \mathbb{R}^d \times B_1$. Then, for any $l, m = \{0, 1, 2, \ldots\}$, there exists a constant $\theta = \theta(d, l, m) > 0$ such that*

(i) $\displaystyle\sup_{Q_{1/2}} |D_x^l D_v^m u| + \sup_{Q_{1/2}} |\partial_t D_x^l D_v^m u| \leqq N(d, l, m, R)\delta^{-\theta} \|u\|_{L_2(Q_R)},$

(ii) $\displaystyle\sup_{Q_{1/2}}(|D_x^l D_v^{m+1}(-\Delta_x)^{1/6}u| + |\partial_t D_x^l D_v^{m+1}(-\Delta_x)^{1/6}u|)$

$$\leqq N(d, l, m)\delta^{-\theta}(\|D_v(-\Delta_x)^{1/6}u\|_{L_2(Q_1)} + \sum_{k=0}^{\infty} 2^{-2k}(|(-\Delta_x)^{1/3}u|^2)_{Q_{1,2^k}}^{1/2}),$$

(iii) $\displaystyle\sup_{Q_{1/2}} |D_x^l D_v^{m+2}u| + \sup_{Q_{1/2}} |\partial_t D_x^l D_v^{m+2}u| \leqq N(d, l, m)\delta^{-\theta}(\|D_v^2u\|_{L_2(Q_1)}$

$$+ \sum_{k=0}^{\infty} 2^{-k}(|(-\Delta_x)^{1/3}u|^2)_{Q_{1,2^k}}^{1/2}).$$

Proof. In this proof, N is a constant independent of δ.

(i) Let us fix some number $r \in (1/2, R)$. First, we prove that for any $m \in \{0, 1, 2, \ldots\}$,

$$\|D_v^{m+1} u\|_{L_2(Q_r)} \leqq N(d, r, R, m)\delta^{-\theta}\|u\|_{L_2(Q_R)}, \tag{5.19}$$

where $\theta = \theta(d, m)$.

Proof by induction. First, for $m = 0$ the assertion holds due to Lemma 4.5 (i). For $m > 0$, let $\alpha = (\alpha_1, \ldots \alpha_d)$ be a multi-index of order m. Then, by the product rule, formally we have

$$P_0(D_v^\alpha u) = \sum_{\widetilde{\alpha}: \widetilde{\alpha} < \alpha, |\widetilde{\alpha}| = m-1} c_{\widetilde{\alpha}} D_v^{\widetilde{\alpha}} D_x^{\alpha - \widetilde{\alpha}} u. \tag{5.20}$$

We fix some numbers r_1, r_2 such that $r < r_1 < r_2 < R$. By Lemma 4.5 (i),

$$\|D_v^{m+1} u\|_{L_2(Q_r)} \leqq N\delta\|D_v^{m-1} D_x u\|_{L_2(Q_{r_1})} + N\delta^{-2}\|D_v^m u\|_{L_2(Q_{r_1})}. \tag{5.21}$$

Observe that $D_x u$ satisfies $P_0(D_x u) = 0$ in $(-1, 0) \times \mathbb{R}^d \times B_1$. Hence, by the induction hypothesis and Lemma 5.4,

$$\|D_v^{m-1} D_x u\|_{L_2(Q_{r_1})} \leqq N\delta^{-\theta}\|D_x u\|_{L_2(Q_{r_2})} \leqq N\delta^{-\theta}\|u\|_{L_2(Q_R)}. \tag{5.22}$$

To make the argument above rigorous, we need to use the method of finite-difference quotient. Thus, by induction (5.19) holds.

Next, observe that for any multi-index α, the function $D_x^\alpha u$ satisfies

$$P_0(D_x^\alpha u) = 0 \quad \text{in } (-1, 0) \times \mathbb{R}^d \times B_1.$$

Then, by (5.19),

$$\|D_v^m D_x^l u\|_{L_2(Q_{1/2})} \leqq N\delta^{-\theta}\|D_x^l u\|_{L_2(Q_r)}.$$

Iterating the estimate of Lemma 5.4, we get

$$\|D_v^m D_x^l u\|_{L_2(Q_{1/2})} \leqq N\delta^{-\theta}\|u\|_{L_2(Q_R)}. \tag{5.23}$$

Here again, we need to use the method of finite-difference quotient. Furthermore, the fact that $|a^{ij}| < \delta^{-1}$,

$$\partial_t u = a^{ij} D_{v_i v_j} + v \cdot D_x u \tag{5.24}$$

in Q_1 combined with (5.23) yields the estimate

$$\|\partial_t D_v^m D_x^l u\|_{L_2(Q_{1/2})} \leqq N\delta^{-\theta}\|u\|_{L_2(Q_R)}.$$

By this, (5.23), and the Sobolev embedding theorem, we prove the inequality for the sup-norm of $D_x^l D_v^m u$. The estimate of the second term follows from (5.24).

(ii) First, since $u - (u)_{Q_r}$ satisfies the same equation as u, by the assertion (i) and the Poincaré inequality,

$$J := \sup_{Q_{1/2}} (|D_x^l D_v^{m+1} u| + |\partial_t D_x^l D_v^{m+1} u|) \leqq N\delta^{-\theta} \|u - (u)_{Q_r}\|_{L_2(Q_r)}$$

$$\leqq N\delta^{-\theta} (\|D_v u\|_{L_2(Q_r)} + \|D_x u\|_{L_2(Q_r)} + \|\partial_t u\|_{L_2(Q_r)}).$$

Furthermore, by (5.24) and (5.21), we get

$$\|\partial_t u\|_{L_2(Q_r)} \leqq N\delta^{-1} \|D_v^2 u\|_{L_2(Q_r)} + N\|D_x u\|_{L_2(Q_r)}$$

$$\leqq N\delta^{-\theta} (\|D_x u\|_{L_2(Q_R)} + \|D_v u\|_{L_2(Q_R)}),$$

where $R \in (r, 1)$. By the above,

$$J \leqq N\delta^{-\theta} (\|D_v u\|_{L_2(Q_R)} + \|D_x u\|_{L_2(Q_R)}). \tag{5.25}$$

Finally, note that $P_0((-\Delta_x)^{1/6} u) = 0$ in $(-1, 0) \times \mathbb{R}^d \times B_1$. Substituting $(-\Delta_x)^{1/6} u$ in (5.25) and using Lemma 5.5 (ii), we prove the assertion.

(iii) Denote

$$u_1(z) = u(z) - (u)_{Q_r} - v_i (D_{v_i} u)_{Q_r}$$

and observe that $P_0 u_1 = 0$ in $(-1, 0) \times \mathbb{R}^d \times B_1$. Then, by (i),

$$\sup_{Q_{1/2}} |D_x^l D_v^m D_v^2 u| + \sup_{Q_{1/2}} |\partial_t D_x^l D_v^m D_v^2 u| \leqq N\delta^{-\theta} \|u_1\|_{L_2(Q_r)}. \tag{5.26}$$

Applying the Poincaré inequality twice and using (5.24), we get

$$\|u_1\|_{L_2(Q_r)}$$
$$\leqq N(\|D_x u\|_{L_2(Q_r)} + \|\partial_t u\|_{L_2(Q_r)} + \|D_v u - (D_v u)_{Q_r}\|_{L_2(Q_r)})$$
$$\leqq N(\|D_x u\|_{L_2(Q_r)} + \delta^{-1} \|D_v^2 u\|_{L_2(Q_r)} + \|D_{vx} u\|_{L_2(Q_r)} + \|\partial_t D_v u\|_{L_2(Q_r)}). \tag{5.27}$$

By (5.22) with $m = 2$,

$$\|D_{vx} u\|_{L_2(Q_r)} \leqq N\delta^{-\theta} \|D_x u\|_{L_2(Q_R)}. \tag{5.28}$$

Furthermore, by (5.20) with $|\alpha| = 1$,

$$\|\partial_t D_v u\|_{L_2(Q_r)}$$
$$\leqq N\delta^{-1} \|D_v^3 u\|_{L_2(Q_r)} + N\|D_{vx} u\|_{L_2(Q_r)} + N\|D_x u\|_{L_2(Q_r)}. \tag{5.29}$$

By (5.21) with $m = 2$ and (5.28), we have

$$\|D_v^3 u\|_{L_2(Q_r)} \leqq N\delta^{-\theta} (\|D_x u\|_{L_2(Q_R)} + \|D_v^2 u\|_{L_2(Q_R)}). \tag{5.30}$$

Combining (5.27)–(5.30), we obtain

$$\|u_1\|_{L_2(Q_r)} \leqq N\delta^{-\theta} (\|D_v^2 u\|_{L_2(Q_R)} + \|D_x u\|_{L_2(Q_R)}).$$

Now the assertion follows from (5.26), the above inequality, and Lemma 5.5 (i). □

Proof of Proposition 5.3. Let $\widetilde{u}$ and $\widetilde{P_0}$ be the function and the operator from Lemma 3.6 defined with vr in place of r. Then, by the aforementioned lemma, we have

$$\widetilde{P_0}\widetilde{u} = 0 \quad \text{in } (-1, 0) \times \mathbb{R}^d \times B_1,$$

and, for any $c > 0$,

$$\fint_{Q_{vr,cvr}(z_0)} |(-\Delta_x)^{1/3}u|^2 \, dz = (vr)^{-4} \fint_{Q_{1,c}} |(-\Delta_x)^{1/3}\widetilde{u}|^2 \, dz,$$

$$\fint_{Q_r(z_0)} \left| (-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_r(z_0)} \right|^2 \, dz$$

$$= (vr)^{-4} \fint_{Q_{1/v}} \left| (-\Delta_x)^{1/3}\widetilde{u} - ((-\Delta_x)^{1/3}\widetilde{u})_{Q_{1/v}} \right|^2 \, dz.$$

Similar identities hold for $D_v(-\Delta_x)^{1/6}u$ and $D_v^2 u$. Hence, we may assume that $r = 1/v$ and $z_0 = 0$.

Next, the fact that $P_0((-\Delta_x)^{1/3}u) = 0$ in $(-1, 0) \times \mathbb{R}^d \times B_1$ combined with Lemma 5.6 (i) gives

$$J_1 \leqq \sup_{z_1, z_2 \in Q_{1/v}} |(-\Delta_x)^{1/3}u(z_1) - (-\Delta_x)^{1/3}u(z_2)|$$

$$\leqq N(d)v^{-1}\delta^{-\theta} \left(\fint_{Q_1} |(-\Delta_x)^{1/3}u|^2 \, dz \right)^{1/2}.$$

Similarly, by Lemma 5.6 (ii), we prove the estimate of J_2. Finally, Lemma 5.6 (iii) implies the desired estimate of J_3.　□

5.2. Proof of Theorem 5.1

Let us give an outline of the proof. First, we prove a mean oscillation estimate (see Proposition 5.7), and, as a result, we obtain Theorem 5.1 (i) with $p > 2$ and $\lambda = 0$. Then, by using Agmon's method, we derive the a priori estimate of $\lambda \|u\|_{L_p(\mathbb{R}_T^{1+2d})}$ for the same range of p. Furthermore, we show that $(P_0 + \lambda)C_0^\infty(\mathbb{R}^{1+2d})$ is dense in $L_p(\mathbb{R}^{1+2d})$ for $p > 2$ and $\lambda \geqq 0$. Thus, we prove the unique solvability of Eq. (1.2) for $p > 2$. The results for $p \in (1, 2)$ are obtained by using the duality method.

Proposition 5.7. *Let $r > 0$, $v \geqq 2$ be numbers, $z_0 \in \overline{\mathbb{R}_T^{1+2d}}$, and $u \in S_2(\mathbb{R}_T^{1+2d})$. Suppose that $P_0 u = f$ in $\mathbb{R}_T^{1+2d}$. Then, there exists constants $\theta = \theta(d) > 0$ and $N = N(d) > 0$ such that the following assertions hold.*

(i) $\left(|(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_r(z_0)}|^2 \right)_{Q_r(z_0)}^{1/2}$

$\leqq Nv^{-1}\delta^{-\theta}(|(-\Delta_x)^{1/3}u|^2)_{Q_{vr}(z_0)}^{1/2}$

$$+ N\nu^{1+2d}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-2k}(f^2)^{1/2}_{Q_{2\nu r, 2^{k+1}\nu r/\delta^2}(z_0)},$$

(ii) $\left(\left|D_v(-\Delta_x)^{1/6}u - (D_v(-\Delta_x)^{1/6}u)_{Q_r(z_0)}\right|^2\right)^{1/2}_{Q_r(z_0)}$

$$\leq N\nu^{-1}\delta^{-\theta}(|D_v(-\Delta_x)^{1/6}u|^2)^{1/2}_{Q_{\nu r}(z_0)}$$

$$+ N\nu^{-1}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-2k}(|(-\Delta_x)^{1/3}u|^2)^{1/2}_{Q_{\nu r, 2^k \nu r}(z_0)}$$

$$+ N\nu^{1+2d}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k}(f^2)^{1/2}_{Q_{2\nu r, 2^{k+1}\nu r/\delta^2}(z_0)},$$

(iii) $\left(\left|D_v^2 u - (D_v^2 u)_{Q_r(z_0)}\right|^2\right)^{1/2}_{Q_r(z_0)}$

$$\leq N\nu^{-1}\delta^{-\theta}(|D_v^2 u|^2)^{1/2}_{Q_{\nu r}(z_0)} + N\nu^{-1}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k}(|(-\Delta_x)^{1/3}u|^2)^{1/2}_{Q_{\nu r, 2^k \nu r}(z_0)}$$

$$+ N\nu^{1+2d}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k}(f^2)^{1/2}_{Q_{2\nu r, 2^{k+1}\nu r/\delta^2}(z_0)}.$$

Proof. Here we assume that N depends only on d.

Let $\phi \in C_0^{\infty}((t_0 - (2\nu r)^2, t_0 + (2\nu r)^2) \times B_{2\nu r}(v_0))$ be a function of (t, v) such that $\phi = 1$ on $(t_0 - (\nu r)^2, t_0) \times B_{\nu r}(v_0)$. Then, by Theorem 4.1 there exists a unique solution $g \in S_2((t_0 - (2\nu r)^2, t_0) \times \mathbb{R}^{2d})$ to the Cauchy problem

$$P_0 g = f\phi, \quad g(t_0 - (2\nu r)^2, \cdot) \equiv 0,$$

and, by Lemma 5.2 and the scaling argument (see Lemma 3.6),

$$\fint_{Q_{\nu r}(z_0)} |(-\Delta_x)^{1/3}g|^2 \, dz \leq N\delta^{-\theta}\left(\sum_{k=0}^{\infty} 2^{-2k}(f^2)^{1/2}_{Q_{2\nu r, 2^{k+1}\nu r/\delta^2}(z_0)}\right)^2, \quad (5.31)$$

$$\fint_{Q_r(z_0)} |(-\Delta_x)^{1/3}g|^2 \, dz \leq N\nu^{2+4d} \fint_{Q_{2\nu r}(z_0)} |(-\Delta_x)^{1/3}g|^2 \, dz$$

$$\leq N\nu^{2+4d}\delta^{-\theta}\left(\sum_{k=0}^{\infty} 2^{-2k}(f^2)^{1/2}_{Q_{2\nu r, 2^{k+1}\nu r/\delta^2}(z_0)}\right)^2. \quad (5.32)$$

Furthermore, note that the function $h = u - g \in S_2((t_0 - (2\nu r)^2, t_0) \times \mathbb{R}^{2d})$ satisfies

$$P_0 h = f(1 - \phi) \quad \text{in } (t_0 - (2\nu r)^2, t_0) \times \mathbb{R}^{2d}.$$

Then, by Proposition 5.3 and (5.31),

$$\fint_{Q_r(z_0)} \left|(-\Delta_x)^{1/3}h - ((-\Delta_x)^{1/3}h)_{Q_r(z_0)}\right|^2 \, dz$$

$$\leqq N\nu^{-2}\delta^{-\theta}(|(-\Delta_x)^{1/3}u|^2)_{Q_{vr}(z_0)} + N\nu^{-2}\delta^{-\theta}(|(-\Delta_x)^{1/3}g|^2)_{Q_{vr}(z_0)}$$

$$\leqq N\nu^{-2}\delta^{-\theta}(|(-\Delta_x)^{1/3}u|^2)_{Q_{vr}(z_0)}$$

$$+ N\nu^{-2}\delta^{-\theta}\left(\sum_{k=0}^{\infty} 2^{-2k}(f^2)_{Q_{2vr,2^{k+1}vr/\delta^2}(z_0)}^{1/2}\right)^2.$$

Combining this with (5.32), we prove the assertion (i).

(ii) By Lemmas 5.2 and 3.6,

$$\left(\fint_{Q_{vr}(z_0)} |D_v(-\Delta_x)^{1/6}g|^2\, dz\right)^{1/2}$$

$$\leqq N\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-k}(f^2)_{Q_{2vr,2^{k+1}vr/\delta^2}(z_0)}^{1/2}, \tag{5.33}$$

$$\left(\fint_{Q_r(z_0)} |D_v(-\Delta_x)^{1/6}g|^2\, dz\right)^{1/2}$$

$$\leqq N\delta^{-\theta}\nu^{1+2d}\sum_{k=0}^{\infty} 2^{-k}(f^2)_{Q_{2vr,2^{k+1}vr/\delta^2}(z_0)}^{1/2}. \tag{5.34}$$

Furthermore, by Proposition 5.3 and the triangle inequality,

$$\left(\fint_{Q_r(z_0)} \left|D_v(-\Delta_x)^{1/6}h - (D_v(-\Delta_x)^{1/6}h)_{Q_r(z_0)}\right|^2 dz\right)^{1/2}$$

$$\leqq N\nu^{-1}\delta^{-\theta}(|D_v(-\Delta_x)^{1/6}u|^2)_{Q_{vr}(z_0)}^{1/2} + N\nu^{-1}\delta^{-\theta}(|D_v(-\Delta_x)^{1/6}g|^2)_{Q_{vr}(z_0)}^{1/2}$$

$$+ N\nu^{-1}\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-2k}(|(-\Delta_x)^{1/3}u|^2)_{Q_{vr,2^k vr}(z_0)}^{1/2}$$

$$+ N\nu^{-1}\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-2k}(|(-\Delta_x)^{1/3}g|^2)_{Q_{vr,2^k vr}(z_0)}^{1/2}.$$

By (5.31) and (5.33), we estimate the terms containing g on the right-hand side of the above inequality. We obtain

$$\left(\fint_{Q_r(z_0)} \left|D_v(-\Delta_x)^{\frac{1}{6}}h - (D_v(-\Delta_x)^{\frac{1}{6}}h)_{Q_r(z_0)}\right|^2 dz\right)^{1/2}$$

$$\leqq N\nu^{-1}\delta^{-\theta}(|D_v(-\Delta_x)^{\frac{1}{6}}u|^2)_{Q_{vr}(z_0)}^{1/2}$$

$$+ N\nu^{-1}\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-2k}(|(-\Delta_x)^{\frac{1}{3}}u|^2)_{Q_{vr,2^k vr}(z_0)}^{1/2}$$

$$+ N\nu^{-1}\delta^{-\theta}\sum_{l=0}^{\infty} 2^{-l}(f^2)_{Q_{2vr,2^{l+1}vr/\delta^2}(z_0)}^{1/2}$$

$$+ N\nu^{-1}\delta^{-\theta}\sum_{k,l=0}^{\infty} 2^{-2k-l}(f^2)_{Q_{2vr,2^{k+l+1}vr/\delta^2}(z_0)}^{1/2}.$$

Changing the index of summation $l \to k + l$, we replace the last term with

$$N\nu^{-1}\delta^{-\theta} \sum_{l=0}^{\infty} 2^{-l} (f^2)_{Q_{2\nu r, 2^{l+1}\nu r/\delta^2}(z_0)}^{1/2}.$$

Combining this inequality with (5.34), we prove the assertion (ii).

(iii) This time, by Lemma 5.2 we have

$$\left(\fint_{Q_{\nu r}(z_0)} |D_v^2 g|^2 \, dz \right)^{1/2} \leq N\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k^2/8} (f^2)_{Q_{2\nu r, 2^{k+1}\nu r/\delta^2}(z_0)}^{1/2}, \qquad (5.35)$$

$$\left(\fint_{Q_r(z_0)} |D_v^2 g|^2 \, dz \right)^{1/2} \leq N\delta^{-\theta} \nu^{1+2d} \sum_{k=0}^{\infty} 2^{-k^2/8} (f^2)_{Q_{2\nu r, 2^{k+1}\nu r/\delta^2}(z_0)}^{1/2}. \qquad (5.36)$$

Next, by using Proposition 5.3 and (5.35) and arguing as above, we get

$$\left(\fint_{Q_r(z_0)} \left| D_v^2 h - (D_v^2 h)_{Q_r(z_0)} \right|^2 dz \right)^{1/2}$$

$$\leq N\nu^{-1}\delta^{-\theta} (|D_v^2 u|^2)_{Q_{\nu r}(z_0)}^{1/2} + N\nu^{-1}\delta^{-\theta} (|D_v^2 g|^2)_{Q_{\nu r}(z_0)}^{1/2}$$

$$+ N\nu^{-1}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} \left((|(-\Delta_x)^{1/3} u|^2)_{Q_{\nu r, 2^k \nu r}(z_0)}^{1/2} + (|(-\Delta_x)^{1/3} g|^2)_{Q_{\nu r, 2^k \nu r}(z_0)}^{1/2} \right)$$

$$\leq N\nu^{-1}\delta^{-\theta} (|D_v^2 u|^2)_{Q_{\nu r}(z_0)}^{1/2} + N\nu^{-1}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} (|(-\Delta_x)^{1/3} u|^2)_{Q_{\nu r, 2^k \nu r}(z_0)}^{1/2}$$

$$+ N\nu^{-1}\delta^{-\theta} \sum_{l=0}^{\infty} 2^{-l^2/8} (f^2)_{Q_{2\nu r, 2^{l+1}\nu r/\delta^2}(z_0)}^{1/2}$$

$$+ N\nu^{-1}\delta^{-\theta} \sum_{k,l=0}^{\infty} 2^{-l^2/8-k} (f^2)_{Q_{2\nu r, 2^{k+l+1}\nu r/\delta^2}(z_0)}^{1/2}.$$

As above, we may replace the double sum with the term

$$N\nu^{-1}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} (f^2)_{Q_{2\nu r, 2^{k+1}\nu r/\delta^2}(z_0)}^{1/2}.$$

As before, this inequality and (5.36) imply the estimate of the mean-square oscillation of $D_v^2 u$. $\square$

Proposition 5.8. *For any $p \in (2, \infty)$, $T \in (-\infty, \infty]$, and $u \in S_p(\mathbb{R}_T^{1+2d})$, the estimate (5.1) holds with $\lambda = 0$.*

Proof. By Proposition 5.7, there exist constants $N = N(d)$ and $\theta = \theta(d) > 0$ such that for any $z \in \mathbb{R}_T^{1+2d}$,

$$((-\Delta_x)^{1/3} u)_T^\#(z) \leq N\nu^{-1}\delta^{-\theta} \mathcal{M}_T^{1/2} |(-\Delta_x)^{1/3} u|^2(z)$$

$$+ N\nu^{1+2d}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-2k} \mathbb{M}_{2^k,T}^{1/2} f^2(z),$$

$$(D_v(-\Delta_x)^{1/6}u)_T^{\#}(z) \lesssim N\nu^{-1}\delta^{-\theta} \mathcal{M}_T^{1/2} |D_v(-\Delta_x)^{1/6}u|^2(z)$$

$$+ N\nu^{1+2d}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} \mathbb{M}_{2^k/\delta^2,T}^{1/2} f^2(z)$$

$$+ N\nu^{-1}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-2k} \mathbb{M}_{2^k,T}^{1/2} |(-\Delta_x)^{1/3}u|^2(z),$$

$$(D_v^2 u)_T^{\#}(z) \lesssim N\nu^{-1}\delta^{-\theta} \mathcal{M}_T^{1/2} |D_v^2 u|^2(z) + N\nu^{1+2d}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} \mathbb{M}_{2^k/\delta^2,T}^{1/2} f^2(z)$$

$$+ N\nu^{-1}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} \mathbb{M}_{2^k,T}^{1/2} |(-\Delta_x)^{1/3}u|^2(z).$$

We raise the above inequalities to the p-th power, integrate over $\mathbb{R}_T^{1+2d}$, and use the Minkowski inequality. Furthermore, we apply Corollary 3.2 (i) and (ii) with $p/2 > 1$. We obtain

$$\|(-\Delta_x)^{1/3}u\|_{L_p(\mathbb{R}_T^{1+2d})}$$
$$\lesssim N\nu^{-1}\delta^{-\theta}\|(-\Delta_x)^{1/3}u\|_{L_p(\mathbb{R}_T^{1+2d})} + N\nu^{1+2d}\delta^{-\theta}\|f\|_{L_p(\mathbb{R}_T^{1+2d})},$$
$$\|D_v(-\Delta_x)^{1/6}u\|_{L_p(\mathbb{R}_T^{1+2d})} \lesssim N\nu^{-1}\delta^{-\theta}\|D_v(-\Delta_x)^{1/6}u\|_{L_p(\mathbb{R}_T^{1+2d})}$$
$$+ N\nu^{-1}\delta^{-\theta}\|(-\Delta_x)^{1/3}u\|_{L_p(\mathbb{R}_T^{1+2d})} + N\nu^{1+2d}\delta^{-\theta}\|f\|_{L_p(\mathbb{R}_T^{1+2d})},$$
$$\|D_v^2 u\|_{L_p(\mathbb{R}_T^{1+2d})} \lesssim N\nu^{-1}\delta^{-\theta}\|D_v^2 u\|_{L_p(\mathbb{R}_T^{1+2d})}$$
$$+ N\nu^{-1}\delta^{-\theta}\|(-\Delta_x)^{1/3}u\|_{L_p(\mathbb{R}_T^{1+2d})} + N\nu^{1+2d}\delta^{-\theta}\|f\|_{L_p(\mathbb{R}_T^{1+2d})}. \quad (5.37)$$

By setting $\nu = 2(1 + N\delta^{-\theta})$, we cancel the term containing $(-\Delta_x)^{1/3}u$ on the right-hand side of (5.37). Using this and our choice of ν, we prove the estimates for $D_v(-\Delta_x)^{1/6}u$ and $D_v^2 u$. The theorem is proved. □

Lemma 5.9. *Under the assumptions of Proposition 5.8, for any $\lambda > 0$,*

$$\lambda\|u\|_{L_p(\mathbb{R}_T^{1+2d})} \lesssim N\delta^{-\theta}\|P_0 u + \lambda u\|_{L_p(\mathbb{R}_T^{1+2d})},$$

where $N = N(d, p)$ and $\theta = \theta(d) > 0$.

Proof. We use S. Agmon's method (see, for example, the proof of Lemma 6.3.8 of [27]).
    Denote

$$\hat{x} = (x_1, \ldots, x_{d+1}), \quad \hat{v} = (v_1, \ldots, v_{d+1}), \quad \hat{z} = (t, \hat{x}, \hat{v}),$$
$$\hat{P}_0(\hat{z}) = \partial_t - \sum_{i=1}^{d+1} v_i D_{x_i} - \sum_{i,j=1}^{d} a^{ij} D_{v_i v_j} - D_{v_{d+1} v_{d+1}}.$$

Let ζ be a smooth cutoff function on $\mathbb{R}$ such that $\zeta \not\equiv 0$ and denote
$$\hat{u}(\hat{z}) = u(z)\zeta(v_{d+1})\cos(\lambda^{1/2}v_{d+1}).$$

By direct calculations,

$$\lambda u(z)\zeta(v_{d+1})\cos(\lambda^{1/2}v_{d+1}) = -D_{v_{d+1}v_{d+1}}\hat{u}(\hat{z})$$
$$+ u(z)(\zeta''(v_{d+1})\cos(\lambda^{1/2}v_{d+1}) - 2\lambda^{1/2}\zeta'(v_{d+1})\sin(\lambda^{1/2}v_{d+1})), \quad (5.38)$$
$$\hat{P}_0\hat{u}(\hat{z}) = \zeta(v_{d+1})\cos(\lambda^{1/2}v_{d+1})(P_0 u(z) + \lambda u(z))$$
$$- u(z)(\zeta''(v_{d+1})\cos(\lambda^{1/2}v_{d+1}) - 2\lambda^{1/2}\zeta'(v_{d+1})\sin(\lambda^{1/2}v_{d+1})). \quad (5.39)$$

Note that for any $p > 0$ and $\lambda > 1$

$$\int_{\mathbb{R}} |\zeta(t)\cos(\lambda^{1/2}t)|^p \, dt \geqq N_1(p) > 0.$$

This combined with (5.38) gives

$$\lambda\|u\|_{L_p(\mathbb{R}_T^{1+2d})} \leqq N(p)\|D_{v_{d+1}v_{d+1}}\hat{u}\|_{L_p(\mathbb{R}_T^{3+2d})} + N(p)(1+\lambda^{1/2})\|u\|_{L_p(\mathbb{R}_T^{1+2d})}.$$

Furthermore, by Proposition 5.8 and (5.39),

$$\|D_{v_{d+1}v_{d+1}}\hat{u}\|_{L_p(\mathbb{R}_T^{3+2d})} \leqq N\delta^{-\theta}\|\hat{P}_0\hat{u}\|_{L_p(\mathbb{R}_T^{3+2d})}$$
$$\leqq N\delta^{-\theta}\|P_0 u + \lambda u\|_{L_p(\mathbb{R}_T^{1+2d})} + N\delta^{-\theta}(1+\lambda^{1/2})\|u\|_{L_p(\mathbb{R}_T^{1+2d})}.$$

Thus, by the above,

$$\lambda\|u\|_{L_p(\mathbb{R}_T^{1+2d})} \leqq N\delta^{-\theta}\|P_0 u + \lambda u\|_{L_p(\mathbb{R}_T^{1+2d})} + N\delta^{-\theta}(1+\lambda^{1/2})\|u\|_{L_p(\mathbb{R}_T^{1+2d})}.$$

We note that for any $\lambda \geqq \lambda_0 = 16N^2\delta^{-2\theta} + 1$, one has $\lambda - N\delta^{-\theta}(1+\lambda^{1/2}) > \lambda/2$. This gives the desired estimate for $\lambda \geqq \lambda_0$. This restriction is removed by using a scaling argument (see Lemma 3.6). □

Combining Proposition 5.8 with Lemma 5.9, we prove the following result.

Corollary 5.10. *Under the assumptions of Proposition* 5.8, *the estimate* (5.1) *holds.*

To prove the next lemma, we repeat the argument of Lemma 4.5 and replace Theorem 4.1 with Corollary 5.10.

Lemma 5.11. *Let $p > 2$, $\lambda \geqq 0$, and $r_1, r_2, R_1, R_2 > 0$ be numbers such that $r_1 < r_2$, and $R_1 < R_2$. Let $u \in S_{p,loc}(\mathbb{R}_0^{1+2d})$ and denote $f = P_0 u + \lambda u$. Then, there exist constants $N = N(d,p)$ and $\theta = \theta(d) > 0$ such that the following assertions hold.*

(i) $\delta^{-\theta/2}(r_2 - r_1)^{-1}\|D_v u\|_{L_p(Q_{r_1,R_1})} + \|D_v^2 u\|_{L_p(Q_{r_1,R_1})} \leqq N\delta^{-\theta}\|f\|_{L_p(Q_{r_2,R_2})}$
$$+ N\delta^{-\theta}((r_2 - r_1)^{-2} + r_2(R_2 - R_1)^{-3})\|u\|_{L_p(Q_{r_2,R_2})}.$$

(ii) *Denote $C_r = (-r^2, 0) \times \mathbb{R}^d \times B_r$. Then we have*
$$\delta^{-\theta/2}(r_2 - r_1)^{-1}\|D_v u\|_{L_p(C_{r_1})} + \|D_v^2 u\|_{L_p(C_{r_1})}$$
$$\leqq N\delta^{-\theta}\big(\|f\|_{L_p(C_{r_2})} + (r_2 - r_1)^{-2}\|u\|_{L_p(C_{r_2})}\big).$$

Lemma 5.12. *For any $\lambda \geqq 0$ and $p > 1$, the set $(P_0 + \lambda)C_0^\infty(\mathbb{R}^{1+2d})$ is dense in $L_p(\mathbb{R}^{1+2d})$.*

Proof. We may assume that $p \neq 2$, since the case $p = 2$ is covered in Lemma 4.6.

Proof by contradiction. Denote $q = p/(p-1)$. If the claim does not hold, there exists a function $u \in L_q(\mathbb{R}_T^{1+2d})$ that is not identically zero and such that for any $\zeta \in C_0^\infty(\mathbb{R}^{1+2d})$,

$$\int (P_0\zeta + \lambda\zeta)u \, dz = 0.$$

Case $p \in (1, 2)$. We repeat the argument of Lemma 4.6 with appropriate modifications. This time, instead of Lemma 4.5, we use Lemma 5.11 (i) with $R_1 = r_1 = r$ and $R_2 = r_2 = 2r$. By this lemma and (4.8) with 2 replaced with q, we conclude

$$\|D_v w_{(\varepsilon)}\|_{L_q(Q_r)} \leqq N(d, \delta, q)(r\|g_\varepsilon\|_{L_q(Q_{2r})} + r^{-1}\|w_{(\varepsilon)}\|_{L_q(Q_{2r})})$$
$$\leqq N(\varepsilon^{1/2}r + r^{-1})\|w\|_{L_q(\mathbb{R}^{1+2d})}.$$

As in the proof of Lemma 4.6, this implies that $w \equiv 0$, which gives a contradiction.

Case $p > 2$. Let $\eta_\varepsilon = \eta_\varepsilon(x)$ be a standard mollifier. For an integer $k \geqq 1$, we denote by $w_{\varepsilon,k}$ the k-fold mollification of the function $w(z) = u(-t, -x, v)$ in the x variable with η_ε. The idea of the proof is to, first, show that $w_{\varepsilon,k} \in L_2(\mathbb{R}^{1+2d}) \cap S_{2,\text{loc}}(\mathbb{R}^{1+2d})$ for some large k, and then conclude that $w_{\varepsilon,k} \equiv 0$ by using the localized S_2-estimate.

Step 1 For $s \in (1, \infty)$ and an open set $G \subset \mathbb{R}^{1+2d}$, denote

$$\|f\|_{\mathcal{W}_s(G)} = \||f| + |\partial_t f| + |D_v f| + |D_v^2 f|\|_{L_s(G)}.$$

Note that $w_{\varepsilon,1} := w * \eta_\varepsilon$ satisfies

$$\partial_t w_{\varepsilon,1} - a^{ij}D_{v_i v_j}w_{\varepsilon,1} + \lambda w_{\varepsilon,1} = v \cdot D_x w_{\varepsilon,1} \tag{5.40}$$

with

$$\|v \cdot D_x w_{\varepsilon,1}\|_{L_q(\tilde{Q}_{1/2})} \leqq N(d)\varepsilon^{-1}\|w\|_{L_q(\tilde{Q}_1)}$$

provided that $\varepsilon \in (0, 1/2)$. Mollifying Eq. (5.40) in the v variable with a standard mollifier and applying the interior estimate for nondegenerate equations for fixed $x \in B_{(1/4)^3}$ (see, for example, Theorem 5.2.5 of [27]), we get

$$\|w_{\varepsilon,1}\|_{\mathcal{W}_q(\tilde{Q}_{1/4})} \leqq N(d, \delta, q, \varepsilon)\|w\|_{L_q(\tilde{Q}_1)}.$$

By this and the Sobolev embedding theorem for any $q_1 > q$ such that

$$\frac{2}{d+2} \geqq \frac{1}{q} - \frac{1}{q_1},$$

we obtain

$$\|w_{\varepsilon,1}\|_{L_{q_1}(\tilde{Q}_{1/4})} \leqq N(d, \delta, q, q_1, \varepsilon)\|w\|_{L_q(\tilde{Q}_1)}.$$

Step 2 There exists $m = m(d, q) \in \{1, 2, \ldots\}$ and a sequence $\{q_k, k = 0, 1, 2, \ldots\}$, such that $q_0 = q$, $q_m = 2$, and

$$\frac{2}{d+2} \geqq \frac{1}{q_{k-1}} - \frac{1}{q_k} > 0, \ k = 1, \ldots, m.$$

We set $w_{\varepsilon, 0} = w$,

$$w_{\varepsilon, k} = w_{\varepsilon, k-1} * \eta_\varepsilon, k = 1, \ldots, m.$$

We claim that for $k = 1, \ldots, m$,

$$\|w_{\varepsilon, k}\|_{L_{q_k}(\tilde{Q}_{2^{-2k}})} \leqq N(d, \delta, q_{k-1}, q_k, k, \varepsilon) \|w_{\varepsilon, k-1}\|_{L_{q_{k-1}}(\tilde{Q}_{2^{-2(k-1)}})}.$$

To prove this, we repeat the argument of Step 1 with w replaced with $w_{\varepsilon, k-1}$, q with q_{k-1}, and q_1 with q_k. Iterating the above estimate, we conclude that

$$\|w_{\varepsilon, m}\|_{L_2(\tilde{Q}_{2^{-2m}})} \leqq N(d, \delta, q, \varepsilon) \|w\|_{L_q(\tilde{Q}_1)}.$$

By this and the argument of Step 1 again,

$$\|w_{\varepsilon, m+1}\|_{\mathcal{W}_2(\tilde{Q}_{2^{-2(m+1)}})} \leqq N(d, \delta, q, \varepsilon) \|w\|_{L_q(\tilde{Q}_1)}. \tag{5.41}$$

Shifting the center of coordinates and using Lemma 3.6 give

$$\|w_{\varepsilon, m+1}\|_{L_2(\tilde{Q}_{2^{-2(m+1)}}(z))} \leqq N(d, q, \delta, \varepsilon) \|w\|_{L_q(\tilde{Q}_1(z))}, \ \forall z \in \mathbb{R}^{1+2d}. \tag{5.42}$$

Next, by the argument of Lemma 21 of [9], there exists a sequence of points $z_n \in \mathbb{R}^{1+2d}$, $n \geq 1$, such that

$$\bigcup_{n=1}^{\infty} \tilde{Q}_{2^{-2(m+1)}}(z_n) = \mathbb{R}^{1+2d}, \quad \sum_{n=1}^{\infty} 1_{\tilde{Q}_1(z_n)} \leqq N_0(d, m).$$

Then, by this, (5.42), and the inequality

$$\sum_{k=1}^{\infty} b_k^\alpha \leqq \left(\sum_{k=1}^{\infty} b_k \right)^\alpha, \ \alpha \geqq 1, \ b_k \geqq 0, \ k \geqq 1$$

with $\alpha = 2/q > 1$, we obtain

$$\int_{\mathbb{R}^{1+2d}} |w_{\varepsilon, m+1}|^2 \, dz \leqq \sum_{n=1}^{\infty} \int_{\tilde{Q}_{2^{-2(m+1)}}(z_n)} |w_{\varepsilon, m+1}|^2 \, dz$$

$$\leqq N \sum_{n=1}^{\infty} \left(\int_{\tilde{Q}_1(z_n)} |w|^q \, dz \right)^{2/q} \leqq N \|w\|_{L_q(\mathbb{R}^{1+2d})}^2 < \infty.$$

Furthermore, generalizing (5.41) for double cylinders $\tilde{Q}_r$ and $\tilde{Q}_{2^{-2(m+1)}r}$, we show that $w_{\varepsilon, m+1} \in S_{2, \text{loc}}(\mathbb{R}^{1+2d})$. Therefore, by Lemma 4.5 (i), for any $r > 0$,

$$\|D_v w_{\varepsilon, m+1}\|_{L_2(Q_r)} \leqq N(d, \delta) r^{-1} \|w_{\varepsilon, m+1}\|_{L_2(\mathbb{R}^{1+2d})}.$$

Passing to the limit as $r \to \infty$ gives $D_v w_{\varepsilon, m+1} \equiv 0$ in $\mathbb{R}_0^{1+2d}$. Shifting in the t variable, we prove that $D_v w_{\varepsilon, m+1} \equiv 0$ in $\mathbb{R}^{1+2d}$, and hence, $w_{\varepsilon, m+1} = 0$, which implies $w \equiv 0$. $\square$

Proof of Theorem 5.1. We consider three cases:

$$p > 2, \quad p \in (1, 2), T = \infty, \quad \text{and } p \in (1, 2), \, T < \infty.$$

Case 1 $p > 2$. Thanks to Corollary 5.10, it remains to prove the assertion (ii). The latter follows from the a priori estimate in Corollary 5.10 and the denseness result (see Lemma 5.12).

Case 2 $p \in (1, 2)$, $T = \infty$. (i) We use the standard duality argument (see, for example, Theorem 4.3.8 of [27]). Throughout the proof, we assume that $N = N(d, p)$.

Let $U \in C_0^\infty(\mathbb{R}^{1+2d})$ and $u \in S_p(\mathbb{R}^{1+2d})$. Denote

$$f = P_0 u + \lambda u.$$

For $h \in L_{1,\text{loc}}(\mathbb{R}^{1+2d})$, by h_ε we denote the mollification in the x variable with the standard mollifier.

Estimate of $(-\Delta_x)^{1/3} u$. It is well known that

$$(-\Delta_x)^{1/3} U, \ (-\Delta_x)^{1/3} \partial_t U, \ (-\Delta_x)^{1/3}(v_i D_{x_i} U), \ (-\Delta_x)^{1/3} D_v^2 U$$
$$\in C_{\text{loc}}^\infty(\mathbb{R}^{1+2d}) \cap L_1(\mathbb{R}^{1+2d}).$$

Then, by using duality and integrating by parts, we get

$$J := \int ((-\Delta_x)^{1/3} u_\varepsilon)(-\partial_t U + v_i D_{x_i} U - a^{ij} D_{v_i v_j} U + \lambda U) \, dz$$

$$= \int u_\varepsilon(-\partial_t + v_i D_{x_i} - a^{ij} D_{v_i v_j} + \lambda)((-\Delta_x)^{1/3} U) \, dz$$

$$= \int ((-\Delta_x)^{1/3} U)(P_0 u_\varepsilon + \lambda u_\varepsilon) \, dz.$$

Furthermore, by Hölder's inequality, Corollary 5.10, and the change of variables $t \to -t, x \to -x$, one has

$$|J| \leqq \|(-\Delta_x)^{1/3} U\|_{L_q(\mathbb{R}^{1+2d})} \|f_\varepsilon\|_{L_p(\mathbb{R}^{1+2d})}$$
$$\leqq N\delta^{-\theta} \| - \partial_t U + v_i D_{x_i} U - a^{ij} D_{v_i v_j} U + \lambda U\|_{L_q(\mathbb{R}^{1+2d})} \|f_\varepsilon\|_{L_p(\mathbb{R}^{1+2d})},$$

where $q = p/(p-1)$. By Lemma 5.12 and the same change of variables, we conclude that $(-\partial_t - a^{ij} D_{v_i v_j} + v_i D_{x_i} + \lambda)C_0^\infty(\mathbb{R}^{1+2d})$ is dense in $L_q(\mathbb{R}^{1+2d})$. Thus, we obtain

$$\|(-\Delta_x)^{1/3} u_\varepsilon\|_{L_p(\mathbb{R}^{1+2d})} \leqq N\delta^{-\theta} \|f_\varepsilon\|_{L_p(\mathbb{R}^{1+2d})}.$$

Taking the limit as $\varepsilon \to 0$, we prove the desired estimate.

Estimate of $D_v(-\Delta_x)^{1/6} u$. As above, by using duality and integration by parts, we get

$$\mathcal{I} = \int (D_v(-\Delta_x)^{1/6} u_\varepsilon)(-\partial_t U + v \cdot D_x U - a^{ij}(t) D_{v_i v_j} U + \lambda U) \, dz$$

$$= -\int (P_0 u_\varepsilon + \lambda u_\varepsilon) D_v(-\Delta_x)^{1/6} U \, dz$$

$$- \int ((-\Delta_x)^{1/6} u_\varepsilon) \, D_x U \, dz =: \mathcal{I}_1 + \mathcal{I}_2.$$

Furthermore, by Corollary 5.10 with $q > 2$,

$$\mathcal{I}_1 \leqq N\delta^{-\theta} \| - \partial_t U + v_i D_{x_i} U - a^{ij} D_{v_i v_j} U + \lambda U \|_{L_q(\mathbb{R}^{1+2d})} \| f_\varepsilon \|_{L_p(\mathbb{R}^{1+2d})}.$$

Next, by Hölder's inequality,

$$\mathcal{I}_2 \leqq N \| (-\Delta_x)^{1/3} u_\varepsilon \|_{L_p(\mathbb{R}^{1+2d})} \| \mathcal{R}_x (-\Delta_x)^{1/3} U \|_{L_q(\mathbb{R}^{1+2d})} =: \mathcal{I}_{2,1} \mathcal{I}_{2,2},$$

where $\mathcal{R}_x = D_x(-\Delta_x)^{-1/2}$ is the Riesz transform. Due to the L_p estimate of $(-\Delta_x)^{1/3} u_\varepsilon$,

$$\mathcal{I}_{2,1} \leqq N\delta^{-\theta} \| f_\varepsilon \|_{L_p(\mathbb{R}^{1+2d})}.$$

By the L_q boundedness of the Riesz transform and Corollary 5.10 combined with the change of variables as before, we get

$$\mathcal{I}_{2,2} \leqq N\delta^{-\theta} \| - \partial_t U + v_i D_{x_i} U - a^{ij} D_{v_i v_j} U + \lambda U \|_{L_q(\mathbb{R}^{1+2d})}.$$

These estimates imply the desired inequality for $D_v(-\Delta_x)^{1/6} u$.

Estimate of $D_v^2 u$. For any $k, l \in \{1, \dots, d\}$, we have

$$I = \int D_{v_k v_l} u_\varepsilon (-\partial_t U + v_i D_{x_i} U - a^{ij} D_{v_i v_j} U + \lambda U) \, dz =: I_1 + I_2,$$

where

$$I_1 = \int D_{v_k v_l} U (P_0 u_\varepsilon + \lambda u_\varepsilon) \, dz,$$

$$I_2 = - \int (\delta_{ik} D_{v_l} U + \delta_{il} D_{v_k} U) D_{x_i} u_\varepsilon \, dz.$$

We only need to show that

$$|I_1| + |I_2|$$
$$\leqq N\delta^{-\theta} \| - \partial_t U + v_i D_{x_i} U - a^{ij} D_{v_i v_j} U + \lambda U \|_{L_q(\mathbb{R}^{1+2d})} \| f_\varepsilon \|_{L_p(\mathbb{R}^{1+2d})}. \tag{5.43}$$

The estimate of I_1 follows from Corollary 5.10. Furthermore, Hölder's inequality yields

$$|I_2| \leqq \| (-\Delta_x)^{1/6} D_v U \|_{L_q(\mathbb{R}^{1+2d})} \| \mathcal{R}_x (-\Delta_x)^{1/3} u_\varepsilon \|_{L_p(\mathbb{R}^{1+2d})} =: I_{2,1} I_{2,2}.$$

By Corollary 5.10,

$$I_{2,1} \leqq N\delta^{-\theta} \| - \partial_t U + v_i D_{x_i} U - a^{ij} D_{v_i v_j} U + \lambda U \|_{L_q(\mathbb{R}^{1+2d})}.$$

Then, by the L_p-boundedness of $\mathcal{R}_x$ and the L_p estimate of $(-\Delta_x)^{1/3} u_\varepsilon$, we obtain

$$I_{2,2} \leqq N\delta^{-\theta} \| f_\varepsilon \|_{L_p(\mathbb{R}^{1+2d})}.$$

Combining these estimates, we prove (5.43).

Estimate of u. For $\lambda > 0$, integration by parts gives

$$\int \lambda u_\varepsilon (-\partial_t U + v_i D_{x_i} U - a^{ij} D_{v_i v_j} U + \lambda U) \, dz = \int \lambda U f_\varepsilon \, dz.$$

Now the L_p estimate of λu follows from Hölder's inequality and Corollary 5.10. Thus, the assertion (i) is proved. The claim (ii) now follows from (i) and Lemma 5.12.

Case 3 $p \in (1, 2)$, $T < \infty$. (i) We fix an arbitrary function $\phi \in L_q(\mathbb{R}_T^{1+2d})$, where $q = p/(p-1)$, and extend it by zero for $t > T$. By the assertion (ii) in the case $p > 2$ and the change of variables $t \to -t$, $x \to -x$, the equation

$$-\partial_t U + v \cdot D_x U - a^{ij} D_{v_i v_j} U + \lambda U = \phi$$

has a unique solution U such that $U, (-\partial_t + v \cdot D_x)U, D_v U, D_v^2 U \in L_q(\mathbb{R}^{1+2d})$. Note that by Lemma 5.9 and the aforementioned change of variables, $U = 0$ a.e. on $(T, \infty) \times \mathbb{R}^{2d}$.

Furthermore, for a measurable function h on $\mathbb{R}^{1+2d}$, we denote

$$Th(z) = h(t, x - vt, v).$$

By the change of variables $x \to x - vt$ and the chain rule for distributions, we have

$$\int u(-\partial_t U + v \cdot D_x U) \, dz = -\int Tu \, T(\partial_t U - v \cdot D_x U) \, dz$$
$$= -\int Tu \, \partial_t (TU) \, dz.$$

Since $Tu, \partial_t(Tu) \in L_p(\mathbb{R}^{1+2d})$, $TU, \partial_t(TU) \in L_q(\mathbb{R}^{1+2d})$, we may integrate by parts and obtain

$$\int u(-\partial_t U + v \cdot D_x U) \, dz = \int (\partial_t Tu) TU \, dz$$
$$= \int T(\partial_t u - v \cdot D_x u) TU \, dz = \int (\partial_t u - v \cdot D_x u) U \, dz.$$

By this and integration by parts, we get

$$\mathfrak{I} := \int_{\mathbb{R}_T^{1+2d}} u\phi \, dz = \int_{\mathbb{R}^{1+2d}} u(-\partial_t U + v \cdot D_x U - a^{ij}(t) D_{v_i v_j} U + \lambda U) \, dz$$
$$= \int_{\mathbb{R}_T^{1+2d}} U(P_0 u + \lambda u) \, dz.$$

By Hölder's inequality and Lemma 5.9,

$$|\mathfrak{I}| \leqq N \delta^{-\theta} \lambda^{-1} \| -\partial_t U + v \cdot D_x U - a^{ij}(t) D_{v_i v_j} U + \lambda U \|_{L_q(\mathbb{R}^{1+2d})}$$
$$\times \| P_0 u + \lambda u \|_{L_p(\mathbb{R}_T^{1+2d})} = N \lambda^{-1} \delta^{-\theta} \| \phi \|_{L_q(\mathbb{R}_T^{1+2d})} \| P_0 u + \lambda u \|_{L_p(\mathbb{R}_T^{1+2d})}.$$

This gives

$$\lambda \|u\|_{L_p(\mathbb{R}_T^{1+2d})} \leqq N\delta^{-\theta} \|P_0 u + \lambda u\|_{L_p(\mathbb{R}_T^{1+2d})}. \tag{5.44}$$

Next, by the solvability when $T = \infty$, the equation

$$P_0 u_1 + \lambda u_1 = (P_0 u + \lambda u) 1_{t < T}$$

has a unique solution $u_1 \in S_p(\mathbb{R}^{1+2d})$. Then, by (5.1) with $T = \infty$,

$$\lambda \|u_1\|_{L_p(\mathbb{R}_T^{1+2d})} + \lambda^{1/2} \|D_v u_1\|_{L_p(\mathbb{R}_T^{1+2d})} + \|D_v^2 u_1\|_{L_p(\mathbb{R}_T^{1+2d})}$$
$$+ \|(-\Delta_x)^{1/3} u_1\|_{L_p(\mathbb{R}_T^{1+2d})} + \|D_v(-\Delta_x)^{1/6} u_1\|_{L_p(\mathbb{R}_T^{1+2d})}$$
$$\leqq N\delta^{-\theta} \|P_0 u + \lambda u\|_{L_p(\mathbb{R}_T^{1+2d})}.$$

It follows from (5.44) that $u_1 = u$ a.e. in $\mathbb{R}_T^{1+2d}$. Thus, the desired estimate holds for u.

(ii) Similar to the proof of Theorem 4.1 (ii), this claim follows from the a priori estimate in the assertion (i) when $p \in (1, 2), T < \infty$ and the solvability in the assertion (ii) when $p \in (1, 2), T = \infty$. $\square$

6. Mixed-Norm Estimate for the Model Equation

In this section, we prove the following theorem, which is Theorem 2.6 for the operator P_0. We follow the argument of Theorem 5.1 and make only minor adjustments.

Theorem 6.1. *Invoke the assumptions of Theorem 2.6 and assume $b \equiv 0, c \equiv 0$. Then, for any number $\lambda \geqq 0$ and $u \in S_{p,r_1,\dots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$, one has*

$$\lambda \|u\| + \lambda^{1/2} \|D_v u\| + \|D_v^2 u\| + \|(-\Delta_x)^{1/3} u\| + \|D_v(-\Delta_x)^{1/6} u\|$$
$$\leqq N\delta^{-\theta} \|P_0 u + \lambda u\|, \tag{6.1}$$

where $\|\cdot\| = \|\cdot\|_{L_{p,r_1,\dots,r_d,q}(\mathbb{R}_T^{1+2d}, w)}$, $N = N(d, p, r_1, \dots, r_d, q, K)$, and $\theta = \theta(d, p, r_1, \dots, r_d, q, K) > 0$. Furthermore, the part of the Theorem 2.6 (iii) concerning the $S_{p;r_1,\dots,r_d}(\mathbb{R}_T^{1+2d}, |x|^\alpha \prod_{i=1}^d w_i(v_i))$ estimates is valid with P_0 in place of P for any $\lambda \geqq 0$.

Here is a generalization of Lemma 5.2.

Lemma 6.2. *Let $p > 1, R \geqq 1$ be numbers and $f \in L_p(\mathbb{R}_0^{1+2d})$ vanish outside $(-1, 0) \times \mathbb{R}^d \times B_1$. Let $u \in S_p((-1, 0) \times \mathbb{R}^{2d})$ be the unique solution to*

$$P_0 u = f, \quad u(-1, \cdot) = 0.$$

Then, there exist positive constants $\theta_0 = \theta_0(d), \theta = \theta(d)$, and $N = N(d, p)$ such that for $c_k = 2^k/\delta^{\theta_0}, k \geqq 0$,

$$\||u| + |D_v u| + |D_v^2 u|\|_{L_p((-1,0) \times B_{R^3} \times B_R)}$$

$$\leq N\delta^{-\theta}\sum_{k=0}^{\infty}2^{-k(k-1)/4}R^{-k}\|f\|_{L_p(Q_{1,2c_kR})},$$

$$(|(-\Delta_x)^{1/3}u|^p)^{1/p}_{Q_{1,R}}\leq N\delta^{-\theta}R^{-2}\sum_{k=0}^{\infty}2^{-2k}(|f|^p)^{1/p}_{Q_{1,c_kR}},$$

$$(|D_v(-\Delta_x)^{1/6}u|^p)^{1/p}_{Q_{1,R}}\leq N\delta^{-\theta}R^{-1}\sum_{k=0}^{\infty}2^{-k}(|f|^p)^{1/p}_{Q_{1,c_kR}}.$$

Proof. We repeat the proof of Lemma 5.2. Let us point out two minor modifications that we make. First, in the definition of cutoff functions $\xi_j(x, v)$, we replace δ^2 with $\delta^{\widetilde{\theta}+1}$, where $\widetilde{\theta} = \widetilde{\theta}(d) > 1$ is the constant from the a priori estimate in Theorem 5.1 (i). Second, one needs to use Theorem 5.1 instead of Theorem 4.1. □

Proposition 6.3. *Let* $p > 1$, $r > 0$, $v \geq 2$ *be numbers,* $z_0 \in \overline{\mathbb{R}^{1+2d}_T}$, *and* $u \in S_{p,loc}(\mathbb{R}^{1+2d}_T)$ *be a function such that* $P_0u = 0$ *in* $(t_0 - v^2r^2, t_0) \times \mathbb{R}^d \times B_{vr}(v_0)$. *Then, there exist positive constants* $N(d, p)$ *and* $\theta = \theta(d)$ *such that*

$$\left(|(-\Delta_x)^{1/3}u - ((-\Delta_x)^{1/3}u)_{Q_r(z_0)}|^p\right)^{1/p}_{Q_r(z_0)}\leq Nv^{-1}\delta^{-\theta}\left(|(-\Delta_x)^{1/3}u|^p\right)^{1/p}_{Q_{vr}(z_0)},$$

$$\left(|D_v(-\Delta_x)^{1/6}u - (D_v(-\Delta_x)^{1/6}u)_{Q_r(z_0)}|^p\right)^{1/p}_{Q_r(z_0)}$$

$$\leq Nv^{-1}\delta^{-\theta}(|D_v(-\Delta_x)^{1/6}u|^p)^{1/p}_{Q_{vr}(z_0)}$$

$$+ Nv^{-1}\delta^{-\theta}\sum_{k=0}^{\infty}2^{-2k}(|(-\Delta_x)^{1/3}u|^p)^{1/p}_{Q_{vr,2^kvr}(z_0)},$$

$$\left(|D_v^2u - (D_v^2u)_{Q_r(z_0)}|^p\right)^{1/p}_{Q_r(z_0)}\leq Nv^{-1}\delta^{-\theta}(|D_v^2u|^p)^{1/p}_{Q_{vr}(z_0)}$$

$$+ Nv^{-1}\delta^{-\theta}\sum_{k=0}^{\infty}2^{-k}(|(-\Delta_x)^{1/3}u|^p)^{1/p}_{Q_{vr,2^kvr}(z_0)}.$$

6.1. Proof of Proposition 6.3

First, we need a localized L_p estimate, which we prove by repeating the argument of Lemma 4.5 and replacing Theorem 4.1 with Theorem 5.1.

Lemma 6.4. *Lemma 5.11 holds for any* $p \in (1, \infty)$.

The next two lemmas are generalizations of Lemmas 5.4 and 5.5 , respectively. Their proofs go along the same lines as in the lemmas in Section 5. One minor adjustment one needs to make is to replace Theorem 4.1 and Lemma 4.5 with Theorem 5.1 and Lemma 6.4, respectively.

Lemma 6.5. *Let $0 < r < R \leq 1$ and $p \in (1, \infty)$ be numbers, and $u \in S_{p,loc}(\mathbb{R}_0^{1+2d})$ be a function such that $P_0 u = 0$ in Q_1. Then, there exist constants $N = N(d, p, r, R)$ and $\theta = \theta(d) > 0$ such that*

$$\|D_x u\|_{L_p(Q_r)} \leq N \delta^{-\theta} \|u\|_{L_p(Q_R)}.$$

Lemma 6.6. *Let $p \in (1, \infty)$ and $u \in S_{p,loc}(\mathbb{R}_0^{1+2d})$ be a function such that $P_0 u = 0$ in $(-1, 0) \times \mathbb{R}^d \times B_1$. Then, for any $r \in (0, 1)$, we have*

$$\|D_x u\|_{L_p(Q_r)} \leq N \delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} (|(-\Delta_x)^{1/3} u|^p)_{Q_{1,2^k}}^{1/p},$$

$$\|D_x u\|_{L_p(Q_r)} \leq N \delta^{-\theta} \sum_{k=0}^{\infty} 2^{-2k} (|(-\Delta_x)^{1/6} u|^p)_{Q_{1,2^k}}^{1/p},$$

where $N = N(d, p, r)$ and $\theta = \theta(d) > 0$.

Lemma 6.7. *Under the assumptions of Lemma 6.6, for any $l, m \in \{0, 1, \ldots\}$, there exists a constant $\theta = \theta(d, p, l, m) > 0$ such that the following assertions hold:*

(i) *For any $R \in (1/2, 1]$,*

$$\sup_{Q_{1/2}} |D_x^l D_v^m u| + \sup_{Q_{1/2}} |\partial_t D_x^l D_v^m u| \leq N(d, p, l, m, R) \delta^{-\theta} \|u\|_{L_p(Q_R)},$$

(ii) $\sup_{Q_{1/2}} (|D_x^l D_v^{m+1} (-\Delta_x)^{1/6} u| + |\partial_t D_x^l D_v^{m+1} (-\Delta_x)^{1/6} u|)$

$$\leq N(d, p, l, m) \delta^{-\theta} \left(\|D_v (-\Delta_x)^{1/6} u\|_{L_p(Q_1)} \right.$$

$$\left. + \sum_{k=0}^{\infty} 2^{-2k} (|(-\Delta_x)^{1/3} u|^p)_{Q_{1,2^k}}^{1/p} \right),$$

(iii) $\sup_{Q_{1/2}} |D_x^l D_v^{m+2} u| + \sup_{Q_{1/2}} |\partial_t D_x^l D_v^{m+2} u| \leq N(d, p, l, m) \delta^{-\theta} \|D_v^2 u\|_{L_p(Q_1)}$

$$+ N(d, p, l, m) \delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} (|(-\Delta_x)^{1/3} u|^p)_{Q_{1,2^k}}^{1/p}.$$

Proof. (i) The proof is almost identical to that of Lemma 5.6 (i). By the induction argument, for any $j \in \{0, 1\}$, we get

$$\|\partial_t^j D_x^l D_v^m u\|_{L_p(Q_{1/2})} \leq N(d, p, j, l, m, R) \delta^{-\theta} \|u\|_{L_p(Q_R)},$$

where $\theta = \theta(d, j, l, m) > 0$. Now the assertion follows from the last inequality, (5.24) and the Sobolev embedding theorem.

(ii), (iii) The proof is almost the same as the one of Lemma 5.6 (ii), (iii). One merely needs to replace Lemma 5.5 with Lemma 6.6 in this argument. □

Proof of Proposition 6.3. The assertion follows from Lemma 6.7 and the scaling argument (see Lemma 3.6). □

6.2. Proof of Theorem 6.1

Proposition 6.8. *Let $p > 1$, $r > 0$, $\nu \geqq 2$ be numbers, $z_0 \in \overline{\mathbb{R}_T^{1+2d}}$, and $u \in S_p(\mathbb{R}_T^{1+2d})$.*

Then, there exist positive constants $N = N(d, p)$, $\theta_0 = \theta_0(d)$, and $\theta = \theta(d, p)$ such that for $c_k = 2^k/\delta^{\theta_0}$, $k \geqq 0$, and any $z_0 \in \overline{\mathbb{R}_T^{1+2d}}$,

$$\left(\left| (-\Delta_x)^{1/3} u - ((-\Delta_x)^{1/3} u)_{Q_r(z_0)} \right|^p \right)_{Q_r(z_0)}^{1/p}$$

$$\leqq N\nu^{-1}\delta^{-\theta} \left(|(-\Delta_x)^{1/3} u|^p \right)_{Q_{\nu r}(z_0)}^{1/p}$$

$$+ N\nu^{(4d+2)/p}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-2k} (|P_0 u|^p)_{Q_{2\nu r, 2\nu r c_k}(z_0)}^{1/p},$$

$$\left(\left| D_v(-\Delta_x)^{1/6} u - (D_v(-\Delta_x)^{1/6} u)_{Q_r(z_0)} \right|^p \right)_{Q_r(z_0)}^{1/p}$$

$$\leqq N\nu^{-1}\delta^{-\theta} (|D_v(-\Delta_x)^{1/6} u|^p)_{Q_{\nu r}(z_0)}^{1/p}$$

$$+ N\nu^{-1}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-2k} (|(-\Delta_x)^{1/3} u|^p)_{Q_{\nu r, 2^k \nu r}(z_0)}^{1/p}$$

$$+ N\nu^{(4d+2)/p}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} (|P_0 u|^p)_{Q_{2\nu r, 2\nu r c_k}(z_0)}^{1/p},$$

$$(|D_v^2 u - (D_v^2 u)_{Q_r(z_0)}|^p)_{Q_r(z_0)}^{1/p} \leqq N\nu^{-1}\delta^{-\theta} (|D_v^2 u|^p)_{Q_{\nu r}(z_0)}^{1/p}$$

$$+ N\nu^{-1}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} (|(-\Delta_x)^{1/3} u|^p)_{Q_{\nu r, 2^k \nu r}(z_0)}^{1/p}$$

$$+ N\nu^{(4d+2)/p}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} (|P_0 u|^p)_{Q_{2\nu r, 2\nu r c_k}(z_0)}^{1/p}.$$

Proof. The assertion is derived from Lemma 6.2 and Proposition 6.3. See the proof of Proposition 5.7 in Section 5.1. □

Proof of Theorem 6.1. First, we consider the case when the weight depends on t, v variables only. In this proof, N is a constant depending only on d, p, $r_1, \ldots, r_d$, q, and K.

Step 1: Case $\lambda = 0$ and u vanishing for large $|z|$. By Lemma A.2 and the self-improving property of the A_p-weights (see, for instance, Corollary 7.2.6 of [19]), there exists a number

$$p_0 = p_0(d, p, r_1, \ldots, r_d, q, K), \quad 1 < p_0 < \min\{p, r_1, \ldots, r_d, q\},$$

such that $u \in S_{p_0, \text{loc}}(\mathbb{R}_T^{1+2d})$,

$$w_0 \in A_{q/p_0}(\mathbb{R}), \quad w_i \in A_{r_i/p_0}(\mathbb{R}), \quad i = 1, \ldots, d. \tag{6.2}$$

Denote $f = P_0 u$. By what was said above and the fact that u is compactly supported, we have $u \in S_{p_0}(\mathbb{R}_T^{1+2d})$. Then, by Proposition 6.8 with p replaced with p_0, for any $z_0 \in \overline{\mathbb{R}_T^{1+2d}}$, we get

$$((-\Delta_x)^{1/3} u)_T^\#(z_0) \leq N\nu^{-1}\delta^{-\theta} \mathcal{M}_T^{1/p_0} |(-\Delta_x)^{1/3} u|^{p_0}(z_0)$$
$$+ N\nu^{(4d+2)/p_0}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-2k} \mathbb{M}_{2^k/\delta^{\theta_0},T}^{1/p_0} |f|^{p_0}(z_0),$$

$$(D_v(-\Delta_x)^{1/6} u)_T^\#(z_0) \leq N\nu^{-1}\delta^{-\theta} \mathcal{M}_T^{1/p_0} |D_v(-\Delta_x)^{1/6} u|^{p_0}(z_0)$$
$$+ N\nu^{-1}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-2k} \mathbb{M}_{2^k,T}^{1/p_0} |(-\Delta_x)^{1/3} u|^{p_0}(z_0)$$
$$+ N\nu^{(4d+2)/p_0}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} \mathbb{M}_{2^k/\delta^{\theta_0},T}^{1/p_0} |f|^{p_0}(z_0),$$

and

$$(D_v^2 u)_T^\#(z_0) \leq N\nu^{-1}\delta^{-\theta} \mathcal{M}_T^{1/p_0} |D_v^2 u|^{p_0}(z_0)$$
$$+ N\nu^{-1}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} \mathbb{M}_{2^k,T}^{1/p_0} |(-\Delta_x)^{1/3} u|^{p_0}(z_0)$$
$$+ N\nu^{(4d+2)/p_0}\delta^{-\theta} \sum_{k=0}^{\infty} 2^{-k} \mathbb{M}_{2^k/\delta^{\theta_0},T}^{1/p_0} |f|^{p_0}(z_0).$$

We take the $\|\cdot\|$-norm on both sides of the above inequalities and use the Minkowski inequality. After that we apply Corollary 3.2 (i) with $p/p_0, r_1/p_0, \ldots, r_d/p_0,$ $q/p_0 > 1$ and (6.2) combined with Corollary 3.2 (ii). We obtain

$$\|(-\Delta_x)^{\frac{1}{3}} u\| \leq N\nu^{-1}\delta^{-\theta} \|(-\Delta_x)^{\frac{1}{3}} u\| + N\nu^{(4d+2)/p_0}\delta^{-\theta} \|f\|,$$
$$\|D_v(-\Delta_x)^{\frac{1}{6}} u\| \leq N\nu^{-1}\delta^{-\theta} (\|D_v(-\Delta_x)^{\frac{1}{6}} u\| + \|(-\Delta_x)^{\frac{1}{3}} u\|)$$
$$+ N\nu^{(4d+2)/p_0}\delta^{-\theta} \|f\|,$$
$$\|D_v^2 u\| \leq N\nu^{-1}\delta^{-\theta} (\|D_v^2 u\| + \|(-\Delta_x)^{\frac{1}{3}} u\|) + N\nu^{(4d+2)/p_0}\delta^{-\theta} \|f\|.$$

Taking $\nu = 2 + 2N\delta^{-\theta}$, we prove the desired estimate.

Step 2: Case $\lambda = 0$ and arbitrary $u \in S_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$. Let $\phi \in C_0^\infty(\mathbb{R}^{1+2d})$ be a function such that $\phi = 1$ on $\widetilde{Q}_1$ and denote for $n \geq 1$,

$$\phi_n(z) = \phi(t/n^2, x/n^3, v/n), \quad u_n = u\phi_n.$$

By the result of Step 1,

$$\|D_v^2 u_n\| + \|(-\Delta_x)^{1/3} u_n\| + \|D_v(-\Delta_x)^{1/6} u_n\| \leq N\delta^{-\theta} \|P_0 u_n\|. \quad (6.3)$$

Estimate of $D_v^2 u$. By (6.3),

$$\|D_v^2 u\|_{L_{p,r_1,\ldots,r_d,q}(\widetilde{Q}_n \cap \mathbb{R}_T^{1+2d}, w)} \leq N\delta^{-\theta} (\|P_0 u\| + A_1 + A_2 + A_3),$$

where

$$A_1 = \|P_0 u(\phi_n - 1)\|, \quad A_2 = \|(P_0 \phi_n)u\|, \quad A_3 = 2\|(a D_v u) \cdot D_v \phi_n\|.$$

Since $u \in S_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}, w)$, by the dominated convergence theorem,

$$A_1 \to 0 \quad \text{as } n \to \infty. \tag{6.4}$$

Furthermore,

$$A_2 + A_3 \leq N\delta^{-1}n^{-1}(\|u\| + \|D_v u\|) \to 0 \quad \text{as } n \to \infty. \tag{6.5}$$

Thus, the estimate for $D_v^2 u$ holds.

Estimate of $(-\Delta_x)^{1/3}u$ and $D_v(-\Delta_x)^{1/6}u$. We use the duality argument as in the proof of Theorem 4.1 (i). It follows from (6.3)–(6.5) that

$$\|(-\Delta_x)^{1/3}u_n\| + \|D_v(-\Delta_x)^{1/6}u_n\| \leq N\delta^{-\theta}(\|P_0 u\| + n^{-1}\|\|u\| + |D_v u|\|). \tag{6.6}$$

Next, for any $\eta \in C_0^\infty(\mathbb{R}_T^{1+2d})$, we have

$$|(-\Delta_x)^{1/3}\eta|(z) \leq N(d)(1 + |x|)^{-d-2/3}, \tag{6.7}$$

and by this $(-\Delta_x)^{1/3}\eta \in L_{p^*,r_1^*,\ldots,r_d^*,q^*}(\mathbb{R}_T^{1+2d}, w^*)$ of $p, r_1, \ldots, r_d, q$ respectively, where $p^*, r_1^* \ldots, r_d^*, q^*$ are Hölder conjugates, and

$$w^* := w_0^{-1/(q-1)}(t) \prod_{i=1}^d w_i^{-1/(r_i-1)}(v_i).$$

Then, since $u_n \to u$ in $L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$, we have

$$\left| \int_{\mathbb{R}_T^{1+2d}} \eta(-\Delta_x)^{1/3}u \, dz \right| \leq \|\eta\|_{L_{p^*,r_1^*,\ldots,r_d^*,q^*}(\mathbb{R}_T^{1+2d},w^*)} \varlimsup_{n\to\infty} \|(-\Delta_x)^{1/3}u_n\|.$$

This combined with (6.6) implies (6.1) for $(-\Delta_x)^{1/3}u$. In the same way, we prove the estimate for $D_v(-\Delta_x)^{1/6}u$.

Step 3: Case $\lambda > 0$. We set $r_{d+1} = r_d$ and $w_{d+1} \equiv 1$. Repeating the proof of Lemma 5.9 with the spaces $L_p(\mathbb{R}_T^{1+2d})$ and $L_p(\mathbb{R}_T^{3+2d})$ replaced with

$$L_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w) \quad \text{and} \quad L_{p,r_1,\ldots,r_{d+1},q}(\mathbb{R}_T^{3+2d}, w),$$

respectively, we conclude that for $\lambda \geq \lambda_0 = 16N^2\delta^{-2\theta} + 1 > 0$,

$$\lambda\|u\| \leq N\delta^{-\theta}\|P_0 u + \lambda u\|.$$

To show that the desired estimate holds for $\lambda \geq 0$, we use a scaling argument (see Lemma 3.6) and the fact that the map $x \to \lambda x$ preserves the A_p constant. This combined with (6.1) with $\lambda = 0$ proves the desired estimate.

For the estimate in $S_{p;r_1,...,r_d}(\mathbb{R}_T^{1+2d}, |x|^\alpha \prod_{i=1}^d w_i(v_i))$ (see p. 5), the proof goes along the lines of the above argument. Let us point out one minor modification: due to (6.7) and the fact that $\alpha \in (-1, p - 1)$, one has

$$(-\Delta_x)^{1/3}\eta \in L_{p^*;r_1^*,...,r_d^*}(\mathbb{R}_T^{1+2d}, |x|^{-\alpha/(p-1)} \prod_{i=1}^d w_i^{-1/(r_i-1)}(v_i))$$

for any $\eta \in C_0^\infty(\mathbb{R}_T^{1+2d})$.   □

7. Proof of Theorem 2.6

First, we prove a key lemma analogous to Proposition 6.8, which will imply the a priori estimate of $D_v^2 u$ for $u \in S_{p,r_1,...,r_d,q}(\mathbb{R}_T^{1+2d}, w)$.

Lemma 7.1. *Let $\gamma_0 > 0$, $v \geqq 2$, $p_0 \in (1, \infty)$, $\alpha \in (1, 3/2)$ be numbers, $T \in (-\infty, \infty]$, R_0 be the constant from Assumption 2.3 (γ_0), and $u \in S_{p_0}(\mathbb{R}_T^{1+2d})$. Then, under Assumptions 2.2–2.3 (γ_0), there exist positive constants $\theta_0 = \theta_0(d)$, $\theta = \theta(d, p_0)$, $N = N(d, p_0, \alpha)$, and a sequence of positive numbers $\{a_k, k \geqq 0\}$ such that*

$$\sum_{k=0}^\infty a_k \leqq N,$$

and for $c_k = 2^k/\delta^{\theta_0}$ and any $z \in \overline{\mathbb{R}_T^{1+2d}}$, and $r \in (0, R_0/(4v))$,

$$
\begin{aligned}
(|D_v^2 u - (D_v^2 u)_{Q_r(z)}|^{p_0})_{Q_r(z)}^{1/p_0} &\leqq Nv^{-1}\delta^{-\theta}(|D_v^2 u|^{p_0})_{Q_{vr}(z)}^{1/p_0} \\
&+ Nv^{-1}\delta^{-\theta} \sum_{k=0}^\infty 2^{-k}(|(-\Delta_x)^{1/3}u|^{p_0})_{Q_{vr,2^k vr}(z)}^{1/p_0} \\
&+ Nv^{(2+4d)/p_0}\delta^{-\theta} \sum_{k=0}^\infty 2^{-k}(|Pu|^{p_0})_{Q_{2vr,2vrc_k}(z)}^{1/p_0} \\
&+ Nv^{(2+4d)/p_0}\delta^{-\theta}\gamma_0^{(\alpha-1)/(p_0\alpha)} \sum_{k=0}^\infty a_k(|D_v^2 u|^{p_0\alpha})_{Q_{2vr,2vrc_k}(z)}^{1/(p_0\alpha)}.
\end{aligned}
\tag{7.1}
$$

To prove the above lemma we need the following result:

Lemma 7.2. *Let $\gamma_0 > 0$ be a number and R_0 be the constant from Assumption 2.3 (γ_0). Let $r \in (0, R_0/2)$, $c > 0$ be numbers. Then, one has*

$$I := \fint_{Q_{r,cr}} |a(t, x, v) - (a(t, \cdot, \cdot))_{B_{r^3} \times B_r}| \, dz \leqq N(d)c^3\gamma_0.$$

Proof. Let $X \subset B_{(cr)^3}$ be a finite set such that $\{B_{r^3/2}(x), x \in X\}$ is a maximal family of disjoint balls. Then, since $B_{(cr)^3} \subset \cup_{x \in X} B_{r^3}(x)$, we have

$$
\begin{aligned}
I &\leqq |Q_{r,cr}|^{-1} \sum_{x \in X} \int_{Q_r(0,x,0)} |a(z) - (a(t,\cdot,\cdot))_{B_{r^3} \times B_r}| \, dz \\
&\leqq |Q_{r,cr}|^{-1} \sum_{x \in X} (B(x) + C(x)),
\end{aligned}
$$

where

$$
B(x) = \int_{Q_r(0,x,0)} |a(z) - (a(t,\cdot,\cdot))_{B_{r^3}(x) \times B_r}| \, dz,
$$

$$
C(x) = \int_{Q_r(0,x,0)} |(a(t,\cdot,\cdot))_{B_{r^3}(x) \times B_r} - (a(t,\cdot,\cdot))_{B_{r^3} \times B_r}| \, dz.
$$

By the facts that $D_r((0,x,0),t) = B_{r^3}(x) \times B_r$ and $r < R_0$, and Assumption 2.3 (γ_0), we have

$$
B(x) \leqq |Q_r| \gamma_0. \tag{7.2}
$$

Furthermore, for any $x \in X$ such that $x \neq 0$, let x_j, $j = 0, 1, \ldots, m$, be a sequence of points such that $x_0 = 0$, $x_m = x$, and $|x_j - x_{j+1}| \leqq r^3$ for $j = 0, \ldots, m-1$, where $m \leqq N(d)c^3$. Then, by the triangle inequality, we have

$$
\begin{aligned}
C(x) &\leqq \sum_{j=0}^{m-1} \int_{Q_r(0,x,0)} |(a(t,\cdot,\cdot))_{B_{r^3}(x_{j+1}) \times B_r} - (a(t,\cdot,\cdot))_{B_{r^3}(x_j) \times B_r}| \, dz \\
&\leqq \sum_{j=0}^{m-1} \int_{Q_r(0,x,0)} |(a(t,\cdot,\cdot))_{B_{r^3}(x_{j+1}) \times B_r} - (a(t,\cdot,\cdot))_{B_{8r^3}(x_j) \times B_{2r}}| \, dz \\
&+ \sum_{j=0}^{m-1} \int_{Q_r(0,x,0)} |(a(t,\cdot,\cdot))_{B_{8r^3}(x_j) \times B_{2r}} - (a(t,\cdot,\cdot))_{B_{r^3}(x_j) \times B_r}| \, dz.
\end{aligned}
$$

It is easy to check that for any two sets $A \subset A' \subset \mathbb{R}^d$ of positive finite Lebesgue measure and any $f \in L_{1,\text{loc}}(\mathbb{R}^d)$, one has

$$
|(f)_A - (f)_{A'}| \leqq \frac{|A'|}{|A|} (|f - (f)_{A'}|)_{A'}.
$$

By this, the fact that $2r < R_0$, and Assumption 2.3 (γ_0),

$$
\begin{aligned}
C(x) &\leqq |Q_r| N(d) \sum_{j=0}^{m-1} \fint_{Q_{2r}(0,x,0)} |a(z) - (a(t,\cdot,\cdot))_{B_{8r^3}(x_j) \times B_{2r}}| \, dz \\
&\leqq N(d) |Q_r| m \gamma_0 \leqq N(d) |Q_r| c^3 \gamma_0.
\end{aligned}
$$

Combining this with (7.2) and using the fact that $|X| \leqq N(d)c^{3d}$, we prove the lemma. □

Proof of Lemma 7.1. Clearly, we may assume that for any $k \geq 0$,

$$D_v^2 u \in L_{p_0 \alpha}(Q_{2vr, 2vrc_k})$$

because otherwise the sum on the right-hand side of (7.1) is infinite, and the inequality holds trivially. Furthermore, by Lemma 3.6, it suffices to prove (7.1) for $z = 0$. We denote

$$\bar{a}(t) = (a(t, \cdot, \cdot))_{B_{v^3 r^3} \times B_{vr}} \quad \text{and} \quad \bar{P} = \partial_t - v \cdot D_x - \bar{a}^{ij} D_{v_i v_j}.$$

By Proposition 6.8, there exist positive numbers $\theta_0 = \theta_0(d)$, $\theta = \theta(d, p)$, and $N = N(d, p_0)$ such that

$$
\begin{aligned}
(|D_v^2 u - (D_v^2 u)_{Q_r}|^{p_0})_{Q_r}^{1/p_0} &\leq N\nu^{-1}\delta^{-\theta}\left(|D_v^2 u|^{p_0}\right)_{Q_{vr}}^{1/p_0} \\
&+ N\nu^{-1}\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-k}(|(-\Delta_x)^{1/3} u|^{p_0})_{Q_{vr,2^k vr}}^{1/p_0} \\
&+ N\nu^{(4d+2)/p_0}\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-k}(|Pu|^{p_0})_{Q_{2vr,2vrc_k}}^{1/p_0} \\
&+ N\nu^{(4d+2)/p_0}\delta^{-\theta}\sum_{k=0}^{\infty} 2^{-k}(|a - \bar{a}|^{p_0}|D_v^2 u|^{p_0})_{Q_{2vr,2vrc_k}}^{1/p_0},
\end{aligned}
\tag{7.3}
$$

where $c_k = 2^k/\delta^{\theta_0}$.

Fix any $k \in \{0, 1, 2, \ldots\}$ and denote $\alpha_1 = \alpha/(\alpha - 1)(> 3)$. Then, by Hölder's inequality,

$$
\begin{aligned}
(|a - \bar{a}|^{p_0}|D_v^2 u|^{p_0})_{Q_{2vr,2vrc_k}}^{1/p_0} &\\
\leq (|a - \bar{a}|^{p_0 \alpha_1})_{Q_{2vr,2vrc_k}}^{1/(p_0 \alpha_1)} (|D_v^2 u|^{p_0 \alpha})_{Q_{2vr,2vrc_k}}^{1/(p_0 \alpha)} &\\
=: A_1^{1/(p_0 \alpha_1)} A_2^{1/(p_0 \alpha)}.
\end{aligned}
\tag{7.4}
$$

Since a is a bounded function, we have

$$A_1 \leq N\delta^{-\theta} \fint_{Q_{2vr,2vrc_k}} |a - \bar{a}|\, dz.$$

Then, by Lemma 7.2 with r replaced with $2vr$ and $c = 2^k/\delta^{\theta_0}$,

$$A_1 \leq N(d)2^{3k}\delta^{-3\theta_0}\gamma_0.$$

By this and (7.4), we conclude that

$$
\begin{aligned}
2^{-k}(|a - \bar{a}|^{p_0}|D_v^2 u|^{p_0})_{Q_{2vr,2vrc_k}}^{1/p_0} &\\
\leq N\delta^{-\theta}\gamma_0^{1/(p_0 \alpha_1)}2^{-k+3k/(p_0\alpha_1)}(|D_v^2 u|^{p_0\alpha})_{Q_{2vr,2vrc_k}}^{1/(p_0\alpha)}.
\end{aligned}
$$

With $a_k := 2^{-k+3k/(p_0\alpha_1)}$, the series converges since $p_0\alpha_1 > 3$. Now the assertion follows from this and (7.3). $\square$

Proof of Theorem 2.6 (i). We will focus on the case when the weight depends only on t and v. The estimate in $S_{p;r_1,\ldots,r_d}(\mathbb{R}_T^{1+2d}, |x|^\alpha \prod_{i=1}^d w_i(v_i))$ is proved in the same way by using a partition of unity in the v_d variable and the interpolation inequality in the weighted mixed-norm space to absorb the norm of the gradient on the right-hand side of the a priori estimate.

First, we consider the case $T = \infty$. We follow the idea of Section 5.1 of [15]. In the first step, we prove a priori estimate for a function u with a sufficiently small support in the temporal variable. Then, we use partition of unity to handle an arbitrary function $u \in S_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$. Throughout the proof, we assume that $N = N(d, p, r_1, \ldots, r_d, q, K, L)$.

Step 1 We show that there exists $\beta = \beta(d, p, r_1, \ldots, r_d, q, K) > 0$,

$$R_1 = \delta^\beta \widetilde{R}_1(d, p, r_1, \ldots, r_d, q, K) > 0, \quad \gamma_0 = \delta^\beta \widetilde{\gamma}_0(d, p, r_1, \ldots, r_d, q, K) > 0,$$
$$\text{and} \quad \theta = \theta(d, p, r_1, \ldots, r_d, q, K) > 0, \tag{7.5}$$

such that for any $t_0 \in \mathbb{R}$ and $u \in S_{p,r_1,\ldots,r_d,q}(\mathbb{R}_T^{1+2d}, w)$ vanishing outside $(t_0 - (R_0 R_1)^2, t_0) \times \mathbb{R}^{2d}$,

$$\|D_v^2 u\| + \|(-\Delta_x)^{1/3} u\| + \|D_v(-\Delta_x)^{1/6} u\| \leqq N(d, p, r_1, \ldots, r_d, q, K)\delta^{-\theta}\|Pu\|.$$

By Lemma A.2, there exists a number

$$p_0 = p_0(d, p, r_1, \ldots, r_d, q, K)$$

such that

$$1 < p_0 < \min\{p, r_1, \ldots, r_d, q\}$$

and $u \in S_{p_0,\text{loc}}(\mathbb{R}_T^{1+2d})$. Since u is assumed to be compactly supported, this gives $u \in S_{p_0}(\mathbb{R}_T^{1+2d})$. By the self-improving property of the A_p-weights (see, for instance, Corollary 7.2.6 of [19]), we also fix a number α and p_0 further smaller such that

$$1 < \alpha < \min\left\{\frac{3}{2}, \frac{p}{p_0}, \frac{r_1}{p_0}, \ldots, \frac{r_d}{p_0}, \frac{q}{p_0}\right\}$$

and

$$w_0 \in A_{\frac{q}{\alpha p_0}}(\mathbb{R}), \quad w_i \in A_{\frac{r_i}{\alpha p_0}}(\mathbb{R}), \quad i = 1, \ldots, d. \tag{7.6}$$

Let $\nu \geqq 2$, γ_0, and R_1 be some numbers which will be chosen later.

If $4\nu r \geqq R_0$, then by Hölder's inequality with α and $\alpha_1 = \alpha/(\alpha - 1)$, for any $z \in \overline{\mathbb{R}_T^{1+2d}}$ we have

$$\begin{aligned}
(|D_v^2 u - (D_v^2 u)_{Q_r(z)}|^{p_0})_{Q_r(z)}^{1/p_0} &\leqq 2(|D_v^2 u|^{p_0})_{Q_r(z)}^{1/p_0} \\
&\leqq 2(I_{(t_0-(R_0 R_1)^2, t_0)})_{Q_r(z)}^{1/(p_0\alpha_1)}(|D_v^2 u|^{p_0\alpha})_{Q_r(z)}^{1/(p_0\alpha)} \\
&\leqq 2(R_1 R_0 r^{-1})^{2/(p_0\alpha_1)}\mathcal{M}_T^{1/(p_0\alpha)}|D_v^2 u|^{p_0\alpha}(z)
\end{aligned}$$

$$\leq N \nu^{2/(p_0\alpha_1)} R_1^{2/(p_0\alpha_1)} \mathcal{M}_T^{1/(p_0\alpha)} |D_v^2 u|^{p_0\alpha}(z).$$

In the case when $4\nu r < R_0$, we use Lemma 7.1 with p replaced with p_0. Combining these cases, we get in $\overline{\mathbb{R}_T^{1+2d}}$,

$$
\begin{aligned}
(D_v^2 u)_T^\# &\leq N(\nu^{-1}\delta^{-\theta} + \nu^{2/(p_0\alpha_1)} R_1^{2/(p_0\alpha_1)}) \mathcal{M}_T^{1/p_0} |D_v^2 u|^{p_0} \\
&\quad + N\nu^{-1}\delta^{-\theta} \sum_{k=0}^\infty 2^{-k} \mathbb{M}_{2^k,T}^{1/p_0} |(-\Delta_x)^{1/3}u|^{p_0} \\
&\quad + N\nu^{(4d+2)/p_0}\delta^{-\theta}\gamma_0^{1/(p_0\alpha_1)} \sum_{k=0}^\infty a_k \mathbb{M}_{2^k/\delta^{\theta_0},T}^{1/(p_0\alpha)} |D_v^2 u|^{p_0\alpha} \\
&\quad + N\nu^{(4d+2)/p_0}\delta^{-\theta} \sum_{k=0}^\infty 2^{-k} \mathbb{M}_{2^k/\delta^{\theta_0},T}^{1/p_0} |Pu|^{p_0}.
\end{aligned}
$$

We take the $\|\cdot\|$-norm of both sides of this inequality, use Corollary 3.2 (ii) with $p, r_1, \ldots, r_d, q > 1$ and Corollary 3.2 (i) with

$$p/(p_0\alpha), r_1/(p_0\alpha), \ldots, r_d/(p_0\alpha), q/(p_0\alpha) > 1$$

and (7.6). An application of the Minkowski inequality gives

$$
\begin{aligned}
\|D_v^2 u\| &\leq N(\nu^{-1}\delta^{-\theta} + \nu^{2/(p_0\alpha_1)} R_1^{2/(p_0\alpha_1)} + \nu^{(4d+2)/p_0}\delta^{-\theta}\gamma_0^{1/(p_0\alpha_1)})\|D_v^2 u\| \\
&\quad + N\nu^{-1}\delta^{-\theta}\|(-\Delta_x)^{1/3}u\| + N\nu^{(4d+2)/p_0}\delta^{-\theta}\|Pu\|. \tag{7.7}
\end{aligned}
$$

Furthermore, note that u solves the equation

$$\partial_t u - v \cdot D_x u - \Delta_v u = f, \tag{7.8}$$

where $f = Pu + (a^{ij} - \delta_{ij})D_{v_i v_j}$. Then, by Theorem 6.1 applied to (7.8),

$$\|(-\Delta_x)^{1/3}u\| + \|D_v(-\Delta_x)^{1/6}u\| \leq N\delta^{-\theta}(\|Pu\| + \|D_v^2 u\|). \tag{7.9}$$

Combining this with (7.7), we get

$$
\begin{aligned}
\|D_v^2 u\| &\leq N(\nu^{-1} + \nu^{(4d+2)/p_0}\gamma_0^{1/(p_0\alpha_1)})\delta^{-\theta}\|D_v^2 u\| \\
&\quad + N\nu^{2/(p_0\alpha_1)} R_1^{2/(p_0\alpha_1)}\|D_v^2 u\| + N(\nu^{-1} + \nu^{(4d+2)/p_0})\delta^{-\theta}\|Pu\|. \tag{7.10}
\end{aligned}
$$

Furthermore, we set $\nu = 2 + 4N\delta^{-\theta}$. Then we choose $\gamma_0 > 0$ and $R_1 > 0$ such that (7.5) holds and

$$N\nu^{(4d+2)/p_0}\delta^{-\theta}\gamma_0^{1/(p_0\alpha_1)} \leq 1/4, \quad N\nu^{2/(p_0\alpha_1)} R_1^{2/(p_0\alpha_1)} \leq 1/4.$$

Thus, we can cancel the term containing $D_v^2 u$ on the right-hand side of (7.10). By (7.9), we also obtain the estimate of $(-\Delta_x)^{1/3}u$ and $D_v(-\Delta_x)^{1/6}u$.

Step 2 Let $\zeta \in C_0^\infty((-(R_0 R_1)^2, 0))$ be a nonnegative function such that

$$\int \zeta^q(t)\, dt = 1, \quad |\zeta'| \leqq N(R_0 R_1)^{-2-2/q}. \tag{7.11}$$

Observe that for any $t \in \mathbb{R}$,

$$\|D_v^2 u(t, \cdot)\|_{L_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{2d}, \prod_{i=1}^d w_i)}^q$$
$$= \int_\mathbb{R} \|D_v^2 u(t, \cdot)\|_{L_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{2d}, \prod_{i=1}^d w_i)}^q \zeta^q(t - s)\, ds.$$

Multiplying the above inequality by w_0 and integrating over $\mathbb{R}$ give

$$\|D_v^2 u\|^q = \int_\mathbb{R} \|D_v^2\big(u\zeta(\cdot - s)\big)\|^q\, ds.$$

A similar identity holds for $\|(-\Delta_x)^{1/3} u\|$ and $\|D_v(-\Delta_x)^{1/6} u\|$. Furthermore, we fix arbitrary $s \in \mathbb{R}$ and note that $u_s(z) := u(z)\zeta(t - s)$ vanishes outside $(s - (R_0 R_1)^2, s)$ and satisfies the equation

$$P u_s(z) = \zeta(t - s) P u(z) + u\zeta'(t - s).$$

Then, by the conclusion of Step 1 and (7.11),

$$\|D_v^2 u_s\| + \|(-\Delta_x)^{1/3} u_s\| + \|D_v(-\Delta_x)^{1/6} u_s\|$$
$$\leqq N\delta^{-\theta} \|(Pu)\zeta(\cdot - s)\| + N\delta^{-\theta}(R_0 R_1)^{-2-2/q} \|u\phi(\cdot - s)\|,$$

where $\phi \in C_0^\infty((-(R_0 R_1)^2, 0))$ is such that $\phi = 1$ on the support of ζ and $\int |\phi|^q\, dt = N(R_0 R_1)^2$. Raising the above inequality to the q-th power and integrating with respect to s, we get

$$\|D_v^2 u\| + \|(-\Delta_x)^{1/3} u\| + \|D_v(-\Delta_x)^{1/6} u\|$$
$$\leqq N\delta^{-\theta} \|Pu\| + N\delta^{-\theta}(R_0 R_1)^{-2} \|u\|.$$

Due to (7.5), we may replace the last term with $N\delta^{-\theta} R_0^{-2} \|u\|$. By using Agmon's method (see the proof of Theorem 6.1), we conclude that, for any $\lambda \geqq 1$,

$$\lambda \|u\| + \|D_v^2 u\| + \|(-\Delta_x)^{1/3} u\| + \|D_v(-\Delta_x)^{1/6} u\|$$
$$\leqq N\delta^{-\theta} \|Pu + \lambda u\| + N\delta^{-\theta}(R_0^{-2} + \lambda^{1/2}) \|u\|.$$

Setting $\lambda_0 = 16(N\delta^{-\theta} R_0^{-2})^2 + 1$ and canceling the term containing $\|u\|$ on the right-hand side, we prove the desired a priori estimate.

Finally, in the case when b and c are not identically zero, we use the a priori estimate that we just proved. We get

$$\lambda \|u\| + \|D_v^2 u\| + \|(-\Delta_x)^{1/3} u\| + \|D_v(-\Delta_x)^{1/6} u\|$$
$$\leqq N\delta^{-\theta}(\|Pu + b^i D_{v_i} u + cu + \lambda u\| + \|D_v u\| + \|u\|).$$

By using the interpolation inequality in the weighted mixed-norm Sobolev spaces (see Lemma A.3) and further increasing λ_0, we prove the assertion. $\quad\square$

Next, we show the existence part.

Proposition 7.3. *Theorem 2.6 holds if $p = q = r_i$, $i = 1, \ldots, d$, and $w = 1$.*

Proof. By the method of continuity, the assertion follows from Theorems 2.6 (i) and 5.1 (ii).   □

Lemma 7.4. *Invoke the assumptions of Theorem 2.6 (ii) and assume, additionally, that f vanishes outside $\widetilde{Q}_R$ for some $R \geqq 1$. Let $\lambda_0 > 1$ be the constant from Theorem 2.6 (i). Fix any $\lambda \geqq \lambda_0$ and let $u \in S_p(\mathbb{R}^{1+2d})$ be the unique solution to Eq. (2.6) (see Proposition 7.3). Then, for any $j \in \{0, 1, 2, \ldots\}$ one has*

$$\lambda \|u\|_{L_p(\widetilde{Q}_{2^{j+1}R} \setminus \widetilde{Q}_{2^j R})} + \lambda^{1/2} \|D_v u\|_{L_p(\widetilde{Q}_{2^{j+1}R} \setminus \widetilde{Q}_{2^j R})} + \|D_v^2 u\|_{L_p(\widetilde{Q}_{2^{j+1}R} \setminus \widetilde{Q}_{2^j R})}$$
$$\leqq N 2^{-j(j-1)/4} R^{-j} \|f\|_{L_p(\mathbb{R}^{1+2d})},$$

where $N = N(d, \delta, p)$.

Proof. We follow the argument of Section 8 of [15], which is somewhat similar to the one of Lemma 5.2.

First, by Theorem 2.6 (i),

$$\lambda \|u\|_{L_p(\mathbb{R}^{1+2d})} + \lambda^{1/2} \|D_v u\|_{L_p(\mathbb{R}^{1+2d})} + \|D_v^2 u\|_{L_p(\mathbb{R}^{1+2d})} \qquad (7.12)$$
$$\leqq N \|f\|_{L_p(\mathbb{R}^{1+2d})}.$$

Furthermore, let $\{\eta_j, \ j = 0, 1, 2, \ldots\}$ be a sequence of smooth functions such that $\eta_j = 0$ in $\widetilde{Q}_{2^j R}$, $\eta_j = 1$ outside $\widetilde{Q}_{2^{j+1}R}$,

$$|\eta_j| \leqq 1, \quad |D_v \eta_j| \leqq N 2^{-j} R^{-1}, \quad |D_v^2 \eta_j| \leqq N 2^{-2j} R^{-2},$$
$$|D_x \eta_j| \leqq N 2^{-3j} R^{-3}, \quad |\partial_t \eta_j| \leqq N 2^{-2j} R^{-2}. \qquad (7.13)$$

Then, since $f = 0$ outside $\widetilde{Q}_R$, the function $u_j = u\eta_j$ satisfies the equation

$$Pu_j + b^i D_{v_i} u_j + cu_j + \lambda u_j = u(P\eta_j + b^i D_{v_i} \eta_j) - 2(aD_v \eta_j) \cdot D_v u.$$

By Theorem 2.6 (i), interpolation inequality, and (7.13), we get

$$\|\lambda |u| + \lambda^{1/2} |D_v u| + |D_v^2 u|\|_{L_p(\widetilde{Q}_{2^{j+2}R} \setminus \widetilde{Q}_{2^{j+1}R})}$$
$$\leqq \lambda \|u_j\|_{L_p(\mathbb{R}^{1+2d})} + \lambda^{1/2} \|D_v u_j\|_{L_p(\mathbb{R}^{1+2d})} + \|D_v^2 u_j\|_{L_p(\mathbb{R}^{1+2d})}$$
$$\leqq N 2^{-j} R^{-1} \||u| + |D_v u|\|_{L_p(\widetilde{Q}_{2^{j+1}R} \setminus \widetilde{Q}_{2^j R})}$$
$$\leqq N 2^{-j} R^{-1} (\lambda \|u\|_{L_p(\widetilde{Q}_{2^{j+1}R} \setminus \widetilde{Q}_{2^j R})} + \lambda^{1/2} \|D_v u\|_{L_p(\widetilde{Q}_{2^{j+1}R} \setminus \widetilde{Q}_{2^j R})}).$$

Iterating this estimate and using (7.12), we prove the assertion.   □

Proof of Theorem 2.6 (ii). We will only consider the case when the weight is a function of t, v because the result in

$$S_{p;r_1,\ldots,r_d}(\mathbb{R}_T^{1+2d}, |x|^\alpha \prod_{i=1}^d w_i(v_i))$$

is established in the same way. The uniqueness follows from Theorem 2.6 (i).

To prove the existence part, we first consider the case when $T = \infty$. By the reverse Hölder inequality for A_p-weights (see, for example, Theorem 7.2.5 of [19]) and the scaling property of A_p-weights (see Proposition 7.1.5 (9) of [19]), there exist a large constant $p_1 > 1$ and small constants $\varepsilon_i > 0, i = 0, \ldots, d$ depending only on $d, \delta, p, r_1, \ldots, r_d, q$, and K such that for any $R > 1$,

$$\frac{p_1}{q} = \frac{1+\varepsilon_0}{\varepsilon_0}, \quad \frac{p_1}{r_i} = \frac{1+\varepsilon_i}{\varepsilon_i}, \quad i = 1, \ldots, d,$$

$$\left(\fint_{-R^2}^0 w_0^{1+\varepsilon_0} \, dt \right)^{\frac{1}{1+\varepsilon_0}} \leqq N(q, K) \fint_{-R^2}^0 w_0 \, dt \leqq N(q, K) R^{2q-2} \int_{-1}^1 w_0 \, dt,$$

$$\left(\fint_{-R}^R w_i^{1+\varepsilon_i} \, dv_i \right)^{\frac{1}{1+\varepsilon_i}} \leqq N(r_i, K) \fint_{-R}^R w_i \, dv_i$$

$$\leqq N(r_i, K) R^{r_i-1} \int_{-1}^1 w_i \, dv_i, \quad i = 1, \ldots, d.$$

Then, applying Hölder's inequality repeatedly, we prove that, for any $R > 1$ and $h \in L_{p_1}(\widetilde{Q}_R)$,

$$\|h\|_{L_{p,r_1,\ldots,r_d,q}(\widetilde{Q}_R, w)} \leqq N R^\kappa \|h\|_{L_{p_1}(\widetilde{Q}_R)}, \tag{7.14}$$

where $\kappa = \kappa(d, p, r_1, \ldots, r_d, q, K) > 0$ and

$$N = N(d, \delta, p, r_1, \ldots, r_d, q, w_0, w_1, \ldots, w_d) > 0.$$

Note that (7.14) also holds if we replace $\widetilde{Q}_R$ with $\widetilde{Q}_{2R} \setminus \widetilde{Q}_R$.

Next, let $f_n \in C_0^\infty(\mathbb{R}^{1+2d})$ be a sequence of functions such that $f_n \to f$ in $L_{p,r_1,\ldots,r_d,q}(\mathbb{R}^{1+2d}, w)$ as $n \to \infty$. Now we fix some $n \in \{1, 2, \ldots\}$. We may assume that f_n vanishes outside $\widetilde{Q}_R$ for some $R > 1$ depending on n. Let us consider the equation

$$Pu_n + b^i D_{v_i} u_n + c u_n + \lambda u_n = f_n. \tag{7.15}$$

By Proposition 7.3, this equation has a unique solution $u_n \in L_{p_1}(\mathbb{R}^{1+2d})$. Then, by (7.14), $u_n \in S_{p,r_1,\ldots,r_d,q,\text{loc}}(\mathbb{R}^{1+2d}, w)$. Furthermore, denote

$$X_j = L_{p,r_1,\ldots,r_d,q}(\widetilde{Q}_{2^{j+1}R} \setminus \widetilde{Q}_{2^j R}, w), \quad j \geqq 0.$$

Then, by Lemma 7.4 and (7.14) with R replaced with $2^j R$, we have

$$\lambda \|u_n\|_{X_j} + \lambda^{1/2} \|D_v u_n\|_{X_j} + \|D_v^2 u_n\|_{X_j}$$
$$\leqq N 2^{j\kappa} R^\kappa \left(\lambda \|u_n\|_{L_{p_1}(\widetilde{Q}_{2^{j+1}R} \setminus \widetilde{Q}_{2^j R})} + \lambda^{1/2} \|D_v u_n\|_{L_{p_1}(\widetilde{Q}_{2^{j+1}R} \setminus \widetilde{Q}_{2^j R})} \right)$$

$$+ \ \|D_v^2 u_n\|_{L_{p_1}(\widetilde{Q}_{2^{j+1}R} \setminus \widetilde{Q}_{2^j R})})$$
$$\leq N 2^{-j(j-1)/4 + j\kappa} R^{\kappa-j} \|f_n\|_{L_{p_1}(\mathbb{R}^{1+2d})}.$$

Summing up the above inequality with respect to j, we conclude that

$$u_n \in S_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}, w).$$

Then, by the a priori estimate of Theorem 2.6 (i), $\{u_n, n \geq 1\}$ is a Cauchy sequence in $S_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}, w)$. Therefore, this sequence has a limit

$$u \in S_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}, w).$$

Finally, passing to the limit in (7.15), we prove that u is the solution to Eq. (2.6).

In the case when $T < \infty$, we consider the equation

$$Pu_1 + b^i D_{v_i} u_1 + cu_1 + \lambda u_1 = f 1_{t < T}.$$

By the above, this equation has a unique solution $u_1 \in S_{p,r_1,\dots,r_d,q}(\mathbb{R}^{1+2d}, w)$, which is the solution to Eq. (2.6). The theorem is proved.   □

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Appendix A

Lemma A.1. *Let $\sigma > 0$, $R > 0$, $p \geq 1$ be numbers, and $f \in L_{p,loc}(\mathbb{R}^d)$. Denote*

$$g(x) = \int_{|y| > R^3} f(x + y)|y|^{-(d+\sigma)} \, \mathrm{d}y.$$

Then,

$$(|g|^p)_{B_{R^3}}^{1/p} \leq N(d, \sigma) R^{-3\sigma} \sum_{k=0}^{\infty} 2^{-3k\sigma} (|f|^p)_{B_{(2^k R)^3}}^{1/p}.$$

Proof. By Hölder's inequality for any $x \in B_{R^3}$, we have

$$|g(x)| \leqq \sum_{k=0}^{\infty} \int_{2^{3k} R^3 < |y| < 2^{3(k+1)} R^3} |f|(x+y)|y|^{-(d+\sigma)} \, \mathrm{d}y$$

$$\leqq N(d) \sum_{k=0}^{\infty} 2^{-3k\sigma} R^{-3\sigma} \left(\fint_{2^{3k} R^3 < |y| < 2^{3(k+1)} R^3} |f|^p(x+y) \, \mathrm{d}y \right)^{1/p}.$$

Taking the L_p-average of the both sides of the last inequality over B_{R^3} and using the Minkowski inequality, we prove the assertion of this lemma. $\quad\square$

Lemma A.2. *Let $p > 1$ be a number, $w \in A_p(\mathbb{R}^d)$, and $f \in L_p(\mathbb{R}^d, w)$. Then, there exists a number $p_0 > 1$ depending only on d, p, and $[w]_{A_p}$ such that $f \in L_{p_0,loc}(\mathbb{R}^d)$.*

Proof. By Corollary 7.2.6 of [19], there exists $q \in (1, p)$ depending only on p, d, and $[w]_{A_p(\mathbb{R}^d)}$ such that $w \in A_q(\mathbb{R}^d)$. Let $p_0 = p/q$. Then, by this and Hölder's inequality for any cube C,

$$\int_C |f|^{p_0} \, \mathrm{d}x \leqq \left(\int_C |f|^p w \, \mathrm{d}x \right)^{1/q} \left(\int_C w^{-1/(q-1)} \, \mathrm{d}x \right)^{(q-1)/q} < \infty.$$

The lemma is proved. $\quad\square$

For numbers $p_1, \ldots, p_d \in (1, \infty)$, by $L_{p_1,\ldots,p_d}(w_1, \ldots, w_d)$ we denote the space of measurable functions with the finite norm

$$\|f\|_{L_{p_1,\ldots,p_d}(w_1,\ldots,w_d)}$$
$$= \Big| \int_{\mathbb{R}} | \ldots | \int_{\mathbb{R}} \Big| \int_{\mathbb{R}} |f|^{p_1}(x) \, w_1(x_1) \mathrm{d}x_1 \Big|^{\frac{p_2}{p_1}} \ldots w_d(x_d) \mathrm{d}x_d \Big|^{\frac{1}{p_d}}.$$

Furthermore, by $W^2_{p_1,\ldots,p_d}(w_1, \ldots, w_d)$ we mean the Sobolev space of all functions $u \in L_{p_1,\ldots,p_d}(w_1, \ldots, w_d)$ such that $D_x u, D_x^2 u \in L_{p_1,\ldots,p_d}(w_1, \ldots, w_d)$.

Lemma A.3. *(Interpolation inequality) Let $p_1, \ldots, p_d \in (1, \infty)$ be arbitrary numbers and $w_i \in A_{p_i}(\mathbb{R})$, $i = 1, \ldots, d$, such that $[w_i]_{A_{p_i}(\mathbb{R})} \leqq K$, $i = 1, \ldots, d$, for some $K \geqq 1$. Then, for any $u \in W^2_{p_1,\ldots,p_d}(w_1, \ldots, w_d)$ and $\varepsilon > 0$, we have*

$$\|D_x u\| \leqq \varepsilon \|D_x^2 u\| + N\varepsilon^{-1} \|u\|,$$

where $\| \cdot \| = \| \cdot \|_{L_{p_1,\ldots,p_d}(w_1,\ldots,w_d)}$ and $N = N(d, p_1, \ldots, p_d, K)$.

Proof. First, by Lemma 3.8 (iii) of [17], for any $w \in A_{p_1}(\mathbb{R}^d)$ and $\varepsilon > 0$, one has

$$\int |D_x u|^{p_1} w(x) \, \mathrm{d}x \leqq N \int |g|^{p_1} w(x) \, \mathrm{d}x,$$

where

$$g(x) = \varepsilon |D_x^2 u| + \varepsilon^{-1} |u|$$

and $N = N(d, p_1, [w]_{A_{p_1}(\mathbb{R}^d)})$. Applying a variant of the Rubio de Francia extrapolation theorem (see, for example, Theorem 7.11 of [17] and also [29]), we prove the lemma. $\quad\square$

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