PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 151, Number 5, May 2023, Pages 2045–2055
https://doi.org/10.1090/proc/16326
Article electronically published on February 17, 2023

NOTE ON GREEN'S FUNCTIONS OF NON-DIVERGENCE ELLIPTIC OPERATORS WITH CONTINUOUS COEFFICIENTS

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(Communicated by Ryan Hynd)

Abstract. We improve a result in Kim and Lee [Ann. Appl. Math. 37 (2021), pp. 111–130], showing that if the coefficients of an elliptic operator in non-divergence form are of Dini mean oscillation, then its Green's function has the same asymptotic behavior near the pole x_0 as that of the corresponding Green's function for the elliptic equation with constant coefficients frozen at x_0.

1. Introduction and main results

We consider an elliptic operator L in non-divergence form

$$Lu = a^{ij}(x)D_{ij}u,$$

where the coefficients $\mathbf{A} := (a^{ij})$ are symmetric and satisfy the ellipticity condition:

$$\lambda|\xi|^2 \le a^{ij}(x)\xi^i\xi^j \le \Lambda|\xi|^2, \quad \forall x \in \Omega, \tag{1.1}$$

where λ and Λ are positive constants and Ω is a domain in $\mathbb{R}^n$ with $n \ge 3$. Here and below, we use the usual summation convention over repeated indices.

It is well known that the Green's function $G(x,y)$ for an elliptic operator in divergence form has the pointwise bound

$$G(x,y) \le C|x-y|^{2-n} \qquad (x, y \in \Omega,\ x \ne y) \tag{1.2}$$

even when the coefficients are merely measurable; see e.g. [13, 14, 18]. However, the Green's function for an elliptic operator in non-divergence form does not necessarily enjoy the pointwise bound (1.2) even in the case when the coefficients $\mathbf{A}$ are uniformly continuous; see [1]. Recently, it is shown in [15] that if the coefficients $\mathbf{A}$ are of Dini mean oscillation and if the domain is bounded and has $C^{1,1}$ boundary, then the Green's function exists and satisfies the pointwise bound (1.2). Let us recall the definition of Dini mean oscillation. For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball with radius r centered at x, and write $\Omega(x,r) := \Omega \cap B(x,r)$.

Received by the editors January 9, 2022, and, in revised form, September 3, 2022.

2020 *Mathematics Subject Classification.* Primary 35J08, 35B45; Secondary 35J47.

Key words and phrases. Green's function, non-divergence elliptic equation, Dini mean oscillation.

The first author was partially supported by the Simons Foundation, grant no. 709545, a Simons fellowship, grant no. 007638, and the NSF under agreement DMS-2055244.

The second author was partially supported by National Research Foundation of Korea (NRF) Grant No. NRF-2019R1A2C2002724 and No. NRF-2022R1A2C1003322.

We denote

$$\omega_{\mathbf{A}}(r,x) := \fint_{\Omega(x,r)} |\mathbf{A}(y) - \bar{\mathbf{A}}_{\Omega(x,r)}|\,dy, \quad \text{where } \bar{\mathbf{A}}_{\Omega(x,r)} := \fint_{\Omega(x,r)} \mathbf{A},$$

and we write

$$\omega_{\mathbf{A}}(r,D) := \sup_{x\in D} \omega_{\mathbf{A}}(r,x) \quad \text{and} \quad \omega_{\mathbf{A}}(r) = \omega_{\mathbf{A}}(r,\overline{\Omega}).$$

We say that $\mathbf{A}$ is of Dini mean oscillation in Ω if $\omega_{\mathbf{A}}(r)$ satisfies the Dini's condition

$$\int_0^1 \frac{\omega_{\mathbf{A}}(t)}{t}\,dt < +\infty. \tag{1.3}$$

Before proceeding further, let us clarify the relationship between Dini continuity and Dini mean oscillation. We say that $\mathbf{A}$ is Dini continuous at $x_0 \in \Omega$ if

$$\varrho_{\mathbf{A}}(r,x_0) = \sup\{|\mathbf{A}(x) - \mathbf{A}(x_0)| : x \in \Omega,\ |x-x_0| \le r\} \tag{1.4}$$

satisfies the Dini's condition

$$\int_0^1 \frac{\varrho_{\mathbf{A}}(t,x_0)}{t}\,dt < +\infty$$

and that $\mathbf{A}$ is uniformly Dini continuous in Ω if

$$\varrho_{\mathbf{A}}(r) := \sup_{x\in\Omega} \varrho_{\mathbf{A}}(r,x) \quad \text{satisfies} \quad \int_0^1 \frac{\varrho_{\mathbf{A}}(t)}{t}\,dt < +\infty.$$

It is clear that if $\mathbf{A}$ is uniformly Dini continuous in Ω, then $\mathbf{A}$ is of Dini mean oscillation in Ω. If $\mathbf{A}$ is of Dini mean oscillation in Ω, then there is a modification $\bar{\mathbf{A}}$ of $\mathbf{A}$ (i.e., $\bar{\mathbf{A}} = \mathbf{A}$ a.e.) such that $\bar{\mathbf{A}}$ is uniformly continuous in Ω with its modulus of continuity controlled by $\omega_{\mathbf{A}}$. See [15, Appendix] for a proof. However, a function of Dini mean oscillation is not necessarily Dini continuous; see [7] for an example.

In a recent article [16], we gave an alternative proof of the above-mentioned result in [15] and established a sharp comparison with the Green's function for a constant coefficient operator. In particular, we showed that

$$G(x_0,x) - G_{x_0}(x_0,x) = o(|x-x_0|^{2-n}) \text{ as } x \to x_0, \tag{1.5}$$

where G_{x_0} is the Green's function of the constant coefficient operator L_{x_0} given by

$$L_{x_0}u := a^{ij}(x_0)D_{ij}u, \tag{1.6}$$

provided that the mean oscillation of $\mathbf{A}$ satisfies so-called "double Dini condition" near x_0, that is, we have

$$\int_0^1 \frac{1}{s}\int_0^s \frac{\omega_{\mathbf{A}}(t,\Omega(x_0,r_0))}{t}\,dt\,ds = \int_0^1 \frac{\omega_{\mathbf{A}}(t,\Omega(x_0,r_0))\ln\frac{1}{t}}{t}\,dt < +\infty \tag{1.7}$$

for some $r_0 > 0$. It should be noted that the argument in [15, Appendix] reveals that if $\mathbf{A}$ satisfies the double Dini mean oscillation condition (1.7), then there exists a modification of $\mathbf{A}$ that is Dini continuous in a neighborhood of x_0. The asymptotic behavior (1.5) is well known for the Green's functions for elliptic operators in divergence form with continuous coefficients; see [5]. However, in the non-divergence form setting, this is a new result and it is one of the novelties in [16].

The aim of this note is to show that one can replace the condition (1.7) with a simple condition that $\mathbf{A}$ is continuous at x_0, that is,

$$\lim_{r\to 0} \varrho_{\mathbf{A}}(r,x_0)=0. \tag{1.8}$$

Notice that the coefficient matrix $\mathbf{A}=(a^{ij})$, which is assumed to be of Dini mean oscillation in Ω, has a modification that is uniformly continuous in Ω. Therefore, without loss of generality, we shall hereafter assume that $\mathbf{A}$ is uniformly continuous in Ω so that (1.8) holds for all $x_0\in\Omega$.

We now state our main theorems.

Theorem 1.9. *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ with $n\ge 3$. Assume the coefficients $\mathbf{A}=(a^{ij})$ of the operator L satisfy the condition* (1.1) *and are of Dini mean oscillation in Ω. Then there exists the Green's function $G(x,y)$ of the operator L in Ω and it satisfies the pointwise bound* (1.2). *Moreover, for any $x_0\in\Omega$, we have*

$$\lim_{x\to x_0} \lvert x-x_0\rvert^{n-2}\,\lvert G(x_0,x)-G_{x_0}(x_0,x)\rvert=0, \tag{1.10}$$

where G_{x_0} is the Green's function of the constant coefficient operator L_{x_0} as in (1.6).

Theorem 1.11. *Under the same hypothesis of Theorem* 1.9, *we also have*

$$\lim_{x\to x_0} \lvert x-x_0\rvert^{n-2}\,\lvert G(x,x_0)-G_{x_0}(x,x_0)\rvert=0, \tag{1.12}$$

$$\lim_{x\to x_0} \lvert x-x_0\rvert^{n-1}\,\lvert D_xG(x,x_0)-D_xG_{x_0}(x,x_0)\rvert=0, \tag{1.13}$$

$$\lim_{x\to x_0} \lvert x-x_0\rvert^{n}\,\lvert D_x^2G(x,x_0)-D_x^2G_{x_0}(x,x_0)\rvert=0, \tag{1.14}$$

for any $x_0\in\Omega$.

Remark 1.15. As a matter of fact, the proof of Theorem 1.9 shall reveal that we have

$$\lim_{x\to x_0} \lvert x-x_0\rvert^{n-2}\,\lvert G^*(x,x_0)-G^*_{x_0}(x,x_0)\rvert=0,$$

where G^* and $G^*_{x_0}$ are the Green's functions for the adjoint operators L^* and $L^*_{x_0}$, respectively. Therefore, (1.12) can be regarded as the adjoint version of (1.10), and vice versa.

Remark 1.16. We note that the estimates (1.10) and (1.12) are reminiscent of [5, Eq. (23)]. However, the method used in [5] is not applicable to our setting because a local L^∞ estimate corresponding to [5, Lemma 4] is not available via L^p theory for adjoint solutions of $L^*u=\operatorname{div}^2\mathbf{f}$. Nevertheless, under the stronger condition that $\mathbf{A}\in C^\beta$ for some $\beta\in(0,1)$, similar to [5, Theorem 2], we obtain

$$\lvert G(x_0,x)-G_{x_0}(x_0,x)\rvert = O\left(\lvert x-x_0\rvert^{2-n+\beta}\right) \quad \text{as} \quad x\to x_0,$$
$$\lvert D_x^kG(x,x_0)-D_x^kG_{x_0}(x,x_0)\rvert = O\left(\lvert x-x_0\rvert^{2-n-k+\beta}\right) \quad \text{as} \quad x\to x_0 \quad (k=0,1,2).$$

Indeed, by modifying the proof of [5, Theorem 2] and using the global Lorentz type estimate [5, Lemmas 1 and 2] and interior Schauder estimates for the solutions of $L^*u=\operatorname{div}^2\mathbf{f}$ and $Lu=f$, one can get the above estimates.

A few further remarks are in order. We point out that the proofs for Theorems 1.9 and 1.11 are readily extendable to elliptic systems, but they do not work in the two-dimensional setting. See [6] for the results regarding the two-dimensional case as well as for a discussion on how to extend them to elliptic systems. In [16], we assume that Ω is a bounded $C^{2,\alpha}$ domain. This assumption was adopted there to obtain a global pointwise estimate for $D_x^2 G(x,y)$ only. To prove that the pointwise bound (1.2) holds globally in Ω, we just need a local L^∞ estimate (valid up to the boundary) for solutions of the adjoint equation

$$L^* u := D_{ij}(a^{ij} u) = 0,$$

which has been established earlier in [7,8]. We particularly refer to [8, Theorem 1.8], where Ω is assumed to be a bounded $C^{1,1}$ domain. However, in the scalar case, it is not even necessary to assume that Ω is a bounded $C^{1,1}$ domain for the global pointwise bound (1.2) to hold. As it will be indicated in the proof of Theorem 1.9, in the scalar case, it is enough to assume that Ω is a bounded $C^{1,\alpha}$ domain with $\alpha > \frac{n-1}{n}$ in order to utilize the L^p solvability of the problem (2.1) for some $p \in (1, \frac{n}{n-2})$. Also, since we deal with the asymptotic estimates (1.10) and (1.12), which are interior estimates in nature, we only need an interior L^∞ estimate for adjoint solutions to establish these estimates.

Finally, we should mention that there are many interesting papers dealing with the Green's functions or the fundamental solutions of non-divergence form elliptic and parabolic operators. See, for example, [2–4, 9, 11, 12, 17, 19] and references therein.

2. Preliminary lemmas

In this section, we present some lemmas which will be used in the proof of Theorems 1.9 and 1.11. We need to consider the boundary value problem

$$L^* v = \operatorname{div}^2 \mathbf{g} + f \ \text{ in } \Omega, \quad v = \frac{\mathbf{g}\nu \cdot \nu}{\mathbf{A}\nu \cdot \nu} \ \text{ on } \partial\Omega, \tag{2.1}$$

where $\mathbf{g} = (g^{ij})$ is an $n \times n$ matrix-valued function,

$$\operatorname{div}^2 \mathbf{g} := D_{ij} g^{ij},$$

and ν is the unit exterior normal vector of $\partial\Omega$. For $\mathbf{g} \in L^p(\Omega)$ and $f \in L^p(\Omega)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, we say that v in $L^p(\Omega)$ is an adjoint solution of (2.1) if v satisfies

$$\int_\Omega vLu = \int_\Omega \operatorname{tr}(\mathbf{g} D^2 u) + \int_\Omega fu$$

for any u in $W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega)$.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain. Let $1 < p < \infty$ and assume that $\mathbf{g} \in L^p(\Omega)$ and $f \in L^p(\Omega)$. Then there exists a unique adjoint solution u in $L^p(\Omega)$. Moreover, we have the estimate*

$$\|u\|_{L^p(\Omega)} \le C \left(\|\mathbf{g}\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} \right),$$

where the constant C depends on Ω, p, n, λ, Λ, and $\omega_{\mathbf{A}}$.

Proof. See [10, Lemma 2]. ∎

Lemma 2.3. *Let $R_0 > 0$ and $\mathbf{g} = (g^{ij})$ be of Dini mean oscillation in $B(x_0, R_0)$. Suppose u is an L^2 solution of*

$$L^* u = \operatorname{div}^2 \mathbf{g} \ \textit{in } B(x_0, 2r),$$

where $0 < r \le \frac{1}{2}R_0$. *Then we have*

$$\|u\|_{L^\infty(B(x_0,r))} \le C \left(\fint_{B(x_0,2r)} |u| + \int_0^r \frac{\omega_{\mathbf{g}}(t, B(x_0,2r))}{t}\, dt \right),$$

where $C = C(n, \lambda, \Lambda, \omega_{\mathbf{A}}, R_0)$.

Proof. See [16, Lemma 2.2]. □

3. Proof of Theorem 1.9

We slightly modify the argument in [16, §3.5]. Let us denote

$$\mathbf{A}_{x_0} = \mathbf{A}(x_0), \quad L_{x_0}u := a^{ij}(x_0)D_{ij}u = \operatorname{tr}(\mathbf{A}_{x_0}D^2u),$$

and let $G_{x_0}(x,y)$ be the Green's function for L_{x_0}. Since L_{x_0} is an elliptic operator with constant coefficients, the existence of G_{x_0} as well as the following pointwise bound is well known:

$$|G_{x_0}(x,y)| \le C|x-y|^{2-n} \qquad (x \ne y), \tag{3.1}$$

where $C = C(n, \lambda, \Lambda)$. Moreover, since $\mathbf{A}_{x_0}$ is constant and symmetric, we have $L_{x_0} = L^*_{x_0}$ and G_{x_0} is also symmetric, i.e.,

$$G_{x_0}(x,y) = G^*_{x_0}(y,x) = G_{x_0}(y,x) \qquad (x \ne y). \tag{3.2}$$

Let us set

$$\mathbf{g} = -(\mathbf{A} - \mathbf{A}_{x_0})G_{x_0}(\cdot, x_0)$$

and consider the problem

$$L^*v = \operatorname{div}^2 \mathbf{g} \text{ in } \Omega, \quad v = \frac{\mathbf{g}\nu \cdot \nu}{\mathbf{A}\nu \cdot \nu} \text{ on } \partial\Omega. \tag{3.3}$$

Notice that the boundary condition in (3.3) simply reads that $v = 0$ in $\partial\Omega$ since the Green's function $G_{x_0}(\cdot, x_0)$ vanishes on $\partial\Omega$.

By using that $\mathbf{A}$ is of Dini mean oscillation in Ω, it is easy to verify that $\mathbf{g} \in L^q(\Omega)$ for $q \in (1, \frac{n}{n-2})$; see [16, Lemma 3.1]. Therefore, by Lemma 2.2, there is a unique v that belongs to $L^q(\Omega)$ for $q \in (1, \frac{n}{n-2})$. Then, by the same argument as in [16, §3.1], we find that

$$G^*(\cdot, x_0) := G_{x_0}(\cdot, x_0) + v \tag{3.4}$$

becomes the Green's function of L^* in Ω with a pole at x_0. Also, by following the same argument as in [16, §3.4], we find that the function $G(x,y)$ given by

$$G(x,y) = G^*(y,x) \qquad (x \ne y)$$

is the Green's function for L in Ω. Note that we only need the assumption that Ω is a bounded $C^{1,1}$ domain to get these conclusions. In fact, in the scalar case, we can even relax this assumption further to Ω being a $C^{1,\alpha}$ domain for some $\alpha > \frac{n-1}{n}$. See [6, Appendix].

Now, for $y_0 \in \Omega$ with $y_0 \ne x_0$, let

$$r = \tfrac{1}{5}|y_0 - x_0| \quad \text{and} \quad \delta = \min(\kappa r, \operatorname{diam}\Omega), \tag{3.5}$$

where $\kappa > 0$ is to be determined. We assume that $r < \frac{1}{7}\operatorname{dist}(x_0, \partial\Omega)$ is so that $B(y_0, 2r) \subset \Omega$. Let ζ be a smooth function on $\mathbb{R}^n$ such that

$$0 \le \zeta \le 1, \quad \zeta = 0 \text{ in } B(x_0, \delta/2), \quad \zeta = 1 \text{ in } \mathbb{R}^n \setminus B(x_0, \delta), \quad |D\zeta| \le 4/\delta.$$

We then define $\mathbf{g}_1$ and $\mathbf{g}_2$ by

$$\mathbf{g}_1 = -\zeta(\mathbf{A} - \mathbf{A}_{x_0})G_{x_0}(\cdot, x_0) \quad \text{and} \quad \mathbf{g}_2 = -(1-\zeta)(\mathbf{A} - \mathbf{A}_{x_0})G_{x_0}(\cdot, x_0).$$

Recall the definition (1.4) and note that $\|\mathbf{A} - \mathbf{A}_{x_0}\|_\infty \le C(n, \Lambda)$.

Choose $p_1 \in (\frac{n}{n-2}, \infty)$ and $p_2 \in (1, \frac{n}{n-2})$; e.g., take $p_1 = \frac{2n}{n-2}$ and $p_2 = \frac{n-1}{n-2}$. By (3.1) and properties of ζ, we then have

$$\int_\Omega |\mathbf{g}_1|^{p_1} \le C \int_{\Omega\setminus\Omega(x_0,\delta/2)} |x-x_0|^{(2-n)p_1}\,dx \le C\delta^{(2-n)p_1+n}, \tag{3.6}$$

$$\int_\Omega |\mathbf{g}_2|^{p_2} \le C\varrho_{\mathbf{A}}(\delta, x_0)^{p_2} \int_{\Omega(x_0,\delta)} |x-x_0|^{(2-n)p_2} dx \le C\varrho_{\mathbf{A}}(\delta,x_0)^{p_2}\delta^{(2-n)p_2+n}. \tag{3.7}$$

Let v_i be the solutions of the problems

$$L^* v_i = \operatorname{div}^2 \mathbf{g}_i \text{ in } \Omega, \quad v_i = \frac{\mathbf{g}_i\nu\cdot\nu}{\mathbf{A}\nu\cdot\nu} \text{ on } \partial\Omega \qquad (i=1,2).$$

By Lemma 2.2 together with (3.6) and (3.7), we have

$$\|v_1\|_{L^{p_1}(\Omega)} \le C\delta^{2-n+\frac{n}{p_1}} \qquad \text{and} \qquad \|v_2\|_{L^{p_2}(\Omega)} \le C\varrho_{\mathbf{A}}(\delta,x_0)\delta^{2-n+\frac{n}{p_2}}. \tag{3.8}$$

Since $\mathbf{g}_1$ and $\mathbf{g}_2$ also belong to $L^q(\Omega)$ for $q \in (1, \frac{n}{n-2})$, we have $v_1 + v_2 \in L^q(\Omega)$ for $q \in (1, \frac{n}{n-2})$. Therefore, we conclude that

$$v = v_1 + v_2$$

by the uniqueness of solutions to the problem (3.3).

Now, we estimate $v_1(y_0)$ and $v_2(y_0)$ as follows. By Lemma 2.3, we have

$$|v_i(y_0)| \le C \fint_{B(y_0,2r)} |v_i| + C\int_0^r \frac{\omega_{\mathbf{g}_i}(t, B(y_0,2r))}{t}\,dt \qquad (i=1,2). \tag{3.9}$$

Using (3.8) together with Hölder's inequalities, we have

$$\begin{aligned} \fint_{B(y_0,2r)} |v_1| &\le Cr^{-\frac{n}{p_1}}\|v_1\|_{L^{p_1}(B(y_0,2r))} \le Cr^{-\frac{n}{p_1}}\delta^{2-n+\frac{n}{p_1}}, \\ \fint_{B(y_0,2r)} |v_2| &\le Cr^{-\frac{n}{p_2}}\|v_2\|_{L^{p_2}(B(y_0,2r))} \le C\varrho_{\mathbf{A}}(\delta,x_0)r^{-\frac{n}{p_2}}\delta^{2-n+\frac{n}{p_2}}. \end{aligned} \tag{3.10}$$

The following technical lemma is a variant of [16, Lemma 3.2]. In fact, the proof is much shorter under the assumption that $\mathbf{A}$ is continuous at x_0.

Lemma 3.11. *Let η be a Lipschitz function on $\mathbb{R}^n$ such that $0 \le \eta \le 1$ and $|D\eta| \le 4/\delta$ for some $\delta > 0$. Let*

$$\mathbf{g} = -\eta(\mathbf{A} - \mathbf{A}_{x_0})G_{x_0}(\cdot, x_0)$$

and for $y_0 \in \Omega$ with $y_0 \ne x_0$, let $r = \frac{1}{5}|x_0 - y_0|$. Then, for any $t \in (0, r]$ we have

$$\omega_{\mathbf{g}}(t, \Omega(y_0, 2r)) \le Cr^{2-n}\left(\omega_{\mathbf{A}}(t) + \frac{r+\delta}{\delta r}\,\varrho_{\mathbf{A}}(8r, x_0)t\right),$$

where $C = C(n, \lambda, \Lambda, \Omega)$.

Proof. For $\bar{x} \in \Omega(y_0, 2r)$ and $0 < t \le r$, we have

$$\begin{aligned}
\omega_{\mathbf{g}}(t, \bar{x}) &= \fint_{\Omega(\bar{x},t)} \left|(\mathbf{A} - \mathbf{A}_{x_0})G_{x_0}(\cdot, x_0)\eta - \overline{((\mathbf{A} - \mathbf{A}_{x_0})G_{x_0}(\cdot, x_0)\eta)}_{\Omega(\bar{x},t)}\right| \\
&\le \fint_{\Omega(\bar{x},t)} \left|(\mathbf{A} - \mathbf{A}_{x_0})G_{x_0}(\cdot, x_0)\eta - \overline{(\mathbf{A} - \mathbf{A}_{x_0})}_{\Omega(\bar{x},t)} G_{x_0}(\cdot, x_0)\eta\right| \\
&\quad + \fint_{\Omega(\bar{x},t)} \left|\overline{(\mathbf{A} - \mathbf{A}_{x_0})}_{\Omega(\bar{x},t)} G_{x_0}(\cdot, x_0)\eta - \overline{((\mathbf{A} - \mathbf{A}_{x_0})G_{x_0}(\cdot, x_0)\eta)}_{\Omega(\bar{x},t)}\right| \\
&=: I + II.
\end{aligned}$$

Observe that we have $\operatorname{dist}(x_0, \Omega(\bar{x}, t)) \ge 2r$ and thus for x, $y \in \Omega(\bar{x}, t)$, by (3.1) and the interior gradient estimate for elliptic equations with constant coefficients, we have

$$|G_{x_0}(x, x_0) - G_{x_0}(y, x_0)| \le Ctr^{1-n}, \tag{3.12}$$

where $C = C(n, \lambda, \Lambda, \Omega)$. Since $\operatorname{dist}(x_0, \Omega(\bar{x}, t)) \ge 2r$, by using (3.1) we obtain

$$\begin{aligned}
I &\le \fint_{\Omega(\bar{x},t)} \left|(\mathbf{A} - \mathbf{A}_{x_0}) - \overline{(\mathbf{A} - \mathbf{A}_{x_0})}_{\Omega(\bar{x},t)}\right| |G_{x_0}(\cdot, x_0)| \\
&\le \fint_{\Omega(\bar{x},t)} Cr^{2-n} |\mathbf{A} - \bar{\mathbf{A}}_{\Omega(\bar{x},t)}| \le Cr^{2-n} \omega_{\mathbf{A}}(t).
\end{aligned} \tag{3.13}$$

Also, we have

$$\begin{aligned}
II &\le \fint_{\Omega(\bar{x},t)} \left| \fint_{\Omega(\bar{x},t)} (\mathbf{A}(y) - \mathbf{A}(x_0)) \left(G_{x_0}(x, x_0)\eta(x) - G_{x_0}(y, x_0)\eta(y)\right) dy \right| dx \\
&\le \fint_{\Omega(\bar{x},t)} \fint_{\Omega(\bar{x},t)} |\mathbf{A}(y) - \mathbf{A}(x_0)| \, |G_{x_0}(x, x_0)\eta(x) - G_{x_0}(y, x_0)\eta(y)| \, dy \, dx.
\end{aligned} \tag{3.14}$$

By using (3.1), (3.12) and $|D\eta| \le 4/\delta$, we have for x, $y \in \Omega(\bar{x}, t)$ that

$$\begin{aligned}
&|G_{x_0}(x, x_0)\eta(x) - G_{x_0}(y, x_0)\eta(y)| \\
&\quad \le |G_{x_0}(x, x_0) - G_{x_0}(y, x_0)| \, |\eta(x)| + |G_{x_0}(y, x_0)| \, |\eta(x) - \eta(y)| \\
&\quad \le Ctr^{1-n} + Cr^{2-n} t/\delta \le Ctr^{1-n}(1 + r/\delta).
\end{aligned} \tag{3.15}$$

Plugging (3.15) into (3.14) and noting that for $y \in \Omega(\bar{x}, t)$, we have

$$|y - x_0| \le |y - \bar{x}| + |\bar{x} - y_0| + |y_0 - x_0| < t + 2r + 5r = 8r,$$

we obtain

$$II \le Ctr^{1-n}(1 + r/\delta)\varrho_{\mathbf{A}}(8r, x_0). \tag{3.16}$$

Combining (3.13) and (3.16), we have (recall $t \le r$)

$$\omega_{\mathbf{g}}(t, \bar{x}) \le I + II \le Cr^{2-n} \left(\omega_{\mathbf{A}}(t) + t(1/r + 1/\delta)\varrho_{\mathbf{A}}(8r, x_0)\right).$$

The lemma is proved by taking supremum over $\bar{x} \in \Omega(y_0, 2r)$. $\square$

By applying Lemma 3.11 with $\eta = \zeta$ and $\eta = 1 - \zeta$, respectively, we have

$$\int_0^r \frac{\omega_{\mathbf{g}_i}(t, B(y_0, 2r))}{t} \, dt \le Cr^{2-n} \left(\int_0^r \frac{\omega_{\mathbf{A}}(t)}{t} \, dt + \frac{\delta + r}{\delta} \varrho_{\mathbf{A}}(8r, x_0) \right) \quad (i = 1, 2). \tag{3.17}$$

Now we substitute (3.10) and (3.17) back to (3.9) to obtain (recall $v = v_1 + v_2$)

$$
|v(y_0)| \le C r^{2-n} \left(r^{n-2-\frac{n}{p_1}} \delta^{2-n+\frac{n}{p_1}} + \varrho_{\mathbf{A}}(\delta, x_0) r^{n-2-\frac{n}{p_2}} \delta^{2-n+\frac{n}{p_2}} \right.
\left. + \int_0^r \frac{\omega_{\mathbf{A}}(t)}{t}\, dt + \frac{\delta + r}{\delta} \varrho_{\mathbf{A}}(8r, x_0) \right). \tag{3.18}
$$

For any $\epsilon > 0$, we can choose κ sufficiently large so that $\kappa^{2-n+\frac{n}{p_1}} < \epsilon/2$. Then choose r_0 sufficiently small so that $r_0 < \min\left(\frac{1}{7}\operatorname{dist}(x_0, \partial\Omega), \frac{5}{\kappa} \operatorname{diam}\Omega\right)$ and

$$
\varrho_{\mathbf{A}}(\kappa r_0, x_0) \kappa^{2-n+\frac{n}{p_2}} + \int_0^{r_0} \frac{\omega_{\mathbf{A}}(t)}{t}\, dt + \frac{\kappa+1}{\kappa} \varrho_{\mathbf{A}}(8r_0, x_0) < \frac{1}{2}\epsilon,
$$

which is possible due to (1.8) and (1.3). Then, it follows from (3.4), (3.18), and (3.5) that if $|y_0 - x_0| < r_0$, then we have

$$
|G^*(y_0, x_0) - G_{x_0}(y_0, x_0)| \le C\epsilon |y_0 - x_0|^{2-n}. \tag{3.19}
$$

Therefore, we conclude that

$$
\lim_{x \to x_0} |x - x_0|^{n-2} \, |G^*(x, x_0) - G_{x_0}(x, x_0)| = 0
$$

because $\epsilon > 0$ is arbitrary and (3.19) holds for any $y_0 \in \Omega$ satisfying $0 < |y_0 - x_0| < r_0$. Finally, we obtain (1.10) from the above by noting (3.2) and the relation $G(x, y) = G^*(y, x)$ for $x \ne y$. $\square$

4. Proof of Theorem 1.11

We modify the proof of Theorem 1.9 as follows. Notice that for $x_0 \ne y_0$ we have

$$
G(y_0, x_0) - G_{x_0}(y_0, x_0) = G^*(x_0, y_0) - G_{x_0}(x_0, y_0),
$$

which suggests that we should consider

$$
\tilde{v} = \tilde{v}(x) = G^*(x, y_0) - G_{x_0}(x, y_0).
$$

Note that $\tilde{v}$ satisfies

$$
L^* \tilde{v} = \operatorname{div}^2 \tilde{\mathbf{g}} \text{ in } \Omega, \quad v = \frac{\tilde{\mathbf{g}}\nu \cdot \nu}{\mathbf{A}\nu \cdot \nu} \text{ on } \partial\Omega,
$$

where

$$
\tilde{\mathbf{g}} = -(\mathbf{A} - \mathbf{A}_{x_0}) G_{x_0}(\cdot, y_0).
$$

Let r and δ be the same as in (3.5), and let $\tilde{\zeta}$ be a smooth function on $\mathbb{R}^n$ such that

$$
0 \le \tilde{\zeta} \le 1, \quad \tilde{\zeta} = 0 \text{ in } B(y_0, \delta/2), \quad \tilde{\zeta} = 1 \text{ in } \mathbb{R}^n \setminus B(y_0, \delta), \quad |D\tilde{\zeta}| \le 4/\delta.
$$

We define $\tilde{\mathbf{g}}_1$ and $\tilde{\mathbf{g}}_2$ by

$$
\tilde{\mathbf{g}}_1 = -\tilde{\zeta}(\mathbf{A} - \mathbf{A}_{x_0}) G_{x_0}(\cdot, y_0) \quad \text{and} \quad \tilde{\mathbf{g}}_2 = -(1 - \tilde{\zeta})(\mathbf{A} - \mathbf{A}_{x_0}) G_{x_0}(\cdot, y_0).
$$

Similar to (3.6) and (3.7), we have

$$
\begin{aligned}
\int_\Omega |\tilde{\mathbf{g}}_1|^{p_1} &\le C \int_{\Omega \setminus \Omega(y_0, \delta/2)} |x - y_0|^{(2-n)p_1}\, dx \le C \delta^{(2-n)p_1 + n}, \\
\int_\Omega |\tilde{\mathbf{g}}_2|^{p_2} &\le C \varrho_{\mathbf{A}}(\delta + 5r, x_0)^{p_2} \int_{\Omega(y_0, \delta)} |x - y_0|^{(2-n)p_2} dx \\
&\le C \varrho_{\mathbf{A}}(\delta + 5r, x_0)^{p_2} \delta^{(2-n)p_2 + n}.
\end{aligned}
$$

We shall assume that $r < \frac{1}{4}\mathrm{dist}(x_0, \partial\Omega)$ is so that $B_{2r}(x_0) \subset \Omega$. Then by essentially the same proof of Lemma 3.11, we have

$$\omega_{\tilde{\mathbf{g}}_i}(t, B(x_0, 2r)) \le Cr^{2-n}\left(\omega_{\mathbf{A}}(t) + \frac{r+\delta}{\delta r}\,\varrho_{\mathbf{A}}(3r, x_0)t\right).$$

Then, similar to (3.18), we get

$$\begin{aligned}|\tilde{v}(x_0)| \le Cr^{2-n}\Bigg(r^{n-2-\frac{n}{p_1}}\delta^{2-n+\frac{n}{p_1}} + \varrho_{\mathbf{A}}(\delta+5r, x_0)r^{n-2-\frac{n}{p_2}}\delta^{2-n+\frac{n}{p_2}} \\ + \int_0^r \frac{\omega_{\mathbf{A}}(t)}{t}\,dt + \frac{\delta+r}{\delta}\varrho_{\mathbf{A}}(3r, x_0)\Bigg).\end{aligned}$$

By using the last inequality in place of (3.18) and proceeding similarly as in the proof of Theorem 1.9, we find that, similar to (3.19), for any $\epsilon > 0$, there exists a positive $r_0 < \frac{1}{4}\mathrm{dist}(x_0, \partial\Omega)$ such that if $|x_0 - y_0| < r_0$, then we have

$$|G^*(x_0, y_0) - G_{x_0}(x_0, y_0)| \le C\epsilon|y_0 - x_0|^{2-n}.$$

Since $\epsilon > 0$ is arbitrary and the last inequality is true for any $y_0 \in \Omega$ satisfying $0 < |y_0 - x_0| < r_0$, we obtain (1.12) from the last inequality by using the relations $G(x, y) = G^*(y, x)$ and $G_{x_0}(x, y) = G_{x_0}(y, x)$ for $x \ne y$.

To establish (1.13), we note that $v := G(\cdot, x_0) - G_{x_0}(\cdot, x_0)$ satisfies

$$Lv = -(\mathbf{A} - \mathbf{A}_{x_0})D^2G_{x_0}(\cdot, x_0) =: f.$$

For any $y_0 \in \Omega$ satisfying $0 < |y_0 - x_0| < \frac{1}{4}\,\mathrm{dist}(x_0, \partial\Omega)$, let $r = \frac{1}{5}|y_0 - x_0|$. Since G_{x_0} is the Green's function of constant coefficient operator L_{x_0} and $B(x_0, 20r) \subset \Omega$, we have

$$|D_x^k G_{x_0}(x, x_0)| \le C|x - x_0|^{2-n-k}, \quad \forall x \in B(x_0, 10r) \setminus \{x_0\},\ k = 1, 2, \ldots.$$

Then we have

$$\|f\|_{L^\infty(B(y_0,3r))} \le C\varrho_{\mathbf{A}}(8r, x_0)r^{-n}. \tag{4.1}$$

Also, by a similar computation as in the proof of Lemma 3.11, we have

$$\omega_f(t, B(y_0, 2r)) \le Cr^{-n}\left(\omega_{\mathbf{A}}(t) + \frac{t}{r}\,\varrho_{\mathbf{A}}(8r, x_0)\right). \tag{4.2}$$

Therefore, by using (1.12), (4.1), the local $W^{2,p}$ estimate

$$r^2\|D^2v\|_{L^p(B_{2r}(y_0))} + r\|Dv\|_{L^p(B_{2r}(y_0))} \le C\|v\|_{L^p(B_{3r}(y_0))} + Cr^2\|f\|_{L^p(B_{3r}(y_0))}, \tag{4.3}$$

and the Sobolev embedding, we have

$$r^{n-1}|Dv(y_0)| = o(r), \tag{4.4}$$

where we use $o(r)$ to denote some bounded quantity that tends to 0 as $r \to 0$.

Moreover, similar to Lemma 2.3, the proof of [7, Theorem 1.6] reveals that

$$\|D^2v\|_{L^\infty(B(y_0,r))} \le C\left(\fint_{B(y_0,2r)}|D^2v| + \int_0^r \frac{\omega_f(t, B(y_0, 2r))}{t}\,dt\right).$$

Therefore, by (4.1), (4.3), and (4.2), we have

$$r^n|D^2v(y_0)| = o(r). \tag{4.5}$$

Now, (1.13) and (1.14) follow from (4.4) and (4.5), respectively, recalling that $v = G(\cdot, x_0) - G_{x_0}(\cdot, x_0)$ and $r = \frac{1}{5}|x_0 - y_0|$. ∎

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